Distributed optimal coordination across multiple decision-makers and its application to demand response in smart grids

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**DISTRIBUTED OPTIMAL COORDINATION ACROSS
MULTIPLE DECISION-MAKERS AND ITS APPLICATION TO
DEMAND RESPONSE IN SMART GRIDS**

YE MAOJIAO

SCHOOL OF ELECTRICAL & ELECTRONIC ENGINEERING

2016

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RESPONSE IN SMART GRIDS**

YE MAOJIAO

YE MAOJIAO

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

Date

YE MAOJIAO

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Abstract

With the prevalence of coupling engineering systems such as multi-agent systems, wireless networks and the smart grid, solving optimization problems across multiple decision-makers emerges to be of theoretical significance and practical relevance. Resource allocation for systems with limited resources, energy consumption control for electricity users, and economic dispatch among a network of generators in smart grids are typical examples of multi-decision-maker optimization problems. These optimization problems are explored from the perspectives of non-cooperative games and networked optimization in this dissertation. The background, introduction and motivation for this topic are included in the first two chapters. The specific problems considered in the dissertation are summarized as follows.

Energy consumption control among a network of electricity users in smart grids is considered. An aggregate game approach is proposed to solve it by noticing that the aggregate energy consumption is unknown to the electricity users during the energy consumption scheduling process. Average consensus based updating strategies are developed to search for the Nash equilibrium of the energy consumption game. A general energy consumption game with possibly multiple Nash equilibria is firstly explored. In this case, the Nash equilibrium is exponentially stable under the given strategy. In particular, energy consumption game (for heating ventilation and air conditioning systems) that admits a unique Nash equilibrium is then investigated. Based on the uniqueness of the Nash equilibrium, a non-local convergence result is

derived. Furthermore, energy consumption games with stubborn players are studied. With the existence of stubborn players, the rational players' actions converge to a neighborhood of the best response strategies. By utilizing the proposed methods, the electricity users only need to communicate with their neighbors about their estimates on the averaged aggregate energy consumption. No private information is exchanged among the electricity users. Hence, the proposed methods are free of privacy concern.

Since the Nash equilibrium may not be efficient from the system-level perspective, distributed optimization problems are investigated. In Chapter 4, a distributed extremum seeking framework is proposed to solve networked optimization problems. Under some conditions, the proposed method enables the agents' strategies to converge to the optimal solution without using the explicit expressions of the constraints functions, the cost functions or their gradients.

Noticing that many engineering systems exhibit time-varying characteristic while continuous-time time-varying distributed optimization remains as an open problem, we consider it for systems with quadratic objective functions in this dissertation. Time-varying distributed optimization with neighboring coupled objective functions is firstly considered. A robust gradient based method is proposed to solve it. More generally coupled distributed optimization problems are then investigated by penalty function based methods. The penalty function based methods can be utilized to approximate the solution of the distributed optimization problem.

The last problem considered is seeking the solution to non-cooperative games without using explicit model information. Different from existing works that regard the Nash equilibrium as a "fixed point", Nash equilibrium is treated as a time-varying trajectory in this dissertation. The time-varying Nash equilibrium seeking problem is addressed by using an extremum seeking approach. Sinusoidal excitation signals are utilized in the extremum seeking loop to modulate and demodulate the

players' actions. A delay-based subsystem is employed to estimate the gradient and a robust tracking strategy is designed to search for the Nash equilibrium based on the estimated gradients. For symmetric quadratic games, it is shown that the players' actions converge asymptotically to the Nash equilibrium trajectory. For more general quadratic games, a bounded convergence result is presented.

In summary, this dissertation studies optimization problems across multiple decision-makers. If the decision-makers are self-interested, the problem may be modeled as non-cooperative games. Nash equilibrium seeking for non-cooperative games are considered. To achieve the system-level objective, networked optimization is investigated. An extremum seeking approach is proposed to solve it which is followed by some explorations on time-varying distributed optimization problems. To demonstrate the practical relevance of the non-cooperative games and the networked optimization problems, energy consumption control for electricity users in smart grids is studied.

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Symbols and Acronyms

$\text{diag}\{\lambda_{ij}\}$	For $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$, it denotes an $mn \times mn$ dimensional diagonal matrix defined as
$\text{diag}\{\lambda_{ij}\} =$	$\begin{bmatrix} \lambda_{11} & 0 & \cdots & 0 \\ 0 & \lambda_{12} & & \\ & & \ddots & \vdots \\ \vdots & & & \lambda_{1n} \\ & & & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{mn} \end{bmatrix}$
$[h_i]_{vec}$	For $i \in \{1, 2, \dots, m\}$, it denotes $[h_1, h_2, \dots, h_m]^T$
$[h_{ij}]_{vec}$	For $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$, it denotes $[h_{11}, h_{12}, \dots, h_{1n}, h_{21}, \dots, h_{mn}]^T$
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{++}$	The set of positive real numbers.
$\mathbb{R}_+$	The set of non-negative real numbers.
G^T	The transpose of G .
$\otimes$	Kronecker product.
$\mathbb{N}$	$\{1, 2, \dots, N\}$.

$\mathcal{N}_i$	The neighboring set of decision-maker i .
ADMM	Alternating Direction Method of Multipliers
TOU	Time-Of-Use
CPP	Critical Peak Pricing
RTP	Real Time Pricing
EMC	Energy Management Controller
AMI	Advanced Metering Infrastructure
KKT	Karush–Kuhn–Tucker
HVAC	Heating Ventilation and Air Conditioning
ESC	Extremum Seeking Control
FERC	Federal Energy Regulatory Commission
PAR	Peak-to-Average Ratio

Chapter 1

Introduction

1.1 Background, Motivation and Objectives

The prevalence of coupling engineering systems (e.g., multi-agent systems, wireless communication networks and the smart grid) endows the researchers with strong motivations and incentives to study optimization problems across multiple decision-makers. Roughly speaking, optimization problems fall basically into centralized optimization, decomposition optimization, distributed optimization and non-cooperative games as depicted in Figure 1.1 [1]. Among these four categories of optimization problems, we focus on non-cooperative games and distributed optimization problems in this dissertation.

Game theory, which studies the cooperation and confliction among multiple rational decision-makers, can be utilized to analyze a large class of engineering systems (e.g., wireless communication networks, smart grids). The three components that construct a game in a normal form are the players, the players' actions and their payoffs/costs [85]. In particular, if no cooperation other than self-enforcing cooperation exists among the players, the game is a non-cooperative game. In non-cooperative games, the decision-makers are modeled as players that are self-

Figure 1.1: Categorization of optimization problems [1].

interested to minimize/maximize their own cost/payoff values. Non-cooperative games have been widely adopted to analyze engineering systems with limited resources (see, e.g., [2–8]). Related problems include resource allocation [2], power control in networks [3], network defence strategy design [4], intrusion detection [5], charging coordination among plug-in electric vehicles [6], and energy management control [7,8].

For non-cooperative games, the solution is the Nash equilibrium. Nash equilibrium is an action profile on which no player can gain more payoff by unilaterally changing their own actions [9]. However, Nash equilibrium is usually not efficient from the system-level perspective and the inefficiency is captured by the price of anarchy [9]. If coordination and cooperation are allowed across the decision-makers, a more efficient solution is the socially optimal solution.

The objective of distributed optimization is to find the solution to the minimization of the sum of the agents' costs by using local communication and coordination among the agents, i.e., a network of N agents are coordinating with each other to

solve

$$\min_x f(x) = \sum_{i=1}^N f_i(x), \quad (1.1)$$

where $f_i(x)$ is the local cost function for agent i and x is the vector of the decision variables. This problem is common in multi-agent systems where the agents can communicate with their neighbors. Flight formation [10], range localization in networked systems [11], economic dispatch among a network of generators in a smart grid, consensus problems in multi-agent systems [12], can be formulated as distributed optimization problems.

Despite the wide applications of both the non-cooperative game theory and distributed optimization techniques, we focus on their applications to the analysis and design of the price-based demand response schemes in smart grids. Demand response schemes are developed to motivate the consumers to adjust their electricity consumption modes under smart pricing tariffs in smart grids [13]. Time-Of-Use (TOU), Critical Peak Pricing (CPP) and Real-Time Pricing (RTP) are typical price-based demand response schemes among which RTP requires maximal customer participation [13]. The effectiveness of the demand response schemes on reducing or shifting electricity consumption motivates many researchers to design more effective price-based demand response schemes.

The simplified illustration of the electricity market is shown in Figure 1.2. The energy providers buy electricity from the generation's side in the wholesale market and sells it to the electricity consumers in the retail market. The electricity consumers can be categorized as price takers and price-anticipating users (see e.g., [15] and the references therein). The price takers do not consider the effect of their energy consumptions on the electricity price while the price-anticipating users take the effect of their energy consumptions on the price into account. Considering the effect of the aggregate energy consumption on the electricity price gives a coupled optimization problem among the electricity users. Hence, non-cooperative games and distributed

Figure 1.2: Simplified illustration of the electricity buying and selling model.

optimization algorithms are effective tools for the design and analysis of demand response programs in smart grids.

Motivated by both the theoretical and practical importance of studying optimal coordination problems across multiple decision-makers, the main focuses and objectives of this dissertation are summarized as follows.

1. The analysis of non-cooperative games and their solution seeking, i.e., Nash equilibrium seeking. This problem is mainly considered for circumstances where the decision-makers are self-interested to minimize/maximize their own cost/payoff functions.
2. The development of networked optimization algorithms and their convergence analysis. Different from the players in non-cooperative games, the decision-makers involved in distributed optimization problems coordinate with each other to achieve the system-level objective. The system-level objective function is unavailable to the decision-makers but their own local cost/payoff func-

tions can be utilized. The coordination and communication are achieved via a communication graph across the decision-makers.

3. To demonstrate the practical relevance of the considered optimization problems and the effectiveness of the proposed methods, energy consumption control for electricity consumers in smart grids is investigated from the perspectives of game theory and distributed optimization, respectively. Game theory is a useful tool to analyze energy consumption control. However, the efficiency of the Nash equilibrium, which is the solution of the non-cooperative games, is not ensured from the system-level perspective. Hence, when coordination and cooperation are allowed, socially optimal solution is more preferable from the system-level perspective. The electricity users are self-interested in nature, but the cooperation can be driven by designing some incentive provoking mechanisms.

In summary, the main aim of this dissertation is to study optimal coordination techniques across multiple decision-makers including non-cooperative game theory and distributed optimization problems. With this objective, non-cooperative games, distributed optimization and their applications to energy consumption control for the electricity consumers in the electricity market are considered.

1.2 Literature Review

In this section, relating works on extremum seeking control (ESC), Nash equilibrium seeking in non-cooperative games, distributed optimization and game based demand response in smart grids are reviewed.

1.2.1 Extremum Seeking Control

Extremum seeking is an online optimization method which produces control input that is capable of driving the objective function to the extremum value without using explicit model information [16]. The schematic outline of the classical extremum seeking scheme is shown in Figure 1.3 ¹. Sinusoidal dither signals are integrated in the control loop such that the gradient of the objective function is extractable. Singular perturbation analysis and averaging analysis are leveraged for the convergence analysis of the extremum seeking scheme. ESC has been an active topic among the researchers ever since the seminal work in [16]. Various application fields of ESC have been reported in the literature including pressure rise maximization in an axial-flow compressor [17], optimization of variable cam timing engine operation [18], vehicle target tracking [19], design of the anti-lock braking systems [20] and source tracing [21].

Great efforts have been dedicated to study extremum seeking schemes in the past decades. The authors in [22] studied a global extremum seeking scheme for systems with local extrema. The effect on the choice of the dither signals on the performance of the extremum seeking scheme was explored in [23]. The works in [24,25] provided a new view on the analysis of extremum seeking by using Lie bracket approximation. Different from the classical integrator-type extremum seeking scheme, the analysis for the extremum seeking scheme in [24,25] was based on the Lie bracket approximation instead of averaging analysis. A sampled-data framework was studied in [26–29]. A sampler and a zero-order-hold were utilized in the scheme such that nonlinear programming methods can be implemented in the extremum seeking loop. The main advantage lies in the usage of existing nonlinear programming methods and its capability to manage possibly non-convex optimization problem-

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¹More generally, a low-pass filter and a high-pass filter are integrated in the loop to improve the performance of the extremum seeker.

s. The objects for sampled-data extremum scheme are systems with equilibria or systems with periodic attractors [27]. Numerical-optimization based extremum seeking schemes are based on an interconnection structure of a numerical optimization method and a state regulator [20,30,31]. The numerical method is implemented to achieve optimization and the state regulator is utilized to handle system dynamics. State feedback linearizable systems and input-output feedback linearizable systems are mainly considered. The authors in [32–34] studied an extremum seeking scheme based on the sliding mode control. These methods contain a reference model that determines the convergence speed. Finite time convergence was ensured by using a periodic switching control law.

Figure 1.3: Schematic outline of the classical integrator-type extremum seeking scheme.

To accommodate systems with time-varying trajectories, a class of nonlinear systems with periodic orbit were considered in [35]. In [36], a more general nonlinear system with a periodic steady-state output was considered. The authors in [38] proposed a time-varying extremum seeking scheme without using averaging analysis. Constraints were further considered in [39] by using the method in [38]. In [37], the

authors provided an approach to address systems with a time-varying extremum input by designing a delay-based gradient estimation subsystem and a gradient search subsystem.

In summary, various extremum seeking schemes have been developed in the existing literature, e.g., the classical integrator-type extremum seeking scheme [22,23], Lie bracket approximation based extremum seeking [24,25], sampled-data extremum seeking [26–29], numerical optimization-based extremum seeking [20,30,31], and sliding mode based extremum seeking scheme [32–34]. These methods explore extremum seeking controllers from different perspectives leading to great improvement and prosperity of extremum seeking schemes in the past decade.

1.2.2 Nash Equilibrium Seeking in Non-cooperative Games

Nash equilibrium seeking problem embodies its importance as non-cooperative games are widely adopted to handle the optimization problems in engineering systems. Many learning strategies have been proposed to address Nash equilibrium seeking for non-cooperative games in the literature (see, e.g., [40–43,46–49]). The best response strategy, employed by [40] to analyze the non-myopic Cournot competition problem was proposed under the assumption that each player has access to the model information and other players' actions. Fictitious play mechanism, applied in [44] to optimize data allocation performance, is based on the history of the other players' strategies. Both the best response strategy and fictitious play can be implemented based on the observation over their opponents' actions. A dynamic version of the fictitious play was presented in [42], which was extended to the case where the players have only restricted access to measured utility without knowledge of opposing actions [45]. Summary statistics of the players' payoffs are essential for regret testing in [43]. Communication about the players' tentative decision is necessary to implement the distributed iterative algorithm in [47].

Nash equilibrium seeking for potential games, which are a special class of games, has caught great attention as well [50–56,86]. Spatial adaptive play was proposed in [50] to ensure the convergence to the Nash equilibrium with an arbitrarily high probability. A joint strategy fictitious play mechanism was integrated with inertia in [51] to solve large scale repeated potential games. Better/best response dynamics [52,86], fictitious play [53], and logit response [54] are typical methods that lead to the Nash equilibrium in potential games. In [55], the authors extended the results of better reply with inertia for state-based potential games. The authors in [56] demonstrated that all trajectories satisfying positive correlation and non-complacency led to the Nash equilibrium for potential games with many players.

By using the saddle point dynamics, convergence to the Nash equilibria of a two-network zero-sum games was provided in [95]. Nash equilibrium seeking in generalized convex games was considered in [96]–[98]. The authors in [96] and [97] solved the generalized convex game by a discrete-time distributed algorithm and Lemke’s method was adapted for the computation of generalized Nash equilibrium in convex games with quadratic payoffs subject to linear constraints in [98].

Many existing works contribute to Nash seeking problems by using some kind of model information [57]. Different from model based approaches, extremum seeking based methods provide alternative ways to search for the Nash equilibrium based on the measurements of the payoff functions [57]. In [58], the authors proposed a Nash seeking algorithm based on ESC by utilizing a sliding mode controller. A similar analysis is presented in [59] for N -person linear quadratic dynamic game. The authors in [57] employed averaging theory and the singular perturbation method to verify the convergence to the Nash equilibrium based on ESC. Nash equilibrium seeking in mobile sensor networks was considered in [60]. Other extremum seeking methods such as stochastic extremum seeking [60,93], Shahshahani gradient-like extremum seeking [99], Lie bracket extremum seeking [24], were also leveraged for

Nash equilibrium seeking.

By the observations over the existing literatures, it can be concluded that:

1. The players' payoff functions are determined by the players' own actions together with the other players' actions. Hence, most of the existing literatures require the players' observation over its opponents' actions to search for the Nash equilibrium.
2. Most exiting Nash seeking methods rely on some kind of model information for implementation [57]. Extremum seeking based methods, which are based on measurements of the players' payoff functions, are model-free. Furthermore, most of the existing works consider Nash equilibrium as a fixed point. Time-varying Nash equilibrium seeking problem remains to be solved.

1.2.3 Distributed Optimization

Distributed optimization among networked agents has attracted a lot of attention in recent years (e.g., [61–77] and the references therein). Recent works on distributed optimization mainly deal with communication related issues [61–65] or proposing new methods to solve networked optimization problems [66–72, 74–77].

In [61], distributed optimization under a directed graph was discussed. The authors in [62] addressed asynchronous distributed optimization in an event-driven communication graph. Time-varying directed graphs were considered for distributed optimization problems in [64]. Switching communication among the agents was considered in [65] where a policy gradient based method was proposed and distributed optimization over random networks was considered in [63].

Roughly speaking, the methods developed to solve distributed optimization problems can be categorized as consensus based gradient methods, primal-dual methods, and the proportional-integral distributed optimization method [67]. The proportional-

integral distributed optimization method was proposed in [67]. It requires less communication among the agents compared with primal-dual based optimization methods and gives better convergence performance compared with consensus based distributed optimization methods [67]. While most of the existing works are based on discrete-time methods, recent works in [61,66,76,77,124,125] provided continuous-time algorithms. In particular, by using a control perspective, the authors in [76,77] proposed an approach to solve distributed optimization problems under an undirected graph. The method was further generalized to handle directed graphs in [61]. A zero-gradient-sum algorithm was proposed in [124] and positivity constraint was considered in [125]. In [66], distributed optimization was explored by designing state-based potential games.

The aforementioned works contribute to distributed optimization communities by broadening the applicability of distributed optimization algorithms to various communication conditions and by providing new views and methods to solve distributed optimization problems. However, most of the existing methods are model-based and consider time-invariant optimal solutions for distributed optimization problems. The continuous-time time-varying distributed optimization problem, where the variation of the optimal solution can be fast, is still an open question.

1.2.4 Game Theory based Demand Response Schemes in Smart Grids

Demand response is defined as “changes in electric usage by end-user customers from their normal consumption patterns in response to changes in the price of electricity use at times of high wholesale market prices or when system reliability is jeopardized” by the Federal Energy Regulatory Commission (FERC, USA) [78]. Game theory, e.g., Stackelberg game [79,80], non-cooperative game [7,15,81] and differential game [82], has been widely adopted to analyze and design demand response

schemes in smart grids.

In [79], Stackelberg game was leveraged to model the interaction between the demand response aggregators and generators. The Nash equilibrium of the game among the generators was computed via solving a centralized quadratic optimization problem. Stackelberg game was also employed to describe the interactions among multiple utility companies and the end-users [80]. A distributed method was proposed to derive the Stackelberg equilibrium. In [7], the authors considered peak-to-average ratio (PAR) minimization and energy cost minimization problem by designing a non-cooperative energy consumption game. A distributed algorithm was proposed to search for the Nash equilibrium. In the proposed algorithm, the players need to communicate with each other on their daily energy consumption without detailed information on the energy consumption schedule. The authors in [81] studied the interactive behaviors among the electric vehicles and the aggregators by utilizing a non-cooperative game approach. The Nash equilibrium was shown to be the optimal solution by using a specific pricing policy. In [15], the authors considered both price takers and price-anticipating users in which the Nash equilibrium coincides with the socially optimal solution by suitably choosing the price values. Optimality and fairness issues were considered in [83], in which a smart electricity billing mechanism was developed. A coupled-constraint game was employed in [52] for scheduling residential energy consumption. A gradient projection method based on the estimation on the price changing trend was proposed to avoid privacy concern. Multiple utility companies were considered in [109], where the authors used a two-level game approach to manage the interactions. The competition among the utility companies was described by a non-cooperative game and the users were modeled as evolutionary game players. Considering the system dynamics, a differential game approach was proposed in [82]. In [84], the authors proposed a dynamic potential game approach for demand response. The best response strategies were

derived analytically and the efficiency of the equilibrium was studied.

In summary, games, including non-cooperatives, differential games and Stackelberg games, are effective analysis tools to handle the interactions among the electricity consumers and the utility companies. For the various game theoretic methods developed, common concerns for game based demand response methods are the single node congestion problem for centralized methods and the privacy concern of the electricity users.

1.3 Contributions

The main contribution of the dissertation is on the development of optimal coordination methods among multiple decision-makers and their applications to energy consumption management of electricity consumers in the electricity market. The main contribution is achieved by deriving the following results:

1. Energy consumption control among a network of electricity users in smart grids is solved by using an aggregate game approach. Without knowing the actual aggregate energy consumption during the electricity consumption scheduling process, an average consensus protocol is adopted to estimate the averaged energy consumption. By utilizing a game on graph, the users can search for their optimal energy consumptions by communicating with their neighbors on their estimated energy consumptions. No private information is exchanged among the participants and hence, the privacy of the electricity users is protected. Furthermore, the communication between the electricity consumers with the energy provider is reduced thus relieving the single node congestion problem. Nash equilibrium seeking strategies are provided for the electricity users. By utilizing the proposed updating strategy, the Nash equilibria is exponentially stable. In particular, for heating ventilation and air condition-

ing (HVAC) systems, the Nash equilibrium of the associated non-cooperative game is unique. Based on the uniqueness of the Nash equilibrium, non-local convergence results are derived. Energy consumption game with stubborn players is investigated. In this case, the rational players' actions converge to the best response strategies.

2. A distributed extremum seeking approach is proposed to solve networked optimization problems without using explicit model information. The proposed method can be implemented based on the measurement of the local cost functions and the constraints. Neither the explicit expressions of the local cost functions, the constraints, nor their gradients are needed. By utilizing averaging analysis, Lyapunov stability analysis and the saddle point algorithm, the stability analysis is conducted. To demonstrate the practical relevance of the proposed extremum seeking scheme, energy consumption control among the electricity users in smart grids is revisited. Based on the proposed distributed extremum seeking scheme, the electricity users can search for the socially optimal solution without using the explicit expression on the electricity pricing function.
3. Noticing that many engineering systems exhibit time-varying characteristic, time-varying distributed optimization problem for systems with quadratic objective functions is investigated. Based on a robust searching strategy, the agents are able to track the unknown time-varying optimal solution. In particular, for neighboring coupled objective functions, the proposed method guarantees that the users can asymptotically track the optimal solution. Penalty function based methods are proposed to solve the generally coupled distributed optimization problems. The penalty function based methods enable the agents' strategies to approximate the time-varying optimal solution.

4. An N -player non-cooperative game with time-varying Nash equilibrium is considered. The non-cooperative games under consideration admit time-varying Nash equilibrium and payoff values along the Nash equilibrium trajectory. Based on an extremum seeking approach, the proposed Nash equilibrium seeking scheme doesn't require explicit model information on the players' payoff functions. A gradient estimation subsystem and a gradient search subsystem are designed. It is shown that the proposed seeking method enables the players' actions to asymptotically converge to the Nash equilibrium trajectory for symmetric quadratic games. Furthermore, for more general quadratic games, the developed ESC-based seeking method enables the estimated strategy to converge to a neighborhood of the time-varying Nash equilibrium under some conditions.

1.4 Organization of the Thesis

The rest of the dissertation is structured as follows. In **Chapter 2**, some related preliminaries and definitions on game theory, graph theory and Lyapunov stability, etc, are presented. **Chapter 3** considers the application of non-cooperative games to demand response in smart grids. Since the aggregate energy consumption, which affects the electricity pricing function, is unknown to the electricity consumers, an aggregate game approach is proposed. Updating laws for the users are proposed to enable the users' actions to converge to the Nash equilibrium. Nevertheless, Nash equilibrium is usually not the socially optimal solution from the system-level perspective if coordination is allowed across the decision-makers. Therefore, in **Chapter 4**, a distributed extremum seeking scheme is proposed to solve networked optimization problems. Energy consumption control in smart grids is reconsidered by using the proposed distributed extremum seeking approach. Noticing that continuous-time

time-varying networked optimization problem hasn't been addressed yet, in **Chapter 5**, time-varying distributed optimization among networked agents is considered via using a robust searching strategy. A neighboring coupled time-varying distributed optimization problem is firstly considered. By utilizing the penalty function based methods, generally coupled distributed optimization problems are investigated. In **Chapter 6**, time-varying Nash equilibrium seeking for noncooperative games with time-varying payoff functions is studied. A time-varying extremum seeking approach is leveraged to achieve it. For all the methods proposed in the dissertation, numerical examples are provided to illustrate the effectiveness of the methods. Conclusions of the dissertation and future work are included in **Chapter 7**.

Chapter 2

Preliminaries

In this chapter, some preliminaries and definitions on game theory, graph theory, dynamic average consensus, Lyapunov stability and practical stability are given.

2.1 Game Theory

Game theory is a study of cooperation and confliction among rational decision-makers [85]. A normal form game consists of three components, i.e., players, the players' actions and the players' payoffs/costs [85]. Mathematically, a game is defined as follows.

Definition 2.1. [85](Game definition) *A game in a normal form is defined as a triple $\Gamma \triangleq \{\mathbb{N}, X, C\}$ where $\mathbb{N} = \{1, 2, \dots, N\}$ is the set of N players, $X = X_1 \times \dots \times X_N$, $X_i \subset \mathbb{R}$ is the set of actions for player i , and $C = (C_1, \dots, C_N)$ where C_i is the cost function of player i .*

Definition 2.2. (Potential games) [1,51,86] *A game Γ is a potential game if there exists a function P such that*

$$\frac{\partial C_i(l_i, \mathbf{l}_{-i})}{\partial l_i} = \frac{\partial P(l_i, \mathbf{l}_{-i})}{\partial l_i}, \quad (2.1)$$

$\forall i \in \mathbb{N}$. Furthermore, the function P is the potential function. In (2.1), l_i denotes the action of player i and, $\mathbf{l}_{-i}$ denotes all the players' actions other than the action of player i , i.e., $\mathbf{l}_{-i} = [l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_N]^T$.

Definition 2.3. [103,104,106] (Aggregate game) *An aggregate game is a normal form game with the players' cost functions depending only on its own action and a linear aggregate of the full action profile.*

In particular, a non-cooperative game is one in which no cooperation other than self-enforcing cooperation exists in the game. The solution of a non-cooperative game is called Nash equilibrium which is defined as follows.

Definition 2.4. [85] (Nash equilibrium) *Nash equilibrium is an action profile on which no player can reduce its cost by unilaterally changing its own action, i.e., an action profile $(l_i^*, \mathbf{l}_{-i}^*) \in X$ is Nash equilibrium if for all the players*

$$C_i(l_i^*, \mathbf{l}_{-i}^*) \leq C_i(l_i, \mathbf{l}_{-i}^*), \forall l_i \in X_i$$

where $\mathbf{l}_{-i}$ denotes all the players' strategies other than player i .

For aggregate games with the aggregate function being $\bar{l} = \sum_{i=1}^N l_i$, an equilibrium of an aggregate game Γ is an action profile $\mathbf{l}^* \in X$ on which for each player $i \in \mathbb{N}$

$$C_i(l_i^*, l_i^* + \sum_{j \neq i} l_j^*) \leq C_i(l_i, l_i + \sum_{j \neq i} l_j^*), l_i \in X_i$$

and the associated equilibrium aggregate is denoted as $\bar{l}^* = \sum_{i=1}^N l_i^*$.

2.2 Graph Theory and Dynamic Average Consensus

For a graph defined as $G = (V, E)$ where E is the edge set satisfying $E \subset V \times V$ with $V = \{1, 2, \dots, N\}$ being the set of nodes in the network, it is undirected if for every $(i, j) \in E$, $(j, i) \in E$. An undirected graph is connected if there exists a path between any pair of distinct vertices. The elements in the adjacency matrix A_G is defined as $a_{ij} = 1$ if node j is connected with node i , else, $a_{ij} = 0$. The Laplacian matrix for the graph L is defined as $L = D_G - A_G$, where D_G is defined as a diagonal matrix whose i th diagonal element is equal to the out degree of node i , represented by $\sum_{j=1}^N a_{ij}$ [87].

Theorem 2.1. [100,121] *Let G be a connected, undirected graph, L be the Laplacian of G . Then, for any constant $\omega \in \mathbb{R}^N$, the state of the following system*

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} -I - L & -L \\ L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \omega \\ \mathbf{0} \end{bmatrix},$$

with arbitrary initial conditions $\mathbf{x}(0), \mathbf{y}(0) \in \mathbb{R}^N$ remains bounded and $\mathbf{x}(t)$ converges exponentially to $\frac{1}{N} \mathbf{1}^T \omega \mathbf{1}$ as $t \rightarrow \infty$ where $\mathbf{1}$ denotes an N dimensional column vector composed of 1.

2.3 Lyapunov Stability

Definition 2.5. [88] *A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.*

Definition 2.6. [88] *A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s , the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with*

respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Theorem 2.2. [88] Let $x = 0$ be an equilibrium of

$$\dot{x} = f(x)$$

and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V_L : \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function such that $V_L(0) = 0$ and $V_L(x) > 0$ in $\mathcal{D} \setminus \{0\}$, $\dot{V}_L(x) \leq 0$ in $\mathcal{D}$.

Then, $x = 0$ is stable.

Moreover, if $\dot{V}_L(x) < 0$ in $\mathcal{D} \setminus \{0\}$, then $x = 0$ is asymptotically stable.

Theorem 2.3. [88] Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$ and suppose that $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x , uniformly in t on $[0, \infty) \times \mathcal{D}$. Furthermore, suppose $f(t, 0)$ is uniformly bounded for all $t \geq 0$. Let $V_L : [0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$ be continuously differentiable function such that

$$\begin{aligned} W_1(x) &\leq V_L(t, x) \leq W_2(x), \\ \dot{V}_L(t, x) &= \frac{\partial V_L}{\partial t} + \frac{\partial V_L}{\partial x} f(t, x) \leq -W(x), \end{aligned} \tag{2.2}$$

$\forall t \geq 0, \forall x \in \mathcal{D}$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions and $W(x)$ is a continuous positive semi-definite function on $\mathcal{D}$. Choose $r > 0$ such that $B_r \subset \mathcal{D}$ and let $\rho < \min_{\|x\|=r} W(x)$. Then, all solutions of

$$\dot{x} = f(t, x)$$

with $x(0) \in \{x \in B_r | W_2(x) \leq \rho\}$ are bounded and satisfy $W(x(t)) \rightarrow 0$, as $t \rightarrow \infty$. Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, the statement is true for all $x(0) \in \mathbb{R}^n$.

2.4 Practical Stability

Below is a definition on practical stability [24,89,90]. Readers are referred to [24,89,90] for more details.

Definition 2.7. [90] Consider a parameterized family of systems:

$$\dot{x} = f(t, x, \epsilon), \quad (2.3)$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}_+$ and $\epsilon \in \mathbb{R}_{++}^l$ are the state of the system, the time variable and the parameter vector, respectively.

The equilibrium of the system $x = 0$ in (2.3) is said to be practically asymptotically stable with arbitrary bounded initial condition, uniformly in $(\epsilon_1, \epsilon_2, \dots, \epsilon_j)$, $j \in \{1, 2, \dots, l\}$, if there exists $\beta \in \mathcal{KL}$ such that the following holds. For each pair of strictly positive real number (Δ, v) , there exist real numbers $\varepsilon_k^* = \varepsilon_k^*(\Delta, v) > 0$, $k = 1, 2, \dots, j$ and each fixed $\varepsilon_k \in (0, \varepsilon_k^*)$, $k = 1, 2, \dots, j$, there exist $\varepsilon_i = \varepsilon_i(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{i-1}, \Delta, v)$, with $i = j+1, j+2, \dots, l$, such that the solution of (2.3) with the constructed parameters $\epsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l)$ satisfy:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, \varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_l(t - t_0) + v,$$

for all $t \geq t_0 \geq 0$, $\|x(t_0)\| \leq \Delta$. If $j = l$, then, the equilibrium of the system is practically asymptotically stable with arbitrary bounded initial condition, uniformly in ϵ .

2.5 Saddle Point

Definition 2.8. (Saddle point) [91] *The saddle point (x^*, y^*) of function $F(x, y)$ is a point on which the following is satisfied:*

$$F(x^*, y) \leq F(x^*, y^*) \leq F(x, y^*).$$

More generally, $(x^*, y^*, z^*) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^m$ is the saddle point of $F(x, y, z)$ if

$$F(x^*, y, z) \leq F(x^*, y^*, z^*) \leq F(x, y^*, z^*),$$

where $F(x, y, z) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^m \rightarrow \mathbb{R}$.

Throughout this dissertation, we say that a time-varying matrix $R(t)$ is positive definite if there exists a positive constant q_1 such that $q_1 I \leq R(t)$. The positive definite matrix is said to be bounded if there exist a positive constant $q_2 \geq q_1$ such that $R(t) \leq q_2 I$. A time-varying matrix $R(t)$ is negative definite and bounded if $-R(t)$ is positive definite and bounded.

Chapter 3

An Aggregate Game Approach for Energy Consumption Control in Smart Grids

3.1 Introduction

As stated in Chapter 1, considering the effect of the electricity consumption on the electricity price rises to coupled optimization problems and hence, game theory is an effective analysis tool to handle the interactions among the electricity consumers [13].

Stackelberg game [79,80], non-cooperative game [7,15,81], differential game [82] are effective analysis tools for demand response programs in smart grids. More specifically, when multiple utility companies are considered, the demand response programs exhibit a “leader-following” structure, Stackelberg game [79,80] or two-level game [109] approaches can be utilized. If systems dynamics are addressed, dynamics games, e.g., differentiable games [82] or dynamic potential games [84] can be employed in the analysis of demand response in smart grids. If coupled constraints are to be accommodated, coupled-constraint game [52] is an effective

tool. Despite the different game tools that can be used, the users' privacy is a vital concern in the design and analysis of demand response programs in smart grids.

Different from existing game based approaches, in this chapter, energy consumption control is addressed by using an aggregate game approach which does not require the aggregate energy consumption to be known by the electricity users. To the best of our knowledge, this is the first work to deal with energy consumption control via an aggregate game [105,106,110,111] approach. Compared with the existing works, the main contributions of the chapter are summarized as follows.

- An aggregate game is adopted for the modeling and analysis of energy consumption control in smart grids. By using game on graph, the users update their actions based on the communication with their neighbors. This scheme reduces the communication between the electricity users with the centralized agent (e.g., the energy provider). Hence, the single-node congestion problem is relieved.
- An aggregate game that may admit multiple isolated Nash equilibria is firstly considered. Based on an average consensus protocol, a Nash seeking strategy is proposed for the players. An energy consumption game for a network of heating, ventilation and air conditioning (HVAC) systems, in which the Nash equilibrium is unique, is investigated. The Nash seeking strategy is designed based on a primal-dual dynamics. More specifically, if the unique Nash equilibrium is an inner Nash equilibrium, an exponential convergence result is derived. Energy consumption game with stubborn players is studied. It is shown that with the presence of stubborn players, the rational players' actions converge to a neighborhood of their best response strategies. The proposed Nash seeking strategies serve as alternative approaches for the gossip-algorithm in [111] to solve aggregative game on graph.

- The end-users only need to communicate with their neighbors on their estimations of the aggregate energy consumption. They don't need to share their own energy consumptions with their opponents. Hence, the privacy of the electricity users is protected.

The rest of the chapter is organized as follows. System model and the problem formulation are given in Section 3.2 and general energy consumption game that may have multiple Nash equilibrium is studied in Section 3.3. Energy consumption control for a network of HVAC systems is investigated in Section 3.4. In Section 3.5, numerical examples are provided to verify the effectiveness of the proposed methods. Conclusions on this chapter are given in Section 3.6.

3.2 System Model and Problem Formulation

Consider an electricity market with a network of N price-anticipating users. The users are equipped with an energy-management controller (EMC) and an advanced metering infrastructure (AMI) [14]. The EMC is used to schedule the electricity usage for the corresponding user. The AMI enables bidirectional communication among the electricity users and the centralized agent (e.g., the energy provider). The communication between the electricity users and their neighbors can be modeled by an undirected and connected graph. In this chapter, we suppose that the electricity users schedule their energy consumption by minimizing their own costs.

Let l_i be the energy consumption of user i . Then, the cost of user i can be quantified as [14],

$$C_i(l_i, \bar{l}) = V_i(l_i) + P(\bar{l})l_i, \quad (3.1)$$

where $\bar{l}$ is the aggregate energy consumption of all the electricity users, i.e., $\bar{l} = \sum_{j=1}^N l_j$. Furthermore, $V_i(l_i)$ is the load curtailment cost [14]. The term $P(\bar{l})l_i$ represents the billing payment for the consumption of energy l_i where the price $P(\bar{l})$

is a function of the aggregate energy consumption $\bar{l}$.

For each user, the energy consumption should be within its acceptable range, i.e., $l_i \in [l_i^{\min}, l_i^{\max}]$ where $l_i^{\min} < l_i^{\max}$ and $l_i^{\min}, l_i^{\max}$ are the minimal and maximal acceptable energy consumption for user i , respectively. The problem is defined as follows.

Problem 3.1. (Strategy Design for the Electricity Consumers in the Energy Consumption Game) *In the energy consumption game, the electricity users are the players. The energy consumption l_i and $C_i(l_i, \bar{l})$ are the action and cost of player i , respectively. Player i 's objective is defined as*

$$\begin{aligned} \min_{l_i} \quad & C_i(l_i, \bar{l}) \\ \text{subject to} \quad & l_i^{\min} \leq l_i \leq l_i^{\max}, i \in \mathbb{N}, \end{aligned}$$

where $\mathbb{N} = \{1, 2, \dots, N\}$ denotes the set of electricity users. The aggregate energy consumption $\bar{l}$ is unknown to the players. Suppose that the pure-strategy Nash equilibrium exists and is isolated. Furthermore, $C_i, i \in \mathbb{N}$ are smooth functions over $\mathbb{R}^N$. Design a control strategy to search for the Nash equilibrium.

Remark 3.1. *In practice, providing the aggregate energy consumption $\bar{l}$ to all the users is challenging for the centralized agent when the users are dynamically updating their actions. Hence, we consider the Nash equilibrium seeking under the condition that the users have no access to the aggregate energy consumption $\bar{l}$. But the users are allowed to communicate with their neighbors on the estimations of the aggregate energy consumption. Furthermore, we suppose that the centralized agent can broadcast the total number of electricity users N to all the electricity users in the network.*

In summary, the energy consumption control considered in this chapter is based on the following assumption.

Assumption 3.1. *The electricity users can communicate with their neighbors via an undirected and connected graph. Furthermore, the total number of the electricity users, N , is known to all the electricity users.*

3.3 Energy Consumption Game Design and Analysis

In this section, the energy consumption game is considered in the general form (the pricing function is not specified). For simplicity, the constraints $l_i^{\min} \leq l_i \leq l_i^{\max}, i \in \mathbb{N}$, are not considered in this section.

3.3.1 Game Analysis

Before we proceed to facilitate the subsequent analysis, the following assumptions are made.

Assumption 3.2. *[57] There exists isolated, stable Nash equilibrium on which*

$$\begin{aligned} \frac{\partial C_i}{\partial l_i}(\mathbf{l}^*) &= 0, \\ \frac{\partial^2 C_i}{\partial l_i^2}(\mathbf{l}^*) &> 0, \forall i \in \mathbb{N} \end{aligned} \tag{3.2}$$

where $\mathbf{l}^*$ denotes the Nash equilibrium.

Assumption 3.3. *[57] The matrix*

$$B = \begin{bmatrix} \frac{\partial^2 C_1}{\partial l_1^2}(\mathbf{l}^*) & \frac{\partial^2 C_1}{\partial l_1 \partial l_2}(\mathbf{l}^*) & \cdots & \frac{\partial^2 C_1}{\partial l_1 \partial l_N}(\mathbf{l}^*) \\ \frac{\partial^2 C_2}{\partial l_1 \partial l_2}(\mathbf{l}^*) & \frac{\partial^2 C_2}{\partial l_2^2}(\mathbf{l}^*) & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 C_N}{\partial l_1 \partial l_N}(\mathbf{l}^*) & \cdots & & \frac{\partial^2 C_N}{\partial l_N^2}(\mathbf{l}^*) \end{bmatrix} \tag{3.3}$$

is strictly diagonally dominant.

Inspired by [111], let D_i denote player i 's estimation on the aggregate energy consumption. By using the estimations, the players' objectives can be rewritten as:

Problem 3.2. *Player i 's objective is*

$$\min_{l_i} C_i(l_i, D_i), \quad (3.4)$$

where $D_i = \sum_{j=1}^N l_j, \forall i \in \mathbb{N}$.

In the following, a consensus based method will be proposed to search for the Nash equilibrium of the energy consumption game (without considering the constraints).

3.3.2 Nash Equilibrium Seeking for the Aggregate Energy Consumption Game

Based on the consensus protocol in [100] and [121], the Nash seeking strategy for player $i, i \in \mathbb{N}$ is designed as

$$\dot{D}_i = -D_i - \sum_{j \in \mathcal{N}_i} (D_i - D_j) - \sum_{j \in \mathcal{N}_i} (\kappa_i - \kappa_j) + Nl_i \quad (3.5a)$$

$$\dot{\kappa}_i = \sum_{j \in \mathcal{N}_i} (D_i - D_j) \quad (3.5b)$$

$$\dot{l}_i = -\bar{k}_i \left(\frac{\partial V_i}{\partial l_i} + P(D_i) + l_i \frac{\partial P(D_i)}{\partial D_i} \right), \quad (3.5c)$$

where $\mathcal{N}_i$ denotes the neighboring set of player i , $\bar{k}_i = \delta k_i, i \in \mathbb{N}$, δ is a small positive parameter and k_i is a fixed positive parameter. Furthermore, $\kappa_i, i \in \mathbb{N}$ are intermediate variables.

Writing (3.5) in the concatenated form gives

$$\begin{bmatrix} \dot{\mathbf{D}} \\ \dot{\kappa} \end{bmatrix} = \begin{bmatrix} -I - L & -L \\ L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \kappa \end{bmatrix} + \begin{bmatrix} N\mathbf{1} \\ \mathbf{0} \end{bmatrix} \quad (3.6a)$$

$$\dot{\mathbf{l}} = -\delta \mathbf{k} \left(\left[\frac{\partial V_i}{\partial l_i} + P(D_i) + l_i \frac{\partial P(D_i)}{\partial D_i} \right]_{vec} \right), \quad (3.6b)$$

where $\mathbf{k}$ is defined as $\mathbf{k} = \text{diag}\{k_i\}$, $i \in \mathbb{N}$, $\mathbf{D}$, κ , $\mathbf{l}$ are the concatenated vectors of D_i , κ_i , l_i , respectively. Let $U = [\tilde{U}, \mu]$ be an $N \times N$ dimensional orthonormal matrix such that $\mu^T L = 0$ where μ is an N dimensional column vector. Furthermore, let $\kappa = U \begin{bmatrix} \tilde{\kappa} \\ \kappa_\mu \end{bmatrix}$ where $\tilde{\kappa}$ is an $N-1$ dimensional column vector. Then, the closed-loop system can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{D}} \\ \dot{\tilde{\kappa}} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \tilde{\kappa} \end{bmatrix} + \begin{bmatrix} N\mathbf{1} \\ \mathbf{0} \end{bmatrix} \quad (3.7a)$$

$$\dot{\mathbf{l}} = -\delta \mathbf{k} \left(\left[\frac{\partial V_i}{\partial l_i} + P(D_i) + l_i \frac{\partial P(D_i)}{\partial D_i} \right]_{vec} \right), \quad (3.7b)$$

and

$$\dot{\kappa}_\mu = 0, \quad (3.8)$$

which indicates that $\kappa_\mu(t) = \kappa(0)$.

Suppose that $\mathbf{D}^e(\mathbf{l})$, $\tilde{\kappa}^e(\mathbf{l})$ are the quasi-steady states of $\mathbf{D}$ and $\tilde{\kappa}$, respectively, i.e.,

$$\begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}^e(\mathbf{l}) \\ \tilde{\kappa}^e(\mathbf{l}) \end{bmatrix} + \begin{bmatrix} N\mathbf{1} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad (3.9)$$

for fixed $\mathbf{l}$. Note that $\mathbf{D}^e(\mathbf{l})$, $\tilde{\kappa}^e(\mathbf{l})$ is unique for fixed $\mathbf{l}$ as the matrix $\begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix}$ is Hurwitz [121]. Then, by direct calculation, it can be derived that

$$\begin{bmatrix} -I - L & -L \\ L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}^e(\mathbf{l}) \\ U[(\tilde{\kappa}^e(\mathbf{l}))^T, \kappa_\mu]^T \end{bmatrix} + \begin{bmatrix} N\mathbf{l} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad (3.10)$$

which indicates that $\mathbf{D}^e(\mathbf{l})$ is the equilibrium of the original system. Hence, by using Theorem 2.1, it can be concluded that $\mathbf{D}^e(\mathbf{l}) = \mathbf{1} \sum_{i=1}^N l_i$ for fixed $\mathbf{l}$.

Theorem 3.1. *Suppose that Assumptions 3.1-3.3 hold. Then, there exists a positive constant δ^* such that for every $0 < \delta < \delta^*$, $(\mathbf{l}(t), \mathbf{D}(t), \tilde{\kappa}(t))$ converges exponentially to $(\mathbf{l}^*, \mathbf{1} \sum_{i=1}^N l_i^*, \tilde{\kappa}^e(\mathbf{l}^*))$ under (3.7) given that $\|\mathbf{l}(0) - \mathbf{l}^*\|$, $\|\mathbf{D}(0) - \mathbf{1} \sum_{i=1}^N l_i^*\|$, $\|\tilde{\kappa}(0) - \tilde{\kappa}^e(\mathbf{l}^*)\|$ are sufficiently small.*

Proof. Let $\tau = \delta t$. Then, in the τ -time scale, the reduced-system is

$$\begin{bmatrix} \delta \frac{d\mathbf{D}}{d\tau} \\ \delta \frac{d\tilde{\kappa}}{d\tau} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \tilde{\kappa} \end{bmatrix} + \begin{bmatrix} N\mathbf{l} \\ \mathbf{0} \end{bmatrix} \quad (3.11a)$$

$$\frac{d\mathbf{l}}{d\tau} = -\mathbf{k}([\frac{\partial V_i}{\partial l_i} + P(D_i) + l_i \frac{\partial P(D_i)}{\partial D_i}]_{vec}). \quad (3.11b)$$

Quasi-steady state analysis: letting $\delta = 0$ freezes $\mathbf{D}$ and $\tilde{\kappa}$ at the quasi-steady state on which $D_i = \sum_{j=1}^N l_j, \forall i \in \mathbb{N}$. Hence, the reduced-system is

$$\begin{aligned} \frac{d\mathbf{l}}{d\tau} &= -\mathbf{k}([\frac{\partial V_i}{\partial l_i} + P(D_i) + l_i \frac{\partial P(D_i)}{\partial D_i}]_{vec}) \\ &= -\mathbf{k}([\frac{\partial C_i(l_i, \mathbf{l}_{-i})}{\partial l_i}]_{vec}). \end{aligned} \quad (3.12)$$

Linearizing (3.12) at the Nash equilibrium point $\mathbf{l}^*$ gives,

$$\frac{d\mathbf{l}}{d\tau} = -\mathbf{k}B(\mathbf{l} - \mathbf{l}^*), \quad (3.13)$$

where $-\mathbf{k}B$ is Hurwitz by Assumption 3.3 and the Gershgorin Circle Theorem [112]. Hence, the equilibrium point $\mathbf{l}^*$ is locally exponentially stable under (3.12), i.e., there

exist positive constants ϱ_1 and ϱ_2 such that (denote the trajectory of (3.12) as $\mathbf{l}_r(\tau)$)

$$\|\mathbf{l}_r(\tau) - \mathbf{l}^*\| \leq \varrho_1 e^{-\varrho_2 \tau} \|\mathbf{l}_r(0) - \mathbf{l}^*\|, \quad (3.14)$$

given that $\|\mathbf{l}_r(0) - \mathbf{l}^*\|$ is sufficiently small.

Boundary-layer analysis: Since $(\mathbf{D}^e(\mathbf{l}), \tilde{\kappa}^e(\mathbf{l}))$ satisfies,

$$\begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}^e(\mathbf{l}) \\ \tilde{\kappa}^e(\mathbf{l}) \end{bmatrix} + \begin{bmatrix} N\mathbf{l} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad (3.15)$$

it can be derived that $\mathbf{D}^e(\mathbf{l}), \tilde{\kappa}^e(\mathbf{l})$ are linear functions of $\mathbf{l}$ as the matrix $\begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix}$ is non-singular.

Let

$$\begin{aligned} \mathbf{D}' &= \mathbf{D} - \mathbf{D}^e \\ \tilde{\kappa}' &= \kappa - \kappa^e. \end{aligned} \quad (3.16)$$

Then,

$$\begin{aligned} \delta \begin{bmatrix} \frac{d\mathbf{D}'}{d\tau} \\ \frac{d\tilde{\kappa}'}{d\tau} \end{bmatrix} &= \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}' + \mathbf{D}^e \\ \tilde{\kappa}' + \tilde{\kappa}^e \end{bmatrix} + \begin{bmatrix} N\mathbf{l} \\ \mathbf{0} \end{bmatrix} - \delta \begin{bmatrix} \frac{\partial \mathbf{D}^{eT} T}{\partial \mathbf{l}} \\ \frac{\partial \tilde{\kappa}^{eT} T}{\partial \mathbf{l}} \end{bmatrix} \\ &\quad \times (-\mathbf{k}([\frac{\partial V_i}{\partial l_i} + P(D'_i + \sum_{j=1}^N l_j) + l_i \frac{\partial P(D'_i + \sum_{j=1}^N l_j)}{\partial (D'_i + \sum_{j=1}^N l_j)}]_{vec})) \\ &= \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}' \\ \tilde{\kappa}' \end{bmatrix} \\ &\quad - \delta \begin{bmatrix} \frac{\partial \mathbf{D}^{eT} T}{\partial \mathbf{l}} \\ \frac{\partial \tilde{\kappa}^{eT} T}{\partial \mathbf{l}} \end{bmatrix} (-\mathbf{k}([\frac{\partial V_i}{\partial l_i} + P(D'_i + \sum_{j=1}^N l_j) + l_i \frac{\partial P(D'_i + \sum_{j=1}^N l_j)}{\partial (D'_i + \sum_{j=1}^N l_j)}]_{vec})). \end{aligned} \quad (3.17)$$

Hence, in t -time scale

$$\begin{aligned}
\begin{bmatrix} \frac{d\mathbf{D}'}{dt} \\ \frac{d\tilde{\kappa}'}{dt} \end{bmatrix} &= \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}' \\ \tilde{\kappa}' \end{bmatrix} \\
&\quad - \delta \begin{bmatrix} \frac{\partial \mathbf{D}^{eT}}{\partial \mathbf{l}} \\ \frac{\partial \tilde{\kappa}^{eT}}{\partial \mathbf{l}} \end{bmatrix}^T \left(-\mathbf{k} \left(\frac{\partial V_i}{\partial l_i} \right. \right. \\
&\quad \left. \left. + P(D'_i + \sum_{j=1}^N l_j) \right. \right. \\
&\quad \left. \left. + l_i \frac{\partial P(D'_i + \sum_{j=1}^N l_j)}{\partial (D'_i + \sum_{j=1}^N l_j)} \Big|_{vec} \right) \right). \tag{3.18}
\end{aligned}$$

Letting $\delta = 0$ gives the boundary-layer model of (3.11) as

$$\begin{bmatrix} \frac{d\mathbf{D}'}{dt} \\ \frac{d\tilde{\kappa}'}{dt} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}' \\ \tilde{\kappa}' \end{bmatrix}. \tag{3.19}$$

Since the matrix $\begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix}$ is Hurwitz, the equilibrium point of the boundary-layer model $\mathbf{D}' = 0, \tilde{\kappa}' = 0$ is exponentially stable, uniformly in $(t, \mathbf{l})$.

Therefore, by Theorem 11.4 in [88], it can be concluded that there exists a positive constant δ^* such that for all $0 < \delta < \delta^*$, $(\mathbf{l}^*, \mathbf{1} \sum_{i=1}^N l_i^*, \tilde{\kappa}^e(\mathbf{l}^*))$ is exponentially stable under (3.7).

□

From the analysis, it can be seen that all the states in (3.5) stay bounded and $\mathbf{l}(t)$ produced by (3.5) converges to the Nash equilibrium under the given conditions.

In this section, a Nash seeking strategy is proposed without requiring the uniqueness of the Nash equilibrium. In the following section, energy consumption game for HVAC systems where the Nash equilibrium is unique will be considered.

3.4 Energy Consumption Game among a Network of HVAC Systems

For HVAC systems, the load curtailment cost may be modeled as [14]

$$V_i(l_i) = \nu_i \xi_i^2 (l_i - \hat{l}_i)^2, i \in \mathbb{N}, \quad (3.20)$$

where ν_i and ξ_i are thermal coefficients, $\nu_i \xi_i^2 > 0$ and $\hat{l}_i$ is the energy needed to maintain the indoor temperature of the HVAC system. When the pricing function is

$$P(\bar{l}) = a \sum_{i=1}^N l_i + p_0, \quad (3.21)$$

where a is a non-negative constant and $a < \min_{i \in \mathbb{N}} \frac{2\nu_i \xi_i^2}{N-3}$ for $N > 3$, the uniqueness of the Nash equilibrium is ensured [14].

Based on this given model, a Nash seeking strategy will be proposed in this section for the players to search for the unique Nash equilibrium (by assuming that $a < \min_{i \in \mathbb{N}} \frac{2\nu_i \xi_i^2}{N-3}$ for $N > 3$ in the rest of the chapter).

3.4.1 Nash Equilibrium Seeking for Energy Consumption Game of HVAC Systems

Lemma 3.1. *The energy consumption game is a potential game with a potential function being*

$$Q(\mathbf{l}) = \sum_{i=1}^N \nu_i \xi_i^2 (l_i - \hat{l}_i)^2 + \sum_{i=1}^N a \left(\sum_{j=1, j \neq i}^N l_j \right) l_i + \sum_{i=1}^N (a l_i^2 + p_0 l_i). \quad (3.22)$$

Proof. Noticing that

$$\frac{\partial Q(\mathbf{l})}{\partial l_i} = \frac{\partial C_i(\mathbf{l})}{\partial l_i}, \forall i \in \mathbb{N}, \quad (3.23)$$

the conclusion can be derived by using the definition of potential game. $\square$

Based on the primal-dual dynamics in [108], the Nash seeking strategy for player i is designed as

$$\dot{D}_i = -D_i - \sum_{j \in \mathcal{N}_i} (D_i - D_j) - \sum_{j \in \mathcal{N}_i} (\kappa_i - \kappa_j) + Nl_i \quad (3.24a)$$

$$\dot{\kappa}_i = \sum_{j \in \mathcal{N}_i} (D_i - D_j) \quad (3.24b)$$

$$\dot{l}_i = -\bar{k}_i \left(\frac{\partial V_i}{\partial l_i} + P(D_i) + al_i - \eta_{i1} + \eta_{i2} \right) \quad (3.24c)$$

$$\dot{\eta}_{i1} = \bar{m}_{i1} \eta_{i1} (l_i^{min} - l_i) \quad (3.24d)$$

$$\dot{\eta}_{i2} = \bar{m}_{i2} \eta_{i2} (l_i - l_i^{max}), i \in \mathbb{N} \quad (3.24e)$$

where $\bar{m}_{ij} = \delta m_{ij}$ for all $i \in \mathbb{N}, j \in \{1, 2\}$, m_{ij} are fixed positive parameters and $\eta_{ij}(0) > 0, i \in \mathbb{N}, j \in \{1, 2\}$.

By introducing the orthonormal matrix U as in Section 3.3.2, the system in (3.24) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{D}} \\ \dot{\tilde{\kappa}} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \tilde{\kappa} \end{bmatrix} + \begin{bmatrix} N\mathbf{1} \\ \mathbf{0} \end{bmatrix} \quad (3.25a)$$

$$\dot{l}_i = -\bar{k}_i \left(\frac{\partial V_i}{\partial l_i} + P(D_i) + al_i - \eta_{i1} + \eta_{i2} \right) \quad (3.25b)$$

$$\dot{\eta}_{i1} = \bar{m}_{i1} \eta_{i1} (l_i^{min} - l_i) \quad (3.25c)$$

$$\dot{\eta}_{i2} = \bar{m}_{i2} \eta_{i2} (l_i - l_i^{max}), i \in \mathbb{N} \quad (3.25d)$$

and

$$\dot{\kappa}_\mu = 0. \quad (3.26)$$

Theorem 3.2. *Suppose that Assumption 3.1 is satisfied. Then, there exists $\beta \in \mathcal{KL}$ such that for each pair of strictly positive real number (Δ, v) , there exists $\delta(\Delta, v) > 0$,*

such that

$$\|\chi(t)\| \leq \beta(\|\chi(0)\|, \delta t) + v, \quad (3.27)$$

for all $t \geq 0$, $\|\chi(0)\| \leq \Delta$ under (3.25) given that $\eta_{ij}(0) > 0, i \in \mathbb{N}, j \in \{1, 2\}$. In (3.27), $\chi(t) = [(\mathbf{l}(t) - \mathbf{l}^*)^T, (\mathbf{D}(t) - \mathbf{1} \sum_{i=1}^N l_i^*)^T, (\tilde{\kappa}(t) - \tilde{\kappa}^e(\mathbf{l}^*))^T, (\eta(t) - \eta^*)^T]^T$, and η, η^* are defined in the subsequent proof.

Proof. Following the proof of Theorem 3.1 by using singular perturbation analysis, the reduced-system in τ -time scale is given by

$$\begin{aligned} \frac{dl_i}{d\tau} &= -k_i(2\nu_i\xi_i^2(l_i - \hat{l}_i) + a \sum_{j=1}^N l_j + p_0 + al_i - \eta_{i1} + \eta_{i2}) \\ \frac{d\eta_{i1}}{d\tau} &= m_{i1}\eta_{i1}(l_i^{min} - l_i) \\ \frac{d\eta_{i2}}{d\tau} &= m_{i2}\eta_{i2}(l_i - l_i^{max}), i \in \mathbb{N}. \end{aligned} \quad (3.28)$$

According to Lemma 3.1, the energy consumption game is a potential game. Hence, the Nash seeking can be achieved by solving

$$\begin{aligned} \min_{\mathbf{l}} \quad & Q(\mathbf{l}) \\ \text{subject to} \quad & l_i^{min} \leq l_i \leq l_i^{max}, i \in \mathbb{N}. \end{aligned} \quad (3.29)$$

In the following, we show that (3.28) can be used to solve the problem in (3.29).

Define the Lagrangian function as $L(\mathbf{l}, \eta) = Q(\mathbf{l}) + \sum_{i=1}^N (\eta_{i1}(l_i^{min} - l_i) + \eta_{i2}(l_i - l_i^{max}))$ where $\eta = [\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}, \dots, \eta_{i1}, \eta_{i2}, \dots, \eta_{N1}, \eta_{N2}]^T \in \mathbb{R}_+^{2N}$. The dual problem for the minimization problem in (3.29) can be formulated as

$$\max_{\eta \geq 0} \min_{\mathbf{l}} L(\mathbf{l}, \eta). \quad (3.30)$$

The Hessian matrix of $Q(\mathbf{l})$ is

$$H = \begin{bmatrix} 2\nu_1\xi_1^2 + 2a & a & \cdots & a \\ a & 2\nu_2\xi_2^2 + 2a & \cdots & a \\ \vdots & & \ddots & \vdots \\ a & a & \cdots & 2\nu_N\xi_N^2 + 2a \end{bmatrix}.$$

Since $a < \min_{i \in \mathbb{N}} \frac{2\nu_i\xi_i^2}{N-3}$ for $N > 3$, H is positive definite by the Gershgorin Circle Theorem [112]. Hence, $Q(\mathbf{l})$ is strictly convex in $\mathbf{l}$ as its Hessian matrix is positive definite. Noting that the inequality constraints are linear, the problem in (3.29) has strong duality [150]. Hence, $\mathbf{l}^*$ is the optimal solution to the problem in (3.29) if and only if there exists $\eta^* \in \mathbb{R}_+^{2N}$ such that $(\mathbf{l}^*, \eta^*)$ is the saddle point of $L(\mathbf{l}, \eta)$ by the saddle point theorem [150].

By defining the Lyapunov candidate function as [108]

$$V_L = \frac{1}{2}(\mathbf{l} - \mathbf{l}^*)^T \mathbf{K}^{-1}(\mathbf{l} - \mathbf{l}^*) + \sum_{i=1}^N \sum_{j=1}^2 \frac{1}{m_{ij}} (\eta_{ij} - \eta_{ij}^* - \eta_{ij}^* \log(\eta_{ij}) + \eta_{ij}^* \log(\eta_{ij}^*)), \quad (3.31)$$

where $(\mathbf{l}^*, \eta^*)$ is the saddle point of $L(\mathbf{l}, \eta)$, it can be shown that the saddle point of $L(\mathbf{l}, \eta)$ is globally asymptotically stable under (3.28) by Corollary 2 of [108] given that $\eta_{ij}(0) > 0, i \in \mathbb{N}, j \in \{1, 2\}$.

Hence, the strategy in (3.28) enables $\mathbf{l}$ to converge to the Nash equilibrium of the potential game asymptotically.

Combining this result with the exponential stability of the boundary-layer system (see Theorem 3.1 for boundary-layer analysis), the result can be derived by using Lemma 1 in [90] (see also [102] for more details). $\square$

3.4.2 Nash Equilibrium Seeking for Energy Consumption Game of HVAC Systems with A Unique Inner Nash Equilibrium

In the following, energy consumption game with a unique inner Nash equilibrium is considered ¹. If the constraints do not affect the value of the Nash equilibrium, the Nash seeking strategy can be designed as

$$\dot{D}_i = -D_i - \sum_{j \in \mathcal{N}_i} (D_i - D_j) - \sum_{j \in \mathcal{N}_i} (\kappa_i - \kappa_j) + Nl_i \quad (3.32a)$$

$$\dot{\kappa}_i = \sum_{j \in \mathcal{N}_i} (D_i - D_j) \quad (3.32b)$$

$$\dot{l}_i = -\bar{k}_i \left(\frac{\partial V_i}{\partial l_i} + P(D_i) + al_i \right). \quad (3.32c)$$

By introducing the orthonormal matrix U as in Section 3.3.2, the system in (3.32) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{D}} \\ \dot{\tilde{\kappa}} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \tilde{\kappa} \end{bmatrix} + \begin{bmatrix} N\mathbf{1} \\ \mathbf{0} \end{bmatrix} \quad (3.33a)$$

$$\dot{l}_i = -\bar{k}_i \left(\frac{\partial V_i}{\partial l_i} + P(D_i) + al_i \right), i \in \mathbb{N} \quad (3.33b)$$

and

$$\dot{\kappa}_\mu = 0. \quad (3.34)$$

Theorem 3.3. *Suppose that Assumption 3.1 is satisfied. Then, there exists a positive constant δ^* such that for every $0 < \delta < \delta^*$, $(\mathbf{l}(t), \mathbf{D}(t), \tilde{\kappa}(t))$ converges exponen-*

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¹To make it clear, in this chapter, we say that the Nash equilibrium is an inner Nash equilibrium if the Nash equilibrium satisfies $l_i^{min} < l_i^* < l_i^{max}, \forall i \in \mathbb{N}$, i.e., the Nash equilibrium is achieved at $\frac{\partial C_i}{\partial l_i} = 0, \forall i \in \mathbb{N}$.

tially to $(\mathbf{l}^*, \mathbf{1} \sum_{i=1}^N l_i^*, \tilde{\kappa}^e(\mathbf{l}^*))$ under (3.33).

Proof. Following the proof of Theorem 3.1 by using singular perturbation analysis, the reduced-system at τ -time scale is given by

$$\frac{dl_i}{d\tau} = -k_i(2\nu_i\xi_i^2(l_i - \hat{l}_i) + a \sum_{j=1}^N l_j + p_0 + al_i), i \in \mathbb{N}. \quad (3.35)$$

From (3.35), it can be derived that

$$\dot{\tilde{\mathbf{l}}} = -\mathbf{k}H\tilde{\mathbf{l}}, \quad (3.36)$$

where $\tilde{\mathbf{l}}(t) = \mathbf{l} - \mathbf{l}^*$.

Hence, it can be shown that in τ -time scale

$$\|\tilde{\mathbf{l}}(\tau)\| \leq \sqrt{\frac{\max_{i \in \mathbb{N}}\{k_i\}}{\min_{i \in \mathbb{N}}\{k_i\}}} e^{-\min_{i \in \mathbb{N}}\{k_i\}\lambda_{\min}\{H\}\tau} \|\tilde{\mathbf{l}}(0)\|, \quad (3.37)$$

by defining the Lyapunov candidate function as $V_L = \frac{1}{2}\tilde{\mathbf{l}}^T \mathbf{k}^{-1} \tilde{\mathbf{l}}$ [57].

Combining this result with the exponential stability of the boundary-layer system (see Theorem 3.1 for boundary-layer analysis), the conclusion can be derived by Theorem 11.4 in [88] (see the proof of Theorem 3.1 for more details) . $\square$

Remark 3.2. When the constraints do not affect the value of the Nash equilibrium, the updating strategy in (3.32) is a special case of the one in (3.5). It can be seen that the result in Theorem 3.3 is stronger than the result in Theorem 3.1 under the given HVAC model.

3.4.3 Energy Consumption Game of HVAC Systems with Stubborn Players

In this section, a special case where some players commit to the coordination process while keeping a constant energy consumption is considered. Without loss of generality, we suppose that player i is a stubborn player and updates its action according to

$$\dot{D}_i = -D_i - \sum_{j \in \mathcal{N}_i} (D_i - D_j) - \sum_{j \in \mathcal{N}_i} (\kappa_i - \kappa_j) + Nl_i^s, \quad (3.38a)$$

$$\dot{\kappa}_i = \sum_{j \in \mathcal{N}_i} (D_i - D_j), \quad (3.38b)$$

where l_i^s is the constant energy consumption of player i . Furthermore, all the rational players adopt (3.32) if the constraints do not affect the value of all the players' best response strategies, else, all the rational players adopt (3.24).

By introducing the orthonormal matrix U as in Section 3.3.2, then

$$\begin{bmatrix} \dot{\mathbf{D}} \\ \dot{\tilde{\kappa}} \end{bmatrix} = \begin{bmatrix} -I - L & -L\tilde{U} \\ \tilde{U}^T L & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \tilde{\kappa} \end{bmatrix} + \begin{bmatrix} N\mathbf{l} \\ \mathbf{0} \end{bmatrix}, \quad (3.39)$$

in which the i th component of $\mathbf{l}$ is fixed to be l_i^s , and

$$\dot{\kappa}_\mu = 0. \quad (3.40)$$

Furthermore,

$$\begin{aligned} \dot{l}_j &= -\bar{k}_j \left(\frac{\partial V_j}{\partial l_j} + P(D_j) + al_j - \eta_{j1} + \eta_{j2} \right) \\ \dot{\eta}_{j1} &= \bar{m}_{j1} \eta_{j1} (l_j^{min} - l_j) \\ \dot{\eta}_{j2} &= \bar{m}_{j2} \eta_{j2} (l_j - l_j^{max}), j \in \mathbb{N}, j \neq i \end{aligned} \quad (3.41)$$

if all the rational players adopt (3.24), and

$$\dot{l}_j = -\bar{k}_j \left(\frac{\partial V_j}{\partial l_j} + P(D_j) + al_j \right), j \in \mathbb{N}, j \neq i \quad (3.42)$$

if all the rational players adopt (3.32).

Corollary 3.1. *Suppose that Assumption 3.1 is satisfied. Then, there exists $\beta \in \mathcal{KL}$ such that for each pair of strictly positive real number (Δ, v) , there exists $\delta(\Delta, v) > 0$, such that*

$$\|\chi^{br}(t)\| \leq \beta(\|\chi^{br}(0)\|, \delta t) + v, \quad (3.43)$$

for all $t \geq 0$, $\|\chi^{br}(0)\| \leq \Delta$ under (3.39) and (3.41) given that $\eta_{kj}(0) > 0, k \in \mathbb{N}, k \neq i, j \in \{1, 2\}$. In (3.43), $\chi^{br}(t) = [\check{\mathbf{l}}_{-i}(t)^T, (\mathbf{D}(t) - \mathbf{1}(\sum_{j=1, j \neq i}^N l_j^{br} + l_i^s))^T, (\tilde{\kappa}(t) - \tilde{\kappa}^e(\mathbf{l}_{-i}^{br}, l_i^s))^T, (\eta_{-i}(t) - \eta_{-i}^{br})^T]^T$, $\check{\mathbf{l}}_{-i}(t) = [\check{l}_1(t), \dots, \check{l}_{i-1}(t), \check{l}_{i+1}(t), \dots, \check{l}_N(t)]^T$, $\check{l}_j(t) = l_j(t) - l_j^{br}, j \in \mathbb{N}, j \neq i$ and l_j^{br} denotes the best response strategy of player j , $\mathbf{l}_{-i}^{br}$ is the concatenated vector of $l_j^{br}, i \in \mathbb{N}, j \neq i$, $\eta_{-i}, \eta_{-i}^{br}$ are defined in the subsequent proof.

Furthermore, if the constraints $l_j^{min} \leq l_j \leq l_j^{max}, j \in \mathbb{N}, j \neq i$ do not affect the values of the best response strategies, then, there exists a positive constant δ^* such that for every $0 < \delta < \delta^*$, $(\check{\mathbf{l}}_{-i}(t), \mathbf{D}(t) - \mathbf{1}(\sum_{j=1, j \neq i}^N l_j^{br} + l_i^s), \tilde{\kappa}(t) - \tilde{\kappa}^e(\mathbf{l}_{-i}^{br}, l_i^s))$ converges exponentially to zero under (3.39) and (3.42).

Proof. Following the proof of Theorem 3.1 by using singular perturbation analysis to get the reduced-system for both cases. The first part of the Corollary can be derived by noticing that the following is satisfied,

$$\frac{\partial Q'(\mathbf{l}_{-i})}{\partial l_j} = \frac{\partial C_j}{\partial l_j}, \quad (3.44)$$

for all $j \neq i$, where

$$\begin{aligned} Q'(\mathbf{l}_{-i}) = & \sum_{j=1, j \neq i}^N \nu_j \xi_j^2 (l_j - \hat{l}_j)^2 + \sum_{j=1, j \neq i}^N a \left(\sum_{k=1, k \neq i, k \neq j}^N l_k \right) l_j \\ & + \sum_{j=1, j \neq i}^N (al_j^2 + (p_0 + al_i^s) l_j). \end{aligned} \quad (3.45)$$

Define

$$\begin{aligned} L'(\mathbf{l}_{-i}, \eta_{-i}) = & Q'(\mathbf{l}_{-i}) + \sum_{j=1, j \neq i}^N \eta_{j1} (l_j^{min} - l_j) \\ & + \sum_{j=1, j \neq i}^N \eta_{j2} (l_j - l_j^{max}), \end{aligned} \quad (3.46)$$

where $\eta_{-i} \in \mathbb{R}_+^{2(N-1)}$ denotes the concatenated vector of $\eta_{jk}, j \in \mathbb{N}$ and $j \neq i, k \in \{1, 2\}$.

Then, the dual problem is

$$\max_{\eta_{-i} \geq 0} \min_{\mathbf{l}_{-i}} L'(\mathbf{l}_{-i}, \eta_{-i}).$$

Noticing that $\mathbf{l}_{-i}^{br}$ is the best response strategies of the rational players if and only if there exists $\eta_{-i}^{br} \in \mathbb{R}_+^{2(N-1)}$ such that $(\mathbf{l}_{-i}^{br}, \eta_{-i}^{br})$ is the saddle point of $L'(\mathbf{l}_{-i}, \eta_{-i})$, the rest of the proof follows that of Theorem 3.2.

If the constraints $l_j^{min} \leq l_j \leq l_j^{max}, j \in \mathbb{N}, j \neq i$ do not affect the value of the best response strategies and all the rational players adopt (3.32), then, the reduced-system at the quasi-steady state is

$$\begin{aligned} \frac{dl_j}{d\tau} = & -k_j \frac{\partial C_j(\mathbf{l}_{-i})}{\partial l_j} \\ = & -k_j [2\nu_j \xi_j^2 (l_j - \hat{l}_j) + a \sum_{j=1, j \neq i}^N l_j + p_0 + al_i^s + al_j] \\ = & -k_j [2\nu_j \xi_j^2 \check{l}_j + a \sum_{j=1, j \neq i}^N \check{l}_j + a\check{l}_j], i \in \mathbb{N}. \end{aligned} \quad (3.47)$$

Writing (3.47) in the concatenated form gives

$$\frac{d\check{\mathbf{l}}_{-i}}{d\tau} = -\mathbf{k}H_1\check{\mathbf{l}}_{-i}, \quad (3.48)$$

where $H_1 \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ is defined as

$$H_1 = \begin{bmatrix} h_1 & a & \cdots & & a \\ & \ddots & & & \\ & & h_{i-1} & & \\ & & & h_{i+1} & \vdots \\ \vdots & & & \ddots & \\ a & \cdots & & & h_N \end{bmatrix},$$

with $h_j = 2\nu_j\xi_j^2 + 2a$. Since $a < \min_{i \in \mathbb{N}} \frac{2\nu_i\xi_i^2}{N-3}$ for $N > 3$, it can be derived that the matrix H_1 is symmetric and strictly diagonally dominant with all the diagonal elements positive. Hence, the matrix $-\mathbf{k}H_1$ is Hurwitz by the Gershgorin Circle Theorem [112].

By similar analysis in Theorem 3.3, the conclusion can be derived. $\square$

Remark 3.3. *In this Corollary, only one player (i.e., player i) is supposed to be stubborn. However, this is not restrictive as similar results can be derived if multiple stubborn players exist.*

Remark 3.4. *In the proposed Nash seeking strategy, the players only communicate with its neighbors on D_i and κ_i . They do not communicate on their own energy consumption l_i . Hence, the proposed Nash seeking strategy does not lead to privacy concern for the users. The work in [57] provided an extremum seeking method to seek for the Nash equilibrium in non-cooperative games. However, the method in [57] can't be directly implemented for Nash seeking in the energy consumption game if the aggregate energy consumption is not directly available to the players.*

Figure 3.1: Communication graph for the electricity users

3.5 Numerical Examples

3.5.1 Simulation Setup

In this section, we consider a network of 5 commercial/industrial users that are equipped with HVAC systems. The electricity users communicate with each other via an undirected and connected graph as shown in Figure 3.1.

The cost function for electricity user i is

$$C_i(l_i, \bar{l}) = \nu_i \xi_i^2 (l_i - \hat{l}_i)^2 + P(\bar{l}) l_i, \quad (3.49)$$

where the pricing function $P(\bar{l}) = a \sum_{i=1}^N l_i + p_0$ [14]. Without loss of generality, suppose that $\nu_i \xi_i^2$ for all $i \in \mathbb{N}$ are normalized to 1 in the simulation. For $i \in \{1, 2, 3, 4, 5\}$, $l_i^{min} = 0.8\hat{l}_i$ and $l_i^{max} = 1.2\hat{l}_i$ except that in Section 3.5.2, l_1^{min} and l_1^{max} are set separately. The parameters are listed in Table 3.1².

3.5.2 Energy Consumption Control of HVAC Systems

In this section, we suppose that $l_1^{min} = 45$ and $l_1^{max} = 55$. By direct computation, it can be derived that the Nash equilibrium is $\mathbf{l}^* = (45, 46.4, 51.3, 56.2, 61.1)(\text{kWh})$.

²MU stands for Monetary Unit.

Table 3.1: Parameters in the simulation.

	user 1	user 2	user 3	user 4	user 5
$\hat{l}_i(\text{kWh})$	50	55	60	65	70
$l_i^{\max}(\text{kWh})$	60	66	72	78	84
$l_i^{\min}(\text{kWh})$	40	44	46	52	56
a	0.04				
p_0 (MU/kWh)	5				

Figure 3.2: The users' energy consumptions produced by the proposed seeking strategy in (3.24).

The equilibrium aggregate is $\bar{l}^* = 259.9(\text{kWh})$. Hence, the Nash equilibrium is not an inner Nash equilibrium.

The simulation results by using the seeking strategy in (3.24) are shown in Figures 3.2-3.3.

Figure 3.2 shows the users' electricity energy consumptions produced by the proposed seeking strategy in (3.24) and Figure 3.3 indicates that the users' estimations on the aggregate energy consumptions converge to the actual aggregate energy consumption.

From the simulation results, it can be seen that the energy consumptions produced

Figure 3.3: The actual aggregate energy consumption and the users' estimations on the aggregate energy consumption produced by the proposed seeking strategy in (3.24).

by the proposed method converge to the Nash equilibrium of the energy consumption game.

3.5.3 Energy Consumption Control of HVAC Systems with A Unique inner Nash equilibrium

In this section, we consider the energy consumption game under the setting in Section 3.5.1. By direct calculation, it can be derived that the Nash equilibrium is $\mathbf{I}^* = (41.5, 46.4, 51.3, 56.2, 61.1)(\text{kWh})$. The equilibrium aggregate is $\bar{l}^* = 256.7(\text{kWh})$. The Nash equilibrium is an inner Nash equilibrium and the seeking strategy in (3.32) is used in the simulation. The simulation results produced by the seeking strategy in (3.32) are shown in Figures 3.4-3.5.

From the simulation results, it can be seen that the users' energy consumptions converge to the unique Nash equilibrium.

Figure 3.4: The users' energy consumptions produced by the proposed seeking strategy in (3.32).

Figure 3.5: The actual aggregate energy consumption and the users' estimations on the aggregate energy consumptions produced by the seeking strategy in (3.32).

3.5.4 Energy Consumption Control with Stubborn Players

In this section, we suppose that player 5 is a stubborn player that commits to a constant energy consumption $l_5^s = 100(\text{kWh})$. Then, player 1-4's best response strategies are 40.8(kWh), 45.7(kWh), 50.6(kWh), and 55.5(kWh), respectively. The aggregate energy consumption is 292.7(kWh). In the simulation, the rational players adopt (3.32) to update their actions. The stubborn player uses the seeking strategy in (3.38). The simulation results produced by the proposed method are shown in Figures 3.6-3.7.

Figure 3.6: The users' energy consumptions produced by the proposed method (i.e., the rational players adopt (3.32) and the stubborn player adopts (3.38)).

It can be seen that with the presence of the stubborn player, all the other players' actions converge to the best response strategies with respect to the stubborn action.

3.6 Conclusions

This chapter considers energy consumption control among a network of electricity users. The problem is solved by using an aggregative game on an undirected

Figure 3.7: The actual aggregate energy consumption and the users' estimations on the aggregate energy consumption produced by the proposed method (i.e., the rational players adopt (3.32) and the stubborn player adopts (3.38)).

and connected graph. To estimate the aggregate energy consumption, which is supposed to be unknown to the players during the Nash seeking process, an average consensus protocol is employed. The convergence property is analytically studied via using singular perturbation and Lyapunov stability analysis. A general energy consumption game where multiple Nash equilibria may exist is firstly considered. A Nash seeking strategy based on consensus is proposed to enable the users to search for the Nash equilibrium. Energy consumption control of HVAC systems with linear pricing functions is then studied. Convergence results are provided. Furthermore, stubborn players are investigated and it is shown that the rational players' actions converge to the best response strategies.

Chapter 4

Distributed Extremum Seeking for Constrained Networked Optimization and Its Application to Energy Consumption Control in Smart Grids

4.1 Introduction

As stated in Chapter 1, most of the existing optimization methods require model information. Different from most of the existing model based distributed optimization methods, extremum seeking (e.g., see [16,24,38,57,58,60,90,149]) based methods have an advantage as they only need the measurement of the local cost functions and the constraints while no explicit expressions on them or their gradients are required. To take this advantage, the authors in [57,58,60] used extremum seeking schemes to seek the Nash equilibrium in non-cooperative games. However, since Nash solution

is usually not the pareto-optimal or socially optimal solution, another interesting problem is to solve distributed optimization problems via extremum seeking control.

The authors in [126] proposed an analytic framework based on an extremum seeking control approach to solve the distributed optimization problem in order to locate a socially optimal solution. An interconnection structure of a numerical gradient estimation method and a numerical optimization method with continuous-time system dynamics was proposed and analyzed. This method is based on numerical estimation of the gradients. In [127], the authors proposed a distributed extremum seeking method by using push-sum protocol. The proposed method requires the distributed optimization problem to have a global potential function. Hence, its implementation is restricted to distributed optimization problems with a specific system structure. By using an average consensus protocol, the authors in [121] demonstrated a local convergence result to the solution of a class of unconstrained distributed optimization problem. Similarly, the average consensus protocol was utilized in [122] in the extremum seeking control loop to solve constrained optimization problems with strongly convex social objective functions and constraints. A barrier function was employed to accommodate the constraints and convergence to $O(\mu)$ neighborhood of the optimal solution, where μ is the parameter corresponding to the barrier term, was derived. Distributed resource allocation for multi-agent systems was investigated in [123], where local replicator dynamics and Shahshahani gradient were utilized in the extremum seeker design. The resource allocation was formulated as a distributed optimization problem with decoupled objective functions subject to quasi-static dynamics and a linear simplex constraint. This formulation does not cover general distributed optimization problems. Based on the above observations, we intend to propose a distributed extremum seeking scheme that can be applied to seek for the solution of general convex distributed optimization problems without using model information.

Compared with the existing works, the main contributions of this chapter are summarized as follows.

1. A general non-model based online distributed optimization problem is explored in this chapter. Based on the saddle point dynamics [133], an extremum seeking approach is designed to solve it. In line with standard extremum seeking control, this method does not require the explicit expressions of the local cost functions, the constraint functions and their gradients. By utilizing averaging analysis and Lyapunov stability analysis, the convergence of the proposed method is analytically studied. A general model-free convex optimization problem is firstly investigated which lays foundation for the analysis of distributed optimization problem. A general distributed optimization problem is then studied and a practically asymptotically convergence result is attained.
2. The application of the proposed method to energy consumption control for a network of commercial/industrial energy users is discussed. Several works studied social optimum seeking for demand response in smart grid (e.g., [128–130]). For industrial/commercial consumers with large electricity demand, their energy consumption affects the electricity price. Hence, they need the expression of the electricity pricing function to schedule their energy consumption optimally. Different from model-based methods or centralized methods, the proposed method is model-free, only requiring communication between the users and their neighbors. Using the proposed method, the energy users can search for the social optimum possibly without centralized coordination of the energy provider and the billing payment function can be unknown to the users, which preserves the privacy of the energy provider.

The rest of the chapter is organized as follows. The problem, accompanied by some preliminary results, is formulated in Section 4.2. The main results are presented in

Section 4.4. A non-model based convex optimization problem is firstly explored, followed by the analysis of distributed optimization problems. A brief conclusion is given in Section 6.6.

4.2 Problem Formulation

Problem 4.1. (Non-model Based Distributed Convex Optimization) *Consider a multi-agent system with N agents and the set of agents is denoted by $V = \{1, 2, \dots, N\}$. The agents can communicate with each other via an undirected and connected graph denoted by $G = (V, E)$ with $E \subset V \times V$ representing the edge set. The optimization objective is defined as*

$$\begin{aligned} \min f(x) &= \sum_{i=1}^N f_i(x) \\ \text{s.t. } g_l(x) &\leq 0, l \in \{1, 2, \dots, m\} \end{aligned}$$

where $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ and $f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the cost function for agent i . The explicit expressions of $f_i(x)$, $g_l(x)$, $i \in \{1, 2, \dots, N\}$, $l \in \{1, 2, \dots, m\}$ and their gradients are unknown. Only measurements of the functions $f_i(x)$, $g_l(x)$, $l \in \{1, 2, \dots, m\}$ are available for agent i . The gradients of $f_i(x)$, $g_l(x)$ cannot be measured directly. The objective is to design a distributed updating law for the agents such that they can search for the solution of the optimization problem.

The results of this paper are based on the following assumptions.

Assumption 4.1. *The functions $f_i(x)$, $i \in \{1, 2, \dots, N\}$ are strictly convex in x and sufficiently smooth and $g_l(x)$, $l \in \{1, 2, \dots, m\}$ are convex in x and sufficiently smooth. The solution to the problem in (4.1) exists and is finite. Furthermore, Slater's condition [91] is satisfied.*

Assumption 4.2. Let $g_r(x)$, $r \in \{1, 2, \dots, p\}$, where $p \leq m$, be the active constraints at the optimal solution, i.e., $g_r(x^*) = 0$ and x^* is the optimal solution to Problem 4.1. Then, the matrix $B = [\frac{\partial g_1(x^*)}{\partial x}, \frac{\partial g_2(x^*)}{\partial x}, \dots, \frac{\partial g_p(x^*)}{\partial x}]$ has full column rank.

Remark 4.1. Suppose that each agent uses the proposed updating law to estimate the optimal solution of the distributed optimization problem. Then, Problem 4.1 can be reformulated as [61, 76, 77]

$$\begin{aligned} \min \quad & \check{f}(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}_i) \\ \text{s.t.} \quad & \mathbf{L}\mathbf{x} = \mathbf{0} \\ & g_l(\mathbf{x}_i) \leq 0, l \in \{1, 2, \dots, m\}, i \in \{1, 2, \dots, N\} \end{aligned} \quad (4.1)$$

where $\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{in}]^T \in \mathbb{R}^n$ stands for agent i 's estimation on the optimal solution to Problem 4.1, $\mathbf{L} = L \otimes I_n$, $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T$, L is the Laplacian matrix of the communication topology and I_n is an $n \times n$ dimensional identity matrix.

Lemma 4.1. [61] The solution of the problem shown in (4.1) $\mathbf{x}_i^*, i \in \{1, 2, \dots, N\}$ is equal to the solution of Problem 4.1 x^* under an undirected and connected communication graph.

Based on the above result, a distributed extremum seeking approach will be proposed to seek for the solution of the problem in (4.1) to indirectly solve Problem 4.1.

4.3 Extremum Seeking for Convex Optimization

For presentation convenience and clarity of the main results in Section 4.4, a convex optimization problem is explored by using integrator-type extremum seeking in this section.

The convex optimization problem is considered as,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0, \end{aligned}$$

where $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ is a vector of decision variables. The explicit expressions of $f(x)$, $g(x)$ are unknown. The outputs of $f(x)$, $g(x)$ are measurable while their gradients are not directly measurable.

Assumption 4.3. *The function $f(x)$ is strictly convex in x and sufficiently smooth, and $g(x)$ is convex in x and sufficiently smooth. The solution exists and is finite. Furthermore, Slater's condition is satisfied.*

Remark 4.2. *By this assumption, the convex optimization problem has strong duality. Define $L(x, \lambda) = f(x) + \lambda g(x)$ where $\lambda \in \mathbb{R}_+$ is the Lagrangian multiplier corresponding to the inequality constraint. Then, (x^*, λ^*) is the saddle point of $L(x, \lambda)$ if and only if x^* is the optimal solution to the convex optimization problem by the saddle point theorem [91].*

Based on the saddle point dynamics in [133], the extremum seeker is designed as

$$\begin{aligned} x_i &= \hat{x}_i + b \sin(\omega_i t) \\ \dot{\hat{x}}_i &= -k_i(f(x) + \lambda g(x))b \sin(\omega_i t), \\ \dot{\lambda} &= \alpha \lambda g(x) b^2 \sin^2(\omega_i t), \end{aligned} \tag{4.2}$$

where $k_i = \delta \bar{k}_i$, $\alpha = \delta \bar{\alpha}$, and δ , b are small positive parameters, $\bar{k}_i > 0$, $\bar{\alpha} > 0$ are fixed, $\omega_i > 0$ is the frequency of the sinusoidal dither signal. Furthermore, $\lambda(0) > 0$. In the updating law of λ , i can be chosen as any element in $\{1, 2, \dots, n\}$.

Remark 4.3. *The design of the extremum seeker is motivated by the saddle point dynamics in [133]. It can be shown that the proposed extremum seeker approximates*

the saddle point dynamics by averaging analysis. The details will be given in the proof of the subsequent theorem.

Theorem 4.1. *Suppose that Assumption 4.3 is satisfied, and the set of the saddle point of $L(x, \lambda)$ is a singleton. Then, it is practically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$ under the updating law in (4.2) if the dither frequencies satisfy that $\omega_i \neq \omega_j$, $2\omega_i \neq \omega_j$, $\omega_i \neq \omega_j + \omega_k$ for any $i \neq j \neq k$.*

Proof. See the Appendix (Section 4.8) for the proof. □

Remark 4.4. *In [133], a Lie bracket approximation based extremum seeking method was proposed by utilizing the saddle point dynamics to solve the convex optimization problem. A similar result is derived by using the updating law in (4.2).*

4.4 Main Results: Extremum Seeking for Distributed Optimization

For clarity of presentation, we firstly solve Problem 4.1 without considering the inequality constraints. The general constrained networked optimization problem will be investigated later.

Extremum Seeking for Unconstrained Distributed Optimization

Let $U = [U_1, U_2] \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$, with $U_1 \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn-n}$ and $U_2 \in \mathbb{R}^{Nn} \times \mathbb{R}^n$, be an orthonormal matrix such that $U_2^T \mathbf{L} = \mathbf{0}$ [121]. Then, $\mathbf{L}\mathbf{x} = \mathbf{0}$ if and only if $U^T \mathbf{L}\mathbf{x} = \mathbf{0}$ since U^T is a non-singular matrix. Therefore, Problem 4.1 (without considering the inequality constraints) is equivalent to the following problem

$$\begin{aligned} \min \quad & \check{f}(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}_i) \\ \text{s.t.} \quad & U_1^T \mathbf{L} \mathbf{x} = \mathbf{0}, \end{aligned} \quad (4.3)$$

$$\text{since } U^T \mathbf{L} \mathbf{x} = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \mathbf{L} \mathbf{x} = \begin{bmatrix} U_1^T \mathbf{L} \mathbf{x} \\ \mathbf{0} \end{bmatrix}.$$

Define an augmented Lagrangian function as

$$\tilde{F}(\mathbf{x}, \mathbf{y}) = \check{f}(\mathbf{x}) + \mathbf{x}^T \mathbf{L} U_1 \mathbf{y} + \frac{1}{2} \mathbf{x}^T \mathbf{L} \mathbf{x}, \quad (4.4)$$

where $\mathbf{y}$ is a vector of Lagrangian multipliers defined as $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \dots, \mathbf{y}_{N-1}^T]^T \in \mathbb{R}^{Nn-n}$, $\mathbf{y}_i = [y_{i1}, y_{i2}, \dots, y_{in}]^T \in \mathbb{R}^n$. Then, the following result holds.

Lemma 4.2. *Suppose that Assumption 4.1 holds. If $(\mathbf{x}^*, \mathbf{y}^*) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn-n}$ is a saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$, then, $\mathbf{x}^*$ is a solution of the problem in (4.3). Furthermore, if $\mathbf{x}^*$ is a solution of the problem in (4.3), then, there exists $\mathbf{y}^* \in \mathbb{R}^{Nn-n}$ such that $(\mathbf{x}^*, \mathbf{y}^*) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn-n}$ is a saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$.*

Proof. Noticing that the problem in (4.3) is equivalent to the following problem,

$$\begin{aligned} \min \quad & \tilde{f}(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}_i) + \frac{1}{2} \mathbf{x}^T \mathbf{L} \mathbf{x} \\ \text{s.t.} \quad & U_1^T \mathbf{L} \mathbf{x} = \mathbf{0}, \end{aligned} \quad (4.5)$$

the proof of this Lemma can be completed by the saddle point theorem [91]. $\square$

Remark 4.5. *The Lagrangian function for the problem in (4.3) is*

$$L(\mathbf{x}, \mathbf{y}) = \check{f}(\mathbf{x}) + \mathbf{x}^T \mathbf{L} U_1 \mathbf{y}. \quad (4.6)$$

The benefits of adding $\frac{1}{2} \mathbf{x}^T \mathbf{L} \mathbf{x}$ to the Lagrangian function to construct $\tilde{F}(\mathbf{x}, \mathbf{y})$ were

stated in [77]. According to the Karush–Kuhn–Tucker (KKT) conditions, the following holds at the saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$ [91]:

$$\tilde{F}_{\mathbf{x}}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{0}, U_1^T \mathbf{L} \mathbf{x}^* = \mathbf{0},$$

where $\tilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{y})$ represents for $\frac{\partial \tilde{F}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}}$.

By Lemmas 4.1-4.2, it can be seen that the solution of the distributed problem can be obtained by searching for the saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$.

Inspired by the saddle point dynamics in [61,76,77], the extremum seeking law for agent i is designed as

$$\begin{aligned} x_{ij} &= \hat{x}_{ij} + b_{ij} \sin(\omega_{ij}t) \\ \dot{\hat{x}}_{ij} &= -k_{ij}(f_i(\mathbf{x}_i) \sin(\omega_{ij}t) + \frac{b_{ij}}{2}(\sum_{k=1}^N a_{ik}(\hat{x}_{ij} - \hat{x}_{kj}) + \sum_{k=1}^N a_{ik}(z_{ij} - z_{kj}))) \\ \dot{z}_{ij} &= \theta \sum_{k=1}^N a_{ik}(\hat{x}_{ij} - \hat{x}_{kj}) \end{aligned} \quad (4.7)$$

where $i \in \{1, 2, \dots, N\}$, $j \in \{1, 2, \dots, n\}$ and a_{ik} is the element on the i th row and k th column of the adjacency matrix. The parameters $k_{ij} = \delta b \bar{k}_{ij}$, $\theta = \delta b^2 \bar{\theta}$, $b_{ij} = b \bar{b}_{ij}$ and δ, b are small positive parameters, $\bar{k}_{ij} > 0$, $\bar{\theta} > 0$, $\bar{b}_{ij} > 0$ are fixed, $\omega_{ij} > 0$. Furthermore, $\hat{\mathbf{x}}_i = [\hat{x}_{i1}, \hat{x}_{i2}, \dots, \hat{x}_{in}]^T \in \mathbb{R}^n$, $\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{in}]^T \in \mathbb{R}^n$, and $\mathbf{z} = [z_{ij}]_{vec}$ is a vector of intermediate variables defined as

$$\mathbf{z} = U \begin{bmatrix} \mathbf{y} \\ \mathbf{y}' \end{bmatrix} = [U_1, U_2] \begin{bmatrix} \mathbf{y} \\ \mathbf{y}' \end{bmatrix}, \quad (4.8)$$

where $\mathbf{y}'$ a vector of auxiliary variables.

The concatenated vector form of the updating law in (4.7) is

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -\mathbf{k}([f_i(\mathbf{x}_i) \sin(\omega_{ij}t)]_{vec} + \text{diag}\{\frac{b_{ij}}{2}\}(\mathbf{L}\hat{\mathbf{x}} + \mathbf{L}\mathbf{z})) \\ \dot{\mathbf{z}} &= \theta \mathbf{L}\hat{\mathbf{x}},\end{aligned}\tag{4.9}$$

which is equivalent to

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -\mathbf{k}([f_i(\mathbf{x}_i) \sin(\omega_{ij}t)]_{vec} + \text{diag}\{\frac{b_{ij}}{2}\}(\mathbf{L}\hat{\mathbf{x}} + \mathbf{L}U_1\mathbf{y})) \\ \dot{\mathbf{y}} &= \theta U_1^T \mathbf{L}\hat{\mathbf{x}},\end{aligned}\tag{4.10}$$

and $\mathbf{y}'(t) = \mathbf{y}'(0)$ by substituting (4.8) into (4.9). In (4.9), $\mathbf{k}$ is defined as $\text{diag}\{k_{ij}\}$, $\hat{\mathbf{x}} = [\hat{\mathbf{x}}_1^T, \hat{\mathbf{x}}_2^T, \dots, \hat{\mathbf{x}}_N^T]^T$.

In the subsequent analysis, we show that $(\hat{\mathbf{x}}, \mathbf{y})$ in (4.10) converges to the saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$.

Theorem 4.2. *Suppose that Assumption 4.1 is satisfied, and the set of the saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$ is a singleton. Then, the saddle point of $\tilde{F}(\mathbf{x}, \mathbf{y})$ is practically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) under the updating law in (4.10) if the frequencies satisfy that for every $i \in \{1, 2, \dots, N\}$, $j, k, l \in \{1, 2, \dots, n\}$, $\omega_{ij} \neq \omega_{ik}$, $2\omega_{ij} \neq \omega_{ik}$, $\omega_{ij} \neq \omega_{ik} + \omega_{il}$ if $j \neq k \neq l$.*

Proof. Similar to the proof of Theorem 4.1, we introduce the averaged system as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -\delta b^2 \text{diag}\{\frac{\bar{b}_{ij}\bar{k}_{ij}}{2}\}(\frac{\partial \check{f}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} + \mathbf{L}\hat{\mathbf{x}} + \mathbf{L}U_1\mathbf{y}) \\ \dot{\mathbf{y}} &= \delta b^2 \theta U_1^T \mathbf{L}\hat{\mathbf{x}},\end{aligned}\tag{4.11}$$

which can be rewritten as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -\delta b^2 \text{diag}\left\{\frac{\bar{b}_{ij}\bar{k}_{ij}}{2}\right\} \tilde{F}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}}, \mathbf{y}) \\ \dot{\mathbf{y}} &= \delta b^2 \bar{\theta} U_1^T \mathbf{L} \hat{\mathbf{x}}.\end{aligned}\tag{4.12}$$

Define the Lyapunov candidate function as

$$\begin{aligned}V &= \frac{1}{2}(\hat{\mathbf{x}} - \mathbf{x}^*)^T \left(\text{diag}\left\{\frac{\bar{b}_{ij}\bar{k}_{ij}}{2}\right\} \right)^{-1} (\hat{\mathbf{x}} - \mathbf{x}^*) \\ &\quad + \frac{1}{2\bar{\theta}} (\mathbf{y} - \mathbf{y}^*)^T (\mathbf{y} - \mathbf{y}^*).\end{aligned}$$

Taking the time derivative of V yields

$$\begin{aligned}\dot{V} &= -\delta b^2 (\hat{\mathbf{x}} - \mathbf{x}^*)^T \tilde{F}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}}, \mathbf{y}) + \delta b^2 (\mathbf{y} - \mathbf{y}^*)^T U_1^T \mathbf{L} \hat{\mathbf{x}} \\ &\leq \delta b^2 (\tilde{F}(\mathbf{x}^*, \mathbf{y}) - \tilde{F}(\hat{\mathbf{x}}, \mathbf{y}^*)) \\ &= \delta b^2 (\tilde{F}(\mathbf{x}^*, \mathbf{y}) - \tilde{F}(\mathbf{x}^*, \mathbf{y}^*) + \tilde{F}(\mathbf{x}^*, \mathbf{y}^*) - \tilde{F}(\hat{\mathbf{x}}, \mathbf{y}^*)) \\ &\leq 0.\end{aligned}$$

Hence, the saddle point of $\tilde{F}(\hat{\mathbf{x}}, \mathbf{y})$ is stable. Furthermore, $\dot{V} = 0$ if and only if $\tilde{F}(\mathbf{x}^*, \mathbf{y}) = \tilde{F}(\mathbf{x}^*, \mathbf{y}^*)$ and $\tilde{F}(\hat{\mathbf{x}}, \mathbf{y}^*) = \tilde{F}(\mathbf{x}^*, \mathbf{y}^*)$ by the definition of saddle point. Since $\tilde{F}(\hat{\mathbf{x}}, \mathbf{y})$ is strictly convex in $\hat{\mathbf{x}}$, $\tilde{F}(\hat{\mathbf{x}}, \mathbf{y}^*) = \tilde{F}(\mathbf{x}^*, \mathbf{y}^*)$ if and only if $\hat{\mathbf{x}} = \mathbf{x}^*$. Hence, $\hat{\mathbf{x}} = \mathbf{x}^*$ for $\dot{V} = 0$ which indicates that $\frac{\partial \tilde{F}}{\partial \hat{\mathbf{x}}} = 0$ and $U_1^T \mathbf{L} \hat{\mathbf{x}} = \mathbf{0}$ for $\dot{V} = 0$. Therefore, the KKT conditions are satisfied at $\dot{V} = 0$. Since the set of the saddle point of $\tilde{F}$ is a singleton, $\dot{V} = 0$ if and only if $(\hat{\mathbf{x}}, \mathbf{y}) = (\mathbf{x}^*, \mathbf{y}^*)$. Therefore,

$$\dot{V} < 0$$

for any $(\hat{\mathbf{x}}, \mathbf{y}) \neq (\mathbf{x}^*, \mathbf{y}^*)$ and the saddle point $(\mathbf{x}^*, \mathbf{y}^*)$ is globally asymptotically stable under the averaged system in (4.11).

To quantify the gap between the original system and the averaged system, similar coordinate transformation as in the proof of Theorem 4.1 can be conducted. By similar analysis as in Theorem 4.1, it can be derived that the system in (4.10) is practically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) . The detailed proof is omitted as it follows that of Theorem 4.1. $\square$

Remark 4.6. *By the above analysis, it can be concluded that $(\hat{\mathbf{x}}, \mathbf{z})$ in (4.9) converges to a neighborhood of $\left(\mathbf{x}^*, U \begin{bmatrix} \mathbf{y}^* \\ \mathbf{y}'(0) \end{bmatrix} \right)$. Furthermore, by using Lemmas 4.1-4.2, $\hat{\mathbf{x}}_i, i \in \{1, 2, \dots, N\}$ converge to a small neighborhood of the optimal solution to Problem 4.1 (without considering the inequality constraint).*

Extremum Seeking for Constrained Distributed Optimization

We are now ready to solve the constrained distributed optimization problem.

Define the augmented Lagrangian function as

$$L(\mathbf{x}, \mathbf{y}, \lambda) = \check{f}(\mathbf{x}) + \mathbf{x}^T \mathbf{L} U_1 \mathbf{y} + \frac{1}{2} \mathbf{x}^T \mathbf{L} \mathbf{x} + \sum_{i=1}^N \sum_{j=1}^m \lambda_{ij} g_j(\mathbf{x}_i),$$

where $\lambda \in \mathbb{R}_+^{Nm}$ is the concatenated vector of $\lambda_{il}, i \in \{1, 2, \dots, N\}, l \in \{1, 2, \dots, m\}$.

Design the updating law as

$$\begin{aligned} x_{ij} &= \hat{x}_{ij} + b_{ij} \sin(\omega_{ij} t) \\ \dot{\hat{x}}_{ij} &= -k_{ij} \left((f_i(\mathbf{x}_i) + \sum_{l=1}^m \lambda_{il} g_l(\mathbf{x}_i)) \sin(\omega_{ij} t) \right. \\ &\quad \left. + \frac{b_{ij}}{2} \left(\sum_{k=1}^N a_{ik} (\hat{x}_{ij} - \hat{x}_{kj}) + \sum_{k=1}^N a_{ik} (z_{ij} - z_{kj}) \right) \right) \\ \dot{z}_{ij} &= \theta \sum_{k=1}^N a_{ik} (\hat{x}_{ij} - \hat{x}_{kj}) \\ \dot{\lambda}_{il} &= \alpha_{il} \lambda_{il} g_l(\mathbf{x}_i) b_{ij}^2 \sin^2(\omega_{ij} t), \end{aligned} \tag{4.13}$$

and $\lambda_{il}(0) > 0$, for $i \in \{1, 2, \dots, N\}$, $j \in \{1, 2, \dots, n\}$, $l \in \{1, 2, \dots, m\}$, $\alpha_{il} = \delta \bar{\alpha}_{il}$ where $\bar{\alpha}_{il} > 0$ and is fixed. In the updating law of λ_{il} , j can be chosen as any element in $\{1, 2, \dots, n\}$.

The schematic outline of the proposed distributed extremum seeking method is shown in Figure 4.1 and the updatings of ψ_{ij} and λ_{ij} in Figure 4.1 are shown in Figure 4.2.

Figure 4.1: The schematic outline of the proposed distributed extremum seeking method.

Figure 4.2: The updatings of ψ_{ij} and λ_{ij} in Figure 4.1.

Substituting (4.8) into (4.13) gives the system as

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= -\mathbf{k}([(f_i(\mathbf{x}_i) + \sum_{l=1}^m \lambda_{il} g_l(\mathbf{x}_i)) \sin(\omega_{ij} t)]_{vec} + \text{diag}\{\frac{b_{ij}}{2}\}(\mathbf{L}\hat{\mathbf{x}} + \mathbf{L}U_1\mathbf{y})) \quad (4.14) \\ \dot{\mathbf{y}} &= \theta U_1^T \mathbf{L} \hat{\mathbf{x}} \\ \dot{\lambda}_{il} &= \alpha_{il} \lambda_{il} g_l(\mathbf{x}_i) b_{ij}^2 \sin^2(\omega_{ij} t), \\ i &\in \{1, 2, \dots, N\}, l \in \{1, 2, \dots, m\}, \end{aligned}$$

with $\mathbf{y}'(t) = \mathbf{y}'(0)$.

Theorem 4.3. *Suppose that Assumptions 4.1-4.2 are satisfied, and the set of the saddle point of $L(\mathbf{x}, \mathbf{y}, \lambda)$ is a singleton. Then, the saddle point of $L(\mathbf{x}, \mathbf{y}, \lambda)$ is practically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) with respect to $\mathbb{R}^{Nn} \times \mathbb{R}^{Nn-n} \times \mathbb{R}_{++}^{Nm}$ under the updating law in (4.14) if the frequencies satisfy that for every $i \in \{1, 2, \dots, N\}$, $j, k, l \in \{1, 2, \dots, n\}$, $\omega_{ij} \neq \omega_{ik}$, $2\omega_{ij} \neq \omega_{ik}$, $\omega_{ij} \neq \omega_{ik} + \omega_{il}$ if $j \neq k \neq l$.*

Proof. Similar to the proofs of Theorems 4.1-4.2. □

Remark 4.7. *By using similar result as Lemma 4.2, the convergence to the solution of the problem in (4.1) can be concluded. The proposed extremum seeking scheme achieves the optimization of the objective function by using only measurements of the local cost functions and the constraint functions. The analysis is mainly based on averaging analysis, Lyapunov stability analysis and the saddle point algorithm. In the extremum seeking laws in (4.7) and (4.13), common parameter θ is used through the agents for simplicity. However, this restriction can be easily released by employing consensus protocols to synchronize the parameter in a distributed fashion.*

Remark 4.8. *The design of the proposed extremum seeker is based on the saddle point dynamics. The study of the saddle point dynamics can be traced back to [119, 120, 131] and they were further investigated in [113–118, 132, 133] and the references therein. To achieve the non-negativity of the Lagrangian multipliers corresponding to the inequality constraints, the projection method (see e.g., [118]) and a smooth vector based method proposed in [133] can be used. In this paper, we adopt the smooth vector method in [133] to ensure that the Lagrangian multipliers corresponding to the inequality constraints are non-negative and the updating law is smooth such that averaging analysis can be used in the analysis.*

4.5 Energy Consumption Control in Smart Grids

4.5.1 System Model and Problem Formulation

In this section, a network of N commercial/industrial users whose energy consumptions have dominant effect on the electricity price is considered. The users are supposed to be equipped with an energy-management controller (EMC) and an advanced metering infrastructure (AMI) [14]. The EMC is used to schedule the usage of electricity for the corresponding user. The AMI enables bidirectional communication among the users and the energy provider. For privacy and convenience

concerns of the energy provider who may change the pricing curve according to necessity, we only suppose that the output of the cost function is measurable to the users but the explicit expression on the cost function is unknown. The users are supposed to communicate with their neighbors in real time. The communication graph is assumed to be undirected and connected and is under ideal transmission conditions, i.e., transmission problems such as loss of data and transmission delay are not considered.

The cost of user i is [14]

$$C_i(\mathbf{l}) = \mu_i V_i(l_i) + P\left(\sum_{j=1}^N l_j, l_i\right), \quad (4.15)$$

where l_i denotes the energy consumption of user i , and $\mathbf{l}$ is the concatenated vector of l_i . Furthermore, $V_i(l_i)$ is the cost of consuming l_i . Quadratic functions and logarithmic functions can be used to quantify this cost [109]. For example, it denotes discomfort cost in [14] and in this case, μ_i is a parameter that determines the importance of comfort for the users. The term $P(\sum_{j=1}^N l_j, l_i)$ corresponds to the billing payment for the usage of l_i . The billing function $P(\sum_{j=1}^N l_j, l_i)$ is a function of the aggregate energy consumption $\sum_{j=1}^N l_j$ and its own energy consumption l_i .

For each user, the energy consumption should be within its acceptable range, i.e., $l_i \in [l_i^{\min}, l_i^{\max}]$ where $l_i^{\min}$, $l_i^{\max}$ are the minimal and maximal acceptable energy consumption for user i , respectively. Hence, the social cost minimization problem can be formulated as:

Problem 4.2. (Social cost minimization) *The energy users coordinate with each other to solve*

$$\begin{aligned} \min \quad & C(\mathbf{l}) = \sum_{i=1}^N C_i(\mathbf{l}) \\ \text{s.t.} \quad & l_i^{\min} \leq l_i \leq l_i^{\max}, i \in \{1, 2, \dots, N\}. \end{aligned} \quad (4.16)$$

Remark 4.9. *In this paper, the electricity users are supposed to schedule their electricity consumption by coordinating with each other to solve the social cost minimization problem. Though the electricity users are self-interested in nature, they can be driven to coordinate if they can benefit from the coordination. Incentive provoking mechanisms, such as benefit sharing mechanisms which is beyond the scope of this paper, can be designed to motivate the electricity users to participate in the coordination.*

4.5.2 Social Optimum Seeking Law

Let $\bar{l}_{ij}$ denote user i 's estimation on user j 's energy consumption. By the results in Section 4.4, user i , $i \in \{1, 2, \dots, N\}$ can update its estimations according to ¹

$$\begin{aligned}\dot{\bar{l}}_{ij} &= -k_{ij}(C_i(\mathbf{l}_i) \sin(\omega_{ij}t) + \frac{b_{ij}}{2} \sum_{k=1}^N a_{ik}(\bar{l}_{ij} - \bar{l}_{kj}) + \frac{b_{ij}}{2} \sum_{k=1}^N a_{ik}(z_{ij} - z_{kj})) \\ \dot{z}_{ij} &= \theta_i \sum_{k=1}^N a_{ik}(\bar{l}_{ij} - \bar{l}_{kj}), j \in \{1, 2, \dots, N\},\end{aligned}$$

where $l_{ij} = \bar{l}_{ij} + b_{ij} \sin(\omega_{ij}t)$ and $\mathbf{l}_i = [l_{i1}, l_{i2}, \dots, l_{iN}]^T$, $\bar{\mathbf{l}}_i = [\bar{l}_{i1}, \bar{l}_{i2}, \dots, \bar{l}_{iN}]^T$. Furthermore, θ_i can be either positive constant gains broadcasted by the energy provider such that $\theta_1 = \theta_2 = \dots = \theta_N = \theta$ or synchronized among the electricity users by using consensus protocols (e.g., [100]).

By the result in Section 4.4, $\bar{\mathbf{l}}_i$ for $i \in \{1, 2, \dots, N\}$ converge to a small neighborhood of the socially optimal solution by the proposed updating law suppose that Assumption 4.1 is satisfied. To improve the performance of the extremum seeking controller, a low-pass filter might be added to the extremum seeking loop (e.g., see [16,90]). The low-pass filter might help to deal with high frequency measurement noise and improve transient performance [90]. Readers are referred to [16,90]

¹The local constraint $l_i^{\min} \leq l_i \leq l_i^{\max}$, $i \in \{1, 2, \dots, N\}$ is not considered in the updating law as it can be addressed by a direct projection.

for more details on the integration of the the low-pass filter in the extremum seeking loop.

4.6 A Numerical Example of Consumers with HVAC Systems

In this section, an electricity market with a network of industrial users equipped with heating ventilation and air conditioning (HVAC) systems is considered. For HVAC systems, the discomfort term in (4.15) can be modelled as [14]

$$V_i(l_i) = \rho_i \gamma_i^2 (l_i - \hat{l}_i)^2,$$

where ρ_i is the cost coefficient, γ_i is a parameter that specifies thermal characteristic of the HVAC system and $\hat{l}_i$ is the energy needed to maintain the indoor temperature. Furthermore,²

$$P\left(\sum_{i=1}^N l_i, l_i\right) = \left(k\left(\sum_{i=1}^N l_i - L^*\right) + p_0\right) l_i,$$

where L^* is the expected demand and k is a pricing parameter.

Therefore, the social cost minimization problem is

$$\min_{\mathbf{l}} C(\mathbf{l}) = \sum_{i=1}^5 C_i(\mathbf{l}).$$

In the simulation, the parameters are set according to Table 1 and $L^* = 0.8 \sum_{i=1}^5 \hat{l}_i$.

By direct calculation, it can be derived that the optimal solution is $l_1^* = 82.23$, $l_2^* = 103.63$, $l_3^* = 124.93$, $l_4^* = 146.14$, $l_5^* = 167.27$, $\sum_{i=1}^5 l_i^* = 624.21$. Furthermore, communication graph in Figure. 3.1 is used among the users.

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²Note that under this billing payment function, $C_i(\mathbf{l})$ is not strictly convex in $\mathbf{l}$, indicating that Assumption 4.1 might be conservative for analysis.

user	1	2	3	4	5
$\rho_i \gamma_i^2$	5.2	5.4	5.6	5.8	6
$\hat{l}_i$	120	140	160	180	200
μ_i			0.8		
p_0			10		
k			0.5		

Table 4.1: Parameters in the simulation.

The simulation results by using the proposed extremum seeker with a low-pass filter are shown in Figures. 4.3-4.8 with the initial conditions $\bar{l}_{ij}(0) = 0$, for $i, j \in \{1, 2, \dots, 5\}$.

Figure 4.3: Estimations on user 1's optimal energy consumption l_1^* .

From these figures, it can be seen that the users' estimations on the optimal energy consumption converge to a small neighborhood of the actual optimal solution.

To show the convergence property more clearly, we define e_{ij} as the relative error of user i 's estimation on user j 's optimal energy consumption, i.e.,

$$e_{ij}(t) = \frac{|l_{ij}(t) - l_j^*|}{l_j^*} \times 100\%,$$

for $i, j \in \{1, 2, 3, 4, 5\}$. The relative estimation error in the simulation is shown in

Figure 4.4: Estimations on user 2's optimal energy consumption l_2^* .

Figure 4.5: Estimations on user 3's optimal energy consumption l_3^* .

Figure 4.6: Estimations on user 4's optimal energy consumption l_4^* .

Figure 4.7: Estimations on user 5's optimal energy consumption l_5^* .

Figure 4.8: Users' estimations on the sum of the optimal energy consumptions $\sum_{i=1}^5 l_i^*$.

Figure. 4.9.

Figure 4.9: Relative estimation errors $e_{ij}(t)$, $i, j \in \{1, 2, 3, 4, 5\}$. in the simulation.

Similarly, user i 's relative estimation error on the sum of the optimal energy consumptions is defined as

$$E_i(t) = \frac{\left| \sum_{j=1}^5 l_{ij}(t) - \sum_{i=1}^5 l_i^* \right|}{\sum_{i=1}^5 l_i^*} \times 100\%.$$

User 1-5's relative estimation errors on the sum of the optimal energy consumptions are shown in Figure. 4.10.

Figure 4.10: User 1-5's relative estimation errors on the sum of the optimal energy consumptions.

The relative estimation errors converge to a small neighborhood of 0 in the simulation which implies the convergence of the estimated energy consumptions.

4.7 Conclusion

In this chapter, a distributed extremum seeking method is proposed to seek for the solution of a general convex networked optimization problem. The extremum seeker design is based on the smooth saddle point dynamics in [133]. The conver-

gence analysis for the proposed distributed extremum seeking scheme is provided. A practically stable result is derived without using the explicit expressions on the constraints, the cost functions or their gradients.

4.8 Appendix: Proof of Theorem 4.1

Let T be a the common multiple of the periods of $\sin(\omega_i t)$, $i \in \{1, 2, \dots, n\}$, then, $\frac{1}{T} \int_0^T \sin(\omega_i \tau) d\tau = 0$, $\frac{1}{T} \int_0^T \sin(\omega_i \tau) \sin(\omega_j \tau) d\tau = 0$, for $\omega_i \neq \omega_j$ and $\frac{1}{T} \int_0^T \sin^2(\omega_i \tau) d\tau = \frac{1}{2}$. Furthermore, if $2\omega_i \neq \omega_j$, $\omega_i \neq \omega_j + \omega_k$, Then [57],

$$\begin{aligned} \frac{1}{T} \int_0^T \sin^3(\omega_i \tau) d\tau &= 0, \\ \frac{1}{T} \int_0^T \sin^2(\omega_i \tau) \sin(\omega_j \tau) d\tau &= 0, \\ \frac{1}{T} \int_0^T \sin(\omega_i \tau) \sin(\omega_j \tau) \sin(\omega_k \tau) d\tau &= 0. \end{aligned}$$

From another aspect, by using multi-variable Taylor expansion, it can be derived that

$$\begin{aligned} (f(x) + \lambda g(x)) \sin(\omega_i t) &= f(\hat{x}) \sin(\omega_i t) \\ &+ \sum_{k=1}^n \frac{\partial f(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t) + b^2 R_{i1} + \\ &\lambda (g(\hat{x}) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t) + b^2 R_{i2}) \end{aligned}$$

and

$$\lambda g(x) \sin^2(\omega_i t) = \lambda (g(\hat{x}) \sin^2(\omega_i t) + \sum_{j=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_j} b \sin(\omega_j t) \sin^2(\omega_i t) + b^2 R_{i3}),$$

where R_{i1} , R_{i2} , R_{i3} are the higher-order terms and $\hat{x}$ is the concatenated vector of

$\hat{x}_i$, i.e., $\hat{x} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n]^T$. By using averaging analysis,

$$\eta_i = \frac{1}{T} \int_0^T F_{i1}(\tau, \hat{x}, \lambda) d\tau = \frac{b}{2} \left(\frac{\partial f(\hat{x})}{\partial \hat{x}_i} + \lambda \frac{\partial g(\hat{x})}{\partial \hat{x}_i} \right).$$

where $F_{i1}(t, \hat{x}, \lambda) = f(\hat{x}) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial f(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t) + \lambda (g(\hat{x}) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t))$.

Similarly,

$$v = \frac{1}{T} \int_0^T F_{i2}(\tau, \hat{x}, \lambda) d\tau = \frac{1}{2} \lambda g(\hat{x}), \quad (4.17)$$

where $F_{i2}(t, \hat{x}, \lambda) = \lambda (g(\hat{x}) \sin^2(\omega_i t) + \sum_{j=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_j} b \sin(\omega_j t) \sin^2(\omega_i t))$. Hence, we introduce the averaged system as

$$\begin{aligned} \dot{\hat{x}}_i &= -\frac{\delta b^2 \bar{k}_i}{2} \left(\frac{\partial f(\hat{x})}{\partial \hat{x}_i} + \lambda \frac{\partial g(\hat{x})}{\partial \hat{x}_i} \right) \\ \dot{\lambda} &= \frac{\delta b^2 \bar{\alpha}}{2} \lambda g(\hat{x}). \end{aligned} \quad (4.18)$$

By defining a positive definite and radially unbounded (with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$) Lyapunov function candidate as [133]

$$\begin{aligned} V &= \frac{1}{2} (\hat{x} - x^*)^T \left(\text{diag} \left\{ \frac{\bar{k}_i}{2} \right\} \right)^{-1} (\hat{x} - x^*) \\ &\quad + \frac{2}{\bar{\alpha}} (\lambda - \lambda^* - \lambda^* \log(\lambda) + \lambda^* \log(\lambda^*)), \end{aligned} \quad (4.19)$$

where (x^*, λ^*) denotes the saddle point of $L(x, \lambda)$, it can be derived that

$$\dot{V} \leq \delta b^2 (L(x^*, \lambda) - L(\hat{x}, \lambda^*)) < 0,$$

for every $(\hat{x}, \lambda) \neq (x^*, \lambda^*)$ by Theorem 1 of [133] and the unique saddle point is asymptotically stable under (4.18) with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$.

In the subsequent analysis, we show that the saddle point of $L(x, \lambda)$ is practical-

ly asymptotically stable with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$, uniformly in (δ, b^2) under the updating law in (4.2).

Inspired by Section 10.4 of [88] and the proof of Theorem 1 in [90], define

$$h_i(t, \hat{x}, \lambda) = -\bar{k}_i F_{i1}(t, \hat{x}, \lambda) + \frac{b\bar{k}_i}{2} \left(\frac{\partial f(\hat{x})}{\partial \hat{x}_i} + \lambda \frac{\partial g(\hat{x})}{\partial \hat{x}_i} \right),$$

$i = 1, 2, \dots, n$ and

$$h_{n+1}(t, \hat{x}, \lambda) = \bar{\alpha} b F_{i2}(t, \hat{x}, \lambda) - \frac{\bar{\alpha} b}{2} \lambda g(\hat{x}).$$

Then, $h_i(t, \hat{x}, \lambda)$, $i = 1, 2, \dots, n+1$, is periodic in t and has zero mean for $(\hat{x}, \lambda)$ that belongs to any compact set.

Define

$$u_i(t, \hat{x}, \lambda) = \int_0^t h_i(\tau, \hat{x}, \lambda) d\tau,$$

for all $i = 1, 2, \dots, n+1$. Then, $u_i(t, \hat{x}, \lambda)$ are periodic in t and bounded for $(\hat{x}, \lambda)$ that belongs to any compact set. Furthermore,

$$\begin{aligned} \frac{\partial u_i(t, \hat{x}, \lambda)}{\partial t} &= h_i(t, \hat{x}, \lambda) \\ \frac{\partial u_i(t, \hat{x}, \lambda)}{\partial \hat{x}} &= \int_0^t \frac{\partial h_i(\tau, \hat{x}, \lambda)}{\partial \hat{x}} d\tau \\ \frac{\partial u_i(t, \hat{x}, \lambda)}{\partial \lambda} &= \int_0^t \frac{\partial h_i(\tau, \hat{x}, \lambda)}{\partial \lambda} d\tau, \end{aligned}$$

are periodic and bounded for $(\hat{x}, \lambda)$ that belongs to any compact set.

Define a new coordinate (e, φ) according to

$$\hat{x} = e + \delta b U(t, e, \varphi) \tag{4.20}$$

$$\lambda = \varphi + \delta b u_{n+1}(t, e, \varphi),$$

where $U(t, e, \varphi) = [u_1, u_2, \dots, u_n]^T$. Differentiating both sides of (4.20) gives

$$\begin{aligned}\dot{\hat{x}} &= \dot{e} + \delta b \left(\frac{\partial U(t, e, \varphi)}{\partial t} + \left(\frac{\partial U^T(t, e, \varphi)}{\partial e} \right)^T \dot{e} + \frac{\partial U(t, e, \varphi)}{\partial \varphi} \dot{\varphi} \right) \\ \dot{\lambda} &= \dot{\varphi} + \delta b \left(\frac{\partial u_{n+1}(t, e, \varphi)}{\partial t} + \frac{\partial u_{n+1}(t, e, \varphi)}{\partial e} \dot{e} + \frac{\partial u_{n+1}(t, e, \varphi)}{\partial \varphi} \dot{\varphi} \right).\end{aligned}$$

Therefore,

$$\begin{aligned}& \left(I + \delta b \begin{bmatrix} \left(\frac{\partial U^T(t, e, \varphi)}{\partial e} \right)^T & \frac{\partial U(t, e, \varphi)}{\partial \varphi} \\ \frac{\partial u_{n+1}(t, e, \varphi)}{\partial e} & \frac{\partial u_{n+1}(t, e, \varphi)}{\partial \varphi} \end{bmatrix} \right) \begin{bmatrix} \dot{e} \\ \dot{\varphi} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\hat{x}} \\ \dot{\lambda} \end{bmatrix} - \delta b \begin{bmatrix} \frac{\partial U(t, e, \varphi)}{\partial t} \\ \frac{\partial u_{n+1}(t, e, \varphi)}{\partial t} \end{bmatrix} \\ &= -\delta b^2 \begin{bmatrix} \left[\frac{\bar{k}_i}{2} \left(\frac{\partial f(e)}{\partial e_i} + \varphi \frac{\partial g(e)}{\partial e_i} \right) \right]_{vec} \\ -\frac{\bar{\alpha}}{2} \varphi g(e) \end{bmatrix} + \delta b \begin{bmatrix} q \\ q_{n+1} \end{bmatrix},\end{aligned}$$

where $q = [q_1, q_2, \dots, q_n]^T$ and

$$\begin{aligned}q_i &= -\bar{k}_i (f(\hat{x}) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial f(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t) \\ &\quad + \lambda (g(\hat{x}) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_k} b \sin(\omega_k t) \sin(\omega_i t))) + \bar{k}_i (f(e) \sin(\omega_i t) \\ &\quad + \sum_{k=1}^n \frac{\partial f(e)}{\partial e_k} b \sin(\omega_k t) \sin(\omega_i t) \\ &\quad + \varphi (g(e) \sin(\omega_i t) + \sum_{k=1}^n \frac{\partial g(e)}{\partial e_k} b \sin(\omega_k t) \sin(\omega_i t))) + O(b^2).\end{aligned}$$

Furthermore,

$$\begin{aligned}q_{n+1}(t, e, \varphi) &= b\bar{\alpha} \lambda (g(\hat{x}) \sin^2(\omega_i t) + \sum_{j=1}^n \frac{\partial g(\hat{x})}{\partial \hat{x}_j} b \sin(\omega_j t) \sin^2(\omega_i t)) \\ &\quad - b\bar{\alpha} \varphi (g(e) \sin^2(\omega_i t) + \sum_{j=1}^n \frac{\partial g(e)}{\partial e_j} b \sin(\omega_j t) \sin^2(\omega_i t)) + O(b^3)\end{aligned}$$

as $R_{i1} = O(1)$, $R_{i2} = O(1)$, $R_{i3} = O(1)$ for (e, φ) that belongs to any compact set.

By the mean value theorem, it can be derived that $q_i = O(\delta b) + O(b^2) + O(\delta b^2)$, $i \in \{1, 2, \dots, n+1\}$ for (e, φ) that belongs to any compact set by using (4.20).

$$\begin{aligned} & \text{Since } \begin{bmatrix} \left(\frac{\partial U^T(t, e, \varphi)}{\partial e}\right)^T & \frac{\partial U(t, e, \varphi)}{\partial \varphi} \\ \frac{\partial u_{n+1}(t, e, \varphi)}{\partial e} & \frac{\partial u_{n+1}(t, e, \varphi)}{\partial \varphi} \end{bmatrix} \text{ is bounded for } (e, \varphi) \text{ that belongs to any com-} \\ & \text{pact set, } \left(I + \delta b \begin{bmatrix} \left(\frac{\partial U^T(t, e, \varphi)}{\partial e}\right)^T & \frac{\partial U(t, e, \varphi)}{\partial \varphi} \\ \frac{\partial u_{n+1}(t, e, \varphi)}{\partial e} & \frac{\partial u_{n+1}(t, e, \varphi)}{\partial \varphi} \end{bmatrix} \right) \text{ is non-singular for sufficiently small} \\ & \delta b \text{ and } \left(I + \delta b \begin{bmatrix} \left(\frac{\partial U^T(t, e, \varphi)}{\partial e}\right)^T & \frac{\partial U(t, e, \varphi)}{\partial \varphi} \\ \frac{\partial u_{n+1}(t, e, \varphi)}{\partial e} & \frac{\partial u_{n+1}(t, e, \varphi)}{\partial \varphi} \end{bmatrix} \right)^{-1} = I + O(\delta b). \text{ Therefore,} \\ & \begin{bmatrix} \dot{e} \\ \dot{\varphi} \end{bmatrix} = -\delta b^2 \begin{bmatrix} \left[\frac{\bar{k}_i}{2} \left(\frac{\partial f(e)}{\partial e_i} + \varphi \frac{\partial g(e)}{\partial e_i} \right) \right]_{vec} \\ -\frac{\bar{\alpha}}{2} \varphi g(e) \end{bmatrix} + O(\delta^2 b^2) + O(\delta b^3) + O(\delta^2 b^3). \end{aligned}$$

Hence, by using the same Lyapunov function in (4.19), it can be derived that

$$\dot{V} \leq \delta b^2 (L(e^*, \varphi) - L(e, \varphi^*)) + O(\delta^2 b^2) + O(\delta b^3) + O(\delta^2 b^3).$$

Since $L(e^*, \varphi) - L(e, \varphi^*) < 0$ for every $(e, \varphi) \neq (e^*, \varphi^*)$, there exists a class $\mathcal{K}$ function β_3 such that

$$\dot{V} \leq -\delta b^2 \beta_3(V) + O(\delta^2 b^2) + O(\delta b^3) + O(\delta^2 b^3),$$

by noticing that the Lyapunov function is positive definite and radially unbounded with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$, which indicates that there exist class $\mathcal{K}_\infty$ functions β_1, β_2 such that $\beta_1(\|\sigma\|) \leq V \leq \beta_2(\|\sigma\|)$ where $\sigma = [(e - e^*)^T, \varphi - \varphi^*]^T$.

Therefore, for every $\|\sigma\| > \beta_1^{-1}(\beta_3^{-1}(2(O(\delta) + O(b) + O(\delta b))))$, $\dot{V} < -\frac{\delta b^2}{2} \beta_3(V)$. Hence, the system in (4.21) is practically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$.

Furthermore, by using (4.20), it can be derived that the system in (4.2) is prac-

tically asymptotically stable with arbitrary bounded initial condition, uniformly in (δ, b^2) with respect to $\mathbb{R}^n \times \mathbb{R}_{++}$.

Chapter 5

Distributed Time-varying Optimization for Systems with Quadratic Objective Functions under Undirected Graphs

5.1 Introduction

Most of the existing works on distributed optimization problems contribute to distributed optimization communities by broadening the applicability of distributed optimization techniques to various communication conditions (e.g., undirected graph, directed graph, switching topology and time-varying communication conditions) and by providing new views and methods to solve distributed optimization problems. However, time-varying distributed optimization is rarely studied in the existing literature. The works in [134–141] studied discrete-time dynamic optimization problem where the optimal solution is constant at each time step but is different for different steps. Mathematically, they considered a network of N agents that are

coordinating with each other to solve

$$\min \sum_{i=1}^N f_i^k(x),$$

where $f_i(\cdot)$ denotes the objective function of agent i , k denotes the time instant, and x is the decision variable to be determined. In [135], the authors considered dynamic optimization problem via ADMM and showed that primal and dual variables reach a neighborhood of their optimal value. This bound was related with the condition number of both the primal function and the network, the drift of the optimal primal variable and the gradients. Similar to [135], the works in [136–138] demonstrated that if the variation of the objective functions is sufficiently slow, traditional techniques can be adapted to solve it with minor revision. However, continuous-time time-varying distributed optimization with fast time-varying optimal solution hasn't been addressed yet though many existing engineering systems exhibit continuous time-varying characteristics.

System identification [142], resource allocation in time-varying environment, network traffic management [143] and maximum power point tracking for photovoltaic systems [144] are typical time-varying optimization problems. However, most existing works deal with time-invariant optimization problems and there are only a few works reported on time-varying problems [37,142,145–147]. Compared with time-invariant optimization, the main challenge for time-varying optimization lies in the variation of the optimal solution. For traditional time-invariant optimization problems, the updating of the strategies is designed such that the strategies can be regulated to the optimal solution and stay there thereafter. For time-varying optimization problems, the strategies should have the capability to track the time-varying optimal solution in real time. Therefore, a time-invariant optimization problem can be regarded as a special case of a time-varying optimization problem.

Different from single-agent time-varying optimization problems or centralized time-varying optimization problems where the agents only need to minimize their own costs, the agents for distributed time-varying optimization problems need to coordinate with each other to minimize their aggregate cost. However, they only have knowledge on local cost functions. No centralized knowledge on the other agents' cost functions or the aggregate cost is available. Hence, it is challenging to design updating laws for the agents to achieve the minimization of the aggregate cost in a distributed fashion. Different from tracking problems in control theory, the error signal is not available for the implementation of the updating laws. In addition, the strategy updating laws shall be designed to ensure that the strategies are fast enough to track the unknown optimal solution. The main contributions of the chapter can be summarized as follows. We propose a distributed strategy for the agents to track the unknown time-varying optimal solution for distributed time-varying optimization with quadratic objective functions. Convergence is proven by using Lyapunov analysis. For time-varying optimization problems, an asymptotically tracking result is given. For general distributed optimization problems, a penalty function based method is proposed. The penalty function based method enables the agents' strategies to approximate the time-varying optimal solution. For a special case where the agents' objective functions are only coupled with their neighbors' states, an asymptotic tracking result is presented.

The rest of the chapter is organized as follows. We formulate the problem in Section 5.2. A neighboring decoupled optimization problem is studied in Section 5.3.1, where updating laws based on gradients are proposed. General distributed optimization problems are studied in Section 5.3.2, where a penalty function based method is proposed. Numerical examples are presented to verify the effectiveness of the proposed methods in Section 5.4. Conclusions on this chapter are given in Section 5.5.

5.2 Problem Formulation

The problem is formulated as follows.

Problem 5.1. (Distributed Optimization with Time-varying Optimal Solution)

Consider a multi-agent system with N agents and the set of agents is denoted by $V = \{1, 2, \dots, N\}$. The agents communicate with each other via an undirected and connected graph topology denoted by $G = (V, E)$ where E is the edge set with $E \subset V \times V$. The agents coordinate with each other to achieve the optimization objective defined as

$$\min f(x, t) = \sum_{i=1}^N f_i(x, t) \quad (5.1)$$

where $x = [x_1^T, x_2^T, \dots, x_N^T]^T$ and $x_i = [x_{i1}, x_{i2}, \dots, x_{im_i}]^T \in \mathbb{R}^{m_i}$ denotes the strategy of agent i . The function $f_i(x, t) : \mathbb{R}^{\sum_{i=1}^N m_i} \times \mathbb{R} \rightarrow \mathbb{R}$ is the local objective function of agent i and is sufficiently smooth. The function $f(x, t)$ is the network objective function. The optimal solution $x^*(t)$ is unknown, time-varying, bounded and $f_i(x^*(t), t)$, for all $i \in \{1, 2, \dots, N\}$ are bounded. Each agent has access to its own local objective function only. Design an updating strategy for the agents such that they can track $x^*(t)$ which denotes the solution of (5.1).

Remark 5.1. *The main difference between Problem 5.1 and the problem in (1.1), which has been widely studied in the literature, is that the local objective functions and the network objective function in Problem 5.1 depend on time. Hence, a fast tracking strategy should be designed which makes the problem challenging.*

For simplicity, only quadratic objective functions will be analytically investigated in this chapter. In the subsequent sections, systems with neighboring coupled objective functions¹ will be considered for quadratic network objective function that

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¹To be clear, in this note, we say that the distributed optimization problem is neighboring coupled if, $\forall i$, $f_i(x, t)$ is a function of x_i and x_j where agent j is a neighbor of agent i and if agent j is not a neighbor of agent i , f_i does not depend on x_j .

is strongly convex. Subsequently, solution seeking for generally coupled distributed optimization problems with strongly convex local objective functions will be studied by utilizing penalty function based methods.

5.3 Main Results

Before presenting the proposed methods, the following Lemma is given to facilitate the subsequent Lyapunov stability analysis.

Lemma 5.1. *Suppose that $\bar{A}(t)$ is an $N \times N$ dimensional symmetric, real matrix ($\bar{A}(t)^T = \bar{A}(t)$) that is positive definite and bounded, i.e., there exist constants $\tau_1 > 0$ and $\tau_2 \geq \tau_1$ such that $\tau_1 I \leq \bar{A}(t) \leq \tau_2 I$. Then, $\bar{A}(t)^T \bar{A}(t)$ and $\bar{A}(t)^{-1}$ are symmetric, positive definite and bounded with $\tau_1^2 I \leq \bar{A}(t)^T \bar{A}(t) \leq \tau_2^2 I$, $\frac{1}{\tau_2} I \leq \bar{A}(t)^{-1} \leq \frac{1}{\tau_1} I$. Furthermore, let L be a symmetric positive semi-definite constant matrix ($\lambda_{\min}(L) = 0$). Then, $\bar{A}(t) + \alpha L$ for constant $\alpha > 0$ is symmetric, positive definite and bounded with $\tau_1 I \leq \bar{A}(t) + \alpha L \leq (\tau_2 + \alpha \lambda_{\max}(L)) I$. Furthermore, $\bar{A}(t) + L\beta\beta L$, with β being a positive diagonal constant matrix, is symmetric, positive definite and bounded with $\tau_1 I \leq \bar{A}(t) + L\beta\beta L \leq (\tau_2 + \lambda_{\max}(L\beta\beta L)) I$.*

Proof. See the Appendix for the proof. □

Remark 5.2. *This Lemma will be used in the Lyapunov-based analysis of Theorems 5.1-5.3 in the subsequent sections.*

5.3.1 Distributed optimization with neighboring coupled objective functions

In this section, systems with neighboring coupled objective functions are considered. The result of this section is based on the assumption that $f(x, t)$ is quadratic and strongly convex in x .

Suppose that $f(x, t) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N x_i^T R_{ij}(t) x_j + \sum_{i=1}^N s_i(t)^T x_i + h(t)$, where $R_{ij}(t) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_j}$, $s_i(t) \in \mathbb{R}^{m_i}$, $h(t) \in \mathbb{R}$ are real, time-varying sufficient smooth real coefficients and $R_{ij}(t) = R_{ji}^T(t)$ by definition.

Since $f(x, t)$ is strongly convex, the first-order stationary condition is a sufficient and necessary condition for the objective function to be at the minimum. Hence, $f(x, t)$ reaches the minimum if $\frac{\partial f}{\partial x_i}(x^*(t), t) = \sum_{j=1}^N R_{ij}(t) x_j^*(t) + s_i(t) = 0, \forall i \in \{1, 2, \dots, N\}$, i.e., $R(t)x^*(t) + [s_i(t)]_{vec} = \mathbf{0}$, where $[s_i(t)]_{vec} = [s_1(t)^T, s_2(t)^T, \dots, s_N(t)^T]^T$

$$\text{and } R(t) = \begin{bmatrix} R_{11}(t) & R_{12}(t) & \cdots & R_{1N}(t) \\ R_{21}(t) & & & R_{2N}(t) \\ \vdots & & \ddots & \\ R_{N1}(t) & \cdots & & R_{NN}(t) \end{bmatrix}. \text{ By the strong convexity of } f(x, t),$$

the matrix $R(t)$ is symmetric positive definite, i.e., there exists a positive constant q_1 such that $R(t) \geq q_1 I$ and $R^T(t) = R(t)$ [91].

Inspired by the robust control strategy in [148], the updating strategy is designed as

$$\begin{aligned} \dot{x}_i &= -c_{1i} \frac{\partial}{\partial x_i} (f_i(x, t) + \sum_{j=1}^N a_{ij} f_j(x, t)) + \varphi_i \\ \dot{\varphi}_i &= -c_{1i} k_{1i} \left(\frac{\partial}{\partial x_i} (f_i(x, t) + \sum_{j=1}^N a_{ij} f_j(x, t)) \right) \\ &\quad - k_{2i} \text{sgn} \left(\frac{\partial}{\partial x_i} (f_i(x, t) + \sum_{j=1}^N a_{ij} f_j(x, t)) \right), \end{aligned} \quad (5.2)$$

where a_{ij} is the element on the i th row and j th column of the adjacency matrix, c_{1i} , k_{1i} , k_{2i} are positive diagonal control gain matrices to be determined. Furthermore, $\text{sgn}(\cdot)$ denotes the standard signum function.

Remark 5.3. The strategy in (5.2) is motivated by the robust control strategy in [148] and the references therein. An error signal is used in the control strategies in [148]. However, the error signal is not directly available for distributed op-

timization as the optimal solution is unknown. Hence, the strategy in (5.2) uses gradient instead. In the following, we show that under some conditions, the gradient based strategy in (5.2) enables the agents' strategies to asymptotically converge to the optimal solution.

Theorem 5.1. *For the neighboring coupled distributed optimization problem, if there exist positive constants q_1 and $q_2 \geq q_1$ such that $q_1 I \leq R(t) \leq q_2 I$, and the elements in $R(t)$, $(R(t))^{-1}$, $\dot{R}(t)$, $\ddot{R}(t)$, $\frac{d}{dt}((R(t))^{-1})$, $\frac{d^2}{dt^2}((R(t))^{-1})$ and $\ddot{x}^*(t)$, $\ddot{x}^*(t)$ are bounded, then, $\|x(t) - x^*(t)\| \rightarrow 0$, $\|\dot{x}(t) - \dot{x}^*(t)\| \rightarrow 0$ as $t \rightarrow \infty$ under (5.2) given that the control gain matrices are chosen according to $\lambda_{\min}(k_1) > \frac{q_2^2}{2q_1^2}$, $\lambda_{\min}(k_2) > \frac{\|k_1\|p_1}{\lambda_{\min}(k_1)} + \frac{p_2}{\lambda_{\min}(k_1)}$, the diagonal elements in c_1 are sufficiently large², where p_1 and p_2 are positive constants that satisfy $\|N_c(t)\| \leq p_1$, $\|\dot{N}_c(t)\| \leq p_2$, $N_c(t) = -\ddot{x}^*$ and c_1 , k_1 , k_2 are the concatenated diagonal matrices of c_{1i} , k_{1i} , k_{2i} , respectively.*

Proof. See the Appendix for the proof. □

5.3.2 Distributed optimization with generally coupled objective functions

In this section, a generally coupled distributed optimization problem, in which the local objective functions $f_i(x, t)$ are quadratic and strongly convex in x , is considered. Similar to [61,77], the problem is reformulated as follows.

Problem 5.2. *The agents are coordinating with each other to solve*

$$\begin{aligned} \min f(\mathbf{x}, t) &= \sum_{i=1}^N f_i(\mathbf{x}_i, t), \\ \text{subject to } \mathbf{x}_1 &= \mathbf{x}_2 = \cdots = \mathbf{x}_N, \end{aligned} \tag{5.3}$$

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²Note that the diagonal elements in c_1 do not need to be arbitrarily large. It only needs to be sufficiently large such that $p_4 - \|R(t)^{-1}\| p_3 \geq \delta_3 > 0$ for some constant δ_3 , where p_3, p_4 are defined in the Appendix. Furthermore, similar conclusions are applicable to the choices of the diagonal control gain matrix $\mathbf{c}_1$ in Theorems 5.2-5.3.

where $\mathbf{x}_i$ is an $\sum_{i=1}^N m_i$ dimensional vector defined as $\mathbf{x}_i = [\mathbf{x}_{i1}^T, \mathbf{x}_{i2}^T, \dots, \mathbf{x}_{im_i}^T]^T$, $\mathbf{x}_{ij} = [x_{ij1}, x_{ij2}, \dots, x_{ijm_j}]^T$ and $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T$.

Similar to the time-invariant distributed optimization problem considered in [61], the following result can be derived for the time-varying distributed optimization problem.

Lemma 5.2. *Under an undirected and connected communication graph, Problem 5.2 is equivalent to the following problem*

$$\min f(\mathbf{x}, t) = \sum_{i=1}^N f_i(\mathbf{x}_i, t), \text{ subject to } \mathbf{L}\mathbf{x} = \mathbf{0}, \quad (5.4)$$

where the matrix $\mathbf{L} = L \otimes I$, L is the Laplacian matrix of the communication topology, I is an $\sum_{i=1}^N m_i \times \sum_{i=1}^N m_i$ dimensional identity matrix and $\otimes$ denotes the Kronecker product.

Proof. The proof follows the proof in [61]. □

Based on Lemma 5.2 and the equivalence of Problem 5.2 to Problem 5.1, penalty function based strategies will be proposed to search for the solution of (5.4) to indirectly solve Problem 5.1.

Penalized by $\frac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}$

Define

$$F(\mathbf{x}, t) = \sum_{i=1}^N f_i(\mathbf{x}_i, t) + \frac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}, \quad (5.5)$$

where β is a positive constant penalty parameter and $\frac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}$ denotes the penalty for the violation of the constraint $\mathbf{L}\mathbf{x} = \mathbf{0}$. The penalty term $\frac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x} \geq 0$ and it is equal to 0 if and only if the constraint $\mathbf{L}\mathbf{x} = \mathbf{0}$ is satisfied.

Similar to Section 5.3.1, to seek the minimum solution of (5.5), agent $i, i \in \{1, 2, \dots, N\}$ can update its estimation according to

$$\begin{aligned}\dot{\mathbf{x}}_{ij} &= -\mathbf{c}_{1ij} \left(\frac{\partial f_i(\mathbf{x}_1, t)}{\partial \mathbf{x}_{ij}} + \beta \sum_{k=1}^N a_{ik} (\mathbf{x}_{ij} - \mathbf{x}_{kj}) \right) + \varphi_{ij} \\ \dot{\varphi}_{ij} &= -\mathbf{c}_{1ij} \mathbf{k}_{1ij} \left(\frac{\partial f_i(\mathbf{x}_1, t)}{\partial \mathbf{x}_{ij}} + \beta \sum_{k=1}^N a_{ik} (\mathbf{x}_{ij} - \mathbf{x}_{kj}) \right) \\ &\quad - \mathbf{k}_{2ij} \text{sgn} \left(\frac{\partial f_i(\mathbf{x}_1, t)}{\partial \mathbf{x}_{ij}} + \beta \sum_{k=1}^N a_{ik} (\mathbf{x}_{ij} - \mathbf{x}_{kj}) \right),\end{aligned}\tag{5.6}$$

for all $j \in \{1, 2, \dots, N\}$ where $\mathbf{c}_{1ij}$, $\mathbf{k}_{1ij}$, $\mathbf{k}_{2ij}$ are positive diagonal control gain matrices.

According to the first-order minimum condition, $F(\mathbf{x}, t)$ is at the minimum value if $\frac{\partial \sum_{i=1}^N f_i}{\partial \mathbf{x}}(\mathbf{x}_i^*(t), t) + \beta \mathbf{L} \mathbf{x}^*(t) = \mathbf{0}$. Suppose that $f_i(\mathbf{x}_i, t) = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \mathbf{x}_{ij}^T R_{ijk}(t) \mathbf{x}_{ik} + \sum_{j=1}^N s_{ij}(t)^T \mathbf{x}_{ij} + h_i(t)$, where $R_{ij}(t), s_{ij}(t), h_i(t)$ are coefficients composed of real numbers. Then, $\frac{\partial f_i(\mathbf{x}_1, t)}{\partial \mathbf{x}_{ij}} = \sum_{k=1}^N R_{ijk}(t) \mathbf{x}_{ik} + s_{ij}(t)$. Hence, $R(t) \mathbf{x}^*(t) + [s_{ij}(t)]_{vec} + \beta \mathbf{L} \mathbf{x}^*(t) = \mathbf{0}$, where $R(t)$, defined as $R(t) = \text{diag}\{R_1, R_2, \dots, R_N\}$,

$$R_i(t) = \begin{bmatrix} R_{i11} & R_{i12} & \cdots & R_{i1N} \\ R_{i21} & & & \\ \vdots & & \ddots & \vdots \\ R_{iN1} & R_{iN2} & \cdots & R_{iNN} \end{bmatrix},$$

is symmetric positive definite (there exists a positive constant q_1 such that $R(t) \geq q_1 I$ and $R^T(t) = R(t)$) by the strong convexity condition. Furthermore, $[s_{ij}(t)]_{vec} = [s_{11}, s_{12}, \dots, s_{NN}]^T$.

Define $\mathbf{e} = \mathbf{x} - \mathbf{x}^*(t)$. Then, $\frac{\partial F(\mathbf{x}, t)}{\partial \mathbf{x}} = R(t) \mathbf{x} + [s_{ij}(t)]_{vec} + \beta \mathbf{L} \mathbf{x} = R(t) (\mathbf{x}^*(t) + \mathbf{e}) + [s_{ij}(t)]_{vec} + \beta \mathbf{L} (\mathbf{x}^*(t) + \mathbf{e}) = R(t) \mathbf{e} + \beta \mathbf{L} \mathbf{e}$. Writing (5.6) in the concatenated form

gives

$$\begin{aligned}\dot{\mathbf{x}} &= -\mathbf{c}_1(R(t)+\beta\mathbf{L})\mathbf{e} + \varphi \\ \dot{\varphi} &= -\mathbf{c}_1\mathbf{k}_1(R(t)+\beta\mathbf{L})\mathbf{e} - \mathbf{k}_2\text{sgn}((R(t)+\beta\mathbf{L})\mathbf{e}),\end{aligned}\tag{5.7}$$

where $\mathbf{c}_1, \mathbf{k}_1, \mathbf{k}_2$ are the concatenated diagonal matrices of $\mathbf{c}_{1ij}, \mathbf{k}_{1ij}, \mathbf{k}_{2ij}$, $i, j \in \{1, 2, \dots, N\}$, respectively and φ is the concatenated vector of $\varphi_{ij}, i, j \in \{1, 2, \dots, N\}$.

Theorem 5.2. *The updating strategy in (5.7) enables $\|\mathbf{x}(t) - \mathbf{x}^*(t)\| \rightarrow 0$ and $\|\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^*(t)\| \rightarrow 0$ as $t \rightarrow \infty$ given that there exist positive constants $q_2 \geq q_1$ such that $q_1 I \leq R(t) \leq q_2 I$, the elements in $R(t), \dot{R}(t), \ddot{R}(t), \ddot{\ddot{R}}(t), (R(t)+\beta\mathbf{L})^{-1}, \frac{d}{dt}((R(t)+\beta\mathbf{L})^{-1}), \frac{d^2}{dt^2}((R(t)+\beta\mathbf{L})^{-1})$ and $\ddot{\mathbf{x}}^*(t), \ddot{\ddot{\mathbf{x}}}^*(t)$ are bounded and the control gains are chosen according to $\lambda_{\min}(\mathbf{k}_1) > \frac{(q_2+\beta\lambda_{\max}\{\mathbf{L}\})^2}{2q_1^2}$, $\lambda_{\min}(\mathbf{k}_2) > \frac{\|\mathbf{k}_1\|p_1}{\lambda_{\min}(\mathbf{k}_1)} + \frac{p_2}{\lambda_{\min}(\mathbf{k}_1)}$, the diagonal elements in $\mathbf{c}_1$ are sufficiently large, where p_1, p_2 are constants that satisfy $\|\ddot{\mathbf{x}}^*(t)\| \leq p_1, \|\ddot{\ddot{\mathbf{x}}}^*(t)\| \leq p_2$ and $\mathbf{x}^*(t)$ is the minimum solution of $F(\mathbf{x}, t)$ defined in (5.5).*

Proof. The proof follows that of Theorem 5.1 by defining the Lyapunov function as $V_L = \frac{1}{2}m((R(t)+\beta\mathbf{L})\mathbf{e}(t))^T(R(t)+\beta\mathbf{L})\mathbf{e}(t) + \frac{1}{2}\eta(t)^T((R(t)+\beta\mathbf{L}))^{-1}\eta(t) + W$, where $m \in \mathbb{R}$ is a positive constant, and

$$W = -\mathbf{e}(0)^T(R(0)+\beta\mathbf{L})N_c(0)+\mathbf{e}(0)^T(R(0)+\beta\mathbf{L})^T\mathbf{k}_2\text{sgn}((R(0)+\beta\mathbf{L})\mathbf{e}(0))-\int_0^t \bar{L}(\tau)d\tau,$$

$$\bar{L}(t) \triangleq \eta^T [N_c(t) - \mathbf{k}_2\text{sgn}((R(t)+\beta\mathbf{L})\mathbf{e}(t))] \text{ and } \eta = \frac{d}{dt}((R(t)+\beta\mathbf{L})\mathbf{e})+\mathbf{k}_1(R(t)+\beta\mathbf{L})\mathbf{e},$$

$$N_c(t) = -\ddot{\mathbf{x}}^*(t). \quad \square$$

Remark 5.4. *In the updating strategy design, a common penalty parameter is used among the agents. Though, average consensus protocols have the potential to synchronize it distributively among the agents, analytical stability analysis for the closed-loop system is challenging. In the following, a new penalty function is proposed to*

solve this problem.

Penalized by $\frac{1}{2}\mathbf{x}^T\mathbf{L}\beta\beta\mathbf{L}\mathbf{x}$

Define the penalty function as

$$F(\mathbf{x}, t) = \sum_{i=1}^N f_i(\mathbf{x}_i, t) + \frac{1}{2}\mathbf{x}^T\mathbf{L}\beta\beta\mathbf{L}\mathbf{x}, \quad (5.8)$$

where β is a diagonal constant matrix defined as $\beta = \text{diag}\{\beta_1, \beta_2, \dots, \beta_N\}$, with β_i an $\sum_{i=1}^N m_i \times \sum_{i=1}^N m_i$ dimensional positive diagonal matrix that contains the penalty parameter determined by agent i . For simplicity, let $\beta_i = \varrho_i I$, where ϱ_i is a positive constant. The penalty term $\frac{1}{2}\mathbf{x}^T\mathbf{L}\beta\beta\mathbf{L}\mathbf{x} = \frac{1}{2}\|\beta\mathbf{L}\mathbf{x}\|^2 \geq 0$ and it equals 0 if and only if $\mathbf{L}\mathbf{x} = \mathbf{0}$.

By using the penalty function in (5.8), the updating strategy is designed as

$$\begin{aligned} \dot{\mathbf{x}}_{ij} &= -\mathbf{c}_{1ij} \left(\frac{\partial f_i(\mathbf{x}_i, t)}{\partial \mathbf{x}_{ij}} + \sum_{k=1}^N a_{ik}(\mathbf{y}_{ij} - \mathbf{y}_{kj}) \right) + \varphi_{ij} \\ \dot{\varphi}_{ij} &= -\mathbf{c}_{1ij} \mathbf{k}_{1ij} \left(\frac{\partial f_i(\mathbf{x}_i, t)}{\partial \mathbf{x}_{ij}} + \sum_{k=1}^N a_{ik}(\mathbf{y}_{ij} - \mathbf{y}_{kj}) \right) \\ &\quad - \mathbf{k}_{2ij} \text{sgn} \left(\frac{\partial f_i(\mathbf{x}_i, t)}{\partial \mathbf{x}_{ij}} + \sum_{k=1}^N a_{ik}(\mathbf{y}_{ij} - \mathbf{y}_{kj}) \right), \end{aligned} \quad (5.9)$$

where $\mathbf{y}_{ij} = \varrho_i^2 \sum_{k=1}^N a_{ik}(\mathbf{x}_{ij} - \mathbf{x}_{kj})$. It can be seen that the updating strategy in (5.9) is fully distributed. By similar analysis in Theorem 5.2, the following result can be derived.

Theorem 5.3. *The updating strategy in (5.9) enables $\|\mathbf{x}(t) - \mathbf{x}^*(t)\| \rightarrow 0$, $\|\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^*(t)\| \rightarrow 0$ as $t \rightarrow \infty$ given that there exist positive constants $q_2 \geq q_1$ such that $q_1 I \leq R(t) \leq q_2 I$, the elements in $R(t)$, $\dot{R}(t)$, $\ddot{R}(t)$, $\ddot{\ddot{R}}(t)$, $(R(t) + \mathbf{L}\beta\beta\mathbf{L})^{-1}$, $\frac{d}{dt}((R(t) + \mathbf{L}\beta\beta\mathbf{L})^{-1})$, $\frac{d^2}{dt^2}((R(t) + \mathbf{L}\beta\beta\mathbf{L})^{-1})$ and $\ddot{\mathbf{x}}^*(t)$, $\ddot{\ddot{\mathbf{x}}}^*(t)$ are bounded and the control gains are chosen according to $\lambda_{\min}(\mathbf{k}_1) > \frac{(q_2 + \lambda_{\max}\{\mathbf{L}\beta\beta\mathbf{L}\})^2}{2q_1^2}$, $\lambda_{\min}(\mathbf{k}_2) > \frac{\|\mathbf{k}_1\|p_1}{\lambda_{\min}(\mathbf{k}_1)} + \frac{p_2}{\lambda_{\min}(\mathbf{k}_1)}$, the diagonal*

elements in $\mathbf{c}_1$ are sufficiently large, where p_1, p_2 are positive constants that satisfy $\|\ddot{\mathbf{x}}^*(t)\| \leq p_1, \|\ddot{\mathbf{x}}^*(t)\| \leq p_2$ and $\mathbf{x}^*(t)$ is the minimum solution of $F(\mathbf{x}, t)$ defined in (5.8).

Proof. The proof follows that of Theorem 5.1 by defining the Lyapunov function as $V_L = \frac{1}{2}m((R(t)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e}(t))^T(R(t)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e}(t) + \frac{1}{2}\eta(t)^T(R(t)+\mathbf{L}\beta\beta\mathbf{L})^{-1}\eta(t) + W$, where $m \in \mathbb{R}$ is a positive constant,

$$W = -\mathbf{e}(0)^T(R(0)+\mathbf{L}\beta\beta\mathbf{L})N_c(0) + \mathbf{e}(0)^T(R(0)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{k}_2\text{sgn}((R(0)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e}(0)) - \int_0^t \bar{L}(\tau)d\tau,$$

$$\bar{L}(t) \triangleq \eta^T [N_c(t) - \mathbf{k}_2\text{sgn}(R(t)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e}] \text{ and } \eta = \frac{d}{dt}((R(t)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e})+\mathbf{k}_1(R(t)+\mathbf{L}\beta\beta\mathbf{L})\mathbf{e},$$

$$N_c(t) = -\ddot{\mathbf{x}}^*(t). \quad \square$$

Remark 5.5. By choosing the penalty parameters sufficiently large, the minimization solution to the penalty functions in (5.5) and (5.8) can be used to approximate the solution of (5.4). Hence, $\mathbf{x}_i, i \in \{1, 2, \dots, N\}$ produced by the proposed methods approximate the solution to Problem 5.1 by further using Lemma 5.2 and the equivalence of Problem 1 to Problem 2.

Remark 5.6. The method in (5.2) requires the agents' real-time measurements of the gradients of their own local objective functions and their neighbors' objective functions. The method in (5.7) requires agent $i, i \in \{1, 2, \dots, N\}$ to share $\mathbf{x}_{ij}, j \in \{1, 2, \dots, N\}$ with its neighbors and the method in (5.9) requires agent i to share $\mathbf{x}_{ij}$ and $\mathbf{y}_{ij}, j \in \{1, 2, \dots, N\}$ with its neighbors.

5.4 Numerical Examples

In this section, numerical examples are provided to verify the effectiveness of the proposed methods.

Example 5.1. A network of 6 agents coordinate with each other to solve $f(x, y, t) = \sum_{i=1}^6 f_i(x, y, t)$, where $f_1(x, y, t) = (x_1 - x_2 - 1 - 0.5 \sin(t))^2 + (y_1 - y_2 + 1)^2$, $f_2(x, y, t) = (x_3 - x_2)^2 + (y_3 - y_2 - 2)^2$, $f_3(x, y, t) = (x_4 - x_3 - 1 - 0.5 \sin(t))^2 + (y_4 - y_3 - 1)^2$, $f_4(x, y, t) = (x_4 - x_1)^2 + (y_4 - y_1)^2 + (x_4 + \sin(t))^2 + (y_4 + 2)^2$, $f_5(x, y, t) = (x_5 - x_4)^2 + (y_5 - y_4 - 1)^2$, $f_6(x, y, t) = (x_6 - x_1)^2 + (y_6 - y_1 + 1)^2$, are the local objective functions for agents 1-6, respectively. Let (x_i, y_i) be the strategy of agent i and $x = [x_1, x_2, \dots, x_6]^T$, $y = [y_1, y_2, \dots, y_6]^T$.

The minimum of $f(x, y, t)$ is reached when $(x_1, y_1) = (-\sin(t), -3)$, $(x_2, y_2) = (-1 - 1.5 \sin(t), -3)$, $(x_3, y_3) = (-1 - 1.5 \sin(t), -2)$, $(x_4, y_4) = (-\sin(t), -2)$, $(x_5, y_5) = (-\sin(t), -1)$, $(x_6, y_6) = (-\sin(t), -4)$. Through direct observation, it can be seen that if the minimum is reached, the agents forms a trapezoid in the x - y coordinate. The height of the trapezoid varies with time and is $-1 - 0.5 \sin(t)$. The length of the upper line is fixed to be 1 and the length of the lower line is 3. Furthermore, the position of the trapezoid varies with time.

Figure 5.1 (a) shows the trapezoid at $t = 0$ and the communication graph is shown in Figure 5.1 (b).

The simulation result by using the method in (5.2) is shown in Figure 5.2.

It can be seen from the simulation results, (x, y) converges to $(x^*(t), y^*(t))$ where $(x^*(t), y^*(t))$ is the optimal solution to the network objective.

Example 5.2. A network of 6 agents coordinate with each other to minimize $f(x, y, t) = \frac{1}{2}x^T R x + b(t)^T x + \frac{1}{2}y^T R y + \mu(t)^T y$, where $x = [x_1, x_2, \dots, x_6]^T$. The element on i th row and j th column of R is $R_{ij} = 1$ if $i \neq j$ and $R_{ii} = 8$. The i th element in $b(t)$, $b_i(t) = 14 \sin(t) + 2$ for $i = 1, 4, 5, 6$ and $b_i(t) = 9 + 17.5 \sin(t)$ for $i = 2, 3$. Furthermore, $\mu(t) = [36, 36, 29, 29, 22, 43]^T$. Let agent i 's objective function be $f_i(x, y, t) = \frac{1}{12}x^T R x + b_i(t)x_i + \frac{1}{12}y^T R y + \mu_i(t)y_i$, where $\mu_i(t)$ denotes the i th element in $\mu(t)$, $i = \{1, 2, \dots, 6\}$. By direct calculation, it can be derived that the optimal solution is the same as that of Example 5.1. Same communication graph as

Figure 5.1: (a) is the trapezoid at $t = 0$. (b) is the communication graph for the agents.

Figure 5.2: The formation tracking result derived by using the method in (5.2).

in Example 5.1 is adopted in the simulation.

The simulation result produced by the method in (5.6) is shown in Figure 5.3.

Figure 5.3: The formation tracking result derived by using the method in (5.6).

The simulation result produced by the proposed method in (5.9) is shown in Figure 5.4.

Figure 5.4: The formation tracking result derived by using the method in (5.9).

It can be seen from the simulation results that both the proposed methods in (5.6)

and (5.9) enable the agents to approximate the optimal trajectory.

5.5 Conclusion

In this chapter, a robust control strategy is adapted to solve a time-varying distributed optimization problem. Through Lyapunov analysis, it is shown that the proposed method enables the agents to asymptotically converge to the unknown time-varying optimal solution for systems with neighboring coupled quadratic objective functions. For distributed optimization problems with generally coupled quadratic objective functions, approximated results can be derived by using penalty function based methods.

5.6 Appendix

5.6.1 Proof of Lemma 5.1

Since $\bar{A}(t)$ is a real, symmetric matrix, there exists an orthonormal matrix $T(t)$ such that $\bar{A}(t) = T(t)^T \Lambda(t) T(t)$ where $\Lambda(t)$ is an invertible diagonal matrix.

Since $\tau_1 I \leq \bar{A}(t) \leq \tau_2 I$, $\tau_1 x^T x \leq x^T T(t)^T \Lambda(t) T(t) x \leq \tau_2 x^T x$. Let $y = T(t)x$, i.e., $x = T(t)^T y$. Then $\tau_1 y^T T(t) T(t)^T y \leq y^T T(t) T(t)^T \Lambda(t) T(t) T(t)^T y \leq \tau_2 y^T T(t) T(t)^T y$, i.e., $\tau_1 y^T y \leq y^T \Lambda(t) y \leq \tau_2 y^T y$. Hence $\tau_1 I \leq \Lambda(t) \leq \tau_2 I$. Since $\Lambda(t)$ an invertible diagonal matrix, $\tau_1^2 I \leq \Lambda(t)^T \Lambda(t) \leq \tau_2^2 I$, $\frac{1}{\tau_2} I \leq \Lambda(t)^{-1} \leq \frac{1}{\tau_1} I$. Therefore, $\tau_1^2 y^T y \leq y^T \Lambda(t)^T \Lambda(t) y \leq \tau_2^2 y^T y$, $\frac{1}{\tau_2} y^T y \leq y^T \Lambda(t)^{-1} y \leq \frac{1}{\tau_1} y^T y$. Since $y = T(t)x$, $\tau_1^2 x^T x \leq x^T T(t)^T \Lambda(t)^T \Lambda(t) T(t) x \leq \tau_2^2 x^T x$, $\frac{1}{\tau_2} x^T x \leq x^T T(t)^T \Lambda(t)^{-1} T(t) x \leq \frac{1}{\tau_1} x^T x$. Since $\bar{A}(t) = T(t)^T \Lambda(t) T(t)$, $\bar{A}(t)^{-1} = T(t)^T \Lambda(t)^{-1} T(t)$ and $\bar{A}(t)^T \bar{A}(t) = T(t)^T \Lambda(t)^T \Lambda(t) T(t)$, it can be derived that $\tau_1^2 I \leq \bar{A}(t)^T \bar{A}(t) \leq \tau_2^2 I$, $\frac{1}{\tau_2} I \leq \bar{A}(t)^{-1} \leq \frac{1}{\tau_1} I$.

On the other hand, $\tau_1 x^T x + \alpha \lambda_{\min}(L) x^T x \leq x^T (\bar{A}(t) + \alpha L) x \leq (\tau_2 + \alpha \lambda_{\max}(L)) x^T x$, i.e., $\tau_1 I \leq \bar{A}(t) + \alpha L \leq (\tau_2 + \alpha \lambda_{\max}(L)) I$. The left hand side follows from $\lambda_{\min}(L) = 0$

as L is a positive semi-definite matrix. Furthermore, $\tau_1 x^T x + \lambda_{\min}(L\beta\beta L)x^T x \leq x^T(\bar{A}(t) + L\beta\beta L)x \leq (\tau_2 + \lambda_{\max}(L\beta\beta L))x^T x$. Hence, $\tau_1 I \leq \bar{A}(t) + L\beta\beta L \leq (\tau_2 + \lambda_{\max}(L\beta\beta L))I$. The left hand side follows from $\lambda_{\min}(L\beta\beta L) = 0$ as $x^T L\beta\beta Lx = \|\beta Lx\|^2 \geq 0$, which indicates that the matrix $L\beta\beta L$ is positive semi-definite and hence, $\lambda_{\min}(L\beta\beta L) = 0$.

5.6.2 Proof of Theorem 5.1

If the agents' objective functions are coupled with their neighbors' strategies only, then the following holds

$$\frac{\partial}{\partial x_i}(f_i(x, t) + \sum_{j=1}^N a_{ij}f_j(x, t)) = \frac{\partial f(x, t)}{\partial x_i},$$

for all agents. Hence,

$$\begin{aligned} \dot{x}_i &= -c_{1i} \frac{\partial f(x, t)}{\partial x_i} + \varphi_i \\ \dot{\varphi}_i &= -c_{1i}k_{1i} \frac{\partial f(x, t)}{\partial x_i} - k_{2i} \text{sgn}\left(\frac{\partial f(x, t)}{\partial x_i}\right), \end{aligned} \tag{5.10}$$

Define the error signal as

$$e = x - x^*.$$

Then,

$$\begin{aligned} \frac{\partial f(x, t)}{\partial x} &= R(t)x + [s_i(t)]_{vec} \\ &= R(t)e + R(t)x^* + [s_i(t)]_{vec} \\ &= R(t)e, \end{aligned} \tag{5.11}$$

by using $R(t)x^* + [s_i(t)]_{vec} = \mathbf{0}$.

Substituting (5.11) into (5.10) gives

$$\begin{aligned}\dot{x} &= -c_1 R(t)e + \varphi \\ \dot{\varphi} &= -c_1 k_1 R(t)e - k_2 \text{sgn}(R(t)e),\end{aligned}\tag{5.12}$$

where c_1, k_1, k_2 are the concatenated diagonal matrices of c_{1i}, k_{1i}, k_{2i} , respectively.

Furthermore, φ is the concatenated vector of φ_{ij} .

Define a filtered signal as $\eta = (R(t)e)' + k_1 R(t)e$, where $(R(t)e)'$ denotes the time derivative of $R(t)e$.

Taking the time derivative of η gives

$$\begin{aligned}\dot{\eta} &= (R(t)e)'' + k_1 (R(t)e)' \\ &= R(t)\ddot{e} + \ddot{R}(t)e + 2\dot{R}(t)\dot{e} + k_1 (R(t)e)' \\ &= R(t)(\ddot{x} - \ddot{x}^*) + \ddot{R}(t)e + 2\dot{R}(t)\dot{e} + k_1 (R(t)e)' \\ &= R(t)(-c_1 \eta - k_2 \text{sgn}(R(t)e) - \ddot{x}^*) \\ &\quad + \ddot{R}(t)e + 2\dot{R}(t)(R(t))^{-1}(\eta - k_1 R(t)e - \dot{R}(t)e) \\ &\quad + k_1(\eta - k_1 R(t)e) \\ &= R(t)(-c_1 \eta - k_2 \text{sgn}(R(t)e) + N_c(t)) + \tilde{N}(t),\end{aligned}$$

where $N_c(t) = -\ddot{x}^*$ and $\tilde{N}(t) = \ddot{R}(t)e + 2\dot{R}(t)R(t)^{-1}(\eta - k_1 R(t)e - \dot{R}(t)e) + k_1(\eta - k_1 R(t)e)$. Furthermore, there exist positive constants p_1, p_2, p_3 such that $\|N_c(t)\| \leq p_1$, $\|\dot{N}_c(t)\| \leq p_2$ and $\|\tilde{N}(t)\| \leq p_3 \|\chi\|$ where $\chi = [e^T, \eta^T]^T$ since the elements in $R(t), (R(t))^{-1}, \dot{R}(t), \ddot{R}(t)$ and $\ddot{x}^*(t), \dot{\ddot{x}}^*(t)$ are bounded.

Define a Lyapunov function candidate $V_L(\mathcal{Y}, t) \in \mathbb{R}$ as

$$V_L(\mathcal{Y}, t) \triangleq \frac{1}{2}m(R(t)e(t))^T R(t)e(t) + \frac{1}{2}\eta(t)^T (R(t))^{-1}\eta(t) + W, \tag{5.13}$$

where $\mathcal{Y} = [\chi^T, \sqrt{W}]^T$, $m \in \mathbb{R}$ is a positive constant, and

$$\begin{aligned} W &= -e(0)^T R(0) N_c(0) \\ &\quad + e(0)^T R(0) k_2 \text{sgn}(R(0)e(0)) - \int_0^t \bar{L}(\tau) d\tau, \end{aligned} \quad (5.14)$$

$$\bar{L}(t) \triangleq \eta^T [N_c(t) - k_2 \text{sgn}(R(t)e(t))]. \quad (5.15)$$

The integration of $\bar{L}(t)$ is

$$\begin{aligned} &\int_0^t \bar{L}(\tau) d\tau \\ &= \int_0^t (((R(\tau)e(\tau))')^T + (k_1 R(\tau)e(\tau))^T) [N_c(\tau) \\ &\quad - k_2 \text{sgn}(R(\tau)e(\tau))] d\tau \\ &= \int_0^t ((R(\tau)e(\tau))')^T N_c(\tau) d\tau \\ &\quad - \int_0^t ((R(\tau)e(\tau))')^T k_2 \text{sgn}(R(\tau)e(\tau)) d\tau \\ &\quad + \int_0^t (k_1 R(\tau)e(\tau))^T [N_c(\tau) - k_2 \text{sgn}(R(\tau)e(\tau))] d\tau \\ &\leq e^T(\tau) R(\tau) N_c(\tau) \Big|_0^t - \lambda_{\min}(k_2) \|e^T(t) R(t)\| \\ &\quad + e(0)^T R(0) k_2 \text{sgn}(R(0)e(0)) \\ &\quad + \int_0^t (k_1 R(\tau)e(\tau))^T [N_c(\tau) - k_2 \text{sgn}(R(\tau)e(\tau))] \\ &\quad - e(\tau)^T R(\tau) \dot{N}_c(\tau) d\tau \\ &\leq \|e(t)^T R(t)\| (\|N_c(t)\| - \lambda_{\min}(k_2)) \\ &\quad - e(0)^T R(0) N_c(0) + e(0)^T R(0) k_2 \text{sgn}(R(0)e(0)) \\ &\quad + \int_0^t (\|k_1\| \|N_c(\tau)\| + \|\dot{N}_c(\tau)\| \\ &\quad - \lambda_{\min}(k_1) \lambda_{\min}(k_2)) \|e(\tau)^T R(\tau)\| d\tau. \end{aligned}$$

If $\lambda_{\min}(k_2) > \max\{\frac{\|k_1\|p_1}{\lambda_{\min}(k_1)} + \frac{p_2}{\lambda_{\min}(k_1)}, p_1\}$,

$$\int_0^t \bar{L}(\tau) d\tau \leq -e(0)^T R(0) N_c(0) + e(0)^T R(0) k_2 \text{sgn}(R(0)e(0)).$$

Since $\frac{1}{q_2}I \leq R(t)^{-1} \leq \frac{1}{q_1}I$ and $q_1^2 I \leq R(t)^T R(t) \leq q_2^2 I$, it can be derived that there exist positive constants δ_1, δ_2 such that

$$\delta_1 \|\mathcal{Y}\|^2 \leq V_L(\mathcal{Y}, t) \leq \delta_2 \|\mathcal{Y}\|^2. \quad (5.16)$$

The time derivative of the Lyapunov function candidate satisfies

$$\begin{aligned} \dot{V}_L &= m(R(t)e)^T (R(t)e)' + \frac{1}{2} \eta^T ((R(t))^{-1})' \eta \\ &\quad + \eta^T (-c_1 \eta + (R(t))^{-1} \tilde{N}(t)) \\ &= m(R(t)e)^T (\eta - k_1 R(t)e) + \frac{1}{2} \eta^T ((R(t))^{-1})' \eta \\ &\quad + \eta^T (-c_1 \eta + (R(t))^{-1} \tilde{N}(t)) \\ &\leq -(\lambda_{\min}(k_1) q_1^2 - \frac{q_2^2}{2}) m \|e\|^2 \\ &\quad - (\lambda_{\min}(c_1) - \frac{m}{2} - \frac{1}{2} \|((R(t))^{-1})'\|) \|\eta\|^2 \\ &\quad + \|R(t)^{-1}\| p_3 \|\chi\| \|\eta\| \\ &\leq -(p_4 - \|((R(t))^{-1})'\| p_3) \|\chi\|^2, \end{aligned}$$

where $p_4 = \min\{(\lambda_{\min}(k_1) q_1^2 - \frac{q_2^2}{2}) m, \lambda_{\min}(c_1) - \frac{m}{2} - \frac{1}{2} \varepsilon\}$ (ε is a constant that satisfies $\|((R(t))^{-1})'\| \leq \varepsilon$) which can be positive and sufficiently large (to satisfy $p_4 - \|R(t)^{-1}\| p_3 \geq \delta_3 > 0$ for some constant δ_3) by choosing $\lambda_{\min}(k_1) > \frac{q_2^2}{2q_1^2}$ and c_1 sufficiently large as $\|((R(t))^{-1})'\|, \|((R(t))^{-1})\|$ are bounded.

Therefore, the updating strategy in (5.2) enables $x(t)$ to globally asymptotically track the optimal trajectory $x^*(t)$.

Chapter 6

Time-varying Nash Equilibrium Seeking in Non-cooperative Games

6.1 Introduction

As stated in Chapter 1, most existing Nash equilibrium seeking methods require some kind of model information to be implemented [57]. Motivated by the desire to weaken the dependence on model information, the extremum seeking control (ESC) method [16,22,92] has been employed for Nash equilibrium seeking [57–59,93,94]. In [58,59], a Nash seeking algorithm based on ESC and sliding mode control was designed to drive the system to the sliding boundary layer and stay there thereafter. A multi-input ESC with stochastic perturbations was introduced to improve the performance of ESC on Nash equilibrium seeking in [93]. Based on the fact that the determination of Nash equilibria and the solution of the bilinear non-convex optimization problem are equivalent, an approach to bimatrix games was developed in [94]. In [57], averaging theory and a singular perturbation method were employed to enable local convergence to Nash equilibrium based on first-order ESC. However, the existing works consider only time-invariant Nash equilibrium.

In this chapter, we consider a time-varying Nash equilibrium seeking problem for N -player non-cooperative games without explicit model information, where each player's objective is to maximize its own payoff value. A non-model-based seeking scheme is proposed to achieve time-varying Nash equilibrium seeking. The strategy of each player is updated using a new extremum seeking scheme, where a gradient estimation algorithm and a gradient search algorithm are proposed to ensure convergence to the extremum. By utilizing the proposed seeking scheme, the players are able to track the time-varying Nash equilibrium trajectory without having explicit model information.

In comparison to previous works on Nash equilibrium seeking, the main contributions of this chapter can be summarized as follows.

- The N -player non-cooperative games under consideration admit time-varying Nash equilibrium and payoff values. The proposed seeking scheme doesn't require explicit model information on both the Nash equilibrium trajectory and the payoff value at the Nash equilibrium.
- The proposed seeking method to solve the time-varying Nash seeking problem consists of a gradient estimation part and gradient search part that can be designed separately. The modular design provides more freedom to design each part. For example, the averaging method that is usually necessary for extremum seeking analysis is not required, and methods in the area of robust control can be leveraged to design the gradient search part.
- The developed ESC-based seeking method enables the estimated strategy to converge to a neighborhood of the time-varying Nash equilibrium for the quadratic games.

The rest of this chapter is organized as follows. In Section 6.2, a motivating example is used to demonstrate the existence of time-varying Nash equilibrium.

The problem is formulated in Section 6.3 followed by the main result part in Section 6.4. A numerical example is given in Section 6.5 to help verify the effectiveness of the proposed method. Brief conclusions of this chapter are presented in Section 6.6.

6.2 A Motivating Example

The Nash equilibrium may be time-varying. Take the classical Cournot quantity game as an example. The participants of the game are two firms producing the same products, the strategies for the players are the quantities of the products they produce and the payoffs for the players are the profits they gain.

In the game setting, the price for their product is

$$p(Tot) = a(t) - Tot, \quad (6.1)$$

where $a(t)$ is a time-varying factor and Tot is the total quantity of product that the two firms produce, i.e., $Tot = q_1 + q_2$ where $q_i, i \in \{1, 2\}$, denotes the quantity of the product that firm i produces. The profit for each player is

$$Q_i = p(Tot)q_i - c_i(t)q_i, i \in \{1, 2\}, \quad (6.2)$$

where $c_i(t)$ represents the marginal cost that may be time-varying. Based on the two profit functions, the Nash equilibrium of this game can be given by

$$(q_1^*, q_2^*) = \left(\frac{2a_1'(t) - a_2'(t)}{3}, \frac{-a_1'(t) + 2a_2'(t)}{3} \right), \quad (6.3)$$

where $a_1'(t) = a(t) - c_1(t)$ and $a_2'(t) = a(t) - c_2(t)$. The Nash equilibrium is time-varying.

While most of the Nash equilibrium seeking algorithms need model information

for implementation, ESC is a promising method to enable the Nash equilibrium seeking without model information. Each user's updating law by using first-order ESC-based Nash seeking scheme is [57]

$$\begin{aligned}\dot{\hat{q}}_i &= m_i \sin(\omega_i t) Q_i, \\ q_i &= \hat{q}_i + a_i \sin(\omega_i t), i \in \{1, 2\},\end{aligned}\tag{6.4}$$

where m_i is the integrator gain, a_i and ω_i are the amplitude and frequency of the dither signal of player i , respectively. However, the first-order ESC-based Nash equilibrium seeking scheme will suffer from degraded performance if directly applied to the case of time-varying Nash equilibrium.

The simulation results generated by the first-order ESC and the proposed method are shown in Figures 6.1 and 6.2, respectively. It can be seen that the simulation result generated by the seeking strategy built on first-order ESC displays bounded error and large chattering for the case of time-varying Nash equilibrium. In contrast, the result generated by the proposed method displays much smaller error and much less chattering.

Figure 6.1: The black line and black dashed line denote q_1^* and q_2^* , respectively. The blue and red dashed lines respectively denote $\hat{q}_1(t)$ and $\hat{q}_2(t)$ generated by the first-order ESC-based Nash seeking method.

Figure 6.2: The black line and black dashed line denote q_1^* and q_2^* , respectively. The blue and red dashed lines respectively denote $\hat{q}_1(t)$ and $\hat{q}_2(t)$ generated by the proposed ESC-based Nash seeking method.

6.3 Problem Statement

To facilitate the subsequent design and convergence analysis, several definitions related to the Nash equilibrium seeking problem for non-cooperative games are provided as below.

Definition 6.1. The game to be considered is defined as $\Gamma \triangleq \{\mathbb{N}, (u_i)_{i \in \mathbb{N}}, (Q_i)_{i \in \mathbb{N}}\}$ where $\mathbb{N}$ is the set of N players, $u_i(t)$ is the strategy for player i , $U \subset \mathbb{R}^{\mathbb{N}} = \{u_i \mid i \in \mathbb{N}\}$ denotes the strategy space, and Q_i is the payoff function for player i . Define $Q_i \triangleq Q_i(u_i(t), u_{-i}(t), \varsigma_i(t))$ where $\varsigma_i(t)$ is a time-varying unknown vector which makes the Nash equilibrium time-varying and $u_{-i}(t)$ denotes the strategies for all the players other than player i .

Definition 6.2. The strategy vector of the game defined in Definition 6.1 is said to be at the Nash equilibrium if any unilateral change of a player's strategy gives no more payoff value in the sense that $Q_i(u_i^*(t), u_{-i}^*(t), \varsigma_i(t)) \geq Q_i(u_i(t), u_{-i}^*(t), \varsigma_i(t)), \forall i \in \mathbb{N}$.

To simplify the notations, we define

$$\begin{aligned}
\hat{u} &\triangleq [[\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N]^T, \\
u &\triangleq [[u_1, u_2, \dots, u_N]^T, \\
u^* &\triangleq [[u_1^*, u_2^*, \dots, u_N^*]^T, \\
E_\xi &= [\xi_1, \xi_2, \dots, \xi_N]^T, \\
\xi_i &= \hat{u}_i - u_i^*(t), \\
u_i &= \hat{u}_i + a_i \sin(\omega_i t), i \in \mathbb{N},
\end{aligned} \tag{6.5}$$

and the parameter a_i is assumed to be small and positive.

Problem 6.1. (*Nash equilibrium seeking with time-varying payoff functions*) Consider a time-varying payoff function for each player in the game as $Q_i(u_i(t), u_{-i}(t), \varsigma_i(t))$, $i \in \mathbb{N}$. The mapping $Q_i(u_i(t), u_{-i}(t), \varsigma_i(t))$, the Nash equilibrium trajectory $u^*(t)$, and the mapping at the Nash equilibrium denoted by $Q_i^*(t) \triangleq Q_i(u_i^*(t), u_{-i}^*(t), \varsigma_i(t))$ are all unknown. The Nash equilibrium is bounded and the players' payoff functions at the Nash equilibrium are bounded. Design a seeking control scheme such that the strategy vector $\hat{u}(t)$ tracks the time-varying Nash equilibrium $u^*(t)$.

Remark 6.1. In classical extremum seeking, an effective update law for $\hat{u}_i(t)$ is the continuous steepest descent method, which enables $\hat{u}_i(t)$ to converge to a constant optimal point using the gradient measurements. However, in the case of time-varying Nash equilibrium seeking using classic extremum seeking methods, the error and chattering can be large. The simulation result shown in Figure 6.1 helps to verify this point.

Assumption 6.1. The first three derivatives of the time-varying Nash equilibrium trajectory exist and are all bounded, i.e., $\dot{u}^*(t), \ddot{u}^*(t), \dddot{u}^*(t) \in \mathcal{L}_\infty$.

Assumption 6.2. The non-cooperative game in consideration admits at least one

Nash equilibrium at which $\frac{\partial Q_i(u^*(t), s_i(t))}{\partial u_i} = 0$ and $\frac{\partial^2 Q_i(u^*(t), s_i(t))}{\partial u_i^2}$ is uniformly negative for all $t \geq 0$ and for all $i \in \mathbb{N}$.

Assumption 6.3. The payoff functions are smooth and their partial derivatives with respect to $u_j(t)$ for $j \in \mathbb{N}$ exist and are all bounded along the Nash equilibrium trajectory $u^*(t)$.

Assumption 6.4. The frequencies of the dither signals are such that the variations of the dither signals are much faster than the variations of the payoff functions as well as their partial derivatives with respect to $u_j(t)$ for $j \in \mathbb{N}$ at $\hat{u}(t)$.

Assumption 6.5. Define

$$A(t) = \begin{bmatrix} \frac{\partial^2 Q_1(u^*(t), s_1)}{\partial u_1^2} & \frac{\partial^2 Q_1(u^*(t), s_1)}{\partial u_1 \partial u_2} & \dots & \frac{\partial^2 Q_1(u^*(t), s_1)}{\partial u_1 \partial u_N} \\ \frac{\partial^2 Q_2(u^*(t), s_2)}{\partial u_1 \partial u_2} & \frac{\partial^2 Q_2(u^*(t), s_2)}{\partial u_2^2} & \dots & \frac{\partial^2 Q_2(u^*(t), s_2)}{\partial u_2 \partial u_N} \\ \vdots & & \ddots & \\ \frac{\partial^2 Q_N(u^*(t), s_N)}{\partial u_1 \partial u_N} & \dots & & \frac{\partial^2 Q_N(u^*(t), s_N)}{\partial u_N^2} \end{bmatrix}. \quad (6.6)$$

Then $A(t)$ is invertible, $A(t)^T A(t)$ is positive definite, bounded and there exists a positive constant δ that satisfies $A(t)\mathbf{k}_1 + \mathbf{k}_1 A^T(t) \leq -\delta I$ and δ can be made positive and sufficiently large by tuning the positive diagonal matrix $\mathbf{k}_1$.

Remark 6.2. Since $\frac{\partial^2 Q_i(u^*(t), s_i(t))}{\partial u_i^2}$ is uniformly negative for all $i \in \mathbb{N}$, if $A(t)$, $A^T(t)$ are strictly diagonally dominant, this assumption can be easily satisfied, for example, by choosing all the nonzero components in $\mathbf{k}_1$ to be the same.

6.4 Time-varying Nash Equilibrium Seeking

In this section, we provide a Nash equilibrium tracking scheme as well as the stability analysis of the proposed scheme. The overall schematic outline is shown in Figure 6.3. Each player employs the proposed extremum seeking method to

Figure 6.3: Schematic diagram for time-varying Nash equilibrium seeking.

update its strategy. In the proposed extremum seeking control scheme, sinusoidal dither signals are used to modulate the players' payoff functions, a delay based gradient estimation subsystem and a robust gradient search subsystem are designed for tracking the time-varying Nash equilibrium.

Inspired by [37], the updating law for player i in the game is designed as

$$\dot{\hat{u}}_i = k_{i1}\mu_i + \Phi_i - c_{i1}\hat{u}_i, \quad (6.7)$$

$$\dot{\Phi}_i = c_{i2}k_{i1}\mu_i + k_{i2}\text{sgn}(\mu_i), \quad (6.8)$$

where $\mu_i(t)$ is generated by

$$\mu_i(u_i, u_{-i}, \varsigma_i) = \frac{2}{a_i} \frac{1-e^{-T_i s}}{T_i s} [Q_i(u_i, u_{-i}, \varsigma_i) \sin(\omega_i t)], \quad (6.9)$$

and $\text{sgn}(\mu_i) = 1$ if $\mu_i > 0$, $\text{sgn}(\mu_i) = -1$ if $\mu_i < 0$, $\text{sgn}(\mu_i) = 0$ if $\mu_i = 0$. The parameters $k_{i1}, k_{i2}, c_{i1}, c_{i2}$ are positive control gains to be determined. The parameter T_i is a common multiple of the periods of the dither signals.

Remark 6.3. *The proposed extremum seeking scheme is based on a modular design where $\mu_i(t)$ defined in (6.9) is used to extract the gradient. The output of this gradient estimation part is fed into the gradient search part (shown in (6.7)-(6.8)). By utilizing this gradient estimation method, the convergence analysis can be conduct-*

ed without using the averaging method which is the main analysis tool for classical extremum seeking methods.

Remark 6.4. *The actual Nash trajectory is unknown and thus the error signal is unknown. Hence, the estimated gradient is used in the gradient search part where the function $\text{sgn}(\cdot)$ is used to eliminate the effect of some bounded terms such that asymptotic convergence can be achieved. Furthermore, the function $\text{sgn}(\cdot)$ is integrated in the control law so that the overall control law is continuous and admits less chattering in comparison with discontinuous update laws such as sliding mode controllers [87].*

6.4.1 Game Analysis

For general quadratic games, the payoff functions are defined as

$$Q_i(u_i, u_{-i}, \varsigma_i) = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N p_{ijk}(\varsigma_i) u_j u_k + \sum_{j=1}^N q_{ij}(\varsigma_i) u_j + s_i(\varsigma_i), i \in \mathbb{N} \quad (6.10)$$

where $p_{ijk}(t)$, $q_{ij}(t)$, and $s_i(t) \in \mathbb{R}$ are coefficients and $p_{ijk}(t)$, $\dot{p}_{ijk}(t)$ and $\ddot{p}_{ijk}(t)$ are all bounded. Taking the partial derivative of Q_i with respect to u_i yields

$$\frac{\partial Q_i}{\partial u_i}(u, \varsigma_i) = p_{iii} u_i(t) + \sum_{j=1, j \neq i}^N p_{iij} u_j(t) + q_{ii}, i \in \mathbb{N}. \quad (6.11)$$

By Assumption 6.2, we have

$$P(t)u^*(t) + q(t) = 0 \quad (6.12)$$

where the matrix $P(t)$ is defined as

$$P(t) = \begin{bmatrix} p_{111} & p_{112} & \cdots & p_{11N} \\ p_{221} & p_{222} & & p_{22N} \\ \vdots & & \ddots & \\ p_{NN1} & p_{NN2} & & p_{NNN} \end{bmatrix}$$

and

$$q(t) = [q_{11}, q_{22}, \dots, q_{NN}]^T$$

is an $N \times 1$ vector.

Lemma 6.1. *Suppose that Assumption 6.2 holds and the matrix $P(t)$ is invertible for all $t \geq 0$. Then, the Nash equilibrium of the quadratic game exists and is unique.*

Proof. By Assumption 6.2, if $P(t)u^*(t) + q(t) = 0$ admits a solution, then the Nash equilibrium exists. Since the matrix $P(t)$ is invertible, the Nash equilibrium exists and is unique with $u^*(t) = -P^{-1}(t)q(t)$. $\square$

Remark 6.5. *With the uniqueness of the Nash equilibrium, global stability result might be obtained for quadratic games.*

Lemma 6.2. *Suppose Assumptions 6.2 and 6.4 hold and the frequencies of the dither signals are chosen such that $\omega_i \neq \omega_j$, $2\omega_i \neq \omega_j$, and $\omega_i \neq \omega_j + \omega_k$, for all $i, j, k \in \mathbb{N}$, $i \neq j \neq k$. Then, for a quadratic game, $\mu_i(t)$ defined in (6.9) can be related to the error signal as $\mu_i = \sum_{j=1}^N p_{ij}(\varsigma_i)\xi_j$.*

Proof. From (6.9), we have

$$\begin{aligned}
\mu_i &= \frac{2}{a_i} \frac{1-e^{-T_i s}}{T_i s} [Q_i(u_i, u_{-i}, \varsigma_i) \sin(\omega_i t)] \\
&= \frac{2}{a_i T_i} \int_{t-T_i}^t \left[\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N p_{ijk}(\varsigma_i) u_j(\tau) u_k(\tau) \right. \\
&\quad \left. + \sum_{j=1}^N q_{ij}(\varsigma_i) u_j + s_i(\varsigma_i) \right] \sin(\omega_i \tau) d\tau \\
&= \frac{2}{a_i T_i} \int_{t-T_i}^t \sin(\omega_i \tau) \left[\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N p_{ijk}(\varsigma_i) (u_j^*(\tau) u_k^*(\tau) \right. \\
&\quad \left. + u_j^*(\tau) \xi_k + a_k u_j^*(\tau) \sin(\omega_k \tau) + \xi_j u_k^*(\tau) + \xi_j \xi_k \right. \\
&\quad \left. + a_k \xi_j \sin(\omega_k \tau) + a_j \sin(\omega_j \tau) u_k^*(\tau) + a_j \sin(\omega_j \tau) \xi_k \right. \\
&\quad \left. + a_j a_k \sin(\omega_j \tau) \sin(\omega_k \tau) \right) + \sum_{j=1}^N q_{ij}(\varsigma_i) (u_j^*(\tau) + \xi_j + a_j \sin(\omega_j \tau)) + s_i(\varsigma_i) \Big] d\tau.
\end{aligned} \tag{6.13}$$

Based on Assumption 6.4, the variations of Q_i at $\hat{u}(t)$ are much slower than $\sin(\omega_i t)$.

Thus, $\mu_i(u_i, u_{-i}, \varsigma_i)$ can be written as

$$\begin{aligned}
\mu_i &= \frac{2}{a_i T_i} \int_{t-T_i}^t \sum_{j=1}^N p_{iij}(\varsigma_i) \left[a_i u_j^*(\tau) \sin^2(\omega_i \tau) \right. \\
&\quad \left. + a_i \xi_j \sin^2(\omega_i \tau) \right] + q_{ii}(\varsigma_i) a_i \sin^2(\omega_i \tau) d\tau = \sum_{j=1}^N p_{iij} \xi_j.
\end{aligned}$$

The detailed calculation to get the above equation is omitted due to the space limitation. $\square$

Remark 6.6. By Assumption 6.4, the function of the delay based gradient estimation module is similar to the averaging method in which the relatively slowly varying components are regarded as constants for analysis [88]. Furthermore, the output of the gradient estimation part μ_i is equal to the gradient of $\frac{\partial Q_i}{\partial u_i}(\hat{u}, \varsigma_i)$ for $i \in \mathbb{N}$.

Remark 6.7. If T_i is not a common multiple of the periods of the dither signals but only positive integer times of the period of player i 's dither signal, then, the output of the gradient estimation module will have an approximation error for gradient

estimation. Through numerical simulation, we see that the proposed method still works if $|\omega_i - \omega_j|$, $|2\omega_i - \omega_j|$, and $|\omega_i - (\omega_j + \omega_k)|$, for $i, j, k \in \mathbb{N}$, $i \neq j \neq k$ are not that small. However, the transient performance may be degraded to some extent due to imperfect gradient estimation.

Substituting the result in Lemma 6.2 into the update law (6.7)-(6.8), we get

$$\begin{aligned}\dot{\hat{u}} &= \mathbf{k}_1 P(t) E_\xi + \Phi - \mathbf{c}_1 \hat{u}, \\ \dot{\Phi} &= \mathbf{c}_2 \mathbf{k}_1 P(t) E_\xi + \mathbf{k}_2 \text{sgn}(P(t) E_\xi),\end{aligned}\tag{6.14}$$

where $\Phi = [\Phi_1, \Phi_2, \dots, \Phi_N]^T$ and the bold notations $\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_1$, and $\mathbf{c}_2$ are used to denote diagonal control gain matrices of k_{i1}, k_{i2}, c_{i1} , and c_{i2} , respectively..

6.4.2 Stability Analysis for Quadratic Games

For stability analysis, define a filtered signal

$$E_\eta = \frac{d(P(t)E_\xi)}{dt} + \mathbf{c}_2 P(t)E_\xi.$$

Then, the derivative of $E_\eta(t)$ is

$$\begin{aligned}\dot{E}_\eta &= P(t)\ddot{E}_\xi + \mathbf{c}_2 P(t)\dot{E}_\xi + \mathbf{c}_2 \dot{P}(t)E_\xi + 2\dot{P}(t)\dot{E}_\xi + \ddot{P}(t)E_\xi \\ &= P(t)(\ddot{\hat{u}} - \ddot{u}^*(t)) + \mathbf{c}_2 P(t)\dot{E}_\xi + \mathbf{c}_2 \dot{P}(t)E_\xi + 2\dot{P}(t)\dot{E}_\xi + \ddot{P}(t)E_\xi \\ &= P(t)(\mathbf{k}_1 E_\eta + \mathbf{k}_2 \text{sgn}(P(t)E_\xi) - \mathbf{c}_1 \dot{u}^*(t) - \ddot{u}^*(t)) \\ &\quad - P(t)\mathbf{c}_1 \dot{E}_\xi + \mathbf{c}_2 P(t)\dot{E}_\xi + \mathbf{c}_2 \dot{P}(t)E_\xi + 2\dot{P}(t)\dot{E}_\xi + \ddot{P}(t)E_\xi \\ &= P(t)(\mathbf{k}_1 E_\eta + \mathbf{k}_2 \text{sgn}(P(t)E_\xi) - N_c(t)) + \tilde{N}(t),\end{aligned}\tag{6.15}$$

where $N_c(t) = \mathbf{c}_1 \dot{u}^*(t) + \ddot{u}^*(t)$ and

$$\begin{aligned} \tilde{N}(t) = & \left(-P(t)\mathbf{c}_1 + 2\dot{P}(t) \right) P^{-1}(t) \left(E_\eta - \mathbf{c}_2 P(t) E_\xi - \dot{P}(t) E_\xi \right) + \mathbf{c}_2 (E_\eta - \mathbf{c}_2 P(t) E_\xi) \\ & + \ddot{P}(t) E_\xi. \end{aligned}$$

Remark 6.8. By Assumption 6.1, $\dot{u}^*(t), \ddot{u}^*(t), \ddot{u}^*(t)$ are all bounded. Hence, $N_c(t), \dot{N}_c(t)$ are bounded. For convenience, we denote the upper bounds of $N_c(t)$ and $\dot{N}_c(t)$ as U_{N_c} and $U_{\dot{N}_c}$, respectively. Since the elements in $P(t), \dot{P}(t)$ and $\ddot{P}(t)$ are bounded, there exists a positive k such that $\|\tilde{N}(t)\| \leq k \|E_h\|$ where $E_h = [E_\xi^T \ E_\eta^T]^T$ if the elements in $P^{-1}(t)$ are bounded.

A Special Class of Quadratic Games

The subsequent analysis considers a special case where the following assumption is satisfied.

Assumption 6.6. The matrix $P(t)$ is symmetric negative definite and bounded.

Remark 6.9. Under this assumption, the game can be shown to be a potential game [86].

To facilitate the subsequent stability analysis of the closed-loop system, define an auxiliary function $H(t)$ as

$$\begin{aligned} H = & \lambda_{\max}(\mathbf{k}_2) \|E_\eta^T(0)P(0)\|_1 - E_\eta^T(0)P(0)N_c(0) \\ & - \int_0^t E_\eta^T [N_c(\tau) - \mathbf{k}_2 \text{sgn}(P(t)E_\xi)] d\tau. \end{aligned} \quad (6.16)$$

Lemma 6.3. Suppose that $\|\mathbf{c}_2\| U_{N_c} + U_{\dot{N}_c} - \lambda_{\min}(\mathbf{c}_2 \mathbf{k}_2) < 0$ and $U_{N_c} - \lambda_{\min}(\mathbf{k}_2) < 0$. Then, $H \geq 0$ under Assumption 6.1.

Proof. See the proof of Theorem 5.1 for the details. $\square$

Theorem 6.1. *Suppose that Assumptions 6.1-6.4 and Assumption 6.6 hold, the elements in $P^{-1}(t)$ and $\frac{dP^{-1}}{dt}(t)$ are bounded and the control gain $\mathbf{k}_2$ is selected such that $H(t) \geq 0$, $\mathbf{c}_2$ and $\mathbf{k}_1$ are large enough, $\omega_i \neq \omega_j$, $2\omega_i \neq \omega_j$, and $\omega_i \neq \omega_j + \omega_k$ for all $i, j, k \in \mathbb{N}$ and $i \neq j \neq k$. Then, the estimated control input $\hat{u}(t)$ produced by (6.14) globally asymptotically converges to the time-varying Nash equilibrium.*

Proof. The proof follows that of Theorem 5.1. □

Remark 6.10. *The tuning of the control gains follows that of Theorem 5.1.*

More General Quadratic Games

In Theorem 6.1, we assume that the matrix $P(t)$ is symmetric negative definite and bounded by which we get a globally asymptotically stable result. If Assumption 6.6 is not satisfied, we can get a boundedness result under a milder condition. Noticing that for quadratic games $P(t) = A(t)$, which is defined in Assumption 6.5, the following conclusion can be derived.

Theorem 6.2. *Suppose that Assumptions 6.1-6.5 hold, the elements in the matrix $P^{-1}(t)$ are bounded, $\omega_i \neq \omega_j$, $2\omega_i \neq \omega_j$, and $\omega_i \neq \omega_j + \omega_k$, for all $i, j, k \in \mathbb{N}$ and $i \neq j \neq k$. Then, the estimated control input $\hat{u}(t)$ globally asymptotically converges to a neighborhood of the time-varying Nash equilibrium by suitably choosing the control gains under (6.14).*

Proof. Define a Lyapunov function candidate

$$V = \frac{m}{2} E_\xi^T P^T(t) P(t) E_\xi + \frac{1}{2} E_\eta^T E_\eta.$$

With a positive m , there exist positive constants β_{11} , and β_{22} such that $\beta_{11} \|E_h\|^2 \leq V \leq \beta_{22} \|E_h\|^2$.

The time derivative of the Lyapunov candidate function is

$$\begin{aligned}
\dot{V} &= mE_\xi^T P^T(t)(E_\eta - \mathbf{c}_2 P(t)E_\xi) + E_\eta^T \tilde{N}(t) \\
&\quad + E_\eta^T (P(t)\mathbf{k}_1 E_\eta + P(t)\mathbf{k}_2 \text{sgn}(P(t)E_\xi) - P(t)N_c(t)) \\
&\leq -m(\lambda_{\min}(\mathbf{c}_2)\delta_1 - \delta_2) \|E_h\|^2 \\
&\quad + m\lambda_{\min}(\mathbf{c}_2)\delta_1 E_\eta^T E_\eta + k_h \|E_\eta\| \\
&\quad - \frac{1}{2}\delta E_\eta^T E_\eta + k \|E_h\| \|E_\eta\|,
\end{aligned}$$

where δ_1, δ_2 are positive constants that satisfy $\delta_1 I \leq P^T(t)P(t)$, $\|P^T(t)\| \leq \delta_2$, and k_h is a positive constant such that $\|P(t)\mathbf{k}_2\| + \|P(t)\| \|N_c(t)\| \leq k_h$. Define $k_{xx} = \lambda_{\min}(\mathbf{c}_2)\delta_1 - \delta_2$, and $k_{yy} = \frac{1}{2}\delta - m\lambda_{\min}(\mathbf{c}_2)\delta_1$. Then,

$$\begin{aligned}
\dot{V} &\leq -m(\lambda_{\min}(\mathbf{c}_2)\delta_1 - \delta_2) \|E_h\|^2 \\
&\quad - \left(\frac{1}{2}\delta - m\lambda_{\min}(\mathbf{c}_2)\delta_1\right) \|E_\eta\|^2 \\
&\quad + k \|E_h\| \|E_\eta\| + k_h \|E_\eta\| \\
&\leq -\left(mk_{xx} - \frac{k^2}{4k_{yy}}\right) \|E_h\|^2 + k_h \|E_\eta\|.
\end{aligned}$$

The positiveness of k_{xx} , k_{yy} and $k_{zz} = mk_{xx} - \frac{k^2}{4k_{yy}}$ can be ensured by choosing the control gains suitably as described in the subsequent remark. Hence $\dot{V} \leq -k_{zz} \|E_h\|^2 + k_h \|E_h\| \leq -k_{zz} \left(\|E_h\| - \frac{k_h}{2k_{zz}}\right)^2 + \frac{k_h^2}{4k_{zz}}$. The ultimate bound term can be made small by choosing the parameters such that k_{zz} is sufficiently large. $\square$

Remark 6.11. *In this theorem, the parameters should be chosen to satisfy the following inequalities: $k_{xx} > 0, k_{yy} > 0$ and $k_{zz} = mk_{xx} - \frac{k^2}{4k_{yy}} \geq \sigma_2 > 0$, for some positive constant σ_2 . With a fixed m , σ_2 can be made large by choosing $\mathbf{c}_2, \mathbf{k}_1$ such that k_{xx} and k_{yy} are both large.*

6.5 A Numerical Example

Consider a three-player game with payoff functions given by

$$\begin{aligned} Q_1 &= - \left(u_1 - \frac{1}{3}u_2 + \frac{1}{2}u_3 - \varsigma_1 + 1 \right)^2 + 2 + \varsigma_2, \\ Q_2 &= - \left(-\frac{1}{4}u_1 + u_2 - \frac{1}{2}u_3 - \varsigma_3 + 3 \right)^2 + 1 + \varsigma_4, \\ Q_3 &= - \left(-\frac{1}{2}u_1 - \frac{1}{5}u_2 + u_3 + 5 \right)^2 + 2 + \varsigma_5, \end{aligned} \quad (6.17)$$

where $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ and ς_5 are unknown time-varying signals.

Solving this game gives the time-varying Nash equilibrium

$$\begin{aligned} u_1^*(t) &= \frac{108}{121}\varsigma_1 + \frac{28}{121}\varsigma_3 + \frac{8}{121}, \\ u_2^*(t) &= \frac{60}{121}\varsigma_1 + \frac{150}{121}\varsigma_3 - \frac{735}{121}, \\ u_3^*(t) &= \frac{6}{11}\varsigma_1 + \frac{4}{11}\varsigma_3 - \frac{68}{11}, \end{aligned} \quad (6.18)$$

and the payoff values for the three players at Nash equilibrium are

$$\begin{aligned} Q_1^*(t) &= 2 + \varsigma_2, \\ Q_2^*(t) &= 1 + \varsigma_4, \\ Q_3^*(t) &= 2 + \varsigma_5. \end{aligned} \quad (6.19)$$

The simulation results with $\omega_1 = 15$, $\omega_2 = 9$, $\omega_3 = 20$, $a_i = 0.1$, $i \in \{1, 2, 3\}$, are shown in Figures 6.4-6.5.

The simulation results show that the proposed extremum seeking based Nash equilibrium seeking scheme enables $\hat{u}(t)$ to reach a neighborhood of the time-varying Nash equilibrium trajectory.

Figure 6.4: The blue, red and green lines denote the actual Nash equilibrium trajectory for player 1, 2, and 3, respectively. The blue, red and green dashed lines denote the estimated control input of player 1, 2, and 3, respectively, which are generated by the proposed seeking method.

Figure 6.5: The blue, red and green lines denote the actual payoff values at the Nash equilibrium trajectory for player 1, 2, and 3, respectively. The blue, red and green dashed lines denote the output values of player 1, 2, and 3, respectively, which are generated by the proposed seeking method.

6.6 Conclusion

In this chapter, a time-varying extremum seeking strategy is leveraged to seek for the time-varying Nash equilibrium. Different from existing works, the Nash equilibrium and the payoff value at the Nash equilibrium considered can be time-varying. By the interconnected structure of a delay-based gradient estimation method and a robust gradient search method, non-model based Nash seeking is achieved. No explicit expressions on the players' payoff functions are needed. Through Lyapunov stability analysis, it can be shown that for quadratic games, the proposed control law enables the estimated game strategy to globally asymptotically converge to the unknown time-varying Nash equilibrium for symmetric games. A boundedness result is derived for more general quadratic games.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this dissertation, optimal coordination across multiple decision-makers is investigated from the perspectives of non-cooperative game theory and distributed optimization, respectively. If the decision-makers are self-interested, the problems can be formulated as games and the decision-makers are modeled as the players in the game. If coordination among the decision-makers is allowed, distributed optimization problems are explored to achieve the system-level objective. Several methods are proposed to solve non-cooperative games and distributed optimization problems, respectively. To demonstrate the practical applications of game theory and distributed optimization methods, energy consumption control among the electricity consumers is considered. As a summary, the optimal coordination problem in this dissertation is studied from the following aspects:

1. Energy consumption control for a network of price-anticipating users via an aggregate game approach;
2. Distributed extremum seeking for networked optimization and its application to energy consumption control in smart grids;

3. Time-varying distributed optimization among networked agents and Nash equilibrium seeking in non-cooperative games.

Firstly, the application of non-cooperative games for energy consumption control for electricity consumers in smart grids is considered. The problem is solved by using an aggregate game approach. To estimate the aggregate energy consumption, which is unknown to the electricity users (players) during the Nash seeking process, an average consensus protocol is employed. The convergence property is analytically studied via using the singular perturbation method and Lyapunov stability analysis. For a general energy consumption game that may have multiple equilibria, it is shown that the Nash equilibrium is locally exponentially stable. A special case, where the energy consumption game admits a unique inner Nash equilibrium, is discussed. Based on the uniqueness of the Nash equilibrium, an updating law based on the saddle point dynamics is proposed by which the Nash equilibrium is practically stable. Energy consumption game with stubborn players is then studied. For this case, the rational players' actions are shown to converge to a neighborhood of the best response strategies.

Since the Nash equilibrium is usually not the socially optimal solution, a distributed optimization problem is considered for networked agents if coordination is allowed among the agents. To solve this problem, a distributed extremum seeking method is designed to seek for the solution of a general convex networked optimization problem. Extremum seeking laws based on the saddle point dynamics is proposed and studied. The convergence analysis for the proposed distributed extremum seeking scheme is provided. A practically stable result is derived without using the explicit expressions on the constraints, the cost functions or their gradients. The proposed method is applied to solve energy consumption control in smart grids.

Thirdly, continuous-time time-varying distributed optimization problems with quadratic objective functions under undirected graphs are considered. Through Lya-

punov analysis, it can be concluded that the proposed method enables the agents to asymptotically track the unknown time-varying optimal solution for systems with neighboring coupled objective functions. For generally coupled objective functions, approximated results can be obtained by using penalty function based methods.

Lastly, time-varying Nash equilibrium seeking for non-cooperative games is addressed. An extremum seeking based scheme is proposed to achieve time-varying Nash equilibrium seeking for games with time-varying output and Nash equilibrium. By using a delay-based gradient estimation method and a robust gradient search method, a seeking scheme is proposed to track the Nash equilibrium without knowing the explicit model information. Quadratic games are investigated. The estimated game strategy is shown to globally asymptotically converge to the unknown time-varying Nash equilibrium for symmetric quadratic games. For general quadratic games, it is shown that the seeking law enables the estimated game strategy to converge to a neighborhood of the Nash equilibrium under some conditions.

7.2 Future Work

The following problems will be considered in the future work.

1. Analysis of generalized convex game and its solution seeking. In Chapter 3, only a local linear constraint is considered. However, in many practical problems, the strategy set is constrained by more general inequality constraints. By considering the inequality constraints, the game covers more general cases to solve practical problems.
2. Exploration and study of continuous-time time-varying distributed optimization problems. In this dissertation, the generally coupled distributed optimization problem is solved via using penalty function based methods. Nevertheless, we can only get approximated solutions by using the penalty function based

methods.

- (a) Is it possible to get an exact solution for general time-varying distributed optimization problems?
  - (b) Is it possible to generalize the existing results to various communication conditions such as undirected graph, switching topologies and random networks?
  - (c) The development of more efficient distributed optimization techniques, e.g., techniques that reduce communication cost and computation cost.
3. Demand response program design and analysis for the smart grid. With the newly developed communication infrastructures (e.g., smart meters) and control algorithms, demand response has great potential to benefit both the electricity consumers and the generators. Below are some interesting problems to be further explored:
- (a) How to design pricing schemes such that when the electricity users self-interested to minimize their own costs, the social cost is also be minimized?
  - (b) How to accommodate the uncertainties if renewable energy resources such as wind power and solar system are integrated in the power system?

As a summary, this dissertation studied non-cooperative game theory and distributed optimization problem from different perspectives. The application of the non-cooperative game theory and distributed optimization methods on energy consumption control for electricity consumers is investigated. By the above explorations, the goal to study optimal coordination among multiple decision makers is achieved to some extent. With the prevalence of complex systems, this active area is of both practical and theoretical significance. Nevertheless, there are still many

open questions. More efforts will be dedicated to study game theory with constraints, time-varying distributed optimization problems as well as the design and analysis of demand response schemes in smart grids.

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