Multiscale maxwell equations : homogenization and high dimensional finite element method

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**NANYANG
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**MULTISCALE MAXWELL EQUATIONS:
HOMOGENIZATION AND HIGH DIMENSIONAL
FINITE ELEMENT METHOD**

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SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

2016

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HOMOGENIZATION AND HIGH DIMENSIONAL
FINITE ELEMENT METHOD**

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Abstract

Solving multiscale partial differential equations is exceedingly complex. Traditional methods have to use a mesh size of at most the order of the smallest scale to produce accurate approximations. Due to the importance of these problems in practical applications, there have been intensive efforts to study multiscale partial differential equations theoretically, and to develop feasible numerical methods to solve them. Although there has been a huge literature on the subject, there are still many open challenging problems that need theoretical study and effective solution methods.

The thesis studies multiscale Maxwell equations. Comparing to other types of equations, such as multiscale elliptic equations and multiscale elasticity equations, multiscale Maxwell equations have attracted far less attention, although they arise from many important applications.

Homogenization of multiscale Maxwell equation possesses features that are not seen in elliptic problems due to the fact that in a domain $D \subset \mathbb{R}^d$, unlike $H_0^1(D)$, the space $H_0(\text{curl}, D)$ is not compactly embedded in $L^2(D)^d$. Homogenization of multiscale Maxwell equations can be significantly more complicated than homogenization of multiscale elliptic equations in many circumstances.

The thesis contributes rigorous study of mathematical homogenization of multiscale Maxwell equations. It includes new homogenization errors when the solution to the homogenized equation possesses low regularity. Such errors (even for elliptic problems) have not been deduced in the literature. We derive the error for two-scale Maxwell equations, but the procedure works the same way for other types of equations such as two-scale elliptic problems, and two-scale elasticity problems. The thesis develops the sparse tensor finite element approach, using edge finite elements, for solving the high dimensional multiscale homogenized Maxwell equations. It obtains the solution to the homogenized equation, which describes the solution to the multiscale equation macroscopically, and the scale interacting (corrector) terms, which encode the microscopic information, at the same time. The method achieves essentially optimal complexity. From the finite element solutions, we construct a numerical corrector for the solution of the multiscale problem, with an explicit error in terms of the homogenization error and the finite element error in the two-scale cases.

In Chapter 2, we consider the time independent multiscale Maxwell equation which is of the same type as that considered in Bensoussan et al.[15], but in the multiscale setting, in a domain $D \subset \mathbb{R}^d$ ($d = 2, 3$). We use multiscale convergence

(see [79, 7]) to derive the multiscale homogenized equation, from which the homogenized equation is obtained. In the two-scale case, when the solution of the homogenized equation $u_0 \in H^1(\text{curl}, D)$, we derive the standard $O(\varepsilon^{1/2})$ homogenization error, where ε is the microscopic scale. However, in polygonal domains which are of interest in finite element discretization, u_0 only belongs to a weaker regularity space $H^s(\text{curl}, D)$ for $0 < s < 1$, for which the standard procedure does not work. We modify the procedure substantially to deduce a new homogenization error in this case. The method is general; It works also for other types of two-scale problems when the solution to the homogenized equation possesses low regularity. For multiscale problems, correctors are derived without an error.

Chapter 3 studies the multiscale Maxwell wave equation. When the coefficient of the second time derivative term is oscillating, derivation of the homogenized equation is much more complicated, as unlike the multiscale wave equation where the homogenized equation contains the average of this coefficient, the homogenized Maxwell wave equation contains the elliptic homogenized coefficient of this coefficient. We use multiscale convergence to derive the multiscale homogenized equation in the generalized sense, from which, with significant technical development, we derive the homogenized Maxwell wave equation and the initial conditions. For the case where the initial condition of the multiscale problem $u^\varepsilon(0, \cdot) = 0$, we derive correctors. For the two-scale problem, if the solution u_0 of the homogenized problem belongs to $L^\infty(0, T; H^1(\text{curl}, D))$, we derive the standard $O(\varepsilon^{1/2})$ homogenization error. As mentioned above, in polygonal domains, this regularity normally does not hold; u_0 only belongs to a weaker regularity space $L^\infty(0, T; H^s(\text{curl}, D))$ for $0 < s < 1$. We develop a new procedure to derive a homogenization error for this case of weaker regularity for u_0 . For multiscale problems, correctors are derived but without an error.

Chapter 4 develops the sparse tensor finite elements for the multiscale Maxwell homogenized equation derived in Chapter 2, using edge finite elements. We solve for the solution of the homogenized equation, and the corrector terms at the same time. The method achieves an accuracy essentially equal to that for the full tensor product finite elements, but with a level of complexity essentially equal to that for solving a problem in $\mathbb{R}^d$ to obtain the same level of accuracy. We show that the regularity required to obtain the sparse tensor finite element error estimates holds. Numerical correctors are deduced from the finite element solutions, with an error in terms of the homogenization error and the finite element error in the two-scale case. The theoretical results are confirmed by some numerical examples of two-scale Maxwell equations in two dimensions.

Chapter 5 develops finite element approximations for the multiscale homogenized Maxwell wave equation derived in Chapter 3. We first study the spatially semi-discretized problem and the fully-discretized problem for general finite ele-

ment spaces. We adapt the approach of Dupont [40] for general wave equations, but need to modify the approach significantly for the Maxwell case as the second time derivatives of the gradient of some corrector terms need to be considered. We deduce the conditions for the schemes to converge, and estimate the errors. We then apply the general framework to the full and sparse tensor product finite elements. Error estimates are derived explicitly with respect to the spatial mesh size (in the semi-discretized problem) and the spatial mesh size and time step (in the fully-discretized problem). From the finite element solutions, we deduced numerical correctors, with an explicit error in terms of the homogenization error and the finite element error in the two-scale case. We solve some two-scale Maxwell wave equations in two dimensions to illustrate the theory.

Chapter 1

Introduction

Solving partial differential equations that depend on multiple scales is exceedingly complicated. To take into account all the microscopic scales, the approximation mesh-width has to be at most of the order of the smallest scale, that is of several orders of magnitude smaller than the macroscopic dimension of the domain of interests. The traditional solution methods thus require prohibitive amounts of resources that often surpass the capacity of currently available computers. Multiscale partial differential equations play highly essential roles in many engineering and practical problems such as composite materials, subsurface flow, oil engineering, biology, etc (see, e.g., [88, 71, 66]). Thus finding feasible computational methods that can solve these problems accurately within practically acceptable computational resources is of utmost important. There has been intensive efforts to study multiscale problems mathematically and to develop practical methods for solving them.

Figure 1.1: Visualization of homogenization limit

Homogenization is a key method for studying multiscale partial differential equations. The problems are approximated by equivalent partial differential equations that only depend on the macroscopic scales and can be solved by traditional

methods. These homogenized problems approximate the original multiscale problems in the average sense, and are obtained when letting the microscopic scales converge to zero. The subject has a long history; it dates back to the work of Rayleigh [87] and Maxwell [67] in the 19th century, but only attracts much attention in the last few decades. Though there has been enormous literature on the subject [15, 62, 37, 13], there is still a large number of important open problems that need theoretical study and effective numerical methods to solve.

In the case where the problems depend on the microscopic scales periodically, for two-scale problems (which depend on one microscopic scale and one macroscopic scale), the homogenized equation can be established by the two-scale asymptotic expansion method (see, e.g., [10, 11, 12, 15, 88, 62, 29, 81, 13]). For multiscale problems, such an asymptotic expansion can be carried out but is much more complicated (see [15]). To establish the coefficients of the homogenized problems (the homogenized coefficients), a set of equations that are posed in the periodic cell (the cell problems) has to be solved. Solving them can be highly complex, especially when the problem is only locally periodic, or depends on many microscopic scales. For general problems, such a limiting homogenized equation can be shown to exist by the abstract G and Γ convergence theories (see, e.g., [39, 62, 27, 6]) but this is not a constructive result and provides no clear way for establishing the homogenized equation.

Further, homogenized problems only provide the macroscopic information on the solution of the multiscale problems. To get the microscopic information, we need the corrector terms. Computing them is an expensive process.

Several numerical methods have been developed to solve directly multiscale problems with reduced complexity. The Multiscale Finite Element method (Ms-FEM) (see [58, 30, 44]) constructs finite element basis functions which contain the microscopic information by solving fine scale equations using fine mesh in each macroscopic simplex. The procedure can be carried out by parallelization. The advantage is that once the multiscale basis has been established, it can be reused for different right hand side. We note the recent work on Generalized Multiscale Finite Element method (see [43]). The Heterogeneous Multiscale Method (HMM) (see [42, 41, 3]) solves one elliptic cell problem for each macroscopic degree of freedom to take into account the microscopic information. The complexity of these methods is less than that for solving directly the multiscale problems using fine mesh, but to achieve an error $O(h)$ for a problem in $\mathbb{R}^d$, the complexity grows superlinearly with respect to $N = O(h^{-d})$, i.e., the number of degrees of freedom needed for solving a macroscopic problem for the same accuracy.

We consider in this thesis multiscale Maxwell equations. Comparing to other types of equations such as multiscale elliptic or elasticity equations, multiscale Maxwell equations have attracted far less attention. Homogenization of two-scale

Maxwell type equation is treated in Bensoussan et al. [15], Sanchez-Palencia [88] and Jikov et al. [62] using two-scale asymptotic expansion and compensated compactness theory. The two-scale convergence method (see [79, 5]) is employed for Maxwell equation in Wellander and Kristensson [98] and Amirat and Shelukhin [8]; see also the references [18, 14] where the related unfolding method is employed. In the contexts of metamaterials, Kohn and Shipman [63] and Bouchitté and Schweizer [24] employed two-scale convergence to study effective properties of Maxwell equations in media with high contrast coefficients. Numerically, there has been very little work on multiscale Maxwell equations. We mention the works by Zhang et al. [106, 28, 105] where cell problems are solved to establish the homogenized equation and the correctors.

Multiscale Maxwell equations arise in many applications. Studying its effective properties have been an important subject in physics, mechanics and engineering literature. The subject dates back to the work of Maxwell [67]. The equations has been studied extensively in the physics and mechanics literature, we refer to the book by Milton [71] for the vast literature. In the applied mathematics and mechanics literature, a large amount of works concentrated on finding appropriate bounds for the effective coefficients (see [16, 69, 70, 99, 97, 86, 94, 95]). Multiscale Maxwell equations play a vital role in studying metamaterials which are composites with negative indices (see [85, 92]). There has been some work in the homogenization theory in this direction, typically Kohn and Shipman [63], and Bouchitté and Schweizer [24] (see also [38, 23, 74, 19, 45, 20, 21, 22, 35]). As shown in the initial work by Bensoussan et al. [15], the homogenized Maxwell equation can be very different from that of an elliptic equation of similar form. This is due to the fact that in a domain $D \subset \mathbb{R}^d$, the space $H_0(\text{curl}, D)$, unlike the Sobolev space $H_0^1(D)$, is not compactly embedded in $L^2(D)^3$. This leads to studying multiscale Maxwell equations significantly more complicated, especially the multiscale Maxwell wave equations.

We study in this thesis the Maxwell type equation (2.4) and the Maxwell wave equation (3.2). However, we note that Helmholtz equations with large wave numbers play an exceedingly important role in practical applications. To study these equations, we need to deal with the fact that they are not positive definite, adding to the already high complex technicalities involving the $H(\text{curl}, D)$ space. We will need to deal with Helmholtz type equations with highly oscillating coefficients. In the literature, even in the H^1 setting, Helmholtz equations with multiple scales have not been adequately studied. Solving a Helmholtz equation that depends on multiple spatial scales, with a large wave number, will be highly complicated. The problem is entirely open in literature. We thus do not think that it is adequate to address this topic within the scope of the thesis, given that analysis and numerical analysis of multiscale Maxwell type equations have not been much studied, and

that the two problems we consider here are already very complicated. Multiscale Helmholtz equations deserve another full separate research project, for which the current project may serve as a useful foundation.

In Chapter 2, we consider the Maxwell type equation that was studied in Bensoussan et al. [15], but in the general multiscale setting. Bensoussan et al. [15] essentially employ the two scale expansion in the case where the coefficient b^ε in (2.4) is a constants. For the general problem, they show that in the homogenized equation, the homogenized coefficient b^0 is essentially that of an elliptic problem with b^ε as the multiscale coefficient. The proof relies on the compensated compactness theorem [75, 76]. In the elliptic case, the homogenized coefficient for b^ε is only the average of $b(x, y)$ in the cell Y with respect to y . This is due to the fact that the space $H_0^1(D)$ is compactly embedded in $L^2(D)$; this is not the case with the space $H(\text{curl}, D)$ that we consider here. We use multiscale convergence to study this problem and derive the homogenized equation that is of the same form as in [15]. Two-scale limit of a bounded sequence in $H(\text{curl}, D)$ is studied in [98]. As our equation depends on multiple scales, and the form of the limit that we use is slightly different from that in [98], we derive the multiscale limit of a bounded sequence in $H(\text{curl}, D)$ in details. We contribute in this chapter the homogenization rate when the solution u_0 of the homogenized equation possesses less regularity. It is well known that for a two-scale elliptic problem, if the solution of the homogenized equation in a domain $D \subset \mathbb{R}^d$ belongs to $H^2(D)$, the homogenization error $O(\varepsilon^{1/2})$ (ε is the microscopic scale) can be deduced. This regularity for u_0 is attainable when the domain is smooth or is a convex polygon (which is of interests in finite element discretization). Using the same procedure, a similar error for two-scale Maxwell equations is achieved when the solution of the homogenized equation is in $H^1(\text{curl}, D)$; we perform this in details in this chapter. However, in polygonal domains, this regularity does not hold normally; u_0 only belongs to a weaker regularity space $H^s(\text{curl}, D)$ where $0 < s < 1$ (we refer to R. A. Adams and J.F Fournier [4] for definition and properties of fractional Sobolev spaces). We develop a procedure to derive homogenization error for the case where $u_0 \in H^s(\text{curl}, D)$. This is a new result in homogenization theory. For two-scale elliptic problems, if the solution u_0 of the homogenized equation is in $H^{1+s}(D)$ for $0 < s < 1$, to the best of our knowledge, a homogenization error has not been deduced in the literature. Our method applies verbatim in this case, and produces the same homogenization error as for the Maxwell equation in this chapter. We thus contribute a new approach to derive homogenization errors when the solution to the homogenized equation possesses low regularity; this approach works for many classes of equations: elliptic equations, elasticity equations, Maxwell equations, etc. and also for time dependent equations (see the paragraph below for Maxwell wave equations). As usual, to deduce homogenization errors, we need sufficient smooth-

ness for the solutions to the cell problems. We prove that the required smoothness holds in this chapter.

In Chapter 3, we consider the Maxwell wave equation. Derivation of the homogenized equation for the Maxwell wave equation when the coefficient b^ε in (3.2) depends on the fast variables is far more complicated than that for the multiscale wave equation considered in Xia and Hoang [101]. This is due to the corrector functions $\mathbf{u}_i$ in (3.5); the time derivatives $\frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_i$ have to be considered which in the general case are only understood in the distribution sense, and do not belong to $L^2(D \times Y_1 \times \cdots \times Y_n)$. The multiscale homogenized problem (3.9) is only understood in the general sense. The derivation of the initial conditions requires the derivations of the representative of the time derivative $\frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i$ in terms of the solutions of the cell problems, which are quite nontrivial. In this chapter, we deduce the homogenization error for the case where the solution u_0 of the homogenized equation belongs to $L^\infty(0, T; H^1(\text{curl}, D))$, and for the case where u_0 belongs to a weaker regularity space $L^\infty(0, T; H^s(\text{curl}, D))$ where $0 < s < 1$ in the two-scale case. As is well known, for two-scale wave equations, a corrector similar to that of two-scale elliptic problems does not always hold as the energy of the multiscale problem, in general, does not converge to the energy of the homogenized problem (see [26]). Therefore for deriving the correctors, we restrict our consideration to the case where the initial condition g_0 in (3.2) equals 0. The derivation of the homogenization error when $u_0 \in L^\infty(0, T; H^1(\text{curl}, D))$ follows from the procedure in Xia and Hoang [101] for two-scale wave equations, which adapts the standard approach for elliptic problems in [62]. As for the time independent equation considered in Chapter 2, in polygonal domains, the regularity $u_0 \in L^\infty(0, T; H^1(\text{curl}, D))$ generally does not hold; u_0 only belongs to a weaker regularity space $u_0 \in L^\infty(0, T; H^s(\text{curl}, D))$ for $0 < s < 1$. With this lower regularity, the approach of Xia and Hoang [101] does not work. We modify the procedure for deriving homogenization errors substantially for this case, using the ideas developed in Chapter 2 for time independent equations, and obtain a new homogenization error for the case where u_0 belongs to this weaker regularity space. This is a new result in the literature. For two-scale wave equations considered in [101], a homogenization error is deduced only when $u_0 \in L^\infty(0, T; H^2(D))$. When the solution of the homogenized wave equation u_0 belongs to a weaker regularity space $L^\infty(0, T; H^{1+s}(D))$ ($0 < s < 1$), their approach does not apply. For this case, our new procedure works exactly the same way and provides the same homogenization error. We show in this chapter that the regularity required to get the homogenization errors hold. For the general multiscale problem, we derive correctors for the solution of the multiscale equation but without an explicit homogenization error. Due to the fact that the homogenized equation (3.23) contains the homogenized coefficient b^0 (which is the usual elliptic homogenized coefficient for b^ε), the derivation is more complicated than for the

multiscale wave equation in [101].

Chapters 4 and 5 develop the sparse tensor finite element (FE) approach to solve the multiscale homogenized Maxwell equations derived in Chapters 2 and 3.

The concept of two-scale convergence is first studied by Nguetseng [79] and explored further by Allaire [5], and is extended to multiple scales by Allaire and Briane [7] (see Arbogast et al. [9] for a related result). Multiscale convergence has since become a very effective tool for homogenization. For elliptic problems, these authors derive the multiscale homogenized equations which contain both the solution to the homogenized equation which approximates the solution of the multiscale equation macroscopically, and the corrector terms which contain the microscopic information. Solving this equation, we get all the necessary information. The setback is that this equation is posed in a high dimensional space: if the original multiscale equation is posed in $\mathbb{R}^d$ and depends on n microscopic scales, the multiscale homogenized equation is posed in $\mathbb{R}^{d(n+1)}$. Solving this problem by straightforward full tensor product FE is prohibitively expensive. Hoang and Schwab [56] proposed the sparse tensor product FE approach, where the corrector terms that depend on the slow and fast variables and are posed in product domains are discretized using a sparse tensor product of separate FE subspaces with respect to each variable. The approach produces an approximation that achieves a level of accuracy essentially equal to that of the plain full tensor product FE approach, but requires only a number of degrees of freedom that is essentially equal to that needed for solving a problem in $\mathbb{R}^d$ to obtain the same level of accuracy. The sparse tensor product FE approach works essentially due to the fact that the corrector terms which are posed in product domains possess regularity with respect to each variable at the same time. The approach has been applied successfully to solving multiscale nonlinear monotone problems in Hoang [54], multiscale wave equations [101], multiscale elasticity equations [102], and multiscale elastic wave equations [103]. It is employed for multiscale elliptic equations using frames in Harbrecht and Schwab [50]. We note that the multiscale homogenized equation for elliptic equations has been solved in the context of the HMM method by Ohlberger [80] and the time dependent advection-diffusion problem by Ohlberger and Henning [51] but the complexity is equivalent to that for using full tensor product elements. We note further that for ergodic homogenization problem, the concept of two-scale convergence is extended to two-scale convergence in the mean by Bourgeat et al. [25]. Further, for stochastic problem which are not necessarily ergodic, multiscale convergence have been applied recently to find macroscopic and microscopic information in [57] and [55]. The sparse tensor finite elements methods may be able to solve these stochastic problems but this has not been explored. Sparse grid and sparse tensor product approximations have a long history. They are first proposed by Smolyak [93] in the context of approximations of integrals. Extensive references

on the subject can be found in [104, 96]. Recently, sparse tensor finite elements are extended to the anisotropic setting by Griebel and Harbrecht in [48, 47]. In the context of stochastic partial differential equations, sparse tensor approximations are employed in Schwab and Todor [90, 91] and Bieri et al. [17].

In Chapter 4, we develop the sparse tensor product FE for the multiscale Maxwell equation that does not depend on time which is considered in Chapter 2. We note that sparse tensor product FEs are employed in Hiptmair et al. [53] in the context of computing the moments of the solution of stochastic Maxwell equations. These authors construct the sparse tensor FE spaces theoretically, without numerical implementation. The construction of the “detail spaces” is quite complicated. In our setting, due to the special structures of our corrector terms, we do not need the detail spaces for the space $H_{\#}(\text{curl}, Y)$ of periodic functions, but only need them for the FE spaces that approximate functions in the Lebesgue spaces L^2 . For the sparse tensor product FE to work, we need regularity of the corrector terms $\mathbf{u}_i$ and u_i in (2.10) with respect to all the independent variables at the same time. We show that the required regularity is achievable. From the FE solution of the multiscale homogenized problem, we construct numerical correctors. In the two-scale case, due to the availability of a homogenization error, an explicit error for this numerical corrector is derived. It is the sum of the homogenization error and the FE error. For the multiscale case, as such a homogenization error is not available, we derive a numerical corrector without an explicit rate of convergence. We solve some two-scale Maxwell equations using sparse tensor product FE to illustrate the theoretical results.

Chapter 5 develops FE approximations for the multiscale Maxwell wave equations. Comparing to multiscale elliptic and parabolic equations, numerical treatment of multiscale wave equations has been paid much less attention. A general method which uses solution of d multiscale elliptic problems to construct a multiscale basis is proposed to solve multiscale wave equations by Owhadi and Zhang see also [82, 84, 83]. Though general, the complexity of this method is similar to that for solving the original multiscale problem using a fine mesh. Jiang et al [61] and Jiang and Efendiev [59, 60] employ the MsFEM method to solve multiscale wave equation with continuum scales, given limited global information. Though general, this method requires a rather high complexity. The HMM method is employed by Engquist et al. [42] to solve multiscale wave equation by finite differences. This paper computes also the solution at large time to show the dispersive behaviour similar to that discovered by Santosa and Symes [89]. Abdulle and Grote employs the HMM method in the finite element setting for multiscale wave equation in [1, 2]. Recently, the Generalized Multiscale Finite Element method has been applied to multiscale wave equations and multiscale elastic wave equations in [33, 46, 32]. The methods developed in these references are general, but their complexity can

be high, which is equivalent to that for solving directly the multiscale equation using fine mesh. When the media are locally periodic, using the sparse tensor finite element method for the multiscale homogenized problem, Xia and Hoang developed essentially optimal method for solving multiscale wave and elastic wave equations in [101, 103]. We adapt the discretization procedure of Dupont [40] for wave equations. This procedure is used to discretize the multiscale homogenized equations for multiscale wave equations in Xia and Hoang [101]. For multiscale homogenized Maxwell equation, due to the corrector terms $\frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_i$, the procedure of Dupont needs to be modified significantly, in comparison to the work [101]. We first consider the spatially semi-discretized problem for general FE spaces. We then consider the fully discretized problem in both space and time variables. For these schemes, we study theoretically the conditions for them to converge, and estimate the errors. The general framework is then applied to the full tensor product FEs and the sparse tensor product FEs. We derive explicitly the error in terms of the spatial mesh (for the spatially semi-discretized problem) and in terms of the time step and the spatial mesh (for the fully discretized problems). We prove that the regularity required to achieve the FE rate of convergence holds under compatibility conditions for the initial conditions. For achieving the error estimate of the sparse tensor FE product approximation, we show that the corrector terms possess the regularity at the same time in terms of each spatial independent variable. When the initial condition g_0 in (3.2) equals 0, we deduce numerical correctors from the FE solutions. In the two-scale case, we derive an explicit error for the numerical corrector in terms of the homogenization error and the FE error. In the multiscale case, a numerical corrector is derived without an error, as a homogenization error is unavailable. In the end of the chapter, we solve some two-scale Maxwell wave equations in two dimensions to illustrate the theory.

Throughout the thesis, by ∇ and curl , without explicitly indicating the variable, we understand the gradient and the curl of a function of the variable x in the domain D , with respect x , while ∇_x and curl_x denote the partial gradient and partial curl with respect to the variable x of a function that depends also on other variables. The notation $\#$ denotes spaces of periodic functions; $H_{\#}^k(Y)$ denotes the Sobolev spaces of periodic functions in the periodic cell Y ; and $H_{\#}^k(\text{curl}, Y)$ denotes the spaces of periodic functions in $H^k(Y)$ whose curl belongs also to $H^k(Y)$. As usual, repeated indices indicate summation.

Chapter 2

Homogenization of stationary Maxwell equations

We study multiscale time independent Maxwell type equation in this chapter. In Section 2.1, we define the basic notations and introduce the setting up of the multiscale problem. We then recall the concepts of multiscale convergence and derive the multiscale limit of a bounded sequence in $H(\text{curl}, D)$. The multiscale homogenized Maxwell equation is then deduced and is proved to be well posed. In Section 2.2, we drive the correctors for the multiscale problems. In the two-scale case, we analyze a homogenization error. When the solution u_0 of the homogenized equation belongs only to a low regularity space, we develop a new procedure to deduce homogenization error. For the homogenization error to hold, we need regularity for the solution of the cell problems, and for the solution of the homogenized equation. These are proved in Section 2.3.

2.1 Problem setting

2.1.1 Multiscale Maxwell problems

Let D be a domain in $\mathbb{R}^d$ ($d = 2, 3$). Let Y be the unit cube in $\mathbb{R}^d$. By $Y_1, \dots, Y_n$ we denote n copies of Y . We denote by $\mathbf{Y}$ the product set $Y_1 \times Y_2 \times \dots \times Y_n$ and by $\mathbf{y} \in \mathbf{Y}$ the vector $\mathbf{y} = (y_1, y_2, \dots, y_n)$. For each $i = 1, \dots, n - 1$, we denote by $\mathbf{Y}_i$ the set of vectors $\mathbf{y}_i = (y_1, \dots, y_i)$ where $y_j \in Y_j$ for $j = 1, \dots, i$.

For $d = 3$, let a and b be functions with symmetric matrix values from $D \times \mathbf{Y}$ to $\mathbb{R}_{sym}^{d \times d}$; a and b are continuous in $D \times \mathbf{Y}$ and are periodic with respect to each variable y_i with the period being Y_i . We assume that for all $x \in D$ and $\mathbf{y} \in \mathbf{Y}$, and all $\xi, \zeta \in \mathbb{R}^d$,

$$\begin{aligned} \alpha|\xi|^2 &\leq a_{ij}(x, \mathbf{y})\xi_i\xi_j, & a_{ij}(x, \mathbf{y})\xi_i\zeta_j &\leq \beta|\xi||\zeta|, \\ \alpha|\xi|^2 &\leq b_{ij}(x, \mathbf{y})\xi_i\xi_j, & b_{ij}(x, \mathbf{y})\xi_i\zeta_j &\leq \beta|\xi||\zeta|, \end{aligned} \quad (2.1)$$

where α and β are positive numbers; $|\cdot|$ denotes the Euclid norm in $\mathbb{R}^3$. Let ε be a small positive value, and $\varepsilon_1, \dots, \varepsilon_n$ be n functions of ε that denote the n microscopic scales that the problem depends on. We assume the following scale separation properties: for all $i = 1, \dots, n-1$

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{i+1}(\varepsilon)}{\varepsilon_i(\varepsilon)} = 0. \quad (2.2)$$

Without loss of generality, we assume that $\varepsilon_1 = \varepsilon$. We define $a^\varepsilon, b^\varepsilon : D \rightarrow \mathbb{R}_{sym}^{d \times d}$ as

$$a^\varepsilon(x) = a\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right), \quad b^\varepsilon(x) = b\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right). \quad (2.3)$$

When $d = 3$ we define the space

$$W = H_0(\text{curl}, D) = \{u \in L^2(D)^3, \quad \text{curl } u \in L^2(D)^3, \quad u \times \nu = 0\}, \quad H = L^2(D)^3$$

where ν denotes the outward normal vector on the boundary ∂D . For notational conciseness, we denote by

$$H = L^2(D)^3, \quad H_i = L^2(D \times \mathbf{Y}_i)^3, \quad i = 1, \dots, n.$$

As W is dense in H and H is dense in W' , we denote by $\langle \cdot, \cdot \rangle_H$ the inner product in H , extending to duality pairing between W' and W , i.e., $\langle \cdot, \cdot \rangle_{W', W}$. Let $f \in W'$. We consider the problem

$$\text{curl}(a^\varepsilon(x)\text{curl } u^\varepsilon(x)) + b^\varepsilon(x)u^\varepsilon(x) = f(x), \quad (2.4)$$

with the boundary condition $u^\varepsilon \times \nu = 0$ on ∂D .

We formulate this problem in the variational form as: Find $u^\varepsilon \in W$ so that

$$\int_D [a^\varepsilon(x)\text{curl } u^\varepsilon(x) \cdot \text{curl } \phi(x) + b^\varepsilon(x)u^\varepsilon(x) \cdot \phi(x)]dx = \int_D f(x) \cdot \phi(x)dx \quad (2.5)$$

for all $\phi \in W$. Lax-Milgram lemma guarantees the existence of a unique solution u^ε that satisfies

$$\|u^\varepsilon\|_W \leq c\|f\|_{W'} \quad (2.6)$$

where the constant c only depends on α and β in (2.1).

Figure 2.1: Visualization of two-scale problem

For $d = 2$, the matrix function $b^\varepsilon : D \times \mathbf{Y} \rightarrow \mathbb{R}^{2 \times 2}$ is defined as above. As $\text{curl } u^\varepsilon$ is now a scalar function, $a(x, \mathbf{y})$ is a continuous function from $D \times \mathbf{Y}$ to $\mathbb{R}$ which is periodic with respect to each variable y_i with the period being Y_i . In the place of (2.1), we have

$$\alpha \leq a(x, \mathbf{y}) \leq \beta, \quad \forall x \in D \text{ and } \mathbf{y} \in \mathbf{Y}.$$

In this case, we define

$$W = H_0(\text{curl}, D) = \{u \in L^2(D)^2, \quad \text{curl } u \in L^2(D), \quad u \times \nu = 0\}, \quad H = L^2(D)^2.$$

The variational formulation in two dimensions becomes:

$$\int_D [a^\varepsilon(x) \text{curl } u^\varepsilon(x) \text{curl } \phi(x) + b^\varepsilon(x) u^\varepsilon(x) \cdot \phi(x)] dx = \int_D f(x) \cdot \phi(x) dx \quad \forall \phi \in W. \quad (2.7)$$

In the rest of the thesis, we present the results for the three dimension case and only mention the two dimension case when necessary; the results for two dimensions are similar.

2.1.2 Multiscale convergence

We use multiscale convergence to derive the homogenized equation. We first recall the definition of multiscale convergence (see Nguetseng [79], Allaire [5] and Allaire and Briane [7]).

Definition 2.1.1. A sequence of functions $\{w^\varepsilon\}_\varepsilon \subset L^2(D)$ $(n+1)$ -scale converges to a function $w^0 \in L^2(D \times \mathbf{Y})$ if for all smooth functions $\phi \in C^\infty(D \times \mathbf{Y})$ which are periodic with respect to y_i with the period being Y_i for $i = 1, \dots, n$,

$$\lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) dx = \int_D \int_{\mathbf{Y}} w^0(x, \mathbf{y}) \phi(x, \mathbf{y}) d\mathbf{y} dx.$$

We have the following result.

Proposition 2.1.2. From a bounded sequence in $L^2(D)$ we can extract an $(n+1)$ -scale convergent subsequence.

For a bounded sequence in $H(\text{curl}, D)$, we have the following results on multiscale convergence. These results are first established in [98] for the two-scale case. We will use these results to study the multiscale equations (2.5) and (2.7). By $\tilde{H}_\#(\text{curl}, Y_i)$ we denote the equivalent classes of functions in $H_\#(\text{curl}, Y_i)$ such that if $\text{curl } v = \text{curl } w$ we regard $v = w$ in $\tilde{H}_\#(\text{curl}, Y_i)$. The multiscale convergence limit is different from that for a bounded sequence in $H^1(D)$ due to the fact that $H(\text{curl}, D)$ is not compactly embedded in $L^2(D)$.

Proposition 2.1.3. Let $\{w^\varepsilon\}_\varepsilon$ be a bounded sequence in $H(\text{curl}, D)$. There is a subsequence (not renumbered), a function $w_0 \in H(\text{curl}, D)$, n functions $\mathbf{w}_i \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$ such that

$$w^\varepsilon \xrightarrow{(n+1)\text{-scale}} w_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{w}_i.$$

Further, there are n functions $w_i \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ such that

$$\text{curl } w^\varepsilon \xrightarrow{(n+1)\text{-scale}} \text{curl } w_0 + \sum_{i=1}^n \text{curl}_{y_i} w_i.$$

Proof Let $\xi \in L^2(D \times \mathbf{Y})^3$ be the $(n+1)$ -scale limit of $\{w^\varepsilon\}_\varepsilon$. Consider the function

$$\phi = \varepsilon_n \Phi(x, y_1, \dots, y_n)$$

where Φ is a function in $C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_n) \dots))^3$ and is periodic with respect to $y_1, \dots, y_n$ with the period being $Y_1, \dots, Y_n$ respectively. We then have

$$\lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \varepsilon_n \Phi\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) dx = 0.$$

On the other hand,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_D \operatorname{curl} w^\varepsilon \cdot \varepsilon_n \Phi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx &= \lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon \cdot \varepsilon_n \operatorname{curl} \Phi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon \cdot \operatorname{curl}_{y_n} \Phi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx \\
&= \int_D \int_{\mathbf{Y}} \xi(x, \mathbf{y}) \cdot \operatorname{curl}_{y_n} \Phi(x, \mathbf{y}) d\mathbf{y} dx.
\end{aligned}$$

Thus there is a function $\xi_{n-1}(x, \mathbf{y}_{n-1}) \in L^2(D \times \mathbf{Y}_{n-1})^3$ and a function $\mathbf{w}_n(x, \mathbf{y}_n) \in L^2(D \times \mathbf{Y}_{n-1}, H_\#^1(Y_n)/\mathbb{R})$ such that

$$\xi(x, \mathbf{y}) = \xi_{n-1}(x, \mathbf{y}_{n-1}) + \nabla_{y_n} \mathbf{w}_n(x, \mathbf{y}).$$

Next we choose

$$\phi = \varepsilon_{n-1} \Phi(x, y_1, \dots, y_{n-1})$$

for a function $\Phi \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_{n-1}) \dots))^3$ which is periodic with respect to $y_1, \dots, y_{n-1}$. We then have

$$\begin{aligned}
0 &= \lim_{\varepsilon \rightarrow 0} \int_D \operatorname{curl} w^\varepsilon \cdot \varepsilon_{n-1} \Phi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}} \right) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon \cdot \operatorname{curl}_{y_{n-1}} \Phi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}} \right) dx \\
&= \int_D \int_{\mathbf{Y}} (\xi_{n-1}(x, \mathbf{y}_{n-1}) + \nabla_{y_n} \mathbf{w}_n(x, \mathbf{y})) \cdot \operatorname{curl}_{y_{n-1}} \Phi(x, y_1, \dots, y_{n-1}) d\mathbf{y} dx \\
&= \int_D \int_{\mathbf{Y}_{n-1}} \xi_{n-1}(x, \mathbf{y}_{n-1}) \cdot \operatorname{curl}_{y_{n-1}} \Phi(x, y_1, \dots, y_{n-1}) d\mathbf{y}_{n-1} dx.
\end{aligned}$$

From this, there is a function $\xi_{n-2}(x, \mathbf{y}_{n-2}) \in L^2(D \times \mathbf{Y}_{n-2})^3$ and a function $\mathbf{w}_{n-1}(x, \mathbf{y}_{n-1}) \in L^2(D \times \mathbf{Y}_{n-2}, H_\#^1(Y_{n-1})/\mathbb{R})$ so that

$$\xi_{n-1}(x, \mathbf{y}_{n-1}) = \xi_{n-2}(x, \mathbf{y}_{n-2}) + \nabla_{y_{n-1}} \mathbf{w}_{n-1}(x, \mathbf{y}_{n-1}),$$

so

$$\xi(x, \mathbf{y}) = \xi_{n-2}(x, \mathbf{y}_{n-2}) + \nabla_{y_{n-1}} \mathbf{w}_{n-1}(x, \mathbf{y}_{n-1}) + \nabla_{y_n} \mathbf{w}_n(x, \mathbf{y}).$$

Continue this process, we have that

$$\xi(x, \mathbf{y}) = w_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{w}_i(x, \mathbf{y}_i)$$

where $w_0 \in H$ and $\mathbf{w}_i(x, \mathbf{y}_i) \in L^2(D \times \mathbf{Y}_{i-1}, H_\#^1(Y_i)/\mathbb{R})$. As $\int_Y \xi(x, \mathbf{y}) d\mathbf{y} = w_0(x)$, w_0 is the weak limit of w^ε in H .

Let $\eta(x, \mathbf{y})$ be the $(n+1)$ -scale convergence limit of $\text{curl } w^\varepsilon$ in $L^2(D \times \mathbf{Y})^3$. Let $\Psi(x, y_1, \dots, y_n) \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_n) \dots))$. We have

$$\begin{aligned} \int_D \text{curl } w^\varepsilon \cdot \nabla \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx &= \int_D w^\varepsilon \cdot \text{curl } \nabla \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx \\ &\quad - \int_{\partial D} (w^\varepsilon \times \nu) \cdot \nabla \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) ds = 0. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \varepsilon_n \nabla \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \nabla_{y_n} \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) dx \\ &= \int_D \int_{\mathbf{Y}} \eta(x, \mathbf{y}) \cdot \nabla_{y_n} \Psi(x, y_1, \dots, y_n) d\mathbf{y} dx. \end{aligned}$$

Therefore there is a function $w_n(x, \mathbf{y}_n) \in L^2(D \times \mathbf{Y}_{n-1}, \tilde{H}_\#(\text{curl}, Y_n))$ and a function $\eta_{n-1}(x, \mathbf{y}_{n-1}) \in L^2(D \times \mathbf{Y}_{n-1})^3$ such that

$$\eta(x, \mathbf{y}) = \eta_{n-1}(x, \mathbf{y}_{n-1}) + \text{curl}_{y_n} w_n(x, \mathbf{y}).$$

Let $\Psi(x, y_1, \dots, y_{n-1}) \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_{n-1}) \dots))$. We have

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \varepsilon_{n-1} \nabla \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}} \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \nabla_{y_{n-1}} \Psi \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}} \right) dx \\ &= \int_D \int_{\mathbf{Y}} (\eta_{n-1}(x, \mathbf{y}_{n-1}) + \text{curl}_{y_n} w_n(x, \mathbf{y})) \cdot \nabla_{y_{n-1}} \Psi(x, y_1, \dots, y_{n-1}) d\mathbf{y} dx \\ &= \int_D \int_{\mathbf{Y}_{n-1}} \eta_{n-1}(x, \mathbf{y}_{n-1}) \cdot \nabla_{y_{n-1}} \Psi(x, y_1, \dots, y_{n-1}) d\mathbf{y}_{n-1} dx. \end{aligned}$$

Therefore there is a function $w_{n-1}(x, \mathbf{y}_{n-1}) \in L^2(D \times \mathbf{Y}_{n-2}, \tilde{H}_\#(\text{curl}, Y_{n-1}))$ and a function $\eta_{n-2}(x, \mathbf{y}_{n-2}) \in L^2(D \times \mathbf{Y}_{n-2})^3$ so that

$$\eta_{n-1}(x, \mathbf{y}_{n-1}) = \eta_{n-2}(x, \mathbf{y}_{n-2}) + \text{curl}_{y_{n-1}} w_{n-1}(x, \mathbf{y}_{n-1})$$

so

$$\eta(x, \mathbf{y}) = \eta_{n-2}(x, \mathbf{y}_{n-2}) + \text{curl}_{y_{n-1}} w_{n-1}(x, \mathbf{y}_{n-1}) + \text{curl}_{y_n} w_n(x, \mathbf{y}).$$

Continuing, we find that there is a function $\eta_0(x) \in H$ and functions $w_i(x, \mathbf{y}_i) \in L^2(D \times \mathbf{Y}_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ so that

$$\eta(x, \mathbf{y}) = \eta_0(x) + \sum_{i=1}^n \text{curl}_{y_i} w_i(x, \mathbf{y}_i).$$

As for all $\phi(x) \in C_0^\infty(D)^3$

$$\lim_{\varepsilon \rightarrow 0} \int_D \text{curl} w^\varepsilon(x) \cdot \phi(x) dx = \int_D \eta_0(x) \cdot \phi(x) dx,$$

η_0 is the weak limit of $\text{curl} w^\varepsilon$ in H . Thus $\eta_0 = \text{curl} w_0$. We then get the conclusion. $\square$

2.1.3 Multiscale homogenized Maxwell problem

From (2.6) and Proposition 2.1.3, we can extract a subsequence (not renumbered), a function $u_0 \in W$, n functions $\mathbf{u}_i \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$ and n functions $u_i \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ such that

$$u^\varepsilon \xrightarrow{(n+1)\text{-scale}} u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i, \quad (2.8)$$

and

$$\text{curl} u^\varepsilon \xrightarrow{(n+1)\text{-scale}} \text{curl} u_0 + \sum_{i=1}^n \text{curl}_{y_i} u_i. \quad (2.9)$$

For $i = 1, \dots, n$, let $W_i = L^2(D \times Y_1 \times \cdots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ and $V_i = L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$. We define the space $\mathbf{V}$

$$\mathbf{V} = W \times W_1 \times \cdots \times W_n \times V_1 \times \cdots \times V_n.$$

The space $\mathbf{V}$ is equipped with the norm

$$|||\mathbf{v}||| = \|v_0\|_{H(\text{curl}, D)} + \sum_{i=1}^n \|v_i\|_{L^2(D \times \mathbf{Y}_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))} + \sum_{i=1}^n \|\mathbf{v}_i\|_{L^2(D \times \mathbf{Y}_{i-1}, H_\#^1(Y_i)/\mathbb{R})}$$

for $\mathbf{v} = (v_0, \{v_i\}, \{\mathbf{v}_i\}) \in \mathbf{V}$. We then have the following result.

Proposition 2.1.4. *We denote by $\mathbf{u} = (u_0, \{u_i\}, \{\mathbf{u}_i\}) \in \mathbf{V}$. Then $\mathbf{u}$ satisfies*

$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) &:= \int_D \int_{\mathbf{Y}} \left[a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \left(\operatorname{curl} v_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i \right) \right. \\ &\quad \left. + b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(v_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i \right) \right] d\mathbf{y} dx \\ &= \int_D f(x) \cdot v_0(x) dx \end{aligned} \quad (2.10)$$

for all $\mathbf{v} = (v_0, \{v_i\}, \{\mathbf{v}_i\}) \in \mathbf{V}$.

Proof Let $v_0 \in C_0^\infty(D)^3$, $v_i \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_i) \dots))^3$ and $\mathbf{v}_i \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_i) \dots))$ for $i = 1, \dots, n$. Let the test function v in (2.5) be

$$v(x) = v_0(x) + \sum_{i=1}^n \varepsilon_i \left(v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \nabla \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right).$$

We have

$$\begin{aligned} &\int_D \left[a^\varepsilon(x) \operatorname{curl} u^\varepsilon(x) \cdot \left(\operatorname{curl} v_0(x) + \sum_{i=1}^n \varepsilon_i \operatorname{curl}_x v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \right. \\ &\quad \left. + \sum_{i=1}^n \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \operatorname{curl}_{y_j} v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \varepsilon_i \operatorname{curl} \nabla \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \\ &\quad + b^\varepsilon(x) u^\varepsilon(x) \cdot \left(v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right. \\ &\quad \left. + \sum_{i=1}^n \varepsilon_i \nabla_x \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j} \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \Big] dx \\ &= \int_D f(x) \cdot \left(v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right. \\ &\quad \left. + \sum_{i=1}^n \varepsilon_i \nabla_x \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j} \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) dx. \end{aligned}$$

Using multiscale convergence and the scale separation (2.2), letting ε go to 0, we have

$$\begin{aligned}
& \int_D \int_{\mathbf{Y}} \left[a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \left(\operatorname{curl} v_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i \right) \right. \\
& \quad \left. + b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(v_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i \right) \right] d\mathbf{y} dx \\
&= \int_D f(x) \cdot v_0(x) dx + \int_D \int_{\mathbf{Y}} f(x) \cdot \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, y_1, \dots, y_i) d\mathbf{y} dx \\
&= \int_D f(x) \cdot v_0(x) dx.
\end{aligned}$$

Using a density argument, we have (2.10). $\square$

Proposition 2.1.5. *The bilinear form $B : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ is coercive and bounded, i.e., there are positive constants c^* and c_* so that*

$$B(\mathbf{u}, \mathbf{v}) \leq c^* |||\mathbf{u}||| |||\mathbf{v}||| \quad \text{and} \quad c_* |||\mathbf{u}||| |||\mathbf{u}||| \leq B(\mathbf{u}, \mathbf{u}) \quad (2.11)$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$. Problem (2.10) thus has a unique solution. The convergence (2.8) and (2.9) hold for the whole sequence $\{u^\varepsilon\}_\varepsilon$.

Proof It is easy to see that there is a positive constant c^* such that

$$B(\mathbf{u}, \mathbf{v}) \leq c^* |||\mathbf{u}||| |||\mathbf{v}|||.$$

Now we show that B is coercive. We have from (2.1)

$$\begin{aligned}
B(\mathbf{u}, \mathbf{u}) &\geq \alpha \int_D \int_{\mathbf{Y}} \left(\left| \operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right|^2 + \left| u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right|^2 \right) d\mathbf{y} dx \\
&\geq \alpha \int_D \int_{\mathbf{Y}} \left(|\operatorname{curl} u_0|^2 + \sum_{i=1}^n |\operatorname{curl}_{y_i} u_i|^2 + |u_0|^2 + \sum_{i=1}^n |\nabla_{y_i} \mathbf{u}_i|^2 \right) d\mathbf{y} dx \\
&\geq c_* |||\mathbf{u}|||^2.
\end{aligned}$$

We then get the conclusion from Lax-Milgram lemma. $\square$

2.2 Correctors and homogenization errors

In the two-scale case, we derive the homogenization rate explicitly in terms of ε . We consider the general case where the solution u_0 of the homogenized problem

belongs to the space $H^s(\text{curl}, D)$ for $0 < s \leq 1$, thus generalizing the standard homogenization rate of convergence $\varepsilon^{1/2}$ for elliptic problems (see, e.g, [15, 62]). This is a new result in homogenization theory. We present it for two-scale Maxwell equations but the procedure works verbatim for two-scale elliptic and elasticity problems where the solutions of the homogenized problems belong to $H^{1+s}(D)$. We present this section for the case $d = 3$; the case $d = 2$ is similar.

2.2.1 Two-scale problems

For the two-scale case, we denote the function $a(x, \mathbf{y})$ by $a(x, y)$. The two-scale homogenized equation becomes

$$\begin{aligned} \int_D \int_Y a(x, y) (\text{curl } u_0 + \text{curl}_y u_1) \cdot (\text{curl } v_0 + \text{curl}_y v_1) dy dx \\ + \int_D \int_Y b(x, y) (u_0 + \nabla_y \mathbf{u}_1) \cdot (v_0 + \nabla_y \mathbf{v}_1) dy dx = \int_D f(x) \cdot v_0(x) dx. \end{aligned}$$

We first let $v_0 = 0$, $v_1 = 0$, and deduce that

$$\int_D \int_Y b(x, y) (u_0 + \nabla_y \mathbf{u}_1) \cdot \nabla_y \mathbf{v}_1 dy dx = 0.$$

For each $r = 1, 2, 3$, let $\omega^r(x, \cdot) \in L^2(D, H_{\#}^1(Y)/\mathbb{R})$ be the solution of the problem

$$\int_D \int_Y b(x, y) (e^r + \nabla_y \omega^r) \cdot \nabla_y \psi dy dx = 0 \quad \forall \psi \in L^2(D, H_{\#}^1(Y)/\mathbb{R}) \quad (2.12)$$

where e^r is the r th unit vector in $\mathbb{R}^3$. This is the standard cell problem in elliptic homogenization. From this we have

$$\mathbf{u}_1(x, y) = \omega^r(x, y) u_{0r}(x).$$

Therefore

$$\int_D \int_Y b(x, y) (u_0 + \nabla_y \mathbf{u}_1) \cdot v_0 dy dx = \int_D b^0(x) u_0(x) \cdot v_0(x) dx,$$

where the positive definite matrix $b^0(x)$ is defined as

$$b_{ij}^0(x) = \int_Y b(x, y) (e^j + \nabla_y \omega^j(x, y)) \cdot (e^i + \nabla_y \omega^i(x, y)) dy$$

which is the usual homogenized coefficient for elliptic problems with the two-scale coefficient matrix b^ε .

Let $v_0 = 0$ and $\mathbf{v}_1 = 0$. We have

$$\int_D \int_Y a(x, y) (\operatorname{curl} u_0 + \operatorname{curl}_y u_1) \cdot \operatorname{curl}_y v_1 dy dx = 0$$

for all $v_1 \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$. For each $r = 1, 2, 3$, let $\chi^r \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$ be the solution of

$$\int_D \int_Y a(x, y) (e^r + \operatorname{curl}_y \chi^r) \cdot \operatorname{curl}_y v dy dx = 0 \quad (2.13)$$

for all $v \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$. We have

$$u_1 = (\operatorname{curl} u_0(x))_r \chi^r(x, y). \quad (2.14)$$

The homogenized coefficient a^0 is determined by

$$a_{ij}^0(x) = \int_Y a(x, y)_{ip} (e_p^j + (\operatorname{curl}_y \chi^j)_p) dy = \int_Y a(x, y) (e^j + \operatorname{curl}_y \chi^j) \cdot (e^i + \operatorname{curl}_y \chi^i) dy. \quad (2.15)$$

We have

$$\int_D \int_Y a(x, y) (\operatorname{curl} u_0 + \operatorname{curl}_y u_1) \cdot \operatorname{curl} v_0 dx dy = \int_D a^0(x) \operatorname{curl} u_0(x) \cdot \operatorname{curl} v_0(x) dx.$$

The homogenized problem is

$$\int_D [a^0(x) \operatorname{curl} u_0(x) \cdot \operatorname{curl} v_0(x) + b^0(x) u_0(x) \cdot v_0(x)] dx = \int_D f(x) \cdot v_0(x) dx \quad (2.16)$$

for all $v_0 \in W$.

Following the procedure for deriving homogenization error ([15, 62]), we have the following homogenization error estimate:

Theorem 2.2.1. *Assume that $u_0 \in H^1(\operatorname{curl}; D)$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\operatorname{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\omega^r \in C^1(\bar{D}, C^1(\bar{Y}))$ ¹ for all $r = 1, 2, 3$, then*

$$\left\| u^\varepsilon - \left[u_0 + \nabla_y \mathbf{u}_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_H \leq c\varepsilon^{1/2}$$

and

$$\left\| \operatorname{curl} u^\varepsilon - \left[\operatorname{curl} u_0 + \operatorname{curl}_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_H \leq c\varepsilon^{1/2}.$$

¹Indeed for Theorems 2.2.1 and 2.2.2, we only need weaker regularity conditions $\chi^r \in W^{1,\infty}(D, L^\infty(Y))$ and $\operatorname{curl}_y \chi^r \in W^{1,\infty}(D, L^\infty(Y))$, and $\omega^r \in W^{1,\infty}(D, W^{1,\infty}(Y))$.

Proof We consider the function

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r + \varepsilon \nabla \left(\omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \right).$$

We have

$$\begin{aligned} \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon &= \operatorname{curl} a \left(x, \frac{x}{\varepsilon} \right) \left[\operatorname{curl} u_0(x) + \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \right. \\ &\quad \left. + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r + \varepsilon \nabla (\operatorname{curl} u_0(x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right] \\ &\quad + b \left(x, \frac{x}{\varepsilon} \right) \left[u_0(x) + \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \right. \\ &\quad \left. + \varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) + \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \nabla u_{0r}(x) \right] \\ &= \operatorname{curl} a^0(x) \operatorname{curl} u_0(x) + b^0(x) u_0(x) + \operatorname{curl} G_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \\ &\quad + g_r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) + \varepsilon \operatorname{curl} I(x) + \varepsilon J(x) \end{aligned}$$

where the vector functions $G_r(x, y)$ and $g_r(x, y)$ are defined by

$$(G_r)_i(x, y) = a_{ir}(x, y) + a_{ij}(x, y) (\operatorname{curl}_y \chi^r(x, y))_j - a_{ir}^0(x), \quad (2.17)$$

$$(g_r)_i(x, y) = b_{ir}(x, y) + b_{ij}(x, y) \frac{\partial \omega^r}{\partial y_j}(x, y) - b_{ir}^0(x); \quad (2.18)$$

and

$$I(x) = a \left(x, \frac{x}{\varepsilon} \right) \left[\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r + \nabla (\operatorname{curl} u_0(x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right]$$

and

$$J(x) = b \left(x, \frac{x}{\varepsilon} \right) \left[\chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r + \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) + \omega^r \left(x, \frac{x}{\varepsilon} \right) \nabla u_{0r}(x) \right].$$

From (2.13), we have that $\operatorname{curl}_y G_r(x, y) = 0$. Further from (2.15), $\int_Y G_r(x, y) dy = 0$. From these we deduce that there are functions $\tilde{G}_r$ such that

$$G_r(x, y) = \nabla_y \tilde{G}_r(x, y). \quad (2.19)$$

From (2.12), we have that $\operatorname{div}_y g_r(x, y) = 0$ and from (3.22), $\int_Y g_r(x, y) dy = 0$. Therefore there are functions $\tilde{g}_r$ such that

$$g_r(x, y) = \operatorname{curl}_y \tilde{g}_r(x, y). \quad (2.20)$$

From (2.19) and (2.20),

$$G_r \left(x, \frac{x}{\varepsilon} \right) = \varepsilon \nabla \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right),$$

and

$$g_r \left(x, \frac{x}{\varepsilon} \right) = \varepsilon \operatorname{curl} \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right).$$

For all $\phi \in \mathcal{D}(D)^3$

$$\begin{aligned} & \langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl} (a^0 \operatorname{curl} u_0) - b^0 u_0, \phi \rangle_H \\ &= \int_D G_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \cdot \operatorname{curl} \phi(x) dx + \int_D g_r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \cdot \phi(x) dx \\ & \quad + \varepsilon \int_D I(x) \cdot \operatorname{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x) dx \\ &= -\varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}((\operatorname{curl} u_0(x))_r \operatorname{curl} \phi(x)) dx \\ & \quad - \varepsilon \int_D \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \cdot \operatorname{curl} \phi(x) dx \\ & \quad + \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} (u_{0r}(x) \phi(x)) dx \\ & \quad - \varepsilon \int_D \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \cdot \operatorname{curl} \phi(x) dx \\ & \quad + \varepsilon \int_D I(x) \cdot \operatorname{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x) dx. \end{aligned}$$

We note that

$$\int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}((\operatorname{curl} u_0(x))_r \operatorname{curl} \phi(x)) dx = \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \nabla(\operatorname{curl} u_0(x))_r \cdot \operatorname{curl} \phi(x) dx$$

and

$$\int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} (u_{0r}(x) \phi(x)) dx = \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) (u_{0r}(x) \operatorname{curl} \phi(x) + \nabla u_{0r}(x) \times \phi(x)) dx.$$

As $\nabla_y \tilde{G}_r(x, \cdot) = G_r(x, \cdot) \in H^1(Y)^3$ so $\Delta_y G_r(x, \cdot) \in L^2(Y)$. Thus $\tilde{G}_r(x, \cdot) \in H^2(Y)^3$ which implies $\tilde{G}_r(x, \cdot) \in C(\bar{Y})^3$. As $G_r(x, y) \in C^1(\bar{D}, H_\#^1(Y))^3$, we deduce that $\tilde{G}_r(x, y) \in C^1(\bar{D}, H^2(\bar{Y}))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$. The construction of function $\tilde{g}_r$ in Jikov et al. [62] implies that $g_r \in C^1(\bar{D}, C(\bar{Y}))^3$ (see also Hoang and Schwab [56]). Thus

$$\left| \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}((\operatorname{curl} u_0(x))_r \operatorname{curl} \phi(x)) dx \right| \leq c \|\operatorname{curl} \phi\|_H,$$

and

$$\left| \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} (u_{0r}(x) \phi(x)) dx \right| \leq c(\|\operatorname{curl} \phi\|_H + \|\phi\|_H).$$

As $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$ and $\omega^r \in C^1(\bar{D}, C(\bar{Y}))$, $\|I\|_H$ and $\|J\|_H$ are uniformly bounded with respect to ε . From these we conclude that

$$|\langle \text{curl}(a^\varepsilon \text{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \text{curl}(a^0 \text{curl} u_0) - b^0 u_0, \phi \rangle_H| \leq c\varepsilon(\|\text{curl} \phi\|_H + \|\phi\|_H).$$

Using a density argument, we have that this holds for all $\phi \in H_0(\text{curl}, D)$, thus

$$\|\text{curl}(a^\varepsilon \text{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \text{curl}(a^0 \text{curl} u_0) - b^0 u_0\|_{W'} \leq c\varepsilon$$

so

$$\|\text{curl}(a^\varepsilon \text{curl}(u_1^\varepsilon - u^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - u^\varepsilon)\|_{W'} \leq c\varepsilon.$$

As u_1^ε is not in $H_0(\text{curl}, D)$, to get a bound for the $H(\text{curl}, D)$ norm of $u^\varepsilon - u_1^\varepsilon$, we employ a boundary layer function τ^ε which makes the function w_1^ε defined below belong to $H_0(\text{curl}, D)$. Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x \in D} \varepsilon |\nabla \tau^\varepsilon(x)| < c$ where c is independent of ε . Let

$$w_1^\varepsilon(x) = u_0(x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) (\text{curl} u_0(x))_r + \varepsilon \nabla \left[\tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \right].$$

The function $w_1^\varepsilon(x)$ belongs to W . We note that

$$u_1^\varepsilon(x) - w_1^\varepsilon(x) = \varepsilon(1 - \tau^\varepsilon(x)) \chi^r \left(x, \frac{x}{\varepsilon} \right) (\text{curl} u_0(x))_r + \varepsilon \nabla \left[(1 - \tau^\varepsilon(x)) \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \right].$$

From this,

$$\begin{aligned} \text{curl}(u_1^\varepsilon(x) - w_1^\varepsilon(x)) &= \varepsilon \text{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\text{curl} u_0(x))_r (1 - \tau^\varepsilon(x)) \\ &\quad + \text{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\text{curl} u_0(x))_r (1 - \tau^\varepsilon(x)) \\ &\quad - \varepsilon (\text{curl} u_0(x))_r \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \nabla (\text{curl} u_0(x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right). \end{aligned}$$

Let D^ε be the ε neighbourhood of the boundary ∂D . We note the result that

Figure 2.2: D^ε neighborhood of ∂D and τ^ε function

$\|\phi\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2 \leq c\varepsilon \|\phi\|_{H^1(D)}^2$,
(see Hoang and Schwab [56]). We therefore deduce that

$$\|(\operatorname{curl} u_0(x))_r\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{1/2}.$$

From these we have

$$\|\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)\|_H \leq c\varepsilon^{1/2}.$$

On the other hand, we have

$$\begin{aligned} u_1^\varepsilon(x) - w_1^\varepsilon(x) = & \varepsilon(1 - \tau^\varepsilon(x))\chi^r\left(x, \frac{x}{\varepsilon}\right)(\operatorname{curl} u_0(x))_r - \varepsilon\nabla\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right)u_{0r}(x) \\ & + \varepsilon(1 - \tau^\varepsilon(x))\nabla_x\omega^r\left(x, \frac{x}{\varepsilon}\right)u_{0r}(x) + (1 - \tau^\varepsilon(x))\nabla_y\omega^r\left(x, \frac{x}{\varepsilon}\right)u_{0r}(x) \\ & + \varepsilon(1 - \tau^\varepsilon(x))\omega^r\left(x, \frac{x}{\varepsilon}\right)\nabla u_{0r}(x). \end{aligned}$$

Using the fact that $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\omega^r \in C^1(\bar{D}, C^1(\bar{Y}))$ and $\|u_{0r}\|_{L^2(D^\varepsilon)} \leq \varepsilon^{1/2}$, we deduce that $\|u_1^\varepsilon - w_1^\varepsilon\|_H \leq c\varepsilon^{1/2}$. Therefore

$$\|\operatorname{curl} a^\varepsilon(\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - w_1^\varepsilon)\|_{W'} \leq c\varepsilon^{1/2}.$$

Thus

$$\|\operatorname{curl} a^\varepsilon(\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u^\varepsilon - w_1^\varepsilon)\|_{W'} \leq c\varepsilon^{1/2}.$$

Since $u^\varepsilon - w_1^\varepsilon \in W$, using the coercive property of a^ε and b^ε we have

$$\begin{aligned} & \alpha(\|\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)\|_H^2 + \|(u^\varepsilon - w_1^\varepsilon)\|_H^2) \\ & \leq \int_D a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon) \cdot \operatorname{curl}(u^\varepsilon - w_1^\varepsilon) + b^\varepsilon(u^\varepsilon - w_1^\varepsilon) \cdot (u^\varepsilon - w_1^\varepsilon) dx \\ & \leq c\varepsilon^{1/2} \|u^\varepsilon - w_1^\varepsilon\|_{H(\operatorname{curl}, D)}. \end{aligned}$$

Thus $\|u^\varepsilon - w_1^\varepsilon\|_{H(\operatorname{curl}, D)} \leq c\varepsilon^{1/2}$. From these we have $\|u^\varepsilon - u_1^\varepsilon\|_{H(\operatorname{curl}, D)} \leq c\varepsilon^{1/2}$. We then get the conclusion. $\square$

The proof above essentially relies on the fact that we can consider the curl operator of the function u_1^ε . This is possible as $\operatorname{curl} u_0$ belongs to $H^1(D)^3$. In the case where $\operatorname{curl} u_0$ only belongs to a weaker regularity space $H^s(D)^3$ for $0 < s < 1$, this is no longer possible. However, in the error estimate that we wish to prove, we only need $\operatorname{curl} u_0$ in $L^2(D)^3$. Therefore, in the proof below for the case where $u_0 \in H^s(\operatorname{curl}, D)$ with $0 < s < 1$, we approximate $\operatorname{curl} u_0$ locally by its mean value inside a small domain, and use a partition of unity to approximate $\operatorname{curl} u_0$ in the $L^2(D)$ norm. By doing this, we get a smooth function which approximates $\operatorname{curl} u_0$ in $L^2(D)^3$ for which the curl operator is well defined.

For $u_0 \in H^s(\operatorname{curl}, D)$ when $0 < s < 1$, we have the following homogenization error estimate.

Theorem 2.2.2. Assume that $u_0 \in H^s(\text{curl}, D)$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\text{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, and $\omega^r \in C^1(\bar{D}, C^1(\bar{Y}))$ for all $r = 1, 2, 3$, then

$$\left\| u^\varepsilon - \left[u_0 + \nabla_y \mathbf{u}_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_H \leq c\varepsilon^{s/(1+s)}$$

and

$$\left\| \text{curl } u^\varepsilon - \left[\text{curl } u_0 + \text{curl}_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_H \leq c\varepsilon^{s/(1+s)}.$$

Proof We consider a set of M open cubes Q_i ($i = 1, \dots, M$) of size ε^{s_1} for $s_1 > 0$ to be chosen later such that $D \subset \bigcup_{i=1}^M Q_i$ and $Q_i \cap D \neq \emptyset$. Each cube Q_i intersects with only a finite number, which does not depend on ε , of other cubes. We consider a partition of unity that consists of M functions ρ_i such that ρ_i has support in Q_i , $\sum_{i=1}^M \rho_i(x) = 1$ for all $x \in D$ and $|\nabla \rho_i(x)| \leq c\varepsilon^{-s_1}$ for all x (indeed such a set of cubes Q_i and a partition of unity can be constructed from a fixed set of cubes of size $O(1)$ by rescaling). For $r = 1, 2, 3$ and $i = 1, \dots, M$, we denote by

$$U_i^r = \frac{1}{|Q_i|} \int_{Q_i} (\text{curl } u_0(x))_r dx \text{ and } V_i^r = \frac{1}{|Q_i|} \int_{Q_i} u_{0r}(x) dx.$$

As $u_0 \in H^s(D)^3$ and $\text{curl } u_0 \in H^s(D)^3$, for the Lipschitz domain D , we can extend each of them, separately, continuously outside D and understand u_0 and $\text{curl } u_0$ as these extensions (see Wloka [100] Theorem 5.6). Let U_i and V_i denote the vector (U_i^1, U_i^2, U_i^3) and (V_i^1, V_i^2, V_i^3) respectively. Let B be the unit cube in $\mathbb{R}^3$. From Poincaré inequality, we have

$$\int_B \left| \phi - \int_B \phi(x) dx \right|^2 dx \leq c \int_B |\nabla \phi(x)|^2 dx \quad \forall \phi \in H^1(B).$$

By translation and scaling, we deduce that

$$\int_{Q_i} \left| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right|^2 dx \leq c\varepsilon^{2s_1} \int_{Q_i} |\nabla \phi(x)|^2 dx \quad \forall \phi \in H^1(Q_i)$$

i.e.,

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c\varepsilon^{s_1} \|\phi\|_{H^1(Q_i)}.$$

Together with

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c \|\phi\|_{L^2(Q_i)}$$

we deduce from interpolation that

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c\varepsilon^{s_1 s} \|\phi\|_{H^s(Q_i)} \quad \forall \phi \in H^s(Q_i).$$

Thus

$$\int_{Q_i} |(\operatorname{curl} u_0(x))_r - U_i^r|^2 dx \leq c\varepsilon^{2s_1 s} \|(\operatorname{curl} u_0)_r\|_{H^s(Q_i)}^2. \quad (2.21)$$

Let

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + \varepsilon \nabla \left[\omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) \right].$$

We have

$$\begin{aligned} & \operatorname{curl} (a^\varepsilon(x) \operatorname{curl} u_1^\varepsilon(x)) + b^\varepsilon(x) u_1^\varepsilon(x) \\ &= \operatorname{curl} a \left(x, \frac{x}{\varepsilon} \right) \left[\operatorname{curl} u_0(x) + \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) \right. \\ & \quad \left. + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + \varepsilon (U_j^r \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right] \\ & \quad + b \left(x, \frac{x}{\varepsilon} \right) \left[u_0(x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + \varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) \right. \\ & \quad \left. + \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) + \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \nabla \rho_j(x) \right] \\ &= \operatorname{curl} (a^0(x) \operatorname{curl} u_0(x)) + b^0(x) u_0(x) + \operatorname{curl} \left[G_r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) \right] \\ & \quad + g_r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) + \varepsilon \operatorname{curl} I(x) + \varepsilon J(x) \\ & \quad + \operatorname{curl} [(a^\varepsilon(x) - a^0(x))(\operatorname{curl} u_0(x) - U_j \rho_j(x))] \\ & \quad + (b^\varepsilon(x) - b^0(x))(u_0(x) - V_j \rho_j(x)), \end{aligned}$$

where the vector functions $G_r(x, y)$ and $g_r(x, y)$ are defined as in (2.17) and (2.18) respectively, and

$$I(x) = a \left(x, \frac{x}{\varepsilon} \right) \left[\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + (U_j^r \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right],$$

$$J(x) = b \left(x, \frac{x}{\varepsilon} \right) \left[\chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) + \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \nabla \rho_j(x) \right].$$

Therefore for $\phi \in H_0(\operatorname{curl}, D)$

$$\begin{aligned} & \langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl} (a^0 \operatorname{curl} u_0) - b^0 u_0, \phi \rangle_H \\ &= \int_D U_j^r \rho_j(x) G_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi dx + \int_D V_j^r \rho_j(x) g_r \left(x, \frac{x}{\varepsilon} \right) \cdot \phi(x) dx \\ & \quad + \varepsilon \int_D I(x) \cdot \operatorname{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x) dx \\ & \quad + \int_D (a^\varepsilon(x) - a^0(x))(\operatorname{curl} u_0(x) - U_j \rho_j(x)) \cdot \operatorname{curl} \phi(x) dx \\ & \quad + \int_D (b^\varepsilon(x) - b^0(x))(u_0(x) - V_j \rho_j(x)) \cdot \phi(x) dx. \end{aligned}$$

Using the function $\tilde{G}_r$ as in (2.19), we have

$$\begin{aligned} & \int_D U_j^r \rho_j(x) G_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi(x) dx \\ &= \int_D U_j^r \rho_j(x) \left[\varepsilon \nabla \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \right] \cdot \operatorname{curl} \phi(x) dx \\ &= -\varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}[(U_j^r \rho_j(x)) \operatorname{curl} \phi(x)] dx \\ &\quad - \varepsilon \int_D U_j^r \rho_j(x) \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi(x) dx. \end{aligned}$$

We note that

$$\left| \int_D U_j^r \rho_j(x) \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi(x) dx \right| \leq c \|U_j^r \rho_j\|_{L^2(D)} \|\operatorname{curl} \phi\|_H.$$

From

$$\|U_j^r \rho_j\|_{L^2(D)}^2 = \int_D (U_j^r)^2 \rho_j(x)^2 dx + \sum_{i \neq j} \int_D U_i^r U_j^r \rho_i(x) \rho_j(x) dx,$$

and the fact that the support of each function ρ_i intersects only with the support of a finite number (which does not depend on ε) of other functions ρ_j in the partition of unity, we deduce

$$\begin{aligned} \|U_j^r \rho_j\|_{L^2(D)}^2 &\leq c \sum_{j=1}^M (U_j^r)^2 |Q_j| = c \sum_{j=1}^M \frac{1}{|Q_j|} \left(\int_{Q_j} (\operatorname{curl} u_0(x))_r dx \right)^2 \\ &\leq c \sum_{j=1}^M \int_{Q_j} (\operatorname{curl} u_0(x))_r^2 dx \leq c \int_D (\operatorname{curl} u_0(x))_r^2 dx. \end{aligned}$$

Thus

$$\left| \varepsilon \int_D U_j^r \rho_j(x) \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi(x) dx \right| \leq c\varepsilon \|\operatorname{curl} \phi\|_H.$$

We have further that

$$\begin{aligned} \varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}[(U_j^r \rho_j(x)) \operatorname{curl} \phi(x)] dx &= \varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \nabla \rho_j(x)) \cdot \operatorname{curl} \phi(x) dx \\ &\leq c\varepsilon \|U_j^r \nabla \rho_j\|_H \|\operatorname{curl} \phi\|_H. \end{aligned}$$

As the support of each function ρ_i intersects with the support of a finite number of other functions ρ_j and $\|\nabla \rho_j\|_{L^\infty(D)} \leq c\varepsilon^{-s_1}$, we have

$$\|U_j^r \nabla \rho_j\|_H^2 \leq c \sum_{j=1}^M (U_j^r)^2 |Q_j| \|\nabla \rho_j\|_{L^\infty(D)}^2 \leq c\varepsilon^{-2s_1} \sum_{j=1}^M (U_j^r)^2 |Q_j| \leq c\varepsilon^{-2s_1},$$

so

$$\begin{aligned} \varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}[(U_j^r \rho_j(x)) \operatorname{curl} \phi(x)] dx &\leq c\varepsilon \|U_j^r \nabla \rho_j\|_H \|\operatorname{curl} \phi\|_H \\ &\leq c\varepsilon^{1-s_1} \|\operatorname{curl} \phi\|_H. \end{aligned}$$

We therefore deduce that

$$\left| \int_D U_j^r \rho_j(x) G_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \phi(x) dx \right| \leq c\varepsilon^{1-s_1} \|\operatorname{curl} \phi\|_H.$$

Using the function $\tilde{g}_r$ as in (2.20), we have

$$\int_D V_j^r \rho_j(x) g_r \left(x, \frac{x}{\varepsilon} \right) \cdot \phi(x) dx = \int_D V_j^r \rho_j(x) \left[\varepsilon \operatorname{curl} \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \right] \cdot \phi(x) dx.$$

Arguing similarly as above, we have

$$\left| \varepsilon \int_D V_j^r \rho_j(x) \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \phi(x) dx \right| \leq c\varepsilon \|V_j^r \rho_j\|_H \|\phi\|_H \leq c\varepsilon \|\phi\|_H,$$

and

$$\begin{aligned} \left| \varepsilon \int_D V_j^r \rho_j(x) \operatorname{curl} \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \phi(x) dx \right| &= \left| \varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} [(V_j^r \rho_j(x)) \phi(x)] dx \right| \\ &= \left| \varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot [(V_j^r \rho_j(x)) \operatorname{curl} \phi(x) + (V_j^r \nabla \rho_j(x)) \times \phi(x)] dx \right| \\ &\leq c (\varepsilon \|\operatorname{curl} \phi\|_H + c\varepsilon^{1-s_1} \|\phi\|_H) \left(\sum_{j=1}^M (V_j^r)^2 |Q_j| \right)^{1/2} \\ &\leq c (\varepsilon \|\operatorname{curl} \phi\|_H + c\varepsilon^{1-s_1} \|\phi\|_H). \end{aligned}$$

We note that

$$\|I\|_H \leq c \sup_r [\|U_j^r \rho_j\|_{L^2(D)} + \|U_j^r \nabla \rho_j\|_{L^2(D)}] \leq c\varepsilon^{-s_1},$$

and

$$\|J\|_H \leq c \sup_r [\|U_j^r \rho_j\|_{L^2(D)} + c\|V_j^r \rho_j\|_{L^2(D)} + c\|V_j^r \nabla \rho_j\|_{L^2(D)}] \leq c\varepsilon^{-s_1}.$$

We have further that

$$\langle \operatorname{curl} ((a^\varepsilon - a^0)(\operatorname{curl} u_0 - U_j \rho_j)), \phi \rangle_H \leq c \|\operatorname{curl} u_0 - U_j \rho_j\|_H \|\operatorname{curl} \phi\|_H.$$

From

$$\int_D |(\operatorname{curl} u_0(x))_r - (U_j^r \rho_j(x))|^2 dx = \int_D \left| \sum_{j=1}^M ((\operatorname{curl} u_0(x))_r - U_j^r) \rho_j(x) \right|^2 dx,$$

using the support property of ρ_j , we have from (2.21)

$$\begin{aligned} \int_D |(\operatorname{curl} u_0(x))_r - (U_j^r \rho_j(x))|^2 dx &\leq c \sum_{j=1}^M \int_{Q_j} |(\operatorname{curl} u_0(x))_r - U_j^r|^2 dx \\ &\leq c\varepsilon^{2ss_1} \sum_{j=1}^M \|(\operatorname{curl} u_0)_r\|_{H^s(Q_j)}^2 \\ &= c\varepsilon^{2ss_1} \sum_{j=1}^M \left[\int_{Q_j} (\operatorname{curl} u_0(x))_r^2 dx + \int_{Q_j \times Q_j} \frac{((\operatorname{curl} u_0(x))_r - (\operatorname{curl} u_0(x'))_r)^2}{|x - x'|^{3+2s}} dx dx' \right] \\ &\leq c\varepsilon^{2ss_1} \left[\|(\operatorname{curl} u_0)_r\|_{L^2(D)}^2 + \int_{D \times D} \frac{((\operatorname{curl} u_0(x))_r - (\operatorname{curl} u_0(x'))_r)^2}{|x - x'|^{3+2s}} dx dx' \right] \\ &= c\varepsilon^{2ss_1} \|(\operatorname{curl} u_0)_r\|_{H^s(D)}^2. \end{aligned} \quad (2.22)$$

Thus

$$\langle \operatorname{curl} ((a^\varepsilon - a^0)(\operatorname{curl} u_0 - U_j \rho_j)), \phi \rangle_H \leq c\varepsilon^{ss_1} \|\operatorname{curl} \phi\|_H.$$

Similarly, we have

$$\int_D |u_{0r}(x) - (V_j^r \rho_j(x))|^2 dx \leq c\varepsilon^{2ss_1} \|u_{0r}\|_{H^s(D)}^2. \quad (2.23)$$

Thus

$$\begin{aligned} \left| \int_D (b^\varepsilon(x) - b^0(x)) \left(u_0(x) - \sum_{j=1}^M V_j \rho_j(x) \right) \cdot \phi(x) dx \right| &\leq c \left\| \sum_{j=1}^M (u_0 - V_j) \rho_j \right\|_H \|\phi\|_H \\ &\leq c\varepsilon^{ss_1} \|\phi\|_H. \end{aligned}$$

Therefore

$$|\langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl} (a^0 \operatorname{curl} u_0) - b^0 u_0, \phi \rangle_H| \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}) \|\phi\|_W,$$

i.e.,

$$\|\operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl} (a^0 \operatorname{curl} u_0) - b^0 u_0\|_{W'} \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}).$$

Thus

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl}(a^\varepsilon \operatorname{curl} u^\varepsilon) - b^\varepsilon u^\varepsilon\|_{W'} \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}). \quad (2.24)$$

As u_1^ε does not belong to $H_0(\operatorname{curl}, D)$, we use a boundary layer function. Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x \in D} \varepsilon |\nabla \tau^\varepsilon(x)| < c$ where c is independent of ε . We consider the function

$$w_1^\varepsilon(x) = u_0(x) + \varepsilon \tau^\varepsilon(x) U_j^r \rho_j(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \nabla \left[V_j^r \rho_j(x) \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \right].$$

We then have

$$u_1^\varepsilon(x) - w_1^\varepsilon(x) = \varepsilon(1 - \tau^\varepsilon(x)) U_j^r \rho_j(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \nabla \left[(1 - \tau^\varepsilon(x)) V_j^r \rho_j(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \right]$$

and

$$\begin{aligned} \operatorname{curl}(u_1^\varepsilon(x) - w_1^\varepsilon(x)) &= \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) (1 - \tau^\varepsilon(x)) \\ &\quad + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) (1 - \tau^\varepsilon(x)) \\ &\quad - \varepsilon U_j^r \rho_j(x) \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) U_j^r \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right). \end{aligned}$$

As shown above $\|U_j^r \rho_j\|_{L^2(D)}$ is uniformly bounded, so

$$\left\| \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) (1 - \tau^\varepsilon(x)) \right\|_H \leq c\varepsilon.$$

Let $\tilde{D}^\varepsilon$ be the $3\varepsilon^{s_1}$ neighbourhood of ∂D . We note that $\operatorname{curl} u_0$ is extended continuously into a function in $H^s(\mathbb{R}^3)$ outside D . As shown in Hoang and Schwab [57], for $\phi \in H^1(\tilde{D}^\varepsilon)$

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq c\varepsilon^{s_1/2} \|\phi\|_{H^1(\tilde{D}^\varepsilon)}.$$

From this and

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq \|\phi\|_{L^2(\tilde{D}^\varepsilon)},$$

using interpolation we get

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq c\varepsilon^{ss_1/2} \|\phi\|_{H^s(\tilde{D}^\varepsilon)} \leq c\varepsilon^{ss_1/2} \|\phi\|_{H^s(D)},$$

for all $\phi \in H^s(D)$ extended continuously outside D . We then have

$$\begin{aligned} \|U_j^r \rho_j\|_{L^2(D^\varepsilon)}^2 &\leq c \sum_{j=1}^M \int_{Q_j \cap D^\varepsilon} (U_j^r)^2 \rho_j^2(x) dx \\ &\leq c \sum_{j=1}^M |Q_j \cap D^\varepsilon| \frac{1}{|Q_j|^2} \left(\int_{Q_j} (\operatorname{curl} u_0(x))_r dx \right)^2 \\ &\leq c \sum_{Q_j \cap D^\varepsilon \neq \emptyset} \frac{|Q_j \cap D^\varepsilon|}{|Q_j|} \int_{Q_j} (\operatorname{curl} u_0(x))_r^2 dx. \end{aligned}$$

As D^ε is the ε neighbourhood of ∂D , ∂D is Lipschitz, and Q_j has size ε^{s_1} , $|Q_j \cap D^\varepsilon| \leq c\varepsilon^{1+2s_1}$ so $|Q_j \cap D^\varepsilon|/|Q_j| \leq c\varepsilon^{1-s_1}$. When $Q_j \cap D^\varepsilon \neq \emptyset$, $Q_j \subset \tilde{D}^\varepsilon$. Thus

$$\|U_j^r \rho_j\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^{1-s_1} \|(\operatorname{curl} u_0)_r\|_{L^2(\tilde{D}^\varepsilon)}^2 \leq c\varepsilon^{1-s_1+ss_1} \|\operatorname{curl} u_0\|_{H^s(D)^3}^2.$$

Therefore

$$\left\| \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) (1 - \tau^\varepsilon(x)) \right\|_H \leq c\varepsilon^{(1-s_1+ss_1)/2}$$

and

$$\left\| \varepsilon (U_j^r \rho_j(x)) \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right\|_H \leq c\varepsilon^{(1-s_1+ss_1)/2}.$$

Similarly we have

$$\begin{aligned} \|U_j^r \nabla \rho_j\|_{L^2(D^\varepsilon)^3}^2 &\leq c\varepsilon^{-2s_1} \sum_{Q_j \cap D^\varepsilon \neq \emptyset} \frac{|Q_j \cap D^\varepsilon|}{|Q_j|} \int_{Q_j} (\operatorname{curl} u_0(x))_r^2 dx \\ &\leq c\varepsilon^{-2s_1+1-s_1} \|\operatorname{curl} u_0\|_{L^2(\tilde{D}^\varepsilon)}^2 \\ &\leq c\varepsilon^{1-3s_1+ss_1} \|\operatorname{curl} u_0\|_{H^s(D)^3}^2. \end{aligned}$$

Thus

$$\left\| \varepsilon (1 - \tau^\varepsilon(x)) (U_j^r \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right\|_H \leq c\varepsilon^{(1-s_1)+(1-s_1+ss_1)/2}.$$

Therefore

$$\|\operatorname{curl} (u_1^\varepsilon - w_1^\varepsilon)\|_H \leq c \left(\varepsilon^{(1-s_1+ss_1)/2} + \varepsilon^{(1-s_1)+(1-s_1+ss_1)/2} \right). \quad (2.25)$$

We further have that

$$\begin{aligned} \varepsilon \nabla \left[(1 - \tau^\varepsilon(x)) \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r \rho_j(x)) \right] &= -\varepsilon \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r \rho_j(x)) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r \rho_j(x)) \\ &\quad + (1 - \tau^\varepsilon(x)) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r \rho_j(x)) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \omega^r \left(x, \frac{x}{\varepsilon} \right) (\nabla_j V_j^r \rho_j(x)). \end{aligned}$$

Arguing as above, we deduce that

$$\|V_j^r \rho_j\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{(1-s_1+ss_1)/2}, \quad \|V_j^r \nabla \rho_j\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{(1-s_1+ss_1)/2-s_1}.$$

Therefore

$$\left\| \varepsilon \nabla \left[(1 - \tau^\varepsilon(x)) \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r \rho_j(x)) \right] \right\|_H \leq c \left(\varepsilon^{(1-s_1+ss_1)/2} + \varepsilon^{1-s_1+(1-s_1+ss_1)/2} \right).$$

Thus

$$\|u_1^\varepsilon - w_1^\varepsilon\|_H \leq c \left(\varepsilon^{(1-s_1+ss_1)/2} + \varepsilon^{(1-s_1)+(1-s_1+ss_1)/2} \right).$$

Choosing $s_1 = 1/(s+1)$ we have

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - w_1^\varepsilon)\|_{W'} \leq c\varepsilon^{s/(s+1)}.$$

This together with (2.24) gives

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u^\varepsilon - w_1^\varepsilon)\|_{W'} \leq c\varepsilon^{s/(s+1)}.$$

Thus

$$\|u^\varepsilon - w_1^\varepsilon\|_W \leq c\varepsilon^{s/(s+1)}.$$

From this and (2.25) and (2.25) we get

$$\|u^\varepsilon - u_1^\varepsilon\|_W \leq c\varepsilon^{s/(s+1)}. \quad (2.26)$$

We note that

$$\begin{aligned} \operatorname{curl} u_1^\varepsilon(x) &= \operatorname{curl} u_0(x) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) \\ &\quad + \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) + \varepsilon (U_j^r \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right). \end{aligned}$$

From

$$\left\| \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) \right\|_H \leq c\varepsilon,$$

and

$$\left\| \varepsilon (U_j^r \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right\|_H \leq c\varepsilon \varepsilon^{-s_1} = c\varepsilon^{s/(1+s)},$$

we deduce that

$$\left\| \operatorname{curl} u_1^\varepsilon(x) - \operatorname{curl} u_0(x) - \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (U_j^r \rho_j(x)) \right\|_H \leq c\varepsilon^{s/(s+1)}.$$

From (2.22)

$$\|\operatorname{curl} u_0 - (U_j \rho_j)\|_H \leq c\varepsilon^{s_1 s} = c\varepsilon^{s/(s+1)},$$

we get

$$\left\| \operatorname{curl} u_1^\varepsilon(x) - \left[\operatorname{curl} u_0(x) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \right] \right\|_H \leq c\varepsilon^{s/(s+1)}.$$

This together with (2.26) implies

$$\left\| \operatorname{curl} u^\varepsilon(x) - \left[\operatorname{curl} u_0(x) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \right] \right\|_H \leq c\varepsilon^{s/(s+1)}.$$

We note that

$$\begin{aligned} u_1^\varepsilon(x) = u_0(x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j(x) + \varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) \\ + \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) + \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \nabla \rho_j(x). \end{aligned}$$

From

$$\left\| \varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) \right\|_H \leq c\varepsilon,$$

and

$$\left\| \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \nabla \rho_j(x) \right\|_H \leq c\varepsilon \varepsilon^{-s_1} = c\varepsilon^{s/(1+s)}$$

we deduce that

$$\left\| u_1^\varepsilon(x) - u_0(x) - \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) V_j^r \rho_j(x) \right\|_H \leq c\varepsilon^{s/(1+s)}.$$

From (2.23)

$$\|u_0 - V_j \rho_j\|_H \leq c\varepsilon^{ss_1} = c\varepsilon^{s/(1+s)},$$

we get

$$\left\| u_1^\varepsilon(x) - u_0(x) - \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \right\|_H \leq c\varepsilon^{s/(1+s)}.$$

This together with (2.26) implies

$$\left\| u^\varepsilon(x) - u_0(x) - \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) u_{0r}(x) \right\|_H \leq c\varepsilon^{s/(1+s)}.$$

We then get the conclusion. $\square$

Remark 2.2.3. *In this thesis, we use the framework of multiscale convergence to derive the homogenized problem, and the correctors. The advantage of this approach is that we can solve for the solution of the homogenized equation, and the scale interacting terms at the same time with essentially optimal complexity. As for two-scale elliptic equations, to derive higher order correctors, we need to use the different method of two-scale asymptotic expansion. It is necessary to assume that the solution of the homogenized problem and the solutions of the cell problems are smooth for the higher order correctors to converge, and to get an explicit error estimate. These are considered for Maxwell equations in [28, 105, 106], using the two scale asymptotic expansion, assuming that the solution u_0 of the homogenized problem and the solution of the cell problems are all smooth. However, such a regularity requirement for u_0 does not hold in general.*

2.2.2 Multiscale problems

For multiscale problems, we do not have an explicit homogenization rate of convergence. However, for the case where $\varepsilon_i/\varepsilon_{i+1}$ is an integer for all $i = 1, \dots, n-1$ we can derive a corrector for the solution u^ε of the multiscale problem. We first derive the homogenized equation.

From (2.10) for all $v_i \in W_i$ we have

$$\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \left(\sum_{i=1}^n \operatorname{curl}_{y_i} v_i \right) d\mathbf{y} dx = 0.$$

Thus

$$\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \operatorname{curl}_{y_n} v_n d\mathbf{y} dx = 0$$

for all $v_n \in W_n$. For each $l = 1, 2, 3$, let $\chi_n^l \in W_n$ be the solution of

$$\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) (e^l + \operatorname{curl}_{y_n} \chi_n^l) \cdot \operatorname{curl}_{y_n} v_n d\mathbf{y} dx = 0$$

for all $v_n \in W_n$. We can write u_n as

$$u_n = \chi_n^l ((\operatorname{curl} u_0)_l + (\operatorname{curl}_{y_1} u_1)_l + \dots + (\operatorname{curl}_{y_{n-1}} u_{n-1})_l).$$

For all $v_{n-1} \in W_{n-1}$,

$$\begin{aligned} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^{n-1} \operatorname{curl}_{y_i} u_i + \operatorname{curl}_{y_n} \chi_n^l \left((\operatorname{curl} u_0)_l + \sum_{i=1}^{n-1} (\operatorname{curl}_{y_i} u_i)_l \right) \right) \\ \cdot \operatorname{curl}_{y_{n-1}} v_{n-1} d\mathbf{y} dx = 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \int_D \int_{\mathbf{Y}} a_{kj}(x, \mathbf{y}) (\delta_{lj} + (\operatorname{curl}_{y_n} \chi_n^l)_j) \\ \left((\operatorname{curl} u_0)_l + \sum_{i=1}^{n-1} (\operatorname{curl}_{y_i} u_i)_l \right) (\operatorname{curl}_{y_{n-1}} v_{n-1})_k d\mathbf{y} dx = 0 \end{aligned}$$

for all $v_{n-1} \in W_{n-1}$. The $(n-1)$ th level homogenized coefficient is defined as:

$$a_{pq}^{n-1}(x, y_1, \dots, y_{n-1}) = \int_{Y_n} a_{pk}(x, \mathbf{y}) (\delta_{kq} + (\operatorname{curl}_{y_n} \chi_n^q)_k) dy_n.$$

Similarly, we have that

$$u_{n-1} = \chi_{(n-1)}^l ((\operatorname{curl} u_0)_l + (\operatorname{curl}_{y_1} u_1)_l + \cdots + (\operatorname{curl}_{y_{n-2}} u_{n-2})_l)$$

where $\chi_{(n-1)}^l$ satisfies the cell problem

$$\int_D \int_{\mathbf{Y}_{n-1}} a^{n-1} (e^l + \operatorname{curl}_{y_{n-1}} \chi_{n-1}^l) \cdot \operatorname{curl}_{y_{n-1}} v_{n-1} d\mathbf{y}_{n-1} dx = 0$$

for all $v_{n-1} \in W_{n-1}$. Letting $a^n = a$, we then have, recursively,

$$u_i = \chi_i^l ((\operatorname{curl} u_0)_l + (\operatorname{curl}_{y_1} u_1)_l + \cdots + (\operatorname{curl}_{y_{i-1}} u_{i-1})_l) \quad (2.27)$$

where $\chi_i^l \in W_i$ satisfies the cell problem

$$\int_D \int_{\mathbf{Y}_i} a^i (e^l + \operatorname{curl}_{y_i} \chi_i^l) \cdot \operatorname{curl}_{y_i} v_i d\mathbf{y}_i = 0$$

for all $v_i \in W_i$. For $i = 1, \dots, n-1$, the i th level homogenized coefficient a^i is defined as

$$a_{pq}^i(x, y_1, \dots, y_i) = \int_{Y_{i+1}} a_{pk}^{i+1} (\delta_{kq} + (\operatorname{curl}_{y_{i+1}} \chi_{i+1}^q)_k) dy_{i+1}.$$

Continuing this process, we finally get the homogenized coefficient $a^0(x)$ as

$$a_{pq}^0(x) = \int_{Y_1} a_{pk}^1 (\delta_{kq} + (\operatorname{curl}_{y_1} \chi_1^q)_k) dy_1.$$

From (2.10) for all $\mathbf{v}_i \in V_i$ we have

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(\sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i \right) d\mathbf{y} dx = 0.$$

Thus

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \nabla_{y_n} \mathbf{v}_n d\mathbf{y} dx = 0$$

for all $\mathbf{v}_n \in V_n$. For each $l = 1, 2, 3$, let $\omega_n^l \in V_n$ be the solution of

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) (e^l + \nabla_{y_n} \omega_n^l) \cdot \nabla_{y_n} \mathbf{v}_n d\mathbf{y} dx = 0$$

for all $\mathbf{v}_n \in V_n$. We can write $\mathbf{u}_n$ as

$$\mathbf{u}_n = \omega_n^l ((u_0)_l + (\nabla_{y_1} \mathbf{u}_1)_l + \cdots + (\nabla_{y_{n-1}} \mathbf{u}_{n-1})_l).$$

For all $\mathbf{v}_{n-1} \in V_{n-1}$,

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i + \nabla_{y_n} \omega_n^l \left((u_0)_l + \sum_{i=1}^{n-1} (\nabla_{y_i} \mathbf{u}_i)_l \right) \right) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx = 0,$$

i.e.,

$$\int_D \int_{\mathbf{Y}} b_{kj}(x, \mathbf{y}) (\delta_{lj} + (\nabla_{y_n} \omega_n^l)_j) \left((u_0)_l + \sum_{i=1}^{n-1} (\nabla_{y_i} \mathbf{u}_i)_l \right) (\nabla_{y_{n-1}} \mathbf{v}_{n-1})_k d\mathbf{y} dx = 0$$

for all $\mathbf{v}_{n-1} \in V_{n-1}$. Let

$$b_{pq}^{n-1}(x, y_1, \dots, y_{n-1}) = \int_{Y_n} b_{pk}(x, \mathbf{y}) (\delta_{kq} + (\nabla_{y_n} \omega_n^q)_k) dy_n.$$

Similarly, we have that

$$\mathbf{u}_{n-1} = \omega_{(n-1)}^l \left((u_0)_l + (\nabla_{y_1} \mathbf{u}_1)_l + \dots + (\nabla_{y_{n-2}} \mathbf{u}_{n-2})_l \right)$$

where $\omega_{(n-1)}^l$ satisfies the cell problem

$$\int_D \int_{\mathbf{Y}_{n-1}} b^{n-1}(e^l + \nabla_{y_{n-1}} \omega_{n-1}^l) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y}_{n-1} dx = 0$$

for all $\mathbf{v}_{n-1} \in V_{n-1}$. Letting $b^n = b$, we then have, recursively,

$$\mathbf{u}_i = \omega_i^l \left((u_0)_l + (\nabla_{y_1} \mathbf{u}_1)_l + \dots + (\nabla_{y_{i-1}} \mathbf{u}_{i-1})_l \right)$$

where $\omega_i^l \in V_i$ satisfies the cell problem

$$\int_D \int_{\mathbf{Y}_i} b^i(e^l + \nabla_{y_i} \omega_i^l) \cdot \nabla_{y_i} \mathbf{v}_i d\mathbf{y}_i = 0$$

for all $\mathbf{v}_i \in V_i$. For $i = 1, \dots, n-1$, the i th level homogenized coefficient b^i is defined as

$$b_{pq}^i(x, y_1, \dots, y_i) = \int_{Y_{i+1}} b_{pk}^{i+1} (\delta_{kq} + (\nabla_{y_{i+1}} \omega_{i+1}^q)_k) dy_{i+1}.$$

Continuing this process, we finally get the homogenized coefficient $b^0(x)$ as

$$b_{pq}^0(x) = \int_{Y_1} b_{pk}^1 (\delta_{kq} + (\nabla_{y_1} \omega_1^q)_k) dy_1.$$

The homogenized equation is

$$\int_D a^0(x) \operatorname{curl} u_0 \cdot \operatorname{curl} v_0 + b^0(x) u_0(x) \cdot v_0(x) dx = \int_D f(x) \cdot v_0(x) dx,$$

i.e.,

$$\operatorname{curl}(a^0(x)\operatorname{curl}u_0(x)) + b^0(x)u_0(x) = f(x).$$

We define a map $\mathcal{T}_n^\varepsilon : L^1(D) \rightarrow L^1(D \times \mathbf{Y})$ such that for each function $\phi \in L^1(D)$ which is understood as 0 outside D

$$\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y}) = \phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_2 \left[\frac{y_1}{\varepsilon_2/\varepsilon_1}\right] + \cdots + \varepsilon_n \left[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\right] + \varepsilon_n y_n\right). \quad (2.28)$$

The purpose of introducing this map is that we can reformulate the $(n+1)$ -scale convergence in terms of the weak convergence in the product domain; this facilitates the establishment of a corrector, which is not obvious from the $(n+1)$ -scale convergence notion.

Letting D^{ε_1} be the $2\varepsilon_1$ neighbourhood of D , we have

$$\int_D \phi dx = \int_{D^{\varepsilon_1}} \int_{Y_1} \cdots \int_{Y_n} \mathcal{T}_n^\varepsilon(\phi) dy_n \cdots dy_1 dx \quad (2.29)$$

for all $\phi \in L^1(D)$. If a sequence $\{\phi^\varepsilon\}_\varepsilon$ $(n+1)$ -scale converges to $\phi(x, y_1, \dots, y_n)$, then

$$\mathcal{T}_n^\varepsilon(\phi) \rightharpoonup \phi(x, y_1, \dots, y_n) \quad (2.30)$$

in $L^2(D \times \mathbf{Y})$. Thus when $\varepsilon \rightarrow 0$,

$$\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) \rightharpoonup \operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \cdots + \operatorname{curl}_{y_n} u_n \quad (2.31)$$

and

$$\mathcal{T}_n^\varepsilon(u^\varepsilon) \rightharpoonup u_0 + \nabla_{y_1} \mathbf{u}_1 + \cdots + \nabla_{y_n} \mathbf{u}_n \quad (2.32)$$

in $L^2(D \times \mathbf{Y})^3$. To deduce a corrector for u^ε in $H(\operatorname{curl}, D)$, we need to map the scale interacting terms u_i and $\mathbf{u}_i$ back to the spaces of functions which depend on x only. We therefore employ the operator $\mathcal{U}_n^\varepsilon : L^1(D \times \mathbf{Y}) \rightarrow L^1(D)$ which approximates the $L^1(D \times \mathbf{Y})$ norm. The operator is defined as

$$\begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi\left(\varepsilon_1 \left[\frac{x}{\varepsilon_1}\right] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1} \left[\frac{\varepsilon_1}{\varepsilon_2} \left\{\frac{x}{\varepsilon_1}\right\}\right] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \dots, \right. \\ \left. \frac{\varepsilon_n}{\varepsilon_{n-1}} \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{\frac{x}{\varepsilon_{n-1}}\right\}\right] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \left\{\frac{x}{\varepsilon_n}\right\}\right) dt_n \cdots dt_1 \end{aligned}$$

for all functions $\Phi \in L^1(D \times \mathbf{Y})$. For each function $\Phi \in L^1(D \times \mathbf{Y})$ we have

$$\int_{D^{\varepsilon_1}} \mathcal{U}_n^\varepsilon(\Phi) dx = \int_D \int_{\mathbf{Y}} \Phi(x, \mathbf{y}) d\mathbf{y} dx. \quad (2.33)$$

The proofs for these facts may be found in [36]. We then have the following corrector result.

Proposition 2.2.4. *The solution u^ε of problem (2.5) and the solution*

$$(u_0, \{u_i\}, \{\mathbf{u}_i\})$$

of problem (2.10) satisfy

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - [u_0 + \mathcal{U}_n^\varepsilon(\nabla_{y_1} \mathbf{u}_1) + \dots + \mathcal{U}_n^\varepsilon(\nabla_{y_n} \mathbf{u}_n)]\|_H = 0, \quad (2.34)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|\operatorname{curl} u^\varepsilon - [\operatorname{curl} u_0 + \mathcal{U}_n^\varepsilon(\operatorname{curl}_{y_1} u_1) + \dots + \mathcal{U}_n^\varepsilon(\operatorname{curl}_{y_n} u_n)]\|_H = 0. \quad (2.35)$$

Proof We consider the expression

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} [\mathcal{T}_n^\varepsilon(a^\varepsilon)(\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n)) \\ & \quad \cdot (\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n)) + \\ & \quad \mathcal{T}_n^\varepsilon(b^\varepsilon)(\mathcal{T}_n^\varepsilon(u^\varepsilon) - (u_0 + \nabla_{y_1} \mathbf{u}_1 + \dots + \nabla_{y_n} \mathbf{u}_n)) \\ & \quad \cdot (\mathcal{T}_n^\varepsilon(u^\varepsilon) - (u_0 + \nabla_{y_1} \mathbf{u}_1 + \dots + \nabla_{y_n} \mathbf{u}_n))] d\mathbf{y} dx. \end{aligned}$$

Using (2.5), (2.10), (2.29), (2.31) and (2.32), we deduce that this expression converges to 0. From (2.1) we have

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n)\|_{H_n} = 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_n^\varepsilon(u^\varepsilon) - (u_0 + \nabla_{y_1} \mathbf{u}_1 + \dots + \nabla_{y_n} \mathbf{u}_n)\|_{H_n} = 0.$$

From (2.33) and the fact that $\mathcal{U}_n^\varepsilon(\Phi)^2 \leq \mathcal{U}_n^\varepsilon(\Phi^2)$, we have

$$\begin{aligned} & \int_D |\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n))(x)|^2 dx \\ & \leq \int_D \mathcal{U}_n^\varepsilon(|\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n)|^2)(x) dx \\ & \leq \int_D \int_{\mathbf{Y}} |\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon) - (\operatorname{curl} u_0 + \operatorname{curl}_{y_1} u_1 + \dots + \operatorname{curl}_{y_n} u_n)|^2 d\mathbf{y} dx \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$. Using $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\Phi)) = \Phi$, we get (2.35). We derive (2.34) similarly. $\square$

2.3 Regularity of χ^r , ω^r and u_0

We show in this section that the regularity requirements for obtaining the homogenization error estimate in the previous sections are achievable. We first prove the results for the two-scale case in details. We first extend the well known result that functions in $H_0(\text{curl}, D) \cap H(\text{div}, D)$ belongs to $H^1(D)^3$ to periodic functions.

Lemma 2.3.1. *Let $\psi \in H_{\#}(\text{curl}, Y) \cap H_{\#}(\text{div}, Y)$. Assume further that*

$$\int_Y \psi(y) dy = 0.$$

Then $\psi \in H_{\#}^1(Y)^3$ and

$$\|\psi\|_{H^1(Y)^3} \leq c(\|\text{curl}_y \psi\|_{L^2(Y)^3} + \|\text{div}_y \psi\|_{L^2(Y)}).$$

Proof Let $\Omega \subset \mathbb{R}^3$ be a smooth domain such that $\Omega \supset Y$. To use the regularity results for functions in $H_0(\text{curl}, \Omega)$, we employ a boundary layer function. Let $\eta \in \mathcal{D}(\Omega)$ be such that $\eta(y) = 1$ when $y \in Y$. We have

$$\text{curl}_y(\eta\psi) = \eta \text{curl}_y \psi + \nabla_y \eta \times \psi \in L^2(\Omega)^3$$

and

$$\text{div}_y(\eta\psi) = \nabla_y \eta \cdot \psi + \eta \text{div}_y \psi \in L^2(\Omega).$$

Together with the boundary condition, from the standard theory (see, e.g., [73]) we conclude that $\eta\psi \in H^1(\Omega)^3$ so $\psi \in H^1(Y)^3$.

We note that

$$\begin{aligned} & \int_Y (\text{div}_y \psi(y)^2 + |\text{curl}_y \psi(y)|^2) dy \\ &= \sum_{i,j=1}^3 \int_Y \left(\frac{\partial \psi_i}{\partial y_j} \right)^2 + \sum_{i \neq j} \int_Y \frac{\partial \psi_i}{\partial y_i} \frac{\partial \psi_j}{\partial y_j} dy - \sum_{i \neq j} \int_Y \frac{\partial \psi_j}{\partial y_i} \frac{\partial \psi_i}{\partial y_j} dy. \end{aligned}$$

Assume that ψ is a smooth periodic function. We have

$$\int_Y \frac{\partial \psi_i}{\partial y_i} \frac{\partial \psi_j}{\partial y_j} dy = \int_Y \left[\frac{\partial}{\partial y_i} \left(\psi_i \frac{\partial \psi_j}{\partial y_j} \right) - \psi_i \frac{\partial^2 \psi_j}{\partial y_i \partial y_j} \right] dy = - \int_Y \psi_i \frac{\partial^2 \psi_j}{\partial y_i \partial y_j} dy$$

as ψ is periodic. Similarly, we have

$$\int_Y \frac{\partial \psi_i}{\partial y_j} \frac{\partial \psi_j}{\partial y_i} dy = \int_Y \left[\frac{\partial}{\partial y_j} \left(\psi_i \frac{\partial \psi_j}{\partial y_i} \right) - \psi_i \frac{\partial^2 \psi_j}{\partial y_j \partial y_i} \right] dy = - \int_Y \psi_i \frac{\partial^2 \psi_j}{\partial y_j \partial y_i} dy.$$

Thus

$$\int_Y \frac{\partial \psi_i}{\partial y_i} \frac{\partial \psi_j}{\partial y_j} dy = \int_Y \frac{\partial \psi_i}{\partial y_j} \frac{\partial \psi_j}{\partial y_i} dy.$$

Therefore

$$\|\nabla_y \psi\|_{L^2(Y)^3}^2 = \|\operatorname{div}_y \psi\|_{L^2(Y)}^2 + \|\operatorname{curl}_y \psi\|_{L^2(Y)^3}^2.$$

Using a density argument, this holds for all $\psi \in H_{\#}^1(Y)^3$. As $\int_Y \psi(y) dy = 0$, from Poincaré inequality we deduce

$$\|\psi\|_{H^1(Y)^3} \leq c(\|\operatorname{div} \psi\|_{L^2(Y)} + \|\operatorname{curl}_y \psi\|_{L^2(Y)^3}).$$

□

We now show regularity properties for periodic solutions of Maxwell equation, using the previous lemma.

Lemma 2.3.2. *Let $\alpha \in C_{\#}^1(\bar{Y})^{3 \times 3}$ be uniformly bounded, positive definite and symmetric for all $y \in \bar{Y}$. Let $F \in L^2(Y)$ extending periodically to $\mathbb{R}^3$. Let $\psi \in H_{\#}^1(Y)^3$ satisfy the equation*

$$\operatorname{curl}_y(\alpha(y) \operatorname{curl}_y \psi(y)) = F(y).$$

Then $\operatorname{curl}_y \psi \in H_{\#}^1(Y)^3$ and

$$\|\operatorname{curl}_y \psi\|_{H^1(Y)^3} \leq c(\|F\|_{L^2(Y)^3} + \|\psi\|_{H^1(Y)^3}).$$

Proof Let $\Omega \supset Y$ be a smooth domain. Let $\eta \in \mathcal{D}(\Omega)$ be such that $\eta(y) = 1$ for $y \in Y$. We have

$$\begin{aligned} \operatorname{curl}_y(\alpha \operatorname{curl}_y(\eta \psi)) &= \operatorname{curl}_y(\alpha \eta \operatorname{curl}_y \psi) + \operatorname{curl}_y(\alpha \nabla_y \eta \times \psi) \\ &= \eta \operatorname{curl}_y(\alpha \operatorname{curl}_y \psi) + \nabla_y \eta \times (\alpha \operatorname{curl}_y \psi) + \operatorname{curl}_y(\alpha \nabla_y \eta \times \psi). \end{aligned}$$

Let $U = \alpha \operatorname{curl}_y(\eta \psi)$. We have

$$\|\operatorname{curl}_y U\|_{L^2(\Omega)^3} \leq c(\|F\|_{L^2(\Omega)^3} + \|\psi\|_{H^1(\Omega)^3}) \leq c(\|F\|_{L^2(Y)^3} + \|\psi\|_{H^1(Y)^3}).$$

Further,

$$\|U\|_{L^2(\Omega)^3} = \|\alpha(\nabla_y \eta \times \psi + \eta \operatorname{curl}_y \psi)\|_{L^2(\Omega)^3} \leq c\|\psi\|_{H^1(\Omega)^3} \leq c\|\psi\|_{H^1(Y)^3}.$$

As $\eta \in \mathcal{D}(\Omega)$, U has a compact support in Ω so U belongs to $H_0(\operatorname{curl}, \Omega)$. Thus we can write

$$U = z + \nabla \Phi$$

$z \in H_0^1(\Omega)^3$ and $\Phi \in H_0^1(\Omega)$ satisfy

$$\|z\|_{H^1(\Omega)^3} \leq c\|U\|_{H(\operatorname{curl}, \Omega)}, \quad \text{and} \quad \|\Phi\|_{H^1(\Omega)} \leq c\|U\|_{H(\operatorname{curl}, \Omega)}.$$

From $\operatorname{div}_y(\alpha^{-1}U) = 0$ we deduce that

$$\operatorname{div}_y(\alpha^{-1}\nabla\Phi) = -\operatorname{div}_y(\alpha^{-1}z) \in L^2(\Omega).$$

Since $\alpha \in C^1(\bar{\Omega})^{3 \times 3}$ and is uniformly bounded and positive definite, $\alpha^{-1} \in C^1(\bar{\Omega})^{3 \times 3}$ and is uniformly positive definite. Therefore $\Phi \in H^2(\Omega)$ and satisfies

$$\|\Phi\|_{H^2(\Omega)} \leq c\|z\|_{H^1(\Omega)^3} \leq c\|U\|_{H(\operatorname{curl}, \Omega)}.$$

Thus $U \in H^1(\Omega)^3$ and $\|U\|_{H^1(\Omega)^3} \leq c\|U\|_{H(\operatorname{curl}, \Omega)} \leq c(\|F\|_{L^2(Y)^3} + \|\psi\|_{H^1(Y)^3})$. From $\operatorname{curl}_y(\eta\psi) = \alpha^{-1}U$, we deduce that $\operatorname{curl}_y(\eta\psi) \in H^1(\Omega)^3$ so $\operatorname{curl}_y\psi \in H^1(Y)^3$ and

$$\|\operatorname{curl}_y\psi\|_{H^1(Y)^3} \leq c(\|F\|_{L^2(Y)^3} + \|\psi\|_{H^1(Y)^3}).$$

□

Using these results, we now prove the following regularity properties of χ^r .

Proposition 2.3.3. *Assume that $a(x, y) \in C^1(\bar{D}, C^2(\bar{Y}))^{3 \times 3}$, then $\operatorname{curl}_y\chi^r(x, y) \in C^1(\bar{D}, C(\bar{Y}))^3$ and we can choose a version of χ^r in $L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$ so that $\chi^r(x, y) \in C^1(\bar{D}, C(\bar{Y}))^3$.*

Proof We can choose a version of χ^r so that $\operatorname{div}_y\chi^r = 0$. Indeed, let $\Phi(x, \cdot) \in L^2(D, H_\#^1(Y))$ be such that $\Delta_y\Phi = -\operatorname{div}_y\chi^r$, then $\operatorname{curl}_y(\chi^r + \nabla_y\Phi) = \operatorname{curl}_y\chi^r$ and $\operatorname{div}_y(\chi^r + \nabla_y\Phi) = 0$. Further we can choose χ^r so that $\int_Y \chi^r(x, y)dy = 0$. From Lemma 2.3.1, we have

$$\|\chi^r(x, \cdot)\|_{H^1(Y)^3} \leq c\|\operatorname{curl}_y\chi^r(x, \cdot)\|_{L^2(Y)^3}$$

which is uniformly bounded with respect to x . From (2.13) and Lemma 2.3.2, we deduce that

$$\|\operatorname{curl}_y\chi^r(x, \cdot)\|_{H^1(Y)^3} \leq c(\|\operatorname{curl}_y(a(x, \cdot)e^r)\|_{L^2(Y)^3} + \|\chi^r(x, \cdot)\|_{H^1(Y)^3})$$

which is uniformly bounded with respect to x .

For each index $q = 1, 2, 3$, we have that $\operatorname{curl}_y \frac{\partial}{\partial y_q} \chi^r(x, \cdot)$ is uniformly bounded in $L^2(Y)^3$ and $\operatorname{div}_y \frac{\partial}{\partial y_q} \chi^r(x, \cdot) = 0$. Therefore from Lemma 2.3.1, $\frac{\partial}{\partial y_q} \chi^r(x, \cdot)$ is uniformly bounded in $H^1(Y)^3$.

We note that

$$\frac{\partial}{\partial y_q} (\operatorname{curl}_y(a(x, \cdot)\operatorname{curl}_y\chi^r)) = -\frac{\partial}{\partial y_q} \operatorname{curl}_y(a(x, \cdot)e^r) \in L^2(Y)^3.$$

Thus

$$\operatorname{curl}_y \left(a \operatorname{curl}_y \frac{\partial \chi^r}{\partial y_q} \right) = \frac{\partial}{\partial y_q} (\operatorname{curl}_y(a(x, y)\operatorname{curl}_y\chi^r)) - \operatorname{curl}_y \left(\frac{\partial a}{\partial y_q} \operatorname{curl}_y\chi^r \right) \in L^2(Y)^3.$$

From Lemma 2.3.2, we deduce that $\text{curl}_y \frac{\partial \chi^r}{\partial y_q}(x, \cdot)$ is uniformly bounded in $H^1(Y)^3$ so that $\text{curl}_y \chi^r(x, \cdot)$ is uniformly bounded in $H^2(Y)^3 \subset C(\bar{Y})^3$.

We now show that $\text{curl}_y \chi^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$. Fix $h \in \mathbb{R}^3$. From (2.13) we have

$$\begin{aligned} \text{curl}_y(a(x, y)\text{curl}_y(\chi^r(x + h, y) - \chi^r(x, y))) &= -\text{curl}_y((a(x + h, y) - a(x, y))e^r) \\ &\quad - \text{curl}_y((a(x + h, y) - a(x, y))\text{curl}_y \chi^r(x + h, y)). \end{aligned}$$

The smoothness of a and the uniform boundedness of $\text{curl}_y \chi^r(x, \cdot)$ in $L^2(Y)^3$ gives

$$\lim_{h \rightarrow 0} \|\text{curl}_y(\chi^r(x + h, \cdot) - \chi^r(x, \cdot))\|_{L^2(Y)^3} = 0. \quad (2.36)$$

From Lemma 2.3.1, we have that $\chi^r(x + h, \cdot) - \chi^r(x, \cdot) \in H^1(Y)^3$ and

$$\|\chi^r(x + h, \cdot) - \chi^r(x, \cdot)\|_{H^1(Y)^3} \leq c \|\text{curl}_y(\chi^r(x + h, \cdot) - \chi^r(x, \cdot))\|_{L^2(Y)^3}$$

which converges to 0 when $|h| \rightarrow 0$. From Lemma 2.3.2, we have

$$\begin{aligned} &\|\text{curl}_y(\chi^r(x + h, \cdot) - \chi^r(x, \cdot))\|_{H^1(Y)^3} \\ &\leq \| -\text{curl}_y((a(x + h, \cdot) - a(x, \cdot))e^r) \\ &\quad - \text{curl}_y((a(x + h, \cdot) - a(x, \cdot))\text{curl}_y \chi^r(x + h, \cdot)) \|_{L^2(Y)^3} \\ &\quad + \|\chi^r(x + h, \cdot) - \chi^r(x, \cdot)\|_{H^1(Y)^3} \end{aligned} \quad (2.37)$$

tends to 0 when $|h| \rightarrow 0$. We have further that

$$\begin{aligned} &\text{curl}_y \left(a(x, y) \text{curl}_y \frac{\partial}{\partial y_q} (\chi^r(x + h, y) - \chi^r(x, y)) \right) \\ &= -\text{curl}_y \left(\frac{\partial a}{\partial y_q}(x, y) \text{curl}_y (\chi^r(x + h, y) - \chi^r(x, y)) \right) \\ &\quad - \frac{\partial}{\partial y_q} \text{curl}_y ((a(x + h, y) - a(x, y))e^r) \\ &\quad - \frac{\partial}{\partial y_q} \text{curl}_y ((a(x + h, y) - a(x, y))\text{curl}_y \chi^r(x + h, y)). \end{aligned} \quad (2.38)$$

From this we have

$$\begin{aligned} \left\| \text{curl}_y \frac{\partial}{\partial y_q} (\chi^r(x + h, y) - \chi^r(x, y)) \right\|_{L^2(Y)^3} &\leq c \|\text{curl}_y(\chi^r(x + h, \cdot) - \chi^r(x, \cdot))\|_{L^2(Y)^3} \\ &\quad + c \|a(x + h, \cdot) - a(x, \cdot)\|_{W^{1,\infty}(Y)^3} \rightarrow 0 \end{aligned}$$

when $|h| \rightarrow 0$. So, from Lemma 2.3.1, we have

$$\left\| \frac{\partial}{\partial y_q} (\chi^r(x + h, y) - \chi^r(x, y)) \right\|_{H^1(Y)^3} \rightarrow 0 \quad \text{when } |h| \rightarrow 0.$$

As the right hand side of (2.38) converges to 0 in the $L^2(Y)^3$ norm when $|h| \rightarrow 0$, we deduce from Lemma 2.3.2 that

$$\left\| \operatorname{curl}_y \frac{\partial}{\partial y_q} (\chi^r(x+h, y) - \chi^r(x, y)) \right\|_{H^1(Y)^3} \rightarrow 0 \quad \text{when } |h| \rightarrow 0. \quad (2.39)$$

To show that we can differentiate χ^r with respect to x , we consider the limit of the difference quotient of χ^r . We have

$$\begin{aligned} \operatorname{curl}_y \left[a(x, y) \operatorname{curl}_y \frac{\chi^r(x+h, y) - \chi^r(x, y)}{h} \right] &= -\operatorname{curl}_y \left(\frac{a(x+h, y) - a(x, y)}{h} e^r \right) \\ &\quad - \operatorname{curl}_y \left(\frac{a(x+h, y) - a(x, y)}{h} \operatorname{curl}_y \chi^r(x+h, y) \right). \end{aligned}$$

Let $\chi_1^r(x, \cdot) \in \tilde{H}_\#(\operatorname{curl}, Y)$ with $\operatorname{div}_y \chi_1^r(x, \cdot) = 0$ be the solution of the problem

$$\operatorname{curl}_y (a(x, y) \operatorname{curl}_y \chi_1^r(x, \cdot)) = -\operatorname{curl}_y \left(\frac{\partial a}{\partial x_q} e^r \right) - \operatorname{curl}_y \left(\frac{\partial a}{\partial x_q} \operatorname{curl}_y \chi^r(x, y) \right).$$

We deduce that

$$\begin{aligned} &\operatorname{curl}_y \left(a(x, y) \operatorname{curl}_y \left(\frac{\chi^r(x+h, y) - \chi^r(x, y)}{h} - \chi_1^r(x, y) \right) \right) \\ &= -\operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q}(x, y) \right) e^r \right) \\ &\quad - \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q}(x, y) \right) \operatorname{curl}_y \chi^r(x+h, y) \right) \\ &\quad - \operatorname{curl}_y \left(\frac{\partial a}{\partial x_q}(x, y) \operatorname{curl}_y (\chi^r(x+h, y) - \chi^r(x, y)) \right) := I_1. \end{aligned}$$

Let $h \in \mathbb{R}^3$ be a vector whose all components are 0 except the q th component. We have

$$\begin{aligned} &\left\| \operatorname{curl}_y \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{L^2(Y)^3} \\ &\leq c \left\| \frac{a(x+h, \cdot) - a(x, \cdot)}{h} - \frac{\partial a}{\partial x_q}(x, \cdot) \right\|_{L^\infty(Y)} \\ &\quad + c \left\| \operatorname{curl}_y (\chi^r(x+h, \cdot) - \chi^r(x, \cdot)) \right\|_{L^2(Y)^3} \end{aligned}$$

which converges to 0 when $|h| \rightarrow 0$ due to (2.36). Thus we deduce from Lemma 2.3.1 that

$$\lim_{|h| \rightarrow 0} \left\| \frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right\|_{H^1(Y)^3} = 0. \quad (2.40)$$

From Lemma 2.3.2, we have

$$\begin{aligned} & \lim_{|h| \rightarrow 0} \left\| \operatorname{curl}_y \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{H^1(Y)^3} \\ & \leq \lim_{|h| \rightarrow 0} \|I_1(x, \cdot)\|_{L^2(Y)^3} + \lim_{|h| \rightarrow 0} \left\| \frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right\|_{H^1(Y)^3} = 0 \end{aligned} \quad (2.41)$$

due to (2.37) and (2.40).

Let $q = 1, 2, 3$. We then have

$$\begin{aligned} & \operatorname{curl}_y \left(a(x, y) \operatorname{curl}_y \frac{\partial}{\partial y_q} \left(\frac{\chi^r(x+h, y) - \chi^r(x, y)}{h} - \chi_1^r(x, y) \right) \right) \\ & = -\operatorname{curl}_y \left(\frac{\partial a}{\partial y_p}(x, y) \operatorname{curl}_y \left(\frac{\chi^r(x+h, y) - \chi^r(x, y)}{h} - \chi_1^r(x, y) \right) \right) \\ & \quad - \frac{\partial}{\partial y_p} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a(x, y)}{\partial x_q} \right) e^r \right) \\ & \quad - \frac{\partial}{\partial y_p} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q}(x, y) \right) \operatorname{curl}_y \chi^r(x+h, y) \right) \\ & \quad - \frac{\partial}{\partial y_p} \operatorname{curl}_y \left(\frac{\partial a}{\partial x_q}(x, y) \operatorname{curl}_y (\chi^r(x+h, y) - \chi^r(x, y)) \right) \end{aligned}$$

which converges to 0 in $L^2(Y)$ for each x due to (2.39), (2.41) and the uniform boundedness of $\|\operatorname{curl} \chi^r(x, \cdot)\|_{H^2(Y)^3}$. We have

$$\begin{aligned} & \left\| \operatorname{curl}_y \frac{\partial}{\partial y_p} \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{L^2(Y)^3} \\ & \leq c \left\| \operatorname{curl}_y \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{L^2(Y)^3} \\ & \quad + c \left\| \frac{a(x+h, \cdot) - a(x, \cdot)}{h} - \frac{\partial a(x, \cdot)}{\partial x_q} \right\|_{W^{1,\infty}(Y)^3} \\ & \quad + c \|\operatorname{curl}_y (\chi^r(x+h, \cdot) - \chi^r(x, \cdot))\|_{H^1(Y)^3} \end{aligned}$$

which converges to 0 when $|h| \rightarrow 0$, so from Lemma 2.3.1

$$\lim_{|h| \rightarrow 0} \left\| \frac{\partial}{\partial y_p} \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0.$$

Therefore $\chi^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$. We then get from Lemma 2.3.2 that

$$\lim_{|h| \rightarrow 0} \left\| \operatorname{curl}_y \frac{\partial}{\partial y_p} \left(\frac{\chi^r(x+h, \cdot) - \chi^r(x, \cdot)}{h} - \chi_1^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0.$$

Thus $\operatorname{curl}_y \chi^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$. $\square$

We now prove the regularity properties for the function ω^r . We employ the well known regularity properties for solutions of elliptic equations.

Proposition 2.3.4. *Assume that $b(x, y) \in C^1(\bar{D}, C^2(\bar{Y}))^{3 \times 3}$. The solution ω^r of cell problem (2.12) belongs to $C^1(\bar{D}, C^1(\bar{Y}))$.*

Proof The cell problem (2.12) can be written as

$$-\nabla_y \cdot (b(x, y) \nabla_y \omega^r(x, y)) = \nabla_y \cdot (b(x, y) e^r).$$

Fixing $x \in \bar{D}$, the right hand side is bounded uniformly in $H^1(Y)$ so $\omega^r(x, \cdot)$ is uniformly bounded in $H^3(Y)$ from elliptic regularity (see McLean [68] Theorem 4.16). For $h \in \mathbb{R}^3$, we note that

$$\begin{aligned} -\nabla_y \cdot [b(x, y) \nabla_y (\omega^r(x + h, y) - \omega^r(x, y))] &= \nabla_y \cdot [(b(x + h, y) - b(x, y)) e^r] \\ &\quad + \nabla_y \cdot [(b(x + h, y) - b(x, y)) \nabla_y \omega^r(x + h, y)] := i_1. \end{aligned}$$

As $\int_Y \omega^r(x, y) dy = 0$, we have

$$\begin{aligned} \|\omega^r(x + h, \cdot) - \omega^r(x, \cdot)\|_{H^1(Y)} &\leq c \|\nabla_y (\omega^r(x + h, \cdot) - \omega^r(x, \cdot))\|_{L^2(Y)} \\ &\leq c \|(b(x + h, \cdot) - b(x, \cdot)) e^r\|_{L^2(Y)} \\ &\quad + c \|(b(x + h, \cdot) - b(x, \cdot)) \nabla_y \omega^r(x + h, \cdot)\|_{L^2(Y)} \end{aligned}$$

which converges to 0 when $|h| \rightarrow 0$. Fixing $x \in \bar{D}$, we then have from Theorem 4.16 of [68] that

$$\|\omega^r(x + h, \cdot) - \omega^r(x, \cdot)\|_{H^3(Y)} \leq \|\omega^r(x + h, \cdot) - \omega^r(x, \cdot)\|_{H^1(Y)} + \|i_1(x, \cdot)\|_{H^1(Y)} \quad (2.42)$$

which converges to 0 when $|h| \rightarrow 0$. Fixing an index $q = 1, 2, 3$, let $h \in \mathbb{R}^3$ be a vector whose components are all zero except the q th component. Let $\eta(x, \cdot) \in H_{\#}^1(Y)/\mathbb{R}$ be the solution of the problem

$$-\nabla_y \cdot [b(x, y) \nabla_y \eta(x, y)] = \nabla_y \cdot \left[\frac{\partial b}{\partial x_q} e^r \right] + \nabla_y \cdot \left[\frac{\partial b}{\partial x_q} \nabla_y \omega^r(x, y) \right].$$

We have

$$\begin{aligned} &-\nabla_y \cdot \left[b(x, y) \nabla_y \left(\frac{\omega^r(x + h, y) - \omega^r(x, y)}{h} - \eta(x, y) \right) \right] \\ &= \nabla_y \cdot \left[\left(\frac{b(x + h, y) - b(x, y)}{h} - \frac{\partial b}{\partial x_q}(x, y) \right) e^r \right] \\ &\quad + \nabla_y \cdot \left[\left(\frac{b(x + h, y) - b(x, y)}{h} - \frac{\partial b}{\partial x_q}(x, y) \right) \nabla_y \omega^r(x + h, y) \right] \\ &\quad + \nabla_y \cdot \left[\frac{\partial b(x, y)}{\partial x_q} (\nabla_y \omega^r(x + h, y) - \nabla_y \omega^r(x, y)) \right] := i_2. \end{aligned}$$

From (2.42) and the regularity of b , $\lim_{|h| \rightarrow 0} \|i_2(x, \cdot)\|_{H^1(Y)} = 0$. As $\int_Y \omega^r(x, y) dy = 0$ and $\int_Y \eta(x, y) dy = 0$, we have that

$$\lim_{|h| \rightarrow 0} \left\| \frac{\omega^r(x + h, \cdot) - \omega^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^1(Y)} = 0.$$

Therefore from Theorem 4.16 of [68], we have

$$\begin{aligned} \left\| \frac{\omega^r(x + h, \cdot) - \omega^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^3(Y)} &\leq \left\| \frac{\omega^r(x + h, \cdot) - \omega^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^1(Y)} \\ &\quad + \|i_2(x, \cdot)\|_{H^1(Y)} \end{aligned}$$

which converges to 0 when $|h| \rightarrow 0$. Thus $\omega^r \in C^1(\bar{D}, H^3(Y)) \subset C^1(\bar{D}, C^1(\bar{Y}))$. $\square$

For the regularity of the solution u_0 of the homogenized problem (2.16) we have the following result.

Proposition 2.3.5. *Assume that D is a Lipschitz polygonal domain, and the coefficient $a(x, y)$, as a function of x , is Lipschitz, uniformly with respect to y , then there is a constant $0 < s < 1$ so that $\text{curl } u_0 \in H^s(D)$. Further, if $\text{div } f \in L^2(D)$, then there is a constant $0 < s < 1$ so that $u_0 \in H^s(D)$.*

Proof When $a(x, y)$ is Lipschitz with respect to x , from (2.13), $\|\text{curl}_y \chi^r(x, \cdot)\|_{L^2(Y)}$ is a Lipschitz function of x , so from (2.15) we have that a^0 is Lipschitz with respect to x . As a^0 is positive definite, $(a^0)^{-1}$ is Lipschitz. Let $U = a^0 \text{curl } u_0$. We have from (2.16) that $U \in H(\text{curl}; D)$, $\text{div}((a^0)^{-1}U) = 0$ and $(a^0)^{-1}U \cdot \nu = 0$ on ∂D where ν is the outward normal vector on ∂D . The conclusion follows from Lemma 4.2 of Hiptmair [52]. If $\text{div } f \in L^2(D)$, $\text{div}(b^0 u_0) = \text{div } f \in L^2(D)$. We also have $\text{curl}(b^0)^{-1}(b^0 u_0) \in L^2(D)$ so from Lemma 4.2 of Hiptmair [52] we have $b^0 u_0 \in H^s(D)$ so $u_0 \in H^s(D)$. $\square$

Remark 2.3.6. *If a^0 is isotropic, we have from (2.16) that*

$$\text{curl } \text{curl } u_0 = -(a^0)^{-1} \nabla a^0 \times \text{curl } u_0 - (a^0)^{-1} b^0 u_0 + (a^0)^{-1} f \in H$$

so $u_0 \in H^1(\text{curl}, D)$ if the domain D is convex. However, even if a is isotropic, a^0 may not be isotropic.

The multiscale case is similar, so we summarize the results here without proof.

Proposition 2.3.7. *If $a(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{3 \times 3}$, then*

$$\text{curl}_{y_i} \chi_i^r \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_{i-1}, H^2(Y_i)) \dots))^3$$

and

$$a^i(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_i) \dots)).$$

Similarly, if $b(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))$, then

$$\omega_i^r \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_{i-1}, H^3(Y_i)) \dots)).$$

Summary

In this chapter, we derive the multiscale homogenized equation and the homogenized equation for the multiscale stationary Maxwell equation. In the two-scale case, we derive the corrector and homogenization error for both cases $u_0 \in H^1(\text{curl}, D)$ and $u_0 \in H^s(\text{curl}, D)$ with $0 < s < 1$. The procedure for deriving the homogenization error when u_0 belongs to the weaker regularity space $H^s(\text{curl}, D)$ is new, and can be applied for many other types of equations. In the multiscale case, correctors are derived without an error. We also prove that the regularity requirements for obtaining the homogenization error estimates are achievable.

Chapter 3

Homogenization of time dependent Maxwell equations

We study the multiscale Maxwell wave equation in this chapter. In section 3.1, we define the multiscale Maxwell wave equations. We extend the concept of multiscale convergence to time dependent functions, and apply it to the multiscale problems. The initial conditions are derived, with which, we show that the multiscale homogenized problem possesses a unique solution. Derivation of the homogenized equation and the initial conditions is quite nontrivial and is performed in Section 3.2. To get a homogenization error estimate, the solution u_0 of the homogenized equation has to be sufficiently smooth. We show this regularity in Section 3.3. In Section 3.4, we derive correctors for the homogenization problem; we restrict our consideration to the case where the initial condition g_0 in (3.2) equals 0. In the two-scale case, an explicit homogenization error estimate is proved. When u_0 belongs only to a weaker regularity space $L^\infty(0, T; H^s(\text{curl}, D))$ for $0 < s < 1$, we develop a new approach to derive the homogenization error.

Performing homogenization for multiscale Maxwell wave equations is far more complicated in comparison to that for multiscale wave equations considered in Xia and Hoang [101]. This is due to the fact that the space $H_0(\text{curl}, D)$ is not compactly embedded in $L^2(D)^3$, unlike the space $H_0^1(D)$. We therefore need to deal with the functions $\mathbf{u}_i$ in (3.5) which depend on time. In particular, we need to deal with the second time derivative of the gradient with respect to y_i of these functions. This derivative is only understood in the generalized sense, and the initial conditions for these functions $\mathbf{u}_i$ are only understood in the generalized sense. It is due to these functions that the homogenized coefficient b^0 for the multiscale coefficient b^ε is the elliptic homogenized coefficient, rather than the average of $b(x, \mathbf{y})$ in $\mathbf{Y}$. Thus, deducing the initial conditions for u_0 is by no means as obvious as in [101], and is quite technically involved, as shown below.

We now define the multiscale Maxwell wave equation.

3.1 Problems setting

3.1.1 Multiscale Maxwell problems

We first recall some notations of Chapter 2. Let D be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$). Let Y be the unit cube in $\mathbb{R}^d$. By $Y_1, \dots, Y_n$ we denote n copies of Y . We denote by $\mathbf{Y}$ the product set $Y_1 \times Y_2 \times \dots \times Y_n$ and by $\mathbf{y} = (y_1, \dots, y_n)$. For $i = 1, \dots, n$, we denote by $\mathbf{Y}_i = Y_1 \times \dots \times Y_i$. Let a and b be functions from $D \times Y_1 \times \dots \times Y_n$ to $\mathbb{R}_{sym}^{d \times d}$ that satisfy the boundedness and coerciveness conditions (2.1). Let ε be a small positive value, and $\varepsilon_1, \dots, \varepsilon_n$ be n functions of ε that denote the n microscopic scales that satisfy the scale separation condition (2.2). Without loss of generality, we assume that $\varepsilon_1(\varepsilon) = \varepsilon$. We define the multiscale coefficient a^ε and b^ε which are functions from D to $\mathbb{R}_{sym}^{d \times d}$ as in (2.3).

As in Chapter 2, when $d = 3$ we define the space

$$W = H_0(\text{curl}, D) = \{u \in L^2(D)^3, \text{curl } u \in L^2(D)^3, u \times \nu = 0\}, \quad H = L^2(D)^3,$$

and when $d = 2$

$$W = H_0(\text{curl}, D) = \{u \in L^2(D)^2, \text{curl } u \in L^2(D), u \times \nu = 0\}, \quad H = L^2(D)^2$$

where ν denotes the outward normal vector on the boundary ∂D . For notational conciseness, we denote by

$$H = L^2(D)^3, \quad H_i = L^2(D \times \mathbf{Y}_i)^3, \quad i = 1, \dots, n. \quad (3.1)$$

We have the Gelfand triple $W \subset H \subset W'$. We denote by $\langle \cdot, \cdot \rangle_H$ the inner product in H , extending to the duality pairing between W' and W . Let $f \in L^2(0, T; H)$, $g_0 \in W$ and $g_1 \in H$. We consider the problem: Find $u^\varepsilon(t, x) \in L^2(0, T; W)$ so that

$$\begin{cases} b^\varepsilon(x) \frac{\partial^2 u^\varepsilon(t, x)}{\partial t^2} + \text{curl}(a^\varepsilon(x) \text{curl} u^\varepsilon(t, x)) = f(t, x), & (t, x) \in (0, T) \times D; \\ u^\varepsilon(0, x) = g_0(x), & x \in D; \\ u_t^\varepsilon(0, x) = g_1(x), & x \in D. \end{cases} \quad (3.2)$$

with the boundary condition $u^\varepsilon \times \nu = 0$ on ∂D . We will present the analysis for the case $d = 3$ and only discuss the case $d = 2$ when there is significant difference.

Figure 3.1: Visualization of three-scale problem

In variational form, this problem becomes: Find $u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H)$ so that

$$\left\langle b^\varepsilon(x) \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi(x) \right\rangle_H + \int_D a^\varepsilon(x) \operatorname{curl} u^\varepsilon(t, x) \cdot \operatorname{curl} \phi(x) dx = \int_D f(t, x) \cdot \phi(x) dx, \quad (3.3)$$

for all $\phi \in W$ when $d = 3$; and when $d = 2$ we need to replace the vector product for curl by the scalar multiplication. Problem (3.3) has a unique solution $u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H) \cap H^2(0, T; W')$ that satisfies

$$\|u^\varepsilon\|_{L^2(0, T; W)} + \|u^\varepsilon\|_{H^1(0, T; H)} + \|u^\varepsilon\|_{H^2(0, T; W')} \leq c(\|f\|_{L^2(0, T; H)} + \|g_0\|_W + \|g_1\|_H) \quad (3.4)$$

where the constant c only depends on the constants α and β in (2.1) and T . This is the standard results for hyperbolic equation with a Gelfand triple $W \subset H \subset W'$ (see Wloka [100] Chapter 5).

We will study this problem via multiscale convergence.

3.1.2 Multiscale convergence

For time dependent functions, we modify the definition of multiscale convergence of Nguetseng [79], Allaire [5] and Allaire and Briane [7] as follows.

Definition 3.1.1. A sequence of functions $\{w^\varepsilon\}_\varepsilon \subset L^2(0, T; H)$ $(n+1)$ -scale converges to a function $w_0 \in L^2(0, T; D \times \mathbf{Y})$ if for all smooth functions $\phi(t, x, \mathbf{y})$

which are Y periodic with respect to y_i for all $i = 1, \dots, n$:

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon(t, x) \phi\left(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}\right) dx dt = \int_0^T \int_D \int_Y w_0(t, x, \mathbf{y}) \phi(t, x, \mathbf{y}) d\mathbf{y} dx dt.$$

We have the following result.

Proposition 3.1.2. *From a bounded sequence in $L^2(0, T; H)$ we can extract an $(n + 1)$ -scale convergent subsequence.*

The proof of this proposition is identical to that for time independent functions in [79], [5] and [7].

For a bounded sequence in $L^2(0, T; W)$, we have the following results which are very similar to those in Chapter 2 for functions which do not depend on t . The proofs for these results are very similar to those in Chapter 2 so we do not present them here. As in Chapter 2, we denote by $\tilde{H}_\#(\text{curl}, Y_i)$ the space of equivalent classes of functions in $H_\#(\text{curl}, Y_i)$ of equal curl.

Proposition 3.1.3. *Let $\{w^\varepsilon\}_\varepsilon$ be a bounded sequence in $L^2(0, T; W)$. There is a subsequence (not renumbered), a function $w_0 \in L^2(0, T; W)$, n functions $\mathbf{w}_i \in L^2((0, T) \times D \times Y_1 \times \dots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$ such that*

$$w^\varepsilon \xrightarrow{(n+1)\text{-scale}} w_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{w}_i.$$

Further, there are n functions $w_i \in L^2((0, T) \times D \times \dots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ such that

$$\text{curl } w^\varepsilon \xrightarrow{(n+1)\text{-scale}} \text{curl } w_0 + \sum_{i=1}^n \text{curl}_{y_i} w_i.$$

3.1.3 Multiscale homogenized time dependent Maxwell problem

From (3.4) and Proposition (3.1.3), we can extract a subsequence (not renumbered), a function $u_0 \in L^2(0, T; W)$, n functions $\mathbf{u}_i \in L^2(0, T; D \times Y_1 \times \dots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$ and n functions $u_i \in L^2(0, T; D \times Y_1 \times \dots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ such that

$$u^\varepsilon \xrightarrow{(n+1)\text{-scale}} u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i, \quad (3.5)$$

and

$$\text{curl } u^\varepsilon \xrightarrow{(n+1)\text{-scale}} \text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y_i} u_i.$$

We recall the definitions of the following spaces in Chapter 2. For $i = 1, \dots, n$, let $W_i = L^2(D \times Y_1 \times \dots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))$ and $V_i = L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_\#^1(Y_i)/\mathbb{R})$. We define the space $\mathbf{V}$

$$\mathbf{V} = W \times W_1 \times \dots \times W_n \times V_1 \times \dots \times V_n. \quad (3.6)$$

The space $\mathbf{V}$ is equipped with the norm

$$|||\mathbf{v}||| = \|v_0\|_{H(\text{curl}, D)} + \sum_{i=1}^n \|v_i\|_{L^2(D \times \mathbf{Y}_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))} + \sum_{i=1}^n \|\mathbf{v}_i\|_{L^2(D \times \mathbf{Y}_{i-1}, H_\#^1(Y_i))}$$

for $\mathbf{v} = (v_0, \{v_i\}, \{\mathbf{v}_i\}) \in \mathbf{V}$. Let $\mathbf{u} = (u_0, \{u_i\}, \{\mathbf{u}_i\}) \in \mathbf{V}$. We define the map

$$\int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}$$

in W' as

$$\begin{aligned} & \left\langle \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \cdot v_0 d\mathbf{y} dx; \end{aligned}$$

and the map $b(x, \mathbf{y}) (u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i))$ in V_j' as

$$\begin{aligned} & \left\langle b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V_j', V_j} \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \cdot \nabla_{y_j} \mathbf{v}_j d\mathbf{y} dx. \end{aligned}$$

We then have the following result. As mentioned above, we need to handle the second time derivative of $\nabla_{y_i} \mathbf{u}_i$ so the results below are understood in the generalized sense.

Proposition 3.1.4. *The function $\mathbf{u} = (u_0, \{u_i\}, \{\mathbf{u}_i\})$ satisfies*

$$\begin{aligned} & \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) (u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i)) d\mathbf{y}, v_0 \right\rangle_H = \int_D f(t, x) \cdot v_0(x) dx \\ & - \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y_i} u_i \right) \cdot \left(\text{curl } v_0 + \sum_{i=1}^n \text{curl}_{y_i} v_i \right) d\mathbf{y} dx \quad (3.7) \end{aligned}$$

and

$$\left\langle \frac{\partial^2}{\partial t^2} \left(b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} = 0, \quad (3.8)$$

i.e., the function $\mathbf{u}$ satisfies the multiscale homogenized equation

$$\begin{aligned} & \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H \\ & + \sum_{j=1}^n \left\langle \frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} \\ & + \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \left(\operatorname{curl} v_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i \right) d\mathbf{y} dx \\ & = \int_D f(t, x) \cdot v_0(x) dx \end{aligned} \quad (3.9)$$

for all $\mathbf{v} = (v_0, \{v_j\}, \{\mathbf{v}_j\}) \in \mathbf{V}$.

Proof Let $q \in \mathcal{D}(0, T)$. Let $v_0 \in \mathcal{D}(D)$, $v_i \in \mathcal{D}(D, C_{\#}^{\infty}(Y_1, \dots, C_{\#}^{\infty}(Y_i) \dots))^3$ and $\mathbf{v}_i \in \mathcal{D}(D, C_{\#}^{\infty}(Y_1, \dots, C_{\#}^{\infty}(Y_i) \dots))$ for $i = 1, \dots, n$. Choosing a test function of the form

$$\phi(t, x) = \left(v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \varepsilon_i \nabla \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) q(t)$$

we obtain

$$\begin{aligned} & \int_0^T \int_D b \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) u^{\varepsilon}(t, x) \cdot \left(v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right. \\ & \quad \left. + \sum_{i=1}^n \left(\varepsilon_i \nabla_x \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j} \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \right) \frac{d^2 q}{dt^2}(t) dx dt \\ & + \int_0^T \int_D a \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n} \right) \operatorname{curl} u^{\varepsilon}(t, x) \cdot \left(\operatorname{curl} v_0(x) \right. \\ & \quad \left. + \sum_{i=1}^n \left(\varepsilon_i \operatorname{curl}_x v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \operatorname{curl}_{y_j} v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) \right) q(t) dx dt \\ & = \int_0^T \int_D f(t, x) \cdot \left(v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \varepsilon_i \nabla_x \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right. \\ & \quad \left. + \sum_{i=1}^n \sum_{j=1}^i \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j} \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) \right) q(t) dx dt \end{aligned}$$

Passing to the multiscale limit, using the scale separation (2.2), we have

$$\begin{aligned}
& \int_0^T \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \\
& \quad \cdot \left(v_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, \mathbf{y}_i) \right) \frac{d^2 q}{dt^2}(t) d\mathbf{y} dx dt \\
& + \int_0^T \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0(t, x) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t, x, \mathbf{y}_i) \right) \\
& \quad \cdot \left(\operatorname{curl} v_0(x) + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i(x, \mathbf{y}_i) \right) q(t) d\mathbf{y} dx dt \\
& = \int_0^T \int_D \int_{\mathbf{Y}} f(t, x) \cdot \left(v_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, \mathbf{y}_i) \right) q(t) d\mathbf{y} dx dt \\
& = \int_0^T \int_D f(t, x) \cdot v_0(x) q(t) dx dt.
\end{aligned}$$

Using a density argument, we find that this equation holds for all $(v_0, \{v_i\}, \{\mathbf{v}_i\}) \in \mathbf{V}$. The conclusion then follows. $\square$

We now establish the initial conditions which understood in the generalized sense.

Proposition 3.1.5. *We have $u_0 \in H^1(0, T; H)$, $\nabla_{y_i} \mathbf{u}_i \in H^1(0, T; H_i)$ for all $i = 1, \dots, n$. Further*

$$u_0(0, \cdot) = g_0, \quad \nabla_{y_i} \mathbf{u}_i(0, \cdot, \cdot) = 0, \quad (3.10)$$

$$\frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \Big|_{t=0} = \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) d\mathbf{y}, \quad \text{in } W' \quad (3.11)$$

and for $j = 1, \dots, n$

$$\frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \Big|_{t=0} = b(x, \mathbf{y}) g_1(x), \quad \text{in } V'_j. \quad (3.12)$$

Proof As u^ε is bounded in $H^1(0, T; H)$, u_0 belongs to $H^1(0, T; H) \subset C([0, T]; H)$ and is the weak limit of u^ε in $H^1(0, T; H)$. Let $\phi \in C^\infty([0, T] \times D)$ with $\phi = 0$

when $t = T$. We have

$$\begin{aligned} \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi(t, x) dx dt &= \int_0^T \int_D \left(\frac{\partial}{\partial t} (u^\varepsilon(t, x) \cdot \phi(t, x)) - u^\varepsilon(t, x) \cdot \frac{\partial \phi}{\partial t}(t, x) \right) dx dt \\ &= - \int_D u^\varepsilon(0, x) \cdot \phi(0, x) dx - \int_0^T \int_D u^\varepsilon(t, x) \cdot \frac{\partial \phi}{\partial t}(t, x) dx dt \\ &\rightarrow - \int_D g_0(x) \cdot \phi(0, x) dx - \int_0^T \int_D u_0(t, x) \cdot \frac{\partial \phi}{\partial t}(t, x) dx dt. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi(t, x) dx dt &\rightarrow \int_0^T \int_D \frac{\partial u_0}{\partial t}(t, x) \cdot \phi(t, x) dx dt \\ &= - \int_D u_0(0, x) \cdot \phi(0, x) dx - \int_0^T \int_D u_0(t, x) \cdot \frac{\partial \phi}{\partial t}(t, x) dx dt. \end{aligned}$$

Thus $u_0(0, x) = g_0$.

As $\{\frac{\partial u^\varepsilon}{\partial t}\}_\varepsilon$ is bounded in $L^2(0, T; H)$ so there is a subsequence that $(n+1)$ -scale converges. Let $\xi \in L^2(0, T; L^2(D \times \mathbf{Y}))^3$ be the $(n+1)$ -scale limit. Let $\phi(t, x, y_1) \in C_0^\infty(0, T, C_0^\infty(D, C_\#^\infty(Y_1)))^3$. We have that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi\left(t, x, \frac{x}{\varepsilon_1}\right) dx dt = \int_0^T \int_D \int_{\mathbf{Y}} \xi(t, x, \mathbf{y}) \cdot \phi(t, x, y_1) d\mathbf{y} dx dt.$$

On the other hand

$$\int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \cdot \phi\left(t, x, \frac{x}{\varepsilon_1}\right) dx dt = - \int_0^T \int_D u^\varepsilon \cdot \frac{\partial \phi}{\partial t}\left(t, x, \frac{x}{\varepsilon_1}\right) dx dt$$

which converges to

$$\begin{aligned} - \int_0^T \int_D \int_{\mathbf{Y}} \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \frac{\partial \phi}{\partial t}(t, x, y_1) d\mathbf{y} dx dt \\ = - \int_0^T \int_D \int_{Y_1} (u_0 + \nabla_{y_1} \mathbf{u}_1) \cdot \frac{\partial \phi}{\partial t}(t, x, y_1) dy_1 dx dt. \end{aligned}$$

Thus

$$\int_{Y_2} \dots \int_{Y_n} \xi(t, x, \mathbf{y}) dy_n \dots dy_2 = \frac{\partial}{\partial t} (u_0 + \nabla_{y_1} \mathbf{u}_1)$$

so

$$\frac{\partial}{\partial t} \nabla_{y_1} \mathbf{u}_1 = \int_{Y_2} \dots \int_{Y_n} \xi(t, x, \mathbf{y}) dy_n \dots dy_2 - \frac{\partial u_0}{\partial t} \in H_1.$$

Similarly, using a function $\phi(t, x, y_1, y_2) \in C_0^\infty(0, T, C_0^\infty(D, C_\#^\infty(Y_1, C_\#^\infty(Y_2))))^3$, we have

$$\frac{\partial}{\partial t} \nabla_{y_2} \mathbf{u}_2 = \int_{Y_3} \dots \int_{Y_n} \xi(t, x, \mathbf{y}) dy_n \dots dy_3 - \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial t} \nabla_{y_1} \mathbf{u}_1 \in H_2.$$

Continuing this process, we have that for all $i = 1, \dots, n$

$$\frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \in H_i.$$

Finally, we have

$$\xi(t, x, \mathbf{y}) = \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i.$$

Let $q \in C^\infty([0, T])$ with $q(T) = 0$. Let

$$\phi(x) = v_0(x) + \sum_{i=1}^n \varepsilon_i v_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right) + \sum_{i=1}^n \varepsilon_i \nabla \mathbf{v}_i \left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i} \right).$$

We have

$$\begin{aligned} \int_0^T \left\langle b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right\rangle_H q(t) dt &= \int_0^T \frac{\partial}{\partial t} \left(\left\langle b^\varepsilon \frac{\partial u^\varepsilon}{\partial t}, \phi \right\rangle_H q(t) \right) dt - \int_0^T \left\langle b^\varepsilon \frac{\partial u^\varepsilon}{\partial t}, \phi \right\rangle_H \frac{dq(t)}{dt} dt \\ &= - \left\langle b^\varepsilon \frac{\partial u^\varepsilon}{\partial t}(0), \phi \right\rangle_H q(0) - \int_0^T \left\langle b^\varepsilon \frac{\partial u^\varepsilon}{\partial t}, \phi \right\rangle_H \frac{dq(t)}{dt} dt \\ &= - \langle b^\varepsilon g_1, \phi \rangle_H q(0) - \int_0^T \left\langle b^\varepsilon \frac{\partial u^\varepsilon}{\partial t}, \phi \right\rangle_H \frac{dq(t)}{dt} dt \\ &\rightarrow - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \left(v_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, \mathbf{y}_i) \right) q(0) d\mathbf{y} dx \\ &\quad - \int_0^T \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(x) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \\ &\quad \cdot \left(v_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, \mathbf{y}_i) \right) \frac{dq(t)}{dt} d\mathbf{y} dx dt \quad (3.13) \end{aligned}$$

when $\varepsilon \rightarrow 0$. On the other hand, let q_n be a sequence in $C_0^\infty(0, T)$ that converges to $q(t)$ in $L^2(0, T)$ when $n \rightarrow \infty$. As $b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}$ is bounded in $L^2(0, T; W')$ so there is a constant $c > 0$ such that

$$\left| \int_0^T \left\langle b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right\rangle_H q_n(t) dt - \int_0^T \left\langle b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right\rangle_H q(t) dt \right| \leq c \|q_n - q\|_{L^2(0, T)}.$$

As $q_n \in C_0^\infty(0, T)$, when $\varepsilon \rightarrow 0$,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right\rangle_H q_n(t) dt &= \int_0^T \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \right. \\
&\quad \cdot \left. \left(v_0(x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i(x, \mathbf{y}_i) \right) \right] \frac{d^2 q_n}{dt^2}(t) d\mathbf{y} dx dt \\
&= \int_0^T \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H q_n(t) dt \\
&\quad + \sum_{j=1}^n \int_0^T \left\langle \frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} q_n(t) dt.
\end{aligned}$$

Passing to the limit when $n \rightarrow \infty$, we have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2}, \phi \right\rangle_H q(t) dt &= \int_0^T \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H q(t) dt \\
&\quad + \sum_{j=1}^n \int_0^T \left\langle \frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} q(t) dt.
\end{aligned} \tag{3.14}$$

From (3.7), as

$$\frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \in L^2(0, T; W'),$$

$$\frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \in C([0, T]; W')$$

so the initial condition $\frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) (u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i)) d\mathbf{y}$ at $t = 0$ is well defined in W' . Similarly, from (3.8), the initial condition $\frac{\partial}{\partial t} b(x, \mathbf{y}) (u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i))$ at $t = 0$ is well defined in V'_j . The right hand side of (3.14)

can be written as

$$\begin{aligned}
& \int_0^T \frac{\partial}{\partial t} \left(\left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H q(t) \right) dt \\
& - \int_0^T \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H \frac{dq(t)}{dt} dt \\
& + \sum_{j=1}^n \int_0^T \frac{\partial}{\partial t} \left(\left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} q(t) \right) dt \\
& - \sum_{j=1}^n \int_0^T \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} \frac{dq(t)}{dt} dt \\
& = - \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) d\mathbf{y} \Big|_{t=0}, v_0 \right\rangle_H q(0) \\
& - \int_0^T \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, v_0 \right\rangle_H \frac{dq(t)}{dt} dt \\
& - \sum_{j=1}^n \left\langle \frac{\partial}{\partial t} \left(b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \Big|_{t=0} \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} q(0) \\
& - \sum_{j=1}^n \int_0^T \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{v}_j \right\rangle_{V'_j, V_j} \frac{dq(t)}{dt} dt. \quad (3.15)
\end{aligned}$$

Comparing (3.13) and (3.15), we have

$$\begin{aligned}
& \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \Big|_{t=0}, v_0 \right\rangle_H \\
& = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot v_0(x) d\mathbf{y} dx,
\end{aligned}$$

and

$$\left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \Big|_{t=0}, \mathbf{v}_j \right\rangle_{V'_j, V_j} = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot \nabla_{y_j} \mathbf{v}_j(x, \mathbf{y}_j) d\mathbf{y} dx.$$

for all $j = 1, \dots, n$.

□

We now show the uniqueness of a solution of the multiscale homogenized equation.

Proposition 3.1.6. *With the initial conditions (3.10), (3.11) and (3.12), problem (3.9) has a unique solution.*

Proof We show that when $f = 0$, $g_0 = 0$ and $g_1 = 0$, the solution of (3.9) is $u_0 = 0$, $u_i = 0$ and $\mathbf{u}_i = 0$ for all $i = 1, \dots, n$.

Following the procedure in [100] Theorem 19.1 for showing the uniqueness of solution of wave equations, fixing $t' \in (0, T)$, we define

$$\begin{aligned} w_0(t) &= \begin{cases} -\int_t^{t'} u_0(\sigma) d\sigma, & t < t', \\ 0, & t \geq t'; \end{cases} \\ \mathbf{w}_i(t) &= \begin{cases} -\int_t^{t'} \mathbf{u}_i(\sigma) d\sigma, & t < t', \\ 0, & t \geq t'; \end{cases} \\ w_i(t) &= \begin{cases} -\int_t^{t'} u_i(\sigma) d\sigma, & t < t', \\ 0, & t \geq t'. \end{cases} \end{aligned} \quad (3.16)$$

We have

$$\begin{aligned} & \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, w_0(t, \cdot) \right\rangle_H \\ &= \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, w_0(t, \cdot) \right\rangle_H \\ & \quad + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(t, x) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \cdot \frac{\partial w_0}{\partial t}(t, x) d\mathbf{y} dx. \end{aligned}$$

and for all $j = 1, \dots, n$

$$\begin{aligned} & \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{w}_j(t, \cdot, \cdot) \right\rangle_{V'_j, V_j} \\ &= \left\langle \frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{w}_j(t, \cdot, \cdot) \right\rangle_{V'_j, V_j} \\ & \quad + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(t, x) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \cdot \frac{\partial}{\partial t} \nabla_{y_j} \mathbf{w}_j(t, x, \mathbf{y}_j) d\mathbf{y} dx. \end{aligned}$$

Thus from (3.9)

$$\begin{aligned}
& \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y}, w_0(t, \cdot) \right\rangle_H \\
& + \sum_{j=1}^n \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right), \mathbf{w}_j(t, \cdot, \cdot) \right\rangle_{V'_j, V_j} \\
& = \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) d\mathbf{y}, w_0 \right\rangle_H \\
& + \sum_{j=1}^n \left\langle \frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right), \mathbf{w}_j \right\rangle_{V'_j, V_j} \\
& + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(\frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{w}_i \right) d\mathbf{y} dx \\
& = - \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0(t, x) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t, x, \mathbf{y}_i) \right) \\
& \quad \cdot \left(\operatorname{curl} w_0(t, x) + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} dx \\
& + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(\frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{w}_i \right) d\mathbf{y} dx.
\end{aligned}$$

Integrating over $(0, t')$, we get

$$\begin{aligned}
& \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \right|_{t=t'}, w_0(t', \cdot) \right\rangle_H \\
& + \sum_{j=1}^n \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \right|_{t=t'}, \mathbf{w}_j(t', \cdot, \cdot) \right\rangle_{V'_j, V_j} \\
& - \left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) d\mathbf{y} \right|_{t=0}, w_0(0, \cdot) \right\rangle_H \\
& - \sum_{j=1}^n \left\langle \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}_i) \right) \right|_{t=0}, \mathbf{w}_j(0, \cdot, \cdot) \right\rangle_{V'_j, V_j}
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{t'} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(\frac{\partial w_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{w}_i \right) d\mathbf{y} dx dt \\
&- \int_0^{t'} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) \cdot \left(\operatorname{curl} w_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i \right) d\mathbf{y} dx dt \\
&= \int_0^{t'} \frac{1}{2} \frac{d}{dt} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) d\mathbf{y} dx dt - \\
&\int_0^{t'} \frac{1}{2} \frac{d}{dt} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} w_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i \right) \cdot \left(\operatorname{curl} w_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i \right) d\mathbf{y} dx dt
\end{aligned}$$

due to $u_0 = \frac{\partial w_0}{\partial t}$, $u_i = \frac{\partial w_i}{\partial t}$ and $\mathbf{u}_i = \frac{\partial \mathbf{w}_i}{\partial t}$ for $i = 1, \dots, n$ from (3.16). Since $w_0(t') = 0, w_i(t') = 0$ and the conditions (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
0 &= \frac{1}{2} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t', x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t', x, \mathbf{y}_i) \right) \cdot \left(u_0(t', x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t', x, \mathbf{y}_i) \right) d\mathbf{y} dx \\
&+ \frac{1}{2} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} w_0(0, x) + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i(0, x, \mathbf{y}_i) \right) \\
&\quad \cdot \left(\operatorname{curl} w_0(0, x) + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i(0, x, \mathbf{y}_i) \right) d\mathbf{y} dx.
\end{aligned}$$

We thus deduce that $u_0(t') = 0, \nabla_{y_i} \mathbf{u}_i(t') = 0 \forall t', \operatorname{curl} w_0(0) = 0$ and $\operatorname{curl}_{y_i} w_i(0) = 0$. This means that

$$\int_0^{t'} \operatorname{curl} u_0(\sigma) d\sigma = 0 \quad \text{and} \quad \int_0^{t'} \operatorname{curl}_{y_i} u_i(\sigma) d\sigma = 0$$

for all t' . Thus for all σ , $\operatorname{curl} u_0(\sigma) = 0$ and $\operatorname{curl}_{y_i} u_i(\sigma) = 0$. $\square$

3.2 Homogenized equation

In this section, we use the multiscale homogenized problem (3.9) to deduce the homogenized equation. The main difficulty in comparison to elliptic equations or the wave equation consider in [101] is that we need to write the functions $\mathbf{u}_i$ in terms of the solution ω_i^r of the cell problems. As the second time derivative of $\nabla_{y_i} \mathbf{u}_i$ which is understood in the generalized sense, we need to do this via a smooth test function $q(t)$. Deriving the initial condition for $\frac{\partial u_0}{\partial t}$ is thus quite complicated. From (3.9), we have that for all $\mathbf{v}_n \in V_n$ and all $q \in \mathcal{D}(0, T)$

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\int_0^T \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) \frac{d^2 q}{dt^2}(t) dt + \nabla_{y_n} \int_0^T \mathbf{u}_n \frac{d^2 q}{dt^2}(t) dt \right) \cdot \nabla_{y_n} \mathbf{v}_n d\mathbf{y} dx = 0.$$

Thus

$$\int_0^T \mathbf{u}_n \frac{d^2 q}{dt^2}(t) dt = \left(\int_0^T \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k \frac{d^2 q}{dt^2}(t) dt \right) \omega_n^k$$

where $\omega_n^k \in L^2(D \times \mathbf{Y}_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ is the solution of the cell problem

$$\nabla_{y_n} \cdot (b(x, \mathbf{y}) (e^k + \nabla_{y_n} \omega_n^k)) = 0; \quad (3.17)$$

e^k is the k th unit vector in $\mathbb{R}^3$. Therefore

$$\int_0^T \left(\nabla_{y_n} \mathbf{u}_n - \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k \nabla_{y_n} \omega_n^k \right) \frac{d^2 q}{dt^2}(t) dt = 0$$

so

$$\int_0^T \left(\frac{\partial}{\partial t} \nabla_{y_n} \mathbf{u}_n - \frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k \nabla_{y_n} \omega_n^k \right) \frac{dq}{dt}(t) dt = 0.$$

From this we have for almost all $t \in [0, T]$

$$\frac{\partial}{\partial t} \nabla_{y_n} \mathbf{u}_n = \frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k \nabla_{y_n} \omega_n^k + G_n(x, \mathbf{y}_n) \quad (3.18)$$

for a function $G_n(x, \mathbf{y}_n)$ in $L^2(D \times \mathbf{Y}_n)$. We then have $\forall \mathbf{v}_{n-1} \in V_{n-1}$

$$\int_0^T \int_D \int_{\mathbf{Y}_{n-1}} b^{n-1}(x, \mathbf{y}_{n-1}) \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} \frac{d^2 q}{dt^2}(t) d\mathbf{y}_{n-1} dx dt = 0,$$

where $b^{n-1}(x, \mathbf{y}_{n-1})$ is the $(n-1)$ th level homogenized coefficient which is defined by

$$b_{ij}^{n-1}(x, \mathbf{y}_{n-1}) = \int_{Y_n} b_{kl}(x, \mathbf{y}) \left(\delta_{jl} + \frac{\partial \omega_n^j}{\partial y_l} \right) \left(\delta_{ik} + \frac{\partial \omega_n^i}{\partial y_k} \right) dy_n. \quad (3.19)$$

Similarly we have

$$\frac{\partial}{\partial t} \nabla_{y_{n-1}} \mathbf{u}_{n-1} = \frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-2} \nabla_{y_i} \mathbf{u}_i \right)_k \nabla_{y_{n-1}} \omega_{n-1}^k + G_{n-1}(x, \mathbf{y}_{n-1}),$$

where $G_{n-1}(x, \mathbf{y}_{n-1}) \in L^2(D \times \mathbf{Y}_{n-1})$, and $\omega_{n-1}^k \in L^2(D \times \mathbf{Y}_{n-2}, H_{\#}^1(Y_{n-1})/\mathbb{R})$ satisfies the cell problem

$$\nabla_{y_{n-1}} \cdot (b^{n-1}(x, \mathbf{y}_{n-1}) (e^k + \nabla_{y_{n-1}} \omega_{n-1}^k)) = 0.$$

Recursively, letting $b^n(x, \mathbf{y}_n) = b(x, \mathbf{y})$, we have for all $i = 0, \dots, n-1$:

$$\int_0^T \mathbf{u}_i \frac{d^2 q}{dt^2}(t) dt = \left(\int_0^T \left(u_0 + \sum_{j=1}^{i-1} \nabla_{y_j} \mathbf{u}_j \right)_k \frac{d^2 q}{dt^2}(t) dt \right) \omega_i^k,$$

where $\omega_i^k \in L^2(D \times \mathbf{Y}_{i-1}, H_{\#}^1(Y_i)/\mathbb{R})$ is the solution of the cell problem

$$\nabla_{y_i} \cdot (b^i(x, \mathbf{y}_i)(e^k + \nabla_{y_i} \omega_i^k)) = 0. \quad (3.20)$$

From an argument as above, we have

$$\frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i = \frac{\partial}{\partial t} \left(u_0 + \sum_{j=1}^{i-1} \nabla_{y_j} \mathbf{u}_j \right)_k \nabla_{y_i} \omega_i^k + G_i(x, \mathbf{y}_i) \quad (3.21)$$

for a function $G_i(x, \mathbf{y}_i) \in L^2(D \times \mathbf{Y}_i)$. The positive definite matrix function $b^0(x)$, which is defined as

$$b_{pq}^0(x) = \int_Y b_{kl}^1(x, y_1) \left(\delta_{ql} + \frac{\partial \omega_1^q}{\partial y_l} \right) \left(\delta_{pk} + \frac{\partial \omega_1^p}{\partial y_k} \right) dy_1 \quad (3.22)$$

is the homogenized coefficient. It satisfies

$$\begin{aligned} \int_0^T \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(v_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i \right) \frac{d^2 q}{dt^2}(t) d\mathbf{y} dx dt \\ = \int_0^T \int_D b^0(x) u_0 \cdot v_0 \frac{d^2 q}{dt^2}(t) dx dt \end{aligned}$$

for all $v_0 \in W$ and $\mathbf{v}_i \in V_i$.

Applying the same procedure as for calculating the homogenized coefficient $a^0(x)$ of stationary Maxwell equation in Section 2.2.2, we get the homogenized coefficient $a^0(x)$ as

$$a_{pq}^0(x) = \int_{Y_1} a_{pk}^1 (\delta_{kq} + (\text{curl}_{y_1} \chi_1^q)_k) dy_1 = \int_{Y_1} a_{kl}^1 (\delta_{kq} + (\text{curl}_{y_1} \chi_1^q)_k) (\delta_{lp} + (\text{curl}_{y_1} \chi_1^p)_l) dy_1.$$

The homogenized equation is

$$\begin{aligned} \int_0^T \int_D b^0(x) u_0(t, x) \cdot v_0(x) \frac{d^2 q}{dt^2}(t) dx dt + \int_0^T \int_D a^0(x) \text{curl } u_0(t, x) \cdot \text{curl } v_0(x) q(t) dx dt \\ = \int_0^T \int_D f(t, x) \cdot v_0(x) q(t) dx dt, \end{aligned}$$

i.e.,

$$b^0(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) + \operatorname{curl}(a^0(x) \operatorname{curl} u_0(t, x)) = f(t, x). \quad (3.23)$$

Now we derive the initial conditions. From (3.10), $u_0(0) = g_0$. From (3.8) as a map in V'_n ,

$$\frac{\partial^2}{\partial t^2} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) = 0,$$

so for all t

$$\frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) = \frac{\partial}{\partial t} b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \Big|_{t=0},$$

i.e.,

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(t) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t) \right) \cdot \nabla_{y_n} \mathbf{v}_n d\mathbf{y} dx = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1 \cdot \nabla_{y_n} \mathbf{v}_n d\mathbf{y} dx$$

for all t . From (3.18) we have

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1 \cdot \nabla_{y_n} \mathbf{v}_n(y_n) d\mathbf{y} dx \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k (e^k + \nabla_{y_n} \omega_n^k) + G_n(x, \mathbf{y}_n) \right) \cdot \nabla_{y_n} \mathbf{v}_n(y_n) d\mathbf{y} dx \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) G_n(x, \mathbf{y}_n) \cdot \nabla_{y_n} \mathbf{v}_n(y_n) d\mathbf{y} dx \end{aligned}$$

due to (3.17). Thus

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) (-g_1 + G_n(x, \mathbf{y}_n)) \cdot \nabla_{y_n} \mathbf{v}_n(y_n) d\mathbf{y} dx = 0.$$

From (3.18) we have

$$\operatorname{curl}_{y_n} G_n(x, \mathbf{y}_n) = 0, \quad \text{and} \quad \int_{Y_n} G_n(x, \mathbf{y}_n) dy_n = 0.$$

Thus there is a function $\tilde{G}_n \in L^2(D \times \mathbf{Y}_{n-1}, H_{\#}^1(Y_n)/\mathbb{R})$ such that $G_n(x, \mathbf{y}_n) = \nabla_{y_n} \tilde{G}_n(x, \mathbf{y}_n)$. From

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) (-g_1 + \nabla_{y_n} \tilde{G}_n(x, \mathbf{y})) \cdot \nabla_{y_n} \mathbf{v}(y_n) d\mathbf{y} dx = 0,$$

we deduce that

$$\tilde{G}_n(x, \mathbf{y}) = -g_{1k}\omega_n^k \quad \text{so} \quad G_n(x, \mathbf{y}) = -g_{1k}\nabla_{y_n}\omega_n^k \quad (3.24)$$

where ω_n^k is the solution of the cell problem (3.17). As a map in V'_{n-1} ,

$$\frac{\partial^2}{\partial t^2}b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) = 0$$

so

$$\frac{\partial}{\partial t}b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) = \frac{\partial}{\partial t}b(x, \mathbf{y}) \left(u_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i \right) \Big|_{t=0}.$$

From (3.12), we have

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1 \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx.$$

From (3.18) and (3.24) we have

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) (e^k + \nabla_{y_n} \omega_n^k) \left(\frac{\partial u_0}{\partial t} + \sum_{j=1}^{n-1} \frac{\partial}{\partial t} \nabla_{y_j} \mathbf{u}_j \right) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx \\ & - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_{1k} \nabla_{y_n} \omega_n^k \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx \\ & = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx. \end{aligned}$$

Thus

$$\begin{aligned} & \int_D \int_{\mathbf{Y}_{n-1}} b^{n-1}(x, \mathbf{y}_{n-1}) \left(\frac{\partial u_0}{\partial t} + \sum_{j=1}^{n-1} \frac{\partial}{\partial t} \nabla_{y_j} \mathbf{u}_j \right) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y}_{n-1} dx \\ & = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_{1k} (e^k + \nabla_{y_n} \omega_n^k) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y} dx \\ & = \int_D \int_{\mathbf{Y}_{n-1}} b^{n-1}(x, \mathbf{y}_{n-1}) g_1(x) \cdot \nabla_{y_{n-1}} \mathbf{v}_{n-1} d\mathbf{y}_{n-1} dx. \end{aligned}$$

Continuing this, we find that for $i = 1, \dots, n$

$$G_i(x, \mathbf{y}_i) = -g_{1k} \nabla_{y_i} \omega_i^k. \quad (3.25)$$

Thus for almost all $t \in [0, T]$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(t, x) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}) \right) \cdot v_0 d\mathbf{y} dx \\ &= \int_D b^0 \frac{\partial u_0}{\partial t}(t, x) \cdot v_0(x) dx - \sum_{i=1}^n \int_D \int_{\mathbf{Y}} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_i^k g_{1k} \cdot v_0 d\mathbf{y} dx. \end{aligned}$$

Since $\frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) (u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y})) d\mathbf{y} \in C([0, T], W')$ and $b^0 \frac{\partial u_0}{\partial t} \in C([0, T], W')$, this holds (in term of the dual pairing $\langle \cdot, \cdot \rangle_{W', W}$) for all $t \in [0, T]$. Therefore for $q \in C^\infty([0, T])$ with $q(T) = 0$, we have

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial^2}{\partial t^2} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}) \right) d\mathbf{y}, v_0 \right\rangle_H q(t) dt \\ &= \int_0^T \frac{\partial}{\partial t} \left(\left\langle \frac{\partial}{\partial t} \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(u_0(t, x) + \sum_{i=1}^n \nabla_{y_i} \mathbf{u}_i(t, x, \mathbf{y}) \right) d\mathbf{y}, v_0 \right\rangle_H q(t) \right) dt \\ &\quad - \int_0^T \left\langle \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right), v_0 \right\rangle_H \frac{dq}{dt}(t) dt \\ &= - \left\langle b^0 \frac{\partial u_0}{\partial t} \Big|_{t=0}, v_0 \right\rangle_H q(0) + \sum_{i=1}^n \int_D \int_{\mathbf{Y}} b^i(x, \mathbf{y}_i) \nabla \omega_i^k g_{1k} \cdot v_0 d\mathbf{y} dx q(0) \\ &\quad - \int_0^T \left\langle \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right), v_0 \right\rangle_H \frac{dq}{dt}(t) dt. \end{aligned}$$

From (3.13) and also (3.14), we deduce

$$\begin{aligned} & - \left\langle b^0 \frac{\partial u_0}{\partial t} \Big|_{t=0}, v_0 \right\rangle_H q(0) + \sum_{i=1}^n \int_D \int_{\mathbf{Y}} b^i(x, \mathbf{y}_i) \nabla \omega_i^k g_{1k} \cdot v_0 d\mathbf{y} dx q(0) \\ &= - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot v_0(x) d\mathbf{y} dx q(0). \end{aligned}$$

We then have

$$\begin{aligned} \left\langle b^0 \frac{\partial u_0}{\partial t} \Big|_{t=0}, v_0 \right\rangle_H &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot v_0(x) d\mathbf{y} dx \\ &\quad + \sum_{i=1}^n \int_D \int_{\mathbf{Y}} b^i(x, \mathbf{y}_i) \nabla \omega_i^k g_{1k} \cdot v_0 d\mathbf{y} dx. \end{aligned}$$

We note that

$$\begin{aligned}
& \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot v_0(x) d\mathbf{y} dx + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \nabla_{y_n} \omega_n^k g_{1k} \cdot v_0 d\mathbf{y} dx \\
&= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) (e^k + \nabla_{y_n} \omega_n^k) g_{1k}(x) \cdot v_0 d\mathbf{y} dx \\
&= \int_D \int_{\mathbf{Y}_{n-1}} b^{n-1}(x, \mathbf{y}_{n-1}) g_1 \cdot v_0 d\mathbf{y}_{n-1} dx.
\end{aligned}$$

Continuing this we have

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1(x) \cdot v_0(x) d\mathbf{y} dx + \sum_{i=1}^n \int_D \int_{\mathbf{Y}} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_i^k g_{1k} d\mathbf{y} dx = \int_D b^0(x) g_1 \cdot v_0 dx$$

Thus as distribution in W'

$$b^0 \frac{\partial u_0}{\partial t} \Big|_{t=0} = b^0 g_1, \quad (3.26)$$

i.e., u_0 is the solution of the problem (3.23) with the initial conditions $u_0(0) = g_0$ and (3.26) which has a unique solution.

From (2.27), the solution $\mathbf{u}$ is written in terms of u_0 as

$$u_i = \chi_i^{r_{i-1}} (\delta_{r_{i-1}r_{i-2}} + \text{curl}_{y_{i-1}} \chi_{i-1}^{r_{i-2}}) \dots (\delta_{r_1r_0} + \text{curl}_{y_1} \chi_1^{r_0}) (\text{curl } u_0)_{r_0},$$

and from (3.21) and (3.25)

$$\begin{aligned}
\frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i &= \left(\frac{\partial u_{0r_0}}{\partial t} - g_{1r_0} \right) \left(\delta_{r_0r_1} + \frac{\partial \omega_1^{r_0}}{\partial y_{1r_1}} \right) \\
&\quad \left(\delta_{r_1r_2} + \frac{\partial \omega_2^{r_1}}{\partial y_{2r_2}} \right) \dots \left(\delta_{r_{i-2}r_{i-1}} + \frac{\partial \omega_{i-1}^{r_{i-2}}}{\partial y_{i-1,r_{i-1}}} \right) \nabla_{y_i} \omega_i^{r_{i-1}}.
\end{aligned}$$

Given that at $t = 0$, $\nabla_{y_i} \mathbf{u}_i = 0$, we then have

$$\nabla_{y_i} \mathbf{u}_i = (u_{0r_0} - g_{1r_0} t - g_{0r_0}) \left(\delta_{r_0r_1} + \frac{\partial \omega_1^{r_0}}{\partial y_{1r_1}} \right) \left(\delta_{r_1r_2} + \frac{\partial \omega_2^{r_1}}{\partial y_{2r_2}} \right) \dots \nabla_{y_i} \omega_i^{r_{i-1}}. \quad (3.27)$$

Without loss of generality, we let

$$\mathbf{u}_i = (u_{0r_0} - g_{1r_0} t - g_{0r_0}) \left(\delta_{r_0r_1} + \frac{\partial \omega_1^{r_0}}{\partial y_{1r_1}} \right) \left(\delta_{r_1r_2} + \frac{\partial \omega_2^{r_1}}{\partial y_{2r_2}} \right) \dots \omega_i^{r_{i-1}}. \quad (3.28)$$

3.3 Regularity of the solution

To establish the homogenization error and to have explicit error estimates for the full and sparse tensor finite element approximating problems in Chapter 5, we now establish the regularity of u_0 and $\nabla_{y_i} \mathbf{u}_i$ with respect to t . We make the following assumption on the smoothness of the matrix functions $a(x, \mathbf{y})$ and $b(x, \mathbf{y})$ as in Proposition 2.3.7.

Assumption 3.3.1. *The matrix functions a and b belong to*

$$C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{3 \times 3}.$$

To show the regularity for the solution u_0 of the Maxwell wave equation, we employ the standard procedure for equations with compatibility initial conditions, as outlined in Wloka [100].

Proposition 3.3.2. *Under Assumption 3.3.1, assume*

$$\begin{cases} f \in H^2(0, T; H), \\ g_1 \in W, \\ (b^0(x))^{-1} [f(0) - \operatorname{curl}(a^0(x) \operatorname{curl} g_0)] \in W, \\ (b^0(x))^{-1} \left[\frac{\partial f}{\partial t}(0) - \operatorname{curl}(a^0(x) \operatorname{curl} g_1) \right] \in H, \end{cases} \quad (3.29)$$

then

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; W), \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H), \quad \text{and} \quad \frac{\partial^3}{\partial t^3} \nabla_{y_i} \mathbf{u}_i \in L^\infty(0, T; H_i).$$

Further, if

$$\begin{cases} f \in H^3(0, T; H), \\ g_1 \in W, \\ (b^0(x))^{-1} [f(0) - \operatorname{curl}(a^0(x) \operatorname{curl} g_0)] \in W, \\ (b^0(x))^{-1} \left[\frac{\partial f}{\partial t}(0) - \operatorname{curl}(a^0(x) \operatorname{curl} g_1) \right] \in W, \\ (b^0(x))^{-1} \left[\frac{\partial^2 f}{\partial t^2}(0) - \operatorname{curl}(a^0(x) \operatorname{curl} ((b^0(x))^{-1} (f(0) - \operatorname{curl}(a^0(x) \operatorname{curl} g_0)))) \right] \in H, \end{cases} \quad (3.30)$$

then

$$\frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; W), \quad \frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0, T; H) \quad \text{and} \quad \frac{\partial^4}{\partial t^4} \nabla_{y_i} \mathbf{u}_i \in L^\infty(0, T; H_i). \quad (3.31)$$

Proof We use the regularity theory of general hyperbolic equations (see, e.g., Wloka [100], Chapter 5). From (3.29) we have that

$$b^0 \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial t} \right) + \operatorname{curl} \left(a^0 \operatorname{curl} \frac{\partial u_0}{\partial t} \right) = \frac{\partial f}{\partial t}$$

with compatibility initial conditions

$$\frac{\partial u_0}{\partial t}(0) = g_1 \in W, \quad \frac{\partial}{\partial t} \frac{\partial u_0}{\partial t}(0) = (b^0)^{-1} [f(0) - \operatorname{curl} (a^0 \operatorname{curl} g_0)] \in W$$

and

$$b^0 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u_0}{\partial t^2} \right) + \operatorname{curl} \left(a^0 \operatorname{curl} \frac{\partial^2 u_0}{\partial t^2} \right) = \frac{\partial^2 f}{\partial t^2}, \quad (3.32)$$

with compatibility initial conditions

$$\frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1} [f(0) - \operatorname{curl} (a^0 \operatorname{curl} g_0)] \in W$$

and

$$\frac{\partial}{\partial t} \frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1} \left[\frac{\partial f}{\partial t}(0) - \operatorname{curl} (a^0 \operatorname{curl} g_1) \right] \in H.$$

We thus deduce that

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; W) \quad \text{and} \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H).$$

From (3.27) and the Proposition 2.3.7, we deduce that

$$\frac{\partial^3}{\partial t^3} \nabla_{y_i} \mathbf{u}_i \in L^\infty(0, T; H_i).$$

Similarly, we deduce regularity (3.31) from (3.30). $\square$

For the regularity of u_0 , we have the following result.

Proposition 3.3.3. *Under Assumption 3.3.1, if D is a Lipschitz polygonal domain, $f \in H^1(0, T; H)$, $g_0 \in H^1(\operatorname{curl}, D)$ and $g_1 \in W$, $\operatorname{div} f \in L^\infty(0, T; L^2(D))$, $\operatorname{div}(b^0 g_0) \in L^2(D)$ and $\operatorname{div}(b^0 g_1) \in L^2(D)$, there is a constant $s \in (0, 1]$ such that $u_0 \in L^\infty(0, T; H^s(\operatorname{curl}, D))$.*

Proof Using Proposition 2.3.7, (2.15) and (3.22), we have that $a^0, b^0 \in C^1(\bar{D})^{3 \times 3}$. As $f \in H^1(0, T; H)$ and $g_0 \in H^1(\operatorname{curl}, D)$, we have that $(b^0)^{-1} [f(0) - \operatorname{curl} (a^0 \operatorname{curl} g^0)] \in H$. The compatibility initial conditions hold so that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H)$. Thus

$$\operatorname{curl} (a^0 \operatorname{curl} u_0) = f - b^0 \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H).$$

Let $U(t) = a^0 \operatorname{curl} u_0(t)$. As $\operatorname{div}((a^0)^{-1}U(t)) = 0$ and $(a^0)^{-1}U(t) \cdot \nu = 0$, from Theorem 4.1 of Hiptmair [52], there is a constant c and a constant $s \in (0, 1]$ which depend on a^0 and the domain D so that

$$\|U(t)\|_{H^s(D)^3} \leq c(\|\operatorname{curl} U(t)\|_H + \|U(t)\|_H)$$

so $U \in L^\infty(0, T; H^s(D))$. As $\operatorname{curl} u_0(t) = (a^0)^{-1}U(t)$ and $(a^0)^{-1} \in C^1(\bar{D})^{3 \times 3}$, $\operatorname{curl} u_0 \in L^\infty(0, T; H^s(D))$. We note that

$$\operatorname{div} \left(b^0 \frac{\partial^2 u_0}{\partial t^2} \right) = \operatorname{div} f,$$

so

$$\operatorname{div}(b^0 u_0(t)) = \int_0^t \int_0^{t'} \operatorname{div} f(\xi) d\xi dt' + t \operatorname{div}(b^0 g_1) + \operatorname{div}(b^0 g_0) \in L^\infty(0, T; L^2(D)).$$

From Theorem 4.1 of Hiptmair [52], we deduce that there is a constant $s \in (0, 1]$ (we take it as the same constant above), so that

$$\|u_0(t)\|_{H^s(D)^3} \leq c(\|u_0(t)\|_{H(\operatorname{curl}, D)} + \|\operatorname{div}(b^0 u_0(t))\|_{L^2(D)}).$$

Thus $u_0 \in L^\infty(0, T; H^s(\operatorname{curl}, D))$. □

Similarly, we can deduce the regularity for $\frac{\partial^2 u_0}{\partial t^2}$.

Proposition 3.3.4. *Under Assumption 3.3.1, if D is a Lipschitz polygonal domain, the compatibility conditions (3.30) hold, $\operatorname{div} f \in L^\infty(0, T; L^2(D))$, there is a constant $s \in (0, 1]$ such that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(\operatorname{curl}, D))$.*

Proof From equation (3.32), we have

$$\operatorname{curl} \left(a^0 \operatorname{curl} \frac{\partial^2 u_0}{\partial t^2} \right) = \frac{\partial^2 f}{\partial t^2} - b^0 \frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0, T; H)$$

as $\frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0, T; H)$ due to (3.30). Following a similar argument as in the proof of Proposition 3.3.3 we deduce that $\operatorname{curl} \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(D)^3)$. We note that

$$\operatorname{div} b^0 \frac{\partial^2 u_0}{\partial t^2} = \operatorname{div} f \in L^\infty(0, T; L^2(D)).$$

From Theorem 4.1 of [52], we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(D))$. □

Remark 3.3.5. *We have*

$$\operatorname{curl} \frac{\partial u_0(t)}{\partial t} = \int_0^t \operatorname{curl} \frac{\partial^2 u_0}{\partial t^2}(t') dt' + \operatorname{curl} g_1, \quad \text{and} \quad \frac{\partial u_0(t)}{\partial t} = \int_0^t \frac{\partial^2 u_0}{\partial t^2}(t') dt' + g_1,$$

$$\operatorname{curl} u_0(t) = \int_0^t \int_0^{t'} \operatorname{curl} \frac{\partial^2 u_0}{\partial t^2}(r) dr dt' + t \operatorname{curl} g_1 + \operatorname{curl} g_0,$$

and

$$u_0(t) = \int_0^t \int_0^s \frac{\partial^2 u_0}{\partial t^2}(r) dr ds + t g_1 + g_0.$$

Thus with the hypothesis of Proposition 3.3.4, together with $g_0 \in H^s(\operatorname{curl}, D)$ and $g_1 \in H^s(\operatorname{curl}, D)$, we deduce that $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^s(\operatorname{curl}, D))$ and $u_0 \in L^\infty(0, T; H^s(\operatorname{curl}, D))$.

3.4 Correctors for the homogenization problems

As shown in Brahim-Otsmane et al. [26], the energy of the multiscale Maxwell wave equation does not converge to the energy of the homogenized equation in the general case, a corrector result similar to that for the time independent case in terms of u_0 and u_1 does not exist in general. We therefore restrict our consideration to the case where the initial condition $g_0 = 0$ to derive correctors for multiscale Maxwell problems in this section. We first consider the two-scale problems for which an explicit error in terms of the microscopic scale ε exists. For multiscale problems, we derive correctors without an error estimate.

3.4.1 Corrector for two-scale problems

For two-scale problems, with sufficient regularity for the solution u_0 of the homogenized equation, we can derive explicit homogenization error. For conciseness, we denote the coefficients $a(x, \mathbf{y})$ and $b(x, \mathbf{y})$ as $a(x, y)$ and $b(x, y)$. The cell problems become

$$\operatorname{curl}_y(a(x, y)(e^r + \operatorname{curl}_y \chi^r(x, y))) = 0,$$

and

$$\nabla_y \cdot (b(x, y)(e^r + \nabla_y \omega^r(x, y))) = 0.$$

The homogenized coefficients are determined by

$$a_{pq}^0(x) = \int_Y a_{pk}(x, y)(\delta_{kq} + (\operatorname{curl}_y \chi^q)_k) dy$$

and

$$b_{pq}^0(x) = \int_Y b_{pl}(x, y) \left(\delta_{ql} + \frac{\partial \omega^q}{\partial y_l} \right) dy.$$

We then have the following result of homogenization error.

Theorem 3.4.1. *For two-scale problems, assume that $g_0 = 0$, $g_1 \in H^1(D)^3 \cap W$, $f \in H^1(0, T; H)$, $u_0 \in L^\infty(0, T; H^1(\text{curl}; D))$, $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^1(\text{curl}, D))$, $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H(\text{curl}, D) \cap H^1(D)^3)$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\text{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$ and $\omega^r \in C^1(\bar{D}, C(\bar{Y}))$ for all $r = 1, 2, 3$. There exists a constant c that does not depend on ε such that*

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t} - \left[\frac{\partial u_0}{\partial t} + \nabla_y \frac{\partial \mathbf{u}_1}{\partial t} \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_{L^\infty(0, T; H)} \\ & + \left\| \text{curl } u^\varepsilon - \left[\text{curl } u_0 + \text{curl}_y u_1 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}. \end{aligned}$$

Proof We note that $\frac{\partial u^\varepsilon}{\partial t}$ satisfies

$$b^\varepsilon \frac{\partial^2}{\partial t^2} \left(\frac{\partial u^\varepsilon}{\partial t} \right) + \text{curl} \left(a^\varepsilon \text{curl} \frac{\partial u^\varepsilon}{\partial t} \right) = \frac{\partial f}{\partial t} \quad (3.33)$$

with the initial condition $\frac{\partial u^\varepsilon}{\partial t}(0) = g_1 \in W$ and $\frac{\partial^2 u^\varepsilon}{\partial t^2}(0) = f(0) \in H$ (due to $g_0 = 0$). We therefore deduce that

$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^\infty(0, T; W)} \leq c \left(\left\| \frac{\partial f}{\partial t} \right\|_{L^2(0, T; H)} + \|g_1\|_W + \|f(0)\|_H \right)$$

where c only depends on the constant α and β in (2.1). Thus $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty(0, T; W)$ for all ε .

We consider the function

$$u_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) (\text{curl } u_0(t, x))_r + \varepsilon \nabla \mathbf{u}_1 \left(t, x, \frac{x}{\varepsilon} \right)$$

where we have from (3.28)

$$\mathbf{u}_1 \left(t, x, \frac{x}{\varepsilon} \right) = \omega^r \left(x, \frac{x}{\varepsilon} \right) (u_{0r}(t, x) - g_{1r}(x)t - g_{0r}(x)).$$

We first show that

$$\left\| \text{curl} (a^\varepsilon \text{curl } u_1^\varepsilon) - \text{curl} (a^0 \text{curl } u_0) \right\|_{L^\infty(0, T; W')} \leq c\varepsilon.$$

We have

$$\begin{aligned} \operatorname{curl} u_1^\varepsilon(t, x) &= \operatorname{curl} u_0(t, x) + \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r \\ &\quad + \varepsilon \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r. \end{aligned} \quad (3.34)$$

Thus

$$\begin{aligned} a^\varepsilon(x) \operatorname{curl} u_1^\varepsilon(t, x) &- a^0(x) \operatorname{curl} u_0(t, x) \\ &= a^\varepsilon(x) \operatorname{curl} u_0(t, x) + \varepsilon a^\varepsilon(x) \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r \\ &\quad + \varepsilon a^\varepsilon(x) \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ &\quad + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r - a^0(x) \operatorname{curl} u_0(t, x) \\ &= a^\varepsilon(x) \operatorname{curl} u_0(t, x) + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r - a^0(x) \operatorname{curl} u_0(t, x) \\ &\quad + \varepsilon a^\varepsilon(x) \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r \\ &\quad + \varepsilon a^\varepsilon(x) \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ &= \operatorname{curl} G_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r + \varepsilon \operatorname{curl} I(t, x) \end{aligned}$$

where the vector functions $G_r(x, y)$ are defined in (2.17) and

$$I(t, x) = a \left(x, \frac{x}{\varepsilon} \right) \left[\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r + \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right]$$

Using the function $\tilde{G}_r$ as in (2.19), for all $\phi \in \mathcal{D}(D)^3$ we have

$$\begin{aligned} \langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon)(t) - \operatorname{curl} a^0 \operatorname{curl} u_0(t), \phi \rangle_H &= \int_D G_r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r \cdot \operatorname{curl} \phi(x) dx \\ &\quad + \varepsilon \int_D I(t, x) \cdot \operatorname{curl} \phi(x) dx \\ &= -\varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}((\operatorname{curl} u_0(t, x))_r \operatorname{curl} \phi(x)) dx \\ &\quad - \varepsilon \int_D \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \cdot [(\operatorname{curl} u_0(t, x))_r \operatorname{curl} \phi(x)] dx \\ &\quad + \varepsilon \int_D I(t, x) \cdot \operatorname{curl} \phi(x) dx. \end{aligned}$$

We note that

$$\begin{aligned} \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}((\operatorname{curl} u_0(t, x))_r \operatorname{curl} \phi(x)) dx \\ = \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \nabla(\operatorname{curl} u_0(t, x))_r \cdot \operatorname{curl} \phi(x) dx. \end{aligned}$$

As $\chi^r \in C^1(\bar{D}, C(\bar{Y})^3)$ and $\operatorname{curl} u_0 \in L^\infty(0, T; H^1(D)^3)$ we deduce that

$$\left| \int_D I(t, x) \cdot \operatorname{curl} \phi(x) dx \right| \leq c \|\operatorname{curl} \phi\|_H$$

where c is independent of t . From these we conclude that

$$|\langle \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon)(t) - \operatorname{curl}(a^0 \operatorname{curl} u_0)(t), \phi \rangle_H| \leq c\varepsilon \|\operatorname{curl} \phi\|_H.$$

Using a density argument, we have that this holds for all $\phi \in W$. Thus

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) - \operatorname{curl}(a^0 \operatorname{curl} u_0)\|_{L^\infty(0, T; W')} \leq c\varepsilon. \quad (3.35)$$

Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x \in D} \varepsilon |\nabla \tau^\varepsilon(x)| < c$ where c is independent of ε . Let

$$w_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r + \varepsilon \nabla \left[\tau^\varepsilon(x) \mathbf{u}_1 \left(t, x, \frac{x}{\varepsilon} \right) \right]$$

where $\mathbf{u}_1(t, x, \frac{x}{\varepsilon}) = \omega^r(x, \frac{x}{\varepsilon}) (u_{0r}(t, x) - g_{1r}(x)t - g_{0r}(x))$. The function $w_1^\varepsilon(t, x)$ belongs to $L^2(0, T; W)$. We note that

$$\begin{aligned} u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x) &= \varepsilon(1 - \tau^\varepsilon(x)) \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r \\ &\quad + \varepsilon \nabla \left[(1 - \tau^\varepsilon(x)) \omega^r \left(x, \frac{x}{\varepsilon} \right) (u_{0r}(t, x) - g_{1r}(x)t - g_{0r}(x)) \right]. \end{aligned}$$

From this,

$$\begin{aligned} \operatorname{curl}(u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x)) &= \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r (1 - \tau^\varepsilon(x)) \\ &\quad + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r (1 - \tau^\varepsilon(x)) \\ &\quad - \varepsilon (\operatorname{curl} u_0(t, x))_r \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ &\quad + \varepsilon (1 - \tau^\varepsilon(x)) \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \end{aligned} \quad (3.36)$$

and

$$\begin{aligned}
& \operatorname{curl} (a^\varepsilon(x) \operatorname{curl} (u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x))) \\
&= \varepsilon \operatorname{curl} \left(a^\varepsilon(x) \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r (1 - \tau^\varepsilon(x)) \right) \\
&+ \operatorname{curl} \left(a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(t, x))_r (1 - \tau^\varepsilon(x)) \right) \\
&- \varepsilon \operatorname{curl} \left(a^\varepsilon(x) (\operatorname{curl} u_0(t, x))_r \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \\
&+ \varepsilon \operatorname{curl} \left(a^\varepsilon(x) (1 - \tau^\varepsilon(x)) \nabla (\operatorname{curl} u_0(t, x))_r \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right). \tag{3.37}
\end{aligned}$$

Since $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^1(D)^3)$, we have

$$\begin{aligned}
\frac{\partial^2 w_1^\varepsilon}{\partial t^2}(t, x) &= \frac{\partial^2 u_0}{\partial t^2}(t, x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} (\operatorname{curl} u_0(t, x))_r \\
&+ \varepsilon \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ \varepsilon \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ \tau^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ \varepsilon \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} \nabla u_{0r}(t, x).
\end{aligned}$$

Thus

$$\begin{aligned}
b^\varepsilon(x) \frac{\partial^2 w_1^\varepsilon}{\partial t^2}(t, x) &= b^\varepsilon(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) + \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} (\operatorname{curl} u_0(t, x))_r \\
&+ \varepsilon b^\varepsilon(x) \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ b^\varepsilon(x) \tau^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&+ \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} \nabla u_{0r}(t, x).
\end{aligned}$$

Therefore

$$\begin{aligned}
& b^0(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x) \frac{\partial^2 w_1^\varepsilon}{\partial t^2}(t, x) \\
&= b^0(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} (\operatorname{curl} u_0(t, x))_r \\
&\quad - \varepsilon b^\varepsilon(x) \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - b^\varepsilon(x) \tau^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} \nabla u_{0r}(t, x) \\
&= b^0(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} (\operatorname{curl} u_0(t, x))_r \\
&\quad - \varepsilon b^\varepsilon(x) \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad + b^\varepsilon(x) (1 - \tau^\varepsilon(x)) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}(t, x) \\
&\quad - \varepsilon b^\varepsilon(x) \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2}{\partial t^2} \nabla u_{0r}(t, x). \tag{3.38}
\end{aligned}$$

For $\phi \in W$, we have

$$\begin{aligned}
& \left\langle b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2}, \phi \right\rangle_H \\
&= \int_D \left(b_{ir}^0 \frac{\partial^2 u_{0r}}{\partial t^2} - b_{ir}^\varepsilon \frac{\partial^2 u_{0r}}{\partial t^2} - b_{ij}^\varepsilon \frac{\partial \omega^r}{\partial y_j} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2} \right) \phi_i dx \\
&= \int_D \left(b_{ir}^0 - b_{ir}^\varepsilon - b_{ij}^\varepsilon \frac{\partial \omega^r}{\partial y_j} \left(x, \frac{x}{\varepsilon} \right) \right) \frac{\partial^2 u_{0r}}{\partial t^2} \phi_i dx \\
&= - \int_D \left(g_r \left(x, \frac{x}{\varepsilon} \right) \right)_i \frac{\partial^2 u_{0r}}{\partial t^2} \phi_i dx
\end{aligned}$$

where the vector function g_r is defined as in (2.18). We use the function $\tilde{g}_r(x, y)$

defined in (2.20) such that $g_r(x, y) = \text{curl}_y \tilde{g}_r(x, y)$. Therefore

$$\begin{aligned}
& \left| \int_D \left(g_r \left(x, \frac{x}{\varepsilon} \right) \right)_i \frac{\partial^2 u_{0r}}{\partial t^2} \phi_i dx \right| \\
&= \left| \int_D \left[\varepsilon \text{curl} \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \text{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \right] \frac{\partial^2 u_{0r}}{\partial t^2} \cdot \phi dx \right| \\
&= \left| \varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \text{curl} \left(\frac{\partial^2 u_{0r}}{\partial t^2} \phi \right) dx - \varepsilon \int_D \text{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2} \cdot \phi dx \right| \\
&= \left| \varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \left(\text{curl} \phi \frac{\partial^2 u_{0r}}{\partial t^2} + \nabla \frac{\partial^2 u_{0r}}{\partial t^2} \times \phi \right) dx - \varepsilon \int_D \text{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 u_{0r}}{\partial t^2} \cdot \phi dx \right| \\
&\leq c\varepsilon (\|\text{curl} \phi\|_H + \|\phi\|_H)
\end{aligned}$$

due to the condition $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^1(D)^3)$. Let $D^\varepsilon \subset D$ be the ε neighbourhood of the boundary ∂D . We know that

$$\|\phi\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2 \leq c\varepsilon \|\phi\|_{H^1(D)}^2$$

(see Hoang and Schwab [56]). From the condition $\text{curl} u_0 \in L^\infty(0, T; H^1(D)^3)$, $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^1(\text{curl}; D))$ and $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^1(D)^3)$ we deduce that:

$$\left\| \frac{\partial}{\partial t} (\text{curl} u_0(t))_r \right\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{\frac{1}{2}}, \quad \|(\text{curl} u_0(t))_r\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{\frac{1}{2}} \quad \text{and} \quad \left\| \frac{\partial^2 u_{0r}}{\partial t^2}(t) \right\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{\frac{1}{2}}. \quad (3.39)$$

From this and (3.38), we get

$$\left| \left\langle b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t^2}, \phi \right\rangle_H \right| \leq c\varepsilon \|\text{curl} \phi\|_H + c\varepsilon^{1/2} \|\phi\|_H. \quad (3.40)$$

Using

$$b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} + \text{curl} (a^\varepsilon \text{curl} u^\varepsilon) = b^0 \frac{\partial^2 u_0}{\partial t^2} + \text{curl} (a^0 \text{curl} u_0)$$

we deduce that

$$\begin{aligned}
b^\varepsilon \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t^2} + \text{curl} (a^\varepsilon \text{curl} (u^\varepsilon - w_1^\varepsilon)) &= b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial w_1^\varepsilon}{\partial t^2} \\
&+ \text{curl} (a^\varepsilon \text{curl} (u_1^\varepsilon - w_1^\varepsilon)) + \text{curl} (a^0 \text{curl} u_0) - \text{curl} (a^\varepsilon \text{curl} u_1^\varepsilon) \quad (3.41)
\end{aligned}$$

We note that

$$\begin{aligned}
\frac{\partial w_1^\varepsilon}{\partial t}(t, x) &= \frac{\partial u_0}{\partial t}(t, x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\text{curl} u_0(t, x))_r + \\
&\left(\varepsilon \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + \tau^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial}{\partial t} u_{0r}(t, x) - g_{1r}(x) \right) \\
&+ \varepsilon \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial}{\partial t} \nabla u_{0r}(t, x) - \nabla g_{1r}(x) \right)
\end{aligned}$$

and

$$\begin{aligned} \operatorname{curl} \frac{\partial w_1^\varepsilon}{\partial t}(t, x) &= \operatorname{curl} \frac{\partial u_0}{\partial t}(t, x) \\ &+ \tau^\varepsilon(x) \left(\varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \frac{\partial}{\partial t} (\operatorname{curl} u_0(t, x))_r \\ &+ \left(\varepsilon \nabla \tau^\varepsilon(x) \frac{\partial}{\partial t} (\operatorname{curl} u_0(t, x))_r + \varepsilon \tau^\varepsilon(x) \nabla \frac{\partial}{\partial t} (\operatorname{curl} u_0(t, x))_r \right) \times \chi^r \left(x, \frac{x}{\varepsilon} \right). \end{aligned}$$

As $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; W \cap H^1(\operatorname{curl}, D))$, we deduce that $\frac{\partial w_1^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty(0, T; W)$. Since $\frac{\partial u^\varepsilon}{\partial t}$ and $\frac{\partial w_1^\varepsilon}{\partial t}$ belong to W and are uniformly bounded in the norm of W , from (3.40), we deduce that

$$\begin{aligned} \int_0^t \left\langle b^0 \frac{\partial^2 u_0}{\partial t'^2} - b^\varepsilon \frac{\partial u^\varepsilon}{\partial t'^2}, \frac{\partial}{\partial t'} (u^\varepsilon(t') - w_1^\varepsilon(t')) \right\rangle_H dt' &\leq c\varepsilon \left\| \frac{\partial}{\partial t} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^\infty(0, T; W)} \\ &+ c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_H \leq c\varepsilon + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_H. \end{aligned}$$

From (3.35) we deduce

$$\begin{aligned} \left| \int_0^t \left\langle \operatorname{curl}(a^0 \operatorname{curl} u_0(t')) - \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon(t')), \frac{\partial}{\partial t'} (u^\varepsilon(t') - w_1^\varepsilon(t')) \right\rangle_H dt' \right| \\ \leq c\varepsilon \left\| \frac{\partial}{\partial t} (u^\varepsilon(t) - w_1^\varepsilon(t)) \right\|_{L^\infty(0, T; W)} \leq c\varepsilon \end{aligned}$$

where c is independent of t . From (3.37) we get

$$\begin{aligned} &\int_0^t \left\langle \operatorname{curl}(a^\varepsilon \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)), \frac{\partial}{\partial t'} (u^\varepsilon(t') - w_1^\varepsilon(t')) \right\rangle_H dt' \\ &\leq c\varepsilon + \int_0^t \int_D a^\varepsilon \left((1 - \tau^\varepsilon) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \\ &\quad \cdot (\operatorname{curl} u_0(t', x))_r \frac{\partial}{\partial t'} \operatorname{curl} (u^\varepsilon(t', x) - w_1^\varepsilon(t', x)) dx dt' \\ &\leq c\varepsilon + \int_0^t \int_D a^\varepsilon \left((1 - \tau^\varepsilon) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \\ &\quad \cdot \left[\frac{\partial}{\partial t'} ((\operatorname{curl} u_0(t', x))_r \operatorname{curl} (u^\varepsilon(t', x) - w_1^\varepsilon(t', x))) \right. \\ &\quad \left. - \operatorname{curl} (u^\varepsilon(t', x) - w_1^\varepsilon(t', x)) \frac{\partial}{\partial t'} (\operatorname{curl} u_0(t', x))_r \right] dx dt' \end{aligned}$$

Using the initial condition $u_0(0) = w_1^\varepsilon(0) = 0$, from (3.39), we deduce that

$$\begin{aligned} & \left| \int_0^t \left\langle \operatorname{curl}(a^\varepsilon \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)), \frac{\partial}{\partial t'}(u^\varepsilon(t') - w_1^\varepsilon(t')) \right\rangle_H dt' \right| \\ & \leq c\varepsilon + c\varepsilon^{1/2} \|\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)\|_H + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)\|_H \\ & \leq c\varepsilon + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)\|_H. \end{aligned}$$

Thus from (3.41)

$$\begin{aligned} & \int_0^t \left\langle b^\varepsilon \frac{\partial^2(u^\varepsilon - w_1^\varepsilon)}{\partial t^2}(t') + \operatorname{curl}(a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t')), \frac{\partial}{\partial t'}(u^\varepsilon - w_1^\varepsilon)(t') \right\rangle_H dt' \\ & \leq c\varepsilon + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t)\|_H + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t}(u^\varepsilon - w_1^\varepsilon)(t) \right\|_H. \end{aligned}$$

The left hand side equals

$$\begin{aligned} & \frac{1}{2} \int_0^t \frac{d}{dt'} \int_D \left[b^\varepsilon \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \cdot \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right. \\ & \quad \left. + a^\varepsilon \operatorname{curl}(u^\varepsilon(t') - w_1^\varepsilon(t')) \cdot \operatorname{curl}(u^\varepsilon(t') - w_1^\varepsilon(t')) \right] dx dt' \\ & = \frac{1}{2} \int_D \left[b^\varepsilon \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \cdot \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right. \\ & \quad \left. + a^\varepsilon \operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t)) \cdot \operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t)) \right] dx \\ & \quad - \frac{1}{2} \int_D \left[b^\varepsilon \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \cdot \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \right. \\ & \quad \left. + a^\varepsilon \operatorname{curl}(u^\varepsilon(0) - w_1^\varepsilon(0)) \cdot \operatorname{curl}(u^\varepsilon(0) - w_1^\varepsilon(0)) \right] dx \\ & \geq \frac{\alpha}{2} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \frac{\alpha}{2} \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H^2 \\ & \quad - \frac{\beta}{2} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \right\|_H^2 - \frac{\beta}{2} \|\operatorname{curl}(u^\varepsilon(0) - w_1^\varepsilon(0))\|_H^2 \\ & = \frac{\alpha}{2} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 - \frac{\beta}{2} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \right\|_H^2 + \frac{\alpha}{2} \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H^2 \end{aligned}$$

due to condition (2.1). We also have

$$\begin{aligned}
& \left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H = \left\| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H \\
& = \left\| \varepsilon \tau^\varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\operatorname{curl} u_0(0, x))_r + \varepsilon \nabla \left[\tau^\varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial u_{0r}}{\partial t}(0) - g_{1r}(x) \right) \right] \right\|_H \\
& = \left\| \varepsilon \tau^\varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\operatorname{curl} u_0(0, x))_r \right\|_H \\
& \leq c\varepsilon \left\| \frac{\partial}{\partial t} \operatorname{curl} u_0(0) \right\|_H = c\varepsilon \|\operatorname{curl} g_1\|_H.
\end{aligned}$$

Thus, we deduce that for all $t \in (0, T)$

$$\begin{aligned}
& \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H^2 \\
& \leq c\varepsilon + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + c\varepsilon^{1/2} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H
\end{aligned}$$

which implies

$$\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H \leq c\varepsilon^{1/2}.$$

From (3.39) and the facts that $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\omega^r \in C^1(\bar{D}, C(\bar{Y}))$ for $r = 1, 2, 3$, we have

$$\begin{aligned}
& \left\| \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_{L^\infty(0, T; H)} \\
& \leq \left\| \varepsilon(1 - \tau^\varepsilon) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\operatorname{curl} u_0(t))_r + \right. \\
& \quad \left(-\varepsilon \nabla \tau^\varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon(1 - \tau^\varepsilon) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + (1 - \tau^\varepsilon) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial u_{0r}}{\partial t} - g_{1r} \right) \\
& \quad \left. + \varepsilon(1 - \tau^\varepsilon) \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\nabla \frac{\partial u_{0r}}{\partial t} - \nabla g_{1r} \right) \right\|_{L^\infty(0, T; H)} \\
& \leq c\varepsilon^{1/2}
\end{aligned}$$

due to

$$\left\| \frac{\partial u_{0r}}{\partial t} \right\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{1/2} \quad \text{and} \quad \|g_{1r}\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{1/2},$$

as $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^1(D)^3)$ and $g_1 \in H^1(D)^3$. Similarly, from (3.36), we have

$$\|\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}.$$

We therefore have

$$\left\| \frac{\partial(u^\varepsilon - u_1^\varepsilon)}{\partial t} \right\|_{L^\infty(0, T; H)} + \|\operatorname{curl}(u^\varepsilon - u_1^\varepsilon)\|_{L^\infty(0, T; H)} \leq c\varepsilon^{1/2}.$$

We note that

$$\begin{aligned} \frac{\partial u_1^\varepsilon}{\partial t}(t, x) &= \frac{\partial u_0}{\partial t}(t, x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial}{\partial t} (\operatorname{curl} u_0(t, x))_r \\ &\quad + \left(\varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial u_{0r}}{\partial t}(t, x) - g_{1r}(x) \right) \\ &\quad + \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\nabla \frac{\partial u_{0r}}{\partial t}(t, x) - \nabla g_{1r}(x) \right). \end{aligned}$$

Thus

$$\left\| \frac{\partial u_1^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - \nabla_y \omega^r \left(\cdot, \frac{\cdot}{\varepsilon} \right) \left(\frac{\partial u_{0r}}{\partial t}(\cdot, \cdot) - g_{1r}(\cdot) \right) \right\|_{L^\infty(0, T; H)} \leq c\varepsilon.$$

Similarly, from (3.34) we have

$$\left\| \operatorname{curl} u_1^\varepsilon - \operatorname{curl} u_0 - \operatorname{curl}_y \chi^r \left(\cdot, \frac{\cdot}{\varepsilon} \right) (\operatorname{curl} u_0)_r \right\|_{L^\infty(0, T; H)} \leq c\varepsilon.$$

We then get the conclusion. $\square$

Next, we derive the homogenization error where u_0 only possesses the weaker regularity $L^\infty(0, T; H^s(\operatorname{curl}, D))$ for $0 < s < 1$. The above proof uses the fact that we can consider the curl operator of $\operatorname{curl} u_0$. However, in this case, this operator is undefined so the proof of the previous theorem breaks down. We note that in the result that we wish to show, we only need $\operatorname{curl} u_0$ in $L^\infty(0, T; H)$ norm. Thus, as for the time independent equation in the previous chapter, we approximate $\operatorname{curl} u_0$ locally by its average in a small domain, and use a partition of unity, for which we can control the derivative of each function, to approximate $\operatorname{curl} u_0$ in $L^2(D)$.

Theorem 3.4.2. *Assume that $g_0 = 0$, $g_1 \in H^1(D)^3 \cap W$, $u_0, \frac{\partial u_0}{\partial t}$ and $\frac{\partial^2 u_0}{\partial t^2}$ belong to $L^\infty(0, T; H^s(\operatorname{curl}; D))$ for $0 < s \leq 1$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\operatorname{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\omega^r \in C^1(\bar{D}, C(\bar{Y}))$ for all $r = 1, 2, 3$. There exists a constant c that does not depend on ε such that*

$$\begin{aligned} &\left\| \frac{\partial u^\varepsilon}{\partial t} - \left[\frac{\partial u_0}{\partial t} + \nabla_y \frac{\partial u_1}{\partial t} \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_{L^\infty(0, T; H)} \\ &\quad + \left\| \operatorname{curl} u^\varepsilon - \left[\operatorname{curl} u_0 + \operatorname{curl}_y u_1 \left(\cdot, \cdot, \frac{\cdot}{\varepsilon} \right) \right] \right\|_{L^\infty(0, T; H)} \leq c\varepsilon^{\frac{s}{1+s}}. \end{aligned}$$

Proof We consider a set of M open cubes Q_i ($i = 1, \dots, M$) of size ε^{s_1} , $s_1 > 0$ to be chosen later such that $D \subset \cup_{i=1}^M Q_i$ and $Q_i \cap D \neq \emptyset$. Each cube Q_i intersects with only a finite number, which does not depend on ε , of other cubes. We consider a partition of unity that consists of M functions ρ_i such that ρ_i has support in Q_i , $\sum_{i=1}^M \rho_i(x) = 1$ for all $x \in D$ and $|\nabla \rho_i| \leq c\varepsilon^{-s_1}$ for all x . For $r = 1, 2, 3$ and $i = 1, \dots, M$, we denote by

$$U_i^r(t) = \frac{1}{|Q_i|} \int_{Q_i} (\operatorname{curl} u_0(t, x))_r dx$$

and

$$V_i^r(t) = \frac{1}{|Q_i|} \int_{Q_i} (u_0(t, x))_r dx.$$

As in Chapter 2, as $u_0(t) \in H^s(D)^3$ and $\operatorname{curl} u_0(t) \in H^s(D)^3$, for the Lipschitz domain D , we can extend each of them, separately, continuously outside D and understand $u_0(t)$ and $\operatorname{curl} u_0(t)$ as these extensions (see Wloka [100] Theorem 5.6). Let $U_i(t)$ and $V_i(t)$ denote the vector $(U_i^1(t), U_i^2(t), U_i^3(t))$ and $(V_i^1(t), V_i^2(t), V_i^3(t))$ respectively. Using the same argument as in the proof of Theorem 2.2.2, we have

$$\int_{Q_i} |(\operatorname{curl} u_0(t, x))_r - U_i^r(t)|^2 dx \leq c\varepsilon^{2ss_1} \|(\operatorname{curl} u_0)_r\|_{H^s(Q_i)}^2.$$

We consider the function

$$\begin{aligned} u_1^\varepsilon(t, x) = & u_0(t, x) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \\ & + \varepsilon \nabla \left[\omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r(t) \rho_j(x) - g_{1r}(x)t - g_{0r}(x)) \right]. \end{aligned}$$

We first show that

$$\| \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon) - \operatorname{curl} (a^0 \operatorname{curl} u_0) \|_{W'} \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}).$$

We have

$$\begin{aligned} \operatorname{curl} u_1^\varepsilon(t, x) = & \operatorname{curl} u_0(t, x) + \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \\ & + \varepsilon U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\ & + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \end{aligned} \quad (3.42)$$

Thus

$$\begin{aligned}
& a^\varepsilon(x) \operatorname{curl} u_1^\varepsilon(t, x) - a^0(x) \operatorname{curl} u_0(t, x) = a^\varepsilon(x) \operatorname{curl} u_0(t, x) \\
& \quad + \varepsilon a^\varepsilon(x) \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \\
& \quad + \varepsilon a^\varepsilon(x) U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\
& \quad + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - a^0(x) \operatorname{curl} u_0(t, x) \\
& = a^\varepsilon(x) U_j(t) \rho_j(x) + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - a^0(x) U_j(t) \rho_j(x) \\
& \quad + \varepsilon a^\varepsilon(x) \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + \varepsilon a^\varepsilon(x) U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \\
& \quad + a^\varepsilon(x) \operatorname{curl} u_0(t, x) + a^0(x) U_j(t) \rho_j(x) \\
& \quad - a^0(x) \operatorname{curl} u_0(t, x) - a^\varepsilon(x) U_j(t) \rho_j(x) \\
& = a^\varepsilon(x) U_j(t) \rho_j(x) + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - a^0(x) U_j(t) \rho_j(x) \\
& \quad + \varepsilon a^\varepsilon(x) \left[\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right] \\
& \quad + (a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j(t) \rho_j(x)).
\end{aligned}$$

We have

$$\begin{aligned}
& \operatorname{curl} (a^\varepsilon(x) \operatorname{curl} u_1^\varepsilon)(t, x) - \operatorname{curl} (a^0(x) \operatorname{curl} u_0)(t, x) \\
& = \operatorname{curl} \left[a^\varepsilon(x) U_j(t) \rho_j(x) + a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) - a^0(x) U_j(t) \rho_j(x) \right] \\
& \quad + \varepsilon \operatorname{curl} \left[a^\varepsilon(x) \left(\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \right] \\
& \quad + \operatorname{curl} [(a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j(t) \rho_j(x))] \\
& = \operatorname{curl} \left[G_r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \right] + \varepsilon \operatorname{curl} I(t, x) \\
& \quad + \operatorname{curl} [(a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j(t) \rho_j(x))]
\end{aligned}$$

where the vector functions $G_r(x, y)$ are defined in (2.17) and

$$I(t, x) = a^\varepsilon(x) \left(\operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) + U_j^r(t) \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right)$$

For all $\phi \in \mathcal{D}(D)^3$ we have

$$\begin{aligned}
\langle \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon)(t) - \operatorname{curl}(a^0 \operatorname{curl} u_0)(t), \phi \rangle_H &= \int_D G_r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \cdot \operatorname{curl} \phi(x) dx \\
&+ \varepsilon \int_D I(t, x) \cdot \operatorname{curl} \phi(x) dx \\
&+ \int_D (a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j^r(t) \rho_j(x)) \cdot \operatorname{curl} \phi(x) dx \\
&= -\varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}(U_j^r(t) \rho_j(x) \operatorname{curl} \phi(x)) dx \\
&- \varepsilon \int_D \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \cdot \operatorname{curl} \phi(x) dx \\
&+ \varepsilon \int_D I(t, x) \cdot \operatorname{curl} \phi(x) dx \\
&+ \int_D (a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j^r(t) \rho_j(x)) \cdot \operatorname{curl} \phi(x) dx,
\end{aligned}$$

where $G_r(x, y) = \nabla_y \tilde{G}_r(x, y)$ as in (2.19). Using the same procedure as in the proof of Theorem 2.2.2, we have

$$\|U_j^r(t) \rho_j\|_{L^2(D)}^2 \leq c \int_D (\operatorname{curl} u_0(t, x))_r^2 dx.$$

Thus

$$\begin{aligned}
\left| \varepsilon \int_D \nabla_x \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \cdot \operatorname{curl} \phi(x) dx \right| &\leq c\varepsilon \|U_j^r(t) \rho_j\|_{L^2(D)} \|\operatorname{curl} \phi\|_H \\
&\leq c\varepsilon \|\operatorname{curl} \phi\|_H.
\end{aligned}$$

We also have

$$\begin{aligned}
&\varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div} [U_j^r(t) \rho_j(x) \operatorname{curl} \phi(x)] dx \\
&= \varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \nabla \rho_j(x) \cdot \operatorname{curl} \phi(x) dx \\
&\leq c\varepsilon \|U_j^r(t) \nabla \rho_j\|_H \|\operatorname{curl} \phi\|_H.
\end{aligned}$$

As the support of each function ρ_i intersects with the support of a finite number of other functions ρ_j and $\|\nabla \rho_j\|_{L^\infty(D)} \leq c\varepsilon^{-s_1}$, following the we have the same procedure as in Theorem 2.2.2

$$\|U_j^r(t) \nabla \rho_j\|_{L^2(D)}^2 \leq c\varepsilon^{-2s_1}.$$

so

$$\begin{aligned} \varepsilon \int_D \tilde{G}_r \left(x, \frac{x}{\varepsilon} \right) \operatorname{div}(U_j(t)\rho_j(x))_r \operatorname{curl} \phi(x) dx &\leq c\varepsilon \|U_j^r(t)\nabla \rho_j\|_H \|\operatorname{curl} \phi\|_H \\ &\leq c\varepsilon^{1-s_1} \|\operatorname{curl} \phi\|_H. \end{aligned}$$

We have further that

$$\begin{aligned} \int_D (a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j(t)\rho_j(x)) \cdot \operatorname{curl} \phi(x) dx \\ \leq c \|\operatorname{curl} u_0(t) - U_j(t)\rho_j\|_H \|\operatorname{curl} \phi\|_H. \end{aligned}$$

An analysis similar to that in the proof of Theorem 2.2.2 shows that

$$\int_D |(\operatorname{curl} u_0(t, x))_r - U_j^r(t)\rho_j(x)|^2 dx \leq c\varepsilon^{2ss_1} \|(\operatorname{curl} u_0(t, x))_r\|_{H^s(D)}^2. \quad (3.43)$$

Thus

$$\left| \int_D (a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(t, x) - U_j(t)\rho_j(x)) \cdot \operatorname{curl} \phi(x) dx \right| \leq c\varepsilon^{ss_1} \|\operatorname{curl} \phi\|_H$$

From these we conclude that

$$|\langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon(t)) - \operatorname{curl} (a^0 \operatorname{curl} u_0(t)), \phi \rangle_H| \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}) \|\phi\|_W.$$

Since $\frac{\partial u_0}{\partial t} \in L^\infty((0, T); H^s(D))$, by an identical argument, we deduce that

$$\left| \left\langle \operatorname{curl} \left(a^\varepsilon \operatorname{curl} \frac{\partial}{\partial t} u_1^\varepsilon(t) \right) - \operatorname{curl} \left(a^0 \operatorname{curl} \frac{\partial}{\partial t} u_0(t) \right), \phi \right\rangle_H \right| \leq c(\varepsilon^{1-s_1} + \varepsilon^{ss_1}) \|\operatorname{curl} \phi\|_H. \quad (3.44)$$

Choose $s_1 = \frac{1}{s+1}$ we have

$$|\langle \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon(t)) - \operatorname{curl} (a^0 \operatorname{curl} u_0(t)), \phi \rangle_H| \leq c\varepsilon^{\frac{s}{s+1}} \|\operatorname{curl} \phi\|_H$$

and

$$\left| \left\langle \operatorname{curl} \left(a^\varepsilon \operatorname{curl} \frac{\partial}{\partial t} u_1^\varepsilon(t) \right) - \operatorname{curl} a^0 \left(\operatorname{curl} \frac{\partial}{\partial t} u_0(t) \right), \phi \right\rangle_H \right| \leq c\varepsilon^{\frac{s}{s+1}} \|\operatorname{curl} \phi\|_H.$$

Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x \in D} \varepsilon |\nabla \tau^\varepsilon(x)| < c$ where c is independent of ε . Let

$$\begin{aligned} w_1^\varepsilon(t, x) &= u_0(t, x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \\ &\quad + \varepsilon \nabla \left[\tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) (V_j^r(t) \rho_j(x) - g_{1r}(x)t - g_{0r}(x)) \right]. \end{aligned}$$

We then have

$$\begin{aligned} u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x) &= \varepsilon(1 - \tau^\varepsilon(x))\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x) \\ &\quad + \varepsilon\nabla\left[(1 - \tau^\varepsilon(x))\omega^r\left(x, \frac{x}{\varepsilon}\right)(V_j^r(t)\rho_j(x) - g_{1r}(x)t - g_{0r}(x))\right]. \end{aligned} \quad (3.45)$$

From this,

$$\begin{aligned} \operatorname{curl}(u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x)) &= \varepsilon\operatorname{curl}_x\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1 - \tau^\varepsilon(x)) \\ &\quad + \operatorname{curl}_y\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1 - \tau^\varepsilon(x)) \\ &\quad - \varepsilon U_j^r(t)\rho_j(x)\nabla\tau^\varepsilon(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right) \\ &\quad + \varepsilon(1 - \tau^\varepsilon(x))U_j^r(t)\nabla\rho_j(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right). \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{curl}(a^\varepsilon(x)\operatorname{curl}(u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x))) &= \varepsilon\operatorname{curl}\left(a^\varepsilon(x)\operatorname{curl}_x\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1 - \tau^\varepsilon(x))\right) \\ &\quad + \operatorname{curl}\left(a^\varepsilon(x)\operatorname{curl}_y\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1 - \tau^\varepsilon(x))\right) \\ &\quad - \varepsilon\operatorname{curl}\left(a^\varepsilon(x)U_j^r(t)\rho_j(x)\nabla\tau^\varepsilon(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right)\right) \\ &\quad + \varepsilon\operatorname{curl}\left(a^\varepsilon(x)(1 - \tau^\varepsilon(x))U_j^r(t)\nabla\rho_j(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right)\right). \end{aligned} \quad (3.46)$$

Let $\tilde{D}^\varepsilon$ and D^ε be the $3\varepsilon^{s_1}$ and ε neighbourhoods of ∂D respectively as in Theorem 2.2.2. Then as in the proof of Theorem 2.2.2, we have

$$\|U_j^r(t)\rho_j\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^{1-s_1+ss_1}\|\operatorname{curl}u_0(t, x)\|_{H^s(D)^3}^2.$$

Therefore, since $u_0 \in L^\infty(0, T; H^s(\operatorname{curl}, D))$

$$\left\|\operatorname{curl}_y\chi^r\left(x, \frac{x}{\varepsilon}\right)U_j^r(t)\rho_j(x)(1 - \tau^\varepsilon(x))\right\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{(1-s_1+ss_1)/2}$$

and

$$\left\|\varepsilon U_j^r(t)\rho_j(x)\nabla\tau^\varepsilon(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{(1-s_1+ss_1)/2}.$$

Similarly, we have

$$\|U_j^r(t)\nabla\rho_j\|_{L^2(D^\varepsilon)^3}^2 \leq c\varepsilon^{1-3s_1+ss_1}\|\operatorname{curl}u_0(t)\|_{H^s(D)^3}^2. \quad (3.47)$$

Thus

$$\left\| \varepsilon(1 - \tau^\varepsilon(x))U_j^r(t)\nabla\rho_j(x) \times \chi^r\left(x, \frac{x}{\varepsilon}\right) \right\|_H \leq c\varepsilon^{(1-s_1)+(1-s_1+ss_1)/2}.$$

Therefore

$$\|\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)\|_H \leq c\left(\varepsilon^{(1-s_1+ss_1)/2} + \varepsilon^{(1-s_1)+(1-s_1+ss_1)/2}\right). \quad (3.48)$$

Arguing as above (with t fixed) we deduce that

$$\|V_j^r(t)\rho_j\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{(1-s_1+ss_1)/2}, \quad \|V_j^r(t)\nabla\rho_j\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{(1-s_1+ss_1)/2-s_1}.$$

We have

$$\begin{aligned} \frac{\partial^2 w_1^\varepsilon}{\partial t^2}(t, x) &= \frac{\partial^2 u_0}{\partial t^2}(t, x) + \varepsilon\tau^\varepsilon(x)\chi^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 U_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad + \varepsilon\nabla\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) + \varepsilon\tau^\varepsilon(x)\nabla_x\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad + \tau^\varepsilon(x)\nabla_y\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) + \varepsilon\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\nabla\rho_j(x). \end{aligned}$$

Thus

$$\begin{aligned} &b^0(x)\frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x)\frac{\partial^2 w_1^\varepsilon}{\partial t^2}(t, x) \\ &= b^0(x)\frac{\partial^2 u_0}{\partial t^2}(t, x) - b^\varepsilon(x)\frac{\partial^2 u_0}{\partial t^2}(t, x) - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\chi^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 U_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad - \varepsilon b^\varepsilon(x)\nabla\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\nabla_x\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad - b^\varepsilon(x)\tau^\varepsilon(x)\nabla_y\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\nabla\rho_j(x) \\ &= b^0(x)\frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) - b^\varepsilon(x)\frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) - b^\varepsilon(x)\nabla_y\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\chi^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 U_j^r}{\partial t^2}(t)\rho_j(x) - \varepsilon b^\varepsilon(x)\nabla\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\nabla_x\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad + b^\varepsilon(x)(1 - \tau^\varepsilon(x))\nabla_y\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x) \\ &\quad - \varepsilon b^\varepsilon(x)\tau^\varepsilon(x)\omega^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^2 V_j^r}{\partial t^2}(t)\nabla\rho_j(x) \\ &\quad + (b^0(x) - b^\varepsilon(x))\left(\frac{\partial^2 u_0}{\partial t^2}(t) - \frac{\partial^2 V_j^r}{\partial t^2}(t)\rho_j(x)\right). \end{aligned}$$

For $\phi \in W$, we consider

$$\begin{aligned}
J &:= \left\langle b^0 \frac{\partial^2 V_j}{\partial t^2}(t) \rho_j - b^\varepsilon \frac{\partial^2 V_j}{\partial t^2}(t) \rho_j - b^\varepsilon \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j, \phi \right\rangle_H \\
&= \int_D \left(b_{ir}^0 \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) - b_{ir}^\varepsilon \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) + b_{ik}^\varepsilon \frac{\partial \omega^r}{\partial y_k} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \right) \phi_i(x) dx \\
&= \int_D \left(b_{ir}^0 - b_{ir}^\varepsilon + b_{ik}^\varepsilon \frac{\partial \omega^r}{\partial y_k} \left(x, \frac{x}{\varepsilon} \right) \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \phi_i(x) dx \\
&= - \int_D \left(g_r \left(x, \frac{x}{\varepsilon} \right) \right)_i \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \phi_i(x) dx
\end{aligned}$$

where $g_r(x, y)$ is defined in (2.18). Using the function $\tilde{g}_r$ defined in (2.20), we have

$$\begin{aligned}
J &= - \int_D \left[\varepsilon \operatorname{curl} \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \right] \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \cdot \phi(x) dx. \\
&= -\varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \left(\frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \phi(x) \right) dx \\
&\quad + \varepsilon \int_D \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \cdot \phi(x) dx \\
&= -\varepsilon \int_D \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \operatorname{curl} \phi(x) + \frac{\partial^2 V_j^r}{\partial t^2}(t) \nabla \rho_j(x) \times \phi(x) \right) dx \\
&\quad + \varepsilon \int_D \operatorname{curl}_x \tilde{g}_r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 V_j^r}{\partial t^2}(t) \rho_j(x) \cdot \phi(x) dx \\
&\leq c\varepsilon \|\operatorname{curl} \phi\|_H + c\varepsilon^{1-s_1} \|\phi\|_H + c\varepsilon \|\phi\|_H
\end{aligned}$$

due to $\frac{\partial^2 V_j^r}{\partial t^2} \rho_j \in L^\infty(0, T; L^2(D))$ as $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H)$. As $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(D)^3)$, there is a constant c such that for all $t \in (0, T)$

$$\begin{aligned}
&\left| \int_D (b^\varepsilon(x) - b^0(x)) \left(\frac{\partial^2 u_0}{\partial t^2}(t) - \frac{\partial^2 V_j}{\partial t^2}(t) \rho_j(x) \right) \cdot \phi(x) dx \right| \\
&\leq c \left\| \sum_{j=1}^M \left(\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 V_j}{\partial t^2}(t) \right) \rho_j \right\|_H \|\phi\|_H \leq c\varepsilon^{ss_1} \|\phi\|_H,
\end{aligned}$$

(see the proof of Theorem (2.2.2)). Thus

$$\begin{aligned}
\left\langle b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t^2}, \phi \right\rangle_H &\leq c\varepsilon \|\operatorname{curl} \phi\|_H + c\varepsilon^{1-s_1} \|\phi\|_H + c\varepsilon^{\frac{1-s_1+ss_1}{2}} \|\phi\|_H + c\varepsilon^{ss_1} \|\phi\|_H \\
&\leq c\varepsilon \|\operatorname{curl} \phi\|_H + c\varepsilon^{\frac{s}{1+s}} \|\phi\|_H
\end{aligned} \tag{3.49}$$

where we choose $s_1 = 1/(1+s)$. Using

$$b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} + \operatorname{curl} (a^\varepsilon \operatorname{curl} u^\varepsilon) = b^0 \frac{\partial^2 u_0}{\partial t^2} + \operatorname{curl} (a^0 \operatorname{curl} u_0)$$

we deduce that

$$\begin{aligned} b^\varepsilon \frac{\partial^2 (u^\varepsilon - w_1^\varepsilon)}{\partial t^2} + \operatorname{curl} (a^\varepsilon \operatorname{curl} (u^\varepsilon - w_1^\varepsilon)) &= b^0 \frac{\partial^2 u_0}{\partial t^2} - b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t^2} \\ &+ \operatorname{curl} (a^\varepsilon \operatorname{curl} (u_1^\varepsilon - w_1^\varepsilon)) + \operatorname{curl} (a^0 \operatorname{curl} u_0) - \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon). \end{aligned} \quad (3.50)$$

As shown in the proof of Theorem 3.4.1, $\frac{\partial u^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty(0, T; W)$ with respect to ε . For w_1^ε , we have

$$\begin{aligned} \frac{\partial w_1^\varepsilon}{\partial t}(t, x) &= \frac{\partial u_0}{\partial t}(t, x) + \varepsilon \tau^\varepsilon(x) \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(t) \rho_j(x) + \\ &\left(\varepsilon \nabla \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \tau^\varepsilon(x) \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + \tau^\varepsilon(x) \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial V_j^r}{\partial t}(t) \rho_j(x) - g_{1r} \right) \\ &+ \varepsilon \tau^\varepsilon(x) \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j(x) - \nabla g_{1r}(x) \right) \end{aligned}$$

and

$$\begin{aligned} \operatorname{curl} \frac{\partial w_1^\varepsilon}{\partial t}(t, x) &= \operatorname{curl} \frac{\partial u_0}{\partial t}(t, x) + \tau^\varepsilon(x) \frac{\partial U_j^r}{\partial t}(t) \rho_j(x) \left(\varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) \right) \\ &+ \frac{\partial U_j^r}{\partial t}(t) \left(\varepsilon \nabla \tau^\varepsilon(x) \rho_j(x) + \varepsilon \tau^\varepsilon(x) \nabla \rho_j(x) \right) \times \chi^r \left(x, \frac{x}{\varepsilon} \right). \end{aligned}$$

As $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^s(\operatorname{curl}, D))$,

$$\left\| \frac{\partial U_j^r}{\partial t}(t) \rho_j \right\|_{L^2(D)} \leq c, \quad \left\| \frac{\partial V_j^r}{\partial t}(t) \rho_j \right\|_{L^2(D)} \leq c, \quad \left\| \frac{\partial U_j^r}{\partial t}(t) \nabla \rho_j \right\|_H \leq c \varepsilon^{-s_1}$$

and

$$\left\| \frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j \right\|_H \leq c \varepsilon^{-s_1}.$$

Therefore $\operatorname{curl} \frac{\partial w_1^\varepsilon}{\partial t}$ and $\frac{\partial w_1^\varepsilon}{\partial t}$ are uniformly bounded in H with respect to ε , i.e., $\frac{\partial w_1^\varepsilon}{\partial t}$ is uniformly bounded in $L^\infty(0, T; W)$. We have from (3.49)

$$\begin{aligned} &\int_0^t \left\langle b^0 \frac{\partial^2 u_0}{\partial t'^2}(t') - b^\varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial t'^2}(t'), \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \\ &\leq \int_0^t \left(c \varepsilon \left\| \operatorname{curl} \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\|_H + c \varepsilon^{\frac{s}{1+s}} \left\| \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\|_H \right) dt' \\ &\leq c \varepsilon + c \varepsilon^{\frac{s}{1+s}} \sup_{0 \leq t \leq T} \left\| \frac{\partial (u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H. \end{aligned}$$

We also have

$$\begin{aligned}
& \int_0^t \left\langle \operatorname{curl} (a^0 \operatorname{curl} u_0(t')) - \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon(t')), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \\
&= \int_0^t \frac{\partial}{\partial t'} \left\langle \operatorname{curl} (a^0 \operatorname{curl} u_0(t')) - \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon(t')), (u^\varepsilon - w_1^\varepsilon)(t') \right\rangle_H dt' \\
&\quad - \int_0^t \left\langle \operatorname{curl} (a^0 \operatorname{curl} \frac{\partial u_0}{\partial t'}(t')) - \operatorname{curl} (a^\varepsilon \operatorname{curl} \frac{\partial u_1^\varepsilon}{\partial t'}(t')), (u^\varepsilon - w_1^\varepsilon)(t') \right\rangle_H dt'.
\end{aligned}$$

Since $u_0(0) = 0$ and $u_1^\varepsilon(0) = 0$, together with (3.44) we have that

$$\begin{aligned}
& \left| \int_0^t \left\langle \operatorname{curl} (a^0 \operatorname{curl} u_0(t')) - \operatorname{curl} (a^\varepsilon \operatorname{curl} u_1^\varepsilon(t')), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \right| \\
&\leq \left| \left\langle \operatorname{curl} (a^0 \operatorname{curl} u_0(t) - a^\varepsilon \operatorname{curl} u_1^\varepsilon(t)), (u^\varepsilon - w_1^\varepsilon)(t) \right\rangle_H \right| \\
&\quad + c \int_0^t \varepsilon^{\frac{s}{s+1}} \|\operatorname{curl} (u^\varepsilon - w_1^\varepsilon)(t')\|_H dt' \\
&\leq c \varepsilon^{\frac{s}{s+1}} \sup_{0 \leq t \leq T} \|\operatorname{curl} (u^\varepsilon - w_1^\varepsilon)(t)\|_H.
\end{aligned}$$

Now we estimate

$$\int_0^t \left\langle \operatorname{curl} (a^\varepsilon \operatorname{curl} (u^\varepsilon(t') - w_1^\varepsilon(t'))), \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt'$$

using (3.46). We have that

$$\begin{aligned}
& \left| \int_0^t \left\langle \varepsilon \operatorname{curl} \left(a^\varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \right| \\
&\leq c \varepsilon \int_0^t \left\| \operatorname{curl} \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\|_H dt' \leq c \varepsilon;
\end{aligned}$$

and from (3.47), we have

$$\begin{aligned}
& \left| \int_0^t \left\langle \varepsilon \operatorname{curl} \left(a^\varepsilon (1 - \tau^\varepsilon(x)) U_j^r(t') \nabla \rho_j(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \right| \\
&\leq c \varepsilon \int_0^t \|U_j^r(t') \nabla \rho_j\|_{L^2(D^\varepsilon)^3} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\|_W dt' \\
&\leq c \varepsilon \varepsilon^{\frac{1-3s_1+ss_1}{2}} = c \varepsilon^{\frac{2s}{s+1}}.
\end{aligned}$$

For the other two terms in (3.46), we have

$$\begin{aligned}
& \int_0^t \left\langle \operatorname{curl} \left(a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \\
&= \int_0^t \frac{\partial}{\partial t'} \int_D a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \operatorname{curl} (u^\varepsilon(t') - w_1^\varepsilon(t')) dx dt' \\
&\quad - \int_0^t \int_D a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t'}(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \operatorname{curl} (u^\varepsilon(t') - w_1^\varepsilon(t')) dx dt' \\
&= \int_D a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \operatorname{curl} (u^\varepsilon(t) - w_1^\varepsilon(t)) dx \\
&\quad - \int_0^t \int_D a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t'}(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \cdot \operatorname{curl} (u^\varepsilon(t') - w_1^\varepsilon(t')) dx dt'.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left| \int_0^t \left\langle \operatorname{curl} \left(a^\varepsilon(x) \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t') \rho_j(x) (1 - \tau^\varepsilon(x)) \right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \right| \\
&\leq \|U_j^r(t) \rho_j\|_{L^2(D^\varepsilon)} \|\operatorname{curl} (u^\varepsilon(t) - w_1^\varepsilon(t))\|_H \\
&\quad + c \int_0^t \left\| \frac{\partial U_j^r}{\partial t'}(t') \rho_j \right\|_{L^2(D^\varepsilon)} \|\operatorname{curl} (u^\varepsilon(t') - w_1^\varepsilon(t'))\|_H dt' \\
&\leq c\varepsilon^{\frac{1-s_1+ss_1}{2}} \sup_{0 \leq t \leq T} \|\operatorname{curl} (u^\varepsilon(t) - w_1^\varepsilon(t))\|_H
\end{aligned}$$

as u_0 and $\frac{\partial u_0}{\partial t}$ belong to $L^\infty(0, T; H^s(D)^3)$. Similarly, we have

$$\begin{aligned}
& \left| \int_0^t \left\langle \varepsilon \operatorname{curl} \left(a^\varepsilon U_j^r(t') \rho_j(x) \nabla \tau^\varepsilon(x) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \right| \\
&\leq c\varepsilon^{\frac{1-s_1+ss_1}{2}} \sup_{0 \leq t \leq T} \|\operatorname{curl} (u^\varepsilon(t) - w_1^\varepsilon(t))\|_H.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left| \int_0^t \left\langle \operatorname{curl} (a^\varepsilon \operatorname{curl} (u_1^\varepsilon(t') - w_1^\varepsilon(t')), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t')) \right\rangle_H \right| \\
&\leq c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{s+1}} \sup_{0 \leq t \leq T} \|\operatorname{curl} (u^\varepsilon(t) - w_1^\varepsilon(t))\|_H.
\end{aligned}$$

We then deduce from (3.50)

$$\begin{aligned} & \int_0^t \left\langle b^\varepsilon \frac{\partial^2(u^\varepsilon - w_1^\varepsilon)}{\partial t'^2}(t') + \operatorname{curl}(a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon))(t'), \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t'}(t') \right\rangle_H dt' \\ & \leq c\varepsilon + c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{1+s}} \sup_{0 \leq t \leq T} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H \\ & \quad + c\varepsilon^{\frac{s}{1+s}} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{2} \int_D b^\varepsilon(x) \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \cdot \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) dx \\ & \quad + \frac{1}{2} \int_D a^\varepsilon(x) \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t) \cdot \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)(t) dx \\ & \leq c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{1+s}} \sup_{0 \leq t \leq T} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H \\ & \quad + c\varepsilon^{\frac{s}{1+s}} \sup_{0 \leq t \leq T} \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H \\ & \quad + \int_D b^\varepsilon(x) \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) \cdot \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(0) dx \end{aligned}$$

(note that $u^\varepsilon(0) = w_1^\varepsilon(0) = 0$). We have

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H = \left\| \frac{\partial u_0}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H \\ & = \left\| \varepsilon \tau^\varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(0) \rho_j(x) + \varepsilon \nabla \left(\tau^\varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial V_j^r}{\partial t}(0) \rho_j(x) - g_{1r}(x) \right) \right) \right\|_H \\ & = \left\| \varepsilon \tau^\varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(0) \rho_j(x) + \right. \\ & \quad \left(\varepsilon \nabla \tau^\varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) + \varepsilon \tau^\varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + \tau^\varepsilon \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial V_j^r}{\partial t}(0) \rho_j(x) - g_{1r}(x) \right) \\ & \quad \left. + \varepsilon \tau^\varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial V_j^r}{\partial t}(0) \nabla \rho_j(x) - \nabla g_{1r}(x) \right) \right\|_H \\ & \leq c\varepsilon + c \left\| \frac{\partial V_j^r}{\partial t}(0) \rho_j(x) - g_{1r}(x) \right\|_{L^2(D)} + c\varepsilon \left\| \frac{\partial V_j^r}{\partial t}(0) \nabla \rho_j(x) \right\|_H. \end{aligned}$$

As $g_1 \in H^1(D)^3$, a similar argument as for showing (3.43) for $s = 1$ shows that

$$\left\| \frac{\partial V_j^r}{\partial t}(0) \rho_j - g_{1r} \right\|_{L^2(D)} \leq \varepsilon^{1/2}.$$

Further,

$$\left\| \frac{\partial V_j^r}{\partial t}(0) \nabla \rho_j(x) \right\|_H \leq c\varepsilon^{-s_1}.$$

Thus

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(0) - \frac{\partial w_1^\varepsilon}{\partial t}(0) \right\|_H \leq c\varepsilon + c\varepsilon^{1/2} + c\varepsilon^{1-s_1} \leq c\varepsilon^{\frac{s}{s+1}}.$$

Using (2.1) we get

$$\begin{aligned} & \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H^2 + \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H^2 \\ & \leq c\varepsilon^{\frac{2s}{s+1}} + c\varepsilon^{\frac{s}{s+1}} \sup_{0 \leq t \leq T} \left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t} \right\|_H + c\varepsilon^{\frac{s}{s+1}} \sup_{0 \leq t \leq T} \|\operatorname{curl} u^\varepsilon(t) - w_1^\varepsilon(t)\|_H. \end{aligned}$$

From this we deduce that for all $t \in (0, T)$

$$\left\| \frac{\partial(u^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H + \|\operatorname{curl}(u^\varepsilon(t) - w_1^\varepsilon(t))\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \quad (3.51)$$

From (3.45) we have

$$\begin{aligned} \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t, x) &= \varepsilon(1 - \tau^\varepsilon(x))\chi^r\left(x, \frac{x}{\varepsilon}\right) \frac{\partial U_j^r}{\partial t}(t)\rho_j(x) \\ &+ \left[-\varepsilon \nabla \tau^\varepsilon(x) \omega^r\left(x, \frac{x}{\varepsilon}\right) + \varepsilon(1 - \tau^\varepsilon(x)) \nabla_x \omega^r\left(x, \frac{x}{\varepsilon}\right) + (1 - \tau^\varepsilon(x)) \nabla_y \omega^r\left(x, \frac{x}{\varepsilon}\right) \right] \\ &\quad \cdot \left(\frac{\partial V_j^r}{\partial t}(t)\rho_j(x) - g_{1r}(x) \right) \\ &+ \varepsilon(1 - \tau^\varepsilon(x)) \omega^r\left(x, \frac{x}{\varepsilon}\right) \left(\frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j(x) - \nabla g_{1r}(x) \right). \end{aligned}$$

Therefore, using $g_1 \in H^1(D)^3$ we get

$$\begin{aligned} \left\| \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H &\leq c\varepsilon \left\| \frac{\partial U_j^r}{\partial t}(t)\rho_j \right\|_{L^2(D^\varepsilon)^3} + c \left\| \frac{\partial V_j^r}{\partial t}(t)\rho_j \right\|_{L^2(D^\varepsilon)^3} \\ &+ c\varepsilon \left\| \frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j \right\|_{L^2(D^\varepsilon)^3} + c\|g_1\|_{L^2(D^\varepsilon)^3} + c\varepsilon \|\nabla g_1\|_{L^2(D^\varepsilon)^3}. \end{aligned}$$

As $\frac{\partial}{\partial t} \operatorname{curl} u_0 \in L^\infty(0, T; H^s(D)^3)$, $\frac{\partial}{\partial t} u_0 \in L^\infty(0, T; H^s(D)^3)$ and $g_1 \in H^1(D)^3$, we deduce that

$$\begin{aligned} \left\| \frac{\partial U_j^r}{\partial t}(t)\rho_j \right\|_{L^2(D^\varepsilon)^3} &\leq c, \quad \left\| \frac{\partial V_j^r}{\partial t}(t)\rho_j \right\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{\frac{1-s_1+ss_1}{2}}, \\ \left\| \frac{\partial V_j^r}{\partial t}(t) \nabla \rho_j \right\|_{L^2(D^\varepsilon)^3} &\leq c\varepsilon^{\frac{1-3s_1+ss_1}{2}}, \quad \|g_1\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{1/2}. \end{aligned}$$

Therefore

$$\left\| \frac{\partial(u_1^\varepsilon - w_1^\varepsilon)}{\partial t}(t) \right\|_H \leq c\varepsilon^{\frac{1-s_1+ss_1}{2}} + \varepsilon\varepsilon^{\frac{1-3s_1+ss_1}{2}} + c\varepsilon^{1/2} \leq c\varepsilon^{\frac{s}{s+1}}. \quad (3.52)$$

From (3.48), (3.51) and (3.52), we deduce that

$$\left\| \frac{\partial(u^\varepsilon - u_1^\varepsilon)}{\partial t}(t) \right\|_H + \|\operatorname{curl}(u^\varepsilon(t) - u_1^\varepsilon(t))\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \quad (3.53)$$

We note that

$$\left\| \varepsilon \operatorname{curl}_x \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r(t) \rho_j(x) \right\|_H \leq c\varepsilon,$$

and

$$\left\| \varepsilon (U_j^r(t) \nabla \rho_j(x)) \times \chi^r \left(x, \frac{x}{\varepsilon} \right) \right\|_H \leq c\varepsilon \varepsilon^{-s_1} = c\varepsilon^{\frac{s}{s+1}}$$

so from (3.42)

$$\left\| \operatorname{curl} u_1^\varepsilon - \left[\operatorname{curl} u_0 + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) U_j^r \rho_j \right] \right\|_H \leq c\varepsilon^{\frac{s}{s+1}}.$$

Thus we deduce from (3.43) that

$$\left\| \operatorname{curl} u_1^\varepsilon - \left[\operatorname{curl} u_0 + \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) (\operatorname{curl} u_0(x))_r \right] \right\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \quad (3.54)$$

We further have

$$\begin{aligned} \frac{\partial u_1^\varepsilon}{\partial t}(t) &= \frac{\partial u_0}{\partial t}(t) + \varepsilon \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(t) \rho_j(x) \\ &\quad + \left(\varepsilon \nabla_x \omega^r \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \right) \left(\frac{\partial V_j^r}{\partial t} \rho_j(x) - g_{1r} \right) \\ &\quad + \varepsilon \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial V_j^r}{\partial t} \nabla \rho_j(x) - \nabla g_{1r}(x) \right). \end{aligned}$$

As

$$\left\| \chi^r \left(x, \frac{x}{\varepsilon} \right) \frac{\partial U_j^r}{\partial t}(t) \rho_j(x) \right\|_H \leq c, \quad \left\| \frac{\partial V_j^r}{\partial t} \rho_j(x) \right\|_H \leq c, \quad \left\| \frac{\partial V_j^r}{\partial t} \nabla \rho_j(x) \right\|_H \leq c\varepsilon^{-s_1}$$

we have that

$$\left\| \frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial V_j^r}{\partial t}(t) \rho_j(x) - g_{1r}(x) \right) \right\|_H \leq c\varepsilon^{\frac{s}{s+1}}.$$

As $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^s(\text{curl}, D))$,

$$\left\| \frac{\partial V_j^r}{\partial t}(t) \rho_j(x) - \frac{\partial u_{0r}}{\partial t} \right\|_H \leq c\varepsilon^{ss_1} = c\varepsilon^{\frac{s}{s+1}}$$

we deduce that

$$\left\| \frac{\partial u_1^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y \omega^r \left(x, \frac{x}{\varepsilon} \right) \left(\frac{\partial u_{0r}}{\partial t}(t) - g_{1r}(x) \right) \right\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \quad (3.55)$$

From (3.53), (3.54) and (3.55), we get

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \frac{\partial u_0}{\partial t}(t) - \nabla_y \omega^r \left(\cdot, \frac{\cdot}{\varepsilon} \right) \left(\frac{\partial u_{0r}}{\partial t}(t) - g_{1r} \right) \right\|_H + \\ & \left\| \text{curl } u^\varepsilon - \text{curl } u_0 - \text{curl}_y \chi^r \left(\cdot, \frac{\cdot}{\varepsilon} \right) (\text{curl } u_0(t, \cdot))_r \right\|_H \leq c\varepsilon^{\frac{s}{s+1}}. \end{aligned}$$

□

3.4.2 Corrector for multiscale problems

For problems that depend on more than two scales, we cannot deduce an explicit homogenization error. However, we can deduce correctors for the case where $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \dots, n$. We use the operator $\mathcal{T}_n^\varepsilon$ and $\mathcal{U}_n^\varepsilon$ defined in (2.28) and (2.33). We first note that from (2.30)

$$\mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) \rightharpoonup \frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i, \quad (3.56)$$

and

$$\mathcal{T}_n^\varepsilon(\text{curl } u^\varepsilon) \rightharpoonup \text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y_i} u_i \quad (3.57)$$

in $L^2(D \times \mathbf{Y})$ when $\varepsilon \rightarrow 0$.

We have the following result.

Proposition 3.4.3. *Assume that $g_0 = 0$, $g_1 \in W$ and $f \in H^1(0, T; H)$. We have*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \nabla_{y_i} \frac{\partial u_i}{\partial t} \right) \right\|_{L^\infty(0, T; H)} \right. \\ & \quad \left. + \left\| \text{curl } u^\varepsilon - \text{curl } u_0 - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \text{curl}_{y_i} u_i \right) \right\|_{L^\infty(0, T; H)} \right) = 0. \end{aligned}$$

Proof We consider

$$\begin{aligned}
E^\varepsilon(t) &= \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(b^\varepsilon) \left(\mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) (t) \right) \\
&\quad \cdot \left(\mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) (t) \right) d\mathbf{y} dx \\
&\quad + \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(a^\varepsilon) \left(\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon)(t) - \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) (t) \right) \\
&\quad \cdot \left(\mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon)(t) - \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) (t) \right) d\mathbf{y} dx.
\end{aligned}$$

We note that for each $t > 0$, from (2.29),

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} \left[\mathcal{T}_n^\varepsilon(b^\varepsilon) \mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) \cdot \mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) \right. \\
&\quad \left. + \mathcal{T}_n^\varepsilon(a^\varepsilon) \mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon)(t) \cdot \mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon)(t) \right] d\mathbf{y} dx \\
&\leq \lim_{\varepsilon \rightarrow 0} \int_D \left[b^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial t}(t) \cdot \frac{\partial u^\varepsilon}{\partial t}(t) + a^\varepsilon(x) \operatorname{curl} u^\varepsilon(t, x) \cdot \operatorname{curl} u^\varepsilon(t, x) \right] dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_D b^\varepsilon(x) g_1(x) \cdot g_1(x) dx + 2 \int_0^t \int_D f(t', x) \frac{\partial u^\varepsilon}{\partial t'} dx dt' \\
&= \int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx + 2 \int_0^t \int_D f(t', x) \frac{\partial u_0}{\partial t'}(t', x) dx dt'
\end{aligned}$$

where we have used the energy formula for wave equation (see Lions and Magenes [65]) and the initial condition $g_0 = 0$. Using (3.56) and (3.57), we have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) &\leq \int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx + 2 \int_0^t \int_D f(t', x) \frac{\partial u_0}{\partial t'}(t', x) dx dt' \\
&\quad - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t}(t) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t) \right) \cdot \left(\frac{\partial u_0}{\partial t}(t) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t) \right) d\mathbf{y} dx \\
&\quad - \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t) \right) \cdot \left(\operatorname{curl} u_0(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t) \right) d\mathbf{y} dx.
\end{aligned}$$

From (3.18) and (3.24), we have

$$\begin{aligned}
& \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) d\mathbf{y} dx \\
&= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left[\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) + \left(\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) - g_{1k} \right) \nabla_{y_n} \omega_n^k \right] \\
&\quad \cdot \left[\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) + \left(\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right) - g_{1k} \right) \nabla_{y_n} \omega_n^k \right] d\mathbf{y} dx \\
&= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left[\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k (e^k + \nabla_{y_n} \omega_n^k) - g_{1k} \nabla_{y_n} \omega_n^k \right] \\
&\quad \cdot \left[\frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_l (e^l + \nabla_{y_n} \omega_n^l) - g_{1l} \nabla_{y_n} \omega_n^l \right] d\mathbf{y} dx.
\end{aligned}$$

From (3.17) and (3.19), this equals

$$\begin{aligned}
& \int_D \int_{\mathbf{Y}_{n-1}} b_{kl}^{n-1} \frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_k \frac{\partial}{\partial t} \left(u_0 + \sum_{i=1}^{n-1} \nabla_{y_i} \mathbf{u}_i \right)_l d\mathbf{y}_{n-1} dx \\
&\quad + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \nabla_{y_n} \omega_n^k \cdot \nabla_{y_n} \omega_n^l g_{1k} g_{1l} d\mathbf{y} dx.
\end{aligned}$$

Continuing this process, the expression equals

$$\int_D b^0(x) \frac{\partial u_0}{\partial t}(t) \cdot \frac{\partial u_0}{\partial t}(t) dx + \sum_{i=1}^n \int_D \int_{\mathbf{Y}_i} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_i^k \cdot \nabla_{y_i} \omega_i^l g_{1k} g_{1l} d\mathbf{y}_i dx.$$

On the other hand we have

$$\begin{aligned}
& \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} u_0(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t) \right) \cdot \left(\operatorname{curl} u_0(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i(t) \right) d\mathbf{y} dx \\
&= \int_D a^0(x) \operatorname{curl} u_0(t) \cdot \operatorname{curl} u_0(t) dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) &\leq \int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx + 2 \int_0^t \int_D f(t', x) \frac{\partial u_0}{\partial t'}(t', x) dx dt' \\
&\quad - \int_D b^0(x) \frac{\partial u_0}{\partial t}(t, x) \cdot \frac{\partial u_0}{\partial t}(t, x) dx \\
&\quad - \sum_{i=1}^n \int_D \int_{\mathbf{Y}_i} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_i^k \cdot \nabla_{y_i} \omega_i^l g_{1k}(x) g_{1l}(x) d\mathbf{y}_i dx \\
&\quad - \int_D a^0(x) \operatorname{curl} u_0(t, x) \cdot \operatorname{curl} u_0(t, x) dx.
\end{aligned}$$

From (3.23), we get

$$\begin{aligned}
&\int_D b^0(x) \frac{\partial u_0}{\partial t}(t, x) \cdot \frac{\partial u_0}{\partial t}(t, x) dx + \int_D a^0(x) \operatorname{curl} u_0(t, x) \cdot \operatorname{curl} u_0(t, x) dx \\
&\quad = \int_D b^0(x) g_1(x) \cdot g_1(x) dx + 2 \int_0^t \int_D f(t', x) \frac{\partial u_0}{\partial t'}(t', x) dx dt'.
\end{aligned}$$

From (3.20) we have

$$\int_{Y_i} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_n^k \cdot \nabla_{y_i} \omega_n^l dy_i = - \int_{Y_i} b^i(x, \mathbf{y}_i) e^k \cdot \nabla_{y_i} \omega_n^l.$$

Thus

$$\begin{aligned}
&\int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx - \sum_{i=1}^n \int_D \int_{\mathbf{Y}_i} b^i(x, \mathbf{y}_i) \nabla_{y_i} \omega_i^k \cdot \nabla_{y_i} \omega_i^l g_{1k} g_{1l} d\mathbf{y}_i dx \\
&\quad = \int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1 \cdot g_1 dx + \sum_{i=1}^n \int_D \int_{\mathbf{Y}_i} b^i(x, \mathbf{y}_i) e^k \cdot \nabla_{y_i} \omega_n^l g_{1l} g_{1k} d\mathbf{y}_i dx.
\end{aligned}$$

We consider

$$\begin{aligned}
&\int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx + \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) e^k \cdot \nabla_{y_n} \omega_n^l g_{1l} g_{1k} d\mathbf{y} dx \\
&\quad = \int_D \left(\int_{\mathbf{Y}} b_{ij}(x, \mathbf{y}) d\mathbf{y} \right) g_{1j}(x) g_{1i}(x) dx \\
&\quad \quad + \int_D \int_{\mathbf{Y}} b_{ki}(x, \mathbf{y}) \frac{\partial \omega_n^j}{\partial y_k}(x, \mathbf{y}) g_{1j}(x) g_{1i}(x) d\mathbf{y} dx \\
&\quad = \int_D \int_{\mathbf{Y}_{n-1}} \left(\int_{Y_n} b_{ik}(\delta_{jk} + \frac{\partial \omega_n^j}{\partial y_k}) dy_n \right) g_{1j}(x) g_{1i}(x) d\mathbf{y}_{n-1} dx \\
&\quad = \int_D \int_{\mathbf{Y}_{n-1}} b_{ij}^{n-1} g_{1j}(x) g_{1i}(x) d\mathbf{y}_{n-1} dx.
\end{aligned}$$

Continuing this, we get

$$\begin{aligned} \int_D \left(\int_{\mathbf{Y}} b(x, \mathbf{y}) d\mathbf{y} \right) g_1(x) \cdot g_1(x) dx + \sum_{i=1}^n \int_D \int_{\mathbf{Y}_i} b^i(x, \mathbf{y}_i) e^k \cdot \nabla_{y_i} \omega_n^l g_{1l} g_{1k} d\mathbf{y}_i dx \\ = \int_D b^0(x) g_1(x) \cdot g_1(x) dx. \end{aligned}$$

Thus

$$\lim_{\varepsilon \rightarrow 0} E^\varepsilon(t) = 0.$$

We show that the convergence is uniform. To make the notation concise, we denote by

$$A(t) = \mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) (t).$$

We consider

$$\begin{aligned} \left| \int_D \int_{\mathbf{Y}} [\mathcal{T}_n^\varepsilon(b^\varepsilon) A(t) \cdot A(t) - \mathcal{T}_n^\varepsilon(b^\varepsilon) A(t') \cdot A(t')] d\mathbf{y} dx \right| \\ = \left| \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(b^\varepsilon) (A(t) - A(t')) \cdot (A(t) + A(t')) d\mathbf{y} dx \right| \\ \leq c \|A(t) - A(t')\|_{H_n} \|A(t) + A(t')\|_{H_n}. \end{aligned}$$

We note that $\|\frac{\partial u^\varepsilon}{\partial t}\|_{L^\infty(0,T;H)}$ is uniformly bounded for all ε , $\frac{\partial u_0}{\partial t} \in L^\infty(0,T;H)$ and $\frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \in L^\infty(0,T;L^2(D \times \mathbf{Y}_i)^3)$. Further

$$A(t) - A(t') = \int_{t'}^t \left[\mathcal{T}_n^\varepsilon \left(\frac{\partial^2 u^\varepsilon}{\partial \tau^2} \right) (\tau) - \left(\frac{\partial^2 u_0}{\partial \tau^2}(\tau) + \sum_{i=1}^n \frac{\partial^2}{\partial \tau^2} \nabla_{y_i} \mathbf{u}_i(\tau) \right) \right] d\tau$$

so

$$\begin{aligned} \|A(t) - A(t')\|_{H_n} \leq \int_{t'}^t \left[\left\| \mathcal{T}_n^\varepsilon \left(\frac{\partial^2 u^\varepsilon}{\partial \tau^2} \right) (\tau) \right\|_{H_n} \right. \\ \left. + \left\| \frac{\partial^2 u_0}{\partial \tau^2}(\tau) \right\|_H + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial \tau^2} \nabla_{y_i} \mathbf{u}_i(\tau) \right\|_{H_i} \right] d\tau. \quad (3.58) \end{aligned}$$

From (3.33), we have that $\frac{\partial^2 u^\varepsilon}{\partial t^2}$ is uniformly bounded in $L^2(0,T;H)$. By a similar argument using the compatible initial condition, we show that $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0,T;H)$

which implies that $\frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_i \in L^2(0, T; H_i)$. We then have

$$\begin{aligned} \int_{t'}^t \left\| \mathcal{T}_n^\varepsilon \left(\frac{\partial^2 u^\varepsilon}{\partial \tau^2} \right) (\tau) \right\|_{H_n} d\tau &\leq (t - t')^{1/2} \left(\int_0^t \left\| \mathcal{T}_n^\varepsilon \left(\frac{\partial^2 u^\varepsilon}{\partial \tau^2} \right) (\tau) \right\|_{H_n}^2 d\tau \right)^{1/2} \\ &\leq c(t - t')^{1/2} \left(\int_0^t \left\| \frac{\partial^2 u^\varepsilon}{\partial \tau^2} (\tau) \right\|_H^2 d\tau \right)^{1/2} \\ &\leq c(t - t')^{1/2}, \end{aligned}$$

where we have used (2.29).

By the same argument, we have similar estimates for the other terms in (3.58). We can perform similarly for the other terms in $E^\varepsilon(t)$. From the Arzelà-Ascoli theorem, we deduce that $E^\varepsilon(t)$ converges to 0 when $\varepsilon \rightarrow 0$ uniformly for all $t \in [0, T]$. The conclusion of the proposition follows from the fact that

$$\begin{aligned} \left\| \frac{\partial u^\varepsilon}{\partial t}(t) - \mathcal{U}_n^\varepsilon \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) (t) \right\|_H \\ \leq \left\| \mathcal{T}_n^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} \right) (t) - \left(\frac{\partial u_0}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) (t) \right\|_{H_n}, \end{aligned}$$

and

$$\begin{aligned} \left\| \operatorname{curl} u^\varepsilon(t) - \mathcal{U}_n^\varepsilon \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) (t) \right\|_H \leq \\ \left\| \mathcal{T}_n^\varepsilon(\operatorname{curl} u^\varepsilon(t)) - \left(\operatorname{curl} u_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} u_i \right) (t) \right\|_{H_n}. \end{aligned}$$

□

Summary

We study multiscale time dependent Maxwell equations in this chapter. We first derive the multiscale homogenized equations which contains the second time derivative of the scale interacting terms $\mathbf{u}_i$ which is understood in the generalized sense. From this equation, with involved technicalities we derive the homogenized time dependent Maxwell equation for u_0 , together with the initial conditions. As the energy of the multiscale Maxwell wave equation generally does not converge to the energy of the homogenized Maxwell wave equation, we restrict our consideration

to the case where $u_0(0, x) = 0$ (in this case, the convergence holds). For two-scale problems, we derive the corrector and homogenization error for both cases $u_0 \in H^1(\text{curl}, D)$ (which is $O(\varepsilon^{1/2})$) and $u_0 \in H^s(\text{curl}, D)$ with $0 < s < 1$ which is $O(\varepsilon^{s/1+s})$. In the multiscale case, correctors are derived without an error. We also prove that the regularity requirements for obtaining the homogenization error estimate are achievable.

Chapter 4

High dimensional finite elements for stationary Maxwell equations

We develop finite element methods for the multiscale homogenized equation derived in Chapter 2. In Section 4.1, we define the full and sparse tensor product finite element spaces, using edge finite element spaces, and prove their approximation properties. We then apply these finite element spaces to solve the multiscale homogenized problems. In Section 4.2, we use the finite element solutions to deduce numerical correctors for the original multiscale problems. In the two-scale case, we prove an explicit error for the numerical corrector in terms of the microscopic scale and the mesh size. In Section 4.3, we solve numerically the two-scale homogenized equation of some two-scale Maxwell equations to confirm the analysis.

4.1 Finite element discretization

Let D be a polygonal domain in $\mathbb{R}^3$. We consider a hierarchy of simplices $\mathcal{T}^l$ ($l = 0, 1, \dots$) where $\mathcal{T}^{l+1}$ is obtained from $\mathcal{T}^l$ by dividing each simplex in $\mathcal{T}^l$ into 8 tetrahedra. The mesh size of $\mathcal{T}^l$ is $h_l = O(2^{-l})$. For each tetrahedron T in $\mathcal{T}^l$, we consider the edge finite element space

$$R(T) = \{v : v = a + b \times x, \quad a, b \in \mathbb{R}^3\}.$$

When $d = 2$, $\mathcal{T}^{l+1}$ is obtained from $\mathcal{T}^l$ by dividing each simplex in $\mathcal{T}^l$ into 4 congruent triangles. For each triangle T , we consider the edge finite element space

$$R(T) = \left\{ v : v = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \right\}$$

where a_1, a_2 and b are constants. Alternatively, if the domain can be partitioned into a set of cubes, we can use edge finite element on cubic mesh instead (see

[77, 78, 72, 64]). We denote by $\mathcal{P}_1(T)$ the set of linear polynomials in each simplex T . In the following, we only present the analysis for the three dimension case, the two dimension case is similar.

For the cube Y , we partition it into a hierarchy of simplices $\mathcal{T}_\#^l$ which are distributed periodically.

We consider the finite element spaces

$$\begin{aligned} W^l &= \{v \in H_0(\text{curl}, D), v|_T \in R(T) \ \forall T \in \mathcal{T}^l\}, \\ V^l &= \{v \in H^1(D), v|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}^l\}, \\ W_\#^l &= \{v \in H_\#(\text{curl}, Y), v|_T \in R(T) \ \forall T \in \mathcal{T}_\#^l\}, \end{aligned}$$

and

$$V_\#^l = \{v \in H_\#^1(Y), v|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_\#^l\}.$$

For $d = 2, 3$, we have the following estimates (see Monk [72] and Ciarlet [34])

$$\inf_{v_l \in W^l} \|v - v_l\|_{H(\text{curl}, D)} \leq ch_l^s (\|v\|_{H^s(D)^d} + \|\text{curl } v\|_{H^s(D)^d})$$

$$\forall v \in H_0(\text{curl}, D) \cap H^s(\text{curl}, D);$$

$$\inf_{v_l \in W_\#^l} \|v - v_l\|_{H_\#(\text{curl}, Y)} \leq ch_l^s (\|v\|_{H^s(Y)^d} + \|\text{curl } v\|_{H^s(Y)^d}) \quad (4.1)$$

$$\forall v \in H_\#(\text{curl}, Y) \cap H^s(\text{curl}, Y);$$

$$\inf_{v_l \in V^l} \|v - v_l\|_{L^2(D)} \leq ch_l^s \|v\|_{H^s(D)} \quad (4.2)$$

$$\forall v \in H^s(D);$$

$$\inf_{v_l \in V_\#^l} \|v - v_l\|_{L^2(Y)} \leq ch_l^s \|v\|_{H^s(Y)} \quad (4.3)$$

$$\forall v \in H_\#^s(Y); \text{ and}$$

$$\inf_{v_l \in V_\#^l} \|v - v_l\|_{H_\#^1(Y)} \leq ch_l^s \|v\|_{H^{1+s}(Y)} \quad (4.4)$$

$$\forall v \in H_\#^1(Y) \cap H^{1+s}(Y).$$

4.1.1 Full tensor product finite elements

As $L^2(D \times \mathbf{Y}_{i-1}, \tilde{H}_\#(\text{curl}, Y_i)) \cong L^2(D) \otimes L^2(Y_1) \otimes \cdots \otimes L^2(Y_{i-1}) \otimes \tilde{H}_\#(\text{curl}, Y_i)$ we use the tensor product finite element space

$$\bar{W}_i^l = V^l \otimes \underbrace{V_\#^l \otimes \cdots \otimes V_\#^l}_{i-1 \text{ times}} \otimes W_\#^l$$

to approximate u_i . Similarly, as $\mathbf{u}_i \in L^2(D \times \mathbf{Y}_{i-1}, H_{\#}^1(Y_i))$, we use the finite element space

$$\bar{V}_i^l = V^l \otimes \underbrace{V_{\#}^l \otimes \cdots \otimes V_{\#}^l}_{i \text{ times}}$$

to approximate $\mathbf{u}_i$. We define the space

$$\bar{\mathbf{V}}^l = W^l \times \bar{W}_1^l \times \cdots \times \bar{W}_n^l \times \bar{V}_1^l \times \cdots \times \bar{V}_n^l.$$

The full tensor product finite element approximating problem is: Find $\bar{\mathbf{u}}^L \in \bar{\mathbf{V}}^L$ so that:

$$B(\bar{\mathbf{u}}^L, \bar{\mathbf{v}}^L) = \int_D f(x) \cdot \bar{v}_0^L(x) dx, \quad \forall \bar{\mathbf{v}}^L = (\bar{v}_0^L, \{\bar{v}_i^L\}, \{\bar{\mathbf{v}}_i^L\}) \in \bar{\mathbf{V}}^L. \quad (4.5)$$

To get an error estimate for the finite element approximating problems, we define the following regularity spaces for $\mathbf{u}_i$ and u_i . For the function u_i we define the space $\bar{\mathcal{H}}_i$ of functions w in $L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^1(\text{curl}, Y_i))$ such that

$$w \in L^2(Y_1 \times \cdots \times Y_{i-1}, H^1(D, \tilde{H}_{\#}(\text{curl}, Y_i)))$$

and for all $j = 1, \dots, i-1$,

$$w \in L^2\left(D \times \prod_{k < i, k \neq j} Y_k, H_{\#}^1(Y_j, \tilde{H}_{\#}(\text{curl}, Y_i))\right).$$

We equip $\bar{\mathcal{H}}_i$ with the norm

$$\begin{aligned} \|w\|_{\bar{\mathcal{H}}_i} &= \|w\|_{L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^1(\text{curl}, Y_i))} + \|w\|_{L^2(Y_1 \times \cdots \times Y_{i-1}, H^1(D, \tilde{H}_{\#}(\text{curl}, Y_i)))} \\ &\quad + \sum_{j=1}^{i-1} \|w\|_{L^2\left(D \times \prod_{k < i, k \neq j} Y_k, H_{\#}^1(Y_j, \tilde{H}_{\#}(\text{curl}, Y_i))\right)}. \end{aligned}$$

By interpolation, we define the space $\bar{\mathcal{H}}_i^s$ of functions w in $L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^s(\text{curl}, Y_i))$ such that

$$w \in L^2(Y_1 \times \cdots \times Y_{i-1}, H^s(D, \tilde{H}_{\#}(\text{curl}, Y_i)))$$

and for all $j = 1, \dots, i-1$,

$$w \in L^2\left(D \times \prod_{k < i, k \neq j} Y_k, H_{\#}^s(Y_j, \tilde{H}_{\#}(\text{curl}, Y_i))\right).$$

We define $\tilde{\mathfrak{H}}_i$ as the space of functions $w \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^2(Y_i))$ such that

$$w \in L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H^1(D, H_{\#}^1(Y_i)))$$

and for all $j = 1, \dots, i-1$,

$$w \in L^2(D \times \prod_{k < i, k \neq j} Y_k, H_{\#}^1(Y_j, H_{\#}^1(Y_i))).$$

We then define the norm

$$\begin{aligned} \|w\|_{\bar{\mathfrak{H}}_i} = & \|w\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i))} + \|w\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H^1(D, H^1(Y_i)))} \\ & + \sum_{j=1}^{i-1} \|w\|_{L^2(D \times \prod_{k < i, k \neq j} Y_k, H^1(Y_j, H^1(Y_i)))}. \end{aligned}$$

By interpolation, we define $\bar{\mathfrak{H}}_i^s$ as the space of functions $w \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^{1+s}(Y_i))$ such that

$$w \in L^2(Y_1 \times \dots \times Y_{i-1}, H^s(D, H_{\#}^1(Y_i)))$$

and for all $j = 1, \dots, i-1$,

$$w \in L^2(D \times \prod_{k < i, k \neq j} Y_k, H_{\#}^s(Y_j, H_{\#}^1(Y_i))).$$

We then define the norm

$$\begin{aligned} \|w\|_{\bar{\mathfrak{H}}_i^s} = & \|w\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^{1+s}(Y_i))} + \|w\|_{L^2(Y_1 \times \dots \times Y_{i-1}, H^s(D, H^1(Y_i)))} \\ & + \sum_{j=1}^{i-1} \|w\|_{L^2(D \times \prod_{k < i, k \neq j} Y_k, H^s(Y_j, H^1(Y_i)))}. \end{aligned}$$

We have the following result.

Lemma 4.1.1. *For $w \in \bar{\mathcal{H}}_i^s$,*

$$\inf_{w^L \in \bar{W}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, \tilde{H}_{\#}(\text{curl}, Y_i))} \leq ch_L^s \|w\|_{\bar{\mathcal{H}}_i^s}.$$

Proof We first define the following orthogonal projections in the norm of $L^2(D)$, $L^2(Y)$ and $H_{\#}(\text{curl}, Y)$, respectively:

$$\begin{aligned} P^{L0} : L^2(D) &\rightarrow V^L, \\ P_{\#}^{L0} : L^2(Y) &\rightarrow V_{\#}^L, \\ P_{\#}^{Lc} : H_{\#}(\text{curl}, Y) &\rightarrow W_{\#}^L. \end{aligned}$$

For $w \in L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))$, we define

$$\bar{w}^L = P^{L0} \otimes \underbrace{P_{\#}^{L0} \otimes \cdots \otimes P_{\#}^{L0}}_{i-1 \text{ times}} \otimes P_{\#}^{Lc} w.$$

We then have

$$\begin{aligned} \|w - \bar{w}^L\|_{L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))} &\leq \|w - id \otimes P_{\#}^{Lc} w\|_{L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))} \\ &+ \|id \otimes P_{\#}^{Lc} w - id \otimes P_{\#}^{L0} \otimes P_{\#}^{Lc} w\|_{L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))} + \cdots \\ &+ \|id \otimes P_{\#}^{L0} \otimes \cdots \otimes P_{\#}^{L0} \otimes P_{\#}^{Lc} w - P^{L0} \otimes P_{\#}^{L0} \otimes \cdots \otimes P_{\#}^{L0} \otimes P_{\#}^{Lc} w\|_{L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))} \\ &\leq ch_L^s \|w\|_{\bar{\mathcal{H}}_i^s} \end{aligned}$$

due to the approximation properties (4.1)-(4.4) and to the boundedness of the projection operators. Thus

$$\begin{aligned} \inf_{w^L \in \bar{W}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \cdots \times Y_{i-1}, \tilde{H}_{\#}(\text{curl}, Y_i))} &\leq \|w - \bar{w}^L\|_{L^2(D \times \mathbf{Y}_{i-1}, H_{\#}(\text{curl}, Y_i))} \\ &\leq ch_L^s \|w\|_{\bar{\mathcal{H}}_i^s}. \end{aligned}$$

□

For the functions in the space $\bar{\mathfrak{H}}_i^s$, we have.

Lemma 4.1.2. For $w \in \bar{\mathfrak{H}}_i^s$,

$$\inf_{w^L \in \bar{V}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \cdots \times Y_{i-1}, H_{\#}^1(Y_i))} \leq ch_L^s \|w\|_{\bar{\mathfrak{H}}_i^s}.$$

The proof of this Lemma is similar to that of Lemma 4.1.1.

We then define the regularity space

$$\bar{\mathcal{H}}^s = H^s(\text{curl}, D) \times \bar{\mathcal{H}}_1^s \times \cdots \times \bar{\mathcal{H}}_n^s \times \bar{\mathfrak{H}}_1^s \times \cdots \times \bar{\mathfrak{H}}_n^s$$

with the norm

$$\|\mathbf{w}\|_{\bar{\mathcal{H}}^s} = \|w_0\|_{H^s(\text{curl}, D)} + \sum_{i=1}^n \|w_i\|_{\bar{\mathcal{H}}_i^s} + \sum_{i=1}^n \|\mathbf{w}_i\|_{\bar{\mathfrak{H}}_i^s}$$

for $\mathbf{w} = (w_0, \{w_i\}, \{\mathbf{w}_i\}) \in \bar{\mathcal{H}}^s$.

From Lemmas 4.1.2 and 4.1.1, we deduce

Lemma 4.1.3. For $\mathbf{w} \in \bar{\mathcal{H}}^s$

$$\inf_{\mathbf{w}^L \in \bar{\mathbf{V}}^L} \|\mathbf{w} - \mathbf{w}^L\|_{\mathbf{V}} \leq ch_L^s \|\mathbf{w}\|_{\bar{\mathcal{H}}^s}.$$

From the boundedness and coerciveness conditions (2.11), using Cea's lemma, we deduce the following result.

Proposition 4.1.4. If $\mathbf{u} \in \bar{\mathcal{H}}^s$, for the full tensor product finite element approximating problem (4.5) we have the error estimate

$$\|\mathbf{u} - \bar{\mathbf{u}}^L\|_{\mathbf{V}} \leq ch_L^s \|\mathbf{u}\|_{\bar{\mathcal{H}}^s}. \quad (4.6)$$

4.1.2 Sparse tensor product finite elements

We employ the following orthogonal projections

$$\begin{aligned} P^{l0} : L^2(D) &\rightarrow V^l, \\ P_{\#}^{l0} : L^2(Y) &\rightarrow V_{\#}^l, \\ P_{\#}^{l1} : H_{\#}^1(Y) &\rightarrow V_{\#}^l, \\ P_{\#}^{lc} : H_{\#}(\text{curl}, Y) &\rightarrow W_{\#}^l \end{aligned}$$

with the convention $P^{-10} = 0$, $P_{\#}^{-10} = 0$, $P_{\#}^{-11} = 0$ and $P_{\#}^{-1c} = 0$. We define the following detail spaces

$$\mathcal{V}^l = (P^{l0} - P^{(l-1)0})V^l, \quad \mathcal{V}_{\#}^l = (P_{\#}^{l0} - P_{\#}^{(l-1)0})V_{\#}^l.$$

We note that

$$V^l = \bigoplus_{0 \leq i \leq l} \mathcal{V}^i \quad \text{and} \quad V_{\#}^l = \bigoplus_{0 \leq i \leq l} \mathcal{V}_{\#}^i.$$

Therefore the full tensor product spaces $\bar{W}_i^L$ and $\bar{V}_i^L$ are defined as

$$\bar{W}_i^L = \left(\bigoplus_{0 \leq l_0, \dots, l_{i-1} \leq L} \mathcal{V}^{l_0} \otimes \mathcal{V}_{\#}^{l_1} \otimes \dots \otimes \mathcal{V}_{\#}^{l_{i-1}} \right) \otimes W_{\#}^L$$

and

$$\bar{V}_i^L = \left(\bigoplus_{0 \leq l_0, \dots, l_{i-1} \leq L} \mathcal{V}^{l_0} \otimes \mathcal{V}_{\#}^{l_1} \otimes \dots \otimes \mathcal{V}_{\#}^{l_{i-1}} \right) \otimes V_{\#}^L.$$

We then define the sparse tensor product FE spaces as

$$\hat{W}_i^L = \bigoplus_{0 \leq l_0 + \dots + l_{i-1} \leq L} \mathcal{V}^{l_0} \otimes \mathcal{V}_{\#}^{l_1} \otimes \dots \otimes \mathcal{V}_{\#}^{l_{i-1}} \otimes W_{\#}^{L-(l_0+\dots+l_{i-1})}$$

and

$$\hat{V}_i^L = \bigoplus_{0 \leq l_0 + \dots + l_{i-1} \leq L} \mathcal{V}^{l_0} \otimes \mathcal{V}_{\#}^{l_1} \otimes \dots \otimes \mathcal{V}_{\#}^{l_{i-1}} \otimes V_{\#}^{L-(l_0+\dots+l_{i-1})}.$$

We equip $\hat{\mathcal{H}}_i$ with the norm

$$\|w\|_{\hat{\mathcal{H}}_i} = \sum_{\substack{\alpha_j \in \mathbb{R}^d, |\alpha_j| \leq 1 \\ 0 \leq j \leq i-1}} \left\| \frac{\partial^{|\alpha_0|+|\alpha_1|+\dots+|\alpha_{i-1}|}}{\partial x^{\alpha_0} \partial y_1^{\alpha_1} \dots \partial y_{i-1}^{\alpha_{i-1}}} w \right\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(\text{curl}, Y_i))}.$$

We can write $\hat{\mathcal{H}}_i$ as $H^1(D, H_{\#}^1(Y_1, \dots, H_{\#}^1(Y_{i-1}, H_{\#}^1(\text{curl}, Y_i)) \dots))$. By interpolation, we define $\hat{\mathcal{H}}_i^s = H^s(D, H_{\#}^s(Y_1, \dots, H_{\#}^s(Y_{i-1}, H_{\#}^s(\text{curl}, Y_i)) \dots))$ for $0 < s < 1$.

We define $\hat{\mathfrak{H}}_i$ as the space of functions $w \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i))$ that are periodic with respect to y_j with the period being Y_j for $j = 1, \dots, i-1$ such that $\alpha_0, \alpha_1, \dots, \alpha_{i-1} \in \mathbb{R}^d$ with $|\alpha_j| \leq 1$ for $j = 0, \dots, i-1$,

$$\frac{\partial^{|\alpha_0|+|\alpha_1|+\dots+|\alpha_{i-1}|}}{\partial x^{\alpha_0} \partial y_1^{\alpha_1} \dots \partial y_{i-1}^{\alpha_{i-1}}} w \in L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i)).$$

The space $\hat{\mathfrak{H}}_i$ is equipped with the norm

$$\|w\|_{\hat{\mathfrak{H}}_i} = \sum_{\substack{\alpha_j \in \mathbb{R}^d, |\alpha_j| \leq 1 \\ 0 \leq j \leq i-1}} \left\| \frac{\partial^{|\alpha_0|+|\alpha_1|+\dots+|\alpha_{i-1}|}}{\partial x^{\alpha_0} \partial y_1^{\alpha_1} \dots \partial y_{i-1}^{\alpha_{i-1}}} w \right\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^2(Y_i))}.$$

We can write $\hat{\mathfrak{H}}_i$ as $H^1(D, H_{\#}^1(Y_1, \dots, H_{\#}^1(Y_{i-1}, H_{\#}^2(Y_i)) \dots))$. By interpolation, we define the space $\hat{\mathfrak{H}}_i^s := H^s(D, H_{\#}^s(Y_1, \dots, H_{\#}^s(Y_{i-1}, H_{\#}^{1+s}(Y_i)) \dots))$. The regularity space $\hat{\mathcal{H}}^s$ is defined as

$$\hat{\mathcal{H}}^s = H^s(\text{curl}, D) \times \hat{\mathcal{H}}_1^s \times \dots \times \hat{\mathcal{H}}_n^s \times \hat{\mathfrak{H}}_1^s \times \dots \times \hat{\mathfrak{H}}_n^s.$$

The following regularity results holds.

Proposition 4.1.5. *If*

$$a(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))^{3 \times 3}$$

and $u_0 \in H^s(\text{curl}, D)$ for $0 < s \leq 1$ then $u_i \in \hat{\mathcal{H}}_i^s$.

Similarly, if $b(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_n) \dots))$ and $u_0 \in H^s(D)$ then $\mathbf{u}_i \in \hat{\mathfrak{H}}_i^s$.

Proof. From Proposition 2.3.7

$$\text{curl}_{y_i} \chi_i^r(x, \mathbf{y}) \in C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_{i-1}, H^2(Y_i)) \dots))^3$$

so $u_i \in \hat{\mathcal{H}}_i$. Similarly, $\mathbf{u}_i \in \hat{\mathfrak{H}}_i^s$. □

We have the following approximating result.

Lemma 4.1.6. For $w \in \hat{\mathcal{H}}_i^s$,

$$\inf_{w^L \in \hat{W}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, \tilde{H}_\#(\text{curl}, Y_i))} \leq cL^{i/2} h_L^s \|w\|_{\hat{\mathcal{H}}_i^s}.$$

Proof For $w \in W_i := L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_\#(\text{curl}, Y_i))$, we define

$$\begin{aligned} \hat{w}^L = \sum_{l_0 + \dots + l_{i-1} \leq L} (P^{l_0} - P^{l_0-1}) \otimes (P_\#^{l_1 0} - P_\#^{(l_1-1)0}) \otimes \dots \\ \otimes (P_\#^{l_{i-1} 0} - P_\#^{(l_{i-1}-1)0}) \otimes P_\#^{(L-(l_0+\dots+l_{i-1}))c} w. \end{aligned}$$

We note that

$$\begin{aligned} w - \hat{w}^L = \sum_{l_0 + \dots + l_{i-1} > L} (P^{l_0} - P^{l_0-1}) \otimes (P_\#^{l_1 0} - P_\#^{(l_1-1)0}) \otimes \dots \otimes (P_\#^{l_{i-1} 0} - P_\#^{(l_{i-1}-1)0}) \otimes idw \\ + \sum_{l_0 + \dots + l_{i-1} \leq L} (P^{l_0} - P^{l_0-1}) \otimes (P_\#^{l_1 0} - P_\#^{(l_1-1)0}) \otimes \dots \\ \otimes (P_\#^{l_{i-1} 0} - P_\#^{(l_{i-1}-1)0}) \otimes (id - P_\#^{(L-(l_0+\dots+l_{i-1}))c}) w. \end{aligned}$$

Therefore

$$\begin{aligned} \|w - \hat{w}^L\|_{W_i}^2 &\leq \sum_{l_0 + \dots + l_{i-1} > L} \left\| (P^{l_0} - P^{l_0-1}) \otimes (P_\#^{l_1 0} - P_\#^{(l_1-1)0}) \otimes \dots \right. \\ &\quad \left. \otimes (P_\#^{l_{i-1} 0} - P_\#^{(l_{i-1}-1)0}) \otimes idw \right\|_{W_i}^2 \\ &\quad + \sum_{l_0 + \dots + l_{i-1} \leq L} \left\| (P^{l_0} - P^{l_0-1}) \otimes (P_\#^{l_1 0} - P_\#^{(l_1-1)0}) \otimes \dots \right. \\ &\quad \left. \otimes (P_\#^{l_{i-1} 0} - P_\#^{(l_{i-1}-1)0}) \otimes (id - P_\#^{(L-(l_0+\dots+l_{i-1}))c}) w \right\|_{W_i}^2 \\ &\leq c \sum_{l_0 + \dots + l_{i-1} > L} 2^{-2(l_0+\dots+l_{i-1})} \|w\|_{\hat{\mathcal{H}}_i^s}^2 \\ &\quad + c \sum_{l_0 + \dots + l_{i-1} \leq L} 2^{-2L} \|w\|_{\hat{\mathcal{H}}_i^s}^2 \\ &\leq c \sum_{l_0 + \dots + l_{i-1} > L} 2^{-2(l_0+\dots+l_{i-1})} \|w\|_{\hat{\mathcal{H}}_i^s}^2 \\ &\quad + cL^i 2^{-2L} \|w\|_{\hat{\mathcal{H}}_i^s}^2. \end{aligned}$$

We note that

$$\begin{aligned}
\sum_{l_0+\dots+l_{i-1}>L} 2^{-2(l_0+\dots+l_{i-1})} &= \sum_{l_0,\dots,l_{i-2}\geq 0} 2^{-2(l_0+\dots+l_{i-2})} \sum_{l_{i-1}\geq \max(L+1-(l_0+\dots+l_{i-2}),0)} 2^{-2l_{i-1}} \\
&= \sum_{l_0,\dots,l_{i-2}\geq 0} 2^{-2(l_0+\dots+l_{i-2})} 2^{-2\max(L+1-(l_0+\dots+l_{i-2}),0)} \frac{1}{1-1/4} \\
&= \frac{4}{3} \sum_{l_0,\dots,l_{i-2}\geq 0} 2^{-2\max(L+1,l_0+\dots+l_{i-2})} \\
&= \frac{4}{3} \sum_{l_0+\dots+l_{i-2}\leq L} 2^{-2(L+1)} + \frac{4}{3} \sum_{l_0+\dots+l_{i-2}\geq L+1} 2^{-2(l_0+\dots+l_{i-2})} \\
&\leq \frac{4}{3} L^{i-1} 2^{-2(L+1)} + \frac{4}{3} \sum_{l_0+\dots+l_{i-2}\geq L+1} 2^{-2(l_0+\dots+l_{i-2})} \\
&\leq \frac{4}{3} L^{i-1} 2^{-2(L+1)} \\
&\quad + \left(\frac{4}{3}\right)^2 L^{i-2} 2^{-2(L+1)} + \left(\frac{4}{3}\right)^2 \sum_{l_0+\dots+l_{i-3}\geq L+1} 2^{-2(l_0+\dots+l_{i-3})} \\
&\quad \dots \\
&\leq ciL^i 2^{-2(L+1)}
\end{aligned}$$

where c is independent of L . From this we get the conclusion. $\square$

Lemma 4.1.7. For $w \in \hat{\mathfrak{H}}_i^s$,

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \dots \times Y_{i-1}, H_{\#}^1(Y_i))} \leq cL^{i/2} h_L^s \|w\|_{\hat{\mathfrak{H}}_i^s}.$$

The proof of this Lemma is similar to that for Lemma 4.1.6. From these lemmas we deduce

Lemma 4.1.8. For $w \in \hat{\mathcal{H}}^s$

$$\inf_{w^L \in \hat{V}^L} \|w - w^L\|_{\mathbf{V}} \leq cL^{n/2} h_L^s \|w\|_{\hat{\mathcal{H}}^s}.$$

From this we deduce the following error estimate for the sparse tensor product finite element problem (4.7).

Proposition 4.1.9. If the solution $\mathbf{u}$ of problem (2.10) belongs to $\hat{\mathcal{H}}^s$, then

$$\|\mathbf{u} - \hat{\mathbf{u}}^L\|_{\mathbf{V}} \leq cL^{n/2} h_L^s \|\mathbf{u}\|_{\hat{\mathcal{H}}^s}.$$

Remark 4.1.10. *The dimension of the full tensor product finite element space $\bar{\mathbf{V}}^L$ is $O(2^{d(n+1)L})$ which is very large when L is large. The dimension of the sparse tensor product finite element space $\hat{\mathbf{V}}^L$ is $O(L^n 2^{dL})$ which is essentially equal to the number of degrees of freedom for solving a problem in $\mathbb{R}^d$ for obtaining the same level of accuracy.*

4.2 Numerical correctors

We derive numerical correctors for the solution of the multiscale problem from the finite element in this section. In the two-scale case, an explicit error for the numerical corrector can be deduced from the finite element error and the homogenization error.

4.2.1 Numerical correctors for two-scale problems

To employ the finite element solutions to construct numerical correctors for u^ε , we define the following operator for $\Phi \in L^1(D)$

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\} \right) dz.$$

Let D^ε be a 2ε neighbourhood of D . Regarding Φ as zero when x is outside D , we have

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x) dx = \int_D \int_Y \Phi(x, y) dx dy. \quad (4.8)$$

The proof of (4.8) may be found in [36]. We have the following result.

Lemma 4.2.1. *Assume that for $r = 1, 2, 3$, $\text{curl}_y \chi^r(x, y) \in C^1(\bar{D}, C(\bar{Y}))^3$ and $u_0 \in H^s(\text{curl}, D)$, then*

$$\left\| \text{curl}_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) - \mathcal{U}^\varepsilon(\text{curl}_y u_1) \right\|_H \leq c\varepsilon^s.$$

Proof We adapt the proof of Lemma 5.5 in [57]. As

$$u_1(x, y) = \sum_{r=1}^3 (\text{curl } u_0(x))_r \chi^r(x, y),$$

it is sufficient to show that for each $r = 1, 2, 3$

$$\begin{aligned} & \int_D \left| (\text{curl } u_0(x))_r \text{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) \right. \\ & \quad \left. - \int_Y \left(\text{curl } u_0 \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right)_r \text{curl}_y \chi^r \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t, \frac{x}{\varepsilon} \right) dt \right|^2 dx \leq c\varepsilon^{2s}. \end{aligned}$$

The expression on the left hand side is bounded by

$$\begin{aligned} & \int_D \int_Y \left| (\operatorname{curl} u_0(x))_r \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \left(\operatorname{curl} u_0 \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right)_r \operatorname{curl}_y \chi^r \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t, \frac{x}{\varepsilon} \right) \right|^2 dt dx \\ & \leq 2 \int_D \int_Y \left| \left((\operatorname{curl} u_0(x))_r - \left(\operatorname{curl} u_0 \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right)_r \right) \operatorname{curl}_y \chi^r \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t, \frac{x}{\varepsilon} \right) \right|^2 dt dx \\ & \quad + 2 \int_D \int_Y \left| (\operatorname{curl} u_0(x))_r \right|^2 \left| \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \operatorname{curl}_y \chi^r \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t, \frac{x}{\varepsilon} \right) \right|^2 dt dx . \end{aligned}$$

As $\operatorname{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, there exists a constant c such that

$$\sup_{x \in D} \sup_{t \in Y} \left| \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \operatorname{curl}_y \chi^r \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t, \frac{x}{\varepsilon} \right) \right| \leq c\varepsilon.$$

From this we have

$$\begin{aligned} & \int_D \left| (\operatorname{curl} u_0(x))_r \operatorname{curl}_y \chi^r \left(x, \frac{x}{\varepsilon} \right) - \mathcal{U}^\varepsilon \left((\operatorname{curl} u_0(\cdot))_r \operatorname{curl}_y \chi^r(\cdot, \cdot) \right)(x) \right|^2 dx \\ & \leq c \int_D \int_Y \left| (\operatorname{curl} u_0(x))_r - \left(\operatorname{curl} u_0 \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right)_r \right|^2 dt dx + c\varepsilon^2 . \end{aligned}$$

We now show that for $\operatorname{curl} u_0 \in H^s(D)$,

$$\int_D \int_Y \left| (\operatorname{curl} u_0(x))_r - \left(\operatorname{curl} u_0 \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right)_r \right|^2 dt dx \leq c\varepsilon^{2s}. \quad (4.9)$$

Letting $\phi(x)$ be a smooth function, we have

$$\begin{aligned} & \int_D \int_Y \left| \phi(x) - \phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon t \right) \right|^2 dt \\ & \leq \sum_{i=1}^3 \int_D \int_Y \left| \phi \left(\varepsilon \left[\frac{x_1}{\varepsilon} \right] + \varepsilon t_1, \dots, \varepsilon \left[\frac{x_{i-1}}{\varepsilon} \right] + \varepsilon t_{i-1}, x_i, \dots, x_3 \right) \right. \\ & \quad \left. - \phi \left(\varepsilon \left[\frac{x_1}{\varepsilon} \right] + \varepsilon t_1, \dots, \varepsilon \left[\frac{x_i}{\varepsilon} \right] + \varepsilon t_i, x_{i+1}, \dots, x_3 \right) \right|^2 dt dx \\ & \leq \sum_{i=1}^3 \int_D \int_Y \left| \varepsilon \int_{t_i}^{\left\{ \frac{x_i}{\varepsilon} \right\}} \frac{\partial \phi}{\partial x_i} \left(\varepsilon \left[\frac{x_1}{\varepsilon} \right] + \varepsilon t_1, \dots, \varepsilon \left[\frac{x_i}{\varepsilon} \right] + \varepsilon \zeta_i, x_{i+1}, \dots, x_3 \right) d\zeta_i \right|^2 dt dx \\ & \leq \varepsilon^2 \sum_{i=1}^3 \int_D \int_Y \int_0^1 \left| \frac{\partial \phi}{\partial x_i} \left(\varepsilon \left[\frac{x_1}{\varepsilon} \right] + \varepsilon t_1, \dots, \varepsilon \left[\frac{x_i}{\varepsilon} \right] + \varepsilon \zeta_i, x_{i+1}, \dots, x_3 \right) \right|^2 d\zeta_i dt dx \\ & \leq \varepsilon^2 \sum_{i=1}^3 \int_D \left| \frac{\partial \phi}{\partial x_i} \right|^2 dx. \end{aligned}$$

which follows from (2.33); here we freeze the variables $x_{i+1}, \dots, x_3$. Let $\psi \in H^1(D)$. We consider a sequence $\{\phi_n\}_n \subset C^\infty(\bar{D})$ which converges to ψ in $H^1(D)$. Using the fact that $(\mathcal{U}^\varepsilon(\Phi))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)$ and (4.8), we have

$$\begin{aligned} \int_D |\mathcal{U}^\varepsilon(\phi_n)(x) - \mathcal{U}^\varepsilon(\psi)(x)|^2 dx &\leq \int_D \mathcal{U}^\varepsilon((\phi_n - \psi)^2)(x) dx \\ &\leq \int_D (\phi_n(x) - \psi(x))^2 dx \rightarrow 0 \end{aligned}$$

when $n \rightarrow \infty$. Therefore

$$\begin{aligned} \int_D (\psi(x) - \mathcal{U}^\varepsilon(\psi)(x))^2 dx &\leq 3 \int_D (\psi - \phi_n)^2 dx + 3 \int_D (\phi_n - \mathcal{U}^\varepsilon(\phi_n))^2 dx \\ &\quad + 3 \int_D (\mathcal{U}^\varepsilon(\phi_n) - \mathcal{U}^\varepsilon(\psi))^2 dx \\ &\leq 6 \int_D (\psi - \phi_n)^2 dx + 3\varepsilon^2 \sum_{i=1}^3 \int_D \left| \frac{\partial \phi_n}{\partial x_i} \right|^2 dx. \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\int_D (\psi(x) - \mathcal{U}^\varepsilon(\psi)(x))^2 dx \leq 3\varepsilon^2 \sum_{i=1}^3 \int_D \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx.$$

Let T be the linear map from $L^2(D)$ to $L^2(D \times Y)$ so that

$$T(\phi)(x, y) = \phi(x) - \phi\left(\varepsilon \left\lceil \frac{x}{\varepsilon} \right\rceil + \varepsilon y\right).$$

We thus have that

$$\|T\|_{H^1(D) \rightarrow L^2(D \times Y)} \leq c\varepsilon.$$

On the other hand

$$\|T\|_{L^2(D) \rightarrow L^2(D \times Y)} \leq c.$$

From interpolation theory, we deduce that

$$\|T\|_{H^s(D) \rightarrow L^2(D \times Y)} \leq c\varepsilon^s.$$

We then get (4.9). The conclusion then follows. $\square$

We have the following corrector result.

Theorem 4.2.2. *Assume that $u_0 \in H^s(\text{curl}, D)$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\text{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$ and $\omega^r \in C^1(\bar{D}, C^1(\bar{Y}))$ for all $r = 1, 2, 3$. Then for the full tensor product finite element solution $(\bar{u}_0^L, \bar{u}_1^L, \bar{u}_1^L)$ of problem (4.5) we have*

$$\|u^\varepsilon - \bar{u}_0^L - \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + h_L^s)$$

and

$$\|\text{curl } u^\varepsilon - \text{curl } \bar{u}_0^L - \mathcal{U}^\varepsilon(\text{curl}_y \bar{u}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + h_L^s).$$

Proof From Lemma 4.2.1, we have

$$\begin{aligned} \|\operatorname{curl} u^\varepsilon - \operatorname{curl} \bar{u}_0^L - \mathcal{U}^\varepsilon(\operatorname{curl}_y \bar{u}_1^L)\|_H &\leq \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0 - \operatorname{curl}_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right\|_H \\ &\quad + \|\operatorname{curl} u_0 - \operatorname{curl} \bar{u}_0^L\|_H \\ &\quad + \left\| \operatorname{curl}_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) - \mathcal{U}^\varepsilon(\operatorname{curl}_y u_1) \right\|_H \\ &\quad + \|\mathcal{U}^\varepsilon(\operatorname{curl}_y u_1) - \mathcal{U}^\varepsilon(\operatorname{curl}_y \bar{u}_1^L)\|_H. \end{aligned}$$

Using the fact that $(\mathcal{U}^\varepsilon(\Phi))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)$ and (4.8), we have

$$\|\mathcal{U}^\varepsilon(\operatorname{curl}_y u_1) - \mathcal{U}^\varepsilon(\operatorname{curl}_y \bar{u}_1^L)\|_H \leq \|\operatorname{curl}_y u_1 - \operatorname{curl}_y \bar{u}_1^L\|_{L^2(D \times Y)^3} \leq ch_L^s.$$

This together with (4.6), Theorem 2.2.2 and Lemma 4.2.1 gives

$$\|\operatorname{curl} u^\varepsilon - \operatorname{curl} \bar{u}_0^L - \mathcal{U}^\varepsilon(\operatorname{curl}_y \bar{u}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + h_L^s).$$

Similarly, we have

$$\|u^\varepsilon - \bar{u}_0^L - \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + h_L^s).$$

□

For the sparse tensor product finite element approximation, we have

Theorem 4.2.3. *Assume that $u_0 \in H^s(\operatorname{curl}, D)$, $\chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\operatorname{curl}_y \chi^r \in C^1(\bar{D}, C(\bar{Y}))^3$ and $\omega^r \in C^1(\bar{D}, C^1(\bar{Y}))$ for all $r = 1, 2, 3$. Then for the sparse tensor product finite element solution $(\hat{u}_0^L, \hat{u}_1^L, \hat{\mathbf{u}}_1^L)$ of problem (4.7) we have*

$$\|u^\varepsilon - \hat{u}_0^L - \mathcal{U}^\varepsilon(\nabla_y \hat{\mathbf{u}}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + L^{1/2} h_L^s)$$

and

$$\|\operatorname{curl} u^\varepsilon - \operatorname{curl} \hat{u}_0^L - \mathcal{U}^\varepsilon(\operatorname{curl}_y \hat{u}_1^L)\|_H \leq c(\varepsilon^{s/(1+s)} + L^{1/2} h_L^s).$$

The proof is identical to that of Theorem 4.2.2.

4.2.2 Numerical corrector for multiscale problems

As a homogenization error estimate is not available for the multiscale case, we will not be able to derive an explicit error for the numerical corrector. We therefore do not distinguish the full and sparse tensor FEs and denote the FE solution as $\mathbf{u}^L = (u_0^L, \{u_i^L\}, \{\mathbf{u}_i^L\})$ for both cases.

Theorem 4.2.4. *For the full and sparse tensor product finite element approximation solutions $\mathbf{u}^L = (u_0^L, \{u_i^L\}, \{\mathbf{u}_i^L\})$ in (4.5) and (4.7), we have*

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|u^\varepsilon - [u_0^L + \mathcal{U}_n^\varepsilon(\nabla_{y_1} \mathbf{u}_1^L) + \cdots + \mathcal{U}_n^\varepsilon(\nabla_{y_n} \mathbf{u}_n^L)]\|_H = 0, \quad (4.10)$$

and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|\operatorname{curl} u^\varepsilon - [\operatorname{curl} u_0^L + \mathcal{U}_n^\varepsilon(\operatorname{curl}_{y_1} u_1^L) + \cdots + \mathcal{U}_n^\varepsilon(\operatorname{curl}_{y_n} u_n^L)]\|_H = 0. \quad (4.11)$$

Proof We note that

$$\begin{aligned} & \left\| \mathcal{U}_n^\varepsilon \left(\operatorname{curl} u_0 + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k \right) - \mathcal{U}_n^\varepsilon \left(\operatorname{curl} u_0^L + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k^L \right) \right\|_H \\ & \leq \int_D \mathcal{U}_n^\varepsilon \left(\left\| \left(\operatorname{curl} u_0 + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k \right) - \left(\operatorname{curl} u_0^L + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k^L \right) \right\|^2 \right) (x) dx \\ & \leq \int_D \int_{\mathbf{Y}} \left\| \left(\operatorname{curl} u_0 + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k \right) - \left(\operatorname{curl} u_0^L + \sum_{k=1}^n \operatorname{curl}_{y_k} u_k^L \right) \right\|^2 d\mathbf{y} dx \end{aligned}$$

which converges to 0 when $L \rightarrow \infty$. From this and (2.35), we get (4.11). We obtain (4.10) in the same way. $\square$

4.3 Numerical results

The detail spaces $\mathcal{V}^l$ and $\mathcal{V}_\#^l$ are defined via orthogonal projection in Section 4.1.2, which are difficult to construct in numerical implementation. We employ Riesz basis functions and define equivalent norms, which facilitate the construction of these spaces. We make the following assumption.

Assumption 4.3.1. *(i) For each multidimensional vector $j \in \mathbb{N}_0^d$, there exists a set of indices $I^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi^{jk} \in L^2(D)$ for $k \in I^j$, such that $V^l = \operatorname{span} \{\phi^{jk} : |j|_\infty \leq l\}$. There are constants $c_2 > c_1 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I^j} \phi^{jk} c_{jk} \in V^l$, then the following norm equivalences hold:*

$$c_1 \sum_{\substack{|j|_\infty \leq l \\ k \in I^j}} |c_{jk}|^2 \leq \|\phi\|_{L^2(D)}^2 \leq c_2 \sum_{\substack{|j|_\infty \leq l \\ k \in I^j}} |c_{jk}|^2,$$

where c_1 and c_2 are independent of ϕ and l .

(ii) For the space $L^2(Y)$, for each $j \in \mathbb{N}_0^d$, there exists a set of indices $I_0^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_0^{jk} \in L^2(Y)$, $k \in I_0^j$, such that $V_\#^l = \text{span}\{\phi_0^{jk} : |j|_\infty \leq l\}$. There are constants $c_4 > c_3 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I_0^j} \phi_0^{jk} c_{jk} \in V^l$, then

$$c_3 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} |c_{jk}|^2 \leq \|\phi\|_{L^2(Y)}^2 \leq c_4 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} |c_{jk}|^2$$

where c_3 and c_4 are independent of ϕ and l .

Due to the norm equivalence, we can use $\mathcal{V}^l = \text{span}\{\phi^{jk} : |j|_\infty = l\}$ and $\mathcal{V}_\#^l = \text{span}\{\phi_0^{jk} : |j|_\infty = l\}$ to construct the sparse tensor product FE spaces.

Example (i) We can construct a hierarchical basis for $L^2(0, 1)$ as follows. We first take three piecewise linear functions as the basis for level $j = 0$: ψ^{01} obtains values $(1, 0)$ at $(0, 1/2)$ and is 0 in $(1/2, 1)$, ψ^{02} is piecewise linear and obtains values $(0, 1, 0)$ at $(0, 1/2, 1)$, and ψ^{03} obtains values $(0, 1)$ at $(1/2, 1)$ and is 0 in $(0, 1/2)$. The basis functions for other levels are constructed from the wavelet function ψ that takes values $(0, -1, 2, -1, 0)$ at $(0, 1/2, 1, 3/2, 2)$, the left boundary function ψ^{left} taking values $(-2, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$, and the right boundary function ψ^{right} taking values $(0, -1, 2, -2)$ at $(1/2, 1, 3/2, 2)$. For levels $j \geq 1$, $I^j = \{1, 2, \dots, 2^j\}$. The wavelet basis functions are defined as $\psi^{j1}(x) = 2^{j/2} \psi^{left}(2^j x)$, $\psi^{jk}(x) = 2^{j/2} \psi(2^j x - k + 3/2)$ for $k = 2, \dots, 2^j - 1$ and $\psi^{j2^j} = 2^{j/2} \psi^{right}(2^j x - 2^j + 2)$. This base satisfies Assumption 4.3.1 (i).

(ii) For $Y = (0, 1)$, we can construct a hierarchical periodic basis functions for $L^2(Y)$ from those in (i). For level 0, we exclude ψ^{01} , ψ^{03} and include the periodic piecewise linear function that takes values $(1, 0, 1)$ at $(0, 1/2, 1)$ respectively. At other levels, the functions ψ^{left} and ψ^{right} are replaced by the piecewise linear functions that take values $(0, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$ and values $(0, -1, 2, 0)$ at $(1/2, 1, 3/2, 2)$ respectively.

When $D = (0, 1)^d$, the basis functions can be constructed by taking the tensor products of the basis functions in $(0, 1)$. They satisfy Assumption 4.3.1 after appropriate scaling, see [49].

Remark 4.3.2. When the norm equivalence for the basis functions in $L^2(D)$ and in $L^2(Y)$ does not hold, we can still prove a rate of convergence similar to those in Lemmas 4.1.6 and 4.1.7 for the sparse tensor product finite element approximations. For example, with the division of the domain D into sets of triangles $\mathcal{T}^l$, the set of continuous piecewise linear functions with value 1 at one vertex and 0 at all the others form a basis of V^l . Let S^l be the set of vertices of the set of simplices $\mathcal{T}^l$. We can define $\mathcal{V}^l$ as the linear span of functions which are 1 at a vertex in $S^l \setminus S^{l-1}$ and 0 at all the other vertices. We can then construct the sparse tensor product

finite element approximations with these spaces but the norm equivalence does not hold. A rate of convergence for sparse tensor product finite elements similar to those in Lemmas 4.1.6 and 4.1.7 can be deduced (see, e.g., [54]).

In the first example, we consider a two-scale Maxwell equation in the two dimensional domain $D = (0, 1)^2$. The coefficients

$$a(x, y) = \frac{(1 + x_1)(1 + x_2)}{(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)},$$

and

$$b(x, y) = \frac{1}{(1 + x_1)(1 + x_2)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)},$$

where $(x_1, x_2) \in D$ and $(y_1, y_2) \in Y$. To compute a^0 , we note that when $d = 2$ the cell equation (2.13) becomes

$$\int_D \int_Y (a(x, y)(1 + \operatorname{curl}_y \chi)) \operatorname{curl}_y v dy dx = 0, \quad \forall v \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y)).$$

Thus

$$\operatorname{curl}_y(a(x, y)(1 + \operatorname{curl}_y \chi)) = 0.$$

Therefore,

$$a(x, y)(1 + \operatorname{curl}_y \chi) = C(x)$$

where $C(x)$ is a function of x and is independent of y . Then

$$1 + \operatorname{curl}_y \chi = \frac{C(x)}{a(x, y)}.$$

Taking the integral both sides with respect to y we have:

$$1 = C(x) \int_Y (a(x, y))^{-1} dy,$$

so

$$C(x) = \left(\int_Y (a(x, y))^{-1} dy \right)^{-1}.$$

Thus the homogenized coefficient a^0 is

$$\begin{aligned} a^0(x) &= \int_Y a(x, y)(1 + \operatorname{curl}_y \chi) dy \\ &= \int_Y \left(\int_Y (a(x, y))^{-1} dy \right)^{-1} dy = \left(\int_Y (a(x, y))^{-1} dy \right)^{-1} \\ &= \frac{4(1 + x_1)(1 + x_2)}{9}. \end{aligned}$$

To compute the homogenized coefficient b^0 , we follow the formula of Jikov et al. [62] page 17 and obtain

$$b^0(x) = \frac{\sqrt{2}}{3(1+x_1)(1+x_2)}.$$

We choose

$$f = \begin{pmatrix} \frac{4}{9}(1+x_1)(1+2x_2-x_1) + \frac{\sqrt{2}}{3(1+x_1)(1+x_2)}x_1x_2(1-x_2) \\ \frac{4}{9}(1+x_2)(1+2x_1-x_2) + \frac{\sqrt{2}}{3(1+x_1)(1+x_2)}x_1x_2(1-x_1) \end{pmatrix}$$

so that the solution to the homogenized equation is

$$u_0 = \begin{pmatrix} x_1x_2(1-x_2) \\ x_1x_2(1-x_1) \end{pmatrix}.$$

In Figure 4.2, we plot the energy error versus the mesh size for the sparse tensor product finite element approximations of the two-scale homogenized Maxwell problem. The figure agrees with the error estimate in Proposition 4.1.9.

Figure 4.2: The sparse tensor energy error $B(\mathbf{u} - \hat{\mathbf{u}}^L, \mathbf{u} - \hat{\mathbf{u}}^L)$

In the second example, we consider the case where b is the identity matrix, i.e., it does not depend on y . In this case, from (2.10) we note that the function $\mathbf{u}_1 = 0$. We choose

$$a(x, y) = \frac{(1+x_1)(1+x_2)}{(1+\cos^2 2\pi y_1)(1+\cos^2 2\pi y_2)}$$

and

$$f = \begin{pmatrix} \frac{4}{9}[2\pi(1+x_1)(1+x_2)\sin 2\pi x_2 + (1+x_1)(\cos 2\pi x_1 - \cos 2\pi x_2)] + \frac{\sin 2\pi x_2}{2\pi} \\ \frac{4}{9}[2\pi(1+x_1)(1+x_2)\sin 2\pi x_1 - (1+x_2)(\cos 2\pi x_1 - \cos 2\pi x_2)] + \frac{\sin 2\pi x_1}{2\pi} \end{pmatrix}$$

so that the solution to the homogenized problem is

$$u_0 = \begin{pmatrix} \frac{1}{2\pi} \sin 2\pi x_2 \\ \frac{1}{2\pi} \sin 2\pi x_1 \end{pmatrix}.$$

Figure 4.3 plots the energy error versus the mesh size for the sparse tensor product finite element approximations for the two-scale homogenized Maxwell problem. The plot confirms the analysis.

Figure 4.3: The sparse tensor energy error $B(\mathbf{u} - \hat{\mathbf{u}}^L, \mathbf{u} - \hat{\mathbf{u}}^L)$

Summary

In this chapter, we solve the multiscale homogenized time independent Maxwell equations by finite elements. We develop the sparse tensor product for edge finite elements. We solve for the solution of the homogenized equation and the corrector terms at the same time, and get all the necessary macroscopic and microscopic information, with essentially optimal complexity. Numerical correctors are deduced from the finite element solutions, with an error in terms of the homogenization error and the finite element error in the two-scale case. We show that the regularity required for obtaining the sparse tensor product finite element error holds. Some two-scale stationary Maxwell equations in two dimensions are solved to illustrate the theory.

Chapter 5

High dimensional finite elements for time dependent Maxwell equations

We develop finite element approximations for the multiscale Maxwell wave equation established in Chapter 3. In Section 5.1, we consider the spatially semidiscrete problem and the fully discrete problems for general finite element spaces. We establish the conditions for the schemes to converge, and estimate the errors. We then apply the general framework to the full and sparse tensor product finite element spaces in Section 5.2. We show that the regularity required for obtaining the rate of convergence of the sparse tensor product finite element approximation holds. Using the finite element solutions, we construct numerical correctors in Section 5.3, with explicit errors in terms of the mesh size and the microscopic scale in the two-scale case. In Section 5.4, we present some numerical examples for two-scale problems in two dimensions to illustrate the theoretical results.

5.1 Finite element discretization

5.1.1 Spatially semidiscrete problem

We consider in this section the spatial semidiscretization of the homogenized problem (3.9). For approximating u_0 , we suppose that there is a hierarchy of finite dimensional subspaces

$$W^1 \subset W^2 \subset \cdots \subset W^L \cdots \subset W;$$

to approximate u_i , $i = 1, 2, \dots, n$, we assume a hierarchy of finite dimensional subspaces

$$W_i^1 \subset W_i^2 \subset \dots \subset W_i^L \dots \subset W_i;$$

and to approximate $\mathbf{u}_i$, $i = 1, 2, \dots, n$, we assume a hierarchy of finite dimensional subspaces

$$V_i^1 \subset V_i^2 \subset \dots \subset V_i^L \dots \subset V_i.$$

Let

$$\mathbf{V}^L = W^L \times W_1^L \times \dots \times W_n^L \times V_1^L \times \dots \times V_n^L;$$

which is a finite dimensional subspace of $\mathbf{V}$ defined in (3.6). We consider the spatially semidiscrete approximating problems: Find $\mathbf{u}^L(t) = (u_0^L, u_1^L, \dots, u_n^L, \mathbf{u}_1^L, \dots, \mathbf{u}_n^L) \in \mathbf{V}^L$ so that

$$\begin{aligned} \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\frac{\partial^2}{\partial t^2} u_0^L(t, x) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2}{\partial t^2} u_i^L(t, x, \mathbf{y}_i) \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) \right. \\ \left. + a(x, \mathbf{y}) \left(\text{curl } u_0^L + \sum_{i=1}^n \text{curl}_{y_i} u_i^L \right) \cdot \left(\text{curl } v_0^L + \sum_{i=1}^n \text{curl}_{y_i} v_i^L \right) \right] d\mathbf{y} dx \\ = \int_D f(t, x) \cdot v_0^L(x) dx \quad (5.1) \end{aligned}$$

for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L, \mathbf{v}_0^L, \dots, \mathbf{v}_n^L) \in \mathbf{V}^L$. Let $g_0^L \in W^L$, $g_1^L \in W^L$ which are approximations of g_0 and g_1 in W and in H respectively. The initial conditions (3.10) becomes

$$u_0^L(0, \cdot) = g_0^L, \quad \nabla_{y_i} \mathbf{u}_i^L(0, \cdot, \cdot) = 0. \quad (5.2)$$

We approximate the initial conditions (3.11) and (3.12) by

$$\begin{aligned} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0^L}{\partial t}(0) + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L(0) \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx \\ = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) g_1^L(x) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx \end{aligned}$$

for all $v_0^L \in W^L$ and $\mathbf{v}_i^L \in W_i^L$, i.e.,

$$\int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial u_0^L}{\partial t}(0) - g_1^L + \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L(0) \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx = 0.$$

Using the coercivity of the matrix $b(x, \mathbf{y})$, we get

$$\frac{\partial u_0^L}{\partial t}(0) = g_1^L, \quad \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L(0) = 0. \quad (5.3)$$

For $\mathbf{v} = (v_0, v_1, \dots, v_n, \mathbf{v}_1, \dots, \mathbf{v}_n)$ and $\mathbf{w} = (w_0, w_1, \dots, w_n, \mathbf{w}_1, \dots, \mathbf{w}_n)$ in $\mathbf{V} = W \times W_1 \times \dots \times W_n \times V_1 \times \dots \times V_n$, we define the bilinear forms

$$A(\mathbf{v}, \mathbf{w}) = \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} v_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i \right) \cdot \left(\operatorname{curl} w_0 + \sum_{i=1}^n \operatorname{curl}_{y_i} w_i \right) d\mathbf{y} dx,$$

and

$$B(\mathbf{v}, \mathbf{w}) = \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(v_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i \right) \cdot \left(w_0 + \sum_{i=1}^n \nabla_{y_i} \mathbf{w}_i \right) d\mathbf{y} dx.$$

Proposition 5.1.1. *Problem (5.1) together with the initial condition (5.2) and (5.3) has a unique solution.*

Proof In the bilinear form B , let R be the matrix that describes the interaction of the basis functions of W^L with themselves, let N be the matrix that describes the interaction of the basis functions of $V_1^L \times \dots \times V_n^L$ with themselves, and let S be the matrix that describes the interaction of the basis functions of W^L and the basis functions of $V_1^L \times \dots \times V_n^L$. For the bilinear form A , let Q be the matrix that describes the interaction of the basis functions of W^L with themselves, let P be the matrix describing the interaction of the basis functions of W^L and $W_1^L \times \dots \times W_n^L$, and let M be the matrix describing the interactions of the basis functions of $W_1^L \times \dots \times W_n^L$ and themselves. Let F be the column matrix describing the interaction of f and the basis functions of W^L . Let C_0 be the coefficient vector in the expansion of u_0^L with respect to the basis functions of W^L . Let C_1 be the coefficient vector in the expansion of $(u_1^L, \dots, u_n^L)$ with respect to the basis functions of $W_1^L \times \dots \times W_n^L$. Let $\mathfrak{C}_1$ be the coefficient vector in the expansion of $(\mathbf{u}_1^L, \dots, \mathbf{u}_n^L)$ with respect to the basis functions $V_1^L \times \dots \times V_n^L$. We have the following equations

$$\begin{aligned} R \frac{d^2 C_0}{dt^2} + S \frac{d^2 \mathfrak{C}_1}{dt^2} + Q C_0 + P C_1 &= F, \\ P^\top C_0 + M C_1 &= 0, \\ S^\top \frac{d^2 C_0}{dt^2} + N \frac{d^2 \mathfrak{C}_1}{dt^2} &= 0. \end{aligned}$$

Using $C_1 = -M^{-1}P^\top C_0$, we deduce the system

$$\begin{bmatrix} R & S \\ S^\top & N \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} C_0 \\ \mathfrak{C}_1 \end{bmatrix} + \begin{bmatrix} Q - P M^{-1} P^\top & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_0 \\ \mathfrak{C}_1 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

We note that $\begin{bmatrix} R & S \\ S^\top & N \end{bmatrix}$ is the Gram matrix for the interaction of the basis of W^L and $V_1^L \times \dots \times V_n^L$ in the matrix B so is positive definite. The system thus has a unique solution. $\square$

For each $t \in (0, T)$, let $\mathbf{w}^L(t) = (w_0^L, w_1^L, \dots, w_n^L, \mathfrak{w}_1^L, \dots, \mathfrak{w}_n^L) \in \mathbf{V}^L$ be the solution of the problem

$$B(\mathbf{w}^L(t) - \mathbf{u}(t), \mathbf{v}^L) + A(\mathbf{w}^L(t) - \mathbf{u}(t), \mathbf{v}^L) = 0 \quad (5.4)$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. As the coefficients a and b in (2.1) are both uniformly bounded and coercive for all $x \in D$ and $\mathbf{y} \in \mathbf{Y}$, problem (5.4) has a unique solutions. Let $\mathbf{q}^L = \mathbf{w}^L - \mathbf{u}$. We then have the following estimate.

Lemma 5.1.2. *For the solution $\mathbf{w}^L$ of problem (5.4)*

$$\|\mathbf{q}^L(t)\|_{\mathbf{V}} \leq c \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}}.$$

Proof From (5.4), we have

$$B(\mathbf{w}^L - \mathbf{u}, \mathbf{w}^L - \mathbf{u}) + A(\mathbf{w}^L - \mathbf{u}, \mathbf{w}^L - \mathbf{u}) = B(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L - \mathbf{u}) + A(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L - \mathbf{u})$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. From the coerciveness and boundedness of matrices $a(x, \mathbf{y})$ and $b(x, \mathbf{y})$ we get the conclusion. $\square$

When $\mathbf{u}$ is sufficiently regular with respect to t , we have the following estimates.

Lemma 5.1.3. *If $\frac{\partial \mathbf{u}}{\partial t} \in C([0, T], \mathbf{V})$, then*

$$\left\| \frac{\partial \mathbf{q}^L}{\partial t} \right\|_{L^\infty(0, T; \mathbf{V})} \leq c \sup_{t \in [0, T]} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \left\| \frac{\partial \mathbf{u}}{\partial t} - \mathbf{v}^L \right\|_{\mathbf{V}}.$$

If $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T, \mathbf{V})$, then

$$\left\| \frac{\partial^2 \mathbf{q}^L}{\partial t^2} \right\|_{L^2(0, T; \mathbf{V})} \leq c \inf_{\mathbf{v}^L \in L^2(0, T; \mathbf{V}^L)} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{v}^L \right\|_{L^2(0, T; \mathbf{V})}.$$

Proof If $\frac{\partial \mathbf{u}}{\partial t} \in C([0, T], \mathbf{V})$ from (5.4) we have

$$B\left(\frac{\partial}{\partial t} \mathbf{w}^L(t) - \frac{\partial}{\partial t} \mathbf{u}(t), \mathbf{v}^L\right) + A\left(\frac{\partial}{\partial t} \mathbf{w}^L(t) - \frac{\partial}{\partial t} \mathbf{u}(t), \mathbf{v}^L\right) = 0$$

for all $\mathbf{v} \in \mathbf{V}^L$. We then proceed as in the proof of Lemma 5.1.2 to show the first inequality. The proof for the second inequality is similar. $\square$

Let $\mathbf{p}^L = \mathbf{u}^L - \mathbf{w}^L$, i.e., for $i = 1, \dots, n$,

$$p_i^L = u_i^L - w_i^L, \mathfrak{p}_i^L = \mathfrak{u}_i^L - \mathfrak{w}_i^L \text{ and } p_0^L = u_0^L - w_0^L.$$

We recall the definition of the spaces H_i in (3.1)

Proposition 5.1.4. *Assume that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, then there is a constant c depending on T such that for all $t \in (0, T)$*

$$\begin{aligned} & \left\| \frac{\partial p_0^L}{\partial t}(t) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(t) \right\|_{H_n} + \left\| \operatorname{curl} p_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L \right\|_{H_n} \\ & \leq c \left[\left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0, T; H_n)} \right. \\ & \quad \left. + \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(0) \right\|_{H_n} + \left\| \operatorname{curl} p_0^L(0) \right\|_H \right]. \end{aligned}$$

Proof Since $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, from (3.9) and (5.1) we have for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L, \mathbf{v}_1^L, \dots, \mathbf{v}_n^L) \in \mathbf{V}^L$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\frac{\partial^2 p_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{p}_i^L}{\partial t^2} \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) \right. \\ & \quad \left. + a(x, \mathbf{y}) \left(\operatorname{curl} p_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L \right) \cdot \left(\operatorname{curl} v_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i^L \right) \right] d\mathbf{y} dx \\ & = - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx - A(\mathbf{q}^L, \mathbf{v}^L). \end{aligned}$$

From (5.4) we have $A(\mathbf{q}^L, \mathbf{v}^L) = -B(\mathbf{q}^L, \mathbf{v}^L)$. Thus

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\frac{\partial^2 p_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{p}_i^L}{\partial t^2} \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) \right. \\ & \quad \left. + a(x, \mathbf{y}) \left(\operatorname{curl} p_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L \right) \cdot \left(\operatorname{curl} v_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} v_i^L \right) \right] d\mathbf{y} dx \\ & = - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right) \\ & \quad \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx. \end{aligned} \tag{5.5}$$

Let $\mathbf{v}^L = \frac{\partial \mathbf{p}^L}{\partial t}$. We then have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\frac{\partial p_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t} \right) \cdot \left(\frac{\partial p_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t} \right) \right. \\
& \quad \left. + a(x, \mathbf{y}) \left(\operatorname{curl} p_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L \right) \cdot \left(\operatorname{curl} p_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L \right) \right] d\mathbf{y} dx \\
& \leq c \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{H_n} \left\| \frac{\partial p_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t} \right\|_{H_n} \\
& \leq \frac{c}{\gamma} \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{H_n}^2 + c\gamma \left\| \frac{\partial p_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t} \right\|_{H_n}^2
\end{aligned}$$

for a constant $\gamma > 0$. Integrating both sides on $(0, t)$ for $0 < t < T$, and using the coercivity of the matrices a and b , we have

$$\begin{aligned}
& \left\| \frac{\partial p_0^L}{\partial t}(t) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(t) \right\|_{H_n}^2 + \left\| \operatorname{curl} p_0^L(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(t) \right\|_{H_n}^2 \\
& \leq \frac{c}{\gamma} \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0, T; H_n)}^2 \\
& \quad + c\gamma T \sup_{t \in [0, T]} \left\| \frac{\partial p_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t} \right\|_{H_n} \\
& \quad + c \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(0) \right\|_{H_n}^2 + c \left\| \operatorname{curl} p_0^L(0) + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right\|_{H_n}^2.
\end{aligned}$$

Choosing a sufficiently small constant γ , there is a constant c depending on T so that for all $t \in (0, T)$

$$\begin{aligned}
& \left\| \frac{\partial p_0^L}{\partial t}(t) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(t) \right\|_{H_n}^2 + \left\| \operatorname{curl} p_0^L(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(t) \right\|_{H_n}^2 \\
& \leq c \left[\left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0, T; H_n)}^2 \right. \\
& \quad \left. + \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(0) \right\|_{H_n}^2 + \left\| \operatorname{curl} p_0^L(0) + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right\|_{H_n}^2 \right].
\end{aligned}$$

Consider equation (5.5) for $t = 0$. Let $v_0^L = 0$, $\mathbf{v}_i^L = 0$ and $v_i^L = p_i^L$. We then have

$$\int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} p_0^L(0) + \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right) \cdot \left(\sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right) d\mathbf{y} dx = 0,$$

i.e.,

$$\begin{aligned} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right) \cdot \left(\sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right) d\mathbf{y} dx \\ = - \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \operatorname{curl} p_0^L(0) \cdot \left(\sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right) d\mathbf{y} dx. \end{aligned}$$

Using (2.1), we deduce that

$$\left\| \sum_{i=1}^n \operatorname{curl}_{y_i} p_i^L(0) \right\|_{H_n} \leq c \left\| \operatorname{curl} p_0^L(0) \right\|_H.$$

We then get the conclusion. □

Proposition 5.1.5. *Assume that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0, T; \mathbf{V})$, and that*

$$\lim_{L \rightarrow \infty} \|g_0^L - g_0\|_W = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \|g_1^L - g_1\|_H = 0. \quad (5.6)$$

Then

$$\begin{aligned} \lim_{L \rightarrow \infty} \left\{ \left\| \frac{\partial(u_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \left\| \nabla_{y_i} \frac{\partial(\mathbf{u}_i^L - \mathbf{u}_i)}{\partial t} \right\|_{L^\infty(0, T; H_i)} \right. \\ \left. + \left\| \operatorname{curl}(u_0^L - u_0) \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \left\| \operatorname{curl}_{y_i}(u_i^L - u_i) \right\|_{L^\infty(0, T; H_i)} \right\} = 0. \end{aligned}$$

Proof From Proposition 5.1.4, as $\mathbf{u}^L - \mathbf{u} = \mathbf{p}^L + \mathbf{q}^L$, we have

$$\begin{aligned}
 & \left\| \frac{\partial(u_0^L - u_0)}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial(\mathbf{u}_i^L - \mathbf{u}_i)}{\partial t} \right\|_{L^\infty(0,T;H_n)}^2 \\
 & + \left\| \operatorname{curl}(u_0^L - u_0) + \sum_{i=1}^n \operatorname{curl}_{y_i}(u_i^L - u_i) \right\|_{L^\infty(0,T;H_n)}^2 \\
 & \leq c \left[\left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_n)}^2 \right. \\
 & + \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(0) \right\|_{H_n}^2 + \left\| \operatorname{curl} p_0^L(0) \right\|_H^2 \Big] \\
 & + \left\| \frac{\partial q_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{q}_i^L}{\partial t} \right\|_{L^\infty(0,T;H_n)} + \|\mathbf{q}^L\|_{L^\infty(0,T;\mathbf{V})} \quad (5.7)
 \end{aligned}$$

We show that $\lim_{L \rightarrow \infty} \|\mathbf{q}^L\|_{L^\infty(0,T;\mathbf{V})} = 0$. As $\mathbf{u} \in C([0,T];\mathbf{V})$, $\mathbf{u}$ is uniformly continuous as a function from $[0,T]$ to $\mathbf{V}$. For $\varepsilon > 0$, there is a piecewise constant (with respect to t) function $\tilde{\mathbf{u}} \in L^\infty(0,T;\mathbf{V})$ such that $\|\mathbf{u} - \tilde{\mathbf{u}}\|_{L^\infty(0,T;\mathbf{V})} < \varepsilon$. As $\tilde{\mathbf{u}}(t)$ obtains only a finite number of $\mathbf{V}$ -values, when L is sufficiently large, there is $\mathbf{v}^L \in L^\infty(0,T;\mathbf{V})$ such that $\|\tilde{\mathbf{u}} - \mathbf{v}^L\|_{L^\infty(0,T;\mathbf{V})} < \varepsilon$. Thus

$$\lim_{L \rightarrow \infty} \sup_{t \in (0,T)} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{u}(t) - \mathbf{v}^L\|_{\mathbf{V}} = 0.$$

We then apply Lemma 5.1.2. Similarly, we have from Lemmas 5.1.2 and 5.1.3

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \mathbf{q}_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_n)} = 0$$

and

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial q_0^L}{\partial t} + \sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{q}_i^L}{\partial t} \right\|_{L^\infty(0,T;H_n)} = 0.$$

Furthermore, we have that

$$\left\| \operatorname{curl} p_0^L(0) \right\|_H \leq \left\| \operatorname{curl} u_0^L(0) - \operatorname{curl} u_0(0) \right\|_H + \left\| \operatorname{curl} u_0(0) - \operatorname{curl} w_0^L(0) \right\|_H,$$

which converges to 0 due to (5.6) and Lemma 5.1.2. Similarly, we have

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_i \nabla_{y_i} \frac{\partial \mathbf{p}_i^L}{\partial t}(0) \right\|_{H_n} = 0.$$

We then get the conclusion. $\square$

5.1.2 Fully discrete problem

Following the scheme of Dupont [40], we discretize problem (5.1) in both spatial and temporal variables. Let $\Delta t = \frac{T}{M}$ where M is a positive integer. Let $t_m = m\Delta t$. We employ the following notations of Dupont for a function $r \in C([0, T], X)$ where X is a Banach space and $r_m = r(t_m, \cdot)$

$$\begin{aligned} r_{m+1/2} &= \frac{1}{2}(r_{m+1} + r_m), & r_{m,\theta} &= \theta r_{m+1} + (1 - 2\theta)r_m + \theta r_{m-1}, \\ \partial_t r_{m+1/2} &= (r_{m+1} - r_m)/\Delta t, & \partial_t^2 r_m &= (r_{m+1} - 2r_m + r_{m-1})/(\Delta t)^2, \\ \delta_t r_m &= (r_{m+1} - r_{m-1})/(2\Delta t). \end{aligned}$$

We consider the following fully discrete problem:

For $m = 1, \dots, M$ find $\mathbf{u}_m^L = (u_{0,m}^L, u_{1,m}^L, \dots, u_{n,m}^L, \mathbf{u}_{1,m}^L, \dots, \mathbf{u}_{n,m}^L) \in \mathbf{V}^L$ such that for $m = 1, \dots, M - 1$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\partial_t^2 u_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 u_{i,m}^L \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) + \right. \\ & \left. a(x, \mathbf{y}) \left(\operatorname{curl} u_{0,m,1/4}^L + \sum_{i=1}^n \operatorname{curl}_{y_i} u_{i,m,1/4}^L \right) \cdot \left(\operatorname{curl} v_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{v}_i^L \right) \right] d\mathbf{y} dx \\ &= \int_D f_{m,1/4}(t, x) \cdot v_0^L(x) dx, \end{aligned} \quad (5.8)$$

for all $\mathbf{v}^L = (v_0^L, v_1^L, \dots, v_n^L, \mathbf{v}_1^L, \dots, \mathbf{v}_n^L) \in \mathbf{V}^L$.

For continuous functions $r : [0, T] \rightarrow X$, let

$$\|r\|_{\tilde{L}^\infty(0,T;X)} := \max_{0 \leq m < M} \|r_{m+1/2}\|_X.$$

We also denote by

$$\|\partial_t r\|_{\tilde{L}^\infty(0,T;X)} := \max_{0 \leq m < M} \|\partial_t r_{m+1/2}\|_X.$$

Let

$$\mathbf{p}_m^L := \mathbf{u}_m^L - \mathbf{w}_m^L.$$

Lemma 5.1.6. *Assume that $\mathbf{u} \in H^2(0, T; \mathbf{V})$, $\frac{\partial^2 q_0^L}{\partial t^2} \in L^2(0, T; H)$, $\frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \in L^2(0, T; H_i)$. If $\frac{\partial^3 u_0}{\partial t^3} \in L^2(0, T; H)$ and $\frac{\partial^3}{\partial t^3} \nabla_{y_i} \mathbf{u}_i \in L^2(0, T; H_i)$, then there exists a*

constant c independent of Δt and $\mathbf{u}$ such that for each $j = 1, 2, \dots, M-1$

$$\begin{aligned}
 & \|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\
 & \leq c \left[(\Delta t)^2 \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_H^2 + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3} \right\|_{H_i}^2 + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 \right. \\
 & \quad \left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)}^2 + \|q_0^L\|_{L^\infty(0,T;H)}^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)}^2 \right] \\
 & \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).
 \end{aligned}$$

Further, if $\frac{\partial^4 u_0}{\partial t^4} \in L^2(0, T; H)$ and $\frac{\partial^4}{\partial t^4} \nabla_{y_i} \mathbf{u}_i \in L^2(0, T; H_i)$, then there exists a constant c independent of Δt and $\mathbf{u}$ such that for each $j = 1, 2, \dots, M-1$

$$\begin{aligned}
 & \|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\
 & \leq c \left[(\Delta t)^4 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_H^2 + (\Delta t)^4 \sum_{i=1}^n \left\| \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4} \right\|_{H_i}^2 + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 \right. \\
 & \quad \left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)}^2 + \|q_0^L\|_{L^\infty(0,T;H)}^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)}^2 \right] \\
 & \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).
 \end{aligned}$$

Proof From (5.4) and (5.8), we have

$$\begin{aligned}
 A(\mathbf{w}^L, \mathbf{v}^L) &= A(\mathbf{u}, \mathbf{v}^L) - B(\mathbf{w}^L - \mathbf{u}, \mathbf{v}^L) = \int_D f(t, x) \cdot v_0^L(x) dx \\
 &\quad - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial^2 u_0}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_i \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx - B(\mathbf{q}^L, \mathbf{v}^L).
 \end{aligned}$$

Averaging this equation at time t_{m+1} , t_m and t_{m-1} with weights $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ respectively,

and using (5.8), we get

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\partial_t^2 u_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \mathbf{u}_{i,m}^L \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx + A(\mathbf{p}_{m,1/4}^L, \mathbf{v}^L) \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial^2 u_{0,m,1/4}}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_{i,m,1/4} \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx \\ & \quad + B(\mathbf{q}_{m,1/4}^L, \mathbf{v}^L). \end{aligned}$$

Thus

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\partial_t^2 p_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \mathbf{p}_{i,m}^L \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx + A(\mathbf{p}_{m,1/4}^L, \mathbf{v}^L) \\ &= \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\frac{\partial^2 u_{0,m,1/4}}{\partial t^2} - \partial_t^2 u_{0,m} + \sum_{i=1}^n \left(\frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_{i,m,1/4} - \nabla_{y_i} \partial_t^2 \mathbf{u}_{i,m} \right) \right) \\ & \quad \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx \\ & \quad - \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\partial_t^2 q_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L \right) \cdot \left(v_0^L + \sum_{i=1}^n \nabla_{y_i} \mathbf{v}_i^L \right) d\mathbf{y} dx \\ & \quad + B(\mathbf{q}_{m,1/4}^L, \mathbf{v}^L). \end{aligned}$$

We denote by

$$s_{0,m} = \frac{\partial^2 u_{0,m,1/4}}{\partial t^2} - \partial_t^2 u_{0,m}, \quad s_{i,m} = \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{u}_{i,m,1/4} - \nabla_{y_i} \partial_t^2 \mathbf{u}_{i,m}.$$

Let $\mathbf{v}^L = \delta_t \mathbf{p}_m^L$. Using the following relationships:

$$\begin{aligned} \partial_t^2 r_m &= \frac{1}{\Delta t} (\partial_t r_{m+1/2} - \partial_t r_{m-1/2}), & r_{m,1/4} &= \frac{1}{2} (r_{m+1/2} + r_{m-1/2}) \\ \delta_t r_m &= \frac{1}{2} (\partial_t r_{m+1/2} + \partial_t r_{m-1/2}) = \frac{1}{\Delta t} (r_{m+1/2} - r_{m-1/2}), \end{aligned}$$

we have

$$\begin{aligned} & \frac{1}{2\Delta t} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\partial_t p_{0,m+1/2}^L - \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n \nabla_{y_i} (\partial_t \mathbf{p}_{i,m+1/2}^L - \partial_t \mathbf{p}_{i,m-1/2}^L) \right) \\ & \quad \cdot \left(\partial_t p_{0,m+1/2}^L + \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n \nabla_{y_i} (\partial_t \mathbf{p}_{i,m+1/2}^L + \partial_t \mathbf{p}_{i,m-1/2}^L) \right) d\mathbf{y} dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2\Delta t} \int_D \int_{\mathbf{Y}} a(x, \mathbf{y}) \left(\operatorname{curl} (p_{0,m+1/2}^L + p_{0,m-1/2}^L) + \sum_{i=1}^n \operatorname{curl}_{y_i} (p_{i,m+1/2}^L + p_{i,m-1/2}^L) \right) \\
 & \quad \cdot \left(\operatorname{curl} (p_{0,m+1/2}^L - p_{0,m-1/2}^L) + \sum_{i=1}^n \operatorname{curl}_{y_i} (p_{i,m+1/2}^L - p_{i,m-1/2}^L) \right) d\mathbf{y} dx \\
 & = \frac{1}{2} \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(s_{0,m} - \partial_t^2 q_{0,m}^L + q_{0,m,1/4}^L + \sum_{i=1}^n (s_{i,m} - \nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L + \nabla_{y_i} \mathbf{q}_{i,m,1/4}^L) \right) \\
 & \quad \cdot \left(\partial_t p_{0,m+1/2}^L + \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n (\nabla_{y_i} \partial_t \mathbf{p}_{i,m+1/2}^L + \nabla_{y_i} \partial_t \mathbf{p}_{i,m-1/2}^L) \right) d\mathbf{y} dx.
 \end{aligned}$$

We thus have

$$\begin{aligned}
 & \frac{1}{2\Delta t} [B(\partial_t \mathbf{p}_{m+1/2}^L, \partial_t \mathbf{p}_{m+1/2}^L) - B(\partial_t \mathbf{p}_{m-1/2}^L, \partial_t \mathbf{p}_{m-1/2}^L) \\
 & \quad + A(\mathbf{p}_{m+1/2}^L, \mathbf{p}_{m+1/2}^L) - A(\mathbf{p}_{m-1/2}^L, \mathbf{p}_{m-1/2}^L)] \\
 & \leq c \|s_{0,m} - \partial_t^2 q_{0,m}^L + q_{0,m,1/4}^L + \sum_{i=1}^n (s_{i,m} - \nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L + \nabla_{y_i} \mathbf{q}_{i,m,1/4}^L)\|_{H_n} \\
 & \quad \cdot \left\| \partial_t p_{0,m+1/2}^L + \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n (\nabla_{y_i} \partial_t \mathbf{p}_{i,m+1/2}^L + \nabla_{y_i} \partial_t \mathbf{p}_{i,m-1/2}^L) \right\|_{H_n} \\
 & \leq \frac{c}{\gamma} \left(\|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_{H_i}^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L\|_{H_i}^2 \right. \\
 & \quad \left. + \|q_{0,m,1/4}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_{i,m,1/4}^L\|_{H_i}^2 \right) \\
 & \quad + c\gamma \left(\|\partial_t p_{0,m+1/2}^L\|_H^2 + \|\partial_t p_{0,m-1/2}^L\|_H^2 \right. \\
 & \quad \left. + \sum_{i=1}^n \|\nabla_{y_i} \partial_t \mathbf{p}_{i,m+1/2}^L\|_{H_i}^2 + \sum_{i=1}^n \|\nabla_{y_i} \partial_t \mathbf{p}_{i,m-1/2}^L\|_{H_i}^2 \right).
 \end{aligned}$$

Summing this up for all $m = 1, \dots, j$, we deduce

$$\begin{aligned}
 & B(\partial_t \mathbf{p}_{j+1/2}^L, \partial_t \mathbf{p}_{j+1/2}^L) - B(\partial_t \mathbf{p}_{1/2}^L, \partial_t \mathbf{p}_{1/2}^L) + A(\mathbf{p}_{j+1/2}^L, \mathbf{p}_{j+1/2}^L) - A(\mathbf{p}_{1/2}^L, \mathbf{p}_{1/2}^L) \\
 & \leq \frac{c}{\gamma} 2\Delta t \sum_{m=1}^M \left(\|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_{H_i}^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L\|_{H_i}^2 \right. \\
 & \quad \left. + \|q_{0,m,1/4}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_{i,m,1/4}^L\|_{H_i}^2 \right)
 \end{aligned}$$

$$\begin{aligned}
& + c\gamma 2\Delta t M \left(\max_{1 \leq m \leq M} \|\partial_t p_{0,m+1/2}^L\|_H^2 + \sum_{i=1}^n \max_{1 \leq m \leq M} \|\partial_t \nabla_{y_i} \mathbf{p}_{i,m+1/2}^L\|_{H_i}^2 \right) \\
& + c\gamma 2\Delta t \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 \right).
\end{aligned}$$

From (2.1), we have

$$\begin{aligned}
& \|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\
& \leq \frac{c}{\gamma} 2\Delta t \sum_{m=1}^M \left(\|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_{H_i}^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L\|_{H_i}^2 \right. \\
& \quad \left. + \|q_{0,m,1/4}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_{i,m,1/4}^L\|_{H_i}^2 \right) \\
& \quad + c\gamma 2\Delta t M \left(\max_{1 \leq m \leq M} \|\partial_t p_{0,m+1/2}^L\|_H^2 + \sum_{i=1}^n \max_{1 \leq m \leq M} \|\partial_t \nabla_{y_i} \mathbf{p}_{i,m+1/2}^L\|_{H_i}^2 \right) \\
& \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).
\end{aligned}$$

Choosing γ sufficiently small, we deduce that

$$\begin{aligned}
& \|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\
& \leq \frac{c}{\gamma} 2\Delta t \sum_{m=1}^M \left(\|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_{H_i}^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \partial_t^2 \mathbf{q}_{i,m}^L\|_{H_i}^2 \right. \\
& \quad \left. + \|q_{0,m,1/4}^L\|_H^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_{i,m,1/4}^L\|_{H_i}^2 \right) \\
& \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).
\end{aligned}$$

Following Dupont [40], using the integral formula of the remainder of Taylor ex-

pansion, we have,

$$\begin{aligned}
 \partial_t^2 q_{0,m}^L &= \frac{q_{0,m+1}^L - 2q_{0,m}^L + q_{0,m-1}^L}{(\Delta t)^2} \\
 &= \frac{1}{(\Delta t)^2} \left(q_{0,m+1}^L - q_{0,m}^L - \frac{\partial q_{0,m}^L}{\partial t} \Delta t + q_{0,m-1}^L - q_{0,m}^L + \frac{\partial q_{0,m}^L}{\partial t} \Delta t \right) \\
 &= \frac{1}{(\Delta t)^2} \left(q_{0,m+1}^L - q_{0,m}^L - \frac{\partial q_{0,m}^L}{\partial t} (t_{m+1} - t_m) + q_{0,m-1}^L - q_{0,m}^L - \frac{\partial q_{0,m}^L}{\partial t} (t_{m-1} - t_m) \right) \\
 &= \frac{1}{(\Delta t)^2} \left(\int_{t_m}^{t_{m+1}} (t_{m+1} - t) \frac{\partial^2 q_0^L}{\partial t^2}(t) dt + \int_{t_m}^{t_{m-1}} (t_{m-1} - t) \frac{\partial^2 q_0^L}{\partial t^2}(t) dt \right) \\
 &= \frac{1}{(\Delta t)^2} \left(\int_0^{\Delta t} (\Delta t - \tau) \frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) d\tau + \int_0^{-\Delta t} (-\Delta t - \tau) \frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) d\tau \right) \\
 &= (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) d\tau,
 \end{aligned}$$

and similarly, for $i = 1, \dots, n$

$$\partial_t^2 (\nabla_{y_i} \mathbf{q}_{i,m}^L) = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 \nabla_{y_i} \mathbf{q}_i^L}{\partial t^2}(t_m + \tau) d\tau.$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
 \sum_{m=1}^M \|\partial_t^2 q_{0,m}^L\|_H^2 \Delta t &= \sum_{m=1}^M \int_D |\partial_t^2 q_{0,m}^L|^2 dx \Delta t \\
 &= \sum_{m=1}^M \int_D \left| (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) d\tau \right|^2 dx \Delta t \\
 &\leq \sum_{m=1}^M (\Delta t)^{-3} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|)^2 d\tau \int_D \int_{-\Delta t}^{\Delta t} \left(\frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) \right)^2 d\tau dx \\
 &\leq \sum_{m=1}^M (\Delta t)^{-3} \frac{2}{3} (\Delta t)^3 \int_D \int_{-\Delta t}^{\Delta t} \left(\frac{\partial^2 q_0^L}{\partial t^2}(t_m + \tau) \right)^2 d\tau dx \\
 &\leq \frac{2}{3} \sum_{m=1}^M \int_D \int_{t_{m-1}}^{t_{m+1}} \left(\frac{\partial^2 q_0^L}{\partial t^2}(t) \right)^2 dt dx \\
 &\leq \frac{4}{3} \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2,
 \end{aligned}$$

and similarly, we have

$$\sum_{m=1}^M \|\partial_t^2 \nabla_{y_i} \mathbf{q}_{i,m}^L\|_{H_i}^2 \Delta t \leq \frac{4}{3} \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)}^2.$$

We can write

$$\begin{aligned} s_{0,m} &= \frac{1}{4} \int_0^{\Delta t} \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau \\ &\quad - \frac{1}{4} \int_{-\Delta t}^0 \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau. \end{aligned}$$

Indeed,

$$\begin{aligned} s_{0,m} &= \frac{\partial^2 u_{0,m,1/4}}{\partial t^2} - \partial_t^2 u_{0,m} \\ &= \frac{1}{4} \frac{\partial^2 u_{0,m+1}}{\partial t^2} + \frac{1}{2} \frac{\partial^2 u_{0,m}}{\partial t^2} + \frac{1}{4} \frac{\partial^2 u_{0,m-1}}{\partial t^2} - \frac{u_{0,m+1} - 2u_{0,m} + u_{0,m-1}}{(\Delta t)^2} \\ &= \frac{1}{4} \frac{\partial^2 u_{0,m+1}}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u_{0,m}}{\partial t^2} + \frac{1}{4} \frac{\partial^2 u_{0,m-1}}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u_{0,m}}{\partial t^2} \\ &\quad - \frac{1}{(\Delta t)^2} \left(u_{0,m+1} - u_{0,m} - \frac{\partial u_0(t_m)}{\partial t} \Delta t - \frac{\partial^2 u_0(t_m)}{\partial t^2} \frac{(\Delta t)^2}{2!} \right) \\ &\quad - \frac{1}{(\Delta t)^2} \left(u_{0,m-1} - u_{0,m} + \frac{\partial u_0(t_m)}{\partial t} \Delta t - \frac{\partial^2 u_0(t_m)}{\partial t^2} \frac{(\Delta t)^2}{2!} \right) \end{aligned}$$

Using the integral formula of the remainder of Taylor expansion, we have

$$\begin{aligned} s_{0,m} &= \frac{1}{4} \int_{t_m}^{t_{m+1}} \frac{\partial^3 u_0(t)}{\partial t^3} dt + \frac{1}{4} \int_{t_m}^{t_{m-1}} \frac{\partial^3 u_0(t)}{\partial t^3} dt \\ &\quad - \frac{1}{2!(\Delta t)^2} \int_{t_m}^{t_{m+1}} (t_{m+1} - t)^2 \frac{\partial^3 u_0(t)}{\partial t^3} dt - \frac{1}{2!(\Delta t)^2} \int_{t_m}^{t_{m-1}} (t_{m-1} - t)^2 \frac{\partial^3 u_0(t)}{\partial t^3} dt \\ &= \frac{1}{4} \int_0^{\Delta t} \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau - \frac{1}{2!(\Delta t)^2} \int_0^{\Delta t} (\Delta t - |\tau|)^2 \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau \\ &\quad - \frac{1}{4} \int_{-\Delta t}^0 \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau + \frac{1}{2!(\Delta t)^2} \int_{-\Delta t}^0 (\Delta t - |\tau|)^2 \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau \\ &= \frac{1}{4} \int_0^{\Delta t} \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau \\ &\quad - \frac{1}{4} \int_{-\Delta t}^0 \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3}(t_m + \tau) d\tau. \end{aligned}$$

Similarly, we have

$$s_{i,m} = \frac{1}{4} \int_0^{\Delta t} \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3}(t_m + \tau) d\tau \\ - \frac{1}{4} \int_{-\Delta t}^0 \left(1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3}(t_m + \tau) d\tau.$$

Using similar estimate as above, we have

$$\|s_{0,m}\|_H^2 \leq c\Delta t \int_{t_{m-1}}^{t_m+1} \left\| \frac{\partial^3 u_0}{\partial t^3}(\tau) \right\|_H^2 d\tau, \quad \|s_{i,m}\|_{H_i}^2 \leq c\Delta t \int_{t_{m-1}}^{t_m+1} \left\| \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3}(\tau) \right\|_{H_i}^2 d\tau.$$

We also have

$$\|q_{0,m,1/4}^L\|_H \leq \max_{t \in [0,T]} \|q_0^L(t)\|_H \quad \text{and} \quad \|\nabla_{y_i} \mathbf{q}_{i,m,1/4}^L\|_{H_i} \leq \max_{t \in [0,T]} \|\nabla_{y_i} \mathbf{q}_i^L(t)\|_{H_i}.$$

We thus deduce

$$\|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\text{curl } p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\ \leq c \left[(\Delta t)^2 \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_H^2 + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3} \right\|_{H_i}^2 + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 \right. \\ \left. + \sum_{i=1}^n \left\| \frac{\partial^2 \nabla_{y_i} \mathbf{q}_i^L}{\partial t^2} \right\|_{L^2(0,T;H_i)}^2 + \|q_0^L\|_{L^\infty(0,T;H)}^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)}^2 \right] \\ + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\text{curl } p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).$$

When

$$\frac{\partial^4 u_0}{\partial t^4} \in L^2(0, T; H) \quad \text{and} \quad \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4} \in L^2(0, T; H_i),$$

we have

$$s_{0,m} = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left(3 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 u_0}{\partial t^4}(t_m + \tau) d\tau.$$

Indeed,

$$\begin{aligned}
s_{0,m} &= \frac{\partial^2 u_{0,m,1/4}}{\partial t^2} - \partial_t^2 u_{0,m} \\
&= \frac{1}{4} \frac{\partial^2 u_{0,m+1}}{\partial t^2} + \frac{1}{2} \frac{\partial^2 u_{0,m}}{\partial t^2} + \frac{1}{4} \frac{\partial^2 u_{0,m-1}}{\partial t^2} - \frac{u_{0,m+1} - 2u_{0,m} + u_{0,m-1}}{(\Delta t)^2} \\
&= \frac{1}{4} \frac{\partial^2 u_{0,m+1}}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u_{0,m}}{\partial t^2} - \frac{1}{4} \frac{\partial^3 u_0(t_m)}{\partial t^3} \Delta t \\
&\quad + \frac{1}{4} \frac{\partial^2 u_{0,m-1}}{\partial t^2} - \frac{1}{4} \frac{\partial^2 u_{0,m}}{\partial t^2} + \frac{1}{4} \frac{\partial^3 u_0(t_m)}{\partial t^3} \Delta t \\
&\quad - \frac{1}{(\Delta t)^2} \left(u_{0,m+1} - u_{0,m} - \frac{\partial u_0(t_m)}{\partial t} \Delta t - \frac{\partial^2 u_0(t_m)}{\partial t^2} \frac{(\Delta t)^2}{2!} - \frac{\partial^3 u_0(t_m)}{\partial t^3} \frac{(\Delta t)^3}{3!} \right) \\
&\quad - \frac{1}{(\Delta t)^2} \left(u_{0,m-1} - u_{0,m} + \frac{\partial u_0(t_m)}{\partial t} \Delta t - \frac{\partial^2 u_0(t_m)}{\partial t^2} \frac{(\Delta t)^2}{2!} + \frac{\partial^3 u_0(t_m)}{\partial t^3} \frac{(\Delta t)^3}{3!} \right)
\end{aligned}$$

Using the integral formula of the remainder of Taylor expansion, we have

$$\begin{aligned}
s_{0,m} &= \frac{1}{4} \int_{t_m}^{t_{m+1}} (t_{m+1} - t) \frac{\partial^4 u_0(t)}{\partial t^4} dt + \frac{1}{4} \int_{t_m}^{t_{m-1}} (t_{m-1} - t) \frac{\partial^4 u_0(t)}{\partial t^4} dt \\
&\quad - \frac{1}{3!(\Delta t)^2} \int_{t_m}^{t_{m+1}} (t_{m+1} - t)^3 \frac{\partial^4 u_0(t)}{\partial t^4} dt - \frac{1}{3!(\Delta t)^2} \int_{t_m}^{t_{m-1}} (t_{m-1} - t)^3 \frac{\partial^4 u_0(t)}{\partial t^4} dt \\
&= \frac{1}{4} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^4 u_0}{\partial t^4}(t_m + \tau) d\tau - \frac{1}{3!(\Delta t)^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|)^3 \frac{\partial^4 u_0}{\partial t^4}(t_m + \tau) d\tau \\
&= \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left(3 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 u_0}{\partial t^4}(t_m + \tau) d\tau.
\end{aligned}$$

Similarly, we have

$$s_{i,m} = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left(3 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4}(t_m + \tau) d\tau.$$

Using similar estimate as for $q_{0,m}^L$, we have

$$\|s_{0,m}\|_H^2 \leq c(\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 u_0}{\partial t^4}(\tau) \right\|_H^2 d\tau$$

and

$$\|s_{i,m}\|_{H_i}^2 \leq c(\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4}(\tau) \right\|_{H_i}^2 d\tau.$$

Thus we have

$$\begin{aligned}
 & \|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,j+1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,j+1/2}^L\|_{H_i}^2 \\
 & \leq c \left[(\Delta t)^4 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_H^2 + (\Delta t)^4 \sum_{i=1}^n \left\| \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4} \right\|_{H_i}^2 + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2 \right. \\
 & \quad \left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)}^2 + \|q_0^L\|_{L^\infty(0,T;H)}^2 + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)}^2 \right] \\
 & \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i}^2 + \|\operatorname{curl} p_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i}^2 \right).
 \end{aligned}$$

□

We then have the following error estimates.

Proposition 5.1.7. *Assume that $\mathbf{u} \in H^2(0,T;\mathbf{V})$. If $\frac{\partial^3 u_0}{\partial t^3} \in L^2(0,T;H)$ and $\frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3} \in L^2(0,T;H_i)$, then there is a constant c such that*

$$\begin{aligned}
 & \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{u}_i^L - \partial_t \nabla_{y_i} \mathbf{u}_i\|_{\tilde{L}^\infty(0,T;H_i)} \\
 & \quad + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i} u_i^L - \operatorname{curl}_{y_i} u_i\|_{\tilde{L}^\infty(0,T;H_i)} \\
 & \leq c \left[\Delta t \left\| \frac{\partial^3 u_0}{\partial t^3} \right\|_{L^2(0,T;H)} + \Delta t \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3} \right\|_{L^2(0,T;H_i)} + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)} \right. \\
 & \quad \left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)} + \|q_0^L\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)} \right] \\
 & \quad + c \left(\|\partial_t p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i} \right) \\
 & \quad + \|\partial_t q_0^L\|_{\tilde{L}^\infty(0,T;H)} + \|\operatorname{curl} q_0^L\|_{\tilde{L}^\infty(0,T;H)} \\
 & \quad + \sum_{i=1}^n \left(\|\partial_t \nabla_{y_i} \mathbf{q}_i^L\|_{\tilde{L}^\infty(0,T;H_i)} + \|\operatorname{curl}_{y_i} q_i^L\|_{\tilde{L}^\infty(0,T;H_i)} \right).
 \end{aligned}$$

If $\frac{\partial^4 u_0}{\partial t^4} \in L^2(0,T;H)$ and $\frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4} \in L^2(0,T;H_i)$, then there is a constant c such that

$$\begin{aligned}
& \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{u}_i^L - \partial_t \nabla_{y_i} \mathbf{u}_i\|_{\tilde{L}^\infty(0,T;H_i)} \\
& + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i} u_i^L - \operatorname{curl}_{y_i} u_i\|_{\tilde{L}^\infty(0,T;H_i)} \\
& \leq c \left[(\Delta t)^2 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|_{L^2(D)} + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^4 \nabla_{y_i} \mathbf{u}_i}{\partial t^4} \right\|_{H_i} + \left\| \frac{\partial^2 q_0^L}{\partial t^2} \right\|_{L^2(0,T;H)} \right. \\
& \quad \left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla_{y_i} \mathbf{q}_i^L \right\|_{L^2(0,T;H_i)} + \|q_0^L\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \|\nabla_{y_i} \mathbf{q}_i^L\|_{L^\infty(0,T;H_i)} \right] \\
& + c \left(\|\partial_t p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i} \right) \\
& + \|\partial_t q_0^L\|_{\tilde{L}^\infty(0,T;H)} + \|\operatorname{curl} q_0^L\|_{\tilde{L}^\infty(0,T;H)} \\
& + \sum_{i=1}^n \left(\|\partial_t \nabla_{y_i} \mathbf{q}_i^L\|_{\tilde{L}^\infty(0,T;H_i)} + \|\operatorname{curl}_{y_i} q_i^L\|_{\tilde{L}^\infty(0,T;H_i)} \right).
\end{aligned}$$

Proof We note that $\mathbf{u}^L - \mathbf{u} = \mathbf{p}^L + \mathbf{q}^L$. The conclusions follow from Lemma 5.1.6. $\square$

From this, we deduce that

Proposition 5.1.8. *If $\mathbf{u} \in H^2(0, T; \mathbf{V})$, $\frac{\partial^3 u_0}{\partial t^3} \in L^2(0, T; H)$ and $\frac{\partial^3 \nabla_{y_i} \mathbf{u}_i}{\partial t^3} \in L^2(0, T; H_i)$, and if we choose $\mathbf{u}_0^L$ and $\mathbf{u}_1^L$ such that*

$$\lim_{L \rightarrow 0} \|\partial_t p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i} = 0,$$

then

$$\begin{aligned}
& \lim_{L \rightarrow \infty} \|\partial_t u_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{u}_i^L - \partial_t \nabla_{y_i} \mathbf{u}_i\|_{\tilde{L}^\infty(0,T;H_i)} \\
& + \|\operatorname{curl} u_0^L - \operatorname{curl} u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i} u_i^L - \operatorname{curl}_{y_i} u_i\|_{\tilde{L}^\infty(0,T;H_i)} = 0.
\end{aligned}$$

Proof From the hypothesis and Lemma 5.1.3, we have that

$$\lim_{L \rightarrow \infty} \left\| \frac{\partial^2 \mathbf{q}^L}{\partial t^2} \right\|_{L^2(0,T;\mathbf{V})} = 0.$$

As $\mathbf{u} \in C([0, T], \mathbf{V})$, from the proof of Proposition 5.1.5 $\lim_{L \rightarrow \infty} \|\mathbf{q}^L\|_{L^\infty(0, T; \mathbf{V})} = 0$. We have that

$$\|\mathbf{q}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq \|\mathbf{q}^L\|_{L^\infty(0, T; \mathbf{V})}$$

so

$$\lim_{L \rightarrow \infty} \|\mathbf{q}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} = 0.$$

Further, from (5.4), we have that

$$B(\partial_t \mathbf{w}_{m+1/2}^L - \partial_t \mathbf{u}_{m+1/2}, \mathbf{v}^L) + A(\partial_t \mathbf{w}_{m+1/2}^L - \partial_t \mathbf{u}_{m+1/2}, \mathbf{v}^L) = 0$$

for all $\mathbf{v}^L \in \mathbf{V}^L$. We thus have

$$\|\partial_t \mathbf{w}_{m+1/2}^L - \partial_t \mathbf{u}_{m+1/2}\|_{\mathbf{V}} \leq c \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{v}^L - \partial_t \mathbf{u}_{m+1/2}\|_{\mathbf{V}}.$$

As $\partial_t \mathbf{u}_{m+1/2} = \frac{\partial \mathbf{u}}{\partial t}(\xi)$ for $\xi \in (0, T)$, we deduce that

$$\|\partial_t \mathbf{q}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} \leq c \sup_{t \in (0, T)} \inf_{\mathbf{v}^L \in \mathbf{V}^L} \|\mathbf{v}^L - \frac{\partial \mathbf{u}}{\partial t}(t)\|_{\mathbf{V}}.$$

As $\frac{\partial \mathbf{u}}{\partial t} \in C([0, T]; \mathbf{V})$, a proof identical to that for $\|\mathbf{q}^L\|_{L^\infty(0, T; \mathbf{V})}$ in Proposition 5.1.5 shows that

$$\lim_{L \rightarrow \infty} \|\partial_t \mathbf{q}^L\|_{\tilde{L}^\infty(0, T; \mathbf{V})} = 0.$$

We thus get the conclusion. $\square$

5.2 Full and sparse tensor product approximations

We consider the approximations of problem (3.9) using the full and sparse tensor product FE.

5.2.1 Full tensor product finite elements

We recall the definition of full tensor product finite element $\bar{\mathbf{V}}^L$ in Chapter 4

$$\bar{\mathbf{V}}^L = W^L \otimes \bar{W}_1^L \otimes \cdots \otimes \bar{W}_n^L \otimes \bar{V}_1^L \otimes \cdots \otimes \bar{V}_n^L.$$

The spatially semidiscrete full tensor product finite element approximating problem is: Find $\bar{\mathbf{u}}^L(t) \in \bar{\mathbf{V}}^L$ so that for all $\bar{\mathbf{v}}^L \in \bar{\mathbf{V}}^L$:

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left[\left(\frac{\partial^2 \bar{u}_0^L}{\partial t^2}(t) + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla_{y_i} \bar{u}_i^L(t) \right) \cdot \left(\bar{v}_0^L + \sum_{i=1}^n \nabla_{y_i} \bar{v}_i^L \right) d\mathbf{y} dx \right. \\ & \quad \left. + a(x, \mathbf{y}) \left(\operatorname{curl} \bar{u}_0^L(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} \bar{u}_i^L(t) \right) \cdot \left(\operatorname{curl} \bar{v}_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} \bar{v}_i^L \right) \right] d\mathbf{y} dx \\ & = \int_D f(t, x) \cdot \bar{v}_0^L(x) dx \end{aligned} \quad (5.9)$$

for all $\bar{\mathbf{v}}^L = (\bar{v}_0^L, \{\bar{v}_i^L\}, \{\bar{\mathbf{v}}_i^L\}) \in \bar{\mathbf{V}}^L$.

We first state the following regularity for $\mathbf{u}$

Proposition 5.2.1. *Under Assumption 3.3.1, and the hypothesis of Proposition 3.3.3, there is a constant $s \in (0, 1]$ so that $\mathbf{u} \in L^\infty(0, T; \hat{\mathcal{H}}^s)$.*

Proof From Proposition 2.3.7, we have that χ_i^r and $\operatorname{curl} \chi_i^r$ belong to

$$C^1(\bar{D}, C^2(\bar{Y}_1, \dots, C^2(\bar{Y}_2, H^2(Y_i)) \dots))^3.$$

Together with $u_0 \in L^\infty(0, T; H^s(\operatorname{curl}, D))$, this implies $u_i \in L^\infty(0, T; \hat{\mathcal{H}}_i^s)$. Similarly, we have $\mathbf{u}_i \in L^\infty(0, T; \hat{\mathcal{H}}_i^s)$. $\square$

By the same argument, we deduce

Proposition 5.2.2. *Under Assumption 3.3.1, and the hypothesis of Proposition 3.3.4, there is a constant $s \in (0, 1]$ so that $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^\infty(0, T; \hat{\mathcal{H}}^s)$.*

We then have the following result for the spatially semidiscrete approximation.

Proposition 5.2.3. *Assume that condition (3.30) and Assumption 3.3.1 hold, D is a Lipschitz polygonal domain, $\operatorname{div} f \in L^\infty(0, T; L^2(D))$ and g_0, g_1 belong to $H^s(\operatorname{curl}, D)$. If g_0^L and g_1^L are chosen so that*

$$\|g_0^L - g_0\|_V \leq ch_L^s \text{ and } \|g_1^L - g_1\|_H \leq ch_L^s, \quad (5.10)$$

where $s \in (0, 1]$ is the constant in Proposition 3.3.4. Then

$$\begin{aligned} & \left\| \frac{\partial(\bar{u}_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \left\| \nabla_{y_i} \frac{\partial(\bar{u}_i^L - u_i)}{\partial t} \right\|_{L^\infty(0, T; H_i)} \\ & + \|\operatorname{curl}(\bar{u}_0^L - u_0)\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i}(\bar{u}_i^L - u_i)\|_{L^\infty(0, T; H_i)} \leq ch_L^s. \end{aligned}$$

Proof From Proposition 5.2.2 and Remark 3.3.5, we deduce that $\mathbf{u} \in L^\infty(0, T; \bar{\mathcal{H}}^s)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^\infty(0, T; \bar{\mathcal{H}}^s)$ and $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^\infty(0, T; \bar{\mathcal{H}}^s)$. From Lemmas 5.1.2 and 5.1.3, we have

$$\|\mathbf{q}^L\|_{L^\infty(0, T; \mathbf{V})} \leq ch_L^s, \quad \left\| \frac{\partial \mathbf{q}^L}{\partial t} \right\|_{L^\infty(0, T; \mathbf{V})} \leq ch_L^s, \quad \text{and} \quad \left\| \frac{\partial^2 \mathbf{q}^L}{\partial t^2} \right\|_{L^2(0, T; \mathbf{V})} \leq ch_L^s. \quad (5.11)$$

These together with

$$\begin{aligned} \frac{\partial p_0^L}{\partial t}(0) &= \frac{\partial}{\partial t}(u_0^L(0) - u(0)) - \frac{\partial q_0^L}{\partial t}, \\ \nabla_{y_i} \frac{\partial p_i^L}{\partial t}(0) &= \nabla_{y_i} \frac{\partial}{\partial t}(\mathbf{u}_i^L(0) - \mathbf{u}_i(0)) - \nabla_{y_i} \frac{\partial \mathbf{q}_i^L}{\partial t}(0), \\ \text{curl } p_0^L(0) &= \text{curl}(u_0^L(0) - u_0(0)) - \text{curl } q_0^L(0), \end{aligned}$$

and (5.10), we have that

$$\left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial p_i^L}{\partial t}(0) \right\|_{H_n} \leq ch_L^s,$$

and

$$\|\text{curl } p_0^L(0)\|_H \leq ch_L^s.$$

Thus the right hand side of (5.7) is not more than ch_L^s . We thus get the conclusion. $\square$

The fully discrete problem now becomes: For $m = 1, \dots, M$ find

$$\bar{\mathbf{u}}_m^L = (\bar{u}_{0,m}^L, \bar{u}_{1,m}^L, \dots, \bar{u}_{n,m}^L, \bar{\mathbf{u}}_{1,m}^L, \dots, \bar{\mathbf{u}}_{n,m}^L) \in \bar{\mathbf{V}}^L$$

such that for $m = 1, \dots, M - 1$

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\partial_t^2 \bar{u}_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \bar{\mathbf{u}}_{i,m}^L \right) \cdot \left(\bar{v}_0^L + \sum_{i=1}^n \nabla_{y_i} \bar{\mathbf{v}}_i^L \right) \right. \\ & \quad \left. + a(x, \mathbf{y}) \left(\text{curl } \bar{u}_{0,m,1/4}^L + \sum_{i=1}^n \text{curl}_{y_i} \bar{u}_{i,m,1/4}^L \right) \cdot \left(\text{curl } \bar{v}_0^L + \sum_{i=1}^n \text{curl}_{y_i} \bar{v}_i^L \right) \right] d\mathbf{y} dx \\ &= \int_D f_{m,1/4}(x) \cdot \bar{v}_0^L(x) dx \end{aligned} \quad (5.12)$$

for all $\bar{\mathbf{v}}^L = (\bar{v}_0^L, \bar{v}_1^L, \dots, \bar{v}_n^L, \bar{\mathbf{v}}_1^L, \dots, \bar{\mathbf{v}}_n^L) \in \bar{\mathbf{V}}^L$.

Proposition 5.2.4. Assume that condition (3.30) and Assumption 3.3.1 hold, D is a Lipschitz polygonal domain, $\text{div } f \in L^\infty(0, T; L^2(D))$ and g_0, g_1 belong to $H^s(\text{curl}, D)$ where $s \in (0, 1]$ is the constant in Proposition 3.3.4.

If

$$\|\partial_t p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i} \leq c((\Delta t)^2 + h_L^s),$$

then

$$\begin{aligned} & \|\partial_t \bar{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\partial_t \nabla_{y_i} (\bar{\mathbf{u}}_i^L - \mathbf{u}_i)\|_{\tilde{L}^\infty(0,T;H_i)} \\ & + \|\operatorname{curl} (\bar{u}_0^L - u_0)\|_{\tilde{L}^\infty(0,T;H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i} (\bar{\mathbf{u}}_i^L - \mathbf{u}_i)\|_{\tilde{L}^\infty(0,T;H_i)} \leq c((\Delta t)^2 + h_L^s). \end{aligned}$$

Proof The proof is similar to that of Proposition 5.2.3. We note that

$$\|\partial_t q_{0,m+1/2}^L\|_H = \left\| \frac{q_{0,m+1}^L - q_{0,m}^L}{\Delta t} \right\|_H \leq \sup_{t \in (0,T)} \left\| \frac{\partial q_0^L}{\partial t} \right\|_H \leq ch_L^s$$

due to (5.11). Similarly

$$\|\partial_t \nabla_{y_i} \mathbf{q}_{i,m+1/2}^L\|_{H_n} \leq ch_L^s.$$

We then get the conclusion. $\square$

5.2.2 Sparse tensor product finite elements

We recall the definition of sparse tensor product finite element $\hat{\mathbf{V}}^L$ in Chapter 4

$$\hat{\mathbf{V}}^L = W^L \otimes \hat{W}_1^L \otimes \cdots \otimes \hat{W}_n^L \otimes \hat{V}_1^L \otimes \cdots \otimes \hat{V}_n^L.$$

The spatially semidiscrete sparse tensor product finite element approximating problem is: Find $\hat{\mathbf{u}}^L(t) \in \hat{\mathbf{V}}^L$ such that :

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} \left[b(x, \mathbf{y}) \left(\frac{\partial^2 \hat{u}_0^L}{\partial t^2}(t) + \sum_{i=1}^n \nabla_{y_i} \frac{\partial^2 \hat{\mathbf{u}}_i^L}{\partial t^2}(t) \right) \cdot \left(\hat{v}_0^L + \sum_{i=1}^n \nabla_{y_i} \hat{\mathbf{v}}_i^L \right) \right. \\ & \quad \left. + a(x, \mathbf{y}) \left(\operatorname{curl} \hat{u}_0^L(t) + \sum_{i=1}^n \operatorname{curl}_{y_i} \hat{u}_i^L(t) \right) \cdot \left(\operatorname{curl} \hat{v}_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} \hat{v}_i^L \right) \right] d\mathbf{y} dx \\ & = \int_D f(x) \cdot \hat{v}_0^L(x) dx \end{aligned} \tag{5.13}$$

for all $\hat{\mathbf{v}}^L = (\hat{v}_0^L, \hat{v}_1^L, \dots, \hat{v}_n^L, \hat{\mathbf{v}}_1^L, \dots, \hat{\mathbf{v}}_n^L) \in \hat{\mathbf{V}}^L$.

We then have the following result in the error estimate.

Proposition 5.2.5. *Assume that condition (3.30) and Assumption 3.3.1 hold, D is a Lipschitz polygonal domain, $\operatorname{div} f \in L^\infty(0, T; L^2(D))$ and g_0, g_1 belong to $H^s(\operatorname{curl}, D)$. If g_0^L and g_1^L are chosen so that*

$$\|g_0^L - g_0\|_V \leq cL^{n/2}h_L^s \text{ and } \|g_1^L - g_1\|_H \leq cL^{n/2}h_L^s,$$

where $s \in (0, 1]$ is the constant in Proposition 3.3.4. Then

$$\begin{aligned} & \left\| \frac{\partial(\hat{u}_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \left\| \nabla_{y_i} \frac{\partial(\hat{\mathbf{u}}_i^L - \mathbf{u}_i)}{\partial t} \right\|_{L^\infty(0, T; H_i)} \\ & + \|\operatorname{curl}(\hat{u}_0^L - u_0)\|_{L^\infty(0, T; H)} + \sum_{i=1}^n \|\operatorname{curl}_{y_i}(\hat{u}_i^L - u_i)\|_{L^\infty(0, T; H_i)} \leq cL^{n/2}h_L^s. \end{aligned}$$

The proof of this proposition is identical to that of Proposition 5.2.3.

The fully discrete sparse tensor finite element product problem is: For $m = 1, \dots, M$ find $\hat{\mathbf{u}}_m^L = (\hat{u}_{0,m}^L, \hat{u}_{1,m}^L, \dots, \hat{u}_{n,m}^L, \hat{\mathbf{u}}_{1,m}^L, \dots, \hat{\mathbf{u}}_{n,m}^L) \in \hat{\mathbf{V}}^L$ such that

$$\begin{aligned} & \int_D \int_{\mathbf{Y}} b(x, \mathbf{y}) \left(\partial_t^2 \hat{u}_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \hat{\mathbf{u}}_{i,m}^L \right) \cdot \left(\hat{v}_0^L + \sum_{i=1}^n \nabla_{y_i} \hat{\mathbf{v}}_i^L \right) \\ & + a(x, \mathbf{y}) \left(\operatorname{curl} \hat{u}_{0,m,1/4}^L + \sum_{i=1}^n \operatorname{curl}_{y_i} \hat{u}_{i,m,1/4}^L \right) \cdot \left(\operatorname{curl} \hat{v}_0^L + \sum_{i=1}^n \operatorname{curl}_{y_i} \hat{v}_i^L \right) d\mathbf{y} dx \\ & = \int_D f_{m,1/4}(x) \cdot \hat{v}_0^L(x) dx \end{aligned} \quad (5.14)$$

for all $\hat{\mathbf{v}}^L = (\hat{v}_0^L, \hat{v}_1^L, \dots, \hat{v}_n^L, \hat{\mathbf{v}}_1^L, \dots, \hat{\mathbf{v}}_n^L) \in \hat{\mathbf{V}}^L$.

For fully discrete problems, we have the following result

Proposition 5.2.6. *Assume that condition (3.30) and Assumption 3.3.1 hold, D is a Lipschitz polygonal domain, $\operatorname{div} f \in L^\infty(0, T; L^2(D))$ and g_0, g_1 belong to $H^s(\operatorname{curl}, D)$ where $s \in (0, 1]$ is the constant in Proposition 3.3.4. If*

$$\begin{aligned} & \|\partial_t p_{0,1/2}^L\|_{L^2(D)} + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{L^2(D \times \mathbf{Y})} + \|\operatorname{curl} p_{0,1/2}^L\|_{L^2(D)} \\ & + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{L^2(D \times \mathbf{Y})} \leq c((\Delta t)^2 + L^{n/2}h_L^s), \end{aligned}$$

then

$$\begin{aligned} & \|\partial_t \hat{u}_0^L - \partial_t u_0\|_{\tilde{L}^\infty(0, T; H)} + \sum_{i=1}^n \|\nabla_{y_i}(\hat{\mathbf{u}}_i^L - \mathbf{u}_i)\|_{\tilde{L}^\infty(0, T; H_i)} + \|\operatorname{curl}(\hat{u}_0^L - u_0)\|_{\tilde{L}^\infty(0, T; H)} \\ & + \sum_{i=1}^n \|\operatorname{curl}_{y_i}(\hat{u}_i^L - u_i)\|_{\tilde{L}^\infty(0, T; H_i)} \leq c((\Delta t)^2 + L^{n/2}h_L^s). \end{aligned}$$

The proof is identical to that of Proposition 5.2.4.

5.3 Numerical correctors

We define numerical correctors in this section.

5.3.1 Numerical correctors for two-scale problems

To establish the numerical correctors, we note the following result.

Lemma 5.3.1. *Assume that $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^s(D)^3)$ and $\text{curl } u_0 \in L^\infty(0, T; H^s(D)^3)$, $\chi^r \in C^1(\bar{D}, C_\#^1(\bar{Y}))^3$ and $\omega^r \in C^1(\bar{D}, C_\#^1(\bar{Y}))$, $r=1,2,3$. Then*

$$\sup_{t \in [0, T]} \int_D \left| \text{curl}_y u_1 \left(t, x, \frac{x}{\varepsilon} \right) - \mathcal{U}^\varepsilon(\text{curl}_y u_1(t, \cdot, \cdot))(x) \right|^2 dx \leq c\varepsilon^{2s}$$

and

$$\sup_{t \in [0, T]} \int_D \left| \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \left(t, x, \frac{x}{\varepsilon} \right) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t, \cdot, \cdot) \right) (x) \right|^2 dx \leq c\varepsilon^{2s}.$$

Proof The proof is similar to that for Lemma 4.2.1 □

Theorem 5.3.2. *Assume that condition (3.30) and Assumption 3.3.1 hold, with $g_0 = 0$ and $\text{div } f \in L^\infty(0, T; L^2(D))$, D is a Lipschitz polygonal domain, and that g_1^L is chosen so that $\|g_1^L - g_1\|_H \leq ch_L^s$ where $s \in (0, 1]$ is the constant in Proposition 3.3.4. Then for the solution of the semidiscrete problem (5.9) using the full tensor product FE, we have*

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right) \right) \right\|_{L^\infty(0, T; H)} \\ & + \left\| \text{curl } u^\varepsilon - \left(\text{curl } \bar{u}_0^L + \mathcal{U}^\varepsilon (\text{curl}_y \bar{u}_1^L) \right) \right\|_{L^\infty(0, T; H)} \leq c \left(h_L^s + \varepsilon^{\frac{s}{s+1}} \right). \end{aligned}$$

For the semidiscrete problem (5.13) using the sparse tensor product FE, if $\|g_1^L - g_1\|_H \leq cL^{1/2}h_L^s$, we have

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \hat{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \hat{\mathbf{u}}_1^L \right) \right) \right\|_{L^\infty(0, T; H)} \\ & + \left\| \text{curl } u^\varepsilon - \left(\text{curl } \hat{u}_0^L + \mathcal{U}^\varepsilon (\text{curl}_y \hat{u}_1^L) \right) \right\|_{L^\infty(0, T; H)} \leq c \left(L^{1/2}h_L^s + \varepsilon^{\frac{s}{s+1}} \right). \end{aligned}$$

Proof With the hypothesis of the theorem, from Proposition 3.3.4 and Proposition 4.1.5, the conditions of Theorem 3.4.2 holds. We then have

$$\left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L(t) \right) \right\|_H \leq \left\| \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L(t) \right\|_{L^2(D \times Y)^3}$$

and

$$\begin{aligned} & \left\| \mathcal{U}^\varepsilon \left(\operatorname{curl} u_0 + \operatorname{curl}_y u_1 - \operatorname{curl} \bar{u}_0^L - \operatorname{curl}_y \bar{u}_1^L \right) \right\|_H \\ & \leq \left\| \operatorname{curl} u_0 + \operatorname{curl}_y u_1 - \operatorname{curl} \bar{u}_0^L - \operatorname{curl}_y \bar{u}_1^L \right\|_{L^2(D \times Y)^3}. \end{aligned}$$

We note that

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right) \right) \right\|_{L^\infty(0,T;H)} \\ & \leq \left\| \frac{\partial u_0}{\partial t} - \left(\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) \right\|_{L^\infty(0,T;H)} \\ & + \left\| \frac{\partial u_0}{\partial t} - \frac{\partial \bar{u}_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} + \left\| \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) \right\|_{L^\infty(0,T;H)} \\ & + \left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right) \right\|_{L^\infty(0,T;H)} \end{aligned}$$

From Theorem 3.4.2, we have

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) \right\|_{L^\infty(0,T;H)} \leq c\varepsilon^{\frac{s}{s+1}}.$$

From Proposition 5.2.3, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \bar{u}_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} \leq ch_L^s.$$

From Lemma 5.3.1, we have

$$\left\| \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) \right\|_{L^\infty(0,T;H)} \leq c\varepsilon^s.$$

We note that

$$\begin{aligned} \left\| \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 \right) - \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right) \right\|_{L^\infty(0,T;H)} & \leq \left\| \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1 - \frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right\|_{L^\infty(0,T;H)} \\ & \leq ch_L^s. \end{aligned}$$

Thus

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \left(\frac{\partial \bar{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left(\frac{\partial}{\partial t} \nabla_y \bar{\mathbf{u}}_1^L \right) \right) \right\|_{L^\infty(0,T;H)} \leq c\varepsilon^{\frac{s}{s+1}} + ch_L^s + c\varepsilon^s + ch_L^s \leq c(h_L^s + \varepsilon^{\frac{s}{s+1}}).$$

Similarly,

$$\begin{aligned} & \left\| \operatorname{curl} u^\varepsilon - \left(\operatorname{curl} \bar{u}_0^L + \mathcal{U}^\varepsilon \left(\operatorname{curl}_y \bar{\mathbf{u}}_1^L \right) \right) \right\|_{L^\infty(0,T;H)} \\ & \leq \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0 - \operatorname{curl}_y u_1 \right\|_{L^\infty(0,T;H)} \\ & \quad + \left\| \operatorname{curl} u_0 - \operatorname{curl} \bar{u}_0^L \right\|_{L^\infty(0,T;H)} \\ & \quad + \left\| \operatorname{curl}_y u_1 - \mathcal{U}^\varepsilon \left(\operatorname{curl}_y u_1 \right) \right\|_{L^\infty(0,T;H)} \\ & \quad + \left\| \mathcal{U}^\varepsilon \left(\operatorname{curl}_y u_1 \right) - \mathcal{U}^\varepsilon \left(\operatorname{curl}_y \bar{\mathbf{u}}_1^L \right) \right\|_{L^\infty(0,T;H)} \\ & \leq c\varepsilon^{\frac{s}{s+1}} + ch_L^s + c\varepsilon^s + ch_L^s \leq c(h_L^s + \varepsilon^{\frac{s}{s+1}}). \end{aligned}$$

We then have the desired estimate.

The proof for the semidiscrete sparse tensor finite element solution is similar.

□

For fully discrete problems, we have the following results.

Theorem 5.3.3. *Assume that condition (3.30) and Assumption 3.3.1 hold, with $g_0 = 0$, $g_1 \in H^s(\operatorname{curl}, D)$ and $\operatorname{div} f \in L^\infty(0, T; L^2(D))$, D is a Lipschitz polygonal domain ($s \in (0, 1]$ is the constant in Proposition 3.3.4). For the fully discrete full tensor finite element problem (5.12), assume that*

$$\begin{aligned} & \left\| \partial_t p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \partial_t \nabla_y \mathbf{p}_{1,1/2}^L \right\|_{L^2(D \times Y)^3} + \left\| \operatorname{curl} p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \operatorname{curl}_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} \\ & \leq c((\Delta t)^2 + h_L^s), \end{aligned}$$

then

$$\begin{aligned} & \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \bar{u}_{0,m+1/2}^L - \mathcal{U}^\varepsilon(\partial_t \nabla_y \bar{\mathbf{u}}_{1,m+1/2}^L) \right\|_H \\ & \quad + \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} \bar{u}_0^L - \mathcal{U}^\varepsilon(\operatorname{curl}_y \bar{\mathbf{u}}_1^L) \right\|_{\tilde{L}^\infty(0,T;H)} \leq c \left((\Delta t)^2 + h_L^s + \varepsilon^{\frac{s}{s+1}} \right). \end{aligned}$$

For the sparse tensor finite element approximation problem (5.14), if

$$\begin{aligned} & \left\| \partial_t p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \partial_t \nabla_y \mathbf{p}_{1,1/2}^L \right\|_{L^2(D \times Y)^3} + \left\| \operatorname{curl} p_{0,1/2}^L \right\|_{L^2(D)^3} \\ & \quad + \left\| \operatorname{curl}_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} \leq c \left((\Delta t)^2 + L^{1/2} h_L^s \right), \end{aligned}$$

then

$$\begin{aligned} & \Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t \hat{u}_{0,m+1/2}^L - \mathcal{U}^\varepsilon(\partial_t \nabla_y \hat{\mathbf{u}}_{1,m+1/2}^L) \right\|_H \\ & \quad + \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} \hat{u}_0^L - \mathcal{U}^\varepsilon(\operatorname{curl}_y \hat{\mathbf{u}}_1^L) \right\|_{\tilde{L}^\infty(0,T;H)} \leq c \left((\Delta t)^2 + L^{1/2} h_L^s + \varepsilon^{\frac{s}{s+1}} \right). \end{aligned}$$

Proof From the compatibility condition $\frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0, T; H)$ so $\frac{\partial^2 u_0}{\partial t^2} \in C([0, T]; H)$. To use the homogenization estimate in Theorem 3.4.2, we estimate

$$\frac{1}{\Delta t}(u_{0,m+1} - u_{0,m}) - \frac{\partial u_0}{\partial t}(t_m) = \frac{\partial u_0}{\partial t}(\tau) - \frac{\partial u_0}{\partial t}(t_m) = \int_{t_m}^{\tau} \frac{\partial^2 u_0}{\partial t^2}(\sigma) d\sigma,$$

for a value $t_m \leq \tau \leq t_{m+1}$. With the compatibility condition (3.30), we have that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H)$. Thus

$$\sup_{0 \leq m < M} \left\| \partial_t u_{0,m+1/2} - \frac{\partial u_0}{\partial t}(t_m) \right\|_H \leq c\Delta t.$$

Similarly, using the smoothness of χ^r and ω^r for $r = 1, 2, 3$, we have that

$$\begin{aligned} \partial_t \nabla_y \mathbf{u}_{1,m+1/2} - \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t_m) &= \frac{\nabla_y \mathbf{u}_{1,m+1} - \nabla_y \mathbf{u}_{1,m}}{\Delta t} - \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t_m) \\ &= \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(\tau) - \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t_m) \\ &= \int_{t_m}^{\tau} \frac{\partial^2}{\partial t^2} \nabla_y \mathbf{u}_1(\sigma) d\sigma, \end{aligned}$$

for a value $t_m \leq \tau \leq t_{m+1}$. Thus

$$\sup_{0 \leq m < M} \left\| \partial_t \nabla_y \mathbf{u}_{1,m+1/2} - \frac{\partial}{\partial t} \nabla_y \mathbf{u}_1(t_m) \right\|_H \leq c\Delta t.$$

We then get the result from Proposition 5.2.6 and Theorem 3.4.2. $\square$

5.3.2 Numerical correctors for multiscale problems

We do not distinguish the full and sparse tensor FE. We work with general FE spaces instead. For the semidiscrete problem (5.1) we have:

Theorem 5.3.4. *Assume that condition (3.29) holds with $g_0 = 0$, and g_1^L is chosen so that $\lim_{L \rightarrow \infty} \|g_1^L - g_1\|_H = 0$. Then the solution of problem (5.1) satisfies*

$$\begin{aligned} \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} u_i^L \right) \right\|_{L^\infty(0, T; H)} \\ + \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} u_i^L \right) \right\|_{L^\infty(0, T; H)} = 0. \end{aligned}$$

Proof The result is a direct consequence of Propositions 5.1.5 and 3.4.3. Indeed,

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)} \\ & \leq \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \right\|_{L^\infty(0,T;H)} \\ & \quad + \left\| \frac{\partial u_0}{\partial t} - \frac{\partial u_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} + \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)}. \end{aligned}$$

From Proposition 3.4.3, we deduce that

$$\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) \right\|_{L^\infty(0,T;H)} \rightarrow 0$$

as $\varepsilon \rightarrow 0$. From Proposition 5.1.5 we deduce that $\left\| \frac{\partial u_0}{\partial t} - \frac{\partial u_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} \rightarrow 0$ as $L \rightarrow \infty$. The last term

$$\begin{aligned} & \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)} \\ & \leq \left\| \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i - \sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L \right\|_{L^\infty(0,T;H_n)} \rightarrow 0 \end{aligned}$$

as $L \rightarrow \infty, \Delta t \rightarrow 0$. Thus

$$\lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)} = 0.$$

Similarly, we have

$$\begin{aligned} & \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)} \\ & \leq \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0 - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{u}_i \right) \right\|_{L^\infty(0,T;H)} + \left\| \operatorname{curl} u_0 - \operatorname{curl} u_0^L \right\|_{L^\infty(0,T;H)} \\ & \quad + \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{u}_i \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{u}_i^L \right) \right\|_{L^\infty(0,T;H)} \end{aligned}$$

tends to 0 as $L \rightarrow \infty, \Delta t \rightarrow 0, \varepsilon \rightarrow 0$. We then get the conclusion. $\square$

For the fully discrete problem (5.8) we have:

Theorem 5.3.5. *Assume that condition (3.29) holds with $g_0 = 0$, and $\mathbf{u}_0^L$ and $\mathbf{u}_1^L$ are chosen such that*

$$\lim_{L \rightarrow \infty} \|\partial_t p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\partial_t \nabla_{y_i} \mathbf{p}_{i,1/2}^L\|_{H_i} + \|\operatorname{curl} p_{0,1/2}^L\|_H + \sum_{i=1}^n \|\operatorname{curl}_{y_i} p_{i,1/2}^L\|_{H_i} = 0.$$

Then

$$\begin{aligned} \lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u_{0,m+1/2}^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H \\ + \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} \mathbf{u}_i^L \right) \right\|_{\tilde{L}^\infty(0,T;H_i)} = 0. \end{aligned}$$

Proof We have

$$\begin{aligned} & \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u_{0,m+1/2}^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H \\ & \leq \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \frac{\partial u_0}{\partial t}(t_m) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{u}_i}{\partial t}(t_m) \right) \right\|_H \\ & \quad + \left\| \frac{\partial u_0}{\partial t}(t_m) - \partial_t u_{0,m+1/2}^L \right\|_H \\ & \quad + \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{u}_i}{\partial t}(t_m) \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H \\ & \quad + \left\| \partial_t u_{0,m+1/2}^L - \partial_t u_{0,m+1/2}^L \right\|_H \\ & \quad + \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H \end{aligned}$$

From Proposition 3.4.3 we deduce that

$$\left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \frac{\partial u_0}{\partial t}(t_m) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{u}_i}{\partial t}(t_m) \right) \right\|_H \rightarrow 0$$

as $\varepsilon \rightarrow 0$. From $\frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H)$ so

$$\lim_{\Delta t \rightarrow 0} \sup_{0 \leq m < M} \left\| \partial_t u_{0,m+1/2}^L - \frac{\partial u_0}{\partial t}(t_m) \right\|_H = 0;$$

and from (3.28) we have $\frac{\partial^2}{\partial t^2} \mathbf{u}_i \in L^\infty(0, T; H_i)$ so

$$\lim_{\Delta t \rightarrow 0} \sup_{0 \leq m < M} \left\| \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2} - \frac{\partial}{\partial t} \nabla_{y_i} \mathbf{u}_i(t_m) \right\|_H = 0.$$

Thus

$$\begin{aligned} & \sup_{0 \leq m < M} \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \nabla_{y_i} \frac{\partial \mathbf{u}_i}{\partial t}(t_m) \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2} \right) \right\|_H \\ & \leq \sup_{0 \leq m < M} \left\| \sum_{i=1}^n \left(\nabla_{y_i} \frac{\partial \mathbf{u}_i}{\partial t}(t_m) - \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2} \right) \right\|_{L^2(D \times \mathbf{Y})^3} \rightarrow 0 \end{aligned}$$

as $\Delta t \rightarrow 0$. From the FE convergence, we have

$$\sup_{0 \leq m < M} \left\| \partial_t u_{0,m+1/2} - \partial_t u_{0,m+1/2}^L \right\|_H \rightarrow 0$$

as $L \rightarrow \infty, \Delta t \rightarrow 0$, and

$$\begin{aligned} & \sup_{0 \leq m < M} \left\| \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2} \right) - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H \\ & \leq \sup_{0 \leq m < M} \left\| \sum_{i=1}^n (\partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2} - \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L) \right\|_{L^2(D \times \mathbf{Y})^3} \rightarrow 0 \end{aligned}$$

as $L \rightarrow \infty, \Delta t \rightarrow 0$. Thus

$$\lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u_{0,m+1/2}^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \partial_t \nabla_{y_i} \mathbf{u}_{i,m+1/2}^L \right) \right\|_H = 0.$$

Using similar argument, we have

$$\lim_{\substack{L \rightarrow \infty \\ \Delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \left\| \operatorname{curl} u^\varepsilon - \operatorname{curl} u_0^L - \mathcal{U}_n^\varepsilon \left(\sum_{i=1}^n \operatorname{curl}_{y_i} u_i^L \right) \right\|_{\tilde{L}^\infty(0,T;H)} = 0.$$

We then get the conclusion. $\square$

5.4 Numerical results

We present in this section some numerical examples for two-scale problems that confirm our analysis above.

In the first example, we consider a two-scale time dependent Maxwell equation in the two dimension domain $D = (0, 1)^2$.

$$b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} + \operatorname{curl} a^\varepsilon \operatorname{curl} u^\varepsilon = f(t, x), \quad \text{in } D$$

$$u^\varepsilon(t, \cdot) \times \nu = 0, \quad \text{on } \partial D$$

$$u^\varepsilon(0, x) = 0$$

$$u_t^\varepsilon(0, x) = 0$$

The coefficients

$$a(x, y) = \frac{1}{(1 + x_1)(1 + x_2)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)},$$

and

$$b(x, y) = \frac{(1 + x_1)(1 + x_2)}{(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}.$$

We can compute the homogenized coefficients as in Example 1 of Chapter 4. In this case

$$a^0(x) = \frac{4}{9(1 + x_1)(1 + x_2)}$$

and

$$b^0(x) = \frac{\sqrt{2}(1 + x_1)(1 + x_2)}{3}.$$

We choose

$$f(x_1, x_2) = \begin{pmatrix} 2\sqrt{2}(1 + x_1)(1 + x_2)x_1x_2(1 - x_2)t + \frac{4t^3}{9(1 + x_2)^2} \\ 2\sqrt{2}(1 + x_1)(1 + x_2)x_1x_2(1 - x_1)t + \frac{4t^3}{9(1 + x_1)^2} \end{pmatrix}$$

so that the solution to the homogenized equation is

$$u_0 = \begin{pmatrix} x_1x_2(1 - x_2)t^3 \\ x_1x_2(1 - x_1)t^3 \end{pmatrix}.$$

From the relation (2.14), we can calculate the solution $\operatorname{curl}_y u_1$ exactly

$$\begin{aligned} \operatorname{curl} u_1 &= \operatorname{curl} u_0 \operatorname{curl}_y \chi = \left(\frac{a^0(x)}{a(x, y)} - 1 \right) \operatorname{curl} u_0 \\ &= \left(\frac{4(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}{9} - 1 \right) (x_2 - x_1)t^3. \end{aligned}$$

Figure 5.1: The sparse tensor errors $\|u_0 - u_0^L\|_{H(\text{curl}, D)}$ and $\|\text{curl } u_1 - \text{curl } u_1^L\|_{L^2(D)^3}$

In Figure 5.1 we plot the errors $\|u_0 - u_0^L\|_{H(\text{curl}, D)}$ and $\|\text{curl } u_1 - \text{curl } u_1^L\|_{L^2(D)^3}$ versus the mesh size for the sparse tensor product for $(\Delta t, h) = (1/4, 1/4), (1/6, 1/8), (1/8, 1/12)$ and $(1/16, 1/32)$. The result confirm our analysis.

In the second example, we choose

$$a(x, y) = \frac{(1 + x_1)(1 + x_2)}{(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)},$$

$$b(x, y) = \frac{1}{(1 + x_1)(1 + x_2)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}.$$

In this case, the homogenized coefficients are

$$a^0(x) = \frac{4(1 + x_1)(1 + x_2)}{9}$$

and

$$b^0(x) = \frac{\sqrt{2}}{3(1 + x_1)(1 + x_2)}.$$

We choose

$$f(x_1, x_2) = \left(\frac{2\sqrt{2}x_2(1 - x_2)t}{(1 + x_2)} + \frac{4t^3(1 + x_1)(2x_2 - x_1 + 1)}{3}, \right. \\ \left. \frac{2\sqrt{2}x_1(1 - x_1)t}{(1 + x_1)} + \frac{4t^3(1 + x_1)(2x_1 - x_2 + 1)}{3} \right)$$

so that the homogenized solution to homogenization problem is

$$u_0 = \begin{pmatrix} (1 + x_1)x_2(1 - x_2)t^3 \\ (1 + x_2)x_1(1 - x_1)t^3 \end{pmatrix}$$

and

$$\begin{aligned}\operatorname{curl} u_1 &= \operatorname{curl} u_0 \operatorname{curl}_y \chi = \left(\frac{a^0(x)}{a(x, y)} - 1 \right) \operatorname{curl} u_0 \\ &= \left(\frac{4(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}{3} - 1 \right) (x_2 - x_1) t^3.\end{aligned}$$

In Figure 5.2 we plot the errors $\|u_0 - u_0^L\|_{H(\operatorname{curl}, D)}$ and $\|\operatorname{curl} u_1 - \operatorname{curl} u_1^L\|_{L^2(D)^3}$ versus the mesh size for the sparse tensor product for $(\Delta t, h) = (1/4, 1/4), (1/6, 1/8), (1/8, 1/12)$ and $(1/16, 1/32)$. The result confirms once more our analysis.

Figure 5.2: The sparse tensor errors $\|u_0 - u_0^L\|_{H(\operatorname{curl}, D)}$ and $\|\operatorname{curl} u_1 - \operatorname{curl} u_1^L\|_{L^2(D)^3}$

Summary

In this chapter, we develop finite element approximations for the multiscale homogenized Maxwell wave equation. We first study the spatially semi-discretized problem and the fully-discretized problem for general finite element spaces. We modify the approach of Dupont [40] for general wave equations significantly for the high dimensional homogenized Maxwell wave equation. We deduce the conditions for the schemes to converge, and estimate the errors. We then apply the general framework to the full and sparse tensor product finite elements. Error estimates are derived explicitly with respect to the spatial mesh size in the semi-discretized problem and the spatial mesh size and time step in the fully-discretized problem. From the finite element solutions, we deduced numerical correctors, with an explicit error in terms of the homogenization error and the finite element error in the

two-scale case. For the multiscale case, we also derive the numerical correctors, but without a rate of convergence. We solve some two-scale Maxwell wave equations in two dimensions to illustrate the theory.

Chapter 6

Conclusions and future work

In this chapter, we give a summary of the works done in this thesis and explore some relevant topics for further investigation.

6.1 Conclusions

We study in the thesis multiscale Maxwell equations. In particular, we perform the analysis for homogenization of time independent and time dependent Maxwell equations and we develop efficient numerical methods to solve them.

We perform homogenization and derive homogenization error for stationary Maxwell equations in Chapter 2. The homogenization in the multiscale case is derived from the multiscale homogenized equation, which contains all the necessary macroscopic and microscopic information. The solution of this equation is used to derive correctors for the solution of the multiscale problem. As $H_0(\text{curl}, D)$ is not compactly embedded in $L^2(D)^d$, the homogenized coefficient for b^ε is now the elliptic homogenized coefficient. This is completely different from the elliptic case where the average with respect to the variable y_i in the unit cell Y_i is the homogenized coefficient. In the two-scale case, we derive a homogenization error. The standard procedure in the literature is modified for Maxwell equation, which produces the homogenization rate $O(\varepsilon^{1/2})$. However, the required regularity $H^1(\text{curl}, D)$ for the solution u_0 of the homogenized equation is normally not achievable; u_0 only belongs to a weaker regularity space $H^s(\text{curl}, D)$ for a value $0 < s < 1$. We develop a new approach to deduce the new homogenization error $O(\varepsilon^{s/(1+s)})$ in this case. For other types of two-scale problems the method works in the same way when the solution to the homogenized equation possesses low regularity.

Multiscale Maxwell wave equation is studied in Chapter 3. Again, we extend the multiscale convergence concept to time dependent equations and derive the multiscale homogenized Maxwell wave equation that contains all the macroscopic

and microscopic information. This equation is in the generalized sense. We derive the homogenized equation together with the initial conditions from this equation. The derivation is very technical, unlike in the wave equation. For wave equation, the energy of the multiscale equation normally does not converge to the energy of the homogenized equation. We thus restrict our consideration to the case where the initial condition $g_0 = 0$ when deriving homogenization correctors. In the two-scale case, homogenization error estimates can be derived. When the solution u_0 of the homogenized problem is in the stronger regularity space $L^\infty(0, T; H^1(\text{curl}, D))$, we derive the standard homogenization error $O(\varepsilon^{1/2})$. In practice, u_0 only belongs to a weaker regularity space $L^\infty(0, T; H^s(\text{curl}, D))$ for $0 < s < 1$. A new procedure to derive a homogenization error $O(\varepsilon^{s/1+s})$ is obtained.

Finite element approximations are considered in Chapters 4 and 5. In Chapter 4, we develop the sparse tensor product finite elements for the multiscale stationary Maxwell homogenized equation, using edge finite elements. Solving the equation, we obtain the solution of the homogenized equation, and the corrector terms at the same time. The method achieves an accuracy essentially equal to that for the full tensor product finite elements, but requires an essentially optimal level of complexity. We establish numerical correctors from the finite element solution of the multiscale homogenized equation. In the two-scale case, with the available homogenization errors, we derive an explicit error for this numerical corrector, which is the sum of the homogenization and the finite element errors.

We develop finite element approximations for the multiscale homogenized Maxwell wave equation in Chapter 5. We modify significantly the scheme of Dupont [40] for general wave equations for the multiscale homogenized Maxwell wave equation. We prove error estimates for both the spatially semi-discretized problem and the fully-discretized problem for general finite element spaces when the solution is sufficiently regular. We then apply the general framework to the full and sparse tensor product finite elements and derive error estimates explicitly in the semi-discrete problem and in the fully-discretized problem. We show that the regularity required for these error estimates to hold can be achieved under mild conditions. We deduce numerical correctors from the finite element solutions, with an explicit error in terms of the homogenization error and the finite element error in the two-scale case.

6.2 Problems for future work

We explore some relevant topics for further investigation in this section.

- **Time harmonic Maxwell equations:** A natural extension of the research of the thesis is to study the multiscale time harmonic Maxwell equation of

the form

$$\operatorname{curl}(a^\varepsilon(x)\operatorname{curl}u^\varepsilon(x)) - k^2u^\varepsilon(x) = f(x),$$

with the boundary condition $u^\varepsilon \times \nu = 0$ on ∂D .

- **Multiscale random Maxwell equations:** We wish to consider the multi-scale Maxwell wave equation where the coefficient a^ε and b^ε are random. A natural extension of the periodic homogenization problem considered in the thesis is to study the case where the random coefficients are ergodic, i.e., they can be obtained from a dynamical system.

$$\begin{cases} b^\varepsilon(x, \omega) \frac{\partial^2 u^\varepsilon(t, x, \omega)}{\partial t^2} + \operatorname{curl}(a^\varepsilon(x, \omega)\operatorname{curl}u^\varepsilon(t, x, \omega)) = f(t, x), & (0, T) \times D \\ u^\varepsilon(0, x, \omega) = g_0(x), \\ u_t^\varepsilon(0, x, \omega) = g_1(x). \end{cases}$$

with the boundary condition $u^\varepsilon \times \nu = 0$ on ∂D .

- **Maxwell equations with high contrast coefficients:** We wish to extend the methods studied in this thesis to the case where the coefficients are high contrast, i.e., $a^\varepsilon = O(\varepsilon^\alpha)$, $b^\varepsilon = O(\varepsilon^\alpha)$ in a part of the domain, and is $O(1)$ in the rest of the domain. These problems arise from the study of man made metamaterials that possess unusual physical properties that cannot be found in nature.

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