Reliability analysis of transmission components under elastohydrodynamic lubrication conditions

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

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COMPONENTS UNDER ELASTOHYDRODYNAMIC
LUBRICATION CONDITIONS**

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Doctor of Philosophy

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Abstract

Lubricants are commonly used to improve the tribological performance of mechanical components in power transmission systems. The localized high pressure generated within the lubricants would induce elastic surface deformation and increase the lubricant viscosity. Lubrication with the consideration of these effects is commonly known as elastohydrodynamic lubrication (EHL). In engineering practice, surfaces are always flawed by roughness, which may significantly influence hydrodynamic flows and the entraining action. A complete separation of the contacting interfaces by the lubricating fluid is preferred to enhance the reliabilities and reduce the lifetime cost. However, nowadays technology allows modern machines to operate in more severe conditions, e.g., under heavier loads at higher temperatures or lubricated by lower-viscosity oils, which may result in a thinner lubricant film than the magnitude of the roughness and thus lead to the coexistence of the hydrodynamic lubricant film and asperity contact. Such a lubrication regime is referred to as mixed EHL and has long been recognized.

Generally, the materials in contact are assumed to be homogeneous. Many materials, nevertheless, have a heterogeneous structure at a certain level of observation. From an engineering point of view, heterogeneous materials, such as composites, coated materials and other multi-layered materials, are desirable to enhance particular properties of materials. However, the presence of the undesirable defects formed unintentionally during the material manufacturing process, such as inclusions, voids, dislocations and cracks, would result in stress concentrations and consequently lead to fracture and fatigue of materials. Therefore, it is of great significance to take the heterogeneous effects of contact components into consideration for their reliability analysis. In addition, modern machinery is required to operate under severe loading

conditions. Such conditions would lead to the plastic evolution of the materials which plays a significant role in their reliability. Analysis of such plastic evolution would thus provide guidance to improve the design of component structures.

A semi-analytical solution is first developed for materials with inhomogeneous inclusions subjected to EHL point contact with the consideration of surface roughness. In this solution, the inclusions are homogenized according to Eshelby's equivalent inclusion method with unknown eigenstrains to be determined. A surface coating layer is also considered and assumed as an inclusion of finite size located on the surface, and thus the same methodology applies. The disturbed surface deformation due to the presence of surface coating and inclusions is iteratively introduced into the lubricant film thickness upon the realization of convergence. The discrete convolution and fast Fourier transform technology is adopted to improve the computational efficiency. Based on this solution, a three-dimensional model of line-contact EHL with inclusions distributed periodically along the contact length direction has been developed upon the implement of an algorithm based on fast Fourier transform technology.

Based on the elastic solution of the subsurface stress field, a closed-form solution of plasto-EHL line and point contact for materials with inhomogeneous inclusions is further presented. The plastic strains are iteratively obtained by a procedure involving a plasticity loop and an incremental loading process, and the disturbed deformation caused by the plastic strains along with that caused by the equivalent eigenstrains is introduced to update the lubricant film thickness. The solution takes into account the mutual interactions among inclusions and their surrounding plastic regions, thus leading to an accurate description of the surface pressure distributions, film thickness profiles, plastic zones and subsurface stress field.

The solution is then extended to model materials with cracks and inhomogeneous

inclusions under plane-strain condition. The crack is modeled by a distribution of edge dislocations with unknown densities according to the distributed dislocation technique. With an assumption that the crack surfaces are not in contact, free surface traction conditions of cracks are employed to formulate the governing equations. Coupled governing equations with unknown equivalent eigenstrains and dislocation densities are established to describe the stress and strain fields beneath the contact surfaces. Stress intensity factors are obtained based on the solution of the dislocation densities.

It can be predicted that these solutions have potential applications for the reliability analysis of mechanical components made from composites or with near-surface defects beneath their surfaces, and would provide guidance for minimizing the potential damage of heterogeneous materials induced by embedded inclusions and cracks under lubricated contact.

List of Publications

- [1] **Dong Q.**, Zhou K., 2014. Elastohydrodynamic lubrication modeling for materials with multiple cracks. *Acta Mechanica* 225, 3395-408.
- [2] **Dong Q.**, Zhou K., 2014. Multiple inhomogeneous inclusions and cracks in a half space under elastohydrodynamic lubrication contact. *International Journal of Applied Mechanics* 7, 1550003.
- [3] **Dong Q.**, Zhou K., 2015. Modelling heterogeneous materials with multiple inclusions under mixed lubrication contact. *International Journal of Mechanical Sciences* 103, 89-96.
- [4] **Dong Q.**, Zhou K., Wei R., Luo J., Srikanth N., 2015. Analysis of fluid pressure, interface stresses and stress intensity factors for layered materials with cracks and inhomogeneities under elastohydrodynamic lubrication contact. *International Journal of Mechanical Sciences* 93, 48-58.
- [5] **Dong Q.**, Yang J., Wang X., Keer L. M., Zhou K., 2015. Heterogeneous structures with inhomogeneous inclusions under elastohydrodynamic lubrication contact with consideration of surface roughness. *Proceedings of the Institution of Mechanical Engineers, Part J: Journal of Engineering Tribology*, 1350650115606944.
- [6] **Dong Q.**, Zhou K., Chen W. W., Fan Q., 2016. Partial slip contact modeling of heterogeneous elasto-plastic materials. *International Journal of Mechanical Sciences* 114, 98-110.
- [7] **Dong Q.**, Zhou K.. Numerical Modeling of Elastohydrodynamic Lubrication in Point and Line Contact for Heterogeneous Elasto-plastic Materials. *Mechanics of Advanced Materials and Structures*, DOI: 10.1080/15376494.2016.1227506.
- [8] Zhou K., **Dong Q.**, 2016. A Three-Dimensional model of line-Contact elastohydrodynamic lubrication for heterogeneous materials with inclusions. *International Journal of Applied Mechanics*, 1650014.

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List of Symbols

Symbols	Physical meaning
a_0	Hertz contact radius
$A_{i,j}$, $B_{i,j}$, $C_{i,j}$, $F_{i,j}$	Coefficients of the discrete Reynolds equation
C_{ijkl} , C_{ijkl}^c	Elastic modulus of the substrate or coating layer
C_{ijkl}^i , C_{ijkl}^h	Elastic modulus of the inhomogeneous or homogeneous inclusion
$c^\perp$, $d^\perp$, $c^\top$, $d^\top$	Coefficients to approximate the dislocation densities
E_1 , E_2	Young's modulus of the contact body 1 or 2
E_i , E_c	Young's modulus of the inclusion or coating layer
E'	Effective Young's modulus
E_T^s , E_T^Ψ	Elasto-plastic tangential modulus of the substrate or inclusion
$E_{ij}^\perp$, $F_{ij}^\perp$, $E_{ij}^\top$, $F_{ij}^\top$	Influence coefficients related to the stress caused by dislocations
h	Film thickness
H	Dimensionless film thickness
$H^\perp$, $K^\perp$, $H^\top$, $K^\top$	Influence coefficients related to the displacement caused by dislocations
h_0	Initial film thickness
h_c	Coating thickness
$G_{ij}^\perp$, $G_{ij}^\top$	Influence functions related to the stress caused by dislocations
K_I , K_{II} , K_{III}	Stress intensity factor of the crack in mode I, II or III
κ	Kolosov constant
N_x , N_y , N_z	Number of grid points in x-, y- or z-axis
p	Surface pressure
P	Dimensionless surface pressure
p_0	Maximum Hertz pressure
R_1 , R_2	Curvatures radius of the contact ellipsoid 1 or 2
R_x , R_y	Curvatures radius in the x- or y-direction
R	Sphere radius involved in a contact system
O	Origin point of coordinate
S	Deviatoric stress
t	Time
T	Dimensionless time
u_1 , u_2	Velocity of the contact body 1 or 2
u_e	Effective velocity of the lubricant
u	Disturbed surface deformation
u^i , u^c	Surface deformation due to the inclusion or crack
U_{yy} , U_{xy}	Influence functions related to the displacement caused by dislocations

V	Elastic surface deformation
$\bar{V}$	Dimensionless elastic surface deformation
W	Normal loading
x, y, z	Space coordinates
X, Y	Dimensionless of space coordinates x or y
x_s, x_e	Position of the lubricant inlet or outlet on the x -axis
$\pm y_b$	Positions of the boundary line on the y -axis
α	Pressure-viscosity coefficient
$\alpha_0, \beta_0, \gamma_0$	Element index in x -, y - or z -axis
δ_1, δ_2	Roughness amplitude of the surface 1 and 2
σ	Stress
σ_v	The von Mises equivalent stress
σ^0	Eigenstress caused by the initial eigenstrain
σ^*	Eigenstress caused by the equivalent eigenstrain
σ^c	Stress caused by dislocations
σ^p	Stress caused by surface pressure
$\sigma_Y^s, \sigma_Y^\Psi$	Yield strength of the substrate or inclusion
ϵ	Strain
ϵ^0	Initial eigenstrain
ϵ^*	Equivalent eigenstrain
η	Viscosity of the lubricant
$\bar{\eta}$	Dimensionless viscosity of the lubricant
η_0	Lubricant viscosity under the ambient condition
μ	Shear modulus
ν_1, ν_2	Poisson's ratio of the contact body 1 or 2
ν_i	Poisson's ratio of the inclusion
ρ	Density of the lubricant
$\bar{\rho}$	Dimensionless density of the lubricant
ρ_0	Lubricant density under the ambient condition
$\rho^\perp, \rho^\parallel$	Density of climb or glide dislocations
Ω_ψ, Γ_ϕ	Subdomain of the inclusion or crack
ξ, ζ, φ	Element index in x -, y - or z -axis
$\Delta_x, \Delta_y, \Delta_z$	Half width of element
λ	Ratio of the inclusion Young's modulus to that of substrate
λ^p	Effective accumulative plastic strain
$\wedge$	Fast Fourier transform operation

Chapter 1. Introduction

This PhD study aims to study the reliability of transmission components under lubricated contact. In this chapter, the research background is introduced. Afterwards, the objectives of this research are illustrated. Finally, the organization of the whole dissertation is outlined.

1.1. Background

Tribological phenomena occur when the surfaces of mechanical components come in contact with a relative motion. Friction between the contacting surfaces could cause energy consumption and lead to wear and fatigue damage. Meanwhile, the rough contact would cause deterioration in tribological behavior and increase the wear rate of surfaces. In order to improve the component performance and alleviate damage, fluid lubricants are usually used to prevent the surfaces from direct contact. Elastohydrodynamic lubrication (EHL) is commonly realized as a mode of fluid-film lubrication, in which the formation of lubricant film is enhanced by surface elastic deformation and lubricant viscosity increase caused by high pressure (Zhu and Wang, 2011). This regime of lubrication is mainly found in lubricated counterformal contact of power and motion transmission components whose curvatures do not match, and typical examples are rolling element gears, cam-followers and bearings, etc.

In terms of surface geometry, two types of EHL problems are distinguished: the point contact problems and the line contact problems. In point contact problems, the contact takes place between a half space (Fig. 1.1 (b)) and an ellipsoid (Fig. 1.1 (c)) in a finite elliptical region; in the line contact problems, a cylinder of infinitely long (Fig. 1.1 (d)) is assumed to be involved and the contact takes place within an infinitely long strip (Lubrecht, 1987; Ai, 1993).

Fig. 1.1. Schematics of (a) an EHL contact system, (b) a half space, (c) an ellipsoid involved in point contact and (d) a cylinder involved in line contact.

Depending on the lubricant film thickness and the contact conditions, three different lubrication regimes could be identified: full film lubrication, mixed lubrication and boundary lubrication (Zhu and Hu, 1999). Full film lubrication refers to a condition under which the fluid film is significantly thick so that the contacting surfaces are completely separated. Boundary lubrication is a regime in which the lubricant film is so thin that the load is supported mainly or completely by solid-to-solid asperity contacts, and normally it is known as dry contact. Between the full film and boundary lubrication, the lubrication region, in which the external load is shared partially by lubricant film and partially by solid-solid contact, is referred to as mixed lubrication.

The surface finish of engineering components by a machining process may lead

to the topography of a certain preference of surface asperity distribution. These asperities may significantly influence the hydrodynamic lubricant flows and the entraining action and consequently, affect the lubrication performance. When subject to severe operating conditions, the surfaces cannot be separated completely, and thus the external load is shared by both hydrodynamic lubricant film and asperity direct contact (Hu and Zhu, 2000). Such asperity contact in the mixed lubrication regime is the major cause of the surface wear removal in a rolling/sliding system.

Component surfaces involved in counterformal contact result in a localized high pressure due to the limited area of surface interaction. Such pressure would thus lead to a high contact stress beneath, which probably exceeds the yield strengths of the contacting components, and the material thus involves a noticeable plastic zone within it. The surface asperities result in more severe stress fields on the surface and consequently lead their peaks to deform plastically. Surface alteration by the plastic displacement of surface and near-surface materials results in dimensional loss of solids, but no actual decoupling and loss of materials. However, the large plastic deformation in the contact region sometimes forms a wedge-like shape, which is followed by crack initiation and propagation induced by the combined effect of pressure and friction and finally the detachment of particles. Based on the wear model proposed by Archard (Archard, 1953; Arnell *et al.*, 1991), the volume of wear can be correlated with a coefficient which indicates the volume of the plastically deformed zone underneath a worn surface (Chen *et al.*, 2008b). The analyses of the plastic evolution allow the investigation of the surface-related problems and subsurface stress/strain evolutions.

Previous studies conducted on EHL are based on the assumption of homogeneous materials (Dowson and Higginson, 1959; Archard and Cowking, 1965; Cheng, 1970; Oh and Rohde, 1977; Lubrecht, 1987); however, engineering materials are naturally

heterogeneous and contain inhomogeneities, such as inclusions, coating layers, dislocations and cracks (Zhou *et al.*, 2009; Zhou *et al.*, 2011a; Zhou *et al.*, 2011d; Zhou *et al.*, 2012; Zhou and Wei, 2014a, b). From an engineering point of view, particle reinforced composites or multi-layered materials are desirable to enhance particular properties of materials to elongate the fatigue life of materials. However, the presence of undesirable defects would result in stress concentrations and lead to fracture and fatigue damage of materials.

The effects of defects on fatigue strength and fracture behaviors have been investigated previously and it is concluded that components may have different strength depending on the locations, sizes and geometric shapes of the defects and the interactions among them (Zhou, 2013). The material failure induced by the crack growth can cause catastrophic fracture in a part or system, such as a tooth broken from a gear, or non-catastrophic spalling on components, such as material removal of gears and bearings due to wear of contact surfaces. The presence of voids and inclusions can initiate cracks at the contacting surfaces or around the inclusions because of large stress concentration at these locations and eventually lead to the failure of materials. Although many models for crack initiation and growth have been developed to predict fatigue life, but few models considered the coupled effects of inclusion and cracks and the response of inclusions in spalling fatigue is rarely investigated so far (Murakami and Endo, 1994; Newman, 1998; Beden *et al.*, 2009).

The tribological failure may cause enormous cost due to the large amount of energy and material losses on virtually every mechanical device in operation (Stachowiak and Batchelor, 2013). The UK economy loses £24 billion every year due to problems of friction, wear and lubrication and this figure takes up to 1.6 percent of the country's GDP. Many failures are associated with bearings and bearing failures on

modern generator sets cost about US\$25,000 per day in the U.S.A. while a contingency budget of about £1 million is required to replace a £200,000 bearing in a point mooring on a North Sea Oil Rig. In addition, there are some production losses which can be very costly. The cost of wear for a single US naval aircraft is up to US\$243 per flight hour. More than 1000 Mt of material is excavated in Australia, and most of this is material waste which must be handled in order to retrieve metalliferous ores or coal. The annual production of a large iron ore mining company might be as high as 40 Mt involving a direct cost through the replacement of wearing parts of A\$6 million per year at 1977 values (Perrott, 1978).

Tribology is an interdisciplinary field that involving several branches of continuum and discrete mechanics, materials science, lubricant chemistry and rheology, surface physics and topography/metrology, interfacial physics and chemistry, molecular dynamics, system dynamics, and possibly more. Tribological analysis of mechanical components may provide foundations to reveal the mechanism of interfacial phenomena and thus optimize the design of such components.

1.2. Objectives

This PhD study aims to develop a unified model for the reliability analysis of transmission components operating under lubricated conditions with the interactions between the lubricant and the structure as well as those among inhomogeneities taken into consideration. The solution should be capable of revealing the responses of the inhomogeneities within materials and providing accurate descriptions of local lubricant film, pressure distribution, and stress fields. The main objectives are set as follows:

- To obtain a semi-analytic solution for materials with three-dimensional (3D)

inhomogeneous inclusions under EHL point or line contact with the surface roughness and interactions between the lubricant and the contact bodies as well as those among inclusions taken into account;

- To establish an approach for materials with 3D inhomogeneous inclusions under plasto-elastohydrodynamic lubrication (PEHL) contact with the plastic zones interacting with the lubricants and structures;
- To develop a model for interacting cracks and 2D inhomogeneous inclusions beneath a layered half-plane surface under plane-strain conditions to investigate the effects of surface coating layers and subsurface cracks and inclusions on stress intensity factors (SIFs) at crack tips.

1.3. Dissertation layout

Following this introduction chapter, Chapter 2 gives a brief introduction of the contact problems, and solid materials forming the boundaries of the contact interface with coating layers and subsurface defects, including inclusions, dislocations and cracks are reviewed. In Chapter 3, a semi-analytical solution is developed for materials with 3D inhomogeneous inclusions subjected to EHL of both point and line contact. The accumulative plastic deformation is introduced to extend the model in Chapter 4 and example cases are also presented. In Chapter 5, the model is further developed to simulate cracks of mixed modes I and II within the materials and the effects of the coating layers and subsurface inclusions on SIFs are investigated. In Chapter 6, conclusions are drawn and future works are recommended.

Chapter 2. Literature Review

In this chapter, the contact of homogeneous materials under dry or lubricated conditions is firstly introduced. Previous studies on the inhomogeneities within the materials as well as the interactions between them are then reviewed. Afterwards, the contact of the heterogeneous materials is briefly introduced.

2.1. Point and line contact of homogeneous materials

Contact between the interfaces of mechanical components, under both dry and lubricated conditions, has long been focused on due to its complex mechanism leading to tribological phenomena such as friction, lubrication, adhesion, and wear. Materials forming the boundaries of the interfaces are traditionally assumed to be homogenous, and the relevant solutions provide foundations for the analysis of the interfacial phenomena between contacting components.

2.1.1. Dry contact

Surfaces of components are routinely subjected to contact loading, where large stresses are applied over a highly localized area. These kinds of loading are referred to as dry contact if the lubricants are absent. Depending on the contact area, the contact problems could be categorized into point contact with small dimensions in both principal directions and line contact with infinite length in one direction.

(a) Dry point contact

The contact of homogeneous materials in dry conditions had been studied since the pioneering work by Hertz (1881), who explored the behavior of two elastic bodies

with curvatures that may be adequately represented by parabolas. The general point contact is assumed to occur between two surfaces of ellipsoidal shapes with local radii of curvatures R_{1x} and R_{2x} in the x -direction and R_{1y} and R_{2y} in the y -direction and such contact could be reduced to a half space indented by an ellipsoid with radii of curvatures R_x and R_y , where

$$\frac{1}{R_x} = \frac{1}{R_{1x}} + \frac{1}{R_{2x}} \quad \text{and} \quad \frac{1}{R_y} = \frac{1}{R_{1y}} + \frac{1}{R_{2y}} \quad . \quad (2.1)$$

The indented bodies create a contact area of an ellipse, and therefore this type of contact is also referred to as elliptical contact. A special case of elliptical contact is the circular contact in which the radii of curvatures in both principal directions are equal $R_1 = R_{1x} = R_{1y}$ and $R_2 = R_{2x} = R_{2y}$, and the reduced geometry corresponds to a contact between a sphere with the radius R and a half space, where R could be derived by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad . \quad (2.2)$$

In determining the solutions of the Hertz contact problems, it is assumed that (i) the dimensions of the contact area must remain small compared with the radii of curvatures of the bodies, (ii) the bodies are in frictionless contact and (iii) no plastic evolution is involved. Provided the moduli of the half space and the sphere are E_1 and E_2 , Poisson's ratios ν_1 and ν_2 and the normal loading W , the elastically deformed bodies result in a contact area of a Hertz radius a_0 and a maximum Hertz pressure p_0 , where

$$a_0 = \left(\frac{3}{4} \frac{WR}{E'} \right)^{1/3} \quad \text{and} \quad p_0 = \left(\frac{6WE'^2}{\pi^3 R^2} \right)^{1/3} \quad (2.3)$$

with the effective modulus E' given by

$$\frac{1}{E'} = \frac{(1 - \nu_1^2)}{E_1} + \frac{(1 - \nu_2^2)}{E_2} . \quad (2.4)$$

Analogous expressions of a_0 and p_0 were also given by Johnson (1985) and Fischer-Cripps (2000).

The Hertz theory is restricted to frictionless contact of elastic solids; however, mechanical components are always frictionally loaded and the presence of friction increases the maximum von Mises stress and shifts its position towards the interfaces (Wang *et al.*, 2012). Later progress in contact mechanics has been made to remove the restrictions. Cattaneo (1938) derived an analytical partial slip solution of cylindrical contact for similar materials subject to a tangential loading, in which the shear stress distribution and stick-slip characteristics were determined at the onset of relative motion of contact bodies. His solution later turned out to be valid for a general elliptical contact (Johnson, 1985). Mindlin and Deresiewicz (1953) presented a general procedure for tangential contact problems, and it was concluded that state of stress depends not only upon the initial state of loading but also upon the loading history. Hamilton (1983) developed explicit formulae for the stresses beneath a sliding contact and the maximum von Mises stresses were discussed for the prediction of material failure. Sackfield and Hills (1983a) developed a closed form solution of the stress field for the general elliptical contact, and later on, they further considered the tangential force in their solution (1983b). Jäger (1993) showed that the tangential stresses in the tangential contact problem are equivalent to the difference between the actual normal stresses and those within the stick area.

Real surfaces are rough on the microscopic scale and the rough contact, particularly if the sliding involved, leading to the formation of wear and fatigue.

Greenwood and Williamson (1966) described a methodology to involve the measured surface topography which was assumed to follow the Gaussian distribution. Whitehouse and Archard (1970) investigated the effect of random surfaces in contact process. Studies of statistic analysis of engineering surfaces had proven that the surface roughness may not always follow the Gaussian distribution, and thus the digital optical apparatus was introduced into the description of surface topography. Hu and Tonder (1992) simulated the engineering surfaces composed of both random and periodic parts with the assistance of a digital filter. Polonsky and Keer (1999) solved rough contact problems based upon the conjugate gradient method (CGM), which had turned out to be robust and efficient for the linear equation system.

The non-uniform temperatures of contacting surfaces would cause thermoelastic deformations and thus result in a different contact pressure distribution from that of pure elastic contact. Barber (1973) presented an exact solution for the problem of a half space compressed by a hot sphere. Dundurs and Comninou (1976) introduced a resistance to heat flow at the interface to eliminate the pathological aspects caused by the overmuch idealized boundary conditions. Duvaut (1979) established an existence theorem for a more realistic boundary condition in which the thermal contact resistance at the interface varies inversely with the contact pressure. Yevtushenko and Kulchytsky-Zhyhailo (1996) involved the frictional heating in their solution, and then they derived the formulae which relate the contact tractions and the flux of heat through the contact region (Kulchytsky-Zhyhailo *et al.*, 2001). Jang *et al.* (2009) developed an approximate analytical solution for general thermoelastic Hertzian contact. Liu *et al.* (2001) took the effects of frictional heating and thermoelastic deformations into their consideration. Kim *et al.* (2006) simulated the sliding contact under steady- and unsteady-state thermal conditions.

Above the elastic limit, the contact may induce a range of alternating loads; materials would act plastically at a critical position where the stress intensity exceeds the yield strength of materials. Francis (1976) investigated the deformation mechanics of plastic spherical indentation reported previously and empirically derived functions to describe the plastically deformed region. Chang *et al.* (1987) presented an elastic-plastic asperity volume conservation model for the contact of rough surfaces, which turned out to be suitable over a large range of load conditions. Yu and Bhushan (1996) developed a methodology for the analysis of nominally flat on flat rough surface contact and found that the friction increases the stress magnitude in the vicinity of asperity contact point, where materials are likely to deform plastically. Fotiu and Nemat-Nasser (1996) built a universal algorithm for the integration of the elastic-plastic constitutive equations. Jacq *et al.* (2002) developed a semi-analytical method based on the solutions of Chiu (1977, 1978) for uniform initial strains within a half space and an infinite space. Kogut and Etsion (2004) incorporated the results of finite element analyses of adhesion and sliding inception of a single asperity in a statistical representation of surface roughness.

With the advancement in computer technologies, a versatile algorithm of fast Fourier transform (FFT) technology was introduced into the process of contact analyses (Brigham and Brigham, 1974). The implement of FFT requires a continuous function $f(x)$ transformed into a discrete series f_k with the discrete Fourier transform and recovers by an inverse discrete Fourier transform:

$$\hat{f}_{\alpha_0} = \sum_{k=0}^{N-1} f_k \exp(-i2\pi\alpha_0 k / N) \Leftrightarrow f_k = \frac{1}{N} \sum_{\alpha_0=0}^{N-1} \hat{f}_{\alpha_0} \exp(i2\pi\alpha_0 k / N)$$

$$0 \leq k, \alpha_0 \leq N-1, \quad (2.5)$$

where $i = \sqrt{-1}$ and ‘ $\wedge$ ’ denotes a discrete Fourier transform operation. Compared

with simple direct multiplication, the time complexity could be reduced significantly from N^2 to $N \log_2 N$ if N is large enough.

However, the implement of FFT was based on the assumption of periodic contact geometry, and nonperiodic computational domain would result in large errors around the borders. In order to eliminate these errors, a computational physical domain should be at least five (Ju and Farris, 1996) or eight times (Polonsky and Keer, 2000) that of the target domain. Liu *et al.* (2000) proposed a methodology of discrete convolution and FFT (DC-FFT), in which twice domain extension was used in each dimension and the region outside the target domain is filled with the zero padding. Following their studies, Liu and Wang (2001) investigated the contact of rough surfaces subjected to frictional heating, and later on, they conducted an investigation of the stress fields caused by surface tractions (Liu and Wang, 2002). The DC-FFT is also capable of solving elastic-plastic contact problems (Antaluca *et al.*, 2004; Wang and Keer, 2005; Boucly *et al.*, 2005; Nășliș *et al.*, 2007b; Nășliș *et al.*, 2007a; Chen *et al.*, 2008b; Antaluca and Nășliș, 2008; Wang *et al.*, 2009a; Wang *et al.*, 2010).

(b) Dry line contact

At extreme point contact, the contacting elements are assumed to be infinitely long in one of the principal direction, and the contact area appears to be a narrow strip with a constant width over the lengths of these elements. The stress and strain fields are thus not dependent on the shape of the distant from the contact area any longer, and they could be approximated by assuming each element as a semi-infinite solid bounded by a plane surface, which is well known as the plane strain condition.

In such line contact situation, two parabolically shaped surfaces with local radii of curvature R_1 and R_2 are involved. The equivalent geometry corresponds to a

contact between a half plane and a cylinder of the radius R which could be derived from Eq. (2.2). Analogous expressions of the strip width a_0 , also referred to as Hertz radius (Hertz, 1882), and maximum Hertz pressure p_0 could be obtained by (Hertz, 1896; Johnson, 1985; Fischer-Cripps, 2000)

$$a_0 = \left(\frac{4}{\pi} \frac{WR}{E'} \right)^{1/2} \quad \text{and} \quad p_0 = \left(\frac{WE'}{\pi R} \right)^{1/2} . \quad (2.6)$$

Mindlin (1949) showed that the tangential stresses had the same distribution over the contact areas as the normal stresses. Poritsky (1950) presented a solution to stresses and displacements of cylindrical bodies in contact caused by combined tangential and normal loads. The assumption of plane strain conditions enables the satisfaction of the following relationships (Timoshenko and Goodier, 1951):

$$\varepsilon_y = 0 \quad \text{and} \quad \sigma_y = \nu_1(\sigma_x + \sigma_z) \quad , \quad (2.7)$$

where y -axis denotes the infinitely extended direction and ε and σ the strain and stress components, respectively, ν_1 the Poisson's ratio of the half plane. Smith and Liu (1953) examined the modification of the stress field caused by the surface friction and extended their analysis to study the significance of these stresses in causing failure. Loo and Troy (1958) investigated the effect of curvatures for two cylinders in contact. McCallion and Truong (1982) took the surface roughness into their account. Ciavarella (1998a, b) gave a more general solution for partial slip plane contact. Liu and Wang (2000) considered a steady-state heat transfer and involved the asperity distortion due to thermoelastic deformations in their solution. Jagodnik and Müftü (2003) developed a cylindrical contact model for 2D multi-asperity profiles to obtain an accurate description of the contact area and contact force over a single asperity. Kubin *et al.* (2013) accounted for the elastic-plastic behaviors of contact bodies.

The technique of FFT was introduced into the analysis of cylindrical contact for

the purpose of efficiency improvements. Ju and Farris (1996) developed an algebraic relationship between the surface displacement and the contact pressure based upon the spectral analysis. Ai and Sawamiphakdi (1999) converted the contact problem into an unconstrained energy minimization problem and presented an accurate and efficient solution. Almqvist *et al.* (2007) developed a solution for linear elastic perfectly plastic rough surfaces with the consideration of energy dissipation due to plastic deformation. Chen *et al.* (2008b) proposed a mixed FFT algorithm to approach to the solution of line contact, in which the pressure in the infinitely extended direction was duplicated padding and zero padding in the other. The implement of such methodology enables the solution of the 3D line-contact problems with the periodic rough surface in the infinite direction (Liu and Hua, 2009).

2.1.2. Elastohydrodynamic lubrication contact

(a) EHL point contact

In order to reduce friction and wear in mechanical contact, it is desirable to have a low shear resistance layer between the surfaces in contact to prevent the mechanical components from direct contact. The lubricant is often chosen to act as the protective layer through the relative motion. The EHL refers to a fluid-film lubrication mode in which the formation of lubrication film is enhanced by surface elastic deformation and lubricant viscosity increase caused by high pressure.

Lubrication has been long recognized as a mean of reducing friction and wear. Beauchamp Tower (1884) presented his first report on friction experiments within lubricated bearings and provided a detailed description of the lubricant behavior. Reynolds (1886) gave the theoretical analysis of the discovery of Beauchamp Tower and derived the basic differential equation of fluid film lubrication, which was known

as Reynolds equation and given as

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{12\eta} \frac{dp}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{\rho h^3}{12\eta} \frac{\partial p}{\partial y} \right) = \frac{u_1 + u_2}{2} \frac{\partial(\rho h)}{\partial x} + \frac{\partial(\rho h)}{\partial t} \quad , \quad (2.8)$$

where ρ is the density, η the viscosity, p the pressure generated in lubrication, h the film thickness and u_1 and u_2 the velocities of the contact bodies, respectively. The physical interpretation of Reynolds equation is a balance of fluid flow: the pressure flow term on the left-hand side of the equation stands for the lubricant flow due to the hydrodynamic pressure, while the shear flow term and squeeze flow term on the right-hand side indicate the lubricant flow caused by surface motion in both the tangential and normal directions.

The EHL contact involves the interactions between the elastic surface deformation and lubricant film. The elastic deformation formulated for the point contact could be obtained by the following equation (Timoshenko and Goodier, 1951):

$$V(x, y, t) = \frac{2}{\pi E'} \iint_{\Omega} \frac{p(x', y', t)}{\sqrt{(x - x')^2 + (y - y')^2}} dx' dy' \quad , \quad (2.9)$$

where E' is the effective Young's Modulus of contact bodies and Ω the contact area. The film thickness for the point contact could be obtained by

$$h(x, y, t) = h_0 + \frac{x^2}{2R_x} + \frac{y^2}{2R_y} + V(x, y, t) \quad , \quad (2.10)$$

where h_0 indicates the initial film thickness and R_x and R_y the effective radius in the x - z and y - z planes, respectively.

The lubricants exhibit nonlinear behaviors due to the dependence of the viscosity and density on the pressure. The equations in the present dissertation are among the commonly used ones and given as follows:

$$\eta = \eta_0 \exp(\alpha p) \quad \text{and} \quad \rho = \rho_0 \left(1 + \frac{0.6p}{1 + 1.7p} \right), \quad (2.11)$$

where η_0 and ρ_0 are the lubricant viscosity and density under the ambient condition and α the pressure-viscosity coefficient. It should be noted that depending on the properties of lubricants, the equations should be selected accordingly.

The lubricant film between the two surfaces was optically measured by Gohar and Cameron (1963, 1967), and further modified by Foord *et al.* (1969). It was observed that in point contact the film showed a shape of a horseshoe and the minimum film thickness was located on two sides away from the centerline. Archard and Cowking (1965) developed a semi-analytical solution based on the assumption that the shape of the surface was identical with that in the corresponding dry contact. Cheng (1970) derived a solution of the film thickness in an elliptical contact, which turned out to be a bit higher than that measured by the experiment. Ranger *et al.* (1975) firstly presented a full solution from a straightforward iterative procedure and demonstrated numerically the typical point-contact EHL characteristics and confirmed experimental investigations. Hamrock and Dowson (1976a; 1976b; 1977a, b) used a similar methodology to investigate the lubricant behaviors. The Newton-Raphson method was initially introduced into the solution by Oh and Rohde (1977). An inverse solution was developed by Evans and Snidle (1981) and Hou *et al.* (1987), respectively. Holmes *et al.* (2003) developed an approach by means of coupled differential deflection method, which accelerates the computation and stabilized the solution under heavy loading conditions.

The straightforward approach has been largely improved to derive rapidly converged solutions. The multigrid method was developed by Lubrecht (1987), and then followed by Venner (1991) and many others (Lee *et al.*, 1995; Ehret *et al.*, 1997;

Wang *et al.*, 2001). The method was improved by Ai (1993), Zhu and Hu (1999) and Hu and Zhu (2000) to achieve converged solutions under extremely severe conditions; in their solutions, the rough surface asperity was considered. Zhu (2007) further speeded up the solution process and improved the numerical accuracy by a progressive mesh densification method with the assistance of DC-FFT, and this method shows its capability when ultrathin film and mixed lubrication are concerned. Based on this methodology, the solution involving the plastic evolution was developed (Ren *et al.*, 2010; Ren *et al.*, 2011; He *et al.*, 2014).

The finite element method does not show its capability in solving such problems until recently. The computational fluid dynamics (CFD) approach based on the Navier-Stokes equations was employed by Schäfer *et al.* (2000), followed by Almqvist and Larsson (2001; 2002; 2004), Hong *et al.* (2002) and Hartinger *et al.* (2007; 2008). Habchi *et al.* (2008b; 2008a; 2010; 2012; 2013) developed a full-system finite element approach, and stabilization techniques were used to extend their solution to the cases of highly loaded contact.

(b) EHL line contact

The line contact is identical to that of point contact with constant cross-section extended infinitely, along which uniform loading is applied. The corresponding equations governing the pressure and film thickness could be derived accordingly based on the assumption of plane strain conditions:

$$\frac{\partial}{\partial x} \left(\frac{\rho h^3}{12\eta} \frac{dp}{dx} \right) = \frac{u_1 + u_2}{2} \frac{\partial(\rho h)}{\partial x} + \frac{\partial(\rho h)}{\partial t} \quad (2.12)$$

and

$$h(x, t) = h_0 + \frac{x^2}{2R_x} + V(x, t) \quad , \quad (2.13)$$

in which the y-axis was chosen to be infinitely extended. The elastic deformation along the contact line l is calculated by

$$V(x, t) = -\frac{2}{\pi E'} \int_l p(x', t) dx' \quad . \quad (2.14)$$

Attempts had been made to solve such line-contact problems since the pioneering work by Grubin and Vinogradova (1949), who firstly took into account both elastic deformation and viscosity increase simultaneously. In their solution, it was assumed that the shape of the elastically deformed cylinders under lubricated contact is the same as that in the corresponding dry contact and the hydrodynamic pressure approached infinity at the inlet border of the Hertzian contact zone. Petrusevich (1951) firstly developed a solution in the absent of such assumptions. Dowson and Higginson (1959) derived an inverse solution within a small number of iterative cycles. The Newton-Raphson method was initially introduced into the solution by Rohde and Oh (1975). Houpert and Hamrock (1986) improved the numerical method for line contacts based on Okamura's system analysis (1982), but a good initial guess of the pressure distribution was required to obtain a converged solution. Hughes *et al.* (2000) developed an approach with the assistance of coupled differential deflection method. The multigrid method was introduced by Venner (1991) and Ai (1993) for the purpose of computational efficiency and solution stability. The solution of 3D line contact was developed by Ren *et al.* (2009) and Zhu *et al.* (2009), and the plastic evolution was then involved in the analysis (He *et al.*, 2015).

2.2. Inhomogeneities within heterogeneous materials

Generally, materials in contact studies are assumed to be homogeneous. However, many natural and engineering materials have a heterogeneous structure. Composites or

multi-phase materials take advantages of particular properties of each constituent to meet the specific demands. However, the presence of undesirable defects, such as inclusions, voids, dislocations and cracks, would lead to fracture of materials and accelerate the wear and fatigue damage.

2.2.1. Inclusions

Inclusions within materials could be categorized into inhomogeneous inclusions and homogeneous inclusions. An inhomogeneous inclusion is referred to as a subdomain of a material with elastic modulus different from the matrix material; while a homogeneous inclusion is of the same material as the matrix but contained eigenstrain, which is caused by non-elastic strain, such as plastic strain, thermal expansion, misfit strain, or phase transformation (Fig. 2.1).

Fig. 2.2. An inhomogeneous inclusion with elastic modulus C_{ijkl}^i and initial eigenstrain $\epsilon_{ijkl}^{0(i)}$ and a homogeneous inclusion with the elastic modulus C_{ijkl}^m , initial eigenstrain $\epsilon_{ijkl}^{0(h)}$ and equivalent eigenstrain ϵ_{ijkl}^* embedded in the matrix of elastic modulus C_{ijkl} coated by a layer of elastic modulus C_{ijkl}^c .

In practical engineering, the protective layers bonded to the substrate may suffer from the interface slipping and locally detaching, and thus tangential tractions and

surface adhesion would take place. In order to simplify the problem, the coating in this dissertation is assumed to be perfectly attached to the surface, and thus the coating could be assumed as an inhomogeneous inclusion infinitely extended on the surface.

Extensive studies have been conducted on inclusion-related problems. Eshelby (1957) proposed the equivalent inclusion method (EIM) based upon the assumption the inhomogeneous inclusions as homogeneous inclusions with initial eigenstrain and equivalent eigenstrain (Fig. 2.1), and this methodology is utilized to solve the problem of an ellipsoidal inhomogeneous inclusion embedded in an infinite space (Mura, 1987; Ju and Sun, 1999; Kim and Lee, 2010; Jin *et al.*, 2011). This approach had also been employed to develop analytical solutions for the elastic fields of inclusions of cuboid (Chiu, 1977), cylinder (Wu and Du, 1995b; Wu and Du, 1995a) and other shapes (Wu and Du, 1999; Onaka *et al.*, 2002; Onaka, 2005, 2001). Anisotropic mediums (Walpole, 1967; Asaro and Barnett, 1975; Mura and Kinoshita, 1978) and piezoelectric materials (Wang, 1992; Deng and Meguid, 1999; Wang and Shen, 2001) with embedded inclusions were studied as well.

Inclusions are not present as isolated ones in the matrix material, and they interact with each other. The interactions greatly influenced the mechanical and physical properties of materials at the local and global scales. Thus, it is of great significance to study these interactions for optimal design of advanced materials. Based on the EIM, the interaction of two ellipsoidal inclusions within materials was studied by Moschovidis and Mura (1975) by utilizing the Taylor series to approximate equivalent eigenstrains. Following their work, Shodja and Sarvestani (2001) investigated an ellipsoidal inhomogeneous inclusion encapsulated in another ellipsoidal inclusion which is surrounded by an infinite medium. Fond *et al.* (2001) compared the results provided by Moschovidis and Mura (1975) with the numerical results from finite

element method (FEM) and gave an exact solution of two spherical inclusions in an infinite elastic space. Rodin and Hwang (1991) developed an EIM-FEM hybrid model to evaluate the interaction between multiple spherical inclusions.

Recently, Zhou *et al.* (2009) presented a solution to the problem of multiple interacting inclusions in a half space through the application of a combination of 3D FFT and 2D FFT algorithms, and this methodology shows its capability of improving the computational efficiency and solution accuracy. In their solution, the original problem of a cuboidal inclusion containing constant eigenstrain ε^0 could be obtained by the summation of three subproblems: (1) a cuboidal inclusion with $\varepsilon^0 = (\varepsilon_{xx}^0, \varepsilon_{yy}^0, \varepsilon_{zz}^0, \varepsilon_{xy}^0, \varepsilon_{xz}^0, \varepsilon_{yz}^0)^T$ in an infinite space, (2) an image counterpart with $\varepsilon^{0m} = (\varepsilon_{xx}^0, \varepsilon_{yy}^0, \varepsilon_{zz}^0, \varepsilon_{xy}^0, -\varepsilon_{xz}^0, -\varepsilon_{yz}^0)^T$ in the same space and (3) a half space subjected to a normal traction distribution $-\sigma_{zz}$, where σ_{zz} is the stress component along z -axis obtained from the superposition of subproblems (1) and (2). Their solution was also employed to investigate the interactions of multiple inclusions in an infinite space (Zhou *et al.*, 2011d) or a half space (Zhou *et al.*, 2011c; Zhou *et al.*, 2012). Following their studies, similar solutions were developed to investigate the effects of embedded inclusions on the stress and strain fields (Jin *et al.*, 2011; Jin *et al.*, 2014; Zhou *et al.*, 2015; Zhou *et al.*, 2016).

2.2.2. Dislocations

Dislocations are abrupt changes in the regular ordering of atoms along a line (referred to as dislocation line) which represents the slip front of propagation of a line defect) in the solid. They are generated and would move when a stress is applied, and their motion would lead to slip-plastic deformation.

Fig. 2.3. Schematics of (a) edge dislocation and (b) screw dislocation.

The direction and amount of slip are commonly defined by the Burgers vector $\mathbf{b}$. When the Burgers vector is perpendicularly or parallelly oriented with respect to the dislocation line, the corresponding dislocation is of edge or screw character (Fig. 2.2). The edge and screw dislocations are extreme forms of the possible dislocation structures, and most dislocations are probably a hybrid of the edge and screw forms.

Dislocations may interact with inclusions. The solution for the elastic interaction of an edge dislocation with an elliptical was developed by Stagni and Lizzio (1983) and Tsuchida *et al.* (1991). Xiao *et al.* (2004; 2008) studied the interactions the interactions between a dislocation interacting with a coated inclusion and collinear interfacial rigid lines in the piezoelectric materials. The screw or edge dislocations interacting with interfacial cracks along an elliptical elastic inclusion were studied by Fang *et al.* (2003; 2005). The interactions between screw dislocations and nanoscale inclusions were also studied (Fang and Liu, 2006; Fang *et al.*, 2008; Fang *et al.*, 2009). Besides the static problems, the time-dependent elastic field induced by a dislocation

interacting with a circular inclusion was obtained by Wang and Pan (2011).

2.2.3. Cracks

A crack in an elastic solid can be considered to a distribution of dislocation on the crack plane, and it weakens the materials such that fracture occurs at a stress much less than the yield strength. Generally, cracks are categorized into three types according to its loading conditions: mode I (opening mode) crack, mode II (in-plane shear mode) crack and mode III (out-of-plane shear, tearing mode) crack. For mode I crack, a tensile stress is applied normal to the crack plane; for mode II crack, a shear stress is applied parallel to the crack plane and meanwhile perpendicular to the crack front; for mode III crack, a shear stress is applied parallel to both the crack plane and the crack front. In practical engineering, cracks are usually subjected to more than one type of the above loading conditions, and these cracks are termed as mixed mode cracks.

Fig. 2.4. Three crack types: (a) mode I, (b) mode II and (c) mode III.

It has been realized that regardless of the loading modes, the stress distribution near a crack tip could be expressed as a product of $1/\sqrt{r}$ in a polar coordinate $r-\theta$ system with the origin at the crack tip, and a stress singularity could be found at the crack tip $r=0$ (Sneddon, 1946; Erdogan, 1962; Tada *et al.*, 2000). In order to predict the stress state near the tip of a crack, the SIF K is defined to indicate the amplitude

of the crack tip singularity. From asymptotic solutions in modes I, II and III, the SIFs are given as (Hills *et al.*, 1996; Anderson, 2005)

$$K_I = \lim_{r \rightarrow 0} (\sqrt{2\pi r} \sigma_{yy}), \quad K_{II} = \lim_{r \rightarrow 0} (\sqrt{2\pi r} \sigma_{xy}), \quad \text{and} \quad K_{III} = \lim_{r \rightarrow 0} (\sqrt{2\pi r} \sigma_{yz}) \quad . \quad (2.15)$$

The presence of inclusions would cause stress concentrations which may lead to material fracture, and extensive interactions are involved within materials when the inclusions and cracks coexisted. Tamate (1968) investigated a crack interacting with a single circular inhomogeneous inclusion in an infinite plate when subject to tension. Erdogan *et al.* (1974) studied the interaction of an arbitrarily oriented crack and a circular elastic inclusion. Hsu and Shivakumar (1976) focused on the interactions between a circular inclusion and two collinear cracks in an infinite elastic isotropic plane. Karihaloo (1993) provided a numerical method to solve the subsurface inclusion interacting with a surface crack. Chen (1997) and Patton and Santare (1990) studied the effect of an elliptical inclusion on a crack. Xiao (1997) investigated the interactions between a penny-shaped crack and a spherical inclusion using a superposition scheme. Lam *et al.* (1998) solved the interactions between a symmetrically branched crack and a circular inclusion in an infinite elastic medium. Shodja *et al.* (2003) studied the interactions between two penny-shaped cracks. Yang *et al.* (2004) investigated the interactions between a crack and an arbitrarily shaped inclusion. Andreasen and Matysiak *et al.* (2004) investigated the interactions between a cylindrical inclusion and a crack with the heat conduction and thermoelasticity effect considered.

The continuous distribution of dislocation was firstly employed to model cracks by Bilby *et al.* (1963; 1968), and since then the distributed dislocation method has been employed to analyze various crack problems. Erdogan *et al.* (1971) presented a solution to the problems of bonded medium composed of three different materials with

a flaw on one of the interfaces which were idealized as a crack. Comninous and Schmeuser (1979) investigated the interface crack under combined normal and shear tractions. Nowell and Hills (1987) developed a determination of SIFs for plane cracks at or near free surfaces with arbitrary far fields. Dai (2002) extended the distributed dislocation method to model cracks in finite geometries with the boundaries modeled as continuous distributions of dislocation dipoles. Curved cracks embedded in solids of arbitrary shape were investigated by Han and Dhanasekar (2004).

Fig. 2.5. (a) A glide edge dislocation, (b) a climb edge dislocation and (c) modelling a crack using dislocations.

Hill *et al.* (1996) proposed a distributed dislocation technique (DDT) to model a crack of mixed mode I and II as a distribution of glide and climb dislocations, in which a glide edge dislocation moves by gliding in the x -direction with its Burgers vector in the x -direction, and a climb edge dislocation moves by climbing in the y -direction with its Burgers vector in the y -direction. Fig. 2.4 shows the glide and climb edge dislocations; each dislocation shown is situated at the origin of an x - y or r - θ coordinate system. This methodology is employed in this dissertation to investigate the effect of cracks on the lubrication performance and subsurface stress and strain fields.

2.3. Contact of heterogeneous materials

Heterogeneous materials indented by a loading body further complicates the

problem due to the intensive interactions within the contact system under dry or lubricated conditions besides those among the inhomogeneities. When a normal or tangential load causes the spherical body to indent the half space, the contact load is distributed within the contact area. The pressure and tractions would cause the deformation of both the sphere and the half-space matrix. The inhomogeneities within materials would respond to the stresses induced by the pressure and tractions and cause the deflection of the contact surfaces, which in return affects the profiles of pressure and tractions.

2.3.1. Dry contact of heterogeneous materials

The contact of heterogeneous materials is a mathematically complicated problem. Miller and Keer (1983) firstly investigated a rigid indenter sliding on a half plane with a circular void or rigid inclusion. King and O' Sullivan (1987) studied the quasi-static stress in a 2D layered elastic plane under combined normal and sliding cylindrical contact, and later on, their solution was extended into a 3D layered half space indented by a sphere (O'sullivan and King, 1988). Salehizadeh and Saka (1992) investigated the mechanics of crack initiation surrounded by a hard particle debonding from the half space matrix. Liu and Farris (1994) studied the effect of 3D voids and inclusions on the sliding contact subsurface stress distribution and fatigue limit. Goshima and Soda (1997) presented a solution for the 2D thermoelastic contact problem of a rolling rigid cylinder in a half space containing a subsurface crack, and their solution was extended to investigate the mutual interactions of two subsurface cracks (Goshima *et al.*, 2001). Nogi and Kato (1997, 2002) developed the explicit frequency response functions for the layered body with the implement of FFT to reduce the computing time, and their approach had been extensively used in contact analyses of layered materials by other

researchers (Polonsky and Keer, 2000; Liu *et al.*, 2005b; Liu *et al.*, 2005a; 2007). Polonsky and Keer (2001) developed a model to analyze complex 3D crack patterns in layered substrate. Kuo and Chang (2007, 2008) studied the contact stress field with inclusions embedded in a half space. The multigrid method was introduced into the stress analysis of heterogeneous contact by Boffy and Venner (2014, 2015) with coatings and inclusions considered.

Recently, the EIM has shown its capability of solving the inclusion-related problems under contact loading. Leroux *et al.* (2010) presented a numerical solution for spherical inclusions, and this approach was soon extended to study stick-slip contact between a sphere and a composite with cylindrical fibers (Leroux and Nédias, 2011a, b). Chen *et al.* (2010) modeled layered materials subject to 3D plasto-elastic contact by assuming the coatings as infinitely extended inclusions on the surface, and it should be noted that this assumption requires the width and length of the surface inclusion to be much larger than its thickness and the contact area. Zhou *et al.* (2011b) developed a solution for mutually interacting inclusions, and then the approach was extended to analyze the plasto-elastic behaviors of materials (Wang *et al.*, 2013a; Wang *et al.*, 2013b). Zhou and Wei (2014b) simulated the cracks of mixed modes I and II by means of DDT and then took the interactions of cracks and inclusions into account (Zhou and Wei, 2014a). Amuzuga *et al.* (2015) presented a model to investigate the mutual interactions among inclusions and their surrounding plastic zone.

2.3.2. Elastohydrodynamic lubrication of heterogeneous materials

The lubricated contact problems concerning heterogeneous materials have not appeared until recently. Slack *et al.* (2007) presented an approach that captured the

effects of inclusions on the profiles of pressure distribution and fluid film thickness as well as subsurface stresses in a line contact by means of the discrete element method. Liu *et al.* (2007) modeled layered materials in point contacts with surface deformation determined by the influence coefficients from the frequency response function, and later this solution was utilized to investigate the influence of stiff coatings on lubrication characteristics (2008). A similar solution was developed by Wang *et al.* (2009b; 2015) for the stress analysis of coating/substrate system. Wang *et al.* (2014) and Zhang *et al.* (2014) independently employed EIM to model heterogeneous materials with inclusions and functionally graded materials were also approximated by inclusions located on the substrate by Wang *et al.* (2014).

The inhomogeneities in materials may respond to the stresses induced by the pressure and thus disturb the profile of film thickness, which would result in a different pressure distribution. Therefore, in order to accurately predict the performance of lubricants as well as the surface stress and strain fields, it is of great significance to take into account these heterogeneous effects with the interactions between the fluid lubricant and solid structure as well as those within heterogeneous bodies considered in a unified framework.

2.4. Summary

The snapshots of the history of contact problems for homogeneous materials are first briefly introduced. Afterwards, the studies of inhomogeneities within materials, such as coatings, inclusions, dislocations and cracks, and the mutual interactions among these inhomogeneities are literally presented. Lastly, the contact problems concerning heterogeneous materials are reviewed.

Chapter 3. Elastohydrodynamic Lubrication of Materials with Inhomogeneous Inclusions

A semi-analytical solution is developed for materials with 3D inhomogeneous inclusions subjected to EHL contact. In this solution, the inhomogeneous inclusions are homogenized according to EIM with unknown eigenstrains to be determined. The coating is assumed as an inhomogeneous inclusion of finite size and thus simulated with the same methodology. The disturbed surface deformation due to the presence of coatings and inhomogeneous inclusions is iteratively introduced into the lubricant film thickness until a convergence is achieved. A mixed FFT is utilized to approach the 3D line-contact. This solution takes into account the interactions between the loading body, the fluid lubricant and the coating/substrate system with embedded inclusions.

3.1. Problem description and solution approach

3.1.1. Elastohydrodynamic lubrication point contact

Consider the EHL contact system between the surface of a half space and that of an elastic ball with the radius R . The half space has the Young's modulus E_1 and the Poisson's ratio ν_1 and is assumed to have the sliding velocity u_1 . The ball has the material constants E_2 and ν_2 and is assumed to have the rolling velocity u_2 . The effective modulus for the homogeneous contact between the ball and the half space is denoted by E' , and can be obtained by $E' = \left[\left(1 - \nu_1^2\right) / E_1 + \left(1 - \nu_2^2\right) / E_2 \right]^{-1}$; the lubricant has an effective velocity $u_e = (u_1 + u_2) / 2$. The half space contains the coating domain Ω_1 and $n-1$ arbitrarily-shaped subdomains Ω_{ψ} ($\psi = 2, 3, \dots, n$), each of which has material constants different from the matrix. A cavity or pore embedded

within materials could also be assumed as an inhomogeneous inclusion. Based on this assumption, Zhou *et al.* (2011d) have analyzed the stress fields of a cuboid cavity surrounded by a semi-spherical inclusion subject to uniform stresses.

Fig. 3.1. Discretization of the point-contact domain.

In order to formulate the governing equations, the domain under the contacting surfaces is discretized into $N_x \times N_y \times N_z$ cuboidal elements of the same size $2\Delta_x \times 2\Delta_y \times 2\Delta_z$ and each element is indexed by a sequence of three integers $[\alpha_0, \beta_0, \gamma_0]$ ($0 \leq \alpha_0 \leq N_x - 1, 0 \leq \beta_0 \leq N_y - 1, 0 \leq \gamma_0 \leq N_z - 1$), while the contact surface is composed of $N_x \times N_y$ square patches of $2\Delta_x \times 2\Delta_y$ (Fig. 3.1). It should be noted that the stresses along the interfaces of the inclusions and matrix could not be accurately captured whenever the inclusion edges are not regularly located along the meshes. Previous studies show that such errors are acceptable for the analysis of subsurface stresses and have fewer effects on the profiles of fluid film thickness and pressure distribution. It can be predicted that finer meshes would result in a more accurate description of the stress fields.

The lubricated contact couples the fluid lubricant and solid structure in contact. In the present model, the lubricant is considered to be Newtonian fluid under isothermal

conditions. In the computational domain, the pressure p generated in the lubricant flowing at an entrainment velocity u_e is governed by the Reynolds equation, as described by Eq. (2.8) in conjunction with the film thickness, as given by Eq. (2.9), and elastic deformation Eq. (2.10) as well as pressure dependent viscosity and density Eq. (2.11). Due to the fact that the variables are not all independent, in order to reduce the number of independent parameters, dimensionless parameters are used. The dimensionless variables are defined as:

$$X = \frac{x}{a_0}, Y = \frac{y}{a_0}, H = \frac{h}{a_0}, P = \frac{p}{p_0}, \bar{\rho} = \frac{\rho}{\rho_0}, \bar{\eta} = \frac{\eta}{\eta_0} \text{ and } T = \frac{u_e t}{a_0}, \quad (3.1)$$

where a_0 and p_0 are Hertz radius and maximum Hertz pressure of the corresponding homogeneous dry contact, respectively. With the assistance of these variables the Reynolds equation can be rewritten with the following three terms:

$$\frac{\partial}{\partial X} \left(\bar{\varepsilon} \frac{\partial P}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\bar{\varepsilon} \frac{\partial P}{\partial Y} \right) = \frac{\partial(\bar{\rho}H)}{\partial X} + \frac{\partial(\bar{\rho}H)}{\partial T}, \quad (3.2)$$

where $\bar{\varepsilon} = \frac{\bar{\rho}H^3}{\eta\bar{\lambda}}$ and $\bar{\lambda} = \frac{12\eta_0 u R}{a_0^3 p_0}$. The pressure at the edge of the computational

zone should be equal to the ambient pressure, which is significantly small compared with the pressure in the contact area. Therefore, the pressure boundary conditions of Reynolds equation are

$$p(x_s, y) = p(x_e, y) = p(x, y_b) = p(x, y_b) = 0, \quad (3.3)$$

where x_s and x_e are the positions of the lubricant inlet on the x -axis while $\pm y_b$ are the positions of the boundary line on the y -axis.

The dimensionless film thickness of the EHL point contact for homogeneous materials form can be written as

$$H(X_i, Y_j, T) = H_0(T) + \frac{a_0 X_i^2}{2R} + \frac{a_0 Y_j^2}{2R} + V(X_i, Y_j, T) + \frac{\delta_1(x_i, y_j, t)}{a_0} + \frac{\delta_2(x_i, y_j, t)}{a_0}, \quad (3.4)$$

where H_0 is the dimensionless initial film thickness, δ_1 and δ_2 the roughness amplitude of surface 1 and 2 and V the surface deformation calculated by

$$V(X_i, Y_j, T) = \frac{2p_0}{\pi E'} \iint_{\Omega} \frac{P(X_i, Y_j, T)}{\sqrt{(X_i - X')^2 + (Y_j - Y')^2}} dX' dY' \quad (3.5)$$

with Ω the computational area. The surface deformation equation of discrete form enables the implementation of DC-FFT to speed up the calculation (Zhu and Hu, 1999; Liu *et al.*, 2000).

In order to investigate the effect of roughness orientations, sinusoidal surfaces are studied; the surface of layered material contacting with a single asperity is investigated as well. The external force W is balanced by the pressure generated in the lubricants:

$$\iint_{\Omega} P(X_i, Y_j, T) dX dY = \frac{2}{3} \pi. \quad (3.6)$$

The above system of equations (3.2)-(3.6) is strongly non-linear due to the elastic surface deformation and increased viscosity when subject to the fluid pressure. The discretization of the equations and the relaxation iterative scheme has been described by Ai (1993) and others (Zhu and Hu, 1999; Ren *et al.*, 2009), and details could be referred to Appendix A. The pressure at patch (i, j) can be obtained by converting the differential equation into a discrete equation (Ai, 1993):

$$A_{i,j} P_{i-1,j} + B_{i,j} P_{i,j} + C_{i,j} P_{i+1,j} = F_{i,j}, \quad (3.7)$$

where $A_{i,j}$, $B_{i,j}$, $C_{i,j}$ and $F_{i,j}$ are all known numerical coefficients. When the film thickness H approaches 0, the left-hand side of the Reynolds equation, which represents the lubricant flow due to the hydrodynamic pressure, vanishes. Therefore,

the Reynolds equation can be reduced to the following form:(Zhu and Hu, 2001, 1999)

$$\frac{\partial(H)}{\partial X} + \frac{\partial(H)}{\partial T} = 0 \quad . \quad (3.8)$$

Therefore, a unified Reynolds equation system is applied to both the hydrodynamic contact area and the solid-solid contact area without requiring any information about the border conditions of the two types of contact areas.

When the ball is subjected to an external load W , the contact would take place within a circle region. The fluid pressure generated within the lubricant would cause the deformation of both the loading body and the half-space matrix while the coating and embedded inclusions would also respond to the stresses induced by the pressure and cause the deflection of the contact surfaces. These disturbances in return affect the pressure distribution on the surface of the half plane as well as the lubricant film thickness within the lubricated area. As long as the surface roughness is concerned, the interactions between the loading body, the fluid lubrication film, and the subsurface inclusions become even more intensive.

The EIM is utilized in this study to homogenize the heterogeneous materials (Eshelby, 1957; Zhou *et al.*, 2011d; Wang *et al.*, 2014). Each inhomogeneous inclusion is simulated by a homogeneous inclusion with initial eigenstrain ε_{ij}^0 plus unknown equivalent eigenstrain ε_{ij}^* . The solution strategy of modeling the coating is to assume the layer as a cuboid inhomogeneous inclusion of size $L_x \times L_y \times h_c$ with h_c equal to the thickness of the coating layer. However, this treatment would cause unexpected errors induced by the coating outside the simulation domain. In order to make these errors negligible, a sufficiently large simulation domain is provided with the dimensions L_x and L_y much larger than the Hertzian radius a_0 for homogeneous dry contact.

According to Zhou *et al.* (2011a), the disturbed displacement u_{α_0, β_0} at the patch (α_0, β_0) due to the eigenstrains can be obtained by

$$u_{\alpha_0, \beta_0} = \sum_{\varphi=0}^{N_z-1} \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} A_{\alpha_0-\xi, \beta_0-\zeta, \varphi} (\varepsilon_{\xi, \zeta, \varphi}^0 + \varepsilon_{\xi, \zeta, \varphi}^*) \quad (3.9)$$

$$(0 \leq \alpha_0 \leq N_x - 1, 0 \leq \beta_0 \leq N_y - 1) \quad ,$$

where $A_{\alpha_0-\xi, \beta_0-\zeta, \varphi}$ are influence coefficients which relate the displacements due to the inclusions u_{α_0, β_0} at the observation point $(x_{\alpha_0}, y_{\beta_0}, 0)$ to the initial eigenstrain $\varepsilon_{\xi, \zeta, \varphi}^0$ and equivalent eigenstrain $\varepsilon_{\xi, \zeta, \varphi}^*$ at $[x_\xi, y_\zeta, z_\varphi]$. The derivation of $\varepsilon_{\xi, \zeta, \varphi}^*$ can be found in Appendix B.

The entire problem couples the lubricated contact and the homogenization of heterogeneous materials. During the computational process, the disturbed displacements are determined and then iteratively introduced into the film thickness equation. The algorithm would perform until the inclusion surface deformation converges.

3.1.2. Elastohydrodynamic lubrication line contact

Assume a line contact between the surface of a cylinder with the local radius R of curvature across the contact width and that of a half space (Fig. 3.2). The computational domain D is distributed periodically along the y -axis. The subdomains of arbitrary shape Ω_ψ ($\psi = 1, 2, \dots, n$) are embedded beneath the half space, and the external load per unit length applied on the cylinder is set to be W .

Fig. 3.2. Discretization of the line-contact domain.

The geometry of the contact bodies would result in the thickness of the lubricant film as follows:

$$H(X, Y, T) = H_0 + \frac{a_0 X^2}{2R} + \frac{2}{\pi^2} \iint_{\Omega} \frac{P(X, Y)}{\sqrt{(X - X')^2 + (Y - Y')^2}} dX' dY' + \frac{\delta_1(x, y, t)}{a_0} + \frac{\delta_2(x, y, t)}{a_0} \quad (3.10)$$

The governing equations implemented in the previous chapter could describe the line-contact system as well. It should be noted that at the borders of the solution domain, the following boundary condition should be satisfied

$$\frac{\partial P}{\partial Y} = 0 \quad (3.11)$$

The DC-FFT is not applicable for 3D line-contact problem any longer due to the infinitely extended domain. Chen *et al.* (2008a) developed a mixed FFT-based algorithm, in which the pressure is duplicated in the periodic direction due to the repeated pressure. When the inclusions in the periodically distributed domain are concerned, the eigenstrains within the computational domain should be duplicated as well to obtain the disturbed stresses and displacements.

3.2. Numerical results of point-contact problems

3.2.1. A single inclusion under smooth surface

The lubrication system in point contact consists of a rigid ball of $R = 0.105\text{mm}$ rolling at $u_2 = 0.0001\text{mm/s}$ over a half space with Young's modulus $E_1 = 210\text{ GPa}$ and Poisson's ratio $\nu_1 = 0.3$, and the applied load is fixed at $W = 0.65\text{ N}$. In order to demonstrate the inclusion effects, the ambient viscosity of the lubricant $\eta_0 = 0.0239\text{ Pa}\cdot\text{s}$ and pressure-viscosity exponent $\alpha = 19.6\text{ GPa}^{-1}$ are selected the same as those used in previous lubrication study (Zhu and Hu, 1999; Zhu, 2007; Zhu and Hu, 2001; Wang *et al.*, 2014; Liu *et al.*, 2006). The corresponding Hertzian pressure $p_0 = 8469\text{ MPa}$ and Hertzian contact radius $a_0 = 0.0061\text{ mm}$ are utilized to normalize the results obtained. A spherical inhomogeneous inclusion of various radii located beneath the contact surface is investigated. The inclusion is centered at $(0,0,h_i)$ and its properties are set to have $E_i = 420\text{ GPa}$ and $\nu_i = 0.3$. The computational domain is set as $-2.0a_0 \leq x \leq 2.0a_0$, $-2.0a_0 \leq y \leq 2.0a_0$ and $0 \leq z \leq 2.0a_0$ with $129 \times 129 \times 65$ discretization grids equally spaced.

It has been proved that the pressure profiles obtained along the x -axis show great consistency with that of the previous study upon dry contact condition (Leroux *et al.*, 2010) (Fig. 3.3). Fig. 3.4 shows subsurface von Mises stress contours for the spherical inclusion of different radii. It demonstrates that the stress concentrations beneath the surface become more intensive with the incensement of the inclusion radius at the same depth h_i . It is noted that the inclusion in the half space is meshed by squares, and the edge of the sphere thus appears to be not smooth. It can be predicted that the finer mesh would result in a more accuracy description of the stress fields.

Fig. 3.3. Pressure profile along the x -axis for a single inclusion of different radii at the depth $h_i = 0.5a_0$.

Fig. 3.4. Subsurface von Mises stress for a single spherical inclusion of different radii.

3.2.2. Multiple inclusions under rough surface

The numerical approach has been applied to model heterogeneous materials with nine inclusions under lubricated contact. The inclusions are assumed to have the same edge length $l_x = l_y = l_z = 0.5a_0$ and are equally spaced by $d = 0.5a_0$; the depths of their central points are set to be $h_i = 0.25a_0$ (Fig. 3.5). The properties of inclusion materials are $E_i = \lambda E_1$ and $\nu_i = \nu_1$. The ball of $R = 19.05 \text{ mm}$ is rolling at $u_2 = 5 \text{ m/s}$ over a steady surface and has Young's modulus $E_2 = 210 \text{ GPa}$ and Poisson's ratio $\nu_2 = 0.3$. The applied load is fixed at $W = 800 \text{ N}$. Other parameters are assumed to be the same with that of the previous case.

Fig. 3.5. Scheme of nine inclusions embedded within a plane.

Sinusoidal roughness in the transverse pattern is investigated firstly. Its geometric shape is determined by

$$\delta(x, y, t) = A \cos\left(\frac{2\pi x}{\omega_x}\right), \quad (3.12)$$

where A denotes the amplitude of the sinusoidal wave and ω_x stands for its wavelengths in the x directions. The amplitude and the wavelength are fixed at $A = 0.2 \mu\text{m}$ and $\omega_x = 0.25a_0$.

Fig. 3.6. Surface deformation of a surface with transverse sinusoidal wavy roughness due to the presence of compliant or stiff inclusions.

Fig. 3.6 shows the dimensionless surface deformation due to the present of embedded inclusions for $\lambda = 0.5$ and $\lambda = 2.0$. It can be predicted that the embedded compliant inclusions would more likely deform when subject to external loading on the surface than the matrix or the stiff inclusions of the same distribution. Therefore, the disturbed surface deformation due to the compliant inclusions is positive within the contact area and acts as a surface dent. The deformation due to the stiff inclusions is negative and similar to a surface bump. These deformations reach their maximum values for $\lambda = 0.5$ and minimum value for $\lambda = 2.0$ around the center of the contact area. It should be noticed that these deformations would contribute to the film thickness profiles and thus would lead to a different pressure distribution.

Fig. 3.7. Pressure and film thickness profiles along the (a) x - and (b) y -axis for transverse roughness with inclusions of $\lambda = 0.5$, $\lambda = 1.0$ and $\lambda = 2.0$.

Fig. 3.7 demonstrates the pressure and film thickness profiles along the x - and y -axis. Due to the presence of roughness, the pressure along x -axis in the x - z plane appears to be wavy, and so is the film thickness. The dimensionless pressure in this plane reaches its maximum value 1.23 for $\lambda = 0.5$, 1.29 for $\lambda = 1.0$ and 1.40 for $\lambda = 2.0$. Cavities are observed symmetrically in the y - z plane, and the dimensionless pressure reaches 0.69 for $\lambda = 0.5$, 0.73 for $\lambda = 1.0$ and 0.85 for $\lambda = 2.0$ at the center of this plane.

Stress concentrations are found around the edges of the inclusions (Fig. 3.8). The stresses within the compliant inclusions are less than that around the inclusions, and on

the contrary stresses within the stiff inclusions appears to be larger than the outside stresses. The stresses around the inclusions are increased as the inclusion becomes stiff, and crack nucleation would be more likely to occur (Tanaka and Mura, 1982; Laz and Hillberry, 1998; Wang *et al.*, 2002).

Fig. 3.8. Subsurface von Mises stress in the x - z plane and y - z plane for transverse roughness with inclusions of (a) $\lambda = 0.5$, (b) $\lambda = 1.0$ and (c) $\lambda = 2.0$.

When the direction of the asperity ridges is perpendicular to that of the surface relative motion, the pressure and the film thickness profiles would be significantly different than that generated with the parallel roughness discussed previously. The geometric shape of longitudinal roughness is determined by

$$\delta(x, y, t) = A \cos\left(\frac{2\pi y}{\omega_x}\right), \quad (3.13)$$

where the parameters are set to be the same with the previous case.

Fig. 3.9. Pressure and film thickness profiles along the (a) x - and (b) y -axis for longitudinal roughness with inclusions of $\lambda = 0.5$, $\lambda = 1.0$ and $\lambda = 2.0$.

The pressure and film thickness along the y -axis in the y - z plane appears to be wavy (Fig. 3.9). The dimensionless pressure reaches its maximum value 0.83 in the x - z plane and 1.10 in the y - z plane for $\lambda = 0.5$. An increase of more than 20% of the maximum pressure is found in the x - z plane and 15% in the y - z plane when stiff inclusions are involved. The effect of compliant or stiff inclusions could be concluded

to be similar to that of the above case when the sinusoidal roughness in the transverse pattern is considered.

The geometric shape of isotropic roughness is determined by

$$\delta(x, y, t) = A \cos\left(\frac{2\pi x}{\omega_x}\right) \cos\left(\frac{2\pi y}{\omega_y}\right) \quad , \quad (3.14)$$

where ω_x and ω_y stand for its wavelengths in the x and y directions, respectively. The amplitude and the wavelengths are fixed at $A = 0.2 \mu\text{m}$ and $\omega_x = \omega_y = 0.25a_0$. Other parameters are set to be the same with the previous cases.

Fig. 3.10. Pressure and film thickness profiles along the (a) x - and (b) y -axis for isotropic roughness with inclusions of $\lambda = 0.5$, $\lambda = 1.0$ and $\lambda = 2.0$.

Pressure and film thickness profiles along the x -axis in the x - z plane and y -axis in the y - z plane for inclusions of $\lambda = 0.5$, $\lambda = 1.0$ and $\lambda = 2.0$ are shown in Fig. 3.10. It can be seen that the pressure is more sensitive to the inclusion stiffness than the film thickness when the selected inclusion distribution is considered. The dimensionless pressure with the presence of inclusions considered reaches its maximum value 1.73 in the x - z plane and 1.50 in the y - z plane when the stiff inclusions $\lambda = 2.0$ are involved, and it decreases to 1.62 and 1.47 for homogeneous contact $\lambda = 1.0$ and 1.57 and 1.45 for compliant inclusions $\lambda = 0.5$.

3.2.3. Material with a coating layer in contact with a stationary asperity

A single asperity located on a stationary smooth ball surface in contact with a layered substrate moving at 5 m/s is studied in this case. The shape of the asperity is modeled with the equation following

$$\delta(x, y, t) = \begin{cases} h_a \left(2^{-6.25(x^2+y^2)/r_a^2} - 2^{-6.25} \right) & \text{if } \sqrt{x^2 + y^2} \leq r_a \\ 0 & \text{if } \sqrt{x^2 + y^2} > r_a \end{cases}, \quad (3.15)$$

where the asperity height h_a is set to have $0.2 \mu\text{m}$ and width $r_a = 0.1a_0$. The solution strategy for modeling the coating is to assume the layer as a cluster of cuboid inclusions. However, the inclusions outside the target domain cannot be captured, and unexpected errors would occur on the stress fields especially at the domain boundary. In the following cases, the computational domain is extended to $[-4.0a_0 \leq x \leq 4.0a_0, -4.0a_0 \leq y \leq 4.0a_0, 0 \leq z \leq 2.0a_0]$ with $129 \times 129 \times 65$ discretization grids equally spaced to make the errors negligible. The coating of thickness $h_c = 0.5a_0$ is set to have Young's modulus $E_c = 2E_1$ or $E_c = 0.5E_1$ and Poisson's ratio $\nu_c = 0.3$.

Fig. 3.11. Surface deformation due to a compliant or stiff coating.

Fig. 3.11 shows the dimensionless surface deformations due to the presence of compliant coating and embedded inclusions. It can be concluded that the coating acts similar to an inclusion extended infinitely on the surface, and the disturbed surface deformations u due to the compliant coating would be positive. Whenever the compliant inclusions are involved, u would always increase the film thickness. With the thickness h_c of the coating and the inclusion distribution selected, the disturbed surface deformations are always contributed to the film thickness positively when the stiff inclusions are taken into consideration. However, with the same coating attached on the surface, the deformation u decreases as the inclusions become stiff.

Fig. 3.12 demonstrates the pressure and film thickness profiles in the x - z plane and y - z plane. It shows that the pressure peak appears at the center point $(0, 0, 0)$ due to the presence of the asperity 1.12 for $\lambda = 0.5$, 1.21 for $\lambda = 1.0$ and 1.36 for $\lambda = 2.0$. Cavities are observed around the outlet position in the x - z plane and y - z plane.

Fig. 3.12. Pressure and film thickness profiles along the (a) x - and (b) y -axis for a single asperity with a coating of $\lambda = 0.5$, $\lambda = 1.0$ and $\lambda = 2.0$.

The coating would disturb the stress field; stress concentrations can be found along the interface of the coating and substrate (Fig. 3.13). The coating act similarly to the inclusions; stiff layers would result in thinner film thickness and higher local pressure while compliant layers increase the film thickness and decrease the pressure.

Fig. 3.13. Subsurface von Mises stress in the x - z plane and y - z plane for a single asperity with a coating of (a) $\lambda = 0.5$, (b) $\lambda = 1.0$ and (c) $\lambda = 2.0$.

3.2.4. Material with multiple inclusions in contact with a moving asperity

An asperity of semi-spherical shape located on the upper surface passing through the contact zone is further investigated. The lower surface is assumed to be smooth. Both of the two surfaces move at the same velocity $u_1 = u_2 = u_e = 1.5$ m/s.

Its geometric shape is determined by

$$\delta(x, y, t) = \begin{cases} \frac{A_a}{R_s} \sqrt{R_s^2 - [(x - x_a)^2 + y^2]} \\ 0 \text{ when } \sqrt{(x - x_a)^2 + y^2} \geq R_s \end{cases}, \quad (3.16)$$

where the asperity height $A_a = 1.0 \mu\text{m}$, the radius of the spherical asperity $R_s = 0.1a_0$ and $x_a = x_s + u_1 t$ with x_s presenting the position at $t = 0$. Three embedded inclusions are oriented perpendicular to the velocity of the lubricant. The inclusions are assumed to have the same edge length $l_x = l_y = l_z = 3a_0/8$ and are equally spaced by $d = 3a_0/8$; the depths of their central points are set to be $h_i = 15a_0/32$. The time step length is set to be $\Delta t = \Delta x / u_e$ and at each time step the asperity has the same x coordinate as that of the inclusions.

Fig. 3.14. Pressure and film thickness profiles and von Mises stress in the x - z plane for a moving asperity (a_1 - a_4) $\lambda = 0.5$, (b_1 - b_4) $\lambda = 1.0$ and (c_1 - c_4) $\lambda = 2.0$.

Fig. 3.14 shows that before the embedded inclusions move into the contact area, the pressure and film thickness profiles for different λ are almost the same and their effects can be ignored. Whenever the inclusions move into the contact area, their response should always be taken into consideration. The compliant inclusions are more likely to deform compared with the stiff inclusions of the same distribution when subject to the external loading, and therefore the case with the compliant inclusions would result in a thicker film and a smaller asperity contact area. Cavities are observed around the solid direct contact area. At the same time step, the case with stiff inclusions would result in a more intensive stress concentration and would more likely to cause cracks along the inclusion edges and lead to material failure.

The disturbed surface deformation due to the inclusions at the corresponding time steps is presented in Fig. 3.15. It can be seen that as the inclusions stay outside of the contact area, the disturbed deformation u has a small value. Thus the influence of inclusions on the pressure and film thickness profiles at such time steps can be ignored. When the inclusions and the asperity appear around the center of the contact area, u

reaches its maximum value for $\lambda = 0.5$ and minimum value for $\lambda = 2.0$ around the location where the pressure peak occurs.

Fig. 3.15. Surface deformation in the x - z plane for a moving asperity with inclusions of (a) $\lambda = 0.5$ and (b) $\lambda = 2.0$.

3.3. Numerical results of line-contact problems

3.3.1. A single inclusion under smooth surface

The applied load is set to be $W = 2500 \text{ N/mm}$ for line-contact problems and the lubricant properties are $\eta_0 = 0.01119 \text{ Pa} \cdot \text{s}$ and $\alpha = 14.94 \text{ GPa}^{-1}$. The heterogeneous

half space with a single cuboidal inclusion centered at $(0, 0, h_i)$ is studied. The Young's modulus of the inclusion is $E_i = \lambda E_1$ and the Poisson's ratio $\nu_i = \nu_1$. The inclusion is assumed to have the same edge length $l_x = l_y = l_z = a_0$ (Fig. 3.16). The lubricant speed is set to be $u_e = 100 \text{ m/s}$.

Fig. 3.16. A single inclusion embedded within a half space.

Film thickness and pressure profiles for the inclusion located at various depths along the y-axis are shown in Figs. 3.17 and 3.18. When the compliant inclusion $\lambda = 0.5$ is involved, the portion of the lubricant film around the center appears to be thicker than other positions. The inclusion results in a lower surface pressure on the region right above it than other regions. It can be concluded that compliant inclusion causes a similar effect like a surface dent. In contrast, a stiff inclusion would act like a surface bump, which decreases the film thickness but increases the local surface pressure. As the depth of the inclusion increases, its effect on the surface contact and pressure decreases. It could be predicted that when the inclusion is located infinitely far away from the surface ($h_i = \infty$), the effect of the inclusion will become negligible, i.e., a homogeneous half-space contact will be realized.

Fig. 3.17. (a-d) Film thickness profiles and (e-h) pressure profiles along the y-axis for the compliant inclusion $\lambda = 0.5$ located at the depth of $h_i = 0.75a_0, 1.00a_0, 1.25a_0$ and ∞ .

Fig. 3.18. (a-d) Film thickness profiles and (e-h) pressure profiles along the y-axis for the stiff inclusion $\lambda = 2.0$ at the depth of $h_i = 0.75a_0, 1.00a_0, 1.25a_0$ and ∞ .

Fig. 3.19 shows the subsurface von Mises stress contours in the $x = 0$ plane for a compliant or stiff inclusion located at the depths of $h_i = 0.75 a_0, 1.00 a_0$ and $1.25 a_0$.

The presence of the inclusion would disturb the subsurface stress fields and stress concentrations are found along the inclusion edges where the crack nucleation is likely to start. The inclusion which is located closer to the surface would induce more intensive interactions between it and the matrix. When a compliant inclusion is concerned, the stresses within it are smaller than those outside it; while the stresses inside are larger when a stiff inclusion is involved.

Fig. 3.19. Subsurface von Mises stress contours in the $x = 0$ plane for the inclusion (a) $\lambda = 0.5$ or (b) $\lambda = 2.0$ at the depths of $h_i = 0.75 a_0, 1.00 a_0$ and $1.25 a_0$.

The location of the inclusion would lead to disturbances of surface deformation and therefore disturb the lubricant film and surface pressure profiles. The inclusion is centered at the point $(-1.0a_0, 0.0, 0.75a_0)$ on the left side of the $x = 0$ plane or at

$(1.0a_0, 0.0, 0.75a_0)$ on the right side. Fig. 3.20 demonstrates the pressure and film thickness profiles and the subsurface von Mises stresses in the $y = 0$ plane. The compliant inclusion $\lambda = 0.5$ on the left results in a peak pressure $p = 1.00p_0$ at the point $(0.03a_0, 0.0, 0.0)$ on the surface, and the right one induces the same pressure peak at $(-0.03a_0, 0.0, 0.0)$. If a stiff inclusion $\lambda = 2.0$ is considered, the peak pressure occurs at $(-0.09a_0, 0.0, 0.0)$ or $(0.09a_0, 0.0, 0.0)$.

Fig. 3.20. Pressure and film thickness profiles and the subsurface von Mises stresses in the $y = 0$ plane for the inclusion (a) $\lambda = 0.5$ or (b) $\lambda = 2.0$ at $(-1.0a_0, 0.0, 0.75a_0)$ or $(1.0a_0, 0.0, 0.75a_0)$.

3.3.2. Multiple inclusions under rough surface

A sinusoidal surface of the half space beneath which multiple inclusions are distributed is concerned. The inclusions have the same length $l_x = l_y = l_z = 0.5a_0$. The other dimensions are set to be $h_i = 0.25a_0$, $d_1 = 0.5a_0$ and $d_2 = 0.5a_0$ (Fig. 3.21), and the lubricant flows at the speed $u_e = 100\text{m/s}$. The sinusoidal wavy surface is expressed as

$$\delta(x, y, t) = A \cos\left(\frac{2\pi x}{\omega_x}\right) \cos\left(\frac{2\pi y}{\omega_y}\right), \quad (3.17)$$

where A denotes the amplitude of the sinusoidal wavy and ω_x and ω_y stand for its wavelengths in the x and y directions, respectively. In this case, the amplitude and the wavelengths are set to be $A = 0.4\mu\text{m}$ and $\omega_x = \omega_y = 0.25a_0$.

Fig. 3.21. Scheme of multiple inclusions embedded within a half space.

Fig. 3.22 shows the pressure and film thickness profiles and the subsurface von Mises stress in the $y = 0$ plane and $x = 0$ plane for the multiple compliant and stiff inclusions. It shows that compliant inclusions would increase the contact surface gaps and thus result in a smaller local surface pressure above them, while the stiff inclusions would induce a larger local surface pressure above them. The stress contours in the $x = 0$ plane appear to be symmetric about the $y = 0$ plane due to the symmetrically distributed inclusions.

Fig. 3.22. Pressure and film thickness profiles and the von Mises stress in the (a) $y = 0$ plane and (b) $x = 0$ plane for a sinusoidal wavy surface with inclusions.

3.3.3. Material with a coating layer in contact with a smooth surface

The solution strategy for modeling a layer of coating is to assume the layer as a cluster of cuboid inclusions. The computational domain is extended to $[-4.0a_0 \leq x \leq 4.0a_0, -4.0a_0 \leq y \leq 4.0a_0, 0 \leq z \leq 4.0a_0]$ in the coating-related problems.

Fig. 3.23. Pressure distribution and film thickness profiles for (a) a compliant coating $\lambda = 0.5$ and (b) a stiff coating $\lambda = 2.0$.

Fig. 3.24. Pressure and film thickness profiles and subsurface von Mises stress in the x - z plane for (a) a compliant coating $\lambda = 0.5$ and (b) a stiff coating $\lambda = 2.0$.

The lubricant speed is set to be the same as that of the previous case. Coatings of various thicknesses are modeled. It shows that a compliant coating is more likely deformed than a stiff coating, and the contact area is enlarged due to the compliant

coating, and reduced due to the stiff coating. For the case of $h_c = 0.50 a_0$, the presence of a compliant coating induces a maximum pressure $p = 0.91 p_0$ at the center of the domain which is smaller than that of $p = 1.06 p_0$ for a stiff coating, as shown in Fig. 3.23. For the case of $h_c = 1.0 a_0$, the pressure reaches its maximum value $p = 0.87 p_0$ and $1.10 p_0$ for the compliant and the stiff coating, respectively. The maximum von Mises stress $\sigma_v = 0.52 p_0$ is observed at the depth of $h_v = 0.5 a_0$ in the plane $y = 0$ for the compliant coating and $\sigma_v = 0.70 p_0$ at $h_v = 1.0 a_0$ for the stiff coating (Fig. 3.24).

3.4. Summary

A semi-analytical solution is developed for heterogeneous materials with multiple inclusions beneath the surfaces with surface roughness considered. By applying EIM, the inhomogeneous inclusions are homogenized as homogeneous inclusions with unknown eigenstrains to be determined. The disturbed surface deformations due to the eigenstrains are iteratively introduced into the lubricant film thickness to obtain an accurate description of the fluid pressure and lubricant film thickness profiles and the subsurface stress fields. The solution takes the interactions among the loading bodies, the fluid lubricant, and the embedded inclusions into account. A mixed FFT is utilized to approach the 3D line-contact.

The numerical results show that the disturbed surface deformation is positive for the cases of compliant inclusions which would act similar to a surface dent; while stiff inclusions induce negative deformation within the contact domain and act as a surface bump. A near-surface stiff inclusion would result in a larger local surface pressure and a larger neighboring subsurface von Mises stress level than a compliant inclusion, and thus is more likely to cause crack initiation and propagation in its neighborhood. When

an inclusion is located close to a surface subject to contact loading, it has a significant effect on the surface pressure and such an effect decreases as the inclusion depth increases. The effect of the upmost coating is similar to an inclusion extended infinitely. Stress concentrations can be found along the interface of the coating and substrate and around the inclusion edges, and it is essential for the analysis of material fracture.

Chapter 4. Plasto-elastohydrodynamic Lubrication of Materials with Inhomogeneous Inclusions

In this chapter, the evolutions of the plastic region in materials with 3D inhomogeneous inclusions are considered. The plastic strains are iteratively obtained by a procedure involving a plasticity loop and an incremental loading process. The model takes into account the interactions among contact bodies, embedded inclusions, lubricant film and plastic zone, and leads to the solution of pressure distribution, film thickness, plastic zone and subsurface stress fields. This solution is of great importance for the analysis of heterogeneous contact coupling the hydrodynamic lubricant behaviors in the elastic-plastic indentation regime.

4.1. Problem description and solution approach

Counterformal geometry of the contact bodies leads to a high contact pressure and would thus lead to local plastic yielding of materials. In this study, a rigid ball with the radius R or a rigid cylinder with the local radius R of curvature across the contact width is involved in the point or line contact EHL. The substrate with the Young's modulus E_1 and the Poisson's ratio ν_1 contains n arbitrarily-shaped subdomains Ω_ψ ($\psi = 1, 2, 3, \dots, n$), and the lubricant flows at the entrainment velocity u_e . The substrate and subsurface inhomogeneous inclusions may yield as long as the von Mises equivalent stresses exceed their yield strength σ_Y^Ψ for inclusions and σ_Y^s for the substrate. The J -2 criterion for material yield initiation is given as (Hill, 1998):

$$\begin{cases} f = \sigma_v - \sigma_Y^\Psi & \text{in } \Omega_\psi \\ f = \sigma_v - \sigma_Y^s & \text{out } \Omega_\psi \end{cases}, \quad (4.1)$$

where the von Mises stress $\sigma_v = \sqrt{3S_{ij} : S_{ij} / 2}$ and the deviatoric stress $S_{ij} = \sigma_{ij} - \sigma_{kk} \delta_{ij} / 3$. The derivation of the stress components can be found in Appendix B.

Once the trial plastic domain is identified, the actual increment of the effective plastic strain returns the J -2 yield function to zero and can be determined by the function $f(\lambda^p + d\lambda^p) = 0$, where the effective accumulative plastic strain $\lambda^p = \sum d\lambda^p = \sum (\sqrt{2d\varepsilon_{ij}^p d\varepsilon_{ij}^p / 3})$ and ε_{ij}^p denotes the plastic strain. The plastic strain variation is determined by the plastic flow rule (Hill, 1998), given as

$$d\varepsilon_{ij}^p = \frac{\partial f}{\partial \sigma_{ij}} d\lambda^p = d\lambda^p \frac{3S_{ij}}{2\sigma_v} . \quad (4.2)$$

The yield strength of the presented cases is assumed to follow the linear hardening law (Fig. 4.1):

$$\begin{cases} \sigma_Y^\Psi(\lambda^p) = \sigma_Y^\Psi(0) + \frac{E_T^\Psi \lambda^p}{1 - E_T^\Psi / E_\Psi} & \text{in } \Omega_\Psi \\ \sigma_Y^s(\lambda^p) = \sigma_Y^s(0) + \frac{E_T^s \lambda^p}{1 - E_T^s / E_2} & \text{out } \Omega_\Psi \end{cases} , \quad (4.3)$$

where $\sigma_Y^\Psi(0)$ and $\sigma_Y^s(0)$ denote the initial yield strengths of the inclusions and substrate, respectively, E_T^Ψ and E_T^s the elasto-plastic tangential moduli. It should be noted that the plastic zone can be regarded as a homogeneous inclusion composed of multiple cuboidal elements. The plastic strain is uniform in each element, but non-uniform in the total plastic zone. Thus, the residual displacements and stresses can be obtained by the same methodology as those due to the equivalent eigenstrains.

Fig. 4.1. The relationship between the yield strength and the effective accumulative plastic strain following the linear hardening law.

There exist intensive interactions within the contact system. When an initial load causes the loading body to indent the half space, the contact load is distributed on the indentation contact area. The pressure would cause the deformation of both the contact bodies. The equivalent eigenstrains within inclusions would also respond to the stresses induced by the pressure and cause the deflection of the contact surfaces. The J -2 criterion is then employed to check if plastic deformation occurs after the pressure achieves its convergence, and the residual displacements induced by plastic strains in the zones where the half space or inclusions yield are then taken into consideration to update the surface contact geometry.

A closed loop linking the variations of the pressure, equivalent eigenstrains and plastic strains and surface geometry is developed in this study. The numerical process performs until the convergence of the residual displacements. The load is increased gradually until the desired load achieved. The flowchart of the entire numerical procedure is shown in Fig. 4.2.

Fig. 4.2. Flowchart of the numerical procedure.

4.2. Numerical results and discussions

In this section, the plastic performances of contact bodies under an increasing normal load involved in a PEHL of point or line contact are investigated. The sphere is assumed to be rigid and has the radius $R = 20$ mm. Young's modulus of the half space is set to have $E_2 = 100$ GPa and Poisson's ratio $\nu_2 = 0.3$; the yield strength of the half space is fixed at $\sigma_Y^s = 600$ MPa. The lubricant properties are $\eta_0 = 0.096$ Pa·s and $\alpha = 18.2$ GPa⁻¹, and the lubricant velocity is set to be $u_e = 1000$ mm/s if not specified. A domain of $4a_0 \times 4a_0 \times 2a_0$ is supposed for point-contact problems with a stiff inclusion $E_i = 2.0E_2$ or compliant inclusion

$E_1 = 0.5E_2$ of the cubic shape with the dimensions $a_0 \times a_0 \times a_0$ centered at $(0, 0, 0.75a_0)$; a domain of $3a_0 \times 3a_0 \times 2a_0$ with an inclusion of $0.75a_0 \times 0.75a_0 \times a_0$ centered at $(0, 0, 0.75a_0)$ is considered for line-contact problems. The domain is discretized into $128 \times 128 \times 40$ elements. The loading increment of each step is W_c , which is the transitional load indicating the onset of an elasto-plastic contact when the sphere is loaded against the homogeneous half space and given by

$$W_c = \frac{(1.6\pi\sigma_{ys})^3}{6} \left(\frac{R}{E^*} \right)^2, \quad (4.4)$$

where E^* is the equivalent Young's modulus of homogeneous contact without inclusion considered, and the maximum normal load is $W = 10W_c$.

4.2.1. Effect of Young's modulus

(a) A stiff inclusion

A stiff inclusion is focused on firstly for the point contact. The profiles of pressure and film thickness for the inclusion with the same tangential modulus as the matrix $E_T^\Psi = E_T^s = 0.9$ and same yield strength $\sigma_Y^\Psi = \sigma_Y^s$ as the substrate are shown in Fig. 4.3. It can be observed that the increase of the loading would induce a higher pressure and a larger the contact area but a thinner film profile. Under a lighter load $W = 2.0W_c$, the pressure and film thickness show classic characteristics as the elastically lubricated contact. As the load increasing, the inclusion shows the same effects as a surface bump and results in a high pressure gradient right above the inclusion edges.

Fig. 4.3. Profiles of pressure and film thickness along the x -axis at various loading steps for a stiff inclusion of $E_T^\Psi = E_T^s = 0.9$.

Fig. 4.4. The von Mises stress at the depth of $0.25a_0$ along the x -axis at various loading steps for a stiff inclusion of $E_T^\Psi = E_T^s = 0.9$.

Fig. 4.4 shows the von Mises stress at the depth of $0.25a_0$ along the x -axis. It shows that the von Mises stress increases with the loading, and stress concentrations appear at the edges of the inclusion $x = -0.5a_0$ and $x = 0.5a_0$. Flat stress profile has been observed around the center of the computational domain due to the plastic deformation of both the substrate and inclusion. The contour of effective plastic strain

λ^p (%) in the $y = 0$ plane is shown in Fig. 4.5. It can be seen that the effective plastic strain is accumulated around the inclusion edges and its maximum value occurs due to the local stress concentration.

Fig. 4.5. The contour of effective plastic strain λ^p (%) in the $y = 0$ plane for a stiff inclusion of $E_T^\Psi = E_T^s = 0.9$.

Fig. 4.6. Profiles of pressure and film thickness along the x -axis at various loading steps for line contact with a stiff inclusion.

The tangential modulus $E_T^\Psi = E_T^s = 0.9$ is concerned in the line-contact problems. The Fig. 4.6 demonstrates the profiles of pressure and film thickness along the x -axis at each loading step for a stiff inclusion. It can be seen that the inclusion shows the same effects as that of point contact on the pressure profiles and lead to a high pressure gradient above the inclusion edges. A thinner lubricant is formed

between the contact surfaces right above the inclusion. Due to the presence of the symmetrically distributed inclusion, the pressure and film thickness along the y -axis show symmetric profiles about the $x = 0$ plane (Fig. 4.7).

Fig. 4.7. Profiles of (a) pressure and (b) film thickness along the y -axis at various loading steps for line contact with a stiff inclusion.

The stress concentrations can be observed along the x -axis and y -axis at the edges of the inclusion, and due to the fact that the domain with a stiff inclusion embedded is periodically repeated along the y -axis, the stress at the center is smaller than the concentrated stress at the inclusion edges $y = \pm 0.325a_0$ but larger than that at the boundary of the domain shown in Fig. 4.8 (b).

Fig. 4.8. Profiles of von Mises stresses at the depth of $0.25a_0$ along the (a) x -axis and (b) y -axis at various loading steps for line contact with a stiff inclusion.

(b) A compliant inclusion

Fig. 4.9 shows the profiles of pressure and film thickness for the compliant inclusion of $E_T^\Psi = E_T^s = 0.9$. When the load $W = 2.0W_c$ the peak pressure appears at the center of the domain; a concave shape can be observed otherwise and the local pressure where the inclusion edges are located beneath appears to be larger than that at the center of the domain.

Fig. 4.9. Profiles of pressure and film thickness along the x -axis at various loading steps for a compliant inclusion of $E_T^\psi = E_T^s = 0.9$.

Fig. 4.10. Profiles of (a) pressure and (b) film thickness along the x -axis at various steps for line contact with a compliant inclusion.

The profiles of pressure and film thickness along the y -axis are shown in Fig. 4.10 for compliant inclusion of $E_i = 0.5E_2$, and similar effects as a surface damp have been observed. Fig. 4.11 shows a lower stress level at the center of the domain than the concentrated stress.

Fig. 4.11. Profiles of von Mises stresses at the depth of $0.25a_0$ along the (a) x -axis and (b) y -axis at various loading steps for line contact with a compliant inclusion.

It is noted from the above results that due to the fact materials with compliant inclusions are more likely to deform compared with homogeneous materials, those

heterogeneous materials would cause a similar effect like a surface dent. Meanwhile, in the last chapter, it has been concluded that the inclusion would induce extensive interactions of stress fields, and thus plasticity is accumulated from the area around the inclusion. The equivalent eigenstrains within the inclusion along with the plastic strain would together disturb the surface deformation and consequently result in a thicker film and a lower pressure level than that of homogeneous contact. On the contrary, when a stiff inclusion is involved in the analysis and the equivalent eigenstrains dominate the disturbance, the inclusion acts like a surface bump and otherwise a surface dent effect could be observed.

4.2.2 Effect of tangential modulus

(a) An inclusion with same tangential modulus as matrix

The profiles of pressure and film thickness for point contact along the x -axis and the contours of von Mises stress on the $y = 0$ for the stiff inclusion of $E_T^\Psi = E_T^s = 0.0$ are shown in Fig. 4.12. The decrease of the tangential modulus would lead both the substrate and the inclusion more likely to plastically deform, which would results in smaller pressure profiles compared with that of $E_T^\Psi = E_T^s = 0.9$. The pressure around the inlet and outlet positions appears to be larger than that at the center of the domain, and the extensive plastic zone due to a smaller tangential modulus leads to a smaller peak pressure. Same conclusions could be drawn for the line-contact problems.

Fig. 4.12. Pressure and film thickness profiles along the x -axis and the von Mises stress contour in the $y = 0$ plane for a stiff inclusion of $E_T^\Psi = E_T^s = 0.0$.

(b) An inclusion with different tangential modulus from matrix

Inclusions may possess different plastic behaviors than the substrate. Fig. 4.13 shows the contours of effective plastic strain of point contact in the $y = 0$ plane for a stiff inclusion of $\sigma_Y^\Psi = \sigma_Y^s/1.5$ and $\sigma_Y^\Psi = 1.5\sigma_Y^s$ or a compliant inclusion of $\sigma_Y^\Psi = \sigma_Y^s/1.5$ and $\sigma_Y^\Psi = 1.5\sigma_Y^s$. In this case, $E_T^\Psi = E_T^s = 0.9$ is assumed. It shows that little yield strength may lead to an extensive plastic zone and the compliant inclusion would result in a larger accumulated peak strain. Fig. 4.14 shows the symmetrically generated plastic strain for line contact accumulated in the $x = 0$ plane for an inclusion, stiff or compliant, with a yield strength $\sigma_Y^\Psi = \sigma_Y^s/1.5$ or $\sigma_Y^\Psi = 1.5\sigma_Y^s$.

Fig. 4.13. The contours of effective plastic strain λ^p (%) in the $y = 0$ plane for a stiff inclusion of (a) $\sigma_Y^\Psi = \sigma_Y^s / 1.5$ and (b) $\sigma_Y^\Psi = 1.5\sigma_Y^s$ or a compliant inclusion of (c) $\sigma_Y^\Psi = \sigma_Y^s / 1.5$ and (d) $\sigma_Y^\Psi = 1.5\sigma_Y^s$.

Fig. 4.14. The contours of effective plastic strain λ^p (%) in the $x = 0$ plane for a stiff inclusion of (a) $\sigma_Y^\Psi = \sigma_Y^s / 1.5$ and (b) $\sigma_Y^\Psi = 1.5\sigma_Y^s$ or a compliant inclusion of (c) $\sigma_Y^\Psi = \sigma_Y^s / 1.5$ and (d) $\sigma_Y^\Psi = 1.5\sigma_Y^s$.

4.2.3. Effect of lubricant velocity

The effects of lubricant speed are investigated by decreasing the flowing speed of point-contact case to $u_e = 100 \text{ mm/s}$, and the tangential modulus $E_T^\Psi = E_T^s = 0.9$. The profiles of pressure and film thickness in Fig. 4.15 show that no cavitation occurs around the outlet position. The film formed between the contact surfaces is thinner than that of Figs. 4.3 and 4.7 and direct solid contact can be observed.

Fig. 4.15. Profiles of pressure and film thickness along the x -axis at various loading step for the (a) stiff or (b) compliant inclusion of $E_T^\Psi = E_T^s = 0.9$ with the lubricant flowing at $u_e = 100 \text{ mm/s}$.

The traditional solution of homogeneous lubricated contact has been proven that the decrease of the lubricant speed results in a smaller maximum pressure and a thinner film and finally leads to a solution of dry contact, in which flat pressure profile can be observed whenever materials in contact behave plastically. With the stiff inclusion taken into consideration, the pressure profile shows a convex shape and the peak pressure appears at the center of the domain; when a compliant inclusion involved, a concave shape observed otherwise and the peak pressure appears symmetrically around the center.

4.3. Summary

A semi-analytic solution is developed for heterogeneous elasto-plastic materials under 3D point or line contact with the consideration of lubrication effects. The embedded inclusion is homogenized according to EIM with properly determined equivalent eigenstrains. The disturbed surface displacements caused by those eigenstrains are then introduced to update the surface geometry. The plastic strains are determined iteratively by a procedure involving a plasticity loop and an incremental loading process. The interactions within the contact system are fully taken into consideration.

Example cases demonstrate that the plastic zone is accumulated around the edges of the inclusion due to the local stress concentrations. The substrate and inclusions may possess different yield strengths and the tangential moduli, and thus their plastic behaviors may lead to various surface pressure profiles and subsurface stress fields. The smaller yield strengths or tangential moduli would cause the plastic deformations of the substrate and the inclusions to occur more easily and result in a larger plastic zone. The stiff inclusion shows the same effects as a surface bump when the

equivalent eigenstrains dominate the disturbed deformation; otherwise, the plastic zone would lead the stiff inclusion acts similarly as a surface dent. Lower entrainment velocity would result in a thinner film thickness and smaller pressure peak when the substrate and the inclusion possess the same plastic properties. The same conclusions could be drawn for 3D line contact problems.

Chapter 5. Elastohydrodynamic Lubrication of Materials with Cracks and Inhomogeneous Inclusions

In this chapter, a semi-analytical solution for coated materials with multiple cracks and inhomogeneous inclusions under EHL contact is developed. The lubricated contact is considered between an infinitely long cylinder and a half plane. Each crack of mixed modes I and II within the materials beneath the contacting surface is modeled as a continuous distribution of climb and glide dislocations with unknown densities, which are iteratively obtained by a modified CGM. The SIFs are obtained based on the solution and the effects of the layers and underneath inclusions and cracks on the SIFs are investigated.

5.1. Problem description and solution approach

5.1.1. Model development for materials with cracks and inclusions

Considers a 2D plane-strain problem concerning the EHL contact between a half-plane matrix and a rolling body of radius R bounded by the plane surface $z = 0$. The plane slides at the velocity u_1 and the loading body rolls at u_2 ; thus the lubricant has an effective velocity $u_e = (u_1 + u_2)/2$. A layer of elastic modulus C_{ijkl}^1 ($i, j, k, l = x, z$) is assumed to be perfectly attached to the matrix of C_{ijkl} , and multiple inhomogeneous inclusions Ω_ψ ($\psi = 2, \dots, n$) of C_{ijkl}^ψ and cracks Γ_φ ($\varphi = 1, 2, \dots, m$) with separated surfaces are embedded beneath the surface of the half plane. The cracks are specified along horizontal or vertical orientation in this chapter; a slant crack can be modeled as a zigzag crack consisting of many small vertical and horizontal cracks.

The homogenization strategy of the coating layer and inclusions is same as those discussed in previous chapters. The DDT is employed to model each crack of mixed

modes I and II as a continuous distribution of climb and glide dislocations with unknown densities $\rho^\perp$ and $\rho^\parallel$ to be determined (shown in Fig. 5.1).

Fig. 5.1. Description of the EHL problem for coated materials with multiple inhomogeneous inclusions and cracks beneath the contacting surfaces.

When stresses within the equivalent inclusions are concerned, the utilization of Hooke's law and stress superposition would yield the following equation:

$$C_{ijkl}^\psi C_{klmq}^{-1} (\sigma_{mq}^0 + \sigma_{mq}^* + \sigma_{mq}^c + \sigma_{mq}^p) - \sigma_{ij}^0 - \sigma_{ij}^* - \sigma_{ij}^c - \sigma_{ij}^p + C_{ijkl}^\psi \epsilon_{kl}^* = 0$$

$$(\psi = 1, 2, \dots, n_1; i, j, k, l, m, q = x, z) \quad \text{within } \Omega_\psi, \quad (5.1)$$

where σ_{ij}^0 is the eigenstress at a point within an inclusion caused by all the initial eigenstrains ϵ_{ij}^0 within all the inclusions; σ_{ij}^* is the eigenstresses caused by all the equivalent eigenstrains ϵ_{ij}^* ; σ_{ij}^c is the stress caused by the cracks; σ_{ij}^p is the stress caused by prescribed surface loading or external loading. On the other hand, the stresses along the cracks should satisfy the free-surface traction conditions:

$$(\sigma_{ij}^0 + \sigma_{ij}^* + \sigma_{ij}^c + \sigma_{ij}^p) n_j = 0 \quad (\phi = 1, 2, \dots, n_2; i, j = x, z) \quad \text{along } \Gamma_\phi, \quad (5.2)$$

where n_j indicates the normal vector to crack surfaces.

Fig. 5.2. Discretization of the domain into $N_x \times N_z$ elements of the same size.

In order to formulate the governing equations, the domain D involving all the inclusions and cracks is meshed into $N_x \times N_z$ square elements of the same size $2\Delta_x \times 2\Delta_z$. Each element is indexed by a sequence of two integers $[\alpha_0, \gamma_0]$ with $0 \leq \alpha_0 \leq N_x - 1, 0 \leq \gamma_0 \leq N_z - 1$. Provided a crack is horizontally oriented, the dislocation densities could be approximated by linear functions along each crack segment $\rho^\perp = c^\perp x + d^\perp$ and $\rho^\parallel = c^\parallel x + d^\parallel$; given a fine enough discretization, these linear functions can well approximate the practical nonlinear dislocation densities (shown in Fig. 5.3).

Fig. 5.3. Linear approximation of the dislocation density.

The stress $\sigma_{i,j}^c$ at a given point (x, z) along a crack segment of length $2a$ centered at $(0, \eta)$ is obtained by

$$\begin{aligned} \sigma_{ij}^c(x, z, \eta) = & \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a [c^\perp x' + d^\perp] G_{ij}^\perp(x - x', z, \eta) dx' \\ & + \frac{2\mu}{\pi(\kappa + 1)} \int_{-a}^a [c^\parallel x' + d^\parallel] G_{ij}^\parallel(x - x', z, \eta) dx' \\ & (i, j = x, z) \quad x' \in [-a, a] \quad , \end{aligned} \quad (5.3)$$

where κ is the Kolosov constant of the half-space material ($\kappa = 3 - 4\nu$ for the plane-strain problem with ν being the Poisson's ratio) and μ is the shear modulus; the functions $G_{ij}^\perp(x, y, \eta)$ and $G_{ij}^\parallel(x, y, \eta)$ are the influence functions described by Hill *et al.* (1996). The discrete form expression can then be determined by summing up the stress contribution by discrete climb and glide dislocations along the crack length:

$$\begin{aligned} \sigma_{ij}^c(x, z, \eta) = & \frac{2\mu}{\pi(\kappa + 1)} [E_{ij}^\perp(x, z, \eta) c^\perp + F_{ij}^\perp(x, z, \eta) d^\perp \\ & + E_{ij}^\parallel(x, z, \eta) c^\parallel + F_{ij}^\parallel(x, z, \eta) d^\parallel] \quad , \end{aligned} \quad (5.4)$$

where the influence coefficients $E_{ij}^\perp$, $F_{ij}^\perp$, $E_{ij}^\parallel$ and $F_{ij}^\parallel$ relate the stress to the climb and glide dislocations, and the expressions for these influence coefficients refer to Appendix C.

Since the dislocation density distribution is continuous along an entire crack, the piecewise linear approximations of the dislocation densities should be continuous across two neighboring elements. For a horizontal crack Γ_φ starting from the element $[m_1, \gamma_0]$ and ending at $[m_2, \gamma_0]$, the following relationship should be satisfied

$$c_{\alpha_0, \gamma_0}^\parallel \Delta x + d_{\alpha_0, \gamma_0}^\parallel = -c_{\alpha_0+1, \gamma_0}^\parallel \Delta x + d_{\alpha_0+1, \gamma_0}^\parallel \quad , \quad (5.5a)$$

$$c_{\alpha_0, \gamma_0}^\perp \Delta x + d_{\alpha_0, \gamma_0}^\perp = -c_{\alpha_0+1, \gamma_0}^\perp \Delta x + d_{\alpha_0+1, \gamma_0}^\perp \quad , \quad (5.5b)$$

where $m_1 \leq \alpha_0 \leq m_2$. Meanwhile, the openings at the tips for closed cracks are zero, and thus,

$$\sum_{\alpha_0=m_1}^{m_2} d_{\alpha_0, \gamma_0}^{\perp} = 0 \quad \text{and} \quad \sum_{\alpha_0=m_1}^{m_2} d_{\alpha_0, \gamma_0}^{\parallel} = 0 \quad . \quad (5.6)$$

The above equations form a set of governing equations with unknowns $c^{\perp}$, $d^{\perp}$, $c^{\parallel}$ and $d^{\parallel}$, which can be determined by a modified CGM (Zhou and Wei, 2014b). The DC-FFT algorithm is employed to achieve the high computational accuracy and efficiency for the two-dimensional summation (Chen *et al.*, 2010; Zhou and Wei, 2014b).

The displacement induced by distributed edge dislocations can be obtained by

$$\begin{aligned} u^c(x, \eta) &= \frac{1}{2\pi(\kappa + 1)} \int_{-a}^a [\rho^{\perp}(x') U_{yy}(x - x', \eta) + \rho^{\parallel}(x') U_{xy}(x - x', \eta)] dx' \\ &= H^{\perp}(x, z) c^{\perp} + K^{\perp}(x, z) d^{\perp} + H^{\parallel}(x, z) c^{\parallel} + K^{\parallel}(x, z) d^{\parallel} \quad , \end{aligned} \quad (5.7)$$

where $U_{yy}(x, \eta)$ and $U_{xy}(x, \eta)$ are the functions given in the reference (Hills *et al.*, 1996); $H^{\perp}$ and $K^{\perp}$ are influence coefficients related to the climb dislocation and $H^{\parallel}$ and $K^{\parallel}$ are related to the glide dislocation, and their analytical forms refer to Appendix E.

It should be noted that the derivation of the above equations is based on the assumption of horizontally aligned cracks; the influence coefficients for vertical cracks relate the stress to the dislocations could be found in Appendix D and those displacement-related coefficients refer to Appendix F.

5.1.2. Calculation of stress intensity factors

The SIF of a mixed-mode I and II crack of length $2l_0$ in the x - z coordinate system with the origin at the center of the crack and the crack parallel to the x -axis can be obtained by (Anderson, 2005)

$$K = \lim_{x \rightarrow l_0} \left[\sqrt{2\pi(x - l_0)} (\sigma_{xx} + i\sigma_{xz}) \right] \quad . \quad (5.8)$$

According to DDT, the stresses for cracks with the climb and glide dislocation densities $\rho^\perp$ and $\rho^\perp$ are given as

$$\sigma_{xx} + i\sigma_{xz} = \frac{2\mu}{\pi(\kappa + 1)} \int_0^{l_0} \frac{\rho^\perp(\xi) + i\rho^\perp(\xi)}{z - \xi} d\xi + \sigma^\Delta, \quad (5.9)$$

where σ^Δ indicates the stresses induced by the inclusions and fluid pressure, and is determined by $\sigma^\Delta = (\sigma^P + \sigma^* + \sigma^0)_{xx} + i(\sigma^P + \sigma^* + \sigma^0)_{xz}$. It could be predicted that σ^Δ approaches a finite value under contact, and therefore the influence of σ^Δ is omitted when the limit is taken. The SIF can be rewritten as

$$K = \frac{2\mu}{\pi(\kappa + 1)} \lim_{x \rightarrow l_0} \left[\sqrt{2\pi(z - l_0)} \int_0^{l_0} \frac{\rho^\perp(\xi) + i\rho^\perp(\xi)}{z - \xi} d\xi \right]. \quad (5.10)$$

The above equation can be normalized to the interval $[-1, 1]$ by writing $\xi = l_0(1 + t_0)/2$ and $x = l_0(1 + s_0)/2$ with $-1 \leq s_0, t_0 \leq 1$ and setting $\rho^\perp(t_0) = \omega(t_0) \phi^\perp(t_0)$ and $\rho^\perp(t_0) = \omega(t_0) \phi^\perp(t_0)$ where $\omega(t_0)$ is a fundamental function; $\phi^\perp(t_0)$ and $\phi^\perp(t_0)$ are smooth continuous functions. Normally, the opening displacement is parabolic in form, and the dislocation densities would approach infinity at crack tips. Therefore, the fundamental solution is set to be $\omega(t_0) = 1/\sqrt{1 - t_0^2}$ (Wu and Zhou, 1996; Zhou *et al.*, 2006). With the discussion above, the SIF can be obtained by

$$K = \frac{2\mu}{\pi(\kappa + 1)} \lim_{s \rightarrow 1} \left[\sqrt{\pi l_0 (s_0 - 1)} \int_{-1}^1 \frac{\phi^\perp(t_0) + i\phi^\perp(t_0)}{(s_0 - t_0)\sqrt{1 - t_0^2}} dt \right]. \quad (5.11)$$

Take the limit, the following equation could be derived

$$K = \pm \frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} [\phi^\perp(\pm 1) + i\phi^\perp(\pm 1)] \quad (5.12)$$

Provided $\phi^\perp(t_0)$ and $\phi^\perp(t_0)$ could be approached by a polynomial, the densities could be fitted by the following expression (Hills *et al.*, 1996)

$$\rho(t_0) = \frac{1}{\sqrt{1-t_0^2}} \sum_{n=0}^N a_n t_0^n, \quad (5.13)$$

where N is the order of the polynomial proposed, and a_n is determined during the regression analysis upon the method of least squares. In the following cases, results show that $N = 5$ can provide sufficient accuracy. The SIF can be obtained by

$$K = \pm \frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} \sum_{n=0}^N (a_n^\perp + i a_n^\parallel)(\pm 1)^n. \quad (5.14)$$

The SIFs K_I and K_{II} for the cracks of mixed modes I and II can be written as

$$\begin{aligned} K_I^A &= -\frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} \sum_{n=0}^N a_n^\perp (-1)^n, & K_I^B &= \frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} \sum_{n=0}^N a_n^\perp (1)^n, \\ K_{II}^A &= -\frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} \sum_{n=0}^N a_n^\parallel (-1)^n, & K_{II}^B &= \frac{\mu\sqrt{2\pi l_0}}{(\kappa + 1)} \sum_{n=0}^N a_n^\parallel (1)^n, \end{aligned} \quad (5.15)$$

where A and B indicate the crack tips A and B, respectively.

In order to verify the method, the solution of a crack of length l_c beneath an infinite space under uniform tension T is compared with that of analytical solution $K_I = T\sqrt{\pi l_c/2}$ (Anderson, 2005). Fig.5.4 (a) shows the comparison of a crack with a fixed length 100×10^{-6} mm subject to various tension 1~9 GPa, and Fig. 4 (b) demonstrates the comparison of a crack with various length $30 \sim 130 \times 10^{-6}$ mm subject to a tension 5 GPa. A good agreement has been found, thus validating the present method.

It should be noted that crack faces under compressive loading are likely to be closed and therefore the factor K_I is negative. Even so, the surfaces of cracks are assumed not in contact. In this study, the analysis of SIFs K_{II} , which characterize the shear displacements and stresses, is focused on.

Fig. 5.4. Method validation with the analytical solution: (a) K_I for various tensions with a fixed crack length and (b) K_I for various lengths of an embedded crack with a fixed tension.

5.2. Numerical results and discussions

5.2.1. Horizontally aligned cracks

The EHL contact involves a stationary elastic half plane with cracks beneath the surface and a loading body with radius $R=10$ mm. The external load per unit length applied on the cylinder is set to be $W=100$ kN; the lubricant flows at the entrainment velocity u_e . Young's moduli for the half-plane matrix and the loading body are $E_1=E_2=210$ GPa and Poisson's ratio $\nu_1=\nu_2=0.3$. The rheological parameters for the lubricant are $\eta_0=0.0239$ Pa·s and $\alpha=19.6$ GPa⁻¹. The Hertzian pressure $p_0=606.04$ MPa and the Hertzian radius $a_0=0.1050$ mm for homogeneous contacts are used to normalize the results obtained. The contact area is set in $-4a_0 \leq x \leq 4a_0$ with 128×64 unit grids covering the solution domain.

Fig. 5.5 shows the dimensionless pressure and film thickness profiles along the x -axis at $u_e=1$ m/s and subsurface von Mises stress contour in the x - z plane, and in this case, the length of the crack is assumed to be 0. It demonstrates that due to the

presence of the lubricant, the stress fields under the contacting surface appear to be not symmetric about the z -axis. The maximum pressure appears to be around the center of the contact region and the pressure spike is higher than the symmetric point about the z -axis.

Fig. 5.5. Pressure distribution and film thickness along x -axis and subsurface von Mises stress contour in the x - z plane for homogeneous materials.

Fig. 5.6 shows the results for a single crack horizontally aligned beneath contacting surfaces with the length $l = 3a_0$ at the same entrainment velocity as above. The center of the crack is located at $x = 0$ and $z_c = a_0$. It appears that the presence of the crack leads to a drop in the pressure profile around the center of the contact region. The stress concentrations are observed at the tips of the cracks, and it is crucial to analyze the crack nucleation and propagation.

Fig. 5.6. Pressure distribution and film thickness along the x -axis and subsurface von Mises stress contour in the x - z plane for $u_e = 1$ m/s .

Fig. 5.7. Pressure distribution and film thickness along the x -axis and subsurface von Mises stress contour in the x - z plane for $u_e = 10$ m/s .

Fig. 5.7 demonstrates the results at the entrainment speed $u_e = 10$ m/s . Other

parameters are set the same as the above cases. It appears that when the velocity increases from $u_e = 1 \text{ m/s}$ to $u_e = 10 \text{ m/s}$, the film thickness and the pressure around the center of the contact area increase. Compared with the low entrainment speed, the point where pressure spike occurs is closer to the central point. Fig. 5.8 plots the dimensionless pressure and film thickness profiles for the crack of $l = 3a_0$ beneath the surface with the depth of $z_c = a_0$ at various entrainment speeds.

Fig. 5.8. Pressure distribution and film thickness along the x -axis for the crack of $l = 3a_0$ beneath the surface with the depth of $z_c = a_0$ at various speeds.

Fig. 5.9 shows the depth of the crack could affect the pressure distribution and film thickness as well. Here the depth of the crack beneath the surface is set to be $z_c = 2a_0$. The results show that compared with Fig. 5.7, as the depth of the crack increase, the surface displacement due to the crack will have less effect on the fluid pressure as well as the lubricant film thickness. Fig. 5.10 demonstrates the dimensionless pressure and film thickness profiles for a single crack embedded in the half space with various depths at entrainment velocity $u_e = 10 \text{ m/s}$. It shows that when the depth is set to have $z_c = 0.75a_0$, the pressure around the central point appears to

be wavy due to the presence of the crack.

Fig. 5.9. Pressure distribution and film thickness along the x -axis and subsurface von Mises stress contour in the x - z plane with a crack at $z_c = 2a_0$.

Fig. 5.10. Pressure and film thickness for a single crack embedded in the half space with various depths.

Fig. 5.11. Displacements caused by cracks with different depths at various velocities.

The displacements due to the crack of different depths are shown in Fig. 5.11, where the solid lines plot the displacements due to the crack at the entrainment speed $u_e = 1 \text{ m/s}$ while dot lines plot the displacements at $u_e = 10 \text{ m/s}$. It appears that the disturbed displacements due to the crack are similar to a surface dent and as the depth increases or the speed slows down, the crack displacement decreases. When the depth of the crack reaches $z_c = 2.5a_0$, the displacements are almost the same at $u_e = 1 \text{ m/s}$ and $u_e = 10 \text{ m/s}$.

Fig. 5.12 shows the results for the crack of length $l = a_0$ beneath the surface at $z_c = 2a_0$ when the lubricant speed is $u_e = 10 \text{ m/s}$. It appears that compared with Fig. 5.9, when the crack length decreases, the pressure spike would increase, and the film thickness is less sensitive than the fluid pressure. Fig. 5.13 shows the dimensionless pressure and film thickness profiles for the crack with various crack lengths and demonstrates the effect of crack length on the fluid pressure and lubricant film thickness.

Fig. 5.12. Pressure distribution and film thickness along the x -axis and subsurface von Mises stress contour in the x - z plane.

Fig. 5.13. Pressure and film thickness for a single crack of various lengths embedded in the half space with depth $z_c = a_0$.

The following case considers cracks Γ_φ ($\varphi = 1, 2, 3$) embedded in the system, which have the same length of $2a_0$ and are centered at $(-1.5a_0, z_c)$, $(0, z_c)$ and $(1.5a_0, z_c)$. The depth of the cracks z_c may be $z_c = 0.75a_0$ or $1.5a_0$.

Fig. 5.14. Pressure and film thickness along the x -axis and subsurface von Mises stress contour with cracks embedded at various depths z_c with the lubricant flowing at $u_e = 1 \text{ m/s}$.

Fig 5.14 shows the dimensionless pressure and film thickness profiles along the x -axis at $u_e = 1 \text{ m/s}$ and subsurface von Mises stress contours in the x - z plane. It appears that the cracks with smaller depth would affect the fluid pressure profile more significantly, and the film thickness is less sensitive to the depth of cracks than the pressure.

Fig. 5.15. Pressure and film thickness along the x -axis and subsurface von Mises stress contour with cracks embedded at various depths z_c with the lubricant flowing at $u_e = 10$ m/s .

The same trends have been observed when the entrainment velocity reaches $u_e = 10$ m/s (Fig. 5.15). It is concluded that the near-surface cracks should be involved to analyze the fluid pressure distribution and lubricant film thickness, and an accuracy description of subsurface stresses is essential to study the nucleation and propagation for cracks.

5.2.2. Horizontally aligned cracks interacting with inclusions

The following case shows results of a single square inclusion interacting with two horizontal cracks. The contact system is assumed to be the same as that of cases for horizontal crack presented above. The inclusion and cracks have the same length $l = a_0$. The inclusion is centered at $(0, a_0)$ and the cracks are centered at $(-1.5a_0, a_0)$ and $(1.5a_0, a_0)$. The inclusion has the same Poisson's ratio as the matrix; while its Young's modulus is set to be E_i . Fig 5.16 shows the fluid pressure distribution and lubricant film thickness profiles for the inclusion with various elastic moduli at the same entrainment velocity $u_e = 1 \text{ m/s}$. It demonstrates that the stiff inclusion increases the pressure around the inclusion and the minimum film thickness, while the compliant inclusion would decrease the corresponding profiles. The film thickness is less sensitive to the modulus of inclusion than the pressure.

Fig. 5.16. Pressure distribution and film thickness along the x -axis for a half space with a single inclusion and two cracks.

Fig. 5.17 presents the dimensionless subsurface von Mises stress contour in the x - z plane at $u = 1 \text{ m/s}$ for various moduli. Stress concentrations are found at the crack tips and the edges of the inclusion. It appears that as the inclusion becomes stiff, the

stresses around the inclusion increase.

Fig. 5.17. Subsurface von Mises stress contour in the x - z plane for two cracks and an inclusion of (a) $E_i = 0.25E_1$, (b) $E_i = 0.5E_1$, (c) $E_i = 2.0E_1$ and (d) $E_i = 4.0E_1$.

Fig. 5.18. Surface deformation due to the inclusions u^i and cracks u^c for (a) $E_i = 0.25E_1$, (b) $E_i = 0.5E_1$, (c) $E_i = 2.0E_1$ and (d) $E_i = 4.0E_1$.

Fig. 5.18 shows the displacements due to the inhomogeneous inclusion and cracks. The solid lines demonstrate the displacements u^i due to the inclusion, while the dotted lines demonstrate the displacements u^c due to cracks. It can be concluded

that the cracks would always cause surface deformation similar to a surface dent, while the stiff inclusion would act like a surface bump and the soft inclusion would act like a surface dent. The displacement due to the cracks is not symmetric with respect to z -axis due to the asymmetry of the fluid pressure generated within the lubricant.

Solutions of two inhomogeneous inclusions and a single crack are demonstrated in the following case. The inclusions and the crack have the same length $l = a_0$. The crack is centered at $(0, a_0)$, and the inclusions are centered at $(-1.5a_0, a_0)$ and $(1.5a_0, a_0)$.

Fig. 5.19. Surface deformation due to the inclusions and crack along the x -axis at various entrainment velocities for (a) $E_i = 0.25E_1$, (b) $E_i = 0.5E_1$, (c) $E_i = 2.0E_1$ and (d) $E_i = 4.0E_1$.

The displacements due to the inhomogeneous inclusions and crack are shown in Fig. 5.19, where the solid lines are for the entrainment velocity $u_e = 1 \text{ m/s}$, while the dotted lines are for $u_e = 10 \text{ m/s}$. It shows that for the soft inclusions and the single crack, the deformation act like a surface dent, while for the case the stiff inclusions $E_i = 2.0E_1$ involved, the deformation is similar to a wavy dent, and for $E_i = 4.0E_1$

the effect of surface displacements around the centers of the inclusions is similar to surface bumps. For the low velocity $u_e = 1 \text{ m/s}$, the surface displacements are symmetric, while when the entrainment velocity reaches $u_e = 10 \text{ m/s}$, the displacements induced by the fluid pressure lose the symmetry.

Fig. 5.20. Subsurface von Mises stress contour in the x - z plane for a crack and two inclusions of (a) $E_i = 0.25E_1$, (b) $E_i = 0.5E_1$, (c) $E_i = 2.0E_1$ and (d) $E_i = 4.0E_1$.

Fig. 5.20 shows dimensionless subsurface von Mises stress contours in the x - z plane at $u_e = 10 \text{ m/s}$. Stress concentrations also appear at the tips of the cracks and the edges of the inclusions. As the inclusions become stiff, the stresses around the tips of the cracks increase, and an accuracy description of the stress fields is essential to analyze the nucleation and propagation of cracks.

Interactions among multiple circular inclusions and cracks are investigated in the case below. The radius for each inclusion is $R_0 = 0.5a_0$ and is equally spaced by a_0 . The cracks have the same length a_0 , and are centered at $(-a_0, 2a_0)$ and $(a_0, 2a_0)$, respectively.

Fig. 5.21. Schematic of multiple circular inclusions and cracks.

Fig. 5.22. Subsurface von Mises stress contour in the x - z plane for cracks and circular inclusions of (a) $E_i = 0.25E_1$, (b) $E_i = 0.5E_1$, (c) $E_i = 2.0E_1$ and (d) $E_i = 4.0E_1$.

Fig. 5.22 shows dimensionless subsurface von Mises stress contours in the x - z plane at $u_e = 10$ m/s. It shows that as the inclusions become stiff, the stress interactions between the inclusions and the cracks are dramatically increased. This case shows that the solution is capable of solving materials with arbitrarily shaped inclusions and horizontal cracks parallel to the x -axis. It should be noted that the vertical cracks could be solved with the same methodology, and a crack of arbitrary

orientation can be decomposed into a collection of many small horizontal cracks and vertical cracks. Due to the inclusions in the half space are meshed by squares, the edge of the sphere appears to be not smooth. It can be predicted that the finer mesh would result in more accuracy description of the stress fields.

5.2.3. Vertically aligned cracks interacting with inclusions and effect of coating layers

The lubricated system, involving layered materials with interacting vertical cracks and inhomogeneous inclusions, consists of a loading body with the radius $R = 20$ mm and a stationary elastic half plane with the coating layer on the top and two inhomogeneous inclusions and a vertical crack beneath the surface (Fig. 5.23). The entrainment velocity is u_e along the x -axis, and the external load per unit length applied on the cylinder is set to be $W = 100$ kN. The Young's moduli for both the half-space matrix and the cylinder are $E_1 = E_2 = 210$ GPa and Poisson's ratio $\nu_1 = \nu_2 = 0.3$. The rheological parameters for the lubricant are $\eta_0 = 0.0239$ Pa·s and $\alpha = 19.6$ GPa⁻¹. The Hertzian pressure $p_0 = 428.53$ MPa and the Hertzian radius $a_0 = 0.1486$ mm for homogeneous contacts are used to normalize the results obtained. The SIFs are normalized by $K_0 = p_0 \sqrt{2\pi a_0} / N_x$. In order to capture the effect of the extended coating, the computational area is set in the specified area $-8a_0 \leq x \leq 8a_0$ and $0 \leq z \leq 2a_0$ with 256×64 unit grids covering the solution domain. Results from previous studies (Zhou *et al.*, 2011a; Chen *et al.*, 2010) have proved that the trivial influence of the layer structure outside the simulation domain can be neglected in this target domain.

Fig. 5.23. Schematic of layered materials with a vertical crack and two inhomogeneous inclusions involved in the EHL contact system.

The coating thickness h_c is taken to be $0.5a_0$, and the Young's moduli for the coating and inhomogeneous inclusions are set to be E_c and E_i respectively. The Poisson's ratio for both is fixed to $\nu_c = \nu_i = 0.3$. The inclusions and cracks have the same length a_0 . The inclusions are centered at $(-a_0, 1.25a_0)$ and $(a_0, 1.25a_0)$ respectively, and the crack is centered at the point $Q(0, z_c)$.

The case of $z_c = 1.25a_0$ and $E_i = E_1$ with various Young's moduli for the coating layer from $1/4$ to 4 times the value for the substrate at the entrainment velocity $u_e = 1 \text{ m/s}$ is studied. Fig. 5.24 demonstrates the dimensionless fluid pressure and film thickness along the x -axis, and subsurface von Mises stress contours. It shows that as the coating becomes stiff, the maximum pressure within the contacting area increases due to the decreased surface deformation, and the position where the pressure spike occurs moves to the opposite direction as the x -axis. The film thickness is less sensitive to the stiffness of the coating layer compared with the pressure profile. For the compliant coating, the pressure generated reaches its maximum around the center, and the pressure spike is lower than the pressure peak. The presence of the crack causes stress concentration at the tips, and the one close to the surface has more severe stress condition. However, for the stiff coating, the pressure spike is higher than the pressure around the contact center, which may generate intensive stress fields

below the surface. The result demonstrates that stresses, where the pressure spikes occur above, are comparable to the stress concentration at the crack tips, and are likely to cause crack nucleation there.

Fig. 5.24. Pressure and film thickness profiles and subsurface von Mises stress for coating layer with Young's moduli (a) $E_c = 0.25E_1$, (b) $E_c = E_1$ and (c) $E_c = 4.0E_1$ at the entrainment velocity $u_e = 1 \text{ m/s}$.

Fig. 5.25. Interfacial shear stress for coating layer with various Young's moduli.

Shear stresses across the substrate-coating interface due to the fluid pressure are essential for the adhesion strength of the layer. Fig. 5.25 shows that the magnitude of the interfacial shear stress increases with the stiffness of the coating in the pressure

area, and the maximum value appears approximately at $x = \pm a_0$, where the interfacial debonding is likely to start.

Fig. 5.26. Dislocation densities (a) $\rho^\perp(t)$ and (b) $\rho^\perp(t)$ fitting with the polynomial of various orders N .

Fig. 5.26 shows the approximate solution of the dislocation densities for the polynomial of $\phi^\perp(t)$ and $\phi^\perp(t)$ with various orders when the regression analysis is conducted by applying method of least squares. It appears that as the order of the polynomial increases, the densities can be fitted smoothly, and when $N \geq 3$, the equation suggested could capture the densities very well. In the following cases, N is

fixed to be 5.

Fig. 5.27. The SIF K_I for the coating layer with various Young's moduli at $u_e = 1 \text{ m/s}$.

Fig. 5.28. The SIF K_{II} for the coating layer with various Young's moduli at $u_e = 1 \text{ m/s}$.

Fig. 5.27 shows the SIFs K_I for the coating layer with various moduli at the entrainment velocity $u_e = 1 \text{ m/s}$. It appears that the factor increases with the modulus of the layer. The negative SIF value indicates that the crack is likely to be closed and the separated surfaces move towards each other. Fig. 5.28 demonstrates the SIFs K_{II} for the coating layer with various moduli. It appears the factor increases with the

moduli for the coating, and the factor for the top tip is higher than the other one when the coating is less than $E_c = 3E_1$, and beyond this critical value the SIFs for the bottom tip are larger.

The entrainment velocity of the lubricant could affect the fluid pressure and film thickness profiles, and therefore would influence the elastic stress fields beneath the contacting interface. Fig. 5.29 shows that the subsurface von Mises stress contours for the $E_c = 0.25E_1$ and $4.0E_1$ at the entrainment velocity $u_e = 10 \text{ m/s}$. It appears that as the velocity increases, the pressure spike increases and moves to the opposite direction as the x -axis. Therefore, the subsurface stress fields generated at the tip close to the surface would be more intensive. The SIFs K_{II} for $u_e = 10 \text{ m/s}$ show the same trends as $u_e = 1 \text{ m/s}$. However the factor for the top tip is larger when the coating is compliant while the factor for the bottom tip is larger when the coating is stiffer than the matrix.

Fig. 5.29. Pressure and film thickness profiles and subsurface von Mises stress for coating layer with Young's moduli (a) $E_c = 0.25E_1$ and (b) $E_c = 4.0E_1$ at the entrainment velocity $u_e = 10$ m/s.

Fig. 5.30. The SIF K_{II} for the coating layer with various Young's moduli at $u_e = 10$ m/s.

Fig. 5.31. Pressure and film thickness profiles and subsurface von Mises stress for compliant coating with inclusions (a) $E_i = 0.25E_1$ and (b) $E_i = 4.0E_1$ at the entrainment velocity $u_e = 1 \text{ m/s}$.

Inclusions beneath the contacting surface could interact with the coating layer and the crack, and therefore it is of significance to investigate all the interactions. For the

compliant coating, the Young's modulus is fixed to $E_c = 0.5E_1$. Fig. 5.31 shows that the maximum von Mises reaches 1.03 at the point A around the edge of the right inclusion where the pressure spike occurs above. However, when the modulus for the inclusions reaches $E_i = 4.0E_1$, the maximum von Mises stress appears at the point B on the edge of the inclusion, and the value decreases to 0.89.

The interfacial shear stresses for a stiff layer of $E_c = 2.0E_1$ with subsurface inclusions of various moduli are shown in Fig. 5.32. It appears that as the inclusions become stiff, the maximum shear stress decreases and the location of maximum shear stress moves away from the center of the domain. It results from the pressure distribution due to the presence of inclusions. The compliant inclusions may induce more extra elastic deformation, which may increase the maximum pressure and shrink the contacting area.

Fig. 5.32. Interfacial shear stress for a stiff layer with subsurface inclusions of various Young's moduli at the entrainment velocity $u_e = 1 \text{ m/s}$.

Fig. 5.33 demonstrates the SIFs K_{II} for the compliant and stiff coatings with inhomogeneities of various Young's moduli. It appears that the factor for the top tip increases with the inclusion stiffness, while for the bottom one almost remains constant. The SIFs of the top tip for the compliant layer are always higher than the other tip. For the stiff layer, the bottom tip is higher at the beginning and lower when the subsurface inclusions are stiffer than the matrix.

Fig. 5.33. The SIF K_{II} for the compliant (a) and stiff (b) coating with various Young's moduli at $u_e = 1 \text{ m/s}$.

When the thickness of the coating decreases to $h_c = 0.25a_0$, the elastic fields underneath the contacting surface would be affected. Fig. 5.34 shows results for the coating with $E_c = 0.25E_1$ and $E_c = 4.0E_1$ at the entrainment velocity $u_e = 1 \text{ m/s}$. The modulus for the inclusions are fixed to $E_i = 2.0E_1$. Compared with the above case, the stresses around the tips of the crack increase, which would increase the damage rate. The maximum von Mises for compliant inclusions reaches 1.05 at C around the edge of the right one. The maximum von Mises for stiff inclusions appears at D on the edge of one inclusion, and the value decreases to 0.94.

Fig. 5.34. Subsurface von Mises stress for compliant coating with inclusions (a) $E_i = 0.25E_1$ and (b) $E_i = 4.0E_1$ at the entrainment velocity $u_e = 1 \text{ m/s}$.

Fig. 5.35. Subsurface von Mises stress for the crack with various depths (a) $z_c = \infty$, (b) $z_c = 1.25a_0$ and (c) $z_c = 1.0a_0$ at the entrainment velocity $u_e = 1 \text{ m/s}$.

Fig. 5.36. The SIF K_{II} for the coating layer with various crack depths.

Fig. 5.35 demonstrates the effect of the depths of the crack beneath the surface. The results show that as the crack moves to the surface, the interactions among the crack and the coating become intensive, and the stresses around the edges of the

inclusions decrease. The dimensionless von Mises stresses around the top crack tip increase, which will lead to the damage to the coated material. Fig. 5.36 shows that the SIFs K_{II} for the top tip remain constant while the factor for the other tip increases with the depth of the crack.

Fig. 5.37. Pressure and film thickness profiles and subsurface von Mises stress for multiple cracks and two inclusions.

Multi cracks within materials would influent the overall pressure distribution and thus would result in different elastic fields. Fig. 5.37 shows the dimensionless pressure and film thickness profiles and subsurface von Mises stresses for three cracks located at $(-0.25a_0, a_0)$, $(0, a_0)$ and $(0.25a_0, a_0)$. The results show that the von Mises stresses around the top tips of the cracks appears to be different. The stresses around the crack where the pressure spike occurs above are larger than other cracks, and so are the SIFs K_{II} . The normalized factor near the top tip for the left crack reaches -12.4, while for the right one reaches 15.7.

5.3. Summary

A semi-analytic solution is developed for the materials with inhomogeneous inclusions and horizontally or vertically oriented cracks in materials under lubricated contact. The solution fully takes into account the interactions between the fluid and the coating/substrate structure as well as the interactions among the inclusions and cracks.

The investigations show that stiff coatings would induce high pressure spike within the lubricants, and therefore would result in stress concentration below the surface. The inclusion beneath the coatings could cause disturbed surface deformation, and therefore affect the elastic stress fields. Compliant inclusions may induce more extra elastic deformation, which may increase the maximum pressure and shrink the contacting area. The decrease of crack depth would increase the interactions underneath the contact surface and more likely cause the damage of materials. The lubricated contact is likely to close the crack surfaces and results in negative values for the mode I SIFs. The mode II SIFs vary with the lubricant velocities, the stiffness of the layer and the inclusion and the depths of the cracks.

The analysis would provide assistance to the material reinforcement by component partials or coatings and the failure evaluation of mechanical systems of cracked materials. The solution may potentially have applications for reliability analysis of heterogeneous materials concerning inelastic deformation and material dissimilarity subject to lubricated contact.

Chapter 6. Conclusions and Recommendations

6.1. Conclusions

This PhD study develops a semi-analytical solution for the lubricated contact of heterogeneous materials with inhomogeneous inclusions and cracks. The EIM is utilized to simulate the inhomogeneous inclusions as homogeneous inclusions with unknown eigenstrains to be determined, and the coating is assumed as an inhomogeneous inclusion of finite size located on the surface. A mixed FFT is utilized to approach the 3D line contact. The accumulated plastic strains within the substrate, as well as those in the inclusions, are then involved in the solution for the analysis of subsurface stress fields. The embedded crack of mixed modes I and II is modeled by DDT as a continuous distribution of climb and glide dislocations with unknown densities to be determined. The solution fully takes the interactions among the loading bodies, the fluid lubricant, and the embedded inclusions into account to obtain an accurate description of the fluid pressure and lubricant film thickness profiles and the subsurface stress fields. Based on this solution, the following conclusions can be drawn.

A near-surface stiff inclusion would act similarly to a surface bump and result in a larger local surface pressure and a larger neighboring subsurface von Mises stress level than a compliant inclusion, and thus is more likely to cause crack initiation and propagation in the surrounding region of the inclusion. When being located close to a surface subject to contact loading, the inclusion has a significant effect on the surface pressure and such an effect reduces as the inclusion depth increases. The effect of the coating is similar to an inclusion extending infinitely. Stress concentrations can be found along the interface of the coating and substrate and around the inclusion edges.

The plastic zone is accumulated around the edges of the inclusion due to the local

stress concentrations. The substrate and inclusions may possess different yield strengths and tangential moduli, and thus their plastic behaviors may lead to different surface pressure profiles and subsurface stress fields; smaller yield strengths or tangential moduli would cause the plastic deformation of the substrate and the inclusions to occur more easily and thus result in a larger plastic zone. When the equivalent eigenstrains dominate the disturbed deformation, the stiff inclusion shows the same effects as a surface bump. In other cases, the stiff inclusion acts similarly as a surface dent. Lower entrainment velocity would result in a thinner film thickness and smaller pressure peak. The same conclusions could be drawn for 3D line-contact problems.

The effect of cracks on film thickness is similar to that of a surface dent. Both the lengths of cracks and their depths beneath the surfaces influence the fluid pressure and lubricant film thickness. As a result, the subsurface stresses fields are affected. As the length increases or the depth decreases, the disturbance due to the cracks becomes significant. The lubricated contact is likely to close the crack surfaces and thus results in negative values for the mode I SIFs. The mode II SIFs vary with the lubricant velocities, the stiffness of the layers and the inclusions and the depths of the cracks.

The researching findings in this PhD study can provide guidance to minimize the potential damage of materials induced by inclusions and cracks when subjected to lubricated contact. The solutions developed have potential applications for analyzing the effects of heterogeneous materials on the pressure and film thickness profiles and the subsurface elastic fields.

6.2. Recommendations

The model developed in this PhD study is capable of solving lubricated contact of

heterogeneous materials with inhomogeneous inclusions and cracks, and the plastic zones can be identified for the inclusion-related problems based upon the flow law. However, the plastic zone for cracks evolves in a different way due to the stress singularity at the crack tip. The Dugdale model was well developed to estimate the size of the plastic region and has been employed to study the interactions between cracks and inclusions in regular shapes, such as circles or ellipses. However, inclusions within materials are naturally in arbitrary shapes and randomly distributed, and the stress concentrations around the interface between the inclusion and the substrate often result in local plastic regions that would disturb the stress field. Thus, it is of significance to study the elastic-plastic fracture behavior of materials with distributed inclusions of arbitrary shape.

Static contact loading is considered in this dissertation; however, in practice engineering, mechanical components are always subjected to cyclic loading. Energy dissipation during the cyclically varying loading process would lead to crack propagation until the eventual failure of a functional component. The previous studies have investigated the fatigue crack growth directions by the implement of the maximum hoop stress criterion and its rates by the Paris law. However, the coupled effects of lubrication and plastic evolution have not been investigated so far, and these effects would cause the deviation of the fatigue lifetime from that predicted by theories. In order to approach the engineering practices, future study should be extended to predict the fatigue life of heterogeneous elasto-plastic materials subject to cyclic loading with lubricants between the interfaces.

Sliding wear is a significant surface failure caused by the relative motion at the contacting interface. Archard's wear law has been widely utilized to predict the surface evolution of homogeneous materials under dry or lubricated conditions. However, the

presence of embedded inclusions and cracks in materials may change the direction and magnitude of forces acting on the surface and finally lead to a sliding wear rate different from that predicted by the Archard's law. Therefore, the effects of inclusions and cracks on the sliding wear should be taken into account to improve the accuracy of wear prediction.

Appendix A

The following discrete expressions are for of EHL point-contact problems, and the corresponding equations for the 2D and 3D line contact could be derived in a similar way.

The surface elastic deformation at the patch (i, j) in point contact can be obtained by converting Eq. (3.5) into a discrete equation as follow:

$$V_{i,j} = \sum_{l=0}^{N_y-1} \sum_{k=0}^{N_x-1} D_{k,l}^{i,j} P_{k,l} \quad , \quad (\text{A.1})$$

where the influence coefficients $D_{k,l}^{i,j}$ could be obtained by

$$\begin{aligned} D_{k,l}^{i,j} &= \frac{2p_0}{\pi E'} \iint_{\Omega} \frac{P^T(X_i, Y_j)}{\sqrt{(X_i - X')^2 + (Y_j - Y')^2}} dX' dY' \\ &= |X_{|i-k+0.5|}| \ln \left[\frac{f(X_{|i-k+0.5|}, Y_{|j-l+0.5|})}{f(X_{|i-k+0.5|}, Y_{|j-l-0.5|})} \right] + |X_{|i-k-0.5|}| \ln \left[\frac{f(X_{|i-k-0.5|}, Y_{|j-l-0.5|})}{f(X_{|i-k-0.5|}, Y_{|j-l+0.5|})} \right] \\ &\quad + |Y_1| \ln \left[\frac{f(X_{|i-k+0.5|}, Y_{|j-l+0.5|})}{f(X_{|i-k-0.5|}, Y_{|j-l+0.5|})} \right] + |Y_2| \ln \left[\frac{f(X_{|i-k-0.5|}, Y_{|j-l-0.5|})}{f(X_{|i-k+0.5|}, Y_{|j-l-0.5|})} \right] \end{aligned} \quad (\text{A.2})$$

and $f(X, Y) = X + \sqrt{X^2 + Y^2}$.

By using the appropriate differential schemes, the Reynolds equation can be converted into a discrete differential equation at each unknown pressure point:

$$A_{i,j} P_{i-1,j} + B_{i,j} P_{i,j} + C_{i,j} P_{i+1,j} = F_{i,j} \quad . \quad (\text{A.3})$$

The iteratively determined pressure from the present equation is then compared with that from the previous step. The computational process is performed until the convergence of the pressure.

The second-central differential scheme is used for pressure flow term and can be discretized as

$$\left[\frac{\partial}{\partial X} \left(\varepsilon^x \frac{\partial P}{\partial X} \right) \right]_{i,j} = \frac{1}{\Delta X^2} \left(\varepsilon_{i+1/2,j}^x P_{i+1,j} - (\varepsilon_{i+1/2,j}^x + \varepsilon_{i-1/2,j}^x) P_{i,j} + \varepsilon_{i-1/2,j}^x P_{i-1,j} \right), \quad (\text{A.4})$$

$$\left[\frac{\partial}{\partial Y} \left(\varepsilon^y \frac{\partial P}{\partial Y} \right) \right]_{i,j} = \frac{1}{\Delta Y^2} \left(\varepsilon_{i,j+1/2}^y P_{i,j+1} - (\varepsilon_{i,j+1/2}^y + \varepsilon_{i,j-1/2}^y) P_{i,j} + \varepsilon_{i,j-1/2}^y P_{i,j-1} \right). \quad (\text{A.5})$$

Therefore, the pressure flow term contributed to the coefficients of the discrete Reynolds equation can be obtained as

$$\begin{cases} A_{i,j}^P = \frac{\varepsilon_{i-1/2,j}^x}{\Delta X^2} \\ B_{i,j}^P = -\frac{\varepsilon_{i+1/2,j}^x + \varepsilon_{i-1/2,j}^x}{\Delta X^2} - \frac{\varepsilon_{i,j+1/2}^x + \varepsilon_{i,j-1/2}^x}{\Delta Y^2} \\ C_{i,j}^P = \frac{\varepsilon_{i+1/2,j}^x}{\Delta X^2} \\ F_{i,j}^P = -\frac{\varepsilon_{i,j+1/2}^x + \varepsilon_{i,j-1/2}^x}{\Delta Y^2} \end{cases}. \quad (\text{A.6})$$

The second-order backward scheme could be implemented to discretize the shear flow term and its contributions to the Reynolds equation are given as

$$\left[\frac{\partial(\bar{\rho}H)}{\partial X} \right]_{i,j} = \frac{3\bar{\rho}_{i,j}H_{i,j} - 4\bar{\rho}_{i-1,j}H_{i-1,j} + \bar{\rho}_{i-2,j}H_{i-2,j}}{2\Delta X}, \quad (\text{A.7})$$

$$\begin{cases} A_{i,j}^W = -\frac{-2.0\bar{\rho}_{i-1,j}D_{i-1,j}^{i-1,j} + 0.5\bar{\rho}_{i-2,j}D_{i-1,j}^{i-2,j}}{\Delta X} \\ B_{i,j}^W = -\frac{1.5\bar{\rho}_{i,j}D_{i,j}^{i,j} + 0.5\bar{\rho}_{i-2,j}D_{i,j}^{i-2,j}}{\Delta X} \\ C_{i,j}^W = -\frac{1.5\bar{\rho}_{i,j}D_{i+1,j}^{i,j} + 0.5\bar{\rho}_{i-2,j}D_{i+1,j}^{i-2,j}}{\Delta X} \\ F_{i,j}^W = \frac{1}{\Delta X} \begin{bmatrix} 1.5\bar{\rho}_{i,j}[H_{i,j} - (D_{i,j}^{i,j}P_{i,j}^{old} + D_{i+1,j}^{i,j}P_{i+1,j}^{old})] \\ -2.0\bar{\rho}_{i-1,j}[H_{i-1,j} - D_{i-1,j}^{i-1,j}P_{i-1,j}^{old}] \\ +0.5\bar{\rho}_{i-2,j}[H_{i-2,j} - (D_{i-1,j}^{i-2,j}P_{i-1,j}^{old} + D_{i,j}^{i-2,j}P_{i,j}^{old} + D_{i+1,j}^{i-2,j}P_{i+1,j}^{old})] \end{bmatrix} \end{cases}, \quad (\text{A.8})$$

where $P_{i,j}^{old}$ is the pressure at patch (i, j) obtained the previous iteration. The discrete form of the squeeze flow term and its contributions to the Reynolds equation based on the second-order backward are given as

$$\left[\frac{\partial(\bar{\rho}H)}{\partial T} \right]_{i,j} = \frac{3\bar{\rho}_{i,j}^T H_{i,j}^T - 4\bar{\rho}_{i,j}^{T-1} H_{i,j}^{T-1} + \bar{\rho}_{i,j}^{T-2} H_{i,j}^{T-2}}{2\Delta T}, \quad (\text{A.9})$$

$$\begin{cases} A_{i,j}^T = -\frac{1.5\bar{\rho}_{i,j}^T D_{i-1,j}^{i,j}}{\Delta X} \\ B_{i,j}^T = -\frac{1.5\bar{\rho}_{i,j}^T D_{i,j}^{i,j}}{\Delta X} \\ C_{i,j}^T = -\frac{1.5\bar{\rho}_{i,j}^T D_{i+1,j}^{i,j}}{\Delta X} \\ F_{i,j}^T = \frac{1}{\Delta X} \left[1.5\bar{\rho}_{i,j}^T [H_{i,j}^T - (D_{i-1,j}^{i,j} P_{i-1,j}^{old} + D_{i,j}^{i,j} P_{i,j}^{old} + D_{i+1,j}^{i,j} P_{i+1,j}^{old})] \right. \\ \left. - 2.0\bar{\rho}_{i,j}^{T-1} H_{i,j}^{T-1} + 0.2\bar{\rho}_{i,j}^{T-2} H_{i,j}^{T-2} \right] \end{cases} \quad (\text{A.10})$$

The total coefficients of discrete Reynolds equation are the summations of the coefficients from the pressure flow term, the shear flow term and the squeeze flow term as follows:

$$\begin{cases} A_{i,j} = A_{i,j}^P + A_{i,j}^W + A_{i,j}^T \\ B_{i,j} = B_{i,j}^P + B_{i,j}^W + B_{i,j}^T \\ C_{i,j} = C_{i,j}^P + C_{i,j}^W + C_{i,j}^T \\ F_{i,j} = F_{i,j}^P + F_{i,j}^W + F_{i,j}^T \end{cases} \quad (\text{A.11})$$

It should be noted that in order to improve the numerical accuracy, the shear flow term, and the squeeze flow term could be written in separated forms:

$$\frac{\partial(\bar{\rho}H)}{\partial X} = \bar{\rho} \frac{\partial(H)}{\partial X} + H \frac{\partial(\bar{\rho})}{\partial X} \quad \text{and} \quad \frac{\partial(\bar{\rho}H)}{\partial T} = \bar{\rho} \frac{\partial(H)}{\partial T} + H \frac{\partial(\bar{\rho})}{\partial T} \quad (\text{A.12})$$

with

$$\begin{aligned} H(X_i, Y_j, T) = H_0(T) + \frac{a_0 X_i^2}{2R} + \frac{a_0 Y_j^2}{2R} + V(X_i, Y_j, T) \\ + \frac{\delta_1(x_i, y_j, t)}{a_0} + \frac{\delta_2(x_i, y_j, t)}{a_0} \end{aligned} \quad (\text{A.13})$$

and

$$\left[\frac{\partial(H)}{\partial X} \right]_{i,j} = \left[\frac{a_0 X_i}{R} + \frac{\partial}{\partial X} \left(V(X_i, Y_j, T) + \frac{\delta_1(x_i, y_j, t)}{a_0} + \frac{\delta_2(x_i, y_j, t)}{a_0} \right) \right]_{i,j}. \quad (\text{A.14})$$

The separated second-order backward scheme is then employed to discretize Eqs. (A.11-13), and the details refers to the work by Liu *et al.* (2006).

Appendix B

With Hooke's law and stress superposition considered, the stresses in the equivalent homogeneous inclusions satisfy (Zhou *et al.*, 2011d; Zhou *et al.*, 2011a):

$$C_{ijkl}^{\psi} C_{klmq}^{-1} (\sigma_{mq}^0 + \sigma_{mq}^* + \sigma_{mq}^p) - \sigma_{ij}^0 - \sigma_{ij}^* - \sigma_{ij}^p + C_{ijkl}^{\psi} \varepsilon_{kl}^* = 0$$

$$(\psi = 1, 2, \dots, n; i, j, k, l, m, q = x, y, z) \text{ within } \Omega_{\psi}, \quad (\text{B.1})$$

where σ_{ij}^0 is the eigenstress at a point within an inclusion caused by all the initial eigenstrains ε_{ij}^0 within all the inclusions, σ_{ij}^* the eigenstresses caused by all the equivalent eigenstrains ε_{ij}^* , σ_{ij}^p the stress caused by surface pressure.

The discretization expressions for σ_{ij}^0 , σ_{ij}^* and σ_{ij}^p can be obtained with the assistance of influence coefficients $\mathbf{B}_{\alpha_0-\xi, \beta_0-\zeta, \gamma_0-\varphi}$ which relate the eigenstresses $\sigma_{\alpha_0, \beta_0, \gamma_0}^0$ and $\sigma_{\alpha_0, \beta_0, \gamma_0}^*$ at the observation point $(x_{\alpha_0}, y_{\beta_0}, z_{\gamma_0})$ in the square element $[\alpha_0, \beta_0, \gamma_0]$ to the initial eigenstrain $\varepsilon_{\xi, \zeta, \varphi}^0$ and equivalent eigenstrain $\varepsilon_{\xi, \zeta, \varphi}^*$ in $[\xi, \zeta, \varphi]$ and $\mathbf{M}_{\alpha_0-\xi, \beta_0-\zeta, \gamma_0}$ which related $\sigma_{\alpha_0, \beta_0, \gamma_0}^p$ to the surface pressure $p_{\xi, \zeta}$. Therefore, the governing equation (B.1) can then be written as

$$(C_{\alpha_0, \beta_0, \gamma_0} C^{-1} - I) \left[\sum_{\varphi=0}^{N_z-1} \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{B}_{\alpha_0-\xi, \beta_0-\zeta, \gamma_0-\varphi} (\varepsilon_{\xi, \zeta, \varphi}^0 + \varepsilon_{\xi, \zeta, \varphi}^*) \right. \\ \left. + \sum_{\zeta=0}^{N_y-1} \sum_{\xi=0}^{N_x-1} \mathbf{M}_{\alpha_0-\xi, \beta_0-\zeta, \gamma_0} p_{\xi, \zeta} \right] + C_{\alpha_0, \beta_0, \gamma_0} \varepsilon_{\xi, \zeta, \varphi}^* = 0$$

$$(0 \leq \alpha_0 \leq N_x - 1, 0 \leq \beta_0 \leq N_y - 1, 0 \leq \gamma_0 \leq N_z - 1) \text{ within } \Omega_{\psi}. \quad (\text{B. 2})$$

The unknown eigenstrains within the governing equation can be determined by a modified CGM with achievable convergence.

Appendix C

The influence coefficients for horizontal cracks relate the stresses to the climb and glide dislocations are given as

$$\begin{aligned}
 F_{11}^{\perp} = & -\frac{1}{2} \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] + \frac{1}{2} \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \\
 & + \frac{(\eta-y)^2}{(x-a)^2 + (\eta-y)^2} - \frac{(\eta-y)^2}{(x+a)^2 + (\eta-y)^2} \\
 & + \frac{3\eta^2 + 4\eta y - y^2}{(x-a)^2 + (\eta+y)^2} - \frac{3\eta^2 + 4\eta x - y^2}{(x+a)^2 + (\eta+y)^2} \\
 & - \frac{4\eta y(y+\eta)^2}{[(x-a)^2 + (\eta+y)^2]^2} + \frac{4\eta y(y+\eta)^2}{[(x+a)^2 + (\eta+y)^2]^2}
 \end{aligned} \quad , (C.1)$$

$$\begin{aligned}
 E_{11}^{\perp} = & -\frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] - \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \right\} \\
 & - \frac{2\eta y[(a+x)(\eta+y)^2 - (a-x)^3]}{((a-x)^2 + (\eta+y)^2)^2} + \frac{a(\eta-y)^2}{(a+x)^2 + (\eta-y)^2} \\
 & - \frac{2\eta y[(a-x)(\eta+y)^2 - (a+x)^3]}{((a+x)^2 + (\eta+y)^2)^2} + \frac{a(\eta-y)^2}{(a-x)^2 + (\eta-y)^2} \\
 & - 2(\eta-y) \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right. \\
 & \left. + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right] - \frac{a(y^2 - 4\eta y - 3\eta^2)}{(a-x)^2 + (\eta+y)^2} \\
 & - \frac{a(y^2 - 4\eta y - 3\eta^2)}{(a+x)^2 + (\eta+y)^2}
 \end{aligned} \quad , (C.2)$$

$$\begin{aligned}
 F_{22}^{\perp} = & \frac{y^2 + \eta^2}{(x-a)^2 + (\eta+y)^2} - \frac{y^2 + \eta^2}{(x+a)^2 + (\eta+y)^2} - \frac{(\eta-y)^2}{(x-a)^2 + (\eta-y)^2} \\
 & + \frac{1}{2} \ln \left[\frac{(a-x)^2 + (y-\eta)^2}{(a-x)^2 + (y+\eta)^2} \right] - \frac{1}{2} \ln \left[\frac{(a+x)^2 + (y-\eta)^2}{(a+x)^2 + (y+\eta)^2} \right] \\
 & + \frac{(\eta-y)^2}{(x+a)^2 + (\eta-y)^2} + \frac{4\eta y(y+\eta)^2}{[(x-a)^2 + (\eta+y)^2]^2} \\
 & - \frac{4\eta y(y+\eta)^2}{[(x+a)^2 + (\eta+y)^2]^2}
 \end{aligned} \quad , (C.3)$$

$$\begin{aligned}
 E_{22}^{\perp} = & \frac{2\eta y[(a+x)(y+\eta)^2 - (a-x)^3]}{[(a-x)^2 + (\eta+y)^2]^2} \\
 & + \frac{2\eta y[(a-x)(y+\eta)^2 - (a+x)^3]}{[(a+x)^2 + (\eta+y)^2]^2} \\
 & + \frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y-\eta)^2}{(a-x)^2 + (y+\eta)^2} \right] - \ln \left[\frac{(a+x)^2 + (y-\eta)^2}{(a+x)^2 + (y+\eta)^2} \right] \right\} \\
 & - \frac{a(y-\eta)^2}{(a-x)^2 + (\eta-y)^2} - \frac{a(y-\eta)^2}{(a+x)^2 + (\eta-y)^2} \\
 & + \frac{a(y^2 + \eta^2)}{(a-x)^2 + (\eta+y)^2} + \frac{a(y^2 + \eta^2)}{(a+x)^2 + (\eta+y)^2}
 \end{aligned} \quad , (C.4)$$

$$\begin{aligned}
 F_{12}^{\perp} = & - \frac{8\eta^2 y(a-x)[(a-x)^2 + \eta^2 - y^2]}{[(a-x)^2 + (\eta-y)^2][(a-x)^2 + (\eta+y)^2]^2} \\
 & - \frac{8\eta^2 y(a+x)[(a+x)^2 + \eta^2 - y^2]}{[(a+x)^2 + (\eta-y)^2][(a+x)^2 + (\eta+y)^2]^2}
 \end{aligned} \quad , (C.5)$$

$$\begin{aligned}
 E_{12}^{\perp} = & \frac{5\eta y^2 + 7\eta^2 y + \eta x^2 + ax(\eta-y) - yx^2 + \eta^3 - y^3}{(a+x)^2 + (\eta+y)^2} \\
 & - \frac{5\eta y^2 + 7\eta^2 y + \eta x^2 - ax(\eta-y) - yx^2 + \eta^3 - y^3}{(a-x)^2 + (\eta+y)^2} \\
 & + \frac{4\eta y(\eta+y)[(y+\eta)^2 + x^2 - ax]}{[(a-x)^2 + (\eta+y)^2]^2} \\
 & - \frac{4\eta y(\eta+y)[(y+\eta)^2 + x^2 + ax]}{[(a+x)^2 + (\eta+y)^2]^2} \\
 & + \frac{(\eta-y)[x^2 - ax + (\eta-y)^2]}{(a-x)^2 + (\eta-y)^2} \\
 & - \frac{(\eta-y)[x^2 + ax + (\eta-y)^2]}{(a+x)^2 + (\eta-y)^2} \\
 & + \frac{\eta-y}{2} \left\{ \ln \left[\frac{(a-x)^2 + (\eta-y)^2}{(a-x)^2 + (\eta+y)^2} \right] \right. \\
 & \left. - \ln \left[\frac{(a+x)^2 + (\eta-y)^2}{(a+x)^2 + (\eta+y)^2} \right] \right\}
 \end{aligned} \quad , (C.6)$$

$$\begin{aligned}
 F_{11}^+ = & \frac{4\eta(a-x)[(a^2-2ax+\eta^2+x^2)^2-y^4]}{[(a-x)^2+(\eta-y)^2][(a-x)^2+(\eta+y)^2]^2} \\
 & + \frac{4\eta(a+x)[(a^2+2ax+\eta^2+x^2)^2-y^4]}{[(a+x)^2+(\eta-y)^2][(a+x)^2+(\eta+y)^2]^2} \\
 & - 2 \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a-x}{\eta+y} \right) \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right. \\
 & \left. + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right]
 \end{aligned} \tag{C.7}$$

$$\begin{aligned}
 E_{11}^+ = & \frac{4cy(\eta+y)[(y+\eta)^2+x^2-ax]}{[(a-x)^2+(\eta+y)^2]^2} \\
 & - \frac{4\eta y(\eta+y)[(y+\eta)^2+x^2+ax]}{[(a+x)^2+(\eta+y)^2]^2} \\
 & - \frac{11\eta y^2+13\eta^2 y+3\eta x^2-ax(3\eta+y)+yx^2+3\eta^3+y^3}{(a-x)^2+(\eta+y)^2} \\
 & + \frac{11\eta y^2+13\eta^2 y+3\eta x^2+ax(3\eta+y)+yx^2+3\eta^3+y^3}{(a+x)^2+(\eta+y)^2} \\
 & - \frac{(\eta-y)[(\eta-y)^2+x^2-ax]}{(a-x)^2+(\eta-y)^2} \\
 & + \frac{(\eta-y)[(\eta-y)^2+x^2+ax]}{(a+x)^2+(\eta-y)^2} \\
 & - 2x \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right. \\
 & \left. + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right] - \frac{3(\eta-y)}{2} \ln \left[\frac{(a-x)^2+(\eta-y)^2}{(a+x)^2+(\eta-y)^2} \right] \\
 & - \frac{5\eta+3y}{2} \ln \left[\frac{(a-x)^2+(\eta+y)^2}{(a+x)^2+(\eta+y)^2} \right]
 \end{aligned} \tag{C.8}$$

$$\begin{aligned}
 F_{22}^+ &= \frac{8\eta y^2(a-x)[(a-x)^2 + y^2 - \eta^2]}{[(a-x)^2 + (\eta-y)^2][(a-x)^2 + (\eta+y)^2]^2} \\
 &\quad + \frac{8\eta y^2(a+x)[(a+x)^2 + y^2 - \eta^2]}{[(a+x)^2 + (\eta-y)^2][(a+x)^2 + (\eta+y)^2]^2}
 \end{aligned}
 \tag{C.9}$$

$$\begin{aligned}
 E_{22}^+ &= \frac{7\eta y^2 + 5\eta^2 y - \eta x^2 + ax(\eta-y) + yx^2 - \eta^3 + y^3}{(a-x)^2 + (\eta+y)^2} \\
 &\quad - \frac{7\eta y^2 + 5\eta^2 y - \eta x^2 - ax(\eta-y) + yx^2 - \eta^3 + y^3}{(a+x)^2 + (\eta+y)^2} \\
 &\quad + \frac{(\eta-y)[(\eta-y)^2 + x^2 - ax]}{(a-x)^2 + (\eta-y)^2} \\
 &\quad - \frac{(\eta-y)[(\eta-y)^2 + x^2 + ax]}{(a+x)^2 + (\eta-y)^2} \\
 &\quad - \frac{4\eta y(\eta+y)[(y+\eta)^2 + x^2 - ax]}{[(a-x)^2 + (\eta+y)^2]^2} \\
 &\quad + \frac{4\eta y(\eta+y)[(y+\eta)^2 + x^2 + ax]}{[(a+x)^2 + (\eta+y)^2]^2} \\
 &\quad + \frac{\eta-y}{2} \left\{ \ln \left[\frac{(a-x)^2 + (\eta-y)^2}{(a-x)^2 + (\eta+y)^2} \right] \right. \\
 &\quad \left. - \ln \left[\frac{(a+x)^2 + (\eta-y)^2}{(a+x)^2 + (\eta+y)^2} \right] \right\}
 \end{aligned}
 \tag{C.10}$$

$$\begin{aligned}
 F_{12}^+ = & \frac{\eta^2 + 4\eta y + y^2}{(a+x)^2 + (\eta+y)^2} - \frac{\eta^2 + 4\eta y + y^2}{(a-x)^2 + (\eta+y)^2} \\
 & + \frac{4\eta y(\eta+y)^2}{[(a-x)^2 + (\eta+y)^2]^2} \\
 & - \frac{1}{2} \left[\ln \frac{(a-x)^2 + (\eta+y)^2}{(a-x)^2 + (\eta-y)^2} - \ln \frac{(a+x)^2 + (\eta+y)^2}{(a+x)^2 + (\eta-y)^2} \right] \\
 & - \frac{4\eta y(\eta+y)^2}{[(a+x)^2 + (\eta+y)^2]^2} + \frac{(\eta-y)^2}{(a-x)^2 + (\eta-y)^2} \\
 & - \frac{(\eta-y)^2}{(a+x)^2 + (\eta-y)^2}
 \end{aligned} \quad , (C.11)$$

$$\begin{aligned}
 E_{12}^+ = & \frac{2\eta y[(a+x)(\eta+y)^2 - (a-x)^3]}{((a-x)^2 + (\eta+y)^2)^2} \\
 & + \frac{2\eta y[(a-x)(\eta+y)^2 - (a+x)^3]}{((a+x)^2 + (\eta+y)^2)^2} - \frac{a(y^2 + 4\eta y + \eta^2)}{(a-x)^2 + (\eta+y)^2} \\
 & - \frac{a(y^2 + 4\eta y + \eta^2)}{(a+x)^2 + (\eta+y)^2} + \frac{a(\eta-y)^2}{(a-x)^2 + (\eta-y)^2} \\
 & + \frac{a(\eta-y)^2}{(a+x)^2 + (\eta-y)^2} \\
 & - 2(\eta-y) \left[\tan^{-1} \left(\frac{a-x}{\eta-y} \right) + \tan^{-1} \left(\frac{a+x}{\eta-y} \right) \right] \\
 & - \frac{x}{2} \left\{ \ln \left[\frac{(a-x)^2 + (y+\eta)^2}{(a-x)^2 + (y-\eta)^2} \right] - \ln \left[\frac{(a+x)^2 + (y+\eta)^2}{(a+x)^2 + (y-\eta)^2} \right] \right\} \\
 & + 2(\eta+y) \left[\tan^{-1} \left(\frac{a-x}{\eta+y} \right) + \tan^{-1} \left(\frac{a+x}{\eta+y} \right) \right]
 \end{aligned} \quad . (C.12)$$

Appendix D

The influence coefficients for vertical cracks relate the stresses to the climb and glide dislocations are given as

$$\begin{aligned}
 F_{11}^{\perp} = x \bigg\{ & \frac{y - \eta - a}{x^2 + (\eta + a - y)^2} - \frac{y - \eta + a}{x^2 + (\eta - a - y)^2} \\
 & - \frac{4y[x^2 + y(y + \eta + a)]}{[x^2 + (y + \eta + a)^2]^2} + \frac{4y[x^2 + y(y + \eta - a)]}{[x^2 + (y + \eta - a)^2]^2} \\
 & + \frac{7y - 3\eta - 3a}{x^2 + (y + \eta + a)^2} - \frac{7y - 3\eta + 3a}{x^2 + (y + \eta - a)^2} \bigg\} \\
 & + 4 \left[\tan^{-1} \left(\frac{y + \eta + a}{x} \right) - \tan^{-1} \left(\frac{y + \eta - a}{x} \right) \right]
 \end{aligned} \tag{D.1}$$

$$\begin{aligned}
 E_{11}^{\perp} = & \frac{x(\eta + a)(y - \eta - a)}{x^2 + (\eta + a - y)^2} - \frac{x(\eta - a)(y - \eta + a)}{x^2 + (\eta - a - y)^2} \\
 & - \frac{4x(\eta + a)^2[x^2 + (\eta + a)(y + \eta + a)]}{[x^2 + (\eta + a + y)^2]^2} \\
 & + \frac{4x(\eta - a)^2[x^2 + (\eta - a)(y + \eta - a)]}{[x^2 + (\eta - a + y)^2]^2} \\
 & + \frac{x(\eta + a)(\eta + a + 5y)}{x^2 + (\eta + a + y)^2} - \frac{x(\eta - a)(\eta - a + 5y)}{x^2 + (\eta - a + y)^2} \\
 & - 12y \left[\tan^{-1} \left(\frac{y + \eta + a}{x} \right) - \tan^{-1} \left(\frac{y + \eta - a}{x} \right) \right] \\
 & + \frac{x}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{7x}{2} \ln \left[\frac{x^2 + (y + \eta + a)^2}{x^2 + (y + \eta - a)^2} \right]
 \end{aligned} \tag{D.2}$$

$$\begin{aligned}
 F_{22}^{\perp} = & -x \left\{ \frac{y - \eta - a}{x^2 + (\eta + a - y)^2} - \frac{y - \eta + a}{x^2 + (\eta - a - y)^2} \right. \\
 & - \frac{4y[x^2 + y(y + \eta + a)]}{[x^2 + (y + \eta + a)^2]^2} + \frac{4y[x^2 + y(y + \eta - a)]}{[x^2 + (y + \eta - a)^2]^2} \\
 & + \frac{7y + \eta + a}{x^2 + (y + \eta + a)^2} - \frac{7y + \eta - a}{x^2 + (y + \eta - a)^2} \Big\} \\
 & - 2 \tan^{-1} \left(\frac{\eta + a - y}{x} \right) + 2 \tan^{-1} \left(\frac{\eta - a - y}{x} \right) \\
 & + 2 \tan^{-1} \left(\frac{\eta + a + y}{x} \right) - 2 \tan^{-1} \left(\frac{\eta - a + y}{x} \right)
 \end{aligned} \tag{D.3}$$

$$\begin{aligned}
 E_{22}^{\perp} = & - \frac{2(\eta + a)^3 xy [3(\eta + a)^2 - 4(\eta + a)y + 6(x^2 - y^2)]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
 & + \frac{2(\eta - a)^3 xy [3(\eta - a)^2 - 4(\eta - a)y + 6(x^2 - y^2)]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
 & - \frac{2(\eta + a)xy [4(\eta + a)y(x^2 + y^2) + 3(x^2 + y^2)^2]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
 & + \frac{2(\eta - a)xy [4(\eta - a)y(x^2 + y^2) + 3(x^2 + y^2)^2]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
 & - 2y \left[\tan^{-1} \left(\frac{\eta + a - y}{x} \right) - \tan^{-1} \left(\frac{\eta - a - y}{x} \right) \right. \\
 & \left. - \tan^{-1} \left(\frac{\eta + a + y}{x} \right) + \tan^{-1} \left(\frac{\eta - a + y}{x} \right) \right] \\
 & - \frac{3x}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{3x}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
 \end{aligned} \tag{D.4}$$

$$\begin{aligned}
 F_{12}^{\perp} = & - \frac{8(\eta + a)x^2 y [(\eta + a)^2 + x^2 + y^2]}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
 & + \frac{8(\eta - a)x^2 y [(\eta - a)^2 + x^2 + y^2]}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
 & + \frac{2y(3\eta + 3a + 2y)}{x^2 + (y + \eta + a)^2} - \frac{2y(3\eta - 3a + 2y)}{x^2 + (y + \eta - a)^2} \\
 & - \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] + \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
 \end{aligned} \tag{D.5}$$

$$\begin{aligned}
 E_{12}^{\perp} = & 12a - \frac{(\eta + a)x^2}{x^2 + (\eta + a - y)^2} + \frac{(\eta - a)x^2}{x^2 + (\eta - a - y)^2} \\
 & - \frac{4(\eta + a)^2 x^2 y}{[x^2 + (\eta + a + y)^2]^2} + \frac{4(\eta - a)^2 x^2 y}{[x^2 + (\eta - a + y)^2]^2} \\
 & - \frac{6(\eta + a)^3 + 5(\eta + a)x^2 + 4(\eta + a)^2 y}{x^2 + (\eta + a + y)^2} \\
 & + \frac{6(\eta - a)^3 + 5(\eta - a)x^2 + 4(\eta - a)^2 y}{x^2 + (\eta - a + y)^2} \\
 & + 2x \left[\tan^{-1} \left(\frac{\eta + a - y}{x} \right) - \tan^{-1} \left(\frac{\eta - a - y}{x} \right) \right. \\
 & \left. - \tan^{-1} \left(\frac{\eta + a + y}{x} \right) + \tan^{-1} \left(\frac{\eta - a + y}{x} \right) \right] \\
 & - \frac{y}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] - \frac{7y}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
 \end{aligned} \tag{D.6}$$

$$\begin{aligned}
 F_{11}^+ = & \frac{2(\eta - a)(2\eta - 2a + y)}{x^2 + (\eta - a + y)^2} - \frac{2(\eta + a)(2\eta + 2a + y)}{x^2 + (\eta + a + y)^2} \\
 & + \frac{16(\eta + a)^2 x^2 y^2}{[x^2 + (\eta + a - y)^2][x^2 + (\eta + a + y)^2]^2} \\
 & - \frac{16(\eta - a)^2 x^2 y^2}{[x^2 + (\eta - a - y)^2][x^2 + (\eta - a + y)^2]^2} \\
 & + \frac{1}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta + a - y)^2} \right] - \frac{1}{2} \ln \left[\frac{x^2 + (\eta - a + y)^2}{x^2 + (\eta - a - y)^2} \right]
 \end{aligned} \tag{D.7}$$

$$\begin{aligned}
 E_{11}^+ = & 12a + \frac{(\eta + a)x^2}{x^2 + (\eta + a - y)^2} - \frac{(\eta - a)x^2}{x^2 + (\eta - a - y)^2} \\
 & - \frac{4(\eta + a)^2 x^2 y}{[x^2 + (\eta + a + y)^2]^2} + \frac{4(\eta - a)^2 x^2 y}{[x^2 + (\eta - a + y)^2]^2} \\
 & - \frac{6(\eta + a)^3 + 3(\eta + a)x^2 + 4(\eta + a)^2 y}{x^2 + (\eta + a + y)^2} \\
 & + \frac{6(\eta - a)^3 + 3(\eta - a)x^2 + 4(\eta - a)^2 y}{x^2 + (\eta - a + y)^2} \\
 & - 4x \left[\tan^{-1} \left(\frac{\eta + a + y}{x} \right) - \tan^{-1} \left(\frac{\eta - a + y}{x} \right) \right] \\
 & - \frac{y}{2} \ln \left[\frac{x^2 + (\eta + a - y)^2}{x^2 + (\eta - a - y)^2} \right] - \frac{7y}{2} \ln \left[\frac{x^2 + (\eta + a + y)^2}{x^2 + (\eta - a + y)^2} \right]
 \end{aligned} \tag{D.8}$$

$$\begin{aligned}
 F_{22}^+ = & -\frac{2y(\eta+a)^3[(\eta+a)^2+2(x^2-y^2)]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{2y(\eta-a)^3[(\eta-a)^2+2(x^2-y^2)]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
 & -\frac{16x^2y^2(\eta+a)^2}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{16x^2y^2(\eta-a)^2}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
 & -\frac{2y(\eta+a)(x^2+y^2)^2}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{2y(\eta-a)(x^2+y^2)^2}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
 & +\frac{1}{2}\ln\left[\frac{x^2+(\eta+a+y)^2}{x^2+(\eta+a-y)^2}\right]-\frac{1}{2}\ln\left[\frac{x^2+(\eta-a+y)^2}{x^2+(\eta-a-y)^2}\right]
 \end{aligned} \tag{D.9}$$

$$\begin{aligned}
 E_{22}^+ = & \frac{(\eta-a)x^2}{x^2+(\eta-a-y)^2}-\frac{(\eta+a)x^2}{x^2+(\eta+a-y)^2}+\frac{4(\eta+a)^2x^2y}{[x^2+(\eta+a+y)^2]^2} \\
 & -\frac{4(\eta-a)^2x^2y}{[x^2+(\eta-a+y)^2]^2}+\frac{2(\eta+a)^3+3(\eta+a)x^2}{x^2+(\eta+a+y)^2} \\
 & -\frac{2(\eta-a)^3+3(\eta-a)x^2}{x^2+(\eta-a+y)^2}+\frac{y}{2}\ln\left[\frac{x^2+(\eta+a+y)^2}{x^2+(\eta-a+y)^2}\right] \\
 & -\frac{y}{2}\ln\left[\frac{x^2+(\eta-a+y)^2}{x^2+(\eta-a-y)^2}\right]+2x\tan^{-1}\left(\frac{\eta+a-y}{x}\right) \\
 & -2x\tan^{-1}\left(\frac{\eta-a-y}{x}\right)-2x\tan^{-1}\left(\frac{\eta+a+y}{x}\right) \\
 & +2x\tan^{-1}\left(\frac{\eta-a+y}{x}\right)-4a
 \end{aligned} \tag{D.10}$$

$$\begin{aligned}
 F_{12}^+ = & -\frac{8(\eta+a)^2xy[(\eta+a)^2+x^2-y^2]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{8(\eta-a)^2xy[(\eta-a)^2+x^2-y^2]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2}
 \end{aligned} \tag{D.11}$$

$$\begin{aligned}
 E_{12}^+ = & -\frac{10xy(\eta+a)^5}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{10xy(\eta-a)^5}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
 & -\frac{2(\eta+a)xy[6(\eta+a)^2(x^2-y^2)+(x^2+y^2)^2]}{[x^2+(\eta+a-y)^2][x^2+(\eta+a+y)^2]^2} \\
 & +\frac{2(\eta-a)xy[6(\eta-a)^2(x^2-y^2)+(x^2+y^2)^2]}{[x^2+(\eta-a-y)^2][x^2+(\eta-a+y)^2]^2} \\
 & +4y\left[\tan^{-1}\left(\frac{\eta+a+y}{x}\right)-\tan^{-1}\left(\frac{\eta-a+y}{x}\right)\right] \\
 & +\frac{x}{2}\ln\left[\frac{x^2+(\eta+a-y)^2}{x^2+(\eta+a+y)^2}\right]-\frac{x}{2}\ln\left[\frac{x^2+(\eta-a-y)^2}{x^2+(\eta-a+y)^2}\right]
 \end{aligned} \tag{D.12}$$

Appendix E

The influence coefficients for horizontal cracks relate the surface displacements to the climb dislocation and the glide dislocation are given as

$$H^{\perp} = \frac{1}{2\pi(k+1)} \left[\tan^{-1}\left(\frac{a-x}{\eta}\right) + \tan^{-1}\left(\frac{a+x}{\eta}\right) \right] (ka^2 + 3k\eta^2 - kx^2 + a^2 - \eta^2 - x^2) - a\eta \frac{3k-1}{\pi(k+1)} - \frac{k\eta x}{\pi(k+1)} \ln\left(\frac{a^2 - 2ax + x^2 + \eta^2}{a^2 + 2ax + x^2 + \eta^2}\right) , \quad (\text{E.1})$$

$$K^{\perp} = \frac{1}{\pi} \left[(a-x) \tan^{-1}\left(\frac{a-x}{\eta}\right) + (a+x) \tan^{-1}\left(\frac{a+x}{\eta}\right) \right] + \frac{k\eta}{\pi(k+1)} \ln\left(\frac{a^2 + 2ax + x^2 + \eta^2}{a^2 - 2ax + x^2 + \eta^2}\right) + a , \quad (\text{E.2})$$

$$H^{\parallel} = -\frac{\eta x}{\pi} \left[\tan^{-1}\left(\frac{a-x}{\eta}\right) + \tan^{-1}\left(\frac{a+x}{\eta}\right) \right] - \frac{\eta^2}{2\pi} \ln\left(\frac{a^2 - 2ax + x^2 + \eta^2}{a^2 + 2ax + x^2 + \eta^2}\right) , \quad (\text{E.3})$$

$$K^{\parallel} = -\frac{\eta}{\pi} \left[\tan^{-1}\left(\frac{a-x}{\eta}\right) + \tan^{-1}\left(\frac{a+x}{\eta}\right) \right] . \quad (\text{E.4})$$

Appendix F

The influence coefficients for vertical cracks relate the surface displacements to the climb dislocation and the glide dislocation are given as

$$H^{\perp} = \frac{(a^2 - \eta^2)(1+k) + (3-k)x^2}{2\pi(\kappa+1)} \left[\tan^{-1}\left(\frac{x}{a+\eta}\right) - \tan^{-1}\left(\frac{x}{\eta-a}\right) \right] + \frac{\eta x}{\pi(\kappa+1)} \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) - \frac{ax(k-3)}{\pi(\kappa+1)} \quad , (F.1)$$

$$K^{\perp} = a + \frac{1}{\pi} \left[(\eta - a) \tan^{-1}\left(\frac{x}{\eta - a}\right) - (a + \eta) \tan^{-1}\left(\frac{x}{\eta + a}\right) \right] + \frac{x}{\pi(\kappa+1)} \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) \quad , (F.2)$$

$$H^{\parallel} = \frac{x}{2\pi} \left\{ 2\eta \left[\tan^{-1}\left(\frac{a+\eta}{x}\right) - \tan^{-1}\left(\frac{\eta-a}{x}\right) \right] + x \ln\left(\frac{a^2 - 2a\eta + x^2 + \eta^2}{a^2 + 2a\eta + x^2 + \eta^2}\right) \right\} \quad , (F.3)$$

$$K^{\parallel} = -\frac{1}{\pi} \left\{ 2a + x \left[\tan^{-1}\left(\frac{(\eta-a)}{x}\right) - \tan^{-1}\left(\frac{(a+\eta)}{x}\right) \right] \right\} \quad . (F.4)$$

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