Coherent transmission of light through a cold atomic cloud

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NANYANG
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**COHERENT TRANSMISSION OF LIGHT
THROUGH A COLD ATOMIC CLOUD**

KWONG CHANG CHI
SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES
2017

COHERENT TRANSMISSION OF LIGHT THROUGH A COLD ATOMIC CLOUD

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Abstract

We study the coherent transmission of light through cold strontium atomic clouds, using the narrow intercombination line. In the first part of the thesis, the cooperative emission in the coherent transmission is studied experimentally. By applying fast changes to the incident probe field, flashes of light are observed in the forward direction. The peak intensity of the flashes can be much larger than the incident intensity. These flashes are called superflashes. Due to cooperative effect, the time scale of the flashes are shortened with respect to the single scatterer lifetime. By packing these flashes, we generate a pulse train with short repetition time, where the dynamics can almost be fully governed by cooperative effect. The effect of atomic density on the coherent transmission of light is also studied theoretically and numerically. The effective refractive index of the cloud is calculated up to the second order in atomic density. It is eventually validated with a numerical simulation of the coherent light transmission.

The second part of the thesis focuses on the effect of magnetic field on the coherent transmission of light. The strontium intercombination line has a sensitivity in the milligauss regime, which is the typical value of stray magnetic field in the lab. An active magnetic field control system is implemented to cancel the low frequency and AC 50 Hz stray magnetic field. Using this control system, we study the nonlinear magneto-optical rotation (NMOR) effect due to a high intensity light field transmitting through a dilute and thin cloud of strontium atom. Interestingly, at sufficiently high cloud temperature, the NMOR effect can have a higher sensitivity to the magnetic field, compared to its linear counterpart at the same cloud temperature.

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Chapter 1

General Introduction

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1.1 Introduction

Over the past three decades, the field of cold atomic physics has developed rapidly, leading to many interesting applications of cold atoms. In the field of metrology, optical lattice clocks has been developed with an accuracy of 10^{-18} [1–4]. Besides optical clocks, atom interferometric setups have also been used as accelerometers, for instance, to measure the earth’s gravitational field [5, 6]. The high accuracy achievable in cold atomic experiments also allows fundamental tests in physics to be carried out. Among others, cold atomic experiments have been used to test the theory of relativity and the equivalence principle, to measure physical constants, and to search for the permanent electric dipole moments of atoms [7–11]. Cold atomic system has also been proposed as a possible platform to perform quantum computation and quantum information processing [12, 13].

The control over the experimental conditions also makes cold atomic experiments ideal as simulators of other physical systems [14–17]. In an cold atomic experiment, the number of atoms, and the size and temperature of the cold cloud, can be tuned by varying the cooling parameters. The interaction strength between light and the atoms, and the interaction strength among the atoms¹, can also be controlled. Furthermore, the possibility to load atoms into an optical lattice [19, 20], opens up the possibility to study various theoretical models in condensed matter physics using cold atoms. Since it is usually possible to model accurately a cold atomic experiment, a good understanding of the phenomena can be achieved by simulating the physical models with cold atoms.

In a typical cold atomic experiment, the interaction between light and the atoms plays a crucial role [21, 22]. The laser cooling and trapping of the atoms are due to the velocity- and position-dependent forces exerted on the atoms by the light. The control and manipulation of the internal degrees of freedom of the atoms are also performed with laser light. Finally, the read-out of the experimental results is usually achieved by probing the atomic cloud with light. One commonly used method is the absorption imaging of the cold cloud [23]. In this technique, the coherent transmission of light through the atomic cloud is measured to extract important properties of the cold atomic cloud².

In this thesis, we study the coherent transmission of light through the cold atomic clouds. In the steady-state regime, the transmitted light through a dilute

¹For example, an external magnetic field can be applied to tune the atom-atom interaction through the Feshbach resonance [18].

²The absorption imaging technique can be used to measure the size, density and number of atoms in the atomic cloud. With the time-of-flight (TOF) experimental technique, the temperature of the cold atomic cloud can also be measured.

medium³ is exponentially attenuated. The intensity obeys the Beer-Lambert law, i.e.,

$$I_t = I_0 \exp(-b), \quad (1.1)$$

where I_0 is the incident intensity, I_t is the transmitted intensity, and b is the optical thickness of the medium. For an atomic cloud, there is no non-radiative loss. The attenuation in the transmitted intensity is due to the light scattering away from the forward direction. The notation b as optical thickness is common for the multiple scattering community (see for example Refs. [24,25]). In the cold atomic community, another common notation for the optical thickness of an atomic cloud, OD, is used (see for example Ref. [26]). Here, since the quantity optical thickness appears frequently in the equations, a more concise and convenient notation of b is used.

The optical thickness plays an important role in our study. As we show in Chapters 5 and 6, it is a parameter measuring the cooperativity among the atoms [27–29]. In fact, there have been many discussions in the literature regarding the usage of the term “cooperative” (see for instance Ref. [30]). The term cooperative used in this thesis refers to the fact that the atoms act collectively to reduce the time scale of flash emission in the forward direction, in a similar way as the Dicke superradiance [31]. Despite this similarity, the flashes studied in this work are not manifestation of Dicke superradiance. The coherence among the atoms acting cooperatively, is created by the spatial coherence of the incident light. Furthermore, the recurrent scattering of light, which is important at higher atomic density (see Chapter 7), is not necessary for the emission of the flashes.

While the coherent transmission of light is well understood in the steady state regime, the transient regime holds several surprises. As observed by Chalony *et al.*, a flash of light is emitted by an optically thick atomic cloud when a resonant incident probe beam is turned off abruptly [32]. Remarkably, even though the transmission is almost zero during the steady state regime, the flash of light has a peak intensity that is as high as the incident intensity. We built on this work, to further show that the flash peak intensity can be larger than the incident intensity [33]. We achieve this by applying fast changes to either the intensity or the phase of the incident field.

The time scale of the flashes is shown in our work to be governed by the cooperative effects among the atoms in the cloud. Exploiting this cooperative effect, we demonstrate the generation of cooperative pulse trains from an optically thick cold atomic gas [28]. These results are reported in Part I of this thesis.

³Optically dilute medium and optically thick medium have different meanings. A medium is optically dilute when its density is low, with each atom scatters light independently of the others. A medium is optically thick when the light is strongly attenuated in the transmission through the medium.

In our study, the intercombination transition of ^{88}Sr at 689 nm (see Figure 1.1) is used. This transition has a narrow linewidth of $\Gamma/2\pi = 7.5$ kHz, which is three orders of magnitude smaller than typical linewidths of dipole allowed transitions. The properties of this transition is discussed further in Section 1.2.

In Part I, besides studying experimentally the dilute atomic gas, we also explore theoretically the modifications to the coherent transmission of light coming from the density of the atomic cloud. In a dilute medium, the atoms scatter light independently. In this regime, the independent scattering approximation (ISA) for the coherent transmission holds. This changes when the density of the medium increases, as has been shown by Morice *et al.* in Ref. [34]. In that work, they identified the correction terms to the effective refractive index, n_r , that are proportional to ρ_0^2 , where ρ_0 is the atomic density. A similar study was also performed by J. Ruostekoski and J. Javanainen [35]. In these studies, the atoms are considered to be motionless classical particles. We repeat the calculations of these correction terms in this thesis, using a perturbative diagrammatic expansion method. This theoretical study is then validated with numerical simulations of light multiple scattering in an atomic medium (see Chapter 7). The work here lays the foundation for future studies involving moving atoms.

In the Part II of the thesis, we investigate the effect of magnetic field on the steady state coherent transmission of light through a cold atomic cloud. Due to the Zeeman shifts of the substates of $^3\text{P}_1$, the polarization undergoes a magneto-optical rotation (MOR) when the light is transmitted through the medium. This has been widely studied with hot [36–38] and cold atomic gases [39–42]. It has also been considered for possible application in the field of atomic magnetometry [43]. With cold atomic gases, studies have so far been focused mainly on the alkali-metal atoms. Here, we perform our studies on cold clouds of ^{88}Sr atoms, by probing the narrow intercombination transition. Compared to the typical transitions in alkali-metal atoms, this narrow transition is sensitive to the magnetic field at the milligauss level.

The higher sensitivity of the intercombination line to the stray field requires some form of magnetic field compensation to be carried out. In our setup, we implement an active magnetic field control system. This system is calibrated by performing MOR experiments on the ^{88}Sr intercombination transition. The system is then used to study the nonlinear MOR (NMOR) effect [43, 44], which may have a better sensitivity to the magnetic field compared to its linear counterpart at the same temperature.

In the following, we introduce the system that is the focus of our study. The cold strontium clouds that are used to perform the studies are described in Section 1.2. In Section 1.3, we discuss the physical processes that participate in light transport

Figure 1.1: The energy level scheme of the ^{88}Sr atoms, showing the important transitions used in the preparation of the cold atomic cloud, and in the experiments. In particular, the narrow intercombination line at 689 nm (linewidth $\Gamma/2\pi = 7.5$ kHz) is not only important for cooling of the atoms, it is also the transition that is probed for our studies.

through the cold atomic cloud. In this work, only the coherent transmission of light is studied. Finally, we present the organization of the various chapters in this thesis.

1.2 Properties of cold ^{88}Sr cloud

As mentioned before, the work reported in this thesis focuses on the study of the coherent transmission of a quasi-resonant light through the atomic cloud. The light used to study the coherent transmission is tuned close to the ^{88}Sr intercombination line.

To prepare the cold cloud for experimental studies, ^{88}Sr atoms are first cooled and trapped by a two-stage magneto-optical trap (MOT) [45]. The energy level diagram shown in Figure 1.1, depicts the relevant energy levels for the experiment. There are two important transitions for cooling and trapping of the ^{88}Sr atoms. One is the blue transition at 461 nm; the other is a red transition at 689 nm. The details of cooling and trapping of ^{88}Sr is further described in Appendix C and in Ref. [46].

The $^1S_0 \rightarrow ^3P_1$ intercombination line transition of ^{88}Sr at 689 nm possesses several properties suitable for our studies. First, it has a long lifetime of 21 μs , making it suitable to study the coherent transmission of light in the transient regime. It leads to a less stringent experimental conditions to apply fast changes to the optical field⁴. Furthermore, a time-resolved detection of the temporal signal can also be achieved easier.

Second, the narrow transition linewidth makes the atoms more sensitive to the magnetic field. For the $m = \pm 1$ sublevels, a magnetic field of 3.57 mG is already sufficient to shift the energy level by one linewidth. While this leads to a better sensitivity to the magnetic field, it also means that a better control of the stray magnetic field is required. As mentioned previously, an active magnetic field control system is implemented on the experimental setup to cancel the magnetic field at a level that is lower than 1 mG.

Third, the ^{88}Sr atoms have zero nuclear spin. Hence, there is no hyperfine structure in these atoms. The modeling of the coherent transmission process is thus simpler. In particular, in the study of MOR effects, only a consideration of a $J = 0 \rightarrow J = 1$ transition is required.

Finally, even if the temperature of the ^{88}Sr is typically a few microkelvins, the transition is still Doppler broadened. Nevertheless, in the forward direction, the frequency of the transmitted light remains unchanged even if the atoms are moving. Moreover, we see in Chapter 5 that many important results are independent of the temperature. In particular, the cooperative effect among the atoms is not affected by the temperature, as long as the optical thickness is large. In the study of the NMOR effect, we further encounter a situation where the nonlinear contribution, coming from the coherence in the excited Zeeman substates, is unaffected by the temperature of the medium. Therefore, the cold strontium cloud in the experiments can be used to perform our studies without any adverse effect due to Doppler broadening.

Additionally, the recoil frequency for the intercombination line is $\omega_r = 0.64\Gamma$. This value is comparable to the transition linewidth. Hence, the velocity of an atom is modified significantly every time it scatters light. Nevertheless, in the forward direction, there is no net change in the atomic velocity. The recoil of the atoms does not significantly affect the coherent transmission light in this forward direction. In this thesis, we do not consider the recoil effect on the atoms due to light scattering.

⁴By fast changes, we mean that the response time of the changes are shorter compared to the lifetime of the transition.

1.3 Transport of light in atomic clouds

The studies in this thesis are concerned with light transport in the cold atomic cloud, where the atoms are distributed randomly. Each atom is a resonant point scatterer of light. Only the elastic scattering of light by the atoms are taken into account in our studies. With a large number of atoms, the light experiences multiple scattering inside the medium [47–50]⁵. As discussed in Chapter 3, the multiple scattering of light is described by the coupled dipole equations [59, 60]. Two phenomena can then be distinguished for the transport of light in a disordered medium [50]. They are the ballistic and the diffused transport of light.

Usually, the study into disordered media considers quantities which are averaged over an ensemble of the scattering medium with different realizations. Since the atoms are randomly distributed in the cloud, the scattering of light in a general direction does not preserve the phase relation after spatial averaging. This means that, generally, the average of the light field is zero while the average intensity remains nonzero. This leads to the diffused transport of the light. It is also known as the incoherent transport of light⁶. In the forward direction, however, the phase relation with the incident field is preserved even after the ensemble averaging. Thus, the average field in the forward direction is nonzero. The light is coherent with the incident light and is transported ballistically. The work in this thesis focuses only on this particular aspect of light transport.

1.4 Organization of the thesis

The thesis is organized into two parts. In the first part, the study of coherent transmission of light in the absence of magnetic field is presented. For these studies, the atoms are modeled as two-level atoms⁷. Some of the discussions here could be of interest to the communities that study light multiple scattering or collective effect in light scattering. This is discussed further in Chapter 2, where the background to the studies in Part I are presented. In Part II, the study of MOR effect arising from an external static magnetic field is reported. The focus here is mainly on the

⁵The multiple scattering of light has important consequences for atomic gases. It leads to radiation trapping inside the atomic gas [51–55]. The multiple scattering of light also leads to a repulsive force among the atoms. This repulsive force is undesirable for the cooling and trapping of the atoms [56, 57], especially in the path to reach quantum degeneracy. It also leads to a heating of the atomic cloud [58].

⁶The term incoherent is a misnomer. Coherent effects can still be observed, e.g., coherent backscattering (CBS) as a form of weak localization of light under time reversal symmetry [61–63]. This CBS effect has also been observed experimentally in cold atomic clouds [64, 65].

⁷This includes both cases of degenerate and non-degenerate two-level systems.

rotation of a linear polarization of light during its propagation through the medium. The Zeeman sublevels of the transition play an important role for these studies.

In Part I of the thesis, the basic theory of coherent light transmission in the steady state regime, and at zero magnetic field, is reviewed in Chapter 3. The experimental setup for the investigation of the transient coherent transmission is described in Chapter 4. The experimental results are summarized in Chapters 5 and 6, for superflash and pulse train generation studies, respectively. The extension of the basic theory described in Chapter 3 to the regime beyond the ISA is discussed in Chapter 7. A conclusion of these studies reported in Part I are given in Chapter 8.

The introduction to the studies reported in Part II, is given in Chapter 9. The theory of the linear MOR effect is reviewed in Chapter 10. The system to actively control the magnetic field in the milligauss regime is described in Chapter 11. Chapter 12 then describes the integration of this system in the cold atomic experimental setup. This chapter also describes several studies to calibrate the system, and test its performances. Using the active control system, a study of NMOR effects is reported in Chapter 13. Chapter 14 then concludes Part II of this thesis.

Finally, an overall conclusion is given in Chapter 15, the final chapter of this thesis. This is followed by several appendices that supplements the discussions in the thesis.

Despite being separated into two parts, the studies reported in both parts are still related to the coherent transmission light through cold atomic cloud. Thus, some common aspects are expected. From the experimental point of view, the preparation of the cold atoms are the same for the experiments reported in both parts. The preparation of the cold atomic cloud is summarized in Appendix C. The atomic clouds prepared for the experiments are optically dilute, and thus, the coherent transmission can be modeled within the ISA regime. The regime beyond ISA is also explored, but these studies are only performed theoretically and numerically. The 689 nm laser used to probe the intercombination line is introduced in Chapter 4. The same probe beam is also used in the studies reported in Part II. The measurement and imaging systems, however, are different. They are described separately in Chapters 4 and 12, for the studies described in Parts I and II, respectively.

From a theoretical point of view, the basic model of coherent light transmission is discussed in Chapter 3. The extensions to this model are described in Chapter 7 for Part I, and Chapter 10 for Part II. To supplement the theoretical discussion, the description of the coherent transmission in the ISA regime, from the point of view of Maxwell-Bloch equations, is given in Appendix A.

Part I

Coherent Transmission: Transient Effects and Density Dependence

Chapter 2

Background

Contents

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The studies in Part I of this thesis can be separated into the following two topics,

- the experimental observation in the transient regime, of the superflash effect and the generation of high repetition rate pulse train,
- the theoretical and numerical studies of the coherent transmission beyond the independent scattering approximation (ISA), for motionless atoms.

In this chapter, the background studies relevant to these two main topics are introduced.

2.1 Study of the transient regime of coherent transmission

The atomic cloud studied in the experiment is an example of a disordered scattering medium. For one given static realization of the positions of the atoms inside the cloud, the resulting speckle pattern of light indicates that the emission is always coherent in all direction. Here, we are interested in the forward transmitted field averaged over all realizations of the disorder¹. Despite the random positions of the atoms, the ensemble-averaged field in the forward direction is always coherent with the incident field. This coherent field can also be viewed as a result of superposition between the incident field, and the forward scattered field emitted by the atoms. One of the major aim of our work is to study this coherent field in the transient regime.

In the experiments, we study the transmission of light through a dilute, but optically thick atomic cloud. We consider the transmission signal when fast changes are applied to the incident field. This study is performed in the linear response regime, using a probe beam with a weak intensity. The following three types of fast changes to the incident field, E_0 , are considered,

- an abrupt ignition of the probe beam,
- an abrupt extinction of the probe beam,
- a jump in the phase of the incident field.

Since the atomic cloud used in the experiment is dilute, the coherent transmission can be modeled the independent scattering approximation (ISA) regime, where each atom scatters light independently. By considering the atomic cloud as an effective

¹In the experiment, the intensity of the transmission signal is averaged over different realizations. The dominant contribution in the forward direction comes from the coherent part of light.

optical medium with a complex refractive index, the coherent transmission of light through the cold atomic cloud can be modeled by a simple linear dispersion theory [66]. The large optical thickness of the cloud then ensures a strong cooperative effect among the atoms.

In the early 20th century, A. Sommerfeld and L. Brillouin both studied how the leading edge of an optical signal propagates through a dispersive medium [67–69]. Their purpose was to understand the velocity of a signal traveling through a dispersive medium. Their works revealed the optical precursors, which travel through the dispersive medium with the vacuum speed of light, c_0 . This remarkable property was explained by W. Voigt [69]; since the electrons have finite response time, the front of the wave travels as if it is in the vacuum. M. D. Crisp then predicted the profile of the leading edge as it propagates through the medium [70]. That work was carried out in the small pulse area limit². Thus, it is only accurate for a short time following the leading edge of the signal. These optical precursors have since been observed in experiments with microwave in a dispersive waveguide [71], and with optical pulses that are transmitted through a layer of GaAs [72]. The optical precursors were also observed in a cold rubidium cloud with small optical thickness [73].

Unlike the previous studies, the atomic clouds in our experiment are optically thick. Hence, in the steady state regime, the coherent transmission is close to zero. Nevertheless, during an abrupt ignition of the probe beam, a flash of light is transmitted in the forward direction. This flash of light is related to the propagation of the optical precursors through the medium. Since the atoms do not have time to respond to the abrupt change in the incident field, the initially transmitted intensity is always equal to the incident intensity I_0 . This initial flash has been observed in several studies related to cold atomic gases [32, 74], including our work in Ref. [33] (see also section 5.1.2.1).

We now turn our attention to the case of probe extinction, which is a situation complementary to the ignition of the probe beam³. The switching-off of the probe removes the driving field from the atomic dipoles. This is similar to the free induction decay (FID) experiments first studied in nuclear magnetic resonance (NMR) to measure the transverse lifetime of the nuclear spins [75]. The FID phenomenon is not just limited to nuclear spins; its optical analog was observed by G. Brewer and R. L. Shoemaker [76] in a molecular sample. With cold alkali-metal atoms, the observation of an optical FID signal is reported in Refs. [77, 78]. However, in these

²The pulse area at position z , is defined by $\int_{-\infty}^{\infty} \mathcal{E}(z, t) dt$, where $\mathcal{E}(z, t)$ is the envelope of the field.

³The complementary between the abrupt ignition and extinction of the probe beam is discussed in Section 5.2.2.2.

studies, the extinction of the driving was not abrupt with respect to the atoms.

Because of the complementarity with the ignition of the probe beam, one may expect that a flash is also emitted when the incident field is abruptly switched off. Using the long lifetime $^1\text{S}_0 \rightarrow ^3\text{P}_1$ intercombination line of strontium, Chalony *et al.* observed a flash with a peak intensity of I_0 , when a resonant probe beam is abruptly turned off [32].

A question arises on whether the flash peak intensity can take a value that is larger than I_0 . This is one of the main motivation of our work. We abruptly switch off a sufficiently far-detuned probe beam, and observe a surprisingly large peak value of the flash, which is $3.1I_0$ [33]. Due to this large peak intensity, we name it as a superflash. This peak intensity value directly measures the intensity of the field scattered in the forward direction by the atomic cloud. We further show in this work that the upper limit of the peak intensity for the flash emitted during the extinction of the probe, is $4I_0$ (see the discussions in Chapter 5 and Ref. [33] for more details). By measuring only the intensities before and after the probe extinction, we further demonstrate a reconstruction of the *amplitude* and *phase* of the steady state forward scattered field (see Chapter 5).

As it turns out, the abrupt extinction of the probe beam is not the only way to generate a superflash. A superflash can also be generated by shifting rapidly the phase of the incident field [28, 33]. In this case, the incident and forward scattered fields interfere to generate the flash. This interference, in the constructive case, leads to a maximum superflash with peak intensity $9I_0$. In the destructive interference case, this leads to an antflash, which shows up as a dip in the transmission signal.

The flash observed in the experiment, has a decay time that is shorter than the single atom lifetime, τ . This short time scale is associated with the cooperative effect among the atoms. Naively, one may expect that the cooperative time scale is shortened by the effective optical thickness of the cloud, $b_{\bar{v}}(\delta)$, depending on the probe detuning δ . Instead, we find that the time scale of a flash is reduced from the single atom lifetime by typically a factor of $b_0(0)/2$. Here, $b_0(0)$ is the resonant optical thickness of the atomic cloud at zero temperature. It plays the role of the cooperative parameter [27, 29]. We discuss this surprising behavior in Section 5.2.

The strong cooperative effect among the atoms is not observed in the steady state regime, due to the large attenuation of the transmitted light by the cold atomic cloud. It is observed only when the abrupt changes are applied. In a second set of experiments, we apply a periodic phase jump of π to the incident field. This generates a train of flashes with a repetition time T_R . Interestingly, even if $T_R < \tau$, the atoms can still emit a high contrast pulse train in the forward direction (see Chapter 6 and Ref. [28]). In this case, the single atom fluorescence events are suppressed, and the light scattering process is fully governed by the cooperative

effect.

Finally, we note that the light-atom system is not the only system where the flashes can be observed. There have also been studies where pulses are generated through phase modulation of the probe beam [79]. In these studies, a high intensity pulse of light was generated by stacking up the properly timed coherent transient. While a large peak intensity is observed, it does not lead to a periodic train of pulses with equal peak intensities.

In addition, in the study of coherent interaction between γ -rays and nuclear ensembles, there are several works have generated pulses of γ -rays in Mössbauer spectroscopy experiments [80–85]. These works are useful to develop the corresponding quantum optics and information processing capabilities with γ -ray radiation. The pulse are generated by modulating the phase or the frequency of the γ -ray source⁴. Here, the γ -ray emission is not monochromatic, unlike the laser emission in the cold atomic experiments. Furthermore, the pulses observed in the Mössbauer spectroscopy typically have intensities that are not much larger than the incident field.

2.2 Study of coherent transmission beyond ISA

For the studies of superflashes and pulse train generation, the atomic cloud is optically dilute. A dilute medium satisfies the condition $k\ell \gg 1$, where k is the wave vector of the light and ℓ is the optical mean free path at resonance⁵. Under this condition, the wavelength of the light is shorter than the mean free path. Thus, the picture of light propagating as a wave between two subsequent scattering events holds. The ISA regime of the coherent transmission is valid under this assumption. For quasi-resonant atomic scatterers, the criterion at resonance can be applied, giving the following condition for a dilute medium,

$$k_0\ell_0 = \frac{1}{\rho_0\alpha_0} \gg 1, \quad (2.1)$$

where α_0 is the modulus of the resonant two-level atom polarizability⁶. To further explore the cooperative effect among the atoms, the dependence of the coherent

⁴A recent theoretical work extends these techniques used in the γ -ray experiments to the case of optics [86].

⁵The quantity $k\ell$ is related to the Ioffe-Regel criterion for Anderson localization [87]

⁶The value of α_0 depends on the type of electromagnetic (EM) waves considered. The exact meanings of the vector and scalar EM waves are explained in Sections 3.2.1 and 3.2.2. For the vector waves, $\alpha_0 = 6\pi/k_0^3$. For the scalar waves, which approximates the EM waves by ignoring the polarization and the near field, $\alpha_0 = 4\pi/k_0^3$. The quantity $k_0 = \omega_0/c_0$ is the resonant wave vector corresponding to a resonant frequency of ω_0 .

transmission of light on the atomic density has to be understood.

For instance, effects that are beyond the ISA regime have been recognized since the early days of Bose Einstein condensation (BEC) research. In Ref. [34], the coherent transmission of light through a Bose gas was studied to understand its optical signatures. In that study, and subsequently other studies [35, 88], they found three additional terms that are contributing to the refractive index n_r , which are proportional to $\rho_0^2 \alpha_0^2$. These terms originate from the Lorenz-Lorentz correction (or the Clausius-Mossotti relation) [89], the pair correlation function in the atomic position that includes the effect of quantum statistics, and the resonant dipole-dipole interactions between a pair of nearby atoms [34]. The resonant dipole-dipole interaction can be viewed as the recurrent scattering of light, involving photon exchanges between two atoms [90–92]. Recently, the predicted effect of quantum statistics on the coherent transmission of light through a BEC was observed experimentally [93]. A similar theoretical study was also performed for a Fermi gas [94]. A degenerate Fermi gas is interesting since some scattering events may be forbidden due to Pauli blocking. This happens for the scattering events where the recoils bring the atoms into other filled states in the Fermi sea. Remarkably, due to the a density dependent length scale of the pair correlation function, the Fermi-Dirac statistics result in a contribution that is more dominant compared to the Lorenz-Lorentz shift and recurrent scattering contributions. A reduction in the linewidth is predicted. Thus, an extension of the study into the coherent transmission regimes beyond the ISA for quantum gases, may allow us to probe and study its quantum statistics.

More recently, light propagation in dense atomic systems has also received attention for possible application in the field of quantum information processing. For instance, one can take advantage of the localization of light [95], or the subradiant mode in a dense atomic cloud [31, 96, 97], to store information in the atomic medium. Thus, it is not surprising that several recent studies of light scattering have been performed using dense atomic clouds [98, 99]. There have also been studies involving dilute and dense atomic clouds with size comparable to the wavelength of the probing laser [100–102]. Besides cold atomic gas, hot dense atomic vapors are also interesting systems, and have been studied for its light-atom interaction [103].

While there are many theoretical works on light scattering in a dense medium [104–107], this phenomenon is still far from completely understood. In fact, a recent work showed that even at relatively low atomic density, standard optics theory still fails to account correctly for the propagation of light [108]. In particular, an interesting phenomenon was discussed in Ref. [106], where a crossover to the phenomenology of inhomogeneous Doppler broadening is predicted to place at $k\bar{v} \sim \Gamma$. This regime is easily accessible for a Doppler cooled strontium cloud. Thus, it is interesting to explore further the behavior of coherent transmission in this crossover regime.

In the ISA regime, the temperature dependence is easily obtained by averaging the effective refractive index over the thermal velocity distribution of the atoms [109]. This gives the Voigt profile of the absorption spectrum, as described in Chapter 3. However, beyond the ISA regime, the interplay among the different contributions, at different temperature, leads to a more complicated temperature dependence. For example, it is expected that the recurrent scattering contribution will reduce at higher temperature, since the scattered photons quickly goes off resonance due to Doppler shift. On the other hand, the Lorenz-Lorentz shift survives at high temperature, as seen in recent experiments with atomic vapors [103]. Furthermore, the motion of the atoms can results in collisional effects, leading also to a shift in the frequency [110].

As a preliminary step towards the study of the temperature dependence, we compute theoretically and numerically the effective relative permittivity, $\varepsilon_r = n_r^2$, of an atomic cloud taking into account the density effects (see Chapter 7), assuming that the atoms are classical motionless particles. This theoretical study is performed with a diagrammatic expansion of the effective relative permittivity, in orders of $\rho_0\alpha_0$ [111, 112]. We keep only terms up to $\rho_0^2\alpha_0^2$, and find results that are consistent with Refs. [34, 35]. Even if the results do not hold when the density is high, i.e., when $\rho_0\alpha_0 \simeq 1$, this perturbative method has an advantage that the motion of the atoms can be handled within the same framework. Furthermore, one can also systematically include higher order terms in the calculation.

In parallel to the theoretical studies, we also perform numerical simulations of the coherent light transmission. The numerical simulations are based on the coupled dipole equations. The coupled dipole equations are used to describe the multiple scattering of light in the atomic cloud [59, 60]. From the coherent field calculated in the simulation, the value of ε_r can be computed. The correction terms that come in $\rho_0^2\alpha_0^2$ are validated with the results of the numerical simulation. Besides performing the studies with vector waves, we also compute ε_r for scalar waves. In this case, which is simpler to handle theoretically, we find generally better agreement between the theoretical prediction and the numerical simulation result.

It is important also to point out that the quantity ε_r is a bulk property of the atomic medium. Thus, the atomic medium is assumed to have a size larger than the wavelength of the light. Geometric effects like partial reflections [113] from the medium boundaries are not included in our studies. Similarly, the cooperative Lamb shift [103, 110, 114, 115] which is related to the exchange of virtual photons between the atoms, is also not included in our studies. Finally, there has also been a study of the scattering of light by a 2-D layer of atoms [116]. This also lies beyond the scope of our present studies.

Chapter 3

Theory of Coherent Transmission for Two-Level Atoms

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In this chapter, we review the theory underlying the coherent transmission of light. We focus only on the case of a dilute atomic medium, which is relevant to understand the results in Chapters 5 and 6. In these dilute atomic clouds, the independent scattering approximation (ISA) regime is sufficient to describe the coherent transmission of light. As its name suggests, the atoms in this regime scatter light independently of each other. One simple way to model the coherent transmission of light, in the dilute regime, is through the Maxwell-Bloch equation (see Appendix A). Unfortunately, this model breaks down if we consider dipole-dipole interactions between the atoms, which becomes important at higher density. These density effects on coherent transmission are discussed later in Chapter 7. To unify within the same framework, the studies from Chapters 5 to 7 that involve different regimes of atomic density, we use a more general approach to model the coherent transmission of light.

Since the atomic clouds in the experiments contain many atoms, typically 10^8 of them, the multiple scattering of light is important. The multiple scattering of light can be modeled using a set of equations known as the coupled dipole equations [59, 60]. In Sections 3.3.1 and 3.3.2, the coupled dipole equations and the multiple scattering of light are introduced. This set of equations is applicable in a wide range of density regimes, and not limited to the case of dilute medium. For instance, we numerically simulate this set of equations in Chapter 7 to study the density effects that comes in second order, on the coherent transmission. Importantly, this set of equations can be solved for any given realization of the atomic medium.

We further consider the atomic cloud effectively as a dispersive medium. The medium is then associated with an effective relative permittivity, ε_r . Starting from the coupled dipole equations, we arrive at a Dyson equation for the coherent transmission of light, whereby a perturbative diagrammatic expansion method (see for example Ref. [112]) can be used to compute ε_r in orders of $\rho_0\alpha_0$ ¹. Here, unlike the coupled dipole equations, we have to consider an ensemble averaging of the field over many different realizations of the atomic medium. In the expansion, ρ_0 is the atomic density, and α_0 is the modulus of the resonant atomic polarizability (see Section 3.1). We describe this method in Section 3.4. Since we focus only on the dilute atomic medium in this chapter, the expansion is terminated at the linear order of $\rho_0\alpha_0$. The correction terms that are proportional to $\rho_0^2\alpha_0^2$ are discussed in Chapter 7.

To simplify our study, we work with the semiclassical model of light atom interaction, where the radiation field and the external atomic motion are modeled classically. The usual electric dipole approximation is also applied here. We further

¹The quantity $\rho_0\alpha_0$ is equal to $1/(k_0\ell_0)$, where k_0 is the resonant wave vector and ℓ_0 the resonant optical mean free path in the ISA regime.

assume that the probe beam intensity is much lower than the saturation intensity, considering only the linear regime of coherent transmission.

Finally, we are interested in the coherent field that is transmitted through the medium in the forward direction. To compute this coherently transmitted field, $\mathbf{E}_t$, the effective refractive index of the cold atomic cloud, n_r , has to be known. It is given by $n_r = \sqrt{\varepsilon_r}$. The optical thickness of the medium, and the phase shift experienced by the transmitted field can then be computed knowing n_r . This is carried out in Section 3.5. We also derive an expression for the time dependence of the coherently transmitted field, i.e., Equation (3.71). This equation serves as the starting point to compute the time evolution of the flashes in Chapter 5.

While light is well-known to be a vector electromagnetic (EM) wave, it is also possible to approximate the EM wave as a scalar wave (see Section 3.2), by neglecting its polarization and near field. This scalar model is useful, as it describes well the experimental results involving coherent transmission in the ISA regime (for instance, the studies reported in Chapters 5 and 6). Also, the theoretical analysis is considerably simplified for the scalar wave. Nevertheless, the discussions here still include the realistic case of vector waves to set the stage for the calculations involving $\rho_0^2\alpha_0^2$ terms in Chapter 7. There, unlike the case of ISA, the scalar and vector waves behaves differently even for coherent light transmission.

3.1 Atomic polarizability

An atom has a size that is smaller compared to the wavelength of the light. It behaves like a Rayleigh scatterers of light. We model them as two-level atoms, with the polarizability in the linear regime given by,

$$\alpha(\delta) = -\frac{\alpha_0\Gamma/2}{\delta + i\Gamma/2}. \quad (3.1)$$

The detuning of the probe beam is defined as $\delta = \omega_l - \omega_0 \ll \omega_0$. The resonant frequency of the two-level atom is denoted by ω_0 , while the frequency of the incident beam is denoted by ω_l . The linewidth of the excited state is denoted by Γ . For the vector EM wave, $\alpha_0 = 6\pi/k_0^3$. The above expression is written in the rotating wave approximation (RWA) [117], where the counter-rotating term is ignored. Further details of the atomic polarizability can be found in Appendix A.

The polarizability is related to the extinction cross section, $\sigma_e = k_l \text{Im}\{\alpha\}$. The wave vector of the incident light is defined by $k_l = |\mathbf{k}_l| = \omega_l/c_0$, where c_0 is the speed of light in vacuum. There is no true absorption in the atomic cloud; σ_e can

be equated to the scattering cross section $\sigma_s = (k_l^4/6\pi)|\alpha|^2$. This is the well-known optical theorem. In particular, for a probe beam that is detuned by δ , we have

$$\sigma_s(\delta) = \frac{6\pi}{k_l^2} \frac{\Gamma^2/4}{\delta^2 + \Gamma^2/4}. \quad (3.2)$$

We further consider that at position $\mathbf{r}$ and time t , the incident field is given by

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \{ \mathbf{E}_0 \exp(-i\omega_l t + i\mathbf{k}_l \cdot \mathbf{r}) \}, \quad (3.3)$$

where $\mathbf{E}_0$ is the amplitude of the EM wave.

Using the polarizability α , one can relate $\mathbf{E}_0$ to the expectation value of the induced dipole moment operator $\langle \hat{\mathbf{d}} \rangle$, through

$$\langle \hat{\mathbf{d}} \rangle = \varepsilon_0 \alpha \mathbf{E}_0, \quad (3.4)$$

where ε_0 is the electric permittivity in vacuum.

In the scalar case, the same expression in Equation (3.1) holds, with the value of α_0 being $4\pi/k_0^2$.

3.2 Types of EM waves

For the theoretical formulation, two types of EM waves are considered. The vector EM waves display the full vector behavior of light. The scalar EM waves, on the other hand, do not describe an actual light wave, since the light polarization and the near field of the dipole emission are neglected. Nevertheless, the scalar wave is simpler to treat theoretically, and it is usually a good approximation for two-level atoms.

3.2.1 Vector waves

In the frequency domain, the vector wave field denoted by $\mathbf{E}(\omega)$ and emitted by a dipole, $\mathbf{d}(\omega)$, obeys the following vectorial Helmholtz equation,

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r} - \mathbf{r}', \omega) - \left(\frac{\omega}{c_0} \right)^2 \mathbf{E}(\mathbf{r} - \mathbf{r}', \omega) = \mu_0 \omega^2 \mathbf{d}(\omega) \delta(\mathbf{r} - \mathbf{r}'). \quad (3.5)$$

The vacuum magnetic permeability is denoted by μ_0 .

The vector waves free space Green function is a dyadic [118]. It is the solution

of the following equation,

$$\nabla \times \nabla \times \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) - \left(\frac{\omega}{c_0}\right)^2 \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}') \overleftrightarrow{\mathbf{I}}. \quad (3.6)$$

It is found to be

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) = \frac{1}{4\pi} \frac{\omega}{c_0} \left[\beta(\omega R/c_0) \overleftrightarrow{\mathbf{P}} + \gamma(\omega R/c_0) \overleftrightarrow{\mathbf{U}} \right] - \frac{\overleftrightarrow{\mathbf{I}}}{3(\omega/c_0)^2} \delta(\mathbf{R}). \quad (3.7)$$

We have defined $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and $R = |\mathbf{R}|$. The functions β and γ are given by,

$$\beta(x) = \frac{e^{ix}}{x} \left[1 - \frac{1}{ix} - \frac{1}{x^2} \right], \quad (3.8)$$

$$\gamma(x) = \frac{2e^{ix}}{x} \left[\frac{1}{ix} + \frac{1}{x^2} \right]. \quad (3.9)$$

The dyadic $\overleftrightarrow{\mathbf{U}} = \mathbf{u} \otimes \mathbf{u}$, is the projector onto the direction of $\mathbf{u} = \mathbf{R}/R$. The dyadic $\overleftrightarrow{\mathbf{P}} = \overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\mathbf{U}}$ is the projector onto the space perpendicular to $\mathbf{u}$. In $\beta(\omega R/c_0)$, we find the $1/R$ dependence that is associated to the far field dipole radiation. The remaining terms of $\overleftrightarrow{\mathbf{G}}_0$ then describe the near field emission.

In the $\mathbf{k}$ -space, where $\mathbf{k}$ is the Fourier counterpart of $\mathbf{R}$, the dyadic Green function is given by

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega) = \frac{\overleftrightarrow{\mathbf{I}}}{k^2 \overleftrightarrow{\mathbf{P}}_k(\mathbf{k}, \omega) - (\omega/c_0)^2 \overleftrightarrow{\mathbf{I}}}. \quad (3.10)$$

In the equation above, we define $k = |\mathbf{k}|$. The dyadic $\overleftrightarrow{\mathbf{P}}_k = \overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\mathbf{K}}$ is the projector onto the plane orthogonal to $\mathbf{k}$, where $\overleftrightarrow{\mathbf{K}} = \frac{\mathbf{k} \otimes \mathbf{k}}{k^2}$. For convenience, the free space dyadic Green function is decomposed into its transverse and longitudinal components, i.e.,

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega) = \frac{1}{k^2 - (\omega/c_0)^2} \overleftrightarrow{\mathbf{P}}_k - \frac{1}{(\omega/c_0)^2} \overleftrightarrow{\mathbf{K}}. \quad (3.11)$$

The transverse component is proportional to $\overleftrightarrow{\mathbf{P}}_k$, while the longitudinal component is proportional to $\overleftrightarrow{\mathbf{K}}$. The longitudinal component represents a part of the field that does not propagate further than typically one wavelength. This part does not play a role in the coherent transmission of light. A similar separation into the

transverse and longitudinal components can also be made for the mass operator, as we shall see in the following.

Let the dipole moment located at $\mathbf{r}'$, be induced by an external field $\mathbf{E}^{\text{ex}}$, i.e.,

$$\mathbf{d}(\omega) = \varepsilon_0 \alpha(\omega) \mathbf{E}^{\text{ex}}(\mathbf{r}', \omega), \quad (3.12)$$

with the polarizability, $\alpha(\omega)$, that is described in Section 3.1. In terms of the dyadic Green function, the solution to Equation (3.5), is given by

$$\mathbf{E}(\mathbf{r}, \omega) = \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) \cdot \mathbf{E}^{\text{ex}}(\mathbf{r}', \omega). \quad (3.13)$$

This equation is useful for the discussions of the coupled dipole equations in Section 3.3.1.

3.2.2 Scalar waves

we consider now the scalar wave field, E . In the frequency domain, it obeys the following Helmholtz equation,

$$\nabla^2 E(\mathbf{r}, \omega) + \left(\frac{\omega}{c_0} \right)^2 E(\mathbf{r}, \omega) = -\mu_0 \omega^2 d(\omega) \delta(\mathbf{r} - \mathbf{r}'), \quad (3.14)$$

with a source of scalar dipole moment $d(\omega)$ located at $\mathbf{r}'$.

The free space Green function, G_0 , obeys

$$\nabla^2 G_0(\mathbf{r} - \mathbf{r}', \omega) + \left(\frac{\omega}{c_0} \right)^2 G_0(\mathbf{r} - \mathbf{r}', \omega) = -\delta(\mathbf{r} - \mathbf{r}'). \quad (3.15)$$

It is found to be,

$$G_0(\mathbf{R}, \omega) = \frac{1}{4\pi R} e^{i\omega R/c_0}. \quad (3.16)$$

Unlike the case of vector waves, the scalar wave is scattered isotropically by the atom. In the Fourier space, the scalar Green function is

$$G_0(\mathbf{k}, \omega) = \frac{1}{k^2 - (\omega/c_0)^2}. \quad (3.17)$$

For a dipole source of $d(\omega) = \varepsilon_0 \alpha(\omega) E_{\text{ex}}(\mathbf{r}', \omega)$, the solution to Equation (3.14)

is

$$E(\mathbf{r}, \omega) = \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 G_0(\mathbf{r} - \mathbf{r}', \omega) E_{\text{ex}}(\mathbf{r}', \omega). \quad (3.18)$$

The scalar polarizability that appears in the above equation, is modified from the vector case, by taking $\alpha_0 = 4\pi/k_0^2$.

3.3 Light scattering in the atomic cloud

To describe the scattering of light in an atomic medium, a system of coupled dipole equations is used [59, 60]. This system of equation describes microscopically the various light scattering processes for a given configuration of the atoms inside the cloud. The multiple scattering of light inside the atomic medium is included in this set of coupled equations.

3.3.1 Coupled dipole equations

We start by considering the atoms to be motionless, and focus only on the steady state regime of the coherent transmission. We further limit the study to the linear regime without light saturation. We start with the case of the vector waves before moving to the case of the scalar waves.

3.3.1.1 Vector waves

Let $\mathbf{d}_i$ be the induced dipole moment of the i -th atom located at $\mathbf{r}_i$. The external field that induces the dipole moment for the i -th atom is denoted by $\mathbf{E}_i^{\text{ex}}(\omega)$. We have,

$$\mathbf{d}_i(\omega) = \varepsilon_0 \alpha(\omega) \mathbf{E}_i^{\text{ex}}(\omega), \quad (3.19)$$

There are a total of N_a external field vectors $\mathbf{E}_i^{\text{ex}}(\omega)$, one at each atom. The total field, at a position $\mathbf{r}$, is a sum of the incident field and the scattered field emitted by all the atomic scatterers,

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\mathbf{r}, \omega) + \mu_0 \omega^2 \sum_{i=1}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \mathbf{d}_i(\omega). \quad (3.20)$$

Equation (3.19) is inserted in the equation above to give

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\mathbf{r}, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{i=1}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \mathbf{E}_i^{\text{ex}}(\omega), \quad (3.21)$$

To complete the description, we need to calculate $\mathbf{E}_i^{\text{ex}}(\omega)$ for each atom. This is achieved using the following set of coupled dipole equations,

$$\mathbf{E}_i^{\text{ex}}(\omega) = \mathbf{E}_0(\mathbf{r}_i, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \mathbf{E}_j^{\text{ex}}(\omega), \quad (3.22)$$

where the contribution of the dipole emission from the i -th atom itself is excluded. The coupled dipole equations for the vector waves, contain a total of $3N_a$ linear coupled equations, one for each component of the external fields. For a given realization of the atomic positions, the linear set of coupled equations in Equation (3.22) for the N_a atoms, can be solved. Numerical methods are available to solve this kind of coupled equations (see Chapter 7 for more details). These solutions of $\mathbf{E}^{\text{ex}}$ are then used in Equation (3.21) to compute $\mathbf{E}(\mathbf{r}, \omega)$.

3.3.1.2 Scalar waves

A similar set of equations can also be written in the case of scalar EM wave.

$$E(\mathbf{r}, \omega) = E_0(\mathbf{r}, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{i=1}^{N_a} G_0(\mathbf{r} - \mathbf{r}_i, \omega) E_i^{\text{ex}}(\omega), \quad (3.23)$$

$$E_i^{\text{ex}}(\omega) = E_0(\mathbf{r}_i, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{\substack{j=1 \\ j \neq i}}^{N_a} G_0(\mathbf{r}_i - \mathbf{r}_j, \omega) E_j^{\text{ex}}(\omega). \quad (3.24)$$

The solution of this set of equations also start from the coupled dipole equations in Equation (3.24). This gives the N_a values of E_i^{ex} of all the atoms, which are then used in Equation (3.23) to compute the total field.

3.3.2 Multiple scattering of light

The phenomenon of the multiple scattering of light is contained inside the coupled dipole equations. To illustrate this, we consider only the vector waves. The discussion here can be easily extended to the scalar case. We also limit our discussions to the case where the atoms are not moving.

The coupled dipole equations can be rewritten in the following form,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, \omega) = & \mathbf{E}_0(\mathbf{r}, \omega) + T_s(\omega) \sum_{i=1}^{N_a} \vec{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \mathbf{E}_0(\mathbf{r}_i, \omega) \\
& + T_s(\omega)^2 \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \vec{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \mathbf{E}_0(\mathbf{r}_j, \omega) \\
& + T_s(\omega)^3 \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \sum_{\substack{l=1 \\ l \neq j}}^{N_a} \vec{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{r}_j - \mathbf{r}_l, \omega) \cdot \mathbf{E}_0(\mathbf{r}_l, \omega) \\
& + \dots,
\end{aligned} \tag{3.25}$$

by substituting Equation (3.22) repetitively in Equation (3.21). For an atom located at $\mathbf{r}_0$, the scattering operator is given by

$$\mathcal{T}_{\mathbf{r}_0}(\mathbf{r}, \mathbf{r}', \omega) = T_s(\omega) \delta(\mathbf{r} - \mathbf{r}_0) \delta(\mathbf{r}' - \mathbf{r}_0), \tag{3.26}$$

where $T_s(\omega)$ is related to the polarizability $\alpha(\omega)$, i.e.,

$$T_s(\omega) = \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2. \tag{3.27}$$

Equation (3.25) is an infinite series involving different orders of $T_s(\omega)$. Each term represents a particular order of multiple scattering of light. The first term on the right hand side gives the direct contribution of the incident field. The second term, proportional to T_s , gives the contribution of the single scattering events. The third term, proportional to T_s^2 corresponds to the contributions of double scattering events. The fourth term, which comes in T_s^3 , is related to the situation where the light experiences three scattering events inside the medium. This summation extends to all orders of multiple scattering of light inside the scattering medium.

Figure 3.1: The different orders of recurrent scattering involving two atoms. The examples are shown for three, four and five scattering events between the pair of atoms. In chapter 7, we find that the recurrent scattering processes are important for the calculation of ϵ_r beyond the ISA regime.

The T_s^3 term deserves further attention. It can be separated into two terms,

$$\begin{aligned}
& T_s(\omega)^3 \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \sum_{\substack{l=1 \\ l \neq j}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_j - \mathbf{r}_l, \omega) \cdot \mathbf{E}_0(\mathbf{r}_l, \omega) \\
& = T_s(\omega)^3 \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \sum_{\substack{l=1 \\ l \neq j \\ l \neq i}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_j - \mathbf{r}_l, \omega) \cdot \mathbf{E}_0(\mathbf{r}_l, \omega) \\
& + T_s(\omega)^3 \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_j - \mathbf{r}_i, \omega) \cdot \mathbf{E}_0(\mathbf{r}_i, \omega). \quad (3.28)
\end{aligned}$$

The first term gives the contribution from triple scattering events involving three different atomic scatterers. The second term is more interesting, since it involves only a pair of atomic scatterers. The first scattering of light occurs at the i -th atom. The emitted light then scatters off the j -th atom, before it returns to the i -th atom to be scattered again. This is an example of the recurrent scattering of light, where the scattered light returns to one or more of its previous scatterers. Several examples of the recurrent scattering of light between a pair of atoms, are shown pictorially in Figure 3.1. The recurrent scattering of light plays an important role in the calculation of ε_r , we study the coherent transmission beyond the ISA regime (see Chapter 7).

3.4 Perturbative expansion of ε_r

So far, the set of coupled dipole equations describes the scattering of light for each realization of a zero temperature atomic cloud. For a single realization of the atomic cloud, a speckle pattern of light is observed due to the randomly positioned atoms in the cloud.

For the theoretical treatment, the relevant quantities are usually subjected to ensemble averaging. For example, in the case of a cold atomic cloud, it is possible to average the measured intensities of light over many runs of the experiment, under the same conditions. The speckle pattern disappears upon this averaging. We have instead the diffused transport of light which is isotropic. Only in the exact forward direction, the coherent field survives the ensemble averaging.

For simplicity, we consider only a statistically uniform medium. The single particle probability distribution, to find an atom at $\mathbf{r}$, is given by

$$P_1(\mathbf{r}) = \frac{1}{V}, \quad (3.29)$$

where V is the volume of the medium. Here, V is assumed to be very large.

The coherent field can be viewed as propagating through a homogeneous effective optical medium. In the next section, we review the Dyson equation, which is a governing equation for the coherent field. We discuss this for both the vector and scalar wave cases.

3.4.1 Dyson equation

3.4.1.1 Vector waves

The ensemble-averaged field, $\langle \mathbf{E}(\mathbf{r}, \omega) \rangle$, which is also the coherently transmitted field, obeys a Dyson equation [47, 119, 120]. For the vector waves, this is

$$\langle \mathbf{E}(\mathbf{r}, \omega) \rangle = \mathbf{E}_0(\mathbf{r}, \omega) + \iint \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) \cdot \overleftrightarrow{\mathbf{M}}(\mathbf{r}' - \mathbf{r}'', \omega) \cdot \langle \mathbf{E}(\mathbf{r}'', \omega) \rangle d^3\mathbf{r}' d^3\mathbf{r}''. \quad (3.30)$$

The integral is performed on the positions $\mathbf{r}'$ and $\mathbf{r}''$ over the volume of the scattering medium. In analogy with quantum field theory, $\overleftrightarrow{\mathbf{M}}$ is known as the mass operator, or the self energy.

Let us consider an incident wave propagating in the free space, with an external current source of $\mathbf{j}(\mathbf{r}', \omega)$. The incident field is related to this current source by

$$\mathbf{E}_0(\mathbf{r}, \omega) = i\omega\mu_0 \int \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) \mathbf{j}(\mathbf{r}', \omega) d^3\mathbf{r}'. \quad (3.31)$$

The integral is assumed to be extended over the entire volume of the current source. The coherent field is also related to the same current source, but through an average Green function $\langle \overleftrightarrow{\mathbf{G}} \rangle$, i.e.,

$$\langle \mathbf{E}(\mathbf{r}, \omega) \rangle = i\omega\mu_0 \int \langle \overleftrightarrow{\mathbf{G}}(\mathbf{r} - \mathbf{r}', \omega) \rangle \mathbf{j}(\mathbf{r}', \omega) d^3\mathbf{r}'. \quad (3.32)$$

In fact, we can write an equation similar to the Dyson equation for the average Green function,

$$\begin{aligned} \langle \overleftrightarrow{\mathbf{G}}(\mathbf{r} - \mathbf{r}', \omega) \rangle &= \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) \\ &+ \iint \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}'', \omega) \cdot \overleftrightarrow{\mathbf{M}}(\mathbf{r}'' - \mathbf{r}''', \omega) \cdot \langle \overleftrightarrow{\mathbf{G}}(\mathbf{r}''' - \mathbf{r}', \omega) \rangle d^3\mathbf{r}'' d^3\mathbf{r}'''. \end{aligned} \quad (3.33)$$

In the Fourier domain, the above expression becomes

$$\langle \overleftrightarrow{\mathbf{G}}(\mathbf{k}, \omega) \rangle = \overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega) + \overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega) \cdot \overleftrightarrow{\mathbf{M}}(\mathbf{k}, \omega) \cdot \langle \overleftrightarrow{\mathbf{G}}(\mathbf{k}, \omega) \rangle, \quad (3.34)$$

where $\mathbf{k}$ is the Fourier variable corresponding to $\mathbf{r} - \mathbf{r}'$. Rearranging the terms leads to

$$\langle \overleftrightarrow{\mathbf{G}}(\mathbf{k}, \omega) \rangle = \left[\overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega) \cdot \overleftrightarrow{\mathbf{M}}(\mathbf{k}, \omega) \right]^{-1} \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{k}, \omega). \quad (3.35)$$

Inserting Equation (3.10) for the dyadic free space Green function, we arrive at

$$\langle \overleftrightarrow{\mathbf{G}}(\mathbf{k}, \omega) \rangle = \frac{\overleftrightarrow{\mathbf{I}}}{k^2 \overleftrightarrow{\mathbf{P}}_k - (\omega/c_0)^2 \overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\mathbf{M}}(\mathbf{k}, \omega)}. \quad (3.36)$$

The average Green function can also be separated into its transverse and longitudinal components, as done for the free space Green function,

$$\langle \overleftrightarrow{\mathbf{G}}(\mathbf{k}, \omega) \rangle = \frac{1}{k^2 - (\omega/c_0)^2 - M_t(\mathbf{k}, \omega)} \overleftrightarrow{\mathbf{P}}_k - \frac{1}{(\omega/c_0)^2 + M_l(\mathbf{k}, \omega)} \overleftrightarrow{\mathbf{K}}. \quad (3.37)$$

The dyadic mass operator is also split into the transverse and longitudinal components,

$$\overleftrightarrow{\mathbf{M}}(\mathbf{k}, \omega) = M_t(\mathbf{k}, \omega) \overleftrightarrow{\mathbf{P}}_k + M_l(\mathbf{k}, \omega) \overleftrightarrow{\mathbf{K}}. \quad (3.38)$$

For the coherent transmission of light, only the transverse propagating component is relevant. Comparing with $\overleftrightarrow{\mathbf{G}}_0$, we can identify the effective relative permittivity as [47],

$$\varepsilon_r(\mathbf{k}, \omega) = 1 + \frac{M_t(\mathbf{k}, \omega)}{(\omega_0/c_0)^2}. \quad (3.39)$$

3.4.1.2 Scalar waves

A similar Dyson equation can be written for the scalar EM wave,

$$\langle E(\mathbf{r}, \omega) \rangle = E_0(\mathbf{r}, \omega) + \iint G_0(\mathbf{r} - \mathbf{r}', \omega) M(\mathbf{r}' - \mathbf{r}'', \omega) \langle E(\mathbf{r}'', \omega) \rangle d^3\mathbf{r}' d^3\mathbf{r}''. \quad (3.40)$$

In this case, the mass operator, M , is a scalar quantity. The following relation is then found between M and ε_r ,

$$\varepsilon_r(\mathbf{k}, \omega) = 1 + \frac{M(\mathbf{k}, \omega)}{(\omega_0/c_0)^2}. \quad (3.41)$$

In this section, we have related the mass operator to ε_r of the atomic cloud. In the following section, we describe the diagrammatic expansion method used to compute the mass operator.

3.4.2 The diagrammatic expansion method

In the diagrammatic expansion method, the mass operator is expanded in a series of scattering diagrams. There are many diagrams to take into consideration. For our studies, we consider only up to diagrams that come in order of ρ_0^2 . Before we investigate the mass operator, it is helpful to introduce the various elements that appear in the scattering diagrams. They are summarized in Table 3.1.

Diagram element	Description
The open circle represents a single light scattering event by an atom.
The solid line joining two successive light scattering events represents the propagation of light between the two scattering events.
The dashed line between two scatterers in the diagram signifies positional correlation between the two atoms.
The solid line between two non-consecutive light scattering events, means that they occur at the same atom.

Table 3.1: Table summarizing the different elements in the diagrammatic expansion of the mass operator.

The mass operator contains all the scattering diagrams that are non-factorisable [47, 121]. We show this up to terms of the order $\rho_0^2\alpha_0^2$, in Equation (3.42). For each diagrammatic term, ensemble averaging is implied.

$$\begin{aligned}
\vec{\mathbf{M}} = & \text{○} + \text{○} \text{---} \text{○} \\
& + \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \dots \\
& + \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \dots
\end{aligned} \tag{3.42}$$

The first term in Equation (3.42) refers to the ISA contribution to the mass operator. This is the only term that is linear in $\rho_0\alpha_0$. The second term reflects the contribution due to the pair correlation function between two atoms. The terms in the second line of Equation (3.42), consist of purely recurrent scattering contributions. For instance, the first term of this line corresponds to the recurrent scattering process depicted in the leftmost diagram in Figure 3.1. The second term of this line corresponds to the center diagram and the third term corresponds to the rightmost diagram. Finally, the third line accounts for the possibility of positional correlation between that the atomic pair involving in the recurrent scattering.

We denote the ISA term of the mass operator by $\overleftrightarrow{\mathbf{M}}_{\text{ISA}}$ (M_{ISA}) for the case of vector (scalar) waves. The ISA mass operator term is calculated in the following sections. The remaining terms, which are proportional to $\rho_0^2\alpha_0^2$, becomes important only at a higher density (see Chapter 7).

3.4.2.1 Vector waves

In the case of the vector waves, the ISA contribution is calculated from

$$\overleftrightarrow{\mathbf{M}}_{\text{ISA}}(\mathbf{r} - \mathbf{r}', \omega) = \frac{1}{V} \sum_{i=1}^{N_a} \int \mathcal{T}_{\mathbf{r}_i}(\mathbf{r}, \mathbf{r}', \omega) d^3\mathbf{r}_i \overleftrightarrow{\mathbf{I}}. \tag{3.43}$$

This integration is performed over the volume V of the scattering medium. The single scattering event appearing in the scattering diagram of the ISA term in Equation (3.42), is mathematically represented by the scattering operator $\mathcal{T}_{\mathbf{r}_i}(\mathbf{r}, \mathbf{r}', \omega)$. The ensemble average is performed by averaging over the positions of all the N_a atoms. We find the following expression, by performing the integration in Equation (3.43),

$$\overleftrightarrow{\mathbf{M}}_{\text{ISA}}(\mathbf{r} - \mathbf{r}', \omega) = \frac{N_a}{V} \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \delta(\mathbf{r} - \mathbf{r}') \overleftrightarrow{\mathbf{I}}. \tag{3.44}$$

In the $\mathbf{k}$ -space, this becomes

$$\overleftrightarrow{\mathbf{M}}_{\text{ISA}}(\omega) = \rho_0 \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \overleftrightarrow{\mathbf{I}}, \quad (3.45)$$

where the atomic density is defined by $\rho_0 = N_a/V$.

As discussed previously, only the transverse component of the ISA mass operator, $M_{t,\text{ISA}}$, is relevant for the coherent transmission. In the ISA regime, we have

$$M_{t,\text{ISA}}(\omega) = \rho_0 \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2. \quad (3.46)$$

3.4.2.2 Scalar waves

A similar equation for M_{ISA} can also be written for the scalar EM waves,

$$M_{\text{ISA}}(\mathbf{r} - \mathbf{r}', \omega) = \frac{1}{V} \sum_{i=1}^{N_a} \int \mathcal{T}_{\mathbf{r}_i}(\mathbf{r}, \mathbf{r}', \omega) d^3 \mathbf{r}_i. \quad (3.47)$$

The mass operator in the $\mathbf{k}$ -space is found to be

$$M_{\text{ISA}}(\omega) = \rho_0 \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2. \quad (3.48)$$

This is the same expression as $M_{t,\text{ISA}}$.

Hence, for both the scalar and vector EM waves in the ISA regime, we find

$$\varepsilon_r(\omega) = 1 + \rho_0 \alpha(\omega). \quad (3.49)$$

Note that the polarizability in the case of the vector waves is 1.5 times larger than in the case of the scalar waves (see Section 3.1).

The linear dependence on $\rho_0 \alpha_0$ or equivalently on $1/k_0 \ell_0$, can be made explicit as follows,

$$\varepsilon_r(\omega) = 1 + \rho_0 \alpha_0 \tilde{\alpha}(\omega) = 1 + \frac{\tilde{\alpha}(\omega)}{k_0 \ell_0}, \quad (3.50)$$

with $\tilde{\alpha}(\omega)$ given by

$$\tilde{\alpha}(\omega) = -\frac{\Gamma/2}{\omega - \omega_0 + i\Gamma/2}. \quad (3.51)$$

3.4.3 Effect of Doppler shift on the coherent transmission

So far, we have assumed that the atoms are stationary. However, this approximation does not hold for cold ^{88}Sr clouds that are studied in the experiment. Thermal motion of the atoms affects the coherent transmission of light. Consider an atom moving with a velocity component v along the propagation direction of light, the Doppler effect leads to a shift in the frequency by kv . In the ISA regime, the atoms scatter light independently. Hence, the average polarizability of an atom in the cloud is given by an average over the velocity distribution of the atoms. The velocity distribution of an atom is given by the following Gaussian distribution,

$$g_{\bar{v}}(\mathbf{v}) = \frac{1}{(\sqrt{2\pi}\bar{v})^3} \exp\left[-\frac{|\mathbf{v}|^2}{2\bar{v}^2}\right]. \quad (3.52)$$

The root mean square (rms) of the velocity component along each direction, $\bar{v}$, is related to the cloud temperature, T , through $\bar{v} = \sqrt{k_B T/m_a}$. Here, k_B is the Boltzmann constant, and m_a is the mass of an atom. In the Cartesian coordinate, the velocity distribution can be separated into

$$g_{\bar{v}}(\mathbf{v}) = f(v_x)f(v_y)f(v_z), \quad (3.53)$$

where

$$f(v_\nu) = \frac{1}{\sqrt{2\pi}\bar{v}} \exp\left[-\frac{v_\nu^2}{2\bar{v}^2}\right], \quad (3.54)$$

and v_ν is the velocity component along ν -direction, with $\nu = x, y, z$. The velocity averaged polarizability is given by

$$\alpha_{\bar{v}}(\omega) = \int_{-\infty}^{\infty} \alpha(\omega - kv)f(v)dv. \quad (3.55)$$

This average is carried out only for the component of the velocity along the propagation direction.

Similarly, ε_r for a thermal atomic cloud can also be calculated by averaging over the different velocity classes of atoms in the cold cloud. Therefore, we have

$$\varepsilon_r(\omega) = 1 + \rho_0 \alpha_{\bar{v}}(\omega), \quad (3.56)$$

consistent with the result of Pierrat *et al.* [109].

In the next section, the optical thickness of the atomic medium, and the phase shift experienced by the coherent field upon transmission through the medium is discussed. We start by describing the steady state coherent transmission, before

Figure 3.2: The transmission of light through a slab of thickness L . The transmitted field, E_t , is typically defined at $z = L$, which is the out-going surface of the slab. The incident field is physically defined at the in-coming surface of the slab, i.e., $z = 0$. In this thesis, we define a hypothetical incident field at $z = L$, when the slab is absent. The red curve depicts the exponential decay of light intensity, as described by the Beer-Lambert law.

moving on to describe the coherent transmission in the transient regime. The latter is useful for the studies reported in Chapters 5 and 6.

3.5 Coherent transmission in the ISA regime

For the coherent transmission light at zero magnetic field which is considered here, the polarization is preserved upon transmission through the medium. Thus, the polarization can be ignored, allowing us to deal only with the scalar part of the field. Therefore, we can discuss both the cases of the vector and scalar waves together in the following².

To simplify our treatment, the medium is assumed to be a slab geometry with a thickness L and atomic density ρ_0 (see Figure 3.2). The incident beam is assumed to be a plane wave with an intensity much lower than the saturation intensity (linear response regime).

3.5.1 Steady state coherent transmission

The effective refractive index of a dilute cold atomic cloud, at a probe frequency of ω , is given by,

$$n_r(\omega) = 1 + \frac{\rho_0 \alpha_{\bar{v}}(\omega)}{2}. \quad (3.57)$$

²Note that the atomic polarizability is different between the two cases.

The Doppler broadened polarizability of the atom, $\alpha_{\bar{v}}$, can be computed with the help of Ref. [122], to give

$$\alpha_{\bar{v}}(\omega) = \int_{-\infty}^{\infty} \alpha(\omega - kv) f(v) dv = \frac{3i\pi c_0^3}{\omega_0^3} \sqrt{\frac{\pi}{2}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right]. \quad (3.58)$$

The Faddeeva function, w , appears due to the convolution between the Lorentzian profile of the atomic polarizability, and the Gaussian velocity distribution (see Appendix B and Ref. [122] for more details). The effective refractive index is then given by

$$n_r(\omega) = 1 + \frac{3i\rho_0\pi c_0^3}{\omega_0^3} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right], \quad (3.59)$$

Using Equation (B.1), the zero temperature limit ($k\bar{v} \rightarrow 0$) of Equation (3.59) gives

$$\frac{1}{k\bar{v}} w \left[\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] \approx i\sqrt{\frac{2}{\pi}} \frac{1}{\omega - \omega_0 + i\Gamma/2}, \quad (3.60)$$

and we recover the refractive index for an atomic medium at zero temperature,

$$n_r(\omega) = 1 - \frac{3\rho_0\pi c_0^3}{2\omega_0^3} \frac{1}{\omega - \omega_0 + i\Gamma/2}. \quad (3.61)$$

In the steady state transmission of light, the transmitted field is given by

$$E_t(t, L) = E_0(t, 0) \exp \left[\frac{i\omega L}{c_0} \right] \exp \left[\frac{i\rho_0\omega\alpha_{\bar{v}}(\omega)L}{2c_0} \right]. \quad (3.62)$$

The steady state transmitted field is calculated at the out-going surface ($z = L$) of the slab, while the incident field is given at the in-coming surface ($z = 0$) of the slab. In our studies, it is useful to consider both the incident and the transmitted fields at the same location on the out-going surface of the slab. The incident field at the outgoing surface is defined assuming the absence of the slab. In terms of the incident field at $z = 0$, it is given by $E_0(t, L) = E_0(t, 0) \exp(i\omega L/c)$. We have now

$$E_t(t, L) = E_0(t, L) \exp \left[\frac{i\rho_0\omega\alpha_{\bar{v}}(\omega)L}{2c_0} \right] \equiv E_0(t, L) \exp \left[-\frac{b_{\bar{v}}(\delta)}{2} + i\phi_{\bar{v}}(\delta) \right], \quad (3.63)$$

where the steady state transmitted field is written in terms of the slab optical thickness, $b_{\bar{v}}(\delta)$, and the phase shift, $\phi_{\bar{v}}(\delta)$, experienced by the coherently transmitted field. This phase shift is measured between E_t and E_0 on the out-going surface.

Figure 3.3: The Voigt profile of the optical thickness $b_{\bar{v}}(\delta)$ calculated at $b_0(0) = 118$ and $k\bar{v}/\Gamma = 3.4$, using Equation (3.65). The lineshape is Doppler broadened due to the thermal motion of the atoms.

We also define an important parameter $b_0(0)$,

$$b_0(0) = \frac{6\pi\rho_0 L c_0^2}{\omega_0^2} = \rho_0 \sigma_s(0) L, \quad (3.64)$$

which is the resonant optical thickness of the atomic medium measured at zero temperature. This quantity is proportional to the product $\rho_0 L$. The proportionality factor is given by the resonant scattering cross section of an atom, $\sigma_s(0) = 6\pi c_0^2/\omega_0^2 = 3\lambda_0^2/2\pi$ [see Equation (3.2)]. Here, λ_0 is the resonant wavelength of the transition. The same atomic cloud has a same value of $b_0(0)$, regardless of its temperature.

The optical thickness and the phase shift can be written in terms of $b_0(0)$,

$$b_{\bar{v}}(\delta) = b_0(0) \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \operatorname{Re} \left\{ w \left(\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right\}, \quad (3.65)$$

$$\phi_{\bar{v}}(\delta) = -\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \operatorname{Im} \left\{ w \left(\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right\}. \quad (3.66)$$

In Equation (3.65), we have the expression for the Voigt profile of the optical thickness due to Doppler broadening (see Figure 3.3). In Equation (3.66), the dispersive behavior modified by Doppler broadening is found (see Figure 3.4). The above expression for the Doppler broadened $b_{\bar{v}}(\delta)$, can be seen as resulting from the reduction of the scattering cross section, $\sigma_s(\delta)$ in Equation (3.2), resulting from an average over the thermal motion of the atoms.

Figure 3.4: The phase shift $\phi_{\bar{v}}(\delta)$ calculated at $b_0(0) = 118$ and $k\bar{v}/\Gamma = 3.4$, using Equation (3.66).

At resonance, we recover the following expression,

$$b_{\bar{v}}(0) = b_0(0)g(k\bar{v}/\Gamma), \quad (3.67)$$

which relates the resonant optical thickness of the medium, $b_{\bar{v}}(0)$, to $b_0(0)$, through a real function $g(x)$ that is defined in Refs. [28, 32, 33] [see also Equation (B.7) in the appendix]. The function $g(x)$ is defined by

$$g(x) = \sqrt{\frac{\pi}{8}} \frac{1}{x} w \left[\frac{i}{\sqrt{8}x} \right] = \sqrt{\frac{\pi}{8}} \exp \left[\frac{1}{8x^2} \right] \operatorname{erfc} \left[\frac{1}{\sqrt{8}x} \right] \frac{1}{x}. \quad (3.68)$$

For $x \approx 0$, $g(x) = 1$. In the limit of $x \gg 1$, $g(x) \simeq \sqrt{\pi/8}/x$.

From Equation (3.63), the steady state coherently transmitted intensity, I_t , is related to the incident intensity, I_0 , through

$$I_t = I_0 \exp[-b_{\bar{v}}(\delta)]. \quad (3.69)$$

This is the Beer-Lambert law that describes the exponential attenuation of the transmitted light (see Figure 3.2).

3.5.2 Transient coherent transmission

So far we discussed only the steady state of coherent transmission. To consider the transient regime of coherent transmission, we focus our discussion in the frequency domain. The incident field in the frequency domain is computed using a Fourier

transform,

$$E_0(\omega, L) = \int_{-\infty}^{\infty} E_0(t, L) \exp(i\omega t) dt. \quad (3.70)$$

This allows us to interpret the incident field as consisting of various plane wave components at different values of the frequency ω . Since we are in the linear response regime, each frequency component propagates through the medium independently. Equation (3.63) then gives the transmitted field for each frequency component.

The transmitted field in the time domain is then obtained by an inverse Fourier transform

$$E_t(t, L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) \exp(-i\omega t) \times \exp \left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right] d\omega. \quad (3.71)$$

Equation (3.71) serves as the starting point to compute the time dependence of the flashes observed in our experiment.

3.5.3 Coherent transmission on an isolated $m = 0 \rightarrow m = 0$ transition

In the experiment, the $m = 0 \rightarrow m = 0$ transition is isolated by applying a large magnetic field to shift away the $m = \pm 1$ substates of the excited state. The magnetic field is applied along the polarization direction of the light, which isolates a non-degenerate two-level system on the π transition. In this situation, the theoretical development for the coherent transmission of light in the ISA regime, described in this chapter, can be applied. The experimental setup for these studies is presented in the next chapter.

Chapter 4

Experimental setup and procedure

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In this chapter, we discuss the experimental setup and the procedure used to study the coherent transmission of light in the transient regime. A detailed description of the cooling and trapping of the cold ^{88}Sr cloud is given in Appendix C and in Ref. [46]. Here, we focus more on the 689 nm probe beam and a measurement system.

The red 689 nm laser (see Section 4.1) is used to probe the narrow $^{88}\text{Sr } ^1\text{S}_0 \rightarrow ^3\text{P}_1$ intercombination line to study the transient effect of coherent transmission. We implement fast changes to the intensity and the phase of this probe field. This is done using an acousto-optic modulator (AOM) to switch on and off the probe beam, and an electro-optic modulator (EOM) to apply a phase jump on the incident field.

Since the cooperative effect among the atoms reduces the time scale of the flashes, a detection system with a high bandwidth is required. Furthermore, to remain within the linear-optical response regime of the atoms, the power of the probe beam has to be in the order of a few hundred picowatts. This sets a condition on the sensitivity of the detector, as discussed further in Section 4.2

The general experimental procedure to study the transient effects of the coherent transmission is presented in Section 4.3. Finally, in the last two sections of this chapter, i.e., Sections 4.4 and 4.5, we report the properties of the atomic cloud used in the experiments. This includes to important parameters, which are the temperature and the optical thickness of the cloud. In particular, we describe the measurement of the resonant optical thickness of the medium at zero temperature, $b_0(0)$. This quantity is important, since it serves as the cooperative parameter in the study of the transient coherent transmission effects.

4.1 689 nm probe beam

In this section, we describe the red 689 nm laser used to probe the cold atomic clouds. The full red laser system is described in Appendix C and Ref. [46]. Here, the details are given mainly for the part that is related to the probe beam.

An external cavity diode laser (ECDL) is used as the master laser to seed a slave laser, which gives the 689 nm laser radiation. The master laser is locked on an ultra low expansion Fabry-Pérot (FP) cavity. The long term drift of the FP cavity is compensated by a spectroscopy experiment performed on an atomic beam, in a reference cell. The spectroscopy experiment is performed on the magnetically insensitive $m = 0 \rightarrow m = 0$ transition of the ^{88}Sr intercombination transition (see Section 4.1.3). We measure the absorption signal along the direction perpendicular to the atomic beam. The frequency scan in the spectroscopy experiment is achieved using a double-passed AOM configuration not shown here (AOM2 in Figure C.6).

As shown in Figure 4.1, an AOM (AA Opto-electronic MT80-B30A1.5-VIS) is mounted on the path of the zero-order beam exiting from the first AOM. This AOM controls the beam for the red magneto-optical trap (MOT). When the radio frequency driver of the first AOM is switched off, all the optical power goes into the zero order beam, which enters the second AOM. This AOM acts as a switch for the probe beam (see Section 4.1.1). The -1 order beam exiting from the second AOM is used as the 689 nm probe beam. It is transported with a fiber to the optical table where the experiment is performed.

The probe beam is first attenuated from 10(1) mW to 400(40) pW using a series of neutral density (ND) filters. Three plano-convex lenses, with focal lengths $f = 5$ cm, 15 cm and 25 cm, are mounted on flipper mounts after the fiber. This gives a choice to expand the probe beam by 3 or 5 times.

The probe beam then passes through an EOM (Thorlabs EO-PM-NR-C1 Phase Modulator) and a lens before being transmitted through the atomic cloud. The EOM is used to apply phase shifts to the field, as described in Section 4.1.2.

In the experiment, the beam is finally focused by a lens to a waist of $150 \mu\text{m}$. The $1/e^2$ diameter of the beam is $300 \mu\text{m}$. For a power of 400(40) pW, intensity of the probe beam is $0.45(5) I_{\text{sat}}$ at the position of the atoms. For the intercombination line, the saturation intensity is $I_{\text{sat}} = 3 \mu\text{W}/\text{cm}^2$. The final lens is also mounted on

Figure 4.1: Schematic diagram showing the optical setup of the 689 nm probe beam. An AOM is placed in the path of the zero-order beam from the first AOM, which is AOM3 in Figure C.6. The second AOM switches on and off the probe beam. Fast switching is achieved by focusing the beam in the AOM with a $f = 20$ cm lens. With a fiber, the probe beam is brought to the optical table where the experiment is carried out. To keep the probe beam intensity below the saturation intensity of the transition, some ND filters are used to attenuate the probe beam power. The three lenses on flipper mounts provide a choice to expand the beam by either 3 or 5 times. The beam then passes through an EOM which changes the phase of the incident field. Finally, a lens focuses the beam to the position where the atomic cloud is located.

a flipper. This allows us to vary the probe beam between a plane wave illumination configure (for absorption imaging) and a focused illumination configuration (for the experiments).

4.1.1 Response time of AOM operation

To have a short response time for the switch-on and switch-off of the probe beam, a $f = 20$ cm plano-convex lens is placed before the AOM to focus the beam to a waist of ~ 100 μm . This beam is collimated after the AOM, using a plano-convex lens of $f = 30$ cm. With this arrangement, the rising time¹ of the incident probe intensity is measured to be 34 ns using an oscilloscope (LeCroy WaveJet 324A). A similar measurement is also performed for the case of probe switch off, with the falling time measured to be 45 ns.

The rising and falling time of the probe beam are typically 500 times faster than the lifetime of the $^3\text{P}_1$ excited state. Hence, the switching on and off of the probe beam can be considered to happen abruptly with respect to the single atom dynamics.

4.1.2 Response time of EOM operation

The phase shift of the incident field is achieved by applying a high voltage (HV) across the EOM. For the EOM used in our setup, a voltage of 157 V has to be applied to change the phase of the incident field by π . A HV amplifier (Thorlabs HVA200) is used to supply the high voltage. It is used to amplify an output voltage from a function generator (SIGLENT SDG1020). In Figure 4.2, we show in green the profiles of the phase changes at the falling edge and the rising edge due to the amplified output of the function generator.

For most part of the falling edge (left panel of Figure 4.2) and rising edge (inset on the right panel of Figure 4.2), the phase of the incident field changes linearly when the output voltage of the function generator is applied to the EOM. The slew rate at the falling (rising) edge over the linear region, is measured to be -2.35π rad/ μs (2.35π rad/ μs). It is limited by the response time of the HV amplifier. Furthermore, around the falling edge of the phase, there is an undershoot of the phase value before it recovers back to the steady state value.

To avoid this undershoot, and to improve on the response time, a switching circuit incorporating p- and n-channel MOSFETs, is implemented to quickly cut off the HV applied to the EOM. In our setup, this switching circuit is optimized for

¹The rising time is the duration for the intensity during probe ignition, to change from 10% to 90% of the steady state value. The falling time at probe extinction, is the duration for the probe intensity to reduce from 90% to 10% of the steady state value.

Figure 4.2: The phase profiles at the rising and falling edges of the phase jumps as applied by the EOM. On the left panel, a negative phase jump of π is shown. On the right panel, a positive phase jump of π is shown. The green curves correspond to the profiles of the phase jumps applied using the function generator. The blue curves correspond to the profiles of the phase jumps applied using the switching circuit. The switching circuit leads to an improvement in the negative phase jump profile, as shown on the left panel. However, in terms of a positive phase jump shown, only the function generator leads to a good rising profile, as shown in the inset of the right panel.

negative phase jumps. The phase jump profiles are plotted in blue in Figure 4.2. A negative change of the phase by π is achieved within a time of typically 0.3μ s, without displaying any undershoot. Unfortunately, this circuit cannot be used for positive phase jumps, as indicated by the bad rising profile shown on the right panel of Figure 4.2.

With both methods, a typical phase jump of π is usually achievable within a duration of 0.3μ s to 0.5μ s. While this duration is shorter than the lifetime of transition, it is not very short compared to the time scale of the flashes as reported in Chapter 5. Thus, a significant departure of the experimental results is expected from the theoretical predictions using the ideal instantaneous phase jumps. In this case, we can use the actual phase jump profile to numerically simulate the transmission signal, and compare it with the experimental results.

4.1.3 Frequency of the probe

As mentioned before, The frequency of the probe beam is locked on the FP cavity. The drift of the cavity is then compensated by an absorption spectroscopy at the reference cell. Despite this, it is still necessary to validate the absolute frequency

Figure 4.3: A false color image of the cold atomic cloud. The false color scale for I_t/I_0 is shown on the right. One pixel in the figure corresponds to an area of $64 \mu\text{m} \times 64 \mu\text{m}$.

of the probe beam from time to time.

To do so, absorption spectroscopy experiments are also performed on the cold atomic clouds inside the science cell. The number of atoms in the cold cloud is first reduced to improve the transmission signal at resonance². This spectroscopy is performed on the $m = 0 \rightarrow m = 0$ transition using a weak probe beam. The frequency of the red probe beam is varied. For each frequency, an absorption image of the cloud is captured. In Figure 4.3, a typical transmission image of the cloud, which is elliptical in shape, is plotted with a false color scale. Let the incident intensity be denoted by I_0 and the transmitted intensity be denoted by I_t . The figure shows the normalized transmission intensity, i.e., the transmittance, I_t/I_0 , of the cloud³.

The transmission image is fitted with the Beer-Lambert law, assuming that the cloud optical thickness has a 2D Gaussian profile. Defining x_1 and x_2 as the two coordinates along the major and minor axes of the cloud image respectively, the fit for the transmittance through the cloud is given by,

$$I_t(x_1, x_2)/I_0(x_1, x_2) = \exp \left[-b_p \exp \left(-\frac{x_1^2}{2R_1^2} - \frac{x_2^2}{2R_2^2} \right) \right], \quad (4.1)$$

²The repumper beams at 679 nm and 707 nm (see Appendix C for more details) can be turned off to reduce the loading rate of the MOT. The attenuation of the transmitted light is reduced with less atoms in the cloud. This reduces the bias that may appear due to a strong absorption.

³For each cloud, three images are taken with the EMCCD camera. The first image is the absorption image of the cloud, the second is a reference image for the probe beam, and the third one is an image of the background intensity. The exposure time for the EMCCD camera is the same for all three images. The background image is first subtracted from both the absorption image and the reference image, before the transmittance is computed.

Figure 4.4: The optical thickness spectrum measured on the atomic cloud using the 689 nm probe. The experimental data is represented by blue open circles. The resonance frequency offset and the cloud temperature are determined by fitting the experimental data to a Voigt profile (blue curve). In this example, we find an offset of 44.8 kHz in the frequency of the probe beam. This offset has to be corrected when the experiment is performed. Additionally, we find a temperature of 2.7 μ K from the fit, which corresponds to $k\bar{v}/\Gamma = 3.1$.

where R_1 and R_2 are the root mean square (rms) sizes of the cloud, and b_p is the peak value of cloud optical thickness at the center of the cloud image. The measured values of b_p are then plotted against the frequency offset of the red probe beam, giving the optical thickness spectrum as shown in Figure 4.4.

The measured optical thickness spectrum is fitted to the Voigt profile [see Equation (3.65)]. From the fitting procedure, we find the values for the resonant optical thickness $b_{\bar{v}}(0)$, the temperature of the cloud, and value of the probe frequency offset from the transition resonance. For example, in Figure 4.4, we find $b_{\bar{v}}(0) = 5.8$. The cloud temperature is measured to be 2.7 μ K. The resonance of the transition is located at a frequency offset value of -44.8 kHz. This offset value means that frequency as locked by the FP cavity, even with the correction by the reference cell spectroscopy, is 44.8 kHz away from the actual resonance of the transition.

Throughout the superflash and pulse train experiments, the offset frequencies of the probe beam resonance are measured to be within a range of -20 kHz to -50 kHz. The fluctuation in the frequency is measured to be within a range of 30 kHz. These errors are due to the Doppler effect coming from the transverse velocity component of the atomic beam. A large error here comes from the misalignment of the probe beam and its reflected beam at the reference cell spectroscopy setup. To improve

Figure 4.5: Typical fluctuations of the frequency offset values measured using the cold atomic cloud. The measurements are performed after the cat’s eye configuration have been implemented. Initially, since the experiment setup is just turned on, a larger deviation is observed. Once the experimental system is stable, the frequency fluctuations are typically between ± 5 kHz around the zero offset value.

this, a cat’s eye configuration, which retroreflects the laser beam, is introduced to the spectroscopy setup. This leads to a better alignment of the spectroscopy setup, and reduces the broadening and the frequency shift of the spectrum. With this improvement, the offset frequency typically varies in a range of ± 5 kHz around the zero offset value. Figure 4.5 shows the results of the frequency offset measurement after the cat’s eye configuration has been implemented. The experimental studies reported in Part II are performed with this improved reference cell spectroscopy. The first few data points of the resonance frequency show a larger deviation. This can be attributed to the fact that some time is required for temperature of the experimental setup to stabilize when it is first turned on for the day.

4.2 Detection system

We turn our attention now to the detection system used in the study of the transient coherent transmission effects. The detector used in the experiment is a 10 MHz bandwidth APD module (Hamamatsu Photonics C10508), operated at its maximum gain of 250. The bandwidth of the APD module is high enough to resolve the temporal behavior of flashes. It is important to note that these flashes have time scales that are shorter than the natural lifetime of the excited state, as discussed in Section 5.2.

Figure 4.6: Measurement of a 40 pW probe beam switch-off by the APD module. The signal is averaged over 1000 samples. The trigger for probe extinction is plotted in red.

Another important property of the detector is its sensitivity to the transmitted signal, which has a low optical power. The probe beam power in the experiment is 400(40) pW. For a good resolution in the intensity measurement, the photodetector should be sensitive to signals that are at least one order of magnitude smaller than the incident intensity. In Figure 4.6, we show a test of the detector sensitivity using a probe beam of 40 pW. In the figure, we note that the signal corresponding to the trigger is delayed by typically 1 μ s. With an averaging over 1000 samples, we see that the detector has a signal to noise ratio of about 1. assuming that the noise is uncorrelated, the minimum detection limit of one measurement is $40 \times \sqrt{10^3} = 1300$ pW. This corresponds to 10 times the value stated by the constructor's specification. Indeed, we observe also that the noise are uncorrelated. Nevertheless, with sufficient amount of averaging, the detector sensitivity is still suitable for our experiments.

4.3 Experimental procedure

In this section, we present the experimental procedure implemented to perform the experiments on the transient regime of coherent transmission.

After a cold atomic cloud has been cooled and trapped in the red MOT, the cooling beams are switched off. The magnetic field gradient for the MOT is also switched off. The cloud then freely expands for 10 ms, before the experiment is performed. The size of the elliptical cloud is increased to 1.5 mm $\times$ 1.0 mm, in

terms of its $4\text{-}\sigma$ diameters. The atomic cloud is larger than the size of the probe beam.

The probe beam is then aligned to pass through the center of the cloud. The choice of the cloud dimensions and the beam waist in the experiment allows us to model the coherent transmission of light through the cloud, as a propagation of a plane wave through a uniform slab of the atomic medium. The larger cloud size minimizes the spatial variation of the optical thickness in the region encountered by most of the Gaussian probe beam. The Rayleigh length of the probe beam (~ 10 cm), is also much larger than the size of the cloud. This ensures that incident light is mainly propagating along a single direction inside the atomic cloud, closely approximating a plane wave.

One has also to make sure that the cloud size is not too large, to obtain decent optical thickness value to performs the experiment. After 10 ms of ballistic expansion, the optical thickness of the cloud is $b_v(0) = 19$. As discussed in Section 4.5, this corresponds to $b_0(0) = 115$, which is sufficiently large for a significant cooperative effect to be observed.

The three-stage experimental procedure to measure the transmission signal is depicted schematically in Figure 4.7. Stage I involves the probing of the ^{88}Sr intercombination line by the 689 nm laser (intensity $I_0 = 0.45I_{\text{sat}}$). For each experimental parameter, 250 runs of the experiment are averaged to improve the signal-to-noise ratio. Note that the probe beam intensity is still very small compared to the saturation intensity. However, further reducing the power of the probe beam will increase significantly the number of experiment runs required to achieve a similar signal-to-noise ratio. This increases significantly the duration of the experiment. We seek a balance by setting the incident intensity at $0.45I_{\text{sat}}$. In fact, due to the large optical thickness of the medium, the intensity reduces rapidly as it propagates through the medium. To check the experimental condition is consistent with the theory in the linear response regime, the experimental results are compared to theoretical predictions. As reported in Chapters 5 and 6, a good agreement is achieved. This validates our choice of the probe beam intensity.

To avoid any mechanical effect on the atoms, the duration for probing, τ_p , has to be short in the experiment. It is usually between $40\text{ }\mu\text{s}$ to $130\text{ }\mu\text{s}$ ⁴, to ensure that the steady state regime of coherent transmission is achieved during the probing stage. No mechanical effect on the atomic cloud is observed in our experiment.

After this probing stage, a strong 461 nm laser beam (see Appendix C and

⁴We have $\tau_p = 40\text{ }\mu\text{s}$ for the experiments involving probe extinction and ignition. In the case of phase jump experiments, the total probing duration is $\tau_p = 80\text{ }\mu\text{s}$. The phase jump occurs at the middle of the probing stage. For the pulse generation experiments, $\tau = 130\text{ }\mu\text{s}$. The first $30\text{ }\mu\text{s}$ duration is for the steady state coherent transmission to be reached. The periodic phase jump is then applied during the final $100\text{ }\mu\text{s}$ of the probing stage.

Figure 4.7: Three stages of the experiment. The first stage (stage I), is the probing stage. Stage II is the flushing stage; atoms are removed by a strong 461 nm probe. The final stage III is carried out to record a reference signal in the absence of the atoms.

Ref. [46]) is used to push away the atoms, by shining it on the atomic cloud for a duration of 1 ms. Finally, the 689 nm probe beam is turned on again, and the incident power is recorded as the reference signal. We assume that the power of the red probe beam does not drift significantly throughout the three stages, which is typically 1 ms in duration.

During the experiment, a magnetic field of 1.4 G is applied by a pair of coils in the Helmholtz configuration. This shifts the $m = \pm 1$ substates of the $^3\text{P}_1$ excited state by ± 3 MHz. This is much larger than the $^3\text{P}_1$ excited state linewidth. By aligning the magnetic field along the linear polarization direction of the probe beam, the $m = 0 \rightarrow m = 0$ transition of the intercombination line is isolated. The multiple scattering of light is modified due to this magnetic field [123]. However, as mentioned in Section 3.5.3, this magnetic field does not affect the coherent transmission of light for a dilute medium. Hence, the expressions derived in Chapter 3 are applicable in our experiment. The large magnetic field makes sure that the small fluctuations of the stray magnetic field does not affect the coherent transmission of light through the magneto-optical effects (see Part II of this thesis).

4.4 Characterization of the cold cloud

To compare the experimental results with the theoretical models of coherent transmission, the atomic cloud has to be characterized. In particular, we measure its optical thickness, peak density, size and temperature.

A time-of-flight (TOF) experiment is performed on the atomic cloud using the $\lambda_b = 461$ nm probe beam, which probes the broad blue $^1S_0 \rightarrow ^1P_1$ transition (linewidth $\Gamma_b/2\pi = 32$ MHz). The probe beam has a detuning of $\delta_b/2\pi = 53$ MHz. Using this detuned probe, the attenuation of light by the atomic cloud is reduced, leading to a better signal-to-noise ratio of the absorption images. Two absorption images are taken at different times using the EMCCD camera⁵. The first image is taken 10 ms after the red MOT is switched off, coinciding with the time to perform the experiment. The second image is taken another 10 ms later for the temperature measurement. The cloud images are then fitted using Equation (4.1).

From the fit of a cloud image, we find the optical thickness at the center of the cloud, b_p , and the size of the cloud. After of 10 ms of TOF, the rms sizes of the clouds along the x_1 and x_2 directions are measured to be $R_1 = 380(30)$ μm and $R_2 = 240(10)$ μm , respectively. From this measurement, we can estimate the number of atoms in the cloud. The density profile of the atomic cloud has a Gaussian shape with a peak atomic density of ρ_0 at the cloud center. ρ_0 is related to b_p measured from the transmission image, through

$$b_p = \rho_0 \sigma_b(\delta_b) \sqrt{2\pi} R_3, \quad (4.2)$$

where R_3 is the rms size along the propagation direction of the probe beam, and $\sigma_b(\delta_b)$ is the scattering cross section for the blue probe beam [refer to its definition Equation (3.2)]. In considering this blue transition, the Doppler broadening effect can be neglected since it is much smaller than Γ_b . To find the cloud density, we need the value of R_3 , which is not measured. By assuming that $R_3 = R_2$, the peak density can be up to $\rho_0 = 4.6 \times 10^{11} \text{ cm}^{-3}$.

The size of the second image is analyzed using a similar method. Assuming that the cloud expands ballistically, the temperature can be measured with the two sizes of the cold cloud at the two different times. By repeating the TOF experiment several times, the cloud temperature is found to be $3.3(2)$ μK . This temperature corresponds to a Doppler width of $k\bar{v}/\Gamma = 3.4(1)\Gamma$ on the narrow intercombination line. It agrees reasonably well with the temperature extracted from the spectroscopy experiment using the 689 nm probe, which is $k\bar{v}/\Gamma = 3.1$ in Figure 4.4. Since the

⁵ The magnification of the imaging system is carefully calibrated to measure the size of the atomic cloud. The calibration is performed using a $2'' \times 2''$ positive 1957 USAF resolution target.

frequency range of the 689 nm probe is scanned only within a limited range around the resonance, the temperature determined may not be as accurate as the results from the TOF experiment. Thus, only the temperature values measured in the TOF experiments, are used in our theoretical models.

At a temperature of $3.3 \mu\text{K}$, the rms velocity of the atoms is 2 cm/s . The Doppler width of the 461 nm probe associated with this temperature is just $\bar{v}/\lambda_b = 38 \text{ kHz}$. This is almost 1000 times smaller than the linewidth and justifies the assumption to neglect the Doppler broadening effect for the blue transition.

4.5 Determination of $b_0(0)$

As mentioned before, the parameter $b_0(0)$, which is the resonant optical thickness of the cloud at zero temperature, is also the cooperative parameter in our studies. The measurement of $b_0(0)$ at 689 nm is required for the studies reported in the following two chapters.

Previously, we have described the absorption imaging of the atomic cloud using both the red and the blue probe beams. Both methods can also be used to measure the value of $b_0(0)$. Using the blue 461 nm probe to perform absorption imaging of the cold cloud, the optical thickness measured is unaffected by Doppler effect. However, it is still modified by the detuning of the blue probe beam, δ_b . Note that the scattering cross section is proportional to the square of the probe beam wavelength. Thus, to recover $b_0(0)$ at $\lambda_r = 689 \text{ nm}$ from the value at 461 nm, we use the following equation:

$$b_0(0) = b_p \left[1 + \frac{4\delta_b^2}{\Gamma_b^2} \right] \frac{\lambda_r^2}{\lambda_b^2}. \quad (4.3)$$

At the center of the cloud, we find a value of $b_0(0) = 95(5)$. We may convert this to the resonant peak optical thickness of the cloud at 689 nm using Equation (3.67) which is reproduced here,

$$b_{\bar{v}}(0) = b_0(0)g(k\bar{v}/\Gamma), \quad (4.4)$$

where $g(x) = \sqrt{\pi/8} \exp[1/8x^2] \text{erfc}[1/\sqrt{8}x] / x$. We find $b_{\bar{v}}(0) = 15(1)$ at the center of the atomic clouds.

Using the red 689 nm probe beam, we measure several optical thickness spectra of the cold cloud used in the experiment. Each spectrum is fitted to a Voigt profile, which allows the optical thickness at resonance to be measured. We find $b_{\bar{v}}(0) = 19(2)$ at the center of the cloud. Using Equation (4.4), we deduce a value of $b_0(0) = 115(10)$ at the center of the cloud.

The red and blue absorption imaging of the cold cloud leads to similar value of

Figure 4.8: The top figure shows the absorption spectrum of the atomic cloud probed on the 689 nm intercombination line, while the bottom figure shows the residual plot. The experimental data (blue circles) are fitted by a theoretical prediction of the absorption spectrum, considering a weak probe that propagates through the optically dilute atomic cloud. The fit measures an optical thickness value of $b_{\bar{v}}(0) = 19$ for the atomic cloud. As shown by the residual plot, it is mainly the data outside the two dashed lines of the absorption window that is important in the fitting procedure.

$b_0(0)$. However, the values of $b_0(0)$ reported so far is measured at the center of the cloud. Since the transient behavior of the coherent transmission is studied using an APD, we do not have the spatial resolution of the CCD camera. Therefore, the measured signal is integrated over the whole cross section of the probe beam. Using the APD, we measure directly the value of $b_0(0)$. Now, the experimental transmission signal is time resolved (an example is shown in Figure 5.1). The transmittance through the atomic cloud, I_t/I_0 , is measured after the steady state regime of coherent transmission is achieved, and plotted against δ/Γ in Figure 4.8.

Close to the resonance, the transmittance, I_t/I_0 , for an optically thick medium is small. Therefore, it is more susceptible to systematic error or bias, leading to the offset in the transmitted intensity observed in Figure 4.8. In our case, the bias originates from a background light in our setup. In particular, the incoherent background of the probe beam is typically 2% of the total intensity. This has the right magnitude to explain the transmittance offset inside the absorption window. Furthermore, the detection system may also pick up additional background light

due to the scattering of the probe beam by the optical elements in our setup. For a better fit to the theoretical prediction of the absorption spectrum, the width of the absorption window is chosen, instead, as the main criterion to be satisfied by the fit. The theoretical prediction assumes a slab geometry with a uniform optical thickness. From the fit, the resonant optical thickness is found to be $b_{\bar{v}}(0) = 19$, in agreement with the peak optical thickness determined with absorption imaging. At $k\bar{v}/\Gamma = 3.4$, we have $b_0(0) = 115$. This is the value used in our study, since the measurement method is the same as the one used to perform the superflash and pulse train experiments. The residual plot of this fit is also shown in Figure 4.8.

The error bars in Figure 4.8, are the $1\text{-}\sigma$ standard errors obtained by performing 10 measurements under the same experimental conditions. The standard deviation of the transmittance from the ten different measurements is calculated. It is then divided by $\sqrt{10}$ to compute the standard error.

To sum up, the experimental parameters of the cold atomic cloud is the following,

$$b_{\bar{v}}(0) = 19(2), \tag{4.5}$$

$$b_0(0) = 115(10), \tag{4.6}$$

$$k\bar{v}/\Gamma = 3.1(1). \tag{4.7}$$

These values are used in the theoretical modeling for comparison with the experimental results. We go into more details of the experimental results and the comparison with the theoretical predictions in the following two chapters.

Chapter 5

Superflash

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In this chapter, we report the results of our studies on the cooperative emission of superflashes. These studies are performed on the intercombination transition of ^{88}Sr . Basically, we focus on two types of changes to the incident field. They are the abrupt probe extinction, and the phase jump of the incident field. Besides generating superflashes, these two types of experiments can also measure the *amplitude* and the *phase* of the forward scattered field.

In Section 5.1, we concentrate first on the peak intensity values of the flashes. This corresponds to the forward emission from the cloud at the moment when the incident field is changed.

In Section 5.2, the temporal behavior of the flashes is studied. By defining the initial decay time of the flashes, we study the time scale for the emission of the flashes. A key result of this study is the uncovering of cooperative effects that underlies the emission of the flashes. This manifests as a reduction in the flash time scale. The cooperative parameter in our system is found to be $b_0(0)$, i.e., the resonant optical thickness at zero temperature. This is a surprising result for us, but it is actually consistent with several other studies [27, 29]. This cooperative effect among the atoms is also relevant for the generation of pulse trains, as reported Chapter 6.

5.1 Peak intensity of a superflash

We recall that the transmission signal of the 689 nm probe beam through the cold atomic cloud is measured by an avalanche photodiode (APD). A typical signal, recorded using a digital oscilloscope (Teledyne LeCroy WaveRunner 606Zi), is shown in Figure 5.1. Each recorded signal is an average over 250 measurements under the same experimental conditions.

The measured signal is shown in blue in Figure 5.1. It is possible to identify in this signal, the three stages of experimental procedure as illustrated in Figure 4.7. The transmission signal is measured in stage I of the experimental procedure. The atoms are then flushed away in stage II. The background signal is then measured. Finally, assuming that the laser power does not drift significantly during each run of the experiment, the probe beam is turned on again in stage III to measure the reference signal.

The recorded transmission signal is smoothed by performing a moving average with 40 ns window on the transmission signal. It is important keep this window shorter than the response time of the APD. The background signal is then subtracted from both the transmission and the reference signals. After removing the background signal, the transmission signal is normalized by the averaged intensity

Figure 5.1: An example of the experimental signal (blue curve) recorded by the digital oscilloscope, with an average over 250 experiment runs. The red curve is 1/1000 times the voltage of the trigger signal for the probe beam. When the trigger is at 5 V, the 689 nm probe beam is turned on. We label on the figure, next to the recorded signal, the corresponding stages of the experiment.

of the reference signal. This removes any potential error introduced by a fluctuation and long term drift of the probe beam optical power. For the rest of this thesis, we refer to this smoothed and normalized signal as the experimentally measured transmission signal.

In general, at least two flashes are observed for each transmission signal. They are related to the ignition and extinction of the probe beam. During the ignition of the probe beam, the initial flash at $t = 0$ has a peak intensity close to the incident probe intensity. This initial flash then decays within a short time. Another flash is observed at the end of this steady state regime, which is due to the extinction of the probe beam. Between the two flashes, the duration of the probing phase $\tau_p = 40 \mu\text{s}$, is longer than the natural transition lifetime of $\Gamma^{-1} = 21 \mu\text{s}$. This is enough for the steady state transmission to be established.

In the following section, we start to discuss the coherent transmission in the steady state regime, before moving on to discuss the emission of these flashes.

5.1.1 Steady state regime

The coherent transmission of light can be viewed as a superposition between the incident field, $\mathbf{E}_0$, and the forward scattered field, $\mathbf{E}_s$ [124], which occurs between

Figure 5.2: A diagrammatic representation, in the complex plane, of the interference between the steady state incident and forward scattered fields. The optical thickness of the medium is $b_{\bar{v}}(\delta)$. The phase shift experienced by the transmitted field E_t , with respect to the incident field E_0 , is denoted by $\phi_{\bar{v}}(\delta)$. The light blue shaded area indicates the values that are allowed for E_t .

the fields at the same location¹. As discussed in Section 4.3, the $m = 0 \rightarrow m = 0$ transition is isolated by applying a large magnetic field along the light polarization direction. This avoids effects due to small stray magnetic field, and ensures that the polarizations of the coherently transmitted field $\mathbf{E}_t$ and $\mathbf{E}_0$ are the same. This allows us to write the fields as scalar quantities, i.e.,

$$E_t = E_0 + E_s. \quad (5.1)$$

This superposition of the fields in the steady state regime, can be represented in the complex plane, as shown in Figure 5.2.

In the steady state regime, the optical thickness, $b_{\bar{v}}(\delta)$, and phase shift $\phi_{\bar{v}}(\delta)$, of the transmitted field is given by Equations (3.65) and (3.66). These two quantities depend on the cloud temperature, T , through $\bar{v}$, which is the rms value of the atomic velocity component along propagation direction. The transmitted field is given by

$$E_t = E_0 e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)}. \quad (5.2)$$

With Equation (5.1), the forward scattered field is found to be

$$E_s = E_0 e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - E_0 \equiv E_0 e^{i\phi_s(\delta)} \left| e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - 1 \right|. \quad (5.3)$$

The phase of E_s relative to E_0 , denoted by $\phi_s(\delta)$, takes a value within the range

¹The incident field is defined at same point where the transmitted field is measured, assuming that the medium is absent.

$[\pi/2, 3\pi/2]$. Thus, in the steady state regime, E_s always interferes destructively with E_0 . This is imposed by the fact that the energy flux of the light is conserved in the steady state regime, i.e., $I_t \leq I_0$. There is, however, no such constrain on the phase of the transmitted field; $\phi_{\bar{v}}(\delta)$ can take any value.

The steady state transmittance is measured for different probe frequency, and plotted as the absorption spectrum in Figure 4.8. An important property of the absorption spectrum is the width of its absorption window, δ_a . Inside this window, the light is strongly attenuated when transmitted through the atomic cloud. We define the absorption window to be the difference in frequency between the two detuning values (on the blue and on the red sides of the resonance) where $b_{\bar{v}}(\delta_a/2) = 1$. For an atomic cloud with a large optical thickness, it can be expected that $\delta_a \gg k\bar{v}$ and $\delta_a \gg \Gamma$. Indeed, a large detuning is necessary for the effective optical thickness to reduce to one. Here, k is the wave vector of the probe beam. At large probe detuning δ , the Doppler broadened polarizability $\alpha_{\bar{v}}$ [see Equation (3.58)] can be approximated by,

$$\alpha_{\bar{v}}(\delta) \approx -\frac{6\pi c_0^3}{\omega_0^3} \left(\frac{\Gamma}{2\delta} - i \frac{\Gamma^2}{4\delta^2} \right). \quad (5.4)$$

In the equation above, ω_0 is the resonance frequency of the transition, and c_0 is the speed of light in vacuum. For large detuning, $b_{\bar{v}}(\delta)$ and $\phi_{\bar{v}}(\delta)$ becomes

$$b_{\bar{v}}(\delta) \approx b_0(0) \frac{\Gamma^2}{4\delta^2}, \quad (5.5)$$

$$\phi_{\bar{v}}(\delta) \approx -b_0(0) \frac{\Gamma}{4\delta}. \quad (5.6)$$

Using Equation (5.5), the absorption window width of an optically thick atomic medium, is found to be

$$\delta_a = \sqrt{b_0(0)} \Gamma. \quad (5.7)$$

Interestingly, for a medium with a large optical thickness, the width of the absorption window is independent of its temperature. This is due to the behavior of the atomic response to the light, which is independent of the cloud temperature if $\delta \gg k\bar{v}$. A similar behavior is encountered again later in Section 5.2.5.2, when we discuss the time scale of the flash.

For the absorption spectrum in Figure 4.8, Equation (5.7) predicts a value of $\delta_a = 11\Gamma$, whereas, the actual absorption window measured in the experiment is 19Γ . Since the absorption window is not very large compared to $k\bar{v} = 3.4\Gamma$, the assumption that $\delta_a \gg k\bar{v}$ does not hold; Equation (5.5) only gives a rough estimation for our case here. This explains the discrepancy observed.

Figure 5.3: Forward transmission signal showing the initial flash and the flash during the probe extinction. The probe beam is at resonance with the intercombination transition. Due to the finite response time of the measurement system, the flash peak intensities are slightly lower than the incident intensity. Interestingly, the two flashes show similar temporal behavior. This is linked to the complementary between the experiments for ignition and extinction of the probe beam (see the discussion in Section 5.2.2.2).

5.1.2 During abrupt switch on and off

We consider now the emission of the flashes. The transmission signal for a resonant probe is shown in Figure 5.3. The normalized transmission signal is the blue curve plotted in Figure 5.3. The red curve is the trigger to turn on ($=1$) and turn off ($=0$) the probe beam. As expected, the abrupt ignition and extinction of the probe beam both lead to the emission of flashes.

As seen in Figure 5.3, the flash displays a complicated temporal behavior as it decays towards its steady state transmission value. In the process, some oscillations in the intensity are observed. These oscillations occur at large optical thickness, making the temporal behavior non-exponential, in contrast to optically thin atomic medium. Importantly, these oscillations are not Rabi oscillations. Indeed, since the study is performed at low probe intensity, the Rabi frequency is smaller than the linewidth, therefore, it cannot explain the oscillations. Rather, these oscillations arise from an oscillatory behavior of the transmitted field, which depends on the optical thickness and the temperature of the medium. The appearance of these oscillations can be understood in the frequency domain. Due to the abrupt ignition and extinction of the probe beam, the frequency of the incident probe beam is broadened. The different frequency components are then modified by the spectrum

of the atomic response. Consider, for instance, only the absorption spectrum of the atomic medium. At low optical thickness, the absorption spectrum has a Lorentzian shape, while at large optical thickness, it has a shape similar to a box function. As is well-known, the Fourier transform of the box function gives a $\sin x/x$ profile in the time domain. For the atoms, we have also the dispersive effect. Furthermore, the transmission signal has also to obey causality, as is evident in Figure 5.3. Nevertheless, the oscillations after the flashes resemble the oscillations of $\sin x/x$.

The time scale of a flash is short compared to the transition lifetime. This suggests that a cooperative effect is in play. The temporal behavior of a flash is studied in more details in Section 5.2. For now, the discussion is limited to the moment when the incident field is abruptly switched on or switched off, and the forward scattered field does not have time to evolve. This is when the emission of the peak intensities of the flashes is emitted.

5.1.2.1 Initial flash

Let us consider first the initial flash emitted during the abrupt ignition of the probe beam. Initially, during the abrupt ignition of the probe beam, the atoms do not scatter any light. Thus, the atomic medium is initially transparent. The leading front of the wave passes through the medium as though it is propagating through the vacuum. This gives the Sommerfeld and Brillouin precursors [67, 68], as mentioned in Chapter 2. For an optically thick medium, the precursors appears after transmitting through the medium, as an initial flash [32, 33]. An example of the initial flash observed in our experiment, is shown in Figure 5.3. In fact, this argument holds regardless of the probe detuning; the initial transmitted intensity is always equal to the incident intensity.

The experimentally measured peak intensities of the initial flashes, as shown in Figure 5.3, are slightly lower than the ideal value equal to the incident intensity. The slightly low peaks appear because the measurement system does not have enough time resolution to fully resolve the peaks. Following Ref. [32], this limitation is overcome by a linear extrapolation of the transmission signal to the time when the probe beam is abruptly switched on. We fit a 100 ns window with a linear function, 20 ns after the abrupt probe ignition. An example of the extrapolation procedure, carried out on the initial flash, is shown in Figure 5.4.

This extrapolation procedure is performed on the initial flash peak intensities for different values of δ . The results are shown in Figure 5.5 for different values of δ . Before the extrapolation, the normalized intensities (red open circles) of the flash peaks are systematically lower than the expected value of 1. After the linear extrapolation, the average of the normalized peak intensity values (blue open circles)

Figure 5.4: An example of the linear extrapolation procedure carried out for the initial flash. The peak intensity of the flash before the extrapolation is $I_t(t)/I_0 = 0.86$. The red straight line shows the linear extrapolation to the time when the abrupt ignition of the probe beam occurs (indicated by the black dashed line). The extrapolated value of the flash peak intensity is $I_t(t)/I_0 = 0.99$. The same extrapolation procedure is also carried out for the flashes that are emitted during the abrupt probe extinction.

Figure 5.5: The peak intensity of the initial flash measured with and without the extrapolation procedure. Before applying the extrapolation procedure, the peak intensity of the initial flash is systematically lower than the incident intensity. The agreement is better after the extrapolation procedure. The average value of the peak flash intensity is equal to $0.99I_0$. For selected detuning values, the error bars indicating one standard error are shown.

Figure 5.6: The optical fields in the complex plane before and after an abrupt extinction of the probe laser ($\delta = 0$). The abrupt extinction of the probe beam is assumed to occur at $t = 0$. The transmitted field, at the instant when the probe beam is turned off, is the forward scattered field just before the probe extinction.

for different values of probe detuning, is calculated to be 0.99. Thus, this linear extrapolation method allows us to compare directly the experimentally measured data with the ideal theoretical prediction.

At a few selected detuning values, we plot the error bars to indicate one standard error. This is calculated from a set of ten measurements performed under the same experimental conditions.

5.1.2.2 Flash during probe extinction: at resonance

We focus now on emission of the second flash, which occurs during probe extinction. At the moment when the probe beam is abruptly turned off, the atoms still emits the forward scattered field as if the incident field is still present. This leads to the emission of a flash of light in the forward direction.

For simplicity, the resonance case studied by Chalony *et al.*, is discussed first [32]. For an optically thick atomic cloud, the large attenuation of the transmitted light implies that $E_s \approx -E_0$ in the steady state regime. Immediately after the abrupt extinction, i.e., at time $t = 0^+$, the detector measures only an intensity that is related to the steady state forward scattered field. Since $|E_t(t = 0^+)| = |E_s(t = 0^-)| \approx |E_0(t = 0^-)|$ (see the pictorial representation of the fields in Figure 5.6), a normal flash of a peak intensity $I_t(t = 0^+) \approx I_0(t = 0^-)$ is expected in the forward direction.

The flash peak intensity measured in the experiment is also reduced by the finite resolution of the measurement setup. To overcome this limitation, the same linear extrapolation method used for the case of initial flash, can be applied to estimate the ideal flash peak intensity.

5.1.2.3 Flash during probe extinction: off-resonance case

A question arises if the atomic cloud can emit a flash with a peak intensity larger than the intensity of the probe beam. To achieve this, we require that $I_s > I_0$.

Figure 5.7: The condition to observe the maximum superflash peak intensity. Before the extinction of the probe laser, even if the medium is transparent, there is still large phase shift on the transmitted field. The transmitted field is out of phase compared to the incident field. In this situation, the forward scattered field has a magnitude of $2|E_0|$. When the probe beam is abruptly switched off, the peak flash intensity emitted is $4I_0$.

In the steady state regime, we have necessarily that $I_t \leq I_0$, or in other words, $|E_t| < |E_0|$. This region is indicated by the shaded area in Figure 5.2. However, I_s is not limited to be less than I_0 . Indeed, if we can achieve the situation shown in Figure 5.7, i.e., $E_s \sim -2E_0$ in the steady state regime, a superflash peak intensity of close to $4I_0$ will be emitted when the probe beam is abruptly turned off at $t = 0^+$. Therefore, it is interesting to check if the situation in Figure 5.7 is achievable by an atomic system.

For a probe beam with a large detuning, we find that $b_{\bar{v}}(\delta)$ varies as $1/\delta^2$, while $\phi_{\bar{v}}(\delta)$ varies as $1/\delta$ [see Equations (5.5) and (5.6)]. Thus, at large detuning values, the absorptive response of the atomic medium reduces quickly, while the dispersive action could still be significant. Hence, it is possible to achieve the condition for the maximum superflash shown in Figure 5.7, as long as the atomic cloud is optically thick at resonance.

In our experiment, a maximum superflash peak intensity of $3.1I_0$ (after linear extrapolation) is observed at $\delta = -84$ kHz (-11.2Γ) (see the blue curve in Figure 5.8). A larger superflash with peak intensity $4I_0$ is not observed, because the optical thickness of the cloud is still not large enough (see the discussion in Section 5.1.2.4).

By varying the detuning, we study the peak intensity values of the flashes. The experimental results for the flash peak intensity, $I_t(t = 0^+)/I_0$ (corrected with the linear extrapolation procedure), are plotted in Figure 5.9. Using Equation (5.3),

Figure 5.8: The probe trigger and transmission signal at $\delta = -11.2\Gamma$, showing the emission of a superflash when the incident field is abruptly switched off. The peak intensity of the superflash intensity measured directly from the transmission signal is $2.8I_0$. The linear extrapolation procedure leads to a value of $I_t(t = 0^+) = 3.1I_0$, giving a better agreement with the theoretical prediction.

the peak intensity calculated to be

$$I_t(t = 0^+)/I_0 = I_s/I_0 = |\exp(-b_{\bar{v}}/2 + i\phi_{\bar{v}}) - 1|^2. \quad (5.8)$$

The equation above can be used together with Equations (3.65) and (3.66) to compute theoretically the flash peak intensity. This is plotted in Figure 5.9 as the green solid line. We find a good agreement between the experimental results and the theoretical prediction.

The error bars in Figure 5.9 are calculated from a set of ten measurements under the same experimental conditions. For each measurement, the flash peak intensity value is measured by applying the extrapolation procedure. They are then used to compute the $1\text{-}\sigma$ standard errors, which are subsequently plotted as the error bars.

5.1.2.4 Maximal value of the superflash

In the experiment, we measure a maximum flash peak intensity of $3.1I_0$, a value that is limited by the optical thickness of the cloud. In principle, if the optical thickness is higher, the maximum peak intensity of the superflash is also higher. This can be seen from a theoretical calculation of the forward scattered field I_s/I_0 . In Figure 5.10, we plot the forward scattered intensity as a function of $|\delta|/\Gamma$ and

Figure 5.9: The peak flash intensity during the abrupt probe extinction is plotted as a function of probe detuning. The experimental data (open circles) is compared to the theoretical expression in Equation (5.8) for the forward scattered field (solid curve).

$b_{\bar{v}}(0)$. The temperature of the medium in the calculation is chosen to match the experimental condition, i.e., $k\bar{v}/\Gamma = 3.4$. At large optical thickness, there are more than one region where superflashes are emitted. This is because the phase shift of the incident field can increase beyond 2π . This means that condition for a superflash to be emitted, can be met in more than one region of detuning values. In between these superflash regions, there are also regions where the forward scattered field has a low intensity value. In these regions, we have $E_s \sim 0$. This corresponds to the case where $|E_0| \sim |E_t|$, and $\phi_{\bar{v}} \sim 0$, in the steady state regime.

To further illustrate the situation at larger optical thickness, theoretical calculations of $I_t(t=0+)/I_0$ (blue curve) and $\phi_{\bar{v}}(\delta)$ (green curve) values for a $b_{\bar{v}}(0) = 100$ cloud at $k\bar{v}/\Gamma = 3.4$ [which means $b_0(0) = 600$], are plotted in Figure 5.11. The vertical dotted lines in Figure 5.11 indicate the detuning values, outside the absorption window (shaded in pink), where E_t is out-of-phase with E_0 . As expected, the superflash regimes coincide with this condition. Furthermore, we also observe a larger number of oscillations in the peak flash intensity, as expected from Figure 5.10.

Interestingly, this oscillating behavior of $I_t(t=0+)/I_0$ values seen in Figure 5.11 is similar to the intensity oscillations in Figure 4 of Ref. [125]. In that paper, oscillations are observed for the transmission in one of the linear polarization channel, for the case of Faraday rotation. They also observed the oscillations when the probe beam is detuned outside the absorption window. Indeed, the rotation of light polarization takes a dispersive shape (see also the discussion in Chapter 10), with

Figure 5.10: The theoretical prediction of the forward scattered intensity, at $k\bar{v}/\Gamma = 3.4$. The black dashed line indicates an optical thickness of $b_{\bar{v}}(0) = 19$, which corresponds to the experimental condition. As $b_{\bar{v}}(0)$ increases, the maximum value of I_s/I_0 also increases, approaching the maximum value of 4. The red line indicates the detuning values to observe the maximum I_s , and therefore, the maximum superflash for a given optical thickness of the cloud.

rotation angles that exceed 2π . Hence, there are several conditions to have a maximum or maximum in the transmission of one of the linear polarization channel. This results in a similar kind of oscillations as seen here. Importantly, the number of oscillations in that work can be tuned by varying the applied magnetic field, while in our case, it is mainly the optical thickness that determines the number of oscillations.

Among the various superflash regions in Figure 5.11, the maximum superflash occurs when $\phi_{\bar{v}}(\delta) \approx \pm\pi$. Indeed, the peak intensity value associated with this maximum superflash increases with larger the optical thickness. Nevertheless, it cannot exceed $4I_0$. In fact, at large optical thickness, Equation (5.6) can be used to estimate the probe detuning value associated with the maximum superflash. We find $|\delta| \approx b_0(0)\Gamma/4\pi = b_{\bar{v}}(0)\Gamma/4\pi g(k\bar{v}/\Gamma)$. Similar to the absorption window width, for an atomic medium with a large optical thickness, this detuning value is independent of the cloud temperature.

At $k\bar{v}/\Gamma = 3.4$, the detuning value to observe maximum superflash is found to be $|\delta|/\Gamma \approx 0.48b_{\bar{v}}(\delta)$, which is the red line plotted in Figure 5.10. At the experimental condition of $b_{\bar{v}}(0) = 19$, this expression gives $|\delta|/\Gamma \approx 9.1$ for the maximum superflash, while experimentally it is observed at $|\delta|/\Gamma = 11.2$. This discrepancy arises because the optical thickness of the atomic cloud in the experiment is not sufficient large for the underlying assumption to hold true. Consider an optically

Figure 5.11: Theoretical prediction of the flash peak intensity, $I_t(t = 0^+)$ (in blue), and the phase of the transmitted field, $\phi_{\bar{v}}(\delta)$ (in green), at $b_{\bar{v}}(0) = 100$ and $k\bar{v}/\Gamma = 3.4$. This set of parameters corresponds to $b_0(0) = 600$. The absorption window is shaded in pink. The vertical dotted lines are the detuning values located outside the absorption window, where the out-of-phase condition between E_t and E_0 is achieved. At these detuning values, large superflashes are emitted. In addition, some small superflashes are also observed at the edge of the absorption window, since there is still non-negligible absorption there.

thicker medium with $b_{\bar{v}}(0) = 100$, we find $|\delta| \approx 48\Gamma$, which agrees well with the calculation results plotted in Figure 5.11.

At $|\delta| \approx b_0(0)\Gamma/4\pi = b_{\bar{v}}(0)\Gamma/4\pi g(k\bar{v}/\Gamma)$, the maximum superflash has a peak intensity of

$$I_t(t = 0^+)/I_0 \approx 4 \left[1 - \frac{2\pi^2}{b_0(0)} \right] = 4 \left[1 - \frac{2\pi^2 g(k\bar{v}/\Gamma)}{b_{\bar{v}}(0)} \right]. \quad (5.9)$$

Indeed, the peak intensity value of the superflash emitted during abrupt probe extinction increases at larger optical thickness, while still being bounded by $I_t(t = 0^+)/I_0 = 4$. Equation (5.9) predicts well the peak intensity of the maximum superflash plotted in Figure 5.11, finding a value of $I_t(t = 0^+)/I_0 = 3.87$. Interestingly, with a large optical thickness, the maximum flash peak intensity is unaffected by the cloud temperature.

So far, we have shown that the peak intensity of the flash measures directly the value of I_s in the steady state regime. This gives us the steady state value of $|E_s|$. In the following, we further address the question of whether a direct measurement of E_s , including its phase information, is possible using only the measurement of

Figure 5.12: A geometrical method to compute the phase $\phi_s(\delta)$. There is a phase ambiguity in this method. Both the phase values of ϕ_s and $2\pi - \phi_s$ lead to the same intensity values for I_0 , I_s and I_t .

the flash peak intensity values.

5.1.3 Reconstruction of E_s in the complex plane

In the experiment, I_0 is measured from the reference signal. The steady state value of I_t is measured from the transmission signal. Together with the measurement of I_s through the peak intensity values of the flashes, the steady state phase of E_s relative to E_0 can be computed using the cosine rule², i.e.,

$$\phi_s(\delta) = \arccos \left[\frac{I_t(\delta) - I_0 - I_s(\delta)}{2\sqrt{I_0 I_s(\delta)}} \right], \quad (5.10)$$

The cosine function results in an ambiguity in the value of ϕ_s . This is illustrated in Figure 5.12. For the same values of I_0 , I_t and I_s , Equation (5.10) results in two possible angles for the phase of E_s , namely ϕ_s and $2\pi - \phi_s = -\phi$ modulo 2π .

Using Equation (5.10), $\phi_s(\delta)$ can be calculated from the experimental data. This allows a reconstruction of E_s in the complex plane, up to the ambiguity in the sign of ϕ_s . The experimentally measured E_s is plotted as dots on the complex plane in Figure 5.13. The path that E_s traces out, when δ is varied, is shown in the figure. To choose of a value of ϕ_s for each δ value, we decide between the two ambiguous ϕ_s values, for the one having the best agreement with the theoretical prediction (solid curve). The experimental data and theoretical predictions are plotted with

²The cosine rule states that for a triangle of sides a , b , and c , and an angle A opposite the side a , the sides are related by

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Figure 5.13: The path that is traced out by E_s/E_0 in the complex plane as a function of δ . The medium has a temperature of $3.3 \mu\text{K}$ ($k\bar{v}/\Gamma = 3.4$). The theoretical curves for $b_{\bar{v}}(0) = 19$ (solid curve) and $b_{\bar{v}}(0) = 3$ (dashed curve) are plotted. The values of δ along the path is represented by the false color scale on top. The region where the normal flash occurs is shaded in light gray, while the region where superflash occurs is in white. The dark gray area is forbidden by energy conservation. The dots and stars are the experimental values of E_s . The dots are the values computed using the cosine rule of Equation (5.10). The stars are E_s reconstructed by applying abrupt phase jumps to E_0 . The error ellipses, plotted in the figure, are computed from a set of ten measurements under the same experimental condition.

a false color scale that represents the detuning of the probe beam. The agreement is rather good between the experimental data and the theory. Close to resonance, a larger discrepancy is observed. This comes from the larger systematic error in the measurement of I_t (see Figure 4.8 and the discussion in the text around that figure).

It is seen in Figure 5.13 that $|\phi_s(\delta)|$ is limited to the range $[\pi/2, 3\pi/2]$. This is consistent with the discussion in Section 5.1.1. When the probe laser beam is far detuned, $\phi_s(-\infty) \rightarrow \pi/2$ and $\phi_s(\infty) \rightarrow 3\pi/2$. Meanwhile, the values of $|E_s|$ are close to zero in these two cases. Therefore, no flash is emitted by the atomic cloud at very large detuning. As the probe detuning is decreased, $\phi_s(\delta)$ varies towards a value of π . When $\phi_s(\delta) \sim \pi$, and if the probe beam has a sufficiently large detuning (being outside of the absorption window), we find a large value for $|E_s|$, which can

go up to $2|E_0|$. This is the origin of the superflash.

In the experiment, the large optical thickness [$b_{\bar{v}}(0) = 19$], allows E_s to cross between the upper and lower planes. This is caused by the value of $|\phi_{\bar{v}}(\delta)|$ which can exceed π at large optical thickness. The larger the optical thickness, the more number of crossings are expected. This also means that a larger number of regions to observe superflash. Finally, for an optically thick medium, $|E_s/E_0|$ values tend to one inside the absorption window, leading to the emission of the normal flashes.

In Figure 5.13, the theoretical prediction for a medium with small optical thickness [$b_{\bar{v}}(0) = 3$] and $k\bar{v}/\Gamma = 3.4$ is plotted as the dashed curve. In contrast to the case of large optical thickness, the phase of E_s approaches $\phi_s = \pi$ from the red and blue sides of the resonance, without crossing over to the opposite half plane. Furthermore, $|E_s| < 1$ for this optically thin medium. Indeed, the superflash is emitted only when the optical thickness of the atomic cloud is well above 1.

The reconstruction of E_s , as described in this section, has a limitation due to the ambiguity in the sign of $\phi_s(\delta)$. This ambiguity can be removed, by considering a phase jump experiment. Furthermore, the abrupt phase jump in E_0 can also lead to the emission of a stronger superflash. In the following section, the superflash emitted during a change in the phase of E_0 is investigated. After this, the measurement of ϕ_s to reconstruct E_s in the complex plane, is revisited in the context of abrupt phase jump experiments.

5.1.4 Abrupt phase jump experiment

During an abrupt phase jump of E_0 , a new interference condition between E_0 and E_s is established, albeit for a short while. This leads to an emission of a flash by the atomic cloud. An example of the transmission signal due to an abrupt phase jump is shown as the blue curve in Figure 5.14 for $\delta = 10\Gamma$. This detuning value is close to the occurrence of maximum superflash during the probe extinction, i.e., at $\delta = 11.2\Gamma$. A phase jump of $\Delta\phi = -1.02\pi$ is then applied 40 μs after the probe beam is switched on (at $t = 0$). This ensures that the steady state coherent transmission is achieved before the phase jump. After the phase jump, we wait a further 40 μs for the steady state coherent transmission to be achieved, before switching off the probe beam. A superflash is also observed to be emitted during probe extinction. The high voltage (HV) that is applied to the EOM is shown as the red curve. The EOM switching circuit (see Section 4.1.2), which has a slightly shorter response time for negative phase jump, is used in this study.

Interestingly, the superflash peak intensity in Figure 5.14 has a value of $5.5I_0$. This is even before applying any linear extrapolation procedure to the peak. This is much larger than the upper limit for the peak intensity of the superflashes emitted

Figure 5.14: A typical normalized transmission signal (blue curve) measured in the abrupt phase jump experiment. The voltage applied to the EOM is shown in red. The phase jump of $\Delta\phi = -1.02\pi$ is applied in the middle of the 80 μs long probing phase. The green curve is a numerical calculation of the flash temporal profile (see Section 5.2.4), using the EOM voltage measured in experiment. It agrees very well with the measured transmission signal. The inset shows a zoom around the flash peak during the phase jump.

during the abrupt probe extinction. In the following, we discuss the maximum superflash that can be observed in the phase jump experiments.

5.1.4.1 Maximum superflash emitted in the phase jump experiment

Let us assume that the probe beam has a detuning that allows the maximum superflash to be emitted during the probe extinction, as discussed in Section 5.1.2.4. We then have $\phi_s \sim \pi$ and $|E_s/E_0| \sim 2$ in the steady state regime. For a constructive interference after the phase jump, a change of π has to be applied. This leads to a transmitted field with a magnitude of $|E_s| + |E_0| = 3|E_0|$ after the phase jump. Therefore, a maximum superflash intensity of $9I_0$ is achievable when an abrupt phase jump is applied to the incident field.

In our experiment, a lower peak intensity of the superflash is observed, since we are limited by the optical thickness of the atomic cloud. Furthermore, the response time of the EOM is not very short compared to the decay time of the superflashes. This reduces further the measured peak intensity values of the flashes.

5.1.4.2 Antiflash

In addition to the superflash, an antiflash, which appears as a dip in the transmission signal, can also be emitted during a phase jump experiment. To understand this,

Figure 5.15: The superflash and antifiash emitted by an abrupt phase jump of $\Delta\phi = -0.4\pi$. Experimental results are shown for the probe detuning values of (a) -19.3Γ , and (b) 20.7Γ . At $\delta = -19.3\Gamma$, an antifiash is observed when the phase jump is applied. On the other hand, a superflash is observed at $\delta = 20.7\Gamma$. The insets in the figure show the optical fields in the complex plane before and during the abrupt phase jump.

we have to investigate the new interference condition between E_s and E_0 that is established during the phase jump at $t = 0$.

Making use of Equation (5.3), The transmitted field during an abrupt phase jump of $\Delta\phi$ is given by

$$E_t(t = 0^+) = E_0 e^{i\phi_s(\delta)} |e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - 1| + E_0 \exp(i\Delta\phi). \quad (5.11)$$

The transmitted intensity when the phase jump is applied, is given by

$$I_t(t = 0^+)/I_0 = |e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - 1|^2 + 1 + 2 |e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - 1| \cos[\Delta\phi - \phi_s(\delta)]. \quad (5.12)$$

As seen in the equation above, it is possible to have either constructive or the destructive interferences just after the phase jump. An antifiash is emitted when a stronger destructive interference, compared to the one in the steady state regime, occurs.

In Figure 5.15, we show the superflash (with $\delta = 20.7\Gamma$) and the antifiash (with $\delta = -19.3\Gamma$) that are observed experimentally, for the same phase jump of $\Delta\phi = -0.4\pi$. The interference conditions between E_0 and E_s , before and after the phase jump, are shown in the insets. For the case of $\delta = -19.3\Gamma$, a strong destructive interference between E_s and E_0 occurs after the phase jump, leading to an emission of the antifiash. At $\delta = 20.7\Gamma$, the two fields interfere constructively to

Figure 5.16: Measurement of ϕ_s with the abrupt phase jump experiment, by scanning the value of $\Delta\phi$. The transmitted intensity at $t = 0^+$, obtained with the linear extrapolation method, are plotted against $\Delta\phi$. They are compared to the theoretical curves at $\delta = 20.3\Gamma$ (blue solid line) and at $\delta = -19.7\Gamma$ (green dashed line), respectively. The black dotted lines are the fitted curves used to measure ϕ_s (see text for more details).

result in the superflash.

Between the two cases in Figure 5.15, the behavior of the emission during phase jump is qualitatively very different, even if the two detuning values are similar in magnitude. In the context of the reconstruction of E_s , the abrupt phase jump provides an additional information to determine ϕ_s unambiguously.

5.1.4.3 Measurement of ϕ_s with abrupt phase jumps

Equation (5.12) implies that $I_t(t = 0^+)$ is maximum when $\Delta\phi = \phi_s$, and $I_t(t = 0^+)$ is minimum when $\phi_s = \Delta\phi \pm \pi$. This suggests an alternative way to measure ϕ_s . The values of $\Delta\phi$ are varied in the phase jump experiment, and values of $I_t(t = 0^+)$ are measured. We then locate the maximum and the minimum values of $I_t(t = 0^+)$, and deduce the values of ϕ_s from them.

The results are shown in Figure 5.16. The extrapolated values of $I_t(t = 0^+)$ are plotted for $\delta = 20.3\Gamma$ (blue open circles) and $\delta = -19.7\Gamma$ (green stars). The theoretical predictions of the two cases, at $k\bar{v}/\Gamma = 3.4$, are plotted as the blue solid line and green dashed line. Even though the extrapolated values capture the behavior of $I_t(t = 0^+)$ qualitatively, they are systematically lower than the theoretically predicted values, due to the finite response time of the EOM operation.

Nevertheless, it is still possible to measure $\Delta\phi$, by fitting the experimental data to an equation in the form of $a \cos(\Delta\phi - \phi_s) + b$, where a , b and ϕ_s are real-valued

fit parameters. The blue dotted line and the green dotted line are the fit curves at $\Delta = 20.3\Gamma$ and $\Delta = -19.7\Gamma$, respectively. From the fit at $\Delta = 20.3\Gamma$, we find a value of $\phi_s = 1.28\pi$. This is a close agreement in comparison with the theoretical prediction of $\phi_s = 1.25\pi$. For $\Delta = -19.7\Gamma$, we find $\phi_s = 0.67\pi$. This is also a close agreement with the expected value of $\phi_s = 0.76\pi$.

The values of ϕ_s can be combined with the value of $|E_s|$ determined from the flash peak intensity during abrupt probe extinction, to reconstruct the values of E_s in the complex plane. Some examples are represented as the stars in Figure 5.13, showing a good agreement with the theoretical curve.

So far, we have studied the transmitted intensity at the moment when E_0 is changed rapidly. This leads to a simple picture of the emission of the superflashes, by considering the optical fields in the complex plane. In addition, the study also leads to two methods to reconstruct E_s in the steady state, which we demonstrate in our study. However, a complete understanding of the superflash emission also requires a study of its temporal behavior. This is a topic for the next section of this chapter.

5.2 Temporal behavior

In general, the temporal behavior of a flash is non-exponential, and difficult to be described analytically (see the examples in Figures 5.3 and 5.8). The decay profile of a flash can be exponential only for an optically thin medium at zero temperature. The Doppler broadening effect due to the temperature of the medium further complicates the calculation of the flash temporal behavior. In most cases, one has to resort to numerical simulations to calculate the complete temporal behavior.

Numerical simulation results, however, lacks physical insight. Here, we define a quantity called the initial decay time of the flashes, which can be computed analytically. It offers interesting insights on the origin of the short decay time for the flashes. To compute this quantity, we assume that the steady state coherent transmission of light is reached before any abrupt change is applied to E_0 . In cases where the steady state regime is not achieved, the only way is to numerically calculate the time evolution of the transmission signal.

Some simplifications of the problem is required to treat the temporal evolution of the flashes. We consider a medium with a geometry of a slab with thickness L and uniform atomic density ρ_0 . The incident wave is a plane wave of frequency ω_l and wave vector $k = \omega_l/c_0$. We assume that the abrupt change in E_0 occurs only once at $t = 0$.

Our study of the temporal behavior of the flash begins with the formula for the

transmitted field, E_t , as defined in Equation (3.71). This equation is repeated here for convenience.

$$E_t(t, L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) \exp(-i\omega t) \times \exp\left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w\left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}}\right)\right] d\omega. \quad (5.13)$$

The parameter $b_0(0)$ is defined in Equation (3.64). We recall that the w function is the Faddeeva function defined in Appendix B. $E_0(\omega, L)$ is the incident field in the frequency domain located on the out-going surface. In the next section, we introduce the incident field $E_0(t, L)$ for all the three cases that are considered in our studies. $E_0(\omega, L)$ is computed by performing the Fourier transform on $E_0(t, L)$.

5.2.1 Incident field

Using a unit step function $U(t)$, the incident field for an abrupt probe ignition can be written ideally as,

$$E_0(t, L) = E_0 U(t) \exp(-i\omega_l t), \quad (5.14)$$

where

$$U(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases} \quad (5.15)$$

For the case of abrupt probe extinction, we have

$$E_0(t, L) = E_0 U(-t) \exp(-i\omega_l t). \quad (5.16)$$

For a phase jump of $\Delta\phi$, the incident field is given by

$$E_0(t, L) = \begin{cases} E_0 \exp(-i\omega_l t), & t < 0, \\ E_0 \exp(-i\omega_l t + i\Delta\phi), & t \geq 0, \end{cases} \quad (5.17)$$

which in terms of the step function $U(t)$, can be written as

$$E_0(t, L) = [E_0 \exp(i\Delta\phi)U(t) + E_0 U(-t)] \exp(-i\omega_l t). \quad (5.18)$$

By performing the following Fourier transform,

$$E_0(\omega, L) = \int_{-\infty}^{\infty} E_0(t, L) \exp(i\omega t) dt, \quad (5.19)$$

the incident field in the frequency domain for all three cases can be summarized by

	Abrupt Ignition	Abrupt Extinction	Abrupt Phase Jump of $\Delta\phi$
ξ	1	-1	$\exp(i\Delta\phi) - 1$
η	1	1	$\exp(i\Delta\phi) + 1$

Table 5.1: The values of ξ and η for the three cases: abrupt switch on, abrupt switch off and abrupt phase change by $\Delta\phi$.

the following equation:

$$E_0(\omega, L) = i\xi E_0 \text{PV} \frac{1}{\omega - \omega_l} + \eta\pi E_0 \delta(\omega - \omega_l). \quad (5.20)$$

The values taken by ξ and η for all the cases are summarized in Table 5.1. The PV in one of the terms, means that inside an integral over ω , the principal value integration has to be performed for that term.

In the following, we use Equation (5.13) to study the temporal behavior of the flashes. We see that Equation (5.13) can be integrated analytically only in the limit of small optical thickness. We then identify the transient parts of the transmitted field and the intensity, which are associated with the flashes.

5.2.2 Small optical thickness

We assume that $b_0(0) \ll 1$ for a medium of small optical thickness, and linearize the exponential function in Equation (5.13) to have

$$E_t(t, L) = E_0(t, L) - \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) w \left[\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2k\bar{v}}} \right] \exp(-i\omega t) d\omega. \quad (5.21)$$

Following the calculation outlined in Section D.1.1, the transmitted field is found to be

$$E_t(t, L) = E_0(t, L) - \frac{E_0 b_0(0)}{2} \exp(-i\omega_l t) \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \times \begin{cases} \frac{(\eta - \xi)}{2} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right], & t < 0, \\ \frac{(\eta + \xi)}{2} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right] - \xi e^{i\delta t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right], & t \geq 0. \end{cases} \quad (5.22)$$

At $t = 0^+$, the transmitted field is given by

$$E_t(0^+, L) = E_0(0^+, L) - \frac{(\eta - \xi)}{2} E_0 \exp(-i\omega_l t) \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right]. \quad (5.23)$$

From Equation (5.3), the steady state value of the forward scattered field, up to

the linear order in $b_0(0)$, is given by,

$$E_s^{\text{st}} = -\frac{E_0 b_0(0)}{2} \exp(-i\omega_l t) \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right]. \quad (5.24)$$

Hence, Equation (5.23) can be rewritten as

$$E_t(0^+, L) = E_0(0^+, L) + \frac{(\eta - \xi)}{2} E_s^{\text{st}}. \quad (5.25)$$

As expected, in the case of abrupt probe ignition, $E_t(0^+, L) = E_0(0^+, L)$. For both the cases of abrupt probe extinction and abrupt phase jump, $\eta - \xi = 2$. Thus, we have $E_t(0^+, L) = E_0(0^+, L) + E_s^{\text{st}}$. This agrees with the picture of the interference of the fields, as discussed in Section 5.1.1.

The transmitted field can also be understood by separating it into the transient and the steady state parts at $t > 0$. We denote the steady state part of the transmitted field at $t > 0$ as $E_t^{\text{st}}(t, L)$. It is given by

$$E_t^{\text{st}}(t, L) = E_0(t, L) - \frac{(\eta + \xi)}{2} E_0 e^{-i\omega_l t} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right]. \quad (5.26)$$

The time dependence in $E_t^{\text{st}}(t, L)$ is solely due to its oscillation at the optical frequency. The transient part of the field, $E^{\text{tr}}(t)$, is then identified as the difference between $E_t(t, L)$ and $E_t^{\text{st}}(t, L)$, i.e.,

$$\begin{aligned} E^{\text{tr}}(t) &= E_t(t, L) - E_t^{\text{st}}(t, L) \\ &= \xi E_0 e^{-i\omega_l t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right], \end{aligned} \quad (5.27)$$

for $t > 0$. At zero temperature ($k\bar{v}/\Gamma = 0$), the above expression reduces to a simpler form of

$$E^{\text{tr}}(t) = \xi E_0 e^{-i\omega_l t - \Gamma t/2} \frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2}. \quad (5.28)$$

The transient transmitted intensity is defined by

$$\begin{aligned} I^{\text{tr}}(t) &= I_t(t, L) - I_t(\infty, L) \\ &= |\xi|^2 |E_0|^2 e^{-\Gamma t} e^{-(k\bar{v}t)^2} \left[\frac{b_0(0)}{2} \frac{\Gamma}{k\bar{v}} \right]^2 \frac{\pi}{8} \left| w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right] \right|^2 \\ &\quad + 2e^{-\Gamma t/2} e^{-(k\bar{v}t)^2/2} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \\ &\quad \times \text{Re} \left\{ \xi E_0 e^{-i\omega_l t} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right] E_t^*(\infty, L) \right\}, \end{aligned} \quad (5.29)$$

where $I_t(\infty, L)$ is the steady state transmitted intensity at $t > 0$.

In the following, we consider individually the cases of abrupt probe extinction, abrupt probe ignition and abrupt phase jump with medium of small optical thickness.

5.2.2.1 Abrupt probe extinction

We consider first the case of abrupt probe extinction, as it has the simplest expression among the three cases. The transient optical field is given by,

$$E^{\text{tr (ex)}}(t) = -E_0 e^{-i\omega_0 t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i(\Gamma/2 + k^2 \bar{v}^2 t)}{\sqrt{2}k\bar{v}} \right]. \quad (5.30)$$

The transient part of the intensity is

$$I^{\text{tr (ex)}}(t) = |E_0|^2 e^{-\Gamma t} e^{-(k\bar{v}t)^2} \left[\frac{b_0(0)}{2} \frac{\Gamma}{k\bar{v}} \right]^2 \frac{\pi}{8} \left| w \left[\frac{\delta + i(\Gamma/2 + k^2 \bar{v}^2 t)}{\sqrt{2}k\bar{v}} \right] \right|^2. \quad (5.31)$$

Since $E_0(t > 0, L) = 0$, we have $I^{\text{tr (ex)}} = |E^{\text{tr (ex)}}|^2$.

Importantly, even if the optical thickness is small, neither the field nor the intensity takes an exponential form. The decay is exponential only when the temperature of the medium is zero, i.e.,

$$E^{\text{tr (ex)}}(t) = -E_0 e^{-i\omega_0 t - \Gamma t/2} \frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2}, \quad (5.32)$$

$$I^{\text{tr (ex)}}(t) = |E_0|^2 e^{-\Gamma t} \left| \frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right|^2 = |E_0|^2 e^{-\Gamma t} \frac{b_0(0)^2}{4} \frac{\Gamma^2/4}{\delta^2 + \Gamma^2/4}. \quad (5.33)$$

We keep only the lowest order term in $b_0(0)$ in the two equations above. As expected, the decay time of the intensity at low optical thickness is equal to the single atom natural lifetime, Γ^{-1} , of the excited state. Indeed, the cooperative effect disappears at low optical thickness; there is no shortening of the flash lifetime.

5.2.2.2 Abrupt probe ignition

The case of abrupt probe ignition is more complicated than the case of abrupt extinction, since the incident field remains at $t > 0$. The transient field is given by

$$E^{\text{tr (ig)}}(t) = E_0 e^{-i\omega_0 t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i(\Gamma/2 + k^2 \bar{v}^2 t)}{\sqrt{2}k\bar{v}} \right]. \quad (5.34)$$

This is similar to the transient field in the case of abrupt probe extinction, i.e., Equation (5.30), apart from a difference in the sign. In fact, this can be understood

Figure 5.17: An illustration of the complementary between the probe ignition and probe extinction experiments at low optical thickness. (a) The abrupt ignition of the probe beam results in an initial flash. (b) The abrupt extinction of the probe beam leads to the emission of a flash. (c) The sum of the two incident fields in (a) and (b) leads to a constant incident intensity. The sum of the transmitted fields in the two cases then results in the steady state transmitted field. This happens because the transient parts of the transmitted fields in (a) and (b) cancel out.

by noting that the abrupt extinction and ignition experiments are complementary to one another [32]. An example of the complementarity between abrupt probe ignition and abrupt probe extinction experiments is depicted in Figure 5.17, for the case of small optical thickness. In particular, the incident fields of both the abrupt probe extinction and ignition cases sum up to a field of constant amplitude. Since the medium is linear, the sum of the transmitted field from the two cases of probe extinction and ignition, should give the transmitted field in the steady state regime. This means that the transient fields in the two cases have to sum up to zero.

Furthermore, the complementary between these two cases are true in general, and is not limited to optically thin media. An interesting example for an optically thick medium is given in Figure 5.18. The probe beam is detuned by 11.2Γ to put ourselves at the frequency where the superflash is observed. As shown in the figure, the sum of the transmitted fields after the ignition and extinction of the probe ignition gives the steady state transmitted field.

Another interesting implication of the complementary principle is when it is applied to a medium with large optical thickness, using a resonant probe beam. Here, the steady state transmitted field is almost zero. This means that the transient part of the transmitted field, during the probe ignition and extinction, are almost equal up to a difference in their sign. This explains the similar temporal behavior in Figure 5.3 of the flashes observed during probe ignition and extinction.

Figure 5.18: The complementary between the probe ignition and probe extinction experiments at large optical thickness, in the case where a superflash is emitted ($\delta = 11\Gamma$). (a) The initial flash during probe ignition. (b) The superflash during probe extinction. (c) The sum of the two incident fields in (a) and (b) results in a total field with constant intensity. The correspond sum of the transmitted fields results in the steady state field, along with the steady state transmitted intensity.

The transient intensity, keeping only up to the linear term in $b_0(0)$, is given by

$$I^{\text{tr (ig)}}(t) = b_0(0)|E_0|^2 e^{-\Gamma t/2} e^{-(k\bar{v}t)^2/2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \text{Re} \left\{ e^{i\delta t} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right] \right\}. \quad (5.35)$$

Unlike the case of abrupt probe extinction, where $E_t^{\text{st}} = 0$, $E_t^{\text{st}} \neq 0$ for the case of abrupt ignition. Hence, $I^{\text{tr (ex)}} \neq |E^{\text{tr (ex)}}|^2$. Furthermore, the temporal behavior of the transient part is non-exponential, even at zero temperature, as shown by the following expressions,

$$E^{\text{tr (ig)}}(t) = E_0 e^{-i\omega_0 t - \Gamma t/2} \frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2}, \quad (5.36)$$

$$I^{\text{tr (ig)}}(t) = b_0(0)|E_0|^2 e^{-\Gamma t/2} \text{Re} \left\{ e^{i\delta t} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right\}. \quad (5.37)$$

The flash decays exponentially only when the probe beam is at resonance.

5.2.2.3 Abrupt phase jump

For the case of an abrupt jump of $\Delta\phi$ in the phase of E_0 , the transient part of the transmitted field is found to be

$$E^{\text{tr (ph)}}(t) = (e^{i\Delta\phi} - 1) E_0 e^{-i\omega_0 t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2k\bar{v}}} \right]. \quad (5.38)$$

In the above expression, when $\Delta\phi = 2\pi n$ where n is an integer, the transient field is zero as expected.

The transient part of the intensity, up to the term linear in $b_0(0)$ is

$$I^{\text{tr (ph)}}(t) = b_0(0)|E_0|^2 e^{-\Gamma t/2} e^{-(k\bar{v}t)^2/2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \text{Re} \left\{ (1 - e^{-i\Delta\phi}) e^{i\delta t} \omega \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2}k\bar{v}} \right] \right\}. \quad (5.39)$$

At zero temperature, the expression for the transient field and intensity becomes,

$$E^{\text{tr (ph)}}(t) = (e^{i\Delta\phi} - 1) E_0 e^{-i\omega_0 t - \Gamma t/2} \frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2}, \quad (5.40)$$

$$I^{\text{tr (ph)}}(t) = b_0(0)|E_0|^2 e^{-\Gamma t/2} \text{Re} \left\{ (1 - e^{-i\Delta\phi}) e^{i\delta t} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right\}. \quad (5.41)$$

Similar to the case of abrupt ignition, the decay of the flash is exponential only when we further require that $\delta = 0$.

From the discussions so far, we see that at small optical thickness, the temporal behavior of the flash is already rather complicated. In the following, we briefly discuss the temporal behavior for a cloud of large optical thickness.

5.2.3 Large optical thickness

For a medium with large optical thickness, terms with higher orders in $b_0(0)$ must be retained in Equation (5.13). For simplicity, let us consider a medium at zero temperature. The transmitted field is given by

$$E_t(t, L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) \exp \left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\omega - \omega_0 + i\Gamma/2} \right] \exp(-i\omega t) d\omega. \quad (5.42)$$

After performing the calculation as outlined in Section D.1.2 of the appendix, we find

$$E_t(t, L) = \begin{cases} \frac{(\eta - \xi)E_0}{2} \exp \left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right] e^{-i\omega_l t}, & t < 0, \\ \frac{(\eta + \xi)E_0}{2} \exp \left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right] e^{-i\omega_l t} + E^{\text{tr}}, & t \geq 0. \end{cases} \quad (5.43)$$

We find a transient field $E^{\text{tr}}(t)$ of

$$E^{\text{tr}}(t) = \xi E_0 e^{-i\omega_0 t - \Gamma t/2} \sum_{p=1}^{\infty} \left\{ \frac{1}{p!} \left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2} \right]^p \sum_{q=0}^{p-1} \frac{1}{q!} \left[\left(\frac{\Gamma}{2} - i\delta \right) t \right]^q \right\}. \quad (5.44)$$

We see that even at zero temperature, the transient field has a complicated expression.

At non-zero temperature, each $b_0(0)^p$ term after expanding the exponential in Equation (5.42), contains an additional p integrals due to Doppler averaging. Thus, it is not useful to attempt any analytical calculation of the full temporal behavior of the transmission. Instead, it is better to perform a numerical calculation of the temporal behavior of the flashes. This is described in the following.

5.2.4 Numerical calculation with FFT

For the numerical calculation, the forward scattered field, which is continuous, is calculated first. This allows us to avoid the discontinuity in the transmitted field, which is caused by the abrupt change of the incident field. In the frequency domain, the forward scattered field is given by

$$E_s(\omega, L) = E_0(\omega, L) \left\{ \exp \left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right] - 1 \right\}. \quad (5.45)$$

To perform the numerical calculation, a numerical FFT algorithm is used to calculate $E_0(\omega, L)$, given an arbitrary time dependence of the incident field $E_0(t, L)$. The forward scattered field $E_s(\omega, L)$ is then computed using Equation (5.45). Next, an inverse Fourier transform is performed numerically on $E_s(\omega, L)$ to get $E_s(t, L)$. The incident field $E_0(t, L)$ is then added to the forward scattered field $E_s(t, L)$ to compute the coherently transmitted field, $E_t(t, L)$.

The calculations are performed using MATLAB, taking advantage of the in-built `fft` and `ifft` functions. The Faddeeva function for the response of the atoms, is computed with the MATLAB implementation of the `Faddeeva_w` function from the Faddeeva package by S. G Johnson [126].

An example of the numerical calculation for the abrupt phase jump experiment is shown as the green curve in Figure 5.14. It agrees well with the experimental result. Another example for the case of superflash emission during an abrupt probe extinction is shown in Figure 5.19. An excellent agreement is also obtained between the numerical simulation result, and the transmission signal measured experimentally.

While we can correctly compute the transmission signal for the emission of the flashes, the numerical simulation does not lead to any physical insight. In the following, we overcome this limitation by considering a simple quantity that allows us to understand the shortening of the flash decay time.

Figure 5.19: A comparison between the numerical simulation of the superflash time evolution, and the experimentally measured result. The parameters used in the numerical simulation are $b_0(0) = 115$ and $k\bar{v}/\Gamma = 3.4$, consistent with the experimental parameters of $b_{\bar{v}}(0) = 19$ and $T = 3.3 \mu\text{K}$. The detuning of the probe beam is set at -11.2Γ . An excellent agreement is achieved between the numerical and experimental results.

5.2.5 Initial decay time: analytical expression

We introduce in this section the initial decay time, which can be computed analytically. It allows us to estimate the order of magnitude of the flash time scale. It also gives an interesting physical insight into the origin of the short time scale of the flash, showing the relevant quantities in play. The initial decay time of the flash, $\tau_{\bar{v}}$, is defined as

$$\tau_{\bar{v}} = \left| \frac{I_t(t = 0^+) - I_t(t = \infty)}{\frac{dI_t}{dt}(t = 0^+)} \right|. \quad (5.46)$$

This is a generalization of the definition used in Ref. [32], for the case of abrupt phase extinction. The derivation of $\tau_{\bar{v}}$ follows closely the derivation used to arrive at Equation (4) in Ref. [32]. The results for the cases of abrupt probe ignition, abrupt probe extinction and phase jump, are summarized in the following sections.

5.2.5.1 Analytical expressions of $\tau_{\bar{v}}$

An alternative but more convenient expression for the initial decay time is

$$\tau_{\bar{v}}(\delta) = \left| \frac{2}{b_0(0)\Gamma} \frac{|E_t(t = 0^+)|^2 - |E_t(t = \infty)|^2}{\text{Re}\{\xi E_0 E_t(t = 0^+)^*\}} \right|. \quad (5.47)$$

	Abrupt Ignition	Abrupt Extinction	Abrupt Phase Jump of $\Delta\phi$
$E_t(t = 0^+)$	E_0	E_s^{st}	$E_s^{\text{st}} + E_0 e^{i\Delta\phi}$
$ E_t(t = \infty) $	$ E_0 e^{-b_{\bar{v}}(\delta)/2}$	0	$ E_0 e^{-b_{\bar{v}}(\delta)/2}$

Table 5.2: The transmitted field $E_t(t = 0^+)$ at the instance of the abrupt change, and the magnitude of the field at $t = \infty$ after the change has been applied, $|E_t(t = \infty)|$, for the three different cases of interest. $E_s^{\text{st}} = E_0 [e^{-b_{\bar{v}}(\delta)/2 + i\phi_{\bar{v}}(\delta)} - 1]$ is the forward scattered field in the steady state regime.

Its equivalence to Equation (5.46) is demonstrated in Section D.2 of the appendix. The initial decay time depends on the values of $E_t(t = 0^+)$ and $|E_t(t = \infty)|$, which are summarized in Table 5.2 for all the three cases studied here.

For the case of abrupt probe ignition, the initial decay time is calculated to be,

$$\tau_{\bar{v}}(\delta) = \frac{2}{\Gamma b_0(0)} [1 - e^{-b_{\bar{v}}(\delta)}]. \quad (5.48)$$

The value of $\tau_{\bar{v}}$ in this case is bounded by $2/\Gamma b_0(0)$.

For the case of abrupt probe extinction, the initial decay time of the flash is given by

$$\tau_{\bar{v}}(\delta) = \frac{2}{\Gamma b_0(0)} \frac{1 + e^{-b_{\bar{v}}(\delta)} - 2e^{-b_{\bar{v}}(\delta)/2} \cos[\phi_{\bar{v}}(\delta)]}{1 - e^{-b_{\bar{v}}(\delta)/2} \cos[\phi_{\bar{v}}(\delta)]}. \quad (5.49)$$

At resonance, the initial decay time for a flash emitted from an optically thick cloud is $2/\Gamma b_0(0)$. When the probe beam is far detuned, we find a limiting value of $4/\Gamma b_0(0)$.

For an abrupt change of E_0 phase by $\Delta\phi$, the following expression is found,

$$\tau_{\bar{v}}(\delta) = \frac{2}{\Gamma b_0(0)} \frac{2(1 - \cos \Delta\phi) - 2e^{-b_{\bar{v}}(\delta)/2} [\cos \phi_{\bar{v}}(\delta) - \cos(\phi_{\bar{v}}(\delta) - \Delta\phi)]}{2(1 - \cos \Delta\phi) - e^{-b_{\bar{v}}(\delta)/2} [\cos \phi_{\bar{v}}(\delta) - \cos(\phi_{\bar{v}}(\delta) - \Delta\phi)]}. \quad (5.50)$$

At $\Delta\phi = \pi$, we have

$$\tau_{\bar{v}}(\delta) = \frac{2}{\Gamma b_0(0)} \frac{2 - 2e^{-b_{\bar{v}}(\delta)/2} \cos[\phi_{\bar{v}}(\delta)]}{2 - e^{-b_{\bar{v}}(\delta)/2} \cos[\phi_{\bar{v}}(\delta)]}. \quad (5.51)$$

In Figure 5.20, we plot the expressions of $\tau_{\bar{v}}$ using Equations (5.49) and (5.51), for the cases of abrupt probe extinction, and abrupt phase jump by $\Delta\phi = \pi$, respectively. The calculation is performed for the parameters $b_0(0) = 115$ and

Figure 5.20: The experimental values of the initial flash decay time, $\tau_{\bar{v}}(\delta)$, are compared to the analytical expressions, and the numerical calculations using the finite response of the AOM and EOM. Since the behavior is symmetric around the resonant frequency, we plot $\tau_{\bar{v}}$ against $|\delta|$. $\tau_{\bar{v}}$ for the case of abrupt probe extinction (blue solid curve for the analytical result, dotted curve for the numerical curve and open circles for the experimental data), and the case of abrupt phase jump (red dashed curve for the analytical result, dash-dotted curve for the numerical curve and squares for the experimental data) are plotted. The numerical calculations take into account the finite response time of the AOM and EOM.

$k\bar{v}/\Gamma = 3.4$, consistent with the experimental conditions. The vertical axis on the left shows the dimensionless quantity $\tau_{\bar{v}}(\delta)\Gamma b_0(0)$, while the axis on the right shows the rescaled quantity $\tau_{\bar{v}}(\delta)\Gamma$. Within a range of $|\delta| < 25\Gamma$, the decay time varies weakly with the detuning of the probe.

5.2.5.2 Physical interpretation of the short decay time

The physical origin of the short decay time of the flash, $\tau_{\bar{v}}(\delta)$, is discussed in this section. In general, for all cases computed in the previous section, $\tau_{\bar{v}}$ can be separated into two factors. There is a common factor of $2/\Gamma b_0(0)$ among them, which neither depends on the temperature of the atomic medium, nor on the probe detuning. The second factor, however, takes a different form depending on how the incident field is changed. It also depends on the values of $b_{\bar{v}}(\delta)$ and $\phi_{\bar{v}}(\delta)$, meaning that it is related to the details of light propagation in the medium.

The factor $2/\Gamma b_0(0)$ can be interpreted as originating from the cooperative effects among the atoms. At large optical thickness, cooperative effect among the atoms governs the time scale of the flash. This results in a well-known effect of cooperativity, i.e., the shortening of the emission lifetime. A larger value of $b_0(0)$ signifies that more atoms are involved. This leads to an even higher level of cooperativity among the atoms. Furthermore, if the effective optical thickness $b_{\bar{v}}(\delta)$ is

large, $\tau_{\bar{v}}$ for all three cases, as described by Equations (5.48) to (5.51), reduces to $\tau_{\bar{v}} = 2/\Gamma b_0(0)$. Thus, $b_0(0)$ can be identified as the cooperative parameter in our system.

Additionally, in the limit of small optical thickness, i.e. $b_0(0) \ll 1$, we expect that the cooperative effect is negligible and decay time should not be scaled by $b_0(0)$. Let us take the example of abrupt probe ignition. Using Equation (3.65), the decay time at small optical thickness is

$$\tau_{\bar{v}}(\delta) = \frac{2}{\Gamma} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \operatorname{Re} \left\{ w \left[\frac{(\delta + i\Gamma/2)}{\sqrt{2}k\bar{v}} \right] \right\}. \quad (5.52)$$

Indeed, for an optically thin medium, the decay time is not reduced by $b_0(0)$. Nevertheless, we note that it is still being modified by the Doppler broadening of the cloud. The same can be said for the other two cases. For example, for a zero temperature optically thin medium, the abrupt extinction of a resonant probe results in a flash with $\tau_{\bar{v}} = 1/\Gamma$. Again, this time scale is unmodified by $b_0(0)$, which is consistent with the fact that the cooperative effect is related to the optical thickness $b_0(0)$.

5.2.5.3 Comparison with experimental data

The experimental values of the initial decay time are plotted in Figure 5.20. Equation (5.46) is used to calculate directly $\tau_{\bar{v}}(\delta)$ from the experimental data. The blue open circles are the values calculated from the data of the abrupt probe extinction experiment described in Section 5.1.2. The two red squares are calculated from the experimental data for the phase jumps. In practice, the data from the periodic phase jumps in the pulse generation experiments (reported in Chapter 6) is used to compute the initial decay time represented by these two points.

For the case of an abrupt probe extinction, the initial flash peak, i.e., $I_t(t = 0^+)$ is measured by performing the linear extrapolation procedure on the transmission signal³. Additionally, the gradient of the linear fit gives the value of the initial intensity gradient, $dI_t/dt(t = 0^+)$. The window to perform this fit is 200 ns long, and is applied 20 ns after the extinction of the probe beam. Long after the probe beam is switched off, $I_t(t = \infty) = 0$.

For the case of phase jump, the transmission signals with long repetition time T_R are used to ensure that the steady state regime is reached before each phase jump. The linear extrapolation method is also used to find $I_t(t = 0^+)$ and $dI_t/dt(t = 0^+)$. The window used for the extrapolation procedure also has a duration of 200 ns. However, the window has a longer time delay of 570 ns, after the phase jump

³We assume that the abrupt change to the field E_0 occurs at $t = 0$.

occurs. This is necessary because the HV amplifier that drives the EOM has a longer response time compared to the AOM used to switch the probe beam. To measure $I_t(t = \infty)$, the intensity of the transmission signal is averaged over a duration of $T_R/2$, just before the next phase jump is applied.

The initial decay times calculated from the experimental result are systematically higher than the values calculated from the analytical expressions. The main error in the experimental values of the decay time, comes from the larger error in the determination of the intensity gradient. Since the response time of the AOM is finite, the actual transmission at a time $t = 0^+$ immediately after the probe extinction cannot be measured. Because of this, the extrapolation window is applied with a slight delay from the time when the abrupt changes are applied to E_0 . The slope of the intensity is, therefore, expected to be smaller in magnitude at this time. Furthermore, a larger temporal window has to be used for the extrapolation procedure here, as compared to the window used to measure the peak intensity, since the gradient is quite sensitive to the noise in the transmission signal. This further reduces the value of the slope, and leads to an overestimation of $\tau_{\bar{v}}$, as seen in Figure 5.20. The error is larger for the case of abrupt phase jump, because the response time for the phase jump is longer than the response time for the probe extinction.

To further investigate this discrepancy, the time evolution of the flash is numerically computed. We use the experimentally measured response curves of the AOM and EOM to do the calculation. The values of $\tau_{\bar{v}}$ computed from the numerical result, in the case of probe extinction, are shown as the blue dotted curve. For the case of phase jump, the numerical results are shown as the red dash-dotted curve. The agreement of these numerical results with the experimental points is excellent. This confirms our claim that the discrepancy is due to the finite response time of applying the changes to E_0 .

Finally, our studies here show that the cooperative effect is important for the emission of the flash. The cooperative effect leads to a shortening of the decay time. In the next chapter, we exploit this short decay time, to pack many flashes into a pulse train. In particular, with appropriately chosen parameters, the emission of the pulse train can be dominated by the cooperative effect. This is an effect that is not usually associated with a dilute gas of atoms in the free space.

Chapter 6

Generation of a Cooperative Pulse Train

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In this chapter, we report the generation of cooperative pulse trains when periodic phase jumps are applied to the incident field, E_0 . Our experiment still probes the ^{88}Sr intercombination line. From Section 5.2.5.2, we know that the decay time of a flash is typically $b_0(0)/2$ times smaller than Γ^{-1} , due to the cooperative effect among the atoms. The resonant optical thickness at zero temperature, denoted $b_0(0)$, is interpreted as the cooperative parameter. By packing many flashes within a short time, we address the question of whether the forward emission of the pulse train by the atoms can be fully governed by cooperative effect. This question is answered in Section 6.4 by analyzing the performance of the pulse generation under different experimental conditions.

6.1 Experimental observation of the pulse generation

We observe a pulse train that is emitted by the cold ^{88}Sr atomic cloud, when a periodic phase jump is applied to the 689 nm probe beam that probes the intercombination line of the atoms. A function generator output, in the form of a square wave, is amplified to produce the high voltage (HV) square pulses. The 100 μs long HV square pulses, with a period of $2T_R$, are applied to the electro-optic modulator (EOM) described in Section 4.1.2. Within each period, there are two phase jumps, one being $\Delta\phi$, while the other being $-\Delta\phi$. When $\Delta\phi = \pi$, the phase modulation of E_0 becomes effectively a periodic phase jump of π with a period of T_R . This is the case in our experiment, leading to the emission of pulse trains with a repetition time of T_R .

To observe these pulse trains, the experiments are performed at two detuning values of the probe beam; one at resonance ($\delta = 0$) and another off resonance ($\delta = -11.3\Gamma = -85 \text{ kHz}$, where Γ is the transition linewidth). The latter is close to the condition for the atoms to emit a maximum superflash during probe extinction (see the discussion in Section 5.1.2.4). At long T_R , these two detuning values ensure that a maximum constructive interference condition is achieved after a phase jump of π , between E_0 and the forward scattered field E_s .

We study the behavior of the pulse trains generated within a wide range of time scale. The values of T_R are varied between 50 μs and 1.25 μs , allowing us to explore the regimes where $T_R > \Gamma^{-1}$ and $T_R < \Gamma^{-1}$. For the intercombination line, the natural lifetime is $\Gamma^{-1} = 21 \mu\text{s}$. The transmission signals are shown in Figures 6.1 and 6.2, respectively, for $\delta = 0$ and $\delta = -11.3\Gamma$.

At long repetition time, i.e., $T_R > \Gamma^{-1}$, the coherent transmission evolves to the steady state regime before the next phase jump of π is applied. In this regime, we

Figure 6.1: Pulse trains generated at resonance for T_R values of $50 \mu\text{s}$, $25 \mu\text{s}$, $10 \mu\text{s}$, $5 \mu\text{s}$, $2.5 \mu\text{s}$ and $1.25 \mu\text{s}$. The phase profiles applied by EOM is drawn in red. The full transmission signals through the atomic clouds are shown in blue. Each signal includes the two flashes that are emitted during the ignition and extinction of the probe laser.

observe that the pulses generated at $\delta = -11.3\Gamma$ has a larger peak intensity value compared to those generated at $\delta = 0$. For the optically thick medium studied in the experiment, there is a strong attenuation of light at resonance. Thus, at $\delta = 0$, $E_s \sim -E_0$ in the steady state regime. The magnitude of the transmitted field after the phase jump, is ideally given by $|E_t| = 2|E_0|$. The flash peak intensity, thus, has an ideal value of $4I_0$, where I_0 is the incident intensity. On the other hand, at $\delta = -11.3\Gamma$, $|E_s| = 1.8|E_0|$ in the steady state regime. Ideally, after an abrupt

Figure 6.2: Pulse train generated at a probe detuning value of $\delta = -11.3\Gamma$ for several values of T_R as indicated on the plots. The phase shift profiles applied by EOM and the transmission signals, are shown in red and blue, respectively.

phase jump, the superflash peak has an intensity of $(1 + 1.8)^2 I_0 \approx 7.8 I_0$. Due to finite response time of the phase jumps, the actual values of the peak intensity observed in the experiment are lower than these ideal values predicted theoretically (see the discussion in Section 4.1.2).

The peak intensity is not the only interesting behavior of a pulse train. For $T_R > \Gamma^{-1}$, a large part of the emission is dominated by the steady state coherent transmission. At resonance, the steady state transmitted intensity is zero. However, this is not the case at $\delta = -11.3\Gamma$, where the steady state value of the transmitted

intensity is $0.7I_0$. The transmission does not go to zero between two pulses. This suggests that the flash peak intensity, by itself, might not be a good criteria to quantify the performance of the pulse generation. In Section 6.3, two figures of merit are introduced to quantify the performance of the pulse generation.

As T_R decreases, the time interval between two consecutive phase jumps are not large enough for the steady state regime to be established. This is seen when $T_R \leq 10 \mu\text{s}$. In these cases, the repetition time is shorter than the natural transition lifetime. Surprisingly, the pulse trains are still emitted by the atomic cloud, even at $T_R = 1.25 \mu\text{s}$ which is more than one order of magnitude shorter than Γ^{-1} . For short repetition times, the first phase jump is applied to E_0 when the system is in the steady state regime, but this is not the case subsequently. The pulse train takes a time that is typically Γ^{-1} to settle to the stationary state of pulse train emission. This behavior of the pulse train settling to its stationary emission is seen in Figure 6.2, at $T_R = 1.25 \mu\text{s}$.

At $T_R = 1.25 \mu\text{s}$, we also observe a clear asymmetry between the flashes emitted at the rising and the falling edges of the phase jump profile. In fact, this asymmetry exists for all T_R values. It is due to the difference between the rising and the falling edges of the phase profile (see for example the numerically calculated transmission signal shown in Figure 6.3). This asymmetry becomes stronger at $T_R = 1.25 \mu\text{s}$, since there is a larger asymmetry between the rising and the falling edges of the phase profile.

In the following section, we discuss a method that allows us to compute the time evolution of the pulse trains generated in our experiment.

6.2 Computation of the time evolution

In dealing with pulse trains, we encounter a situation where the phase jump is applied even if the steady state regime is not achieved. Thus, the analysis described in Section 5.2 is not valid. Furthermore, more than one phase jump is applied to generate the pulse trains. This cannot be explained by the theoretical model in Chapter 5, where only one phase jump is considered. The numerical fast Fourier transform (FFT) method described in Section 5.2.4 can still be used to study the pulse trains. Importantly, by considering that a periodic abrupt phase jump is applied to the incident field E_0 , it is possible to obtain some analytical expressions related to the transmission signal.

We first start by writing down the incident field experiencing a general phase modulation, i.e.,

$$E_0(t) = E_0 \exp[-i\omega_l t + i\phi(t)]. \quad (6.1)$$

Here, ω_l is the frequency of the probe beam, and $\phi(t)$ is a function that describes the phase modulation applied to the incident field. The following periodic modulation is considered here,

$$\phi(t) = \begin{cases} 0, & -T_R < t < 0, \\ \Delta\phi, & 0 \leq t < T_R, \end{cases} \quad (6.2)$$

with $\phi(t + 2nT_R) = \phi(t)$ for integer values of n . Since $\exp[i\phi(t)]$ is a periodic function, it can be expanded in Fourier series. The incident field has the following Fourier series expansion,

$$E_0(t) = E_0 e^{-i\omega_l t} \sum_{r=-\infty}^{\infty} \mathcal{E}_r \exp\left[-\frac{i\pi r t}{T_R}\right]. \quad (6.3)$$

The Fourier coefficients calculated to be

$$\mathcal{E}_r = \begin{cases} \frac{1}{2} [1 + e^{i\Delta\phi}], & \text{for } r = 0, \\ \frac{i}{2\pi r} [(e^{-i\pi r} - 1) - e^{i\Delta\phi}(e^{i\pi r} - 1)], & \text{otherwise.} \end{cases} \quad (6.4)$$

The r -th term, in the Fourier series expansion, corresponds to a frequency component with a detuning of $\delta_r = \delta + \pi r/T_R$.

Following Section 5.2.4, $E_s(t)$, which is continuous across the phase jump is computed¹. The following expression is found for the forward scattered field,

$$E_s(t) = E_0 e^{-i\omega_l t} \sum_{r=-\infty}^{\infty} \mathcal{E}_r \left\{ \exp\left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w\left(\frac{\delta_r + i\Gamma/2}{\sqrt{2}k\bar{v}}\right)\right] - 1 \right\} e^{-\frac{i\pi r t}{T_R}}. \quad (6.5)$$

The above equation is the discrete analog of Equation (5.45). The cloud temperature is taken into consideration by the parameter $k\bar{v}$, which is the root-mean-square (rms) Doppler shift. k is the wave vector of the probe beam and $\bar{v}$ is rms velocity component of the atoms, along probe propagation direction. The transmitted field is then computed using

$$E_t(t) = E_s(t) + E_0(t). \quad (6.6)$$

¹This avoids issue with Gibbs phenomenon for a discontinuous function.

In practice, the calculation of the Fourier series summation cannot involve all infinite number of orders of the frequency component. It has to be terminated at a sufficiently large value of $|r|$. The cut-off at upper (lower) limit is denoted by $\mathcal{K}$ ($-\mathcal{K}$). We demonstrate by calculating the sum for the case of pulse generation at $\delta = 0$, with $T_R = 2.5 \mu\text{s}$. We use a cut-off of $\mathcal{K} = 500$, which is large enough for the calculation to converge to the correct result. The calculated transmission signal, for the ideal case of periodic abrupt phase jumps, is plotted as the black dashed curve in Figure 6.3. It captures qualitatively the behavior of the experimental signal, which is shown in blue. We observe that the peak intensity of the pulses are not correctly predicted by the calculation in the ideal case. This originates from the finite response time in applying the phase jumps to the incident field. To check this point, the phase profile applied to the incident field in the experiment is recorded. It is then used in the numerical FFT calculation method described in Section 5.2.4 to compute the transmission signal. We find the red dotted curve in Figure 6.3, which is in good agreement with the experimentally measured signal. This is a confirmation that the disagreement between the experimental results, and ideal theoretical signal, is mainly due to the finite response time of the phase jumps.

So far, we have generated experimentally a number of pulse trains at different probe detuning δ and repetition time T_R . As shown in Figures 6.1 and 6.2, these pulse trains display rich behavior. To compare among the pulse trains generated at different conditions, we introduce two figures of merit in the following section.

6.3 Figures of merit

6.3.1 Definitions

The first quantity introduced to measure the performance of the pulse generation, is the temporal average of the intensity of the emitted pulse train, $\langle I_t(t) \rangle_T$. It measures the transfer efficiency of the optical power from the incident probe beam to the pulse train emitted by the atomic cloud. It is defined as

$$\langle I_t(t) \rangle_T = \frac{1}{T_R} \int_{T_R} I_t(t) dt. \quad (6.7)$$

The second quantity is the intensity contrast of the pulse train, I_c , which mea-

Figure 6.3: The transmission signal (blue curve) measured in the experiment at $\delta = 0$, $k\bar{v}/\Gamma = 3.4$, $\Delta\phi = \pi$ and $T_R = 2.5 \mu s$, is compared to the ideal transmission signal (black dashed curve) predicted theoretically, and the numerically computed transmission signal using the measured phase profile (red curve). The calculation of the ideal transmission signal using Equations (6.5) and (6.6) is performed by setting $K = 500$. The numerical calculation curve in red is calculated using the FFT method described in Section 5.2.4. In our calculations, $b_0(0) = 115$, consistent with the experimental condition. The stationary state emission of the pulse train is achieved after typically one single atom lifetime of $21 \mu s$. Even in the stationary state regime, the successive flash peaks have different heights. This is because the phase jump profiles, as shown in Figures 6.1 and 6.2, are asymmetric between the falling and the rising profiles.

sures the quality of the pulses emitted. It is defined as

$$I_c = \max_{T_R} \{I_t(t)\} - \langle I_t \rangle_{T_R}, \quad (6.8)$$

where $\max_{T_R} \{I_t(t)\}$ denotes the maximum transmitted intensity during a single period T_R of the pulse train.

For the case of ideal periodic abrupt phase jumps, the transmitted intensity $I_t(t)$ is calculated to be

$$I_t(t) = I_0 \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \mathcal{E}_r \mathcal{E}_s^* e^{-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}}} \left[w\left(\frac{\delta_r + i\Gamma/2}{\sqrt{2k\bar{v}}}\right) + \left\{ w\left(\frac{\delta_s + i\Gamma/2}{\sqrt{2k\bar{v}}}\right) \right\}^* \right] e^{-i\pi(r-s)t/T_R}. \quad (6.9)$$

The transmission signal can be integrated over a duration that is equal to two

periods of the pulse train, i.e., $2T_R$. We note that,

$$\frac{1}{2T_R} \int_{-T_R}^{T_R} e^{-i\pi(r-s)t/T_R} dt = \delta_{rs}, \quad (6.10)$$

where δ_{rs} is the Kronecker delta. Thus, the average intensity is found to be

$$\langle I_t(t) \rangle_T = I_0 \sum_{r=-\infty}^{\infty} |\mathcal{E}_r|^2 \exp \left[-b_0(0) \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \operatorname{Re} \left\{ w \left(\frac{\delta_r + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right\} \right]. \quad (6.11)$$

For a practical calculation, it can be computed with a cut-off on the summation, i.e. by summing over integer values of r with $|r| \leq \mathcal{K}$.

For the pulse contrast, the peak intensity of the transmission signal, $\max_{T_R} \{I_t(t)\}$, has to be calculated. In general, it is not easy to determine the maximum intensity during one period of the pulse train. In a case where the maximum intensity corresponds to the flash peak intensity that is emitted at $t = 0^+$ during a phase jump, the Fourier series expansion can be used to compute the $\max_{T_R} \{I_t(t)\}$. The transmitted field at $t = 0^+$ is given by

$$E_t(t = 0^+) = E_0 e^{i\Delta\phi} + E_0 \sum_{r=-\infty}^{\infty} \mathcal{E}_r \left\{ \exp \left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left(\frac{\delta_r + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right] - 1 \right\}. \quad (6.12)$$

Hence, the maximum intensity, under the assumption stated above, is

$$\begin{aligned} \max_{T_R} \{I_t(t)\} &= I_t(t = 0^+) \\ &= I_0 \left| e^{i\Delta\phi} + \sum_{r=-\infty}^{\infty} \mathcal{E}_r \left\{ \exp \left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left(\frac{\delta_r + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right] - 1 \right\} \right|^2. \end{aligned} \quad (6.13)$$

Together with Equation (6.11), I_c can be computed.

In the cases that are studied in our experiments, Equation (6.13) gives the correct predictions for the peak intensity values of the pulses. However, in general, it is possible to encounter situations where Equation (6.13) fails to calculate the correct value of the maximum intensity. Consider the pulse emitted from a medium of $b_0(0) = 5000$ at $T_R\Gamma = 0.0283$. The calculated transmission signal in the ideal case is shown as the blue curve in Figure 6.4. During the phase jump, no flash is emitted by the medium. Instead, a train of antiflashes with $I_t(t = 0^+) = 0.59I_0$ is observed. Using Equation (6.13), this leads to a negative I_c value. Indeed, the maximum intensity does not occur at $t = 0$. It occurs after the antiflash, when the

Figure 6.4: Fourier series calculations of the transmission signal at $\delta = 0$ and $k\bar{v}/\Gamma = 3.4$, for a medium with $b_0(0) = 5000$. The predicted transmission signal at $T_R\Gamma = 0.0283$ is shown in blue, showing an antflash. The transmission signal calculated at $T_R\Gamma = 0.0196$ is shown in red, with a large superflash of intensity up to $12I_0$. We set $K = 500$ in this calculations.

intensity recovers to a maximum value of $3.95I_0$.

We compare this to transmission signal at $T_R\Gamma = 0.0196$, with the same values of $b_0(0)$ and $k\bar{v}/\Gamma$. The result is shown as the red curve in Figure 6.4. In this case, a flash is seen with $\max_{T_R} \{I_t(t)\} = 12I_0$. Here, Equation (6.13) correctly calculates the maximum intensity.

Note that for $T_R\Gamma = 0.0283$, the maximum intensity of the transmission signal does not appear as sharp peak with a short decay time, unlike the one at $T_R\Gamma = 0.0196$. In fact, the maximum intensity of the pulse train with $T_R\Gamma = 0.0283$, is different in origin from the the flash emission as discussed in Section 5.2.5². Therefore, it is not the focus of the study here. Nevertheless, the pulse contrast at $T_R\Gamma = 0.0283$, as defined by Equation (6.8) can still be computed. To do so, we have to compute numerically the full transmission signal to locate the maximum intensity of the pulse train.

The high peak intensity at $T_R\Gamma = 0.0196$ is higher than the maximum value of $9I_0$ (see Section 6.4 for a discussion of this high peak intensity). This is not surprising since the maximum value of $9I_0$ is predicted for the situation where the phase jump occurs after the steady state transmission is reached. With the short repetition time here, the steady state regime of coherent transmission is not achieved

²The antflash emitted at $T_R\Gamma = 0.0283$, on the other hand, displays the same time scale as the superflash emitted at $T_R\Gamma = 0.0196$.

before the next phase jump. Thus, some part of the atomic dipole remains between the pulses, which adds up constructively with the next phase jump. In fact, we have not considered all possible phase modulation on E_0 . Perhaps, with a more general phase modulation, the peak intensity of the superflash could be larger than $12I_0$. This is a question that is left for future investigation.

In the following, we compare the theoretical calculation of the figures of merit to the experimental results.

6.3.2 Comparison with experimental data

From the transmission signals shown in Figures 6.1 and 6.2, the figures of merit can be computed using Equations (6.7) and (6.8).

In computing the pulse contrast, we have to use the peak intensity values of the flashes. Since Equations (6.11) and (6.13) are derived for the ideal case of periodic abrupt phase jumps, the linear extrapolation method is used again to measure the peak intensity of the pulses. To do the linear extrapolation, we use a 200 ns long window located at 570 ns after each phase jump. We usually have more than two pulses in the pulse trains, except when $T_R = 50 \mu\text{s}$. The average values of the peak intensity during the stationary regime of pulse emission is computed. It is then used to calculate I_c . The results are plotted as the red dots in Figures 6.5 and 6.6, for $\delta = 0$ and $\delta = -11.3\Gamma$, respectively. For $T_R < 50 \mu\text{s}$, we further plot the error bars of I_c to indicate the standard errors of the extrapolated peak intensities. The average intensity, $\langle I_t(t) \rangle_T$, can be calculated directly from the measurement signal. They are then plotted as the blue open circles in Figures 6.5 and 6.6. For both the resonant and detuned cases, a good agreement is obtained between the figures of merit calculated theoretically using Equations (6.11) and (6.13), and the values measured experimentally.

At $T_R \gg \Gamma^{-1}$, the steady state regime is always achieved before each phase jump. Thus, we expect to have a good pulse contrast at both $\delta = 0$ and $\delta = -11.3\Gamma$. At resonance, ideally we find $I_c \sim \max_{T_R} \{I_t(t)\} \sim 4I_0$, since $I_t \sim 0$ during the steady state coherent transmission. At $\delta = -11.3\Gamma$, we expect an ideal value of $I_c = 7.1I_0$. The I_c value in the detuned case, is lower than the peak intensity of the flash, since the transmitted intensity is large ($0.7I_0$) in the steady state regime. Nevertheless, the contrast is still larger compared to the resonant case.

At the other extreme, i.e., at $T_R < [b_0(0)\Gamma]^{-1}$, the modulation on the phase is faster than the decay time of the flashes. The atoms cannot keep up with the fast modulation in the phase of the incident field, therefore, they interact less with the light. This results in a larger transmission in the forward direction, but with low pulse contrast. As T_R decreases, I_c tends to zero and $\langle I_t(t) \rangle_T$ tends to I_0 . This

Figure 6.5: The theoretical predictions of the pulse contrast and the average intensity using Equations (6.11) and (6.13), compared to the experimental data ($\delta = 0$). The blue dashed curve is $\langle I_t(t) \rangle_T$, and the red solid curve is I_c . The data points (red dots for pulse contrast and blue open circle for average intensity) show good agreement with their respective theoretical predictions. The error bars for I_c show one standard error. We set $b_0(0) = 115$ and $k\bar{v}/\Gamma = 3.4$ in the theoretical calculations, consistent with the experimental condition. The arrows point to the regions where high power transfer efficiency and high pulse contrast are achieved. At $\delta = 0$, these two occur together at a repetition time that is shorter than the natural lifetime.

behavior is observed for both the resonant and off-resonant cases.

In the intermediate time scale of $[b_0(0)\Gamma]^{-1} \lesssim T_R \lesssim \Gamma^{-1}$, the phase modulation occurs faster than the lifetime of the transition. The steady state regime of coherent transmission is not achieved during the pulse train emission. In this regime, the pulse contrast oscillates between small and large values. At resonance, I_c can be larger than the one at $T_R \gg \Gamma^{-1}$ (see Figure 6.5). For the detuned case, the I_c values also oscillate, but with a general decreasing trend as T_R reduces (see Figure 6.6). In this intermediate regime, almost a full transfer of power from the incident probe beam to the forward emitted pulse train is possible, as shown by $\langle I_t(t) \rangle_T$ for both cases. At the same time, the pulse contrast I_c remains high in value. In these situations, the cooperative effect among the atoms dominates the forward emission process of the atoms.

6.4 The role of cooperative effect

In this section, we consider further the situation when the forward emission of the pulse train is governed mainly by cooperative effect among the atoms. Our discussion focuses only on the case of $\delta = 0$.

Figure 6.6: The theoretically predicted values of pulse contrast and average intensity compared to the experimental results at $(\delta = -11.3\Gamma)$. The theoretical predictions of $\langle I_t(t) \rangle_T$ and I_c are plotted as the blue dashed and red solid curves, respectively. The experimental results for pulse contrast and average intensity, are plotted as red dots and blue open circles, respectively. Each error bar of I_c , like in the resonant case, depicts one standard error. We also set $b_0(0) = 115$ and $k\bar{v}/\Gamma = 3.4$ in the theoretical calculations. The arrows point to the regions where high power transfer efficiency and high pulse contrast are achieved. Unlike resonant case, the optimal value of the two figures of merit do not occur at the same T_R region.

In the intermediate regime of $[b_0(0)\Gamma]^{-1} \lesssim T_R \lesssim \Gamma^{-1}$, we can pack many coherent flashes into a duration of T_R . This is possible since the flash time scale is reduced by cooperative effect, a factor of $1/b_0(0)$ (see Section 5.2.5.2) compared to the decay time of a single atom fluorescence event. At the same time, a high power transfer efficiency from the incident probe beam to the forward emission of the pulse train is achieved. Thus, most of the light is emitted in the forward direction. The single atom fluorescence events are quenched. This inhibition of atomic fluorescence is common in a cavity [127–129]. Here, the atomic cloud is located in the free space. It is, therefore, interesting to observe that the cooperative effect among the atoms can still dominate the emission process of the optically dilute atomic cloud.

As the cooperative parameter $b_0(0)$ increases, flashes with shorter time scale are predicted to emit in the forward direction. This allows an even higher repetition rate for the pulse trains. We compute the figures of merit for several values of $b_0(0)$, and show them in Figure 6.7. Increasing the value of $b_0(0)$ allows us to I_c high and the power transfer close to one, at a smaller value of T_R . In fact, for a the pulse contrast is also slightly higher for a medium with larger optical thickness. For $b_0(0) = 5000$, the maximum flash contrast predicted is at $I_c \approx 11I_0$. This is the superflash seen in Figure 6.4.

Figure 6.7: A comparison of the figures of merit for different optical thickness $b_0(0) = 115$, 1000 and 5000 at $k\bar{v}/\Gamma = 3.4$. The curves in different shades of red are the pulse contrast, and the curves in different shades of blue are the mean intensity. Note that the repetition time is normalized by the factor $1/\Gamma b_0(0)$.

This chapter completes our study into the transient behavior of the coherent light transmission through the cold atomic cloud. We find that the cooperative effect among the atoms plays an important role in the emission of the flashes from the atomic cloud. We stress that until now, our study focuses on the ISA regime of coherent transmission, where the atomic cloud is considered to be dilute. In the next chapter, we extend our study to the steady state coherent transmission of light beyond the ISA regime, considering denser atomic media.

Chapter 7

Coherent Transmission through Dense Atomic Cloud

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So far, our studies have been focused on the coherent transmission of light through a dilute cold atomic cloud. In such a medium, the independent scattering approximation (ISA) is valid. The atoms scatter light independently of one another. This can be understood using the basic theory of coherent transmission as described in Chapter 3. It is, however, interesting to go beyond the ISA regime and consider the coherent transmission for a dense atomic cloud.

In this chapter, we focus mainly on the calculation of the effective relative permittivity, ε_r , of a zero temperature atomic cloud with density ρ_0 . The incident probe beam is assumed to have a low intensity, keeping us in the linear response regime of light-atom interaction. We consider the $J = 0 \rightarrow J = 1$ transition for the atoms, and denote the resonant frequency by ω_0 , and the resonant wave vector $k_0 = \omega_0/c_0$. The speed of light in vacuum is denoted by c_0 .

Within the theoretical framework introduced in Chapter 3, the perturbative expansion can be continued beyond first order in $\rho_0\alpha_0$, to compute the correction terms that are proportional to $\rho_0^2\alpha_0^2$ [34,35,112] (see also Section 7.2). We write α_0 as the modulus of the resonant atomic polarizability¹. In fact, the dominant correction term depends on whether the pair correlation function (see Section 7.2.1.1) of the atoms has any density dependence. For instance, it has been shown in Ref. [94] that for an ideal Fermi gas, the dominant correction term comes in the order of $(\rho_0\alpha_0)^{5/3}$. Here, we consider the atoms as classical scatterers that behave like a hard sphere, with no density dependence in the pair correlation function. Therefore, the lowest order correction terms are still proportional to $(\rho_0\alpha_0)^2$.

To validate the theoretical results, we also perform numerical simulations based on the coupled dipole equations (see Section 3.3.1) in the frequency domain. We describe the numerical simulation algorithm in Section 7.3. The results of these numerical simulations are then compared to the theoretical results in Section 7.4.

Even though the effect of cloud temperature on the coherent transmission of light has been studied in the ISA regime by Pierrat *et al.* [112], a similar study beyond the ISA regime, using the same framework, has not been done. In particular, a simple averaging over the velocity distribution does not work beyond the ISA regime. Our work here sets the stage for a future study of the temperature effect on the coherent transmission of light beyond the ISA regime. In particular, the theoretical framework and the numerical simulation method that are used in our study can be adapted easily to include the effect of atomic motion inside the cloud.

Before discussing the correction terms that goes beyond the ISA regime, a brief recap of the coherent transmission in the ISA regime is given in Section 7.1.

¹For the vector waves, $\alpha_0 = 6\pi/k_0^2$. For the scalar waves, $\alpha_0 = 4\pi/k_0^2$.

7.1 Recap of coherent transmission in the ISA regime

As discussed in Chapter 3, the coherent field propagates through an atomic cloud, as if the medium is a homogeneous medium with an effective relative permittivity ε_r . The effective refractive index is related to ε_r by $n_r = \sqrt{\varepsilon_r}$. Using n_r , one can then calculate the coherently transmitted field through the cloud.

The effective relative permittivity is related to the mass operator. In the case of a vector electromagnetic (EM) wave, this is given by Equation (3.39), i.e.,

$$\varepsilon_r(\mathbf{k}, \omega) = 1 + \frac{M_t(\mathbf{k}, \omega)}{(\omega_0/c_0)^2}, \quad (7.1)$$

where M_t is the transverse component of the dyadic mass operator $\overleftrightarrow{\mathbf{M}}$ (see Section 3.4.1.1 for more details). Similarly, for the case of a scalar EM wave², the relation between ε_r and the mass operator M is given by Equation (3.41), i.e.,

$$\varepsilon_r(\mathbf{k}, \omega) = 1 + \frac{M(\mathbf{k}, \omega)}{(\omega_0/c_0)^2}. \quad (7.2)$$

In Section 3.4.2, a diagrammatic expansion method is used to expand the mass operator in increasing orders of $\rho_0\alpha_0$. For the vector waves, expanding up to the second order, we have

$$\begin{aligned} \overleftrightarrow{\mathbf{M}} = & \text{○} + \text{○} \text{---} \text{○} \\ & + \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \dots \\ & + \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} \text{---} \text{○} + \dots \end{aligned} \quad (7.3)$$

The same diagrammatic expansion holds also for the scalar waves. In the ISA regime, only the first term in the right hand side of Equation (7.3) is considered. For both cases of vector and scalar EM waves, the calculation in the ISA regime leads to

$$\varepsilon_r(\delta) = 1 + \rho_0\alpha_0\tilde{\alpha}(\delta), \quad (7.4)$$

²The scalar EM wave is an approximation to the actual EM wave, where the polarization and the near-field dipole emission are neglected. See Section 3.2 for more details.

where the normalized polarizability of the atom is given by

$$\tilde{\alpha}(\delta) = -\frac{\Gamma/2}{\delta + i\Gamma/2}. \quad (7.5)$$

We only consider the coherent transmission of a weak probe beam. Here, δ is the probe detuning and Γ is the transition linewidth.

To extend beyond the ISA regime, one has to compute the remaining terms in Equation (7.3). The result of this computation is reported in the next section.

7.2 Second order terms in mass operator

As discussed in Section 3.4.2, and pointed out in Refs. [34,35,112], the second order contributions that come in $\rho_0^2\alpha_0^2$ are due to the two-body positional correlation, and the recurrent scattering of light between pairs of atoms. The two-body positional correlation term of the mass operator is given by the second scattering diagram in Equation (7.3). This is computed in Section 7.2.1. The second and third lines in Equation (7.3) consist of an infinite number of diagrams related to the different orders of recurrent scattering between a pair of atoms. They are discussed in Section 7.2.2. All these calculations are performed for both the vector and scalar waves.

7.2.1 Two-body correlation term

We denote the two-body correlation contribution of the mass operator, as $\overleftrightarrow{\mathbf{M}}_{\text{cor}}$ (M_{cor}) for the case of vector (scalar) EM wave.

7.2.1.1 Pair correlation function

To discuss the contribution of the positional correlation between two atoms separated by a distance x , we need to introduce the pair correlation function, $g_c(x)$. Let us consider the two-particle probability density $P_2(\mathbf{r}_1, \mathbf{r}_2)$ of locating one particle at $\mathbf{r}_1$, and another at $\mathbf{r}_2$. It can be written as [130]

$$P_2(\mathbf{r}_1, \mathbf{r}_2) = P_1(\mathbf{r}_1)P_2(\mathbf{r}_2) [1 + g_c(|\mathbf{r}_1 - \mathbf{r}_2|)]. \quad (7.6)$$

Since we assume that the medium is homogeneous and isotropic, the pair correlation function depends only on the distance between the two atoms. For a uniform medium of volume V , the single-particle probability distribution is given by $P_1(\mathbf{r}) = 1/V$.

If the positions of all the atoms are independent, the pair correlation function is $g_c(|\mathbf{r}_1 - \mathbf{r}_2|) = 0$, and we have $P_2(\mathbf{r}_1, \mathbf{r}_2) = P_1(\mathbf{r}_1)P_2(\mathbf{r}_2)$. We exclude the possibility to find two scatterers at the same location. The pair correlation function for the classical atomic scatterers considered in our study is

$$g_c(x) = \begin{cases} -1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (7.7)$$

However, it is more convenient to consider the more general case of

$$g_c(x) = \begin{cases} -1, & x < d_{\min}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.8)$$

This results in an exclusion volume around each atom, assuming that each atom is behaving like a hard sphere of radius $d_{\min}$ ³. This is useful for the calculation of recurrent scattering (see Section 7.2.2), to justify an interchange in the order of summation and integration. At the end of the computation, one can then take the limit $d_{\min} \rightarrow 0$ to recover the expressions for the atomic scatterers. Furthermore, in the numerical simulations, these exclusion volumes around the atoms are needed to avoid the situation when two atomic dipoles are too close. In case of motionless atoms, the large value of the field may cause a divergence in the calculation. In Figure 7.9, we further show that the values of $d_{\min}$ applied in our studies do not lead to significant difference in the calculated results of ε_r .

Having knowledge of the pair correlation function, we next look into how it contributes to the mass operator. We start first with the vector waves in the following section.

7.2.1.2 Vector waves

For the vector waves, the mass operator for the two-body positional correlation term is computed by

$$\begin{aligned} \vec{\mathbf{M}}_{\text{cor}}(\mathbf{r} - \mathbf{r}', \omega) = & \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \iiint \mathcal{T}_{\mathbf{r}_j}(\mathbf{r}, \mathbf{r}_2, \omega) \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_2 - \mathbf{r}_1, \omega) \mathcal{T}_{\mathbf{r}_i}(\mathbf{r}_1, \mathbf{r}', \omega) \\ & \times P_1(\mathbf{r}_i) P_1(\mathbf{r}_j) g_c(|\mathbf{r}_j - \mathbf{r}_i|) d^3\mathbf{r}_i d^3\mathbf{r}_j d^3\mathbf{r}_1 d^3\mathbf{r}_2. \end{aligned} \quad (7.9)$$

³The pair correlation function for hard spheres is, in general, more complicated [131, 132]. Nevertheless, Equation (7.8) is true when the filling fraction, $4\pi N_a d_{\min}^3/3V$, is small.

The integration is performed over the volume of the scattering medium, which we assume to be very large. For simplicity, the integration is extended to all space. The scattering of light by an atom located at $\mathbf{r}_0$ is represented by the scattering operator, $\mathcal{T}_{\mathbf{r}_0}(\mathbf{r}, \mathbf{r}', \omega)$, which is defined in Equation (3.26) and repeated here,

$$\mathcal{T}_{\mathbf{r}_0}(\mathbf{r}, \mathbf{r}', \omega) = \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \delta(\mathbf{r} - \mathbf{r}_0) \delta(\mathbf{r}' - \mathbf{r}_0). \quad (7.10)$$

The pair correlation function accounts for the two-body correlation in the position of the atoms. The propagation of light between the two scattering events at $\mathbf{r}_i$ and $\mathbf{r}_j$, are represented by the dyadic Green function $\overleftrightarrow{\mathbf{G}}_0$ (see Section 3.2.1). For the vector waves, it is

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) = \frac{1}{4\pi} \frac{\omega}{c_0} \left[\beta(\omega R/c_0) \overleftrightarrow{\mathbf{P}} + \gamma(\omega R/c_0) \overleftrightarrow{\mathbf{U}} \right] - \frac{\overleftrightarrow{\mathbf{I}}}{3(\omega/c_0)^2} \delta(\mathbf{R}), \quad (7.11)$$

with

$$\beta(x) = \frac{e^{ix}}{x} \left[1 - \frac{1}{ix} - \frac{1}{x^2} \right], \quad (7.12)$$

$$\gamma(x) = \frac{2e^{ix}}{x} \left[\frac{1}{ix} + \frac{1}{x^2} \right]. \quad (7.13)$$

The Dirac delta functions in $\mathcal{T}_{\mathbf{r}_0}(\mathbf{r}, \mathbf{r}', \omega)$ are used to evaluate some of the integrations of $\overleftrightarrow{\mathbf{M}}_{\text{cor}}$ in Equation (7.9). This leads to

$$\overleftrightarrow{\mathbf{M}}_{\text{cor}}(\mathbf{r} - \mathbf{r}', \omega) = \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^4 \frac{1}{V^2} \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) g_c(|\mathbf{r} - \mathbf{r}'|). \quad (7.14)$$

The double summation results in a factor of $N_a(N_a - 1)$. For large number of atoms, it can be approximated as N_a^2 . The atomic density is given by $\rho_0 = N_a/V$. In the Fourier domain, we have

$$\overleftrightarrow{\mathbf{M}}_{\text{cor}}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^4 \int \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) g_c(R) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R}. \quad (7.15)$$

We apply the on-shell approximation by taking $|\mathbf{k}| = \omega/c_0$. This approximation can be applied, since the mass operator has a peak around $|\mathbf{k}| = \omega/c_0$. Following derivation described in Appendix E, the transverse part of $\overleftrightarrow{\mathbf{M}}_{\text{cor}}$ in the vector wave

cases is found to be,

$$M_{t,\text{cor}}(\omega) = (\rho_0\alpha_0)^2\tilde{\alpha}(\omega)^2\left(\frac{\omega}{c_0}\right)^2\mathcal{C}\left(\tilde{d}_{\min}\right), \quad (7.16)$$

where $\tilde{d}_{\min} = \omega d_{\min}/c_0$, and

$$\mathcal{C}\left(\tilde{d}_{\min}\right) = \frac{2i + 2i\tilde{d}_{\min}^2 - \tilde{d}_{\min}^3 - 2i\tilde{d}_{\min}^4 - \left(2i + 4\tilde{d}_{\min} - 2i\tilde{d}_{\min}^2 - \tilde{d}_{\min}^3\right)e^{2i\tilde{d}_{\min}}}{4\tilde{d}_{\min}^3}.$$

(7.17)

As expected, the two-body correlation term comes in second order of $\rho_0\alpha_0$.

In the limit of $d_{\min} \rightarrow 0$, Equation (7.16) becomes

$$M_{t,\text{cor}} = \frac{1}{3}\rho_0^2\alpha(\omega)^2\left(\frac{\omega}{c_0}\right)^2. \quad (7.18)$$

In this limit, the contribution to the mass operator is only due to the Dirac delta function in the dyadic Green function. As pointed out in Ref. [34], this term is related to the Lorenz-Lorentz correction up to the second order in $\rho_0\alpha_0$. Indeed, if we include the ISA term, we have

$$\varepsilon_r(\omega) = 1 + \rho_0\alpha_0\tilde{\alpha}(\omega) + \frac{1}{3}(\rho_0\alpha_0)^2\tilde{\alpha}(\omega)^2. \quad (7.19)$$

This is the second order expansion in $\rho_0\alpha_0$ of the full Lorenz-Lorentz formula for ε_r , i.e.,

$$\varepsilon_r(\omega) = \frac{3 + 2\rho_0\alpha_0\tilde{\alpha}(\omega)}{3 - \rho_0\alpha_0\tilde{\alpha}(\omega)}. \quad (7.20)$$

Within this perturbative expansion framework, the expression above can be recovered by considering the ISA term, and all orders of n-body correlations among the atoms [112, 133, 134]. Equation (7.19) alone, however, is not sufficient to fully describe the second order correction to ε_r , we still need the contribution from recurrent scattering (see Section 7.2.2).

7.2.1.3 Scalar waves

In the case of scalar wave, M_{cor} , is given by

$$M_{\text{cor}}(\mathbf{k}, \omega) = \rho_0^2\alpha(\omega)^2\left(\frac{\omega}{c_0}\right)^4 \int G_0(\mathbf{R}, \omega)g_c(R)e^{-i\mathbf{k}\cdot\mathbf{R}}d^3\mathbf{R}. \quad (7.21)$$

The free space green function for the scalar wave, $G_0(\mathbf{R}, \omega)$, is given in Section 3.2.2 to be

$$G_0(\mathbf{R}, \omega) = \frac{1}{4\pi R} e^{i\omega R/c_0}. \quad (7.22)$$

This leads to the following contribution to the mass operator, i.e.,

$$M_{\text{cor}}(\omega) = -\rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^2 \left[\frac{1}{4} \left(1 - e^{2i\tilde{d}_{\text{min}}} \right) + \frac{i}{2} \tilde{d}_{\text{min}} \right]. \quad (7.23)$$

As expected, the above equation is indeed second order in $\rho_0 \alpha_0$. The details of the calculation can be found in Appendix E.

At $d_{\text{min}} = 0$, the contribution of the positional correlation to the mass operator becomes zero, namely,

$$M_{\text{cor}}(\omega) = 0. \quad (7.24)$$

Thus, considering the scalar wave approximation, we find no Lorenz-Lorentz correction term to ε_r , at the order of $\rho_0^2 \alpha_0^2$

Importantly, the pair correlation function in the form of Equation (7.8) is not the only type of pair correlation function that can be calculated using this diagrammatic expansion method. In the study of Morice *et al.* [34], the pair correlation function of a Bose gas is considered. Its effect on the coherent transmission has recently been observed in Ref. [93]. The case of Fermi gas is even more interesting, since the pair correlation function for fermions leads to a contribution that comes in $(\rho_0 \alpha_0)^{5/3}$, unlike the $\rho_0^2 \alpha_0^2$ dependence in the case of Bose gas [94]. Other types of pair correlation function can also be considered. In certain cases, one may have to resort to numerical integration to compute the contribution from the pair correlation function.

7.2.2 Recurrent scattering terms

In this section, we discuss the mass operator contribution proportional to $\rho_0^2 \alpha_0^2$ due to recurrent scattering. We denote it as $\overleftrightarrow{\mathbf{M}}_{\text{rec}} (M_{\text{rec}})$ for the case of vector (scalar) EM waves.

As shown in Figure 3.1, the recurrent scattering between a pair of atoms comes in different orders, depending on the number of scattering events. The mass operator due to all these contributions are summed together to give the total recurrent scattering contribution, i.e.,

$$\overleftrightarrow{\mathbf{M}}_{\text{rec}}(\omega) = \sum_{p=3}^{\infty} \overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(p)}(\omega). \quad (7.25)$$

Figure 7.1: The classification of the recurrent scattering diagrams into the loops and the boomerangs. We include also the terms involving the pair correlation function between the two atoms. Only the two lowest order diagrams for each type of diagram are shown in this figure.

The $\vec{M}_{\text{rec}}^{(p)}(\omega)$ term expresses the contribution from the recurrent scattering between pairs of atoms, where there is a total of p number of scattering events between the two atoms. Here, p is an integer greater or equal to 3. A similar expression can be written for the scalar EM wave, i.e.,

$$M_{\text{rec}}(\omega) = \sum_{p=3}^{\infty} M_{\text{rec}}^{(p)}(\omega). \quad (7.26)$$

The recurrent scattering terms can be further classified into terms with odd and even numbers of scattering events. We show this classification for the two lowest order recurrent scattering terms of each type in Figure 7.1. We use the terminology introduced in Ref. [121]. The scattering diagrams with odd number of scattering events, where the first and the last scattering events occurring at the same atom are called the loop diagrams. The other scattering diagrams are called the boomerang diagrams. In these diagrams, the first and the last scattering events occur at different atoms.

In the following, the recurrent scattering mass operators for both the vector and scalar EM wave cases are discussed. We present only the results of the calculation, leaving the finer details of the derivation to Appendix E. The on-shell approximation is applied in these calculations.

7.2.2.1 Vector waves

In the case of vector waves, only the transverse component of the mass operator is relevant for coherent transmission. The transverse component of the recurrent scattering mass operator for vector waves, including all orders of loops and boomerangs, is computed to be,

$$M_{t,\text{rec}} = (\rho_0 \alpha_0)^2 \left(\frac{\omega}{c_0} \right)^2 \frac{\tilde{\alpha}(\omega)^3}{2} \mathcal{J}_v(\omega), \quad (7.27)$$

with

$$\mathcal{J}_v(\omega) = \int_{\tilde{d}_{\min}}^{\infty} \frac{2x^2 [\beta(x)^2 + \frac{9}{4}\tilde{\alpha}(\omega)\beta(x)^3 \{j_0(x) - j_1(x)/x\}]}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\beta(x)^2} + x^2 \frac{\gamma(x)^2 + \frac{9}{2}\tilde{\alpha}(\omega)\gamma(x)^3 j_1(x)/x}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\gamma(x)^2} dx. \quad (7.28)$$

In the above equation, we find the two lowest orders of the spherical Bessel functions,

$$j_0(x) = \frac{\sin x}{x}, \quad (7.29)$$

$$j_1(x) = \frac{\sin x - x \cos x}{x^2}. \quad (7.30)$$

The effect of the pair correlation function $g_c(x)$, with the exclusion volume as defined in Equation (7.8), is taken into consideration here. Integrating $\mathcal{J}_v(\omega)$ in Equation (7.28) analytically is not possible. It has to be integrated numerically (see Section 7.2.5).

7.2.2.2 Scalar waves

For the case of scalar waves, we have

$$M_{\text{rec}}(\omega) = (\rho_0 \alpha_0)^2 \left(\frac{\omega}{c_0} \right)^2 \frac{i\tilde{\alpha}(\omega)^3}{2} \left[e^{2i\tilde{d}_{\min}} + \tilde{\alpha}(\omega)\mathcal{I}_s(\omega) \right], \quad (7.31)$$

with

$$\mathcal{I}_s(\omega) = \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{2ix} [1 - (1 + 2i\tilde{\alpha}(\omega)) e^{2ix}]}{x^2 - \tilde{\alpha}(\omega)^2 e^{2ix}} dx. \quad (7.32)$$

The derivation of this expression is presented in Appendix E. Similar to the case of vector waves, $\mathcal{I}_s(\omega)$ has to be computed numerically, as outlined in Section 7.2.5.

7.2.3 Total mass operator and effective relative permittivity

We have now completed the description of all the terms that are proportional to $\rho_0^2 \alpha_0^2$ for both the cases of vector and scalar waves. In order to compute the total mass operator, we add them to the ISA term. For the vector wave, the transverse component of the mass operator $\overleftrightarrow{\mathbf{M}}$ is

$$M_t = M_{t,\text{ISA}} + M_{t,\text{cor}} + M_{t,\text{rec}}. \quad (7.33)$$

We thus have, in the vector wave case,

$$\begin{aligned} \varepsilon_r(\omega) &= 1 + M_t/(\omega_0/c_0)^2 \\ &= 1 + \rho_0 \alpha_0 \tilde{\alpha}(\omega) + (\rho_0 \alpha_0)^2 \tilde{\alpha}(\omega)^2 \mathcal{C}(\tilde{d}_{\min}) + \frac{1}{2} (\rho_0 \alpha_0)^2 \tilde{\alpha}(\omega)^3 \mathcal{J}_v(\omega), \end{aligned} \quad (7.34)$$

consistent with the work of Morice *et al.* [34].

In the case of scalar waves,

$$M = M_{\text{ISA}} + M_{\text{cor}} + M_{\text{rec}}. \quad (7.35)$$

The effective relative permittivity is given by

$$\begin{aligned} \varepsilon_r(\omega) &= 1 + \rho_0 \alpha_0 \tilde{\alpha}(\omega) + (\rho_0 \alpha_0)^2 \frac{\tilde{\alpha}(\omega)^2}{2} \\ &\quad \times \left\{ i \tilde{\alpha}(\omega) \left[e^{2i\tilde{d}_{\min}} + \tilde{\alpha}(\omega) \mathcal{I}_s(\omega) \right] - \left[\frac{1}{2} \left(1 - e^{2i\tilde{d}_{\min}} \right) + i\tilde{d}_{\min} \right] \right\}. \end{aligned} \quad (7.36)$$

Here, we retain the expression for the general case of $\tilde{d}_{\min} \neq 0$, while noting that the limit of $\tilde{d}_{\min} \rightarrow 0$ can be taken to describe the atomic scatterers.

Due to the $1/R$ dependence of the scalar field, there is no contribution from the Lorenz-Lorentz correction term. As $\tilde{d}_{\min} \rightarrow 0$, we are left with only the recurrent scattering contribution. Thus, the scalar wave is an ideal system to extend the study to the case of moving atoms, where the positional correlation between a pair of atoms is not straightforward to define. With scalar waves, one can focus only on the recurrent scattering contribution. The computation is also simpler, since the light polarization and the near field emission from the atoms are neglected.

Figure 7.2: A comparison of the coherent transmission in the ISA and beyond the ISA regime, for vector waves. In the two upper figures, we compare between the two regimes, the optical thickness, $b_{\bar{v}}(\delta)$, and the phase shift, $\phi_{\bar{v}}(\delta)$. In the two figures below, the differences between the predictions in the two regimes are shown.

7.2.4 Effect on coherent transmission

Using the expressions of ε_r written in the previous section, we can understand how the coherent transmission of light is modified by the second order terms. To do so, we look at the difference in the lineshapes of the optical thickness $b_{\bar{v}}(\delta)$ and the phase shift $\phi_{\bar{v}}(\delta)$ of the transmitted field.

We consider first the case of vector waves. In Figure 7.2, the optical thickness and phase shift in the ISA regime are calculated with Equations (3.65) and (3.66) in the limit of zero temperature. For the second order calculation, ε_r in Equation (7.34) is first used to compute the effective refractive index, n_r . From n_r , the optical thickness and phase shift are calculated. The calculation involving the second order contribution, are carried out with $\rho_0\alpha_0 = 1/9.1$. The medium is assumed to have a resonant optical thickness of $b_0(0) = 1$ in the ISA regime.

Indeed, the second order terms contribute rather weakly to the coherent transmission. This is expected from the perturbative nature of our calculations, where $k_0\ell_0 = 1/9.1$ is still a small parameter. To clearly appreciate the difference in these two calculation regimes, the modifications due to the inclusion of the second order terms are plotted, respectively, under the figures of $b_{\bar{v}}(\delta)$ and $\phi_{\bar{v}}(\delta)$ in Figure 7.2.

In the plot of $b_{\bar{v}}(\delta)$, we also see a small shift (around 0.05Γ) in the resonances due to the effect of the atomic density. Furthermore, the phase shift experienced by the transmitted field at resonance, becomes non-zero. This suggests that a superflash

Figure 7.3: A comparison of the coherent transmission in the ISA and beyond the ISA regime, for scalar waves. In the two upper figures, we compare between the two regimes, the optical thickness, $b_{\bar{v}}(\delta)$, and the phase shift, $\phi_{\bar{v}}(\delta)$. In the two figures below, the differences between the predictions in the two regimes are shown.

could be achieved at resonance, when the atomic density is higher. Within the validity of second order approximation, this effect is rather weak. Nevertheless, a future extension to an even denser atomic cloud could eventually bring interesting possibilities for the study of superflashes.

A similar plot is done for the case of scalar waves Figure 7.3. Again, the difference in the optical thickness and phase shift between the two regimes are very small; the dominant contribution comes from the ISA contribution. Similar with the case of vector waves, we computed the curves at $\rho_0\alpha_0 = 1/9.1$, with a resonant optical thickness of $b_0(0) = 1$ in the ISA regime.

In the numerical studies reported in Section 7.3, we compute and normalize the second order contribution to ε , and compare the values to the numerical simulation results.

7.2.5 Numerical integration for $\mathcal{J}_v(\omega)$ and $\mathcal{I}_s(\omega)$

In this section, we describe the numerical methods used to perform the numerical integrations of $\mathcal{J}_v(\omega)$ and $\mathcal{I}_s(\omega)$. The numerical integration is required, since no analytical solution of the integrals.

7.2.5.1 Vector waves

The integral for $\mathcal{J}_v(\omega)$ can be broken into two parts, with the first part being an integral with a cut-off a large value of $x = \mathcal{M}$. This part is denoted by $\mathcal{J}_v^{\mathcal{M}}(\omega)$. The second part, which is denoted by $\mathcal{J}'_v$, is given by

$$\mathcal{J}'_v = \int_{\mathcal{M}}^{\infty} \frac{2x^2 [\beta(x)^2 + \frac{9}{4}\tilde{\alpha}(\omega)\beta(x)^3 \{j_0(x) - j_1(x)/x\}]}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\beta(x)^2} + x^2 \frac{\gamma(x)^2 + \frac{9}{2}\tilde{\alpha}(\omega)\gamma(x)^3 j_1(x)/x}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\gamma(x)^2} dx. \quad (7.37)$$

By introducing a small and positive artificial dissipation of η to the oscillating exponential functions in $\beta(x)$ and $\gamma(x)$, we find for $\mathcal{M} \gg 1$, the following analytical result,

$$\mathcal{J}'_v = \lim_{\eta \rightarrow 0^+} \int_{\mathcal{M}}^{\infty} 2e^{2ix-\eta x} dx = ie^{2i\mathcal{M}}. \quad (7.38)$$

Thus, only $\mathcal{J}_v^{\mathcal{M}}$ as written below has to be computed numerically,

$$\mathcal{J}_v^{\mathcal{M}}(\omega) = \int_{\tilde{d}_{\min}}^{\mathcal{M}} \frac{2x^2 [\beta(x)^2 + \frac{9}{4}\tilde{\alpha}(\omega)\beta(x)^3 \{j_0(x) - j_1(x)/x\}]}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\beta(x)^2} + x^2 \frac{\gamma(x)^2 + \frac{9}{2}\tilde{\alpha}(\omega)\gamma(x)^3 j_1(x)/x}{1 - \frac{9}{4}\tilde{\alpha}(\omega)^2\gamma(x)^2} dx. \quad (7.39)$$

This integral is computed numerically using an adaptive algorithm that can handle the oscillating integrand. The cut-off value $\mathcal{M}$, is chosen to be 100000. Finally, we find $\mathcal{J}_v(\omega)$ by summing the two parts,

$$\mathcal{J}_v(\omega) = \mathcal{J}_v^{\mathcal{M}}(\omega) + ie^{2i\mathcal{M}}. \quad (7.40)$$

7.2.5.2 Scalar waves

In the case of scalar waves, the calculation of $\mathcal{I}_s(\omega)$ proceeds in a similar manner.

We consider the integral with a large cut-off value of $\mathcal{M}$ on the upper limit, and denote it by $\mathcal{I}_s^{\mathcal{M}}(\omega)$. Unlike the case of vector waves, the integration at large values of x contributes negligibly to $\mathcal{I}_s$. Therefore, we have just

$$\mathcal{I}_s(\omega) = \mathcal{I}_s^{\mathcal{M}}(\omega), \quad (7.41)$$

with

$$\mathcal{I}_s^{\mathcal{M}}(\omega) = \int_{\tilde{d}_{\min}}^{\mathcal{M}} \frac{e^{2ix} [1 - (1 + 2i\tilde{\alpha}(\omega)) e^{2ix}]}{x^2 - \tilde{\alpha}(\omega)^2 e^{2ix}} dx. \quad (7.42)$$

We use the same numerical integration algorithm, as described previously for the

vector waves, to compute the above integral. The value of $\mathcal{M}$ is set at 80000.

The numerical integrations described here completes the discussion of the theoretical calculation of ε_r for both the vector and scalar waves.

In the cold atomic experiments, the density required to validate experimentally these $(\rho_0\alpha_0)^2$ terms is reachable. However, the cold atomic clouds that are dense, are also usually rather small. Hence, there may be additional finite size effects that come into play. For instance, are the partial reflections at the boundaries of the cloud, which becomes important for a cloud with small thickness. There is also a diffraction of light by the small atomic cloud. All these affect the transmission of light. Hence, it is difficult to extract some information on ε_r , which is a quantity related to the bulk, from the experimental result. For this reason, we turn to a numerical simulation method to validate the theoretical results. This is described in the following section.

7.3 Numerical simulation method for coherent transmission

The numerical simulation to compute the coherent transmission of light is based on the coupled dipole equations in the frequency domain (see Section 3.3.1). A first use of this coupled-dipole method for atomic scatterers was reported in Ref. [135] to study a 1-D electrodynamics model of light propagation. In that work, the authors were limited by the computational capabilities then, to fully resolve the 3-D calculations. Nevertheless, their calculations show good agreement with theoretical works and give an early insight of the properties of light propagation through a quantum gas.

Our numerical study considers the realistic 3-D case. For the vector waves, the following set of equations is used to compute the field $\mathbf{E}$ at a position $\mathbf{r}$.

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\mathbf{r}, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{i=1}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}_i, \omega) \cdot \mathbf{E}_i^{\text{ex}}(\omega), \quad (7.43)$$

$$\mathbf{E}_i^{\text{ex}}(\omega) = \mathbf{E}_0(\mathbf{r}_i, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_i - \mathbf{r}_j, \omega) \cdot \mathbf{E}_j^{\text{ex}}(\omega). \quad (7.44)$$

For scalar EM waves, we have a similar set of equations,

$$E(\mathbf{r}, \omega) = E_0(\mathbf{r}, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{i=1}^{N_a} G_0(\mathbf{r} - \mathbf{r}_i, \omega) E_i^{\text{ex}}(\omega), \quad (7.45)$$

$$E_i^{\text{ex}}(\omega) = E_0(\mathbf{r}_i, \omega) + \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \sum_{\substack{j=1 \\ j \neq i}}^{N_a} G_0(\mathbf{r}_i - \mathbf{r}_j, \omega) E_j^{\text{ex}}(\omega). \quad (7.46)$$

Our aim is to describe the steady state light scattering for motionless atoms. Before we discuss further the numerical algorithm, a few comments on the numerical simulation are in order.

7.3.1 Settings for the numerical simulation

In the theoretical studies, the bulk quantity ε_r is calculated assuming a very large medium without any boundaries. The number of atoms is assumed to be very large. In the numerical studies, this is not possible to achieve. For instance, the number of atoms N_a cannot be too large to have reasonable computation time. Thus, a proper geometry and density of the medium, has to be chosen. At the same time, we have to minimize the geometric effects on coherent transmission, such as the diffraction of light, and the finite size effect due to a thin sample.

Taking the above concerns into consideration, the atomic medium for the numerical studies takes the shape of a cylinder, as shown in Figure 7.4. The thickness of the cylinder along the propagation direction is denoted by L . The total radius of the cylinder is denoted by r_o . The cylinder is divided into two regions. The region at the center has an inner radius of r_u . Inside this region, the atomic scatterers are distributed uniformly with a density of ρ_0 . Outside this region, the density of the atomic medium decreases linearly to zero at the edge of the cylinder. The density profile $\rho(r)$, is shown in the upper right part of Figure 7.4. This profile reduces the effect of light diffracting from the edges due to a sharp change in the atomic density.

The atoms in the scattering medium are assumed to be ^{88}Sr atoms. The parameters relevant to the narrow intercombination transition of 689 nm is used in the simulation.

Additionally, we simulate the transmission of a Gaussian incident beam, instead of a plane incident wave used in the theoretical studies. This beam of waist w_0 , is located at the center of the slab. We set $w_0 < r_u$ to make sure that the intensity

Figure 7.4: The setup for the numerical simulations. The scattering medium takes the shape of a cylinder with thickness L . The front and side views of the scattering medium are shown in the figure. The cylinder has an outer radius of r_o , and an inner region of uniform density with radius r_u . In the outer region, the density decreases linearly to zero at the edge of the cylinder. The incident light, which is shaded red, is a Gaussian beam of waist w_0 located at the center of the cylinder. The total field is calculated along the propagation axis. The distance between the point P where the field is computed, and the center of slab, is denoted by z .

is weaker towards the edge of the cylinder. This ensures that a large part of the optical power passes through the central uniform density region. To compare with the plane wave used in the theoretical studies, we further require that L is much smaller than the Rayleigh length, $z_R = \pi w_0^2 / \lambda$, of the Gaussian beam. This ensures that the incident beam inside the medium has an almost constant diameter, with its wave vector is directly mainly along the z -direction.

Parameters	Vector Waves	Scalar Waves
$2r_o$	35 μm	35 μm
$2r_u$	22.5 μm	22 μm
$b_0(0)$	1	1
w_0	4.5 μm	4.5 μm
N_a	2897	4277
$\tilde{d}_{\min} = \omega d_{\min}/c_0$	0.455	0.455
N_{conf}	3200	3200
L	1 – 4 μm	1 – 4 μm

Table 7.1: The typical settings of the parameters for the numerical simulation. The values are summarized for both the cases of vector and scalar EM waves.

L (μm)	$k_0\ell_0$	ρ_0 (μm^{-3})	ρ_0/k_0^3
1	9.1	4.4	0.0058
2	18.2	2.2	0.0029
4	36.4	1.1	0.0015

Table 7.2: The $k_0\ell_0$, ρ_0 and ρ_0/k_0^3 values corresponding to the different values of L , for the numerical simulation with vector waves at $b_0(0) = 1$. The total number of atoms in the cylinder is 2897. The other settings are tabulated in Table 7.1.

The typical settings used in the numerical simulation are summarized in Table 7.1 for both the cases of scalar and vector EM waves. In our studies, the geometry of the cylinder (r_o , r_u and L), and the resonant optical thickness of the uniform region at zero temperature, $b_0(0)$, are set as the input parameter. The number of atom N_a is then computed from these quantities using the density profile shown in Figure 7.4. This results in typically 3000-4000 atoms. With this number of atoms, the numerical computation can be completed in a reasonable time. Since $b_0(0)$ is proportional to the product $\rho_0 L$ [see Equation (3.64)], the choice of L determines the value of ρ_0 . This is shown for different values of L at $b_0(0) = 1$ in Table 7.2, for the case of vector waves. For the case of scalar waves, the corresponding values are summarized in Table 7.3 instead. We note that for the same $b_0(0)$ the density in the scalar case is larger compared to the vector case. This is because the scattering cross section of the scalar waves is 1.5 times smaller than the corresponding value for the vector waves. Thus, a larger number of atoms is required for the medium in the scalar waves calculations, if we require the optical thickness to be also $b_0(0) = 1$. The different values of ρ_0 leads to different values of $\rho_0\alpha_0 = 1/k_0\ell_0$ in the numerical simulations. Here, ℓ_0 is the resonant optical mean free path of light in the ISA regime.

As mentioned before, in the numerical studies, it is useful to consider an exclusion volume of radius $d_{\min}$ around each atomic scatterer. As discussed previously,

L (μm)	$k_0\ell_0$	ρ_0 (μm^{-3})	ρ_0/k_0^3
1	9.1	6.6	0.0087
2	18.2	3.3	0.0043
4	36.4	1.7	0.0022

Table 7.3: The $k_0\ell_0$, ρ_0 and ρ_0/k_0^3 values corresponding to the different values of L , for the numerical simulation with scalar waves at $b_0(0) = 1$. The number of atoms in the cylinder is 4277. The other settings are tabulated in Table 7.1.

this avoids the problem of having a diverging electric field when two atoms are close-by. When populating the medium with the atoms, the position of each newly added atom is checked against all existing atoms to ensure that it remains outside of all their exclusion volumes. If an atom is found to be within the exclusion volume of another atom, a new set of coordinate is generated.

The coherent field is the ensemble-averaged field in the forward direction. A total of N_{conf} different realizations of the atomic medium, with the same density profile, are studied numerically. For each realization, the total field is calculated along the propagation axis of the incident probe beam, up to a distance of $z = 500\lambda$. This is done at several positions along the propagation direction of the incident beam. At each position, the field is then averaged over the different realizations that are computed. The detuning, δ of the probe beam is then varied between $-3 \leq \delta/\Gamma \leq 3$, to obtain the spectrum of ε_r .

In the next section, we discuss the numerical calculation of the coherent field in more details.

7.3.2 Numerical calculation of the coherent field

7.3.2.1 Vector waves

Before the coupled dipole equations can be solved, the incident field has to be defined. The incident light is a Gaussian beam with its waist at the center of the cylinder. Its field at a position of $\mathbf{r} = (x, y, z)$ is defined to be

$$\mathbf{E}_0(\mathbf{r}) = E_0(z)e^{i\mathbf{k}_l \cdot \mathbf{r}} \exp\left[-\frac{x^2 + y^2}{w_0^2(1 + iz/z_R)}\right] \boldsymbol{\epsilon}, \quad (7.47)$$

where

$$E_0(z) = \frac{1}{1 + iz/z_R}. \quad (7.48)$$

The origin of the coordinates is also placed at the center of the cylinder. The incident wave vector is denoted by $\mathbf{k}_l = (\omega/c_0)\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is the unit vector along z . The polarization of the incident field is denoted by $\boldsymbol{\epsilon}$. Along the propagation

axis ($x, y = 0$), the incident field is given simply as $\mathbf{E}_0(\mathbf{r}) = E_0(z)e^{i\omega z/c_0}\mathbf{e}$. With a waist of $w_0 = 4.5 \mu\text{m}$ (see Table 7.1), we have $z_R = 91 \mu\text{m}$ that is much longer than L . As discussed before, this allows us to approximate the Gaussian beam as a plane wave, in terms of coherent transmission.

For a given realization, the $3N_a$ equations of the coupled dipole equations [Equation (7.44)] are solved to compute the external fields $\mathbf{E}_i^{\text{ex}}$ for the atoms. This linear system is solved by an LU decomposition algorithm. The solutions are then used in Equation (7.43) to calculate the field $\mathbf{E}(\mathbf{r})$ in the forward direction along the propagation axis.

To find the coherent field, we average the field over the N_{conf} different realizations of the atomic medium. Since the coherent field preserves the polarization of the field, we can consider only the scalar part $E_t(z)$ of the average field $\langle \mathbf{E}(z) \rangle$. From the averaged field, we can calculate $I_t/I_0 = |E_t(z)/E_0(z)|^2$ which is the normalized transmission. Another quantity that can be obtained is the phase shift experienced by the transmitted field with respect to $E_0(z)$, which is denoted by ϕ

We plot $I_t(z)/I_0(z)$ and $\phi(z)$ against z in Figure 7.5, for the vector waves. The calculation is performed at $k_0\ell_0 = 36.4$ and $\delta = 0.5\Gamma$, with the settings tabulated in Table 7.1. In Figure 7.5, the values of $I_t(z)/I_0(z)$ and $\phi(z)$ fluctuate more closer to the atomic medium. Nevertheless, at a distance that is large enough, $I_t(z)/I_0(z)$ and $\phi(z)$ settle to well-defined values that are associated with the coherently transmitted field. Hence, one can use the averaged field at $z = 500\lambda$ to compute the effective relative permittivity, ε_r (see the discussion in Section 7.3.3).

In Figure 7.5, the numerical computation errors are shaded around the curves for the intensity and the phase shift. This error is calculated by dividing 3200 realizations of the atomic medium into 40 partitions. Each partition has 80 realizations, from which we can compute $I_t(z)$ and $\phi(z)$. The standard deviation among these 40 partitions are first calculated. This quantity can be interpreted as the standard deviation of the intensity and phase shift, when we perform computation with 80 realizations. Here, we have 3200 realizations, and we expect a smaller value of the standard deviation. The corresponding standard deviation for 3200 realizations is obtained by dividing a further factor $\sqrt{40}$. The shaded areas in the figure indicates the magnitudes of these standard deviations.

The algorithm for the scalar waves is similar to the method described above. We discuss it in the following section, pointing only the differences with the computation for the vector waves.

Figure 7.5: The intensity $I_t(z)/I_0(z)$, and phase shift $\phi(z)$ plotted against z for the vector waves. After a distance of typically 100λ , the calculated values becomes independent of the distance z . The parameters for the numerical simulation are $k_0\ell_0 = 36.4$ and $\delta = 0.5\Gamma$. The rest of the settings are listed in Table 7.1. The colored areas with lighter shades around the curves represent an estimation of the numerical calculation errors. These areas represent one standard deviation of the intensity and phase shift values, for numerical simulation with 3200 realizations (more details in the text).

7.3.2.2 Scalar waves

The incident field in this case is

$$E_0(\mathbf{r}) = E_0(z)e^{i\mathbf{k}_l \cdot \mathbf{r}} \exp\left[-\frac{x^2 + y^2}{w_0^2(1 + iz/z_R)}\right]. \quad (7.49)$$

Similar to the case of the vector waves, the set of Equation (7.46) for each realization of the atomic medium, is solved for the steady state values of E_i^{ex} . This is done using the same algorithm as described previously. Here, we have only N_a variables to solve from the N_a coupled equations. Equation (7.45) is then used calculate the total field $E(\mathbf{r})$.

The coherent field, E_t , is computed by an average over the N_{conf} realizations. We find a similar behavior in terms of the distance dependence of the average field, as compared to the case of vector waves. Thus, the normalized transmitted intensity I_t/I_0 and the phase shift ϕ are computed from $E_t(z)/E_0(z)$ at $z = 500\lambda$.

7.3.3 Calculating ε_r from the numerical results

With the values of I_t and ϕ , one can calculate ε_r for the medium. The description below applies to both the vector and the scalar waves.

We compute first the optical thickness, $b_0(\delta) = -\log(I_t/I_0)$. The effective re-

Figure 7.6: The calculated real and imaginary parts of ε_r at $k_0\ell_0 = 9.1$ in the case of vector waves. The real and imaginary parts of ε_r , calculated using the numerical simulation in the frequency domain, is shown as the blue and green dots, respectively, in the plot. The solid curve is the real part of ε_{ISA} and the dashed curve is the imaginary part of ε_{ISA} .

fractive index, $n_r(\delta)$, is then given by

$$n_r(\delta) - 1 = \frac{c_0}{\omega L} \left[i \frac{b_0(\delta)}{2} + \phi(\delta) \right]. \quad (7.50)$$

The effective relative permittivity, ε_r , is computed as the square of the refractive index, i.e.,

$$\varepsilon_r(\delta) = n_r(\delta)^2. \quad (7.51)$$

To compute the relevant error bars, ε_r is calculated for each partition. The standard deviations of the real and imaginary parts of ε_r are then calculated for the computations with 80 realizations. The standard deviations relevant to our computations with 3200 realizations are obtained by a further division of $\sqrt{40}$. They are displayed as the error bars in the subsequent plots of this chapter.

We show in Figures 7.6 and 7.7, the values of ε_r which are computed at $k_0\ell_0 = 9.1$ for the vector and scalar EM waves, respectively. They are compared to the theoretical values of ε_{ISA} , calculated in the ISA regime using,

$$\varepsilon_{ISA}(\delta) = 1 + \rho_0 \alpha(\delta). \quad (7.52)$$

The curves in the ISA regime are plotted as the solid lines.

Figure 7.7: The calculated real and imaginary parts of ε_r at $k_0\ell_0 = 9.1$ in the case of scalar waves. The numerical calculation results are shown in dots, while the solid and dashed lines are, respectively, the real and imaginary parts of ε_{ISA} computed theoretically.

7.4 Numerical simulation results

In the numerical simulation, atomic medium with different values of ρ_0 are studied. The following quantity, which is the normalized second order contribution to ε_r , is useful to compare among the results at different values of ρ_0 .

$$(k_0\ell_0)^2 (\varepsilon_r - \varepsilon_{ISA}), \quad (7.53)$$

In the ideal case, this quantity is independent of $k_0\ell_0$, i.e., it does not depend on the density ρ_0 ⁴. Furthermore, this quantity is ideally independent of the cylinder thickness L ⁵, and the optical thickness $b_0(0)$. It only depends on the detuning of the incident field, δ .

We first check that the thickness of the slab does not affect the numerical results of $(k_0\ell_0)^2 (\varepsilon_r - \varepsilon_{ISA})$. By keeping the same value of ρ_0 , the thickness L of the medium is doubled. This is equivalent to doubling the optical thickness $b_0(0)$ (see Section 7.4.1).

Besides the effect of the optical thickness, we also study the effect of the exclusion volume on the second order contribution terms. Typically, the numerical calculation is carried out with an exclusion volume of $\tilde{d}_{\min} = 0.455$. We compare it with another computation at $\tilde{d}_{\min} = 0.0455$. This corresponds to an exclusion volume

⁴This assumes that the atomic density ρ_0 is not too high for the terms higher order in ρ_0 starts to become important.

⁵This assumes that the thickness is large enough for the bulk properties of coherent transmission to be dominant. This point is checked in Section 7.4.1.

Figure 7.8: The comparison between the numerical simulation results at $b_0(0) = 1$ and $b_0(0) = 2$. The blue circles are for $b_0(0) = 1$, while the green smaller dots are for $b_0(0) = 2$. The numerical results are also compared to the theoretical prediction results.

that is three orders of magnitude smaller. The results are reported in Section 7.4.2, allowing us to check if the numerical simulation give a result that is relevant for atomic scatterers in the limit of $\tilde{d}_{\min} \rightarrow 0$.

Finally, we report the results of a study on the density dependence of the second order terms. The results reported in Section 7.4.3, allows us to validate the $\rho_0^2 \alpha_0^2$ dependence of $\epsilon_r - \epsilon_{\text{ISA}}$, in the density regime that we are investigating.

7.4.1 Effect of $b_0(0)$ values

We consider first the effect of $b_0(0)$ on the numerical simulation results. We only study the vector cases. A similar behavior is also expected for the scalar waves. Typically, the value of $b_0(0)$ in the numerical study is set to be 1 (see Table 7.1). This corresponds to $L = 2 \mu\text{m}$. We compare it to another medium with $b_0(0) = 2$, which is twice thicker. For this calculation, we set $k_0 \ell_0 = 18.2$. The results are shown in Figure 7.8, for both the real and imaginary parts of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{\text{ISA}})$. The results for $b_0(0) = 1$ are represented by the blue open circles, while the results for $b_0(0) = 2$ are represented by the green dots. The numerical results obtained from the two different thickness of the slab are similar. This suggests that the $b_0(0) = 1$ medium with $L = 2 \mu\text{m}$, is already sufficiently thick to give the correct value of the bulk quantity ϵ_r .

We further compare the numerical results to the theoretical prediction curves shown by the blue dashed line [following Equation (7.34)]. A good agreement between the numerical results, and the theoretical prediction is found.

Figure 7.9: The comparison between the numerical simulation results and theoretical predictions for $\tilde{d}_{min} = 0.455$ and 0.0455 . The calculation is performed for the vector waves, at $b_0(0) = 1$ and $k_0\ell_0 = 18.2$.

7.4.2 Effect of exclusion volume size

Next, we consider the effect of the exclusion volume size on the coherent transmission of light. The numerical calculations, for vector waves, are carried out for two different values of $\tilde{d}_{min}$, namely 0.455 and 0.0455 . The results are shown in Figure 7.9. The theoretical predictions are also computed for these two values of $\tilde{d}_{min}$, and shown as the dashed lines. The calculations are also performed at $k_0\ell_0 = 18.2$, with a good agreement between the numerical and theoretical results.

The numerical simulations results of $(k_0\ell_0)^2(\epsilon_r - \epsilon_{ISA})$ for the two cases are similar, even though the volume are three orders of magnitude different. In particular, the density of the cloud is $\rho_0 = 4.4 \mu\text{m}^{-3}$, and the typical separation among the atoms has a non-dimensionalized value of $\omega/c_0/\sqrt[3]{\rho_0} = 7.0$. The size of the exclusion volume is smaller than the typical separation of the atoms. Even for $k_0\ell_0 = 9.1$, we still have $\omega/c_0/\sqrt[3]{\rho_0} = 5.5$. This means the choice of $\tilde{d}_{min} = 0.455$ is small enough for it to be a good approximation to the ideal case of atomic scatterers ($d_{min} \rightarrow 0$).

7.4.3 Effect of $k_0\ell_0$ values

We now investigate the effect of varying $k_0\ell_0$. Effectively, the numerical simulation is performed at different values of the atomic density ρ_0 , as summarized in Tables 7.2 and 7.3, for the vector and the scalar cases, respectively. The results from both cases are reported in the following subsections.

Figure 7.10: The numerical simulation results of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{ISA})$, with vector waves, for $k_0 \ell_0 = 9.1, 18.2$ and 36.4 . The numerical results are compared to the theoretical prediction using Equation (7.34).

7.4.3.1 Vector waves

In Figure 7.10, we compare the numerical simulation results to the theoretical predictions, for the real and imaginary parts of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{ISA})$. The calculation results for $k_0 \ell_0 = 9.1, 18.2$ and 36.4 are shown in the figure. The parameters of the calculations are tabulated in Table 7.1. The theoretical predictions of the mass operator are also plotted in the figure as the black dashed curves. All the numerical results show generally good agreement with the theoretical curve.

To a large extent, the curves corresponding to the three different values of $k_0 \ell_0$ all fall on the same curve. This confirms that $\epsilon_r - \epsilon_{ISA}$ is indeed proportional to $\rho_0^2 \alpha_0^2$ in this range of parameters. The slight difference between the values of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{ISA})$ for different $k_0 \ell_0$, do not originate from ϵ_r terms of the order $1/(k_0 \ell_0)^3$. Indeed, as seen on the right panel of Figure 7.10 for the imaginary part of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{ISA})$, the discrepancies are larger for higher values of $k_0 \ell_0$. The opposite should be true, if the $1/(k_0 \ell_0)^3$ term is the dominant contribution of the error. These discrepancies are dominated by other sources of errors, which we have not yet managed to identify in our numerical simulations.

7.4.3.2 Scalar waves

A similar study is also carried out for the scalar waves, as shown in Figure 7.11.

An excellent agreement is achieved among the numerical simulation results at the

Figure 7.11: The scalar wave numerical simulation results of $(k_0 \ell_0)^2 (\epsilon_r - \epsilon_{ISA})$, with scalar waves, for $k_0 \ell_0 = 9.1, 18.2$ and 36.4 . The numerical results are compared to the theoretical prediction, showing an excellent agreement.

different values of $k_0 \ell_0$. Remarkably, the points representing the numerical results fall almost exactly on the black solid lines, which is the curves of the theoretical prediction. This validates the theoretical result of Equation (7.36) for the scalar waves.

Even if the full vector EM wave can be studied, the study of scalar wave is still interesting as it captures the essential behavior of light scattering. In particular, in the ISA regime, it gives the same result as the full vector model. Furthermore, the simplicity of the scalar model also allows us to extend the studies to more complicated situations, such as the recurrent scattering of light in the presence of atomic motion.

7.4.3.3 Future implementation

Overall, the numerical simulation results with the coupled dipole equations agree well with the theoretical results. This validates the perturbative expansion method to compute the ϵ_r . In particular, this result confirms the identification of recurrent scattering and two-body positional correlation as the two scattering diagram contributions to ϵ_r . These correction terms are shown to come in the order of $\rho_0^2 \alpha_0^2$.

Our experimental setup is currently capable of preparing a cloud with $k_0 \ell_0$ value which is of the order of 100. This is far from the regime to observe experimentally the effects studied in this chapter. One future improvement is to load the atoms

into a dipole trap for further compression. Evaporative cooling technique can also be applied to reduce the cloud temperature. With a suitable geometry, we can try to minimize the finite size effect as much as possible. It is then hope that the correction terms to ε_r that comes in $\rho_0^2 \alpha_0^2$, can be measured directly in our cold strontium experimental setup.

Chapter 8

Conclusion to Part I

This part of the thesis focused mainly on the experimental study of the cooperative effect that appears during the transient regime of coherent transmission. In those studies, the cold atomic cloud is dilute. Additionally, we also theoretically study the effect of atomic density on the coherent transmission of light, within a perturbative expansion framework. This study is eventually validated with numerical simulations of light transmitting through an atomic medium.

By applying fast changes to either the amplitude or the phase of the incident probe field, flashes of light have been observed to be emitted in the forward direction by an optically thick atomic cloud. By probing the narrow intercombination transition of ^{88}Sr , these flashes have been observed experimentally. Importantly, we find flashes with high peak intensities, which are greater than the incident intensity. We call them the superflashes. A peak intensity that is more than five times the incident intensity has been observed in our work. We further predicted theoretically that the flash peak intensity could go up to roughly 12 times the incident intensity.

In the emission of superflashes during the abrupt switch off of the probe beam, we find a counterintuitive result that the steady state forward scattered field is larger than the incident field. Furthermore, based only on the measurements of the flash peak intensity, a reconstruction of the steady state forward scattered field was carried out in the complex plane.

In addition to the high peak intensity values, another key feature of the flashes emitted in the experiments, is their short time scale. We define an initial decay time, which has an analytical form, to quantify the time scale of the flash emission. This time scale is shown to be shortened typically by a factor of $b_0(0)/2$, with $b_0(0)$ the resonant optical thickness at zero temperature. The shortening of this time scale is associated to the cooperative effect among the atoms. Interestingly, the cooperative parameter is not $b_{\bar{v}}(0)$, the resonant optical thickness of the cloud at its temperature, as one might suspect initially.

We further investigated the cooperative emission from the atoms when a periodic phase jump is applied to the incident field. Pulse trains were generated when a periodic phase modulation was applied to the incident field. Interestingly, due to the cooperative effect among the atoms, pulse trains with short repetition time are emitted. A full transfer of the optical power from the probe beam to the pulse train is observed at short repetition time. At the same time, the pulse train still exhibits a surprisingly high intensity contrast. Here, we have an example of a pulse train emission that is almost fully governed by the cooperative effect. Remarkably, this is achieved even when the atomic cloud is dilute and located in a free space. This is far from the usual situation to observe cooperative effects, where the atomic cloud is dense or located in a cavity.

As mentioned in Chapter 2, a similar emission of the flashes has been studied

using the Mössbauer spectroscopy. The γ -ray is emitted from a source that has the same isotopes as the in the target. The nuclei are placed inside a crystal lattice, so that recoilless emission and absorption of the γ -ray is achieved. Thus, to vary the probe frequency the source or the target nuclei has to be moved. In a cold atomic experimental setup, thanks to the laser source, one can have a continuous, monochromatic and easily tunable source emission, which is an advantage. Furthermore, cold atomic experiments are known to be easily controllable; many relevant parameters such as the laser power, detuning, and the number of atoms can be varied easily in the experiment. Importantly, the temperature and the density of the atomic cloud can also be varied. This is helpful for future systematic studies of the cooperative emission, as demonstrated by our work here. By using degenerate quantum gases, one can even study different type of non-trivial pair correlation function, which gives a rich behavior to the coherent transmission of light in the atomic systems.

One can model the physical phenomena in a cold atomic gas with good accuracy. This makes cold atomic experiments good simulators for physical effects in other systems. This means that the results on the cooperative emission studied in this thesis, can be applied to other systems with a better prospect for applications. For instance, the cooperative emission of the pulse trains from the atomic medium can be useful as a pulse generator. Even with a passive medium, we have shown that a gain of typically one order of magnitude is possible. Since the cooperative effect is not adversely affected by the temperature of the cloud, optically thick hot atomic gases can be used to generate the pulse trains. For example, a hot vapor of rubidium gas has an optical thickness of $b_v(0) \approx 600$ at 110 °C [136], and is a good candidate for pulse generation. Furthermore, for rubidium atoms, the 6 MHz natural transition linewidth leads to a time scale in the picosecond regime, increasing the repetition rate of the pulse trains.

Importantly, this pulse generating mechanism is not limited to atomic systems. A condensed matter system can also be used to generate pulse trains using the same cooperative emission mechanism. One interesting candidate is a samarium doped fiber (at the telecommunication band 1.55 μm) which has $b_0(0) \approx 100$. It is possible then for this pulse generation mechanism to potentially find some applications in the field of telecommunication. Nevertheless, there are additional decay channels in doped fibers that may lead to a phase decoherence. It is thus important to understand how these channels may affect the this pulse generation mechanism.

So far, our work has predicted only a peak intensity of roughly one order of magnitude larger than the incident intensity. In a dense medium containing N_a atoms, where the Dicke superradiance occurs, the emission is typically amplified by an factor of N_a^2 . In analogy with superradiance, one wonders if an emission

of the order of $b_0(0)^2$ could be achieved in our case. One future extension to this work could be to see if a more general type of phase modulation could lead to flashes with higher peak intensities. In our study, the large optical thickness of the medium means that only a small fraction of the atoms are excited in the steady state regime. One can then also look into ways to increase the number of excited atoms that emitted cooperatively, in order to achieve the emission of a stronger flash with higher peak intensity.

Finally, we have also studied the effect of cloud density on coherent transmission, since cooperative effect is commonplace for a dense medium [99, 103]. In particular, we have validated the theoretical studies of the second order (in density) corrections to the refractive index of the atomic medium. This provides a platform for which future studies into the effect of scatterer density on the cooperative emission of light, can be performed. The preparation of the atomic cloud can also be improved to allow for an experimental observation of the theoretical predictions here.

Part II

**Coherent Transmission under
Magnetic Field**

Chapter 9

Background

In Part II of this thesis, we study the effect of a static external magnetic field on the coherent transmission of light through a dilute cold atomic cloud. Unlike Part I, which includes studies in the transient regime of coherent transmission, we focus only on the steady state regime in this part.

The presence of magnetic field causes Zeeman shifts of the atomic levels. In our work, we consider magnetic field with a magnitude that is typically up to several tens of milligauss. This is well within the linear regime of Zeeman shift for the intercombination line transition studied here. We expect that this Zeeman shift affects the transport of light in a cold atomic cloud. For instance, in the case of the coherent backscattering (CBS) of light, the application of an external magnetic field leads to an improvement in the CBS peak contrast of a rubidium vapor [137]. This applied magnetic field isolates a unique two-level system for the scattering of light. Interestingly, this effect is opposite to the case of the CBS in a typical optically active medium. There, the magnetic field reduces the CBS peak contrast, due to a breaking of the time reversal invariance [138–143]¹.

The coherent transmission of light through an atomic cloud, is also affected by the magnetic field. The magnetic field sets a quantization axis for the atoms. The light polarization components are now resolved with respect to this quantization axis. Due to the Zeeman shift, each component of the light polarization experiences a different attenuation and phase shift, during its propagation through the cold atomic cloud. There are two special cases of magneto-optical rotation (MOR) experienced by the light polarization. The first is the Faraday rotation, which occurs when the magnetic field is aligned along the propagation direction of the light. The second is the Voigt rotation (also known as the Cotton-Mouton effect), which occurs when the magnetic field is perpendicular to the light propagation direction.

Theoretical studies have been performed by various groups to understand the MOR effects in atomic systems [37, 148, 149]. These effects have since been observed experimentally in hot atomic vapors [36–38, 150], and in cold alkali-metal atoms [39–42]. Compared to the typical condensed matter systems, an atomic system is more sensitive to the magnetic field. For example, in Ref. [40], the authors measured a large Verdet constant for cold rubidium gas², which is four orders of magnitude larger than a typical condensed matter systems. With this sensitivity, the atomic systems can be good candidates as magnetometers [151, 152]. The utilization of the MOR effects in atomic magnetometers has been reported in Refs. [43, 153–155]. Besides being useful as atomic magnetometers, the Faraday rotation effect in an

¹A similar situation is encountered in the condensed matter systems for the electronic wave [144–147]. The weak localization effect also reduces when a magnetic field is applied.

²For small magnetic field, the Faraday rotation angle θ is proportional to the magnetic field B and the length of the medium L , i.e., $\theta \propto BL$. The proportionality constant is known as the Verdet constant.

atomic system has been proposed to function as an optical isolator [156]. Using the MOR effects on the narrow resonances of atomic system, there are also proposals for rubidium vapor cells to function as narrow band Faraday filters [157–159].

In this work, we study the MOR effects using the narrow ^{88}Sr intercombination transition ($^1\text{S}_0 \rightarrow ^3\text{P}_1$, linewidth of $\Gamma/2\pi = 7.5$ kHz). A ^{88}Sr atom has a zero nuclear spin, meaning that there is no hyperfine splitting of the states. Therefore, the transition is purely a $J = 0 \rightarrow J = 1$ transition. This is one of the simplest system to study the MOR effects. A simple theoretical model, as used in Chapters 10 (linear case) and 13 (nonlinear case), is sufficient to describe the effects studied here.

Furthermore, the linewidth of the intercombination transition is typically three orders of magnitude smaller than the typical transition linewidth of alkali-metal atoms. Its equivalence in magnetic field is 3.57 mG (or equivalently 357 nT). This level of the sensitivity is comparable to the typical magnitude of stray magnetic field in the lab³. Thus, the outcome of the experiment is affected by the stray magnetic field.

For the study of cooperative emission reported in Part I, the influence of the stray magnetic field is suppressed by applying a large magnetic field parallel to the light polarization direction. Effectively, we isolate the $m = 0 \rightarrow m = 0$ transition. However, this method may not be suitable for any other types of experiments. Thus, we look to other methods to cancel the stray magnetic field experienced by the atomic cloud.

One way to remove the effect of the magnetic field is through a passive shielding of the stray magnetic field [160]. As described in Appendix C, this is employed around the Zeeman slower. Close to the science chamber of the experimental setup, magnetic shielding is not practical since many optical axes are required for the experiments. An alternative is to measure regularly the magnetic field noise around the science chamber, and cancel it with compensation coils [161]. This works for the case of a static stray field. To remove the 50 Hz fluctuations of the magnetic field, one widely used technique is to synchronize the experiment with the 50 Hz phase of the line power supply. However, this still does not compensate the stray magnetic field that originates from human activities, which typically has a low frequency below 1 Hz.

Here, we follow the work of Refs. [162,163] to implement an active magnetic field control system for our cold ^{88}Sr experimental setup (see Chapter 11 for a description

³There are various sources of stray magnetic field in the lab. In addition to the static contribution of the earth’s magnetic field, and the magnetized components around the setup, human activities and the 50 Hz line power supply also produces fluctuations in magnetic field that is of the order of a few milligauss.

of this system). This system allows us to cancel both the low frequency and the 50 Hz frequency component of the stray magnetic field.

While the active control system can work in a standalone manner, we integrate it with the cold atomic experimental setup. Before it can be used, it has to be calibrated. We do so by performing MOR experiments on the intercombination transition of ^{88}Sr . This gives us the required sensitivity to calibrate the measurements subsystem of the control system. Finally we achieve the target accuracy that is lower than 1 mG (see Chapter 12) for our control system. The proportional-integral-derivative (PID) controller that cancels the stray magnetic field is further shown to have a good stability over a month-long duration.

With this control system, we can perform experiments that are sensitive to the magnetic field. One example is the study into the nonlinear MOR (NMOR) effect, reported in Chapter 13. The active magnetic field control system is applied in this study, since a good control of the magnetic field is required. This NMOR effect that is studied in our work, appears when probe beam has an intensity higher than the saturation intensity of the transition. In this case, the coherences between different substates of the atom cannot be neglected [164–166]. In addition to that, the propagation of the probe beam also becomes nonlinear due to the absorption of the light in the medium. We show that at high temperature, the sensitivity of the NMOR effect can be higher than sensitivity of the LMOR effect at the same temperature [167]. Furthermore, the NMOR sensitivity is found to saturate to a nonzero value at high cloud temperature. Thus, our results can also be applied to hot atomic vapors.

In the next chapter, we first review the theoretical background for the MOR effect in the linear regime, following the results reported in Ref. [149].

Chapter 10

Theory of Linear Magneto-Optic Rotation

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This chapter reviews the theory of magneto-optical rotation (MOR) of light polarization, focusing particularly on the linear MOR (LMOR) effect. To discuss the theoretical aspects of LMOR, we use the approach outlined in Refs. [149, 168]. We focus only on a $J = 0 \rightarrow J = 1$ transition, consistent with the intercombination transition of ^{88}Sr studied experimentally. For the LMOR effect, we consider only a probe beam with a weak intensity to stay in the linear-optics regime of light scattering. The theoretical discussion in this chapter provides the basis to understand the LMOR effect used in the calibration of the active magnetic field control system (see Chapter 12). The case of nonlinear magneto-optical rotation (NMOR) is analyzed later in Chapter 13.

In section 10.1, the general theoretical background for the LMOR is discussed. We then focus specifically on the Faraday rotation effect in Section 10.2, and the Voigt rotation in Section 10.3.

10.1 Theoretical background for LMOR

Our studies focus on the linear regime of Zeeman effect for the atoms. In the $J = 0 \rightarrow J = 1$ transition, only the $m = \pm 1$ substates of the excited state are shifted by the magnetic field. For a magnetic field with magnitude B , the Zeeman shift for the $m = \pm 1$ substates, are given by $\pm\delta_B = \pm g_J \mu_B B / \hbar$. Here, g_J is the Landé g-factor of the excited state, μ_B is the Bohr's magneton, and $\hbar$ is the reduced Planck's constant. The Zeeman shifts of the $J = 1$ substates are depicted in Figure 10.1.

Consider now a weak probe beam of frequency ω_l propagating through the atomic medium. The detuning δ of this probe beam is measured from the resonant frequency of the transition at zero magnetic field, ω_0 . For ^{88}Sr , this is also the transition frequency for the $m = 0 \rightarrow m = 0$ transition, since this transition is insensitive to the magnetic field. The direction of the applied magnetic field sets the quantization axis for the atoms inside the medium. With respect to this quantization axis, the light field $\mathbf{E}$ is decomposed into the π and $\sigma_{\pm}$ polarization components¹. Due to the difference in the Zeeman shifts experienced by the different excited substates, the polarization of the light is modified upon propagating through the medium.

¹For a given quantization axis, the polarization components of the light are labeled according to the transitions that they target. The π polarization component is the linear polarization component that is aligned along the magnetic field direction. It is responsible for the $m = 0 \rightarrow m = 0$ transition. The $\sigma_{\pm}$ polarization components are the circular polarization components which are perpendicular to the magnetic field. They are responsible for the $m = 0 \rightarrow m = \pm 1$ transitions.

Figure 10.1: Figure showing the Zeeman shifts of the $m = \pm 1$ substates of the $J = 1$ excited state. The π and $\sigma_{\pm}$ transitions between the excited substates and the ground state are also shown. The specific example of the ^{88}Sr intercombination transition is shown in this figure. The magnetic field has a magnitude of B . The frequency of the incident beam is denoted by ω_l . The resonant frequency at zero magnetic field is denoted by ω_0 . The π transition does not undergo any Zeeman shift.

In the next section, by adapting the results of Ref. [149], we write down the expressions for the effective refractive index n_r of the medium. The two polarization eigenmodes of coherent light transmission are identified along with the refractive indices associated with them. For now, we focus only on a weak probe beam. This means that the two polarization modes propagate through the atomic medium independently. This simplifies tremendously the calculation presented in the following.

10.1.1 Polarization modes

We assume that the probe beam propagates along the y -direction², with the following wave vector,

$$\mathbf{k} = k\mathbf{e}_y = k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (10.1)$$

²The choice of y -axis as the propagation direction, is consistent with the axis labeling in Figure 11.11, where the probe beam is sent along the horizontal direction perpendicular to atomic beam. To obtain the physically equivalent, but more conventional choice of z -axis being the propagation direction, we have to apply a cyclic permutation on the labels, i.e., $y \rightarrow z$, $z \rightarrow x$ and $x \rightarrow y$. Both sets of labeling lead to the same physics. For example, n_r as computed in Equation (10.3), is effectively a 2×2 matrix, as expected for the two component nature of light polarization.

The unit vectors are denoted by $\mathbf{e}_\nu$, with $\nu = x, y, z$. The polar and the azimuthal angles around the propagation direction $\mathbf{k}$, are denoted by ϑ and φ , respectively. By parameterizing the magnetic field using these two angles, we have

$$\mathbf{B} = B(\sin \vartheta \sin \varphi \mathbf{e}_x + \cos \vartheta \mathbf{e}_y + \sin \vartheta \cos \varphi \mathbf{e}_z) = B \begin{pmatrix} \sin \vartheta \sin \varphi \\ \cos \vartheta \\ \sin \vartheta \cos \varphi \end{pmatrix}. \quad (10.2)$$

Adapting the results of Ref. [149] to the geometry studied in our work, the effective refractive index of the medium is found to be

$$n_r = \Delta_k + \frac{\rho}{2} \begin{pmatrix} \beta_{\bar{v}} + [\alpha_{\bar{v}} - \beta_{\bar{v}}] \sin^2 \vartheta \sin^2 \varphi & 0 & \frac{\alpha_{\bar{v}} - \beta_{\bar{v}}}{2} \sin^2 \vartheta \sin 2\varphi + i\gamma_{\bar{v}} \cos \vartheta \\ 0 & 0 & 0 \\ \frac{\alpha_{\bar{v}} - \beta_{\bar{v}}}{2} \sin^2 \vartheta \sin 2\varphi - i\gamma_{\bar{v}} \cos \vartheta & 0 & \beta_{\bar{v}} + [\alpha_{\bar{v}} - \beta_{\bar{v}}] \sin^2 \vartheta \cos^2 \varphi \end{pmatrix}, \quad (10.3)$$

with

$$\alpha_{\bar{v}}(\delta) = \frac{3i\pi}{k^3} \sqrt{\frac{\pi}{2}} \frac{\Gamma}{k\bar{v}} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right], \quad (10.4)$$

$$\begin{aligned} \beta_{\bar{v}}(\delta, B) &= \frac{3i\pi}{2k^3} \sqrt{\frac{\pi}{2}} \frac{\Gamma}{k\bar{v}} \left\{ w \left[\frac{\delta - \delta_B + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] + w \left[\frac{\delta + \delta_B + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] \right\}, \\ &= \frac{1}{2} [\alpha_{\bar{v}}(\delta - \delta_B) + \alpha_{\bar{v}}(\delta + \delta_B)], \end{aligned} \quad (10.5)$$

$$\begin{aligned} \gamma_{\bar{v}}(\delta, B) &= \frac{3i\pi}{2k^3} \sqrt{\frac{\pi}{2}} \frac{\Gamma}{k\bar{v}} \left\{ w \left[\frac{\delta - \delta_B + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] - w \left[\frac{\delta + \delta_B + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] \right\}, \\ &= \frac{1}{2} [\alpha_{\bar{v}}(\delta - \delta_B) - \alpha_{\bar{v}}(\delta + \delta_B)]. \end{aligned} \quad (10.6)$$

Note that effectively n_r is a 2×2 matrix in the $z - x$ plane. The Faddeeva function w , as defined in Appendix B, appears due to a convolution between the Lorentzian response of the atoms and the Gaussian velocity distribution of the atoms. The distribution of the velocity component v along the propagation direction of the light is given by,

$$f(v) = \frac{1}{\sqrt{2\pi}\bar{v}} e^{-v^2/2\bar{v}^2}, \quad (10.7)$$

with $\bar{v}$ being the root-mean-square (rms) velocity of the atoms. The term Δ_k is the projector onto the subspace orthogonal to the direction of propagation,

$$\Delta_k = \mathcal{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.8)$$

where $\mathcal{I}$ is the identity matrix.

In the expression of n_r , we find three quantities, namely $\alpha_{\bar{v}}(\delta)$, $\beta_{\bar{v}}(\delta, B)$ and $\gamma_{\bar{v}}(\delta, B)$. Each term contains the response of the atoms to the various polarization components of the probe beam. $\alpha_{\bar{v}}(\delta)$ is the two-level atomic polarizability associated with the $m = 0 \rightarrow m = 0$ transition (π component) of the atoms. As expected, it is unaffected by the magnetic field. $\beta_{\bar{v}}(\delta, B)$ and $\gamma_{\bar{v}}(\delta, B)$ describes the contributions of the $\sigma_{\pm}$ components. $\alpha(\delta - \delta_B)$ is the two-level atomic polarizability of the σ_+ transition, while $\alpha(\delta + \delta_B)$ is the two-level atomic polarizability of the σ_- transition. These two quantities are dependent on B ; they play important roles for both cases of Faraday and Voigt rotations.

The polarization eigenmodes of the coherent light transmitted through the atomic medium are found by diagonalizing n_r . These eigenmodes are the polarization components that remain unchanged when they are propagating through the medium. The eigenvalues are the refractive indices of the corresponding eigenmodes. There are three eigenvalues for n_r in Equation (10.3). Among these three, one is zero. This mode corresponding to zero eigenvalue is irrelevant as it corresponds to a polarization mode directed the propagation direction. Indeed, since n_r is effectively a 2×2 matrix, only two eigenvalues are relevant for the polarization modes of coherent transmission. They are

$$n_{\pm} = 1 + \frac{\rho}{2} \left[\beta_{\bar{v}} + \frac{\alpha_{\bar{v}} - \beta_{\bar{v}}}{2} \sin^2 \vartheta \pm \sqrt{\gamma_{\bar{v}}^2 \cos^2 \vartheta + \frac{(\alpha_{\bar{v}} - \beta_{\bar{v}})^2}{4} \sin^4 \vartheta} \right]. \quad (10.9)$$

These eigenvalues are associated with the following polarization eigenmodes

$$\epsilon_{\pm} = \frac{1}{\mathcal{N}_{\pm}} \begin{pmatrix} -\frac{\alpha_{\bar{v}} - \beta_{\bar{v}}}{2} \sin^2 \vartheta \cos 2\varphi \pm \sqrt{\gamma_{\bar{v}}^2 \cos^2 \vartheta + \frac{(\alpha_{\bar{v}} - \beta_{\bar{v}})^2}{4} \sin^4 \vartheta} \\ 0 \\ \frac{\alpha_{\bar{v}} - \beta_{\bar{v}}}{2} \sin^2 \vartheta \sin 2\varphi - i\gamma_{\bar{v}} \cos \vartheta \end{pmatrix}, \quad (10.10)$$

In the above, $\mathcal{N}_{\pm}$ are the normalization factors. These eigenmodes are invariant under a spatial inversion symmetry of the magnetic field in the polarization plane, which is given by a transformation of $\varphi \rightarrow \pi + \varphi$. Furthermore, from Equation (10.9), we see that $n_{\pm}$ are independent of φ . This is due to the rotational

Figure 10.2: The light propagation and polarization directions, in the two sets of bases. The propagation direction of the light is along the y -direction, which is depicted here to be directed out of the plane. The polarization makes an angle of ψ with the z -axis. The axes a and b are rotated 45° away from the z and x axes.

symmetry of the magnetic field around the y -axis. However, the eigenvectors are not simply a vector that is linear in $\cos \varphi$ and $\sin \varphi$, as one might have expected. In general, the polarization modes have circular components which remain unchanged under a rotation around the y -axis.

Using these eigenvalues and eigenmodes, the changes in the light polarization as it transmits through a slab of atomic medium can be calculated. This is discussed in the next section.

10.1.2 Coherently transmitted field

As have been done in Part I, we consider a monochromatic plane wave, which propagates through an atomic medium having the geometry of a slab with thickness L . The linearly polarized incident field is,

$$\mathbf{E}_0 = E_0 e^{-i\omega_1 t} \begin{pmatrix} \sin \psi \\ 0 \\ \cos \psi \end{pmatrix} = E_0 e^{-i\omega_1 t} \boldsymbol{\epsilon}. \quad (10.11)$$

Initially, the light is polarized along $\boldsymbol{\epsilon}$, making an angle of ψ with the z -axis (see Figure 10.2). Following the convention in Part I, we define the incident field on the out-going surface of the atomic medium.

The steady state transmitted field is then given by

$$\mathbf{E}_t = \begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} = E_0 e^{-i\omega t} [c_+ e^{ik(n_+-1)L} \boldsymbol{\epsilon}_+ + c_- e^{ik(n_--1)L} \boldsymbol{\epsilon}_-]. \quad (10.12)$$

The complex coefficients $c_{\pm}$ can be computed from the inner product³

$$c_{\pm} = \boldsymbol{\epsilon}_{\pm} \cdot \boldsymbol{\epsilon}. \quad (10.13)$$

One way to describe the light polarization, is to use the Stokes parameters [89],

$$I_t = |E_z|^2 + |E_x|^2, \quad (10.14)$$

$$Q = |E_z|^2 - |E_x|^2, \quad (10.15)$$

$$U = 2\text{Re}\{E_z E_x^*\}, \quad (10.16)$$

$$V = -2\text{Im}\{E_z E_x^*\}. \quad (10.17)$$

The degree of polarization, P , of the transmitted field is computed as

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I_t}. \quad (10.18)$$

When $P = 1$, the light is fully polarized. When $P = 0$, the light is unpolarized⁴. In general, upon transmission through the optically active atomic medium, the polarization of the transmitted light becomes elliptical. Its ellipticity, e , is:

$$e = \tan \left[\frac{1}{2} \sin^{-1} \frac{V}{I_t P} \right]. \quad (10.19)$$

The polarization, is rotated by θ , which is defined as

$$\theta = \frac{1}{2} \tan^{-1} \frac{U}{Q} - \psi. \quad (10.20)$$

³For a pair of complex vectors, $\mathbf{a} = (a_1, a_2, a_3)^T$ and $\mathbf{b} = (b_1, b_2, b_3)^T$, the inner product is defined as $\mathbf{a} \cdot \mathbf{b} = a_1^* b_1 + a_2^* b_2 + a_3^* b_3$.

⁴ In our studies, we assume that the transmitted light is always fully polarized. We do not consider the possibility of having $P < 1$ for the transmission of light in presence of magnetic field, which is discussed in Refs. [40, 169].

When $P = 1$ and $e = 0$, an alternative form of θ can be written,

$$\theta = \frac{1}{2} \sin^{-1} \frac{U}{I_t} - \psi. \quad (10.21)$$

Using a different pair of axes, i.e., the a and b axes in Figure 10.2, we have

$$U = |E_a|^2 - |E_b|^2, \quad (10.22)$$

with the fields E_a and E_b being related to E_x and E_z by

$$E_a = \frac{1}{\sqrt{2}} (E_z + E_x), \quad (10.23)$$

$$E_b = \frac{1}{\sqrt{2}} (E_x - E_z). \quad (10.24)$$

Thus, for $P = 1$ and $e = 0$,

$$\theta = \frac{1}{2} \sin^{-1} \frac{I_a - I_b}{I_a + I_b} - \psi. \quad (10.25)$$

This formula can be directly applied in the balanced polarimeter measurement method [164, 170] implemented in our studies (see Section 12.1.2).

10.1.3 Small magnetic field limit

We consider now the limit of $\delta_B/k\bar{v} \ll 1$, where the effect of the magnetic field is small compared to the Doppler effect experienced by the atoms. We write in the following, the Taylor expansion of $\alpha_{\bar{v}}$ up to the second order in $\delta_B/k\bar{v}$,

$$\begin{aligned} \alpha_{\bar{v}}(\delta \pm \delta_B) &= \alpha_{\bar{v}}(\delta) \mp \frac{1}{(k\bar{v})} \left[-i\frac{\Gamma}{2}\alpha_0(0) + \left(\delta + i\frac{\Gamma}{2} \right) \alpha_{\bar{v}}(\delta) \right] \frac{\delta_B}{k\bar{v}} \\ &+ \frac{1}{2} \frac{1}{(k\bar{v})^2} \left\{ \left[\left(\delta + i\frac{\Gamma}{2} \right)^2 - (k\bar{v})^2 \right] \alpha(\delta) - i\frac{\Gamma}{2} \left(\delta + i\frac{\Gamma}{2} \right) \alpha_0(0) \right\} \left(\frac{\delta_B}{k\bar{v}} \right)^2. \end{aligned} \quad (10.26)$$

In the expression above, $\alpha_0(0) = 6\pi i/k^3 = i\alpha_0^5$. This expansion is performed

⁵We use the same definition as in Part I, where α_0 is the modulus of the resonant two-level atomic polarizability. For a vector wave, $\alpha_0 = 6\pi/k^3$.

with the help of Equation (B.6). Similarly for $\beta_{\bar{v}}$ and $\gamma_{\bar{v}}$, we have

$$\begin{aligned}\beta_{\bar{v}}(\delta, B) &= \alpha_{\bar{v}}(\delta) \\ &+ \frac{1}{2} \frac{1}{(k\bar{v})^2} \left\{ \left[\left(\delta + i\frac{\Gamma}{2} \right)^2 - (k\bar{v})^2 \right] \alpha_{\bar{v}}(\delta) - i\frac{\Gamma}{2} \left(\delta + i\frac{\Gamma}{2} \right) \alpha_0(0) \right\} \left(\frac{\delta_B}{k\bar{v}} \right)^2,\end{aligned}\tag{10.27}$$

$$\gamma_{\bar{v}}(\delta, B) = \frac{1}{(k\bar{v})} \left[-i\frac{\Gamma}{2} \alpha_0(0) + \left(\delta + i\frac{\Gamma}{2} \right) \alpha_{\bar{v}}(\delta) \right] \frac{\delta_B}{k\bar{v}}.\tag{10.28}$$

These expressions are useful when we consider the effects due to small magnetic field for both the Faraday and Voigt rotations.

10.2 Faraday rotation

In the case of the Faraday rotation, the magnetic field is aligned along the propagation direction, i.e., $\vartheta = 0$. The polarization modes of the coherent light transmission are given by the following two circular polarization vectors,

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \\ 0 \\ 1 \end{pmatrix}.\tag{10.29}$$

The refractive indices of the corresponding modes are found to be simply

$$n_{\pm} = 1 + \frac{\rho}{2} (\beta_{\bar{v}} \pm \gamma_{\bar{v}}) = 1 + \frac{1}{2} \rho \alpha_{\bar{v}} (\delta \mp \delta_B).\tag{10.30}$$

From the above expression, we see that the mode associated with n_+ , is the circular polarization of the $m = 0 \rightarrow m = +1$ transition. The other mode is associated with the $m = 0 \rightarrow m = -1$ transition. An illustration of the light polarization eigenmodes and the transitions they couple to, is shown in Figure 10.3.

Using Equation (10.12), the transmitted field for the Faraday rotation is given by

$$\mathbf{E}_t = \frac{E_0}{2} e^{-i\omega_L t} \begin{pmatrix} i \left[e^{ik\rho\alpha_{\bar{v}}(\delta-\delta_B)L/2} - e^{ik\rho\alpha_{\bar{v}}(\delta+\delta_B)L/2} \right] \\ 0 \\ e^{ik\rho\alpha_{\bar{v}}(\delta-\delta_B)L/2} + e^{ik\rho\alpha_{\bar{v}}(\delta+\delta_B)L/2} \end{pmatrix}.\tag{10.31}$$

Faraday Rotation

Figure 10.3: The two polarization eigenmodes of Faraday rotation. The magnetic field B is applied along the propagation direction of incident beam. The incident field is decomposed into the two orthogonal circular polarization eigenmodes. They are labeled by the $\sigma_{\pm}$ transitions that they target. In the linear response regime, these two polarization eigenmodes propagate independently through the medium.

10.2.1 Faraday rotation at small magnetic field

For a small magnetic field of $\delta_B/k\bar{v} \ll 1$, one may expand $n_{\pm}$ in Taylor series. Applying Equation (10.26) up to first order in $\delta_B/k\bar{v}$, we find

$$n_{\pm} = 1 + \frac{1}{2}\rho\alpha_{\bar{v}}(\delta) \pm \frac{\rho}{2} \frac{1}{k\bar{v}} \left[-i\frac{\Gamma}{2}\alpha_0(0) + \left(\delta + i\frac{\Gamma}{2} \right) \alpha_{\bar{v}}(\delta) \right] \frac{\delta_B}{k\bar{v}}. \quad (10.32)$$

At small magnetic field, the rotation angle is a linear function of B ⁶. For convenience, the following function is defined,

$$\mathcal{A}_1(\delta) = \frac{gJ\mu_B}{\hbar(k\bar{v})^2} \left[-i\frac{\Gamma}{2}\alpha_0(0) + \left(\delta + i\frac{\Gamma}{2} \right) \alpha_{\bar{v}}(\delta) \right]. \quad (10.33)$$

The polarization of the incident field is assumed to be aligned along the z -axis,

⁶The second order term proportional to B^2 , is zero.

that is, $\psi = 0$. The transmitted field is then given by

$$\mathbf{E}_t = E_0 e^{-i\omega t} e^{ik\rho\alpha_{\bar{v}}(\delta)L/2} \begin{pmatrix} -\sin[k\rho\mathcal{A}_1(\delta)BL/2] \\ 0 \\ \cos[k\rho\mathcal{A}_1(\delta)BL/2] \end{pmatrix}. \quad (10.34)$$

The transmitted intensity, I_t , to linear order in B , is given simply by

$$I_t = I_0 e^{-b_{\bar{v}}(\delta)}. \quad (10.35)$$

We have identified an optical thickness $b_{\bar{v}}(\delta) = k\rho_0 L \text{Im}\{\alpha_{\bar{v}}(\delta)\}$. This is consistent with Equation (3.65) derived in Part I, i.e.,

$$b_{\bar{v}}(\delta) = b_0(0) \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \text{Re} \left\{ w \left(\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right\}, \quad (10.36)$$

where $b_0(0)$ is the resonant optical thickness of a medium at zero temperature. The modification of a small magnetic field on the total transmitted intensity, comes only in the quadratic order⁷.

The Stokes parameter U , as defined in Equation (10.16), is

$$U = -I_0 e^{-b_{\bar{v}}(\delta)} \sin[k\rho L \text{Re}\{\mathcal{A}_1(\delta)\} B]. \quad (10.37)$$

Equation (10.15) gives the Stokes parameter Q , which is

$$Q = I_0 e^{-b_{\bar{v}}(\delta)} \cos[k\rho L \text{Re}\{\mathcal{A}_1(\delta)\} B]. \quad (10.38)$$

From Equation (10.20), and noting that $\psi = 0$, the rotation angle experienced by the light polarization upon transmission from the medium is

$$\begin{aligned} \theta &= -k\rho L \text{Re}\{\mathcal{A}_1(\delta)\} B/2, \\ &= -\frac{1}{k\bar{v}} \left\{ \frac{\Gamma}{4} [b_0(0) - b_{\bar{v}}(\delta)] + \delta\phi_{\bar{v}}(\delta) \right\} \frac{\delta_B}{k\bar{v}}. \end{aligned} \quad (10.39)$$

In Figure 10.4, we compare the prediction from above equation (black dash-dotted curve) to the full expression of Equation (10.20) (blue curve). The calculation is performed for a medium with $b_0(0) = 8$ and $k\bar{v}/\Gamma = 2$. The detuning of the probe

⁷The next higher order approximation for I_t is given by

$$I_t = I_0 e^{-b_{\bar{v}}(\delta)} e^{\text{Re}\{ik\rho\mathcal{A}_2(\delta)LB^2\}} \cosh[k\rho L \text{Im}\{\mathcal{A}_1(\delta)\} B],$$

with $\mathcal{A}_2(\delta)$ being a function that is defined in Equation (10.64). At small magnetic field, the transmitted intensity has a quadratic dependence on the magnetic field.

Figure 10.4: Comparison of various expressions used to compute the Faraday rotation angle. The blue solid curve is computed using Equation (10.20). An approximation for $V \approx 0$, allows the angle to be calculated using Equation (10.21). This is shown as the red dashed curve. At small values of B , Equation (10.39) can be used. This is shown as the linear black dot-dashed curve, which agrees well with the other two curves in the linear regime near zero magnetic field.

beam is set at $\delta = \Gamma$. Assuming $V \approx 0$, Equation (10.21) valid when e is small, is also used to compute θ . The result is shown as the red dashed curve. We find that all three curves agree well within a range of $|\delta_B| < 1$.

10.2.2 At resonance

We only perform our experiments at $\delta = 0$, where the sensitivity to the magnetic field is the best. The transmitted field in this situation can be written as

$$\mathbf{E}_t = \frac{E_0}{2} e^{-i\omega_l t} \begin{pmatrix} i [e^{ik\rho\alpha_{\bar{v}}(-\delta_B)L/2} - e^{ik\rho\alpha_{\bar{v}}(\delta_B)L/2}] \\ 0 \\ e^{ik\rho\alpha_{\bar{v}}(-\delta_B)L/2} + e^{ik\rho\alpha_{\bar{v}}(\delta_B)L/2} \end{pmatrix}. \quad (10.40)$$

Using the symmetric properties of $b_{\bar{v}}(x)$, which is even in x , and $\phi_{\bar{v}}(x)$, which is odd in x , we find

$$\mathbf{E}_t = E_0 e^{-i\omega_l t} e^{-b_{\bar{v}}(\delta_B)/2} \begin{pmatrix} -\sin[\phi_{\bar{v}}(-\delta_B)] \\ 0 \\ \cos[\phi_{\bar{v}}(-\delta_B)] \end{pmatrix}. \quad (10.41)$$

Here, we find a rotation of the polarization by

$$\theta = -\phi_{\bar{v}}(-\delta_B), \quad (10.42)$$

Besides, we have $V = 0$ at resonance, which means that $e = 0$. Thus, the polarization is rotated, but remains linear, after being transmitted.

The function $\mathcal{A}_1$ at zero detuning, is now a real quantity,

$$\mathcal{A}_1(0) = \frac{g_J \mu_B}{\hbar(k\bar{v})^2} \frac{3\pi\Gamma}{k^3} \left[1 - g\left(\frac{k\bar{v}}{\Gamma}\right) \right], \quad (10.43)$$

giving,

$$\theta = -\frac{k\rho L}{2} \mathcal{A}_1(0) B = -\frac{b_0(0)\Gamma}{4(k\bar{v})^2} \left[1 - g\left(\frac{k\bar{v}}{\Gamma}\right) \right] \delta_b. \quad (10.44)$$

This expression is consistent with Equation (10.42), with the function $g(x)$ defined in Appendix B. It is related to the Faddeeva function.

Since $P = 1$ and $e = 0$ at resonance, Equation (10.25) can also be used to compute θ . We go into the 45° basis along a and b . The field components are

$$E_a = \frac{E_0}{\sqrt{2}} e^{-i\omega_l t} e^{-b_0(0)/2} \{ \cos[k\rho \mathcal{A}_1(0)BL/2] - \sin[k\rho \mathcal{A}_1(0)BL/2] \}, \quad (10.45)$$

$$E_b = -\frac{E_0}{\sqrt{2}} e^{-i\omega_l t} e^{-b_0(0)/2} \{ \cos[k\rho \mathcal{A}_1(0)BL/2] + \sin[k\rho \mathcal{A}_1(0)BL/2] \}. \quad (10.46)$$

Using these expressions, we also arrive at Equation (10.44) for θ .

At $T = 0$, the small temperature limit of $g(x)$ in Equation (B.10) leads to

$$\theta = -b_0(0) \frac{g_J \mu_B B}{\hbar \Gamma}. \quad (10.47)$$

At high temperature, the following asymptotic behavior is found for the rotation angle of the polarization,

$$\theta = -\frac{b_{\bar{v}}(0)}{\sqrt{2\pi}} \frac{1}{k\bar{v}} \frac{g_J \mu_B B}{\hbar}. \quad (10.48)$$

Since $k\bar{v} \propto \sqrt{T}$, the linear slope of θ around $B = 0$, depends on the temperature as $|d\theta/dB/b_{\bar{v}}(0)^{-1}| \propto T^{-1/2}$. Indeed, the linear Faraday rotation is affected by the temperature; its sensitivity decreases as the cloud temperature increases.

10.2.3 Transmission in the polarization channels a and b

In applying Equation (10.25) to compute θ , the transmission in the a and b polarization channels have to be measured. Here, we compute the logarithm, b_a and b_b , of the transmittance for both polarization channels. Let us consider that the incident field is polarized along the z -axis. The incident intensity in both polarization channels are simply $I_0/2$. We can then define

$$b_a = -\log(2I_a/I_0), \quad (10.49)$$

$$b_b = -\log(2I_b/I_0), \quad (10.50)$$

where I_a and I_b are the transmitted intensity values in the polarization channels a and b , respectively.

At resonance, the fields in channels a and b are given by,

$$E_a = E_0 e^{-i\omega_l t} e^{-b_{\bar{v}}(\delta_B)/2} \cos \left[\frac{\pi}{4} + \phi_{\bar{v}}(-\delta_B) \right], \quad (10.51)$$

$$E_b = -E_0 e^{-i\omega_l t} e^{-b_{\bar{v}}(\delta_B)/2} \sin \left[\frac{\pi}{4} + \phi_{\bar{v}}(-\delta_B) \right]. \quad (10.52)$$

Therefore,

$$b_a = b_{\bar{v}}(\delta_B) - \log \left\{ 2 \cos^2 \left[\frac{\pi}{4} + \phi_{\bar{v}}(-\delta_B) \right] \right\}, \quad (10.53)$$

$$b_b = b_{\bar{v}}(\delta_B) - \log \left\{ 2 \sin^2 \left[\frac{\pi}{4} + \phi_{\bar{v}}(-\delta_B) \right] \right\}. \quad (10.54)$$

Two curves of b_a and b_b are plotted in Figure 10.5, at $k\bar{v}/\Gamma = 2$, with two detuning values of $\delta = 0$ and $\delta = \Gamma$. The resonant optical thickness of the cloud, at zero magnetic field and zero temperature, is given by $b_0(0) = 8$. The conditions used in the theoretical predictions are closely matched to the experimental conditions. For small values of magnetic field, b_a and b_b show different behavior for the two values of δ . However, at large value of magnetic field (i.e., $g\mu_B B/\hbar \gg \delta$), the behavior of b_a and b_b become similar.

Due to the rotation of the light polarization, the intensity in one of the polarization channel can be larger than $I_0/2$. This makes either b_a or b_b negative, with a minimum value of $-\log 2$. Therefore, b_a and b_b cannot be interpreted to be optical thickness quantities. Nevertheless, the optical thickness for the total transmission I_t , as given by,

$$b = -\log(I_t) = -\log \left[\frac{1}{2} e^{-b_a} + \frac{1}{2} e^{-b_b} \right], \quad (10.55)$$

is always positive.

Note that under the change of $B \rightarrow -B$, the roles of the $\sigma_{\pm}$ transitions are

Figure 10.5: The logarithm of transmitted intensity, in the case of Faraday rotation, for the two polarization channels a and b . The theoretical calculation is performed at two detuning values of the probe beam, $\delta = 0$ and $\delta = \Gamma$. The parameters of the calculations are $b_0(0) = 8$ and $k\bar{v}/\Gamma = 2$. They are similar to the actual experimental conditions.

swapped. This changes the sign of the rotation angle. Thus, the transmission signals for the channels a and b are also swapped, as is seen in Figure 10.5.

10.3 Voigt rotation

Besides the Faraday rotation, we also study the Voigt rotation, which corresponds to setting $\vartheta = \pi/2$. The MOR effect, now, depends on the angle that the magnetic field makes with the incident polarization ϵ . Similar to the case of Faraday rotation, the incident polarization is directed along z -direction ($\psi = 0$). φ gives the angle between $\mathbf{B}$ and the incident polarization, ϵ .

The two polarization modes in this case are

$$\epsilon_+ = \begin{pmatrix} \sin \varphi \\ 0 \\ \cos \varphi \end{pmatrix}, \quad (10.56)$$

$$\epsilon_- = \begin{pmatrix} -\cos \varphi \\ 0 \\ \sin \varphi \end{pmatrix}, \quad (10.57)$$

Voigt Rotation

Figure 10.6: The two polarization eigenmodes for Voigt rotation. Since the magnetic field is perpendicular to the propagation direction, π and $\sigma_{\pm}$ transitions all play a role here. The π transition gives the linear polarization eigenmode ϵ_+ , which is parallel to $\mathbf{B}$. For the transmitted light, the circular polarization components related to the $\sigma_{\pm}$ transitions, are projected onto the plane perpendicular to $\mathbf{k}$, giving the other linear polarization eigenmode of ϵ_- .

with the corresponding refractive indices,

$$n_+ = 1 + \frac{\rho \alpha_{\bar{v}}(\delta)}{2}, \quad (10.58)$$

$$n_- = 1 + \frac{\rho \beta_{\bar{v}}(\delta, B)}{2}. \quad (10.59)$$

One of the polarization mode, ϵ_+ , is aligned along the magnetic field direction. It targets π -component, which is independent of the magnetic field. The other polarization mode is perpendicular to $\mathbf{B}$. It is a combined effect of both the σ_+ and σ_- transitions (see Figure 10.6). They originate from the circular polarizations of light that are orthogonal to $\mathbf{B}$. In the transmitted light, the two circular polarizations are projected onto the plane perpendicular to the propagation direction to give the linear polarization ϵ_- . As expected, the refractive index for this mode depends on $\beta_{\bar{v}}(\delta, B) = \frac{1}{2} [\alpha_{\bar{v}}(\delta - \delta_B) - \alpha_{\bar{v}}(\delta + \delta_B)]$, which is the difference between the two-level polarizabilities of the $\sigma_{\pm}$ transitions.

Figure 10.7: Comparison of various expressions used to compute the Voigt rotation angle. The blue solid curve is calculated using Equation (10.20). Equation (10.21), valid when $V \approx 0$, is plotted as the red dashed curve. For small values of B , Equation (10.69) is plotted as the black dot-dashed line. The quadratic dependence on B close to zero magnetic field, is well captured by Equation (10.69). The detuning of the probe beam is set to $\delta = \Gamma$. We also set $b_0(0) = 8$, $k\bar{v}/\Gamma = 2$, $\psi = 0$, and $\varphi = 0.37\pi$ in our calculations.

We have, in this case,

$$c_+ = \cos \varphi, \quad (10.60)$$

$$c_- = \sin \varphi. \quad (10.61)$$

The transmitted field is then given by

$$\mathbf{E}_t = E_0 e^{-i\omega_l t} \begin{pmatrix} \cos \varphi \sin \varphi [e^{ik\rho\alpha_{\bar{v}}(\delta)L/2} - e^{ik\rho\beta_{\bar{v}}(\delta,B)L/2}] \\ 0 \\ \cos^2 \varphi e^{ik\rho\alpha_{\bar{v}}(\delta)/2} + \sin^2 \varphi e^{ik\rho\beta_{\bar{v}}(\delta,B)/2} \end{pmatrix}. \quad (10.62)$$

When $\boldsymbol{\epsilon} \parallel \mathbf{B}$, or $\boldsymbol{\epsilon} \perp \mathbf{B}$, the light polarization experiences no rotation. This is expected since the incident polarization is already one of the eigenmodes.

An example of θ for the Voigt rotation is shown in Figure 10.7 as the blue solid line. An approximation of the rotation angle using Equation (10.21) is plotted as the red dashed curve in the figure. The two predictions at $\delta = \Gamma$, $k\bar{v}/\Gamma = 2$ and $b_0(0) = 8$, agree rather well. For the calculations plotted in the figure, we assume that the magnetic field makes an angle of $\varphi = 0.37\pi$ with $\boldsymbol{\epsilon}$.

From Figure 10.7, the Voigt rotation is observed to have a quadratic dependence on B around zero magnetic field. This is a property that is further discussed in the next section. We notice also that θ saturates to a constant value at large magnetic field. To understand this, we first see that the polarization mode, ϵ_+ , which couples to the π transition, is unaffected by the magnetic field. The other polarization mode ϵ_- is perpendicular to this mode. At zero magnetic field, the ϵ_- mode has the same attenuation with the parallel mode, leading to a zero rotation of the polarization. As the magnetic field increases, the attenuation of the perpendicular mode decreases. With a large magnetic field, this polarization mode is fully transmitted. Thus, the value of θ remains unchanged with even higher magnetic field. If the optical thickness is large, i.e., $b_0(0) \gg 1$, the attenuation on the ϵ_+ polarization mode becomes exponentially larger. The polarization of the transmitted light is then rotated to ϵ_- .

10.3.1 Voigt rotation at small magnetic field

We turn our attention now to the Voigt rotation for small values of magnetic field ($g\mu_B B/k\bar{v}\hbar \ll 1$). Since only $\alpha_{\bar{v}}(\delta)$ and $\beta_{\bar{v}}(\delta, B)$ appear in the expressions for the Voigt rotation, it is expected that the lowest order dependence on B is a quadratic one [see Equation (10.27)]. As expected, the refractive index n_+ depends on $\alpha(\delta)$, and is not affected by the magnetic field. For the other polarization mode, we have

$$n_- = 1 + \frac{\rho\alpha_{\bar{v}}(\delta)}{2} + \frac{\rho\mathcal{A}_2(\delta)}{2}B^2, \quad (10.63)$$

with

$$\mathcal{A}_2(\delta) = \frac{1}{2} \left[\frac{gJ\mu_B}{\hbar(k\bar{v})^2} \right]^2 \left\{ \left[\left(\delta + i\frac{\Gamma}{2} \right)^2 - (k\bar{v})^2 \right] \alpha_{\bar{v}}(\delta) - i\frac{\Gamma}{2} \left(\delta + i\frac{\Gamma}{2} \right) \alpha_0(0) \right\}. \quad (10.64)$$

The transmitted field for $\psi = 0$, is given by

$$\mathbf{E}_t = E_0 e^{-i\omega t} e^{ik\rho\alpha_{\bar{v}}(\delta)L/2} \begin{pmatrix} \sin\varphi \cos\varphi \left[1 - e^{ik\rho\mathcal{A}_2(\delta)B^2L/2} \right] \\ 0 \\ \cos^2\varphi + \sin^2\varphi e^{ik\rho\mathcal{A}_2(\delta)B^2L/2} \end{pmatrix}. \quad (10.65)$$

To find a general expression for the rotation angle is difficult in this case. One can

assume $|k\rho\mathcal{A}_2(\delta)B^2L/2| \ll 1$, to find

$$\mathbf{E}_t = E_0 e^{-i\omega_l t} e^{ik\rho\alpha_{\bar{v}}(\delta)L/2} \begin{pmatrix} -i \sin \varphi \cos \varphi k\rho\mathcal{A}_2(\delta)LB^2/2 \\ 0 \\ 1 + i \sin^2 \varphi k\rho\mathcal{A}_2(\delta)LB^2/2 \end{pmatrix}. \quad (10.66)$$

The intensity of the transmitted field, which is also equal to Q , is given by

$$I_t = I_0 e^{-b_{\bar{v}}(\delta)} \left[1 + 2 \sin^2 \varphi \operatorname{Re} \left\{ \frac{ik\rho\mathcal{A}_2(\delta)L}{2} \right\} B^2 \right] = Q. \quad (10.67)$$

The other Stokes parameter, U , is given by

$$U = -2I_0 e^{-b_{\bar{v}}(\delta)} \sin \varphi \cos \varphi \operatorname{Re} \left\{ \frac{ik\rho\mathcal{A}_2(\delta)L}{2} \right\} B^2. \quad (10.68)$$

The rotation angle is then given by

$$\begin{aligned} \theta &= -\sin \varphi \cos \varphi \operatorname{Re} \left\{ \frac{ik\rho\mathcal{A}_2(\delta)L}{2} \right\} B^2, \\ &= \frac{1}{2} \left[\frac{g_J \mu_B}{\hbar(k\bar{v})^2} \right]^2 \sin \varphi \cos \varphi \left\{ \left[\delta^2 - \frac{\Gamma^2}{4} - (k\bar{v})^2 \right] \frac{b_{\bar{v}}(\delta)}{2} + \Gamma \delta \phi_{\bar{v}}(\delta) + \frac{\Gamma^2}{4} \frac{b_0(0)}{2} \right\} B^2. \end{aligned} \quad (10.69)$$

The above expression holds true for a small rotation angle with a small magnetic field, including also the case of low optical thickness. The rotation angle has a quadratic dependence on B at small magnetic field. A plot of the above expression is shown in Figure 10.7 as the black dot-dashed curve. As is evident from the figure, Equation (10.69) approximates well the behavior of θ at small values of B .

10.3.2 At resonance

Experimentally, we study the Voigt rotation at $\delta = 0$. Thus, we have

$$\beta_{\bar{v}}(0, B) = \frac{1}{2} [\alpha_{\bar{v}}(-\delta_B) + \alpha_{\bar{v}}(\delta_B)] = i \operatorname{Im} \{ \alpha_{\bar{v}}(\delta_B) \}, \quad (10.70)$$

and the transmitted field becomes

$$\mathbf{E}_t = E_0 e^{-i\omega_l t} \begin{pmatrix} \cos \varphi \sin \varphi [e^{-b_{\bar{v}}(0)/2} - e^{-b_{\bar{v}}(\delta_B)/2}] \\ 0 \\ \cos^2 \varphi e^{-b_{\bar{v}}(0)/2} + \sin^2 \varphi e^{-b_{\bar{v}}(\delta_B)/2} \end{pmatrix}. \quad (10.71)$$

The rotation of the light polarization at resonance, occurs purely due to the difference in the absorption of the two independent polarization modes. Similar to the resonant case of Faraday rotation, we have $e = 0$, meaning that the polarization remains linear when transmitted through the atomic medium.

For small magnetic field, the angle of rotation becomes

$$\theta = b_0(0) \frac{\sin \varphi \cos \varphi}{4} \left\{ \frac{\Gamma^2}{4(k\bar{v})^2} \left[1 - g \left(\frac{k\bar{v}}{\Gamma} \right) \right] - g \left(\frac{k\bar{v}}{\Gamma} \right) \right\} \left(\frac{\delta_b}{k\bar{v}} \right)^2. \quad (10.72)$$

At zero temperature, Equation (B.11) can be used to find

$$\theta = -2b_0(0) \sin \varphi \cos \varphi \left[\frac{g_J \mu_B B}{\hbar \Gamma} \right]^2, \quad (10.73)$$

with the maximum rotation occurring at $\varphi = \pi/4$, i.e.,

$$\theta = -b_0(0) \left[\frac{g_J \mu_B B}{\hbar \Gamma} \right]^2. \quad (10.74)$$

We note that the above expression holds true at low optical thickness. At resonance, unlike the case of Faraday rotation, the Voigt rotation angle does not increase in an unbounded manner. In fact, we have $|\theta| < \pi/2$ even for a medium with large optical thickness. This is because one of the polarization eigenmode will be completely absorbed by the medium.

Finally, the quadratic dependence on a small magnetic field makes the Voigt rotation less sensitive to the magnetic field, as compared to the case of Faraday rotation. This has an implication for the calibration and performance of the active magnetic field control system, as discussed in Chapter 12.

In this chapter, we have given a brief review of the theory of LMOR effect, focusing on the two special cases of Faraday and Voigt rotations. Their behaviors at small magnetic field are also discussed. We especially discuss also the cases when $\delta = 0$, as this is the detuning for which the experiments are performed. In the following section, we describe a key component to perform the MOR studies on the intercombination line of ^{88}Sr , which is an active control system for the magnetic field.

Chapter 11

Active control of magnetic field

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As mentioned in Chapter 10, a magnetic field of 3.57 mG shifts the $m = \pm 1$ substates of the $^{88}\text{Sr } ^3\text{P}_1$ excited state $\Gamma/2\pi = 7.5$ kHz, which is its linewidth. This means that the atoms are particularly sensitive to the stray magnetic field that can be found in our laboratory. The stray field that has to be compensated, includes the earth's magnetic field at typically 0.5 G. This is many orders of magnitude larger than the milligauss sensitivity of the intercombination transition. Besides, there is also the 50 Hz AC magnetic field from the line power supply. This has a peak-to-peak value of 1-2 mG in the lab. Human activities further introduce magnetic noise in the milligauss range.

Unwanted magnetic field bias can also come from the components used in our experimental apparatus. To reduce this contribution, we mainly use nonmagnetic components close to the science chamber. This will also lead to a more homogeneous field around the science chamber, thus improving the quality of the interpolation of the magnetic field at the position of the atoms by the sensor network (see Section 11.4.1). The remaining stray fields that remain are then compensated by an active control system. This system cancels the magnetic field below the level of 1 mG. This chapter is devoted to a description of this system, independent of the cold atomic experiment. Its performances when integrated with the cold atomic experimental setup are reported in the next chapter.

11.1 Main components of the control system

A flowchart summarizing the central system is shown in Figure 11.1. It contains the following three major components,

- a sensor network to measure the magnetic field,
- a personal computer (PC), with a data acquisition card, for signal processing,
- several pairs of compensation coils to control the magnetic field.

The magnetic field is measured by a network of sensors. The sensor network consists of eight three-axis magnetic field sensors that are arranged in a cuboid geometry around the science chamber.

The readings of the eight sensors are subsequently analyzed in a PC. The communication between the PC and the sensor network is mediated by a USB-CAN adapter. This adapter communicates with the sensor network through the controller area network (CAN) communication protocol, while it communicates with the PC through the universal serial bus (USB) communication protocol. The timing of the

Figure 11.1: A flowchart showing the system for an active control of the magnetic field.

magnetic field measurement by each three-axis sensor is also synchronized by this adapter¹.

In the PC, a user interface is implemented to operate the active system to control the stray magnetic field. It performs the digital processing of the measurement signals. When the magnetic field readings from the sensor network are received by the PC, the program performs an interpolation to find the magnetic field at the position of the trapped atoms. Several proportional-integral-derivative (PID) controllers and frequency filters are also implemented digitally to control the magnetic field. The control variable of the PID is converted by a data acquisition (DAQ) card (National Instruments PCI6733) to an analog voltage output, which is used to change the low noise current flowing through the compensation coils. This current is supplied by a current controller unit (Thorlabs LD8040)².

We have mounted in our system, two types of compensation coils. For the three components of the magnetic field, three pairs of Helmholtz coils are used to independently control the three components of the magnetic field. Three other pairs of coils are connected in the anti-Helmholtz configuration to control the $\partial B_x / \partial x$,

¹A counting cycle is implemented in the USB-CAN adapter. The conversion of the sensor analog reading to a digital signal, is synchronized by the USB-CAN adapter to measure the magnetic field simultaneously.

²Each current controller unit can provide a current ranging from 0 A to 4 A. The applied current can be varied manually, or through a voltage modulation input to the unit. An applied voltage of 1 V on the modulation input corresponds to a change of 0.4 A in the supplied current.

Figure 11.2: A photo of the three-axis magnetic field sensor. Each sensor consists of two commercial Honeywell magnetic field sensors, the HMC1001 and HMC1002 sensors. The two red arrows show the positive directions of the magnetic field components measured by the HMC1002 sensor. The positive direction of the component measured by HMC1001 is in the direction pointing out from the paper.

$\partial B_y/\partial y$, and $\partial B_z/\partial z$ components of the field gradient. In this thesis, only the Helmholtz coils are described. The feature to compensate the magnetic field gradient components, has not been fully implemented and tested. This is a work left for the future.

In the following, we first describe the sensor network. The properties and the performances of the sensors are presented. Next, the Helmholtz coils used to control the magnetic field are discussed in Section 11.3. Finally, we discuss the digital processing of the measurement outcomes in Section 11.4. There, we also describe an important feature that allows us to track the PID action when the PID loop is opened.

11.2 Sensor network

The sensor network consists of eight three-axis sensors. A photo of one such home-made three-axis sensor is shown in Figure 11.2. The positive directions for the x -, y - and z -channels of the three-axis sensor are indicated in the figure.

Figure 11.3 shows the major components of the three-axis sensor. Each three-axis magnetic field sensor consists of two commercial Honeywell magnetic field sensors. They are the HMC1001 and HMC1002 sensors, which are the single- and two-axis magnetic field sensors, respectively (see a schematic plot of the major components of the three-axis sensor in Figure 11.3). The HMC sensors have analog measurement outputs, which are subsequently amplified and converted to digi-

Figure 11.3: Major components of the three-axis sensor used to measure the magnetic field. The analog output from the each Wheatstone bridge, which measures one magnetic field component, is amplified 200 times by an op-amp circuit, and digitalized by an ADC. Finally, the digital signals are sent to the USB-CAN adapter by the CAN transceiver.

tal signals using an analog-to-digital converter (ADC, ADC7682). The digitalized readings are then sent by the CAN transceiver to the USB-CAN adapter. Here, the readings for all the eight three-axis sensors are collected, before being sent to the PC using USB communication.

In the following, we describe the HMC sensors that are used to construct the three-axis sensors.

11.2.1 Honeywell sensors

The commercial Honeywell sensors, HMC1001 and HMC1002, are the components that are responsible to measure the magnetic field. The HMC1001 sensor contains only one Wheatstone bridge, while in HMC1002, there are two Wheatstone bridges. The sensor operates based on the anisotropic magnetoresistive effect, with each Wheatstone bridge measuring one component of the magnetic field. By proper orientation of the HMC sensors (see Figure 11.2) on the printed circuit board (PCB) of the three-axis sensor, the full magnetic field vector can be measured.

There are four resistive strips in each Wheatstone bridge; one strip on each arm of the bridge (see Figure 11.4). The resistive strips are made of nickel-iron (permalloy) thin films. The sensor operates based on the AMR effect. Each permalloy thin film resistive strip is fabricated in a “barber pole” pattern, as shown in Figure 11.4³.

³A general discussion of the permalloy magnetoresistive strips can be found in Ref. [171]

Figure 11.4: The “barber pole” patterned resistive strips (left) are arranged inside the HMC sensor in the form of a Wheatstone bridge (right). Ideally, at zero magnetic field, the resistance of the four strips (I, II, III and IV) are equal, giving a zero potential differential across the output pins **OUT+** and **OUT-**. External magnetic field changes the resistance of the strips and disrupts the balance of the bridge. As a result, a nonzero voltage appears across the pins **OUT+** and **OUT-**. The dotted lines with the resistors are shunt resistors which can be added to reduce the bridge offset (see Section 11.2.3).

The “barber pole” configuration causes the current to flow at an angle of 45° along the resistive strip. The magnetization vector of the resistive strip is aligned to the easy axis along the resistive strip, as shown in Figure 11.4. Thus, the direction of the current through each of the resistive strip makes an angle of 45° with the magnetization vector. Ideally, when there is no external magnetic field, the bridge is balanced, i.e., all the resistive strips have equal resistance⁴. In reality, a zero balancing is never exactly achieved, leading to an offset in the output voltage of the bridge (see Section 11.2.3).

The Wheatstone bridge in this configuration measures the magnetic field component that is along its sensitive axis (see Figure 11.4), which is perpendicular to the easy axis. When an external magnetic field B is applied along the sensitive axis of the Wheatstone bridge, a transverse magnetization component is added to each resistive strips. The angle between the current and the magnetization vector changes in a different manner for the different resistive strips⁵. Since, the magnetoresistance of the strip depends on the angle between the current and the magnetization vector, the balance of the Wheatstone bridge is. A voltage can now be measured across the two output pins of the Wheatstone bridge. For a small magnetic field, the change in the resistance is proportional to the component of the applied magnetic field B along the sensitive axis [172]. The output voltage across the pins **OUT+** and **OUT-** is

$$V_{\text{out}} = \frac{\Delta R}{R} V_w \propto B, \quad (11.1)$$

⁴In the HMC sensors, each strip has a resistance of typically $R = 1100 \, \Omega$.

⁵The magnetic field makes the resistance of strips I and III different from the resistance of strips II and IV.

Figure 11.5: The magnetic domains of the resistive strip in the set state and in the reset state.

where V_w is the voltage applied across the Wheatstone bridge (see Figure 11.4).

11.2.2 Set-reset strap

Each permalloy strip in a HMC sensor has many magnetic domains (see Figure 11.5). When a large magnetic field is applied and removed subsequently, the magnetization vector may not return completely to its initial orientation along (or opposite) the easy axis. This results in a loss of sensitivity for the sensors. For the HMC sensor, a 3 G magnetic field is already sufficient to affect its sensitivity. In a typical cold atomic experiment, a large magnetic field is generated by the coils to carry out a magneto-optical trap (MOT). Since the sensors are in close proximity to the coils, the sensitivities of the sensors are affected by the preparation of the cold atomic cloud.

To restore the sensor sensitivity, we use the set-reset straps of the HMC sensor. These set-reset straps apply a magnetic field to realign the magnetization vectors of the resistive strips. There are two possible directions to align the magnetization vector (see Figure 11.5), depending on the direction of the current in the set/reset strap. If a positive current is flowing in the set-reset strap, the resistive strip is restored along the easy axis. This gives us the set state of the sensor. On the other hand, with a negative current, the resistive strip is put in the reset state. In the reset state, the magnetization vector is opposite to the easy axis. In both states, the sensor can be used to measure the magnetic field. Ideally, the sensitivities of the set and reset states are opposite in sign. A sketch of the expected sensor readout, as a function of the magnetic field component along the sensitive axis, is shown in Figure 11.6 for the two states of the sensor.

To study qualitatively the properties of the sensors, we place them inside a zero gauss chamber (ZGC). The ZGC consists of three layers of mu-metal magnetic shield⁶. Inside the ZGC, we place a solenoid can be used to apply a uniform mag-

⁶The magnetic field inside the ZGC has a value that is smaller than a fraction of milligauss.

Figure 11.6: A sketch of the output voltages in the set and the reset states, as a function of the magnetic field component B along the sensitive axis. Due to the bridge offset, the intersection between the two curves occurs at a nonzero value. The output voltages in the set and reset states are ideally symmetrical around the value of the bridge offset, V_b . A differential measurement of the magnetic field, with the sensor being in the set and reset states alternatingly, allows the bridge offset to be eliminated from the measurement (see the description in Section 11.2.3).

netic field. The sensor is placed at the center of the ZGC, and positioned such that the magnetic field generated by the solenoid is along its x -direction.

Using the ZGC, the set and reset output voltages for the sensors are measured in a low magnetic field environment. An example of the measured magnetic field signal is plotted in Figure 11.7. The signal has already been digitalized to 16-bit (see Section 11.2.4 for more details on the amplification and digitalization of the measurement signals). We apply a sinusoidal current in the solenoid, with a positive offset current⁷. By looking at the peak-to-peak value of the magnetic field signals, we can compare between the sensor sensitivities in the set and reset states. Ideally, they should be the same, as indicated by the equal gradient of the two curves in Figure 11.6. With our sensors, we measure a maximum of 2% relative difference between the magnitudes of the sensor sensitivity in the set and the reset states.

The bridge offset due to a non perfect balancing of the Wheatstone bridge can be seen in Figure 11.7. Ideally, the measurement signal in the set and reset states should be symmetric around the zero output value. Instead, they are measured to

It contains a solenoid, which provides a uniform magnetic field of 12.84 G at the center along its axis, when a current of 1 A flows in the solenoid.

⁷This offset current is required since we use the LD 8040, which can only supply positive-valued current.

Figure 11.7: The x -channel measurement output of a three-axis sensor. The set and reset currents are applied to the sensors in an alternating manner. A sinusoidal magnetic field is inside the ZGC, along the x -channel of the sensor. The calculation of the bridge offset V_b , using Equation (11.2), results in the black curve. A zoom of the plot is shown in the inset. Each reading in the set state is followed by another in the reset state, and vice versa.

be symmetric about the nonzero black line that is the bridge offset⁸. We discuss further the bridge offset voltage in the next section. For now, we point out an interesting observation of the slight oscillation of the black line in Figure 11.7. This is due to the slight difference in the sensitivities of the set and reset states of the sensor.

11.2.3 Bridge offset

Let us assume in the ideal situation of $V_b = 0$, the HMC sensor has a measurement outcome of V_B in the set state, and $-V_B$ in the reset state. As shown in Figure 11.6, when $V_b \neq 0$, the measurement outcomes are modified as $V_{o,\text{set}} = V_B + V_b$ in the set state, and $V_{o,\text{reset}} = -V_B + V_b$ in the reset state. Both measurements are modified by the bridge offset. Averaging these two measurements, gives us bridge offset voltage, i.e.,

$$V_b = \frac{1}{2} [V_{o,\text{set}} + V_{o,\text{reset}}]. \quad (11.2)$$

This equation is used to compute the black V_b curve plotted in Figure 11.7.

The V_b values of the HMC sensors are listed in the datasheets to be typically

⁸For the results in Figure 11.7, the bridge offset value has already been reduced by adding some shunt resistors (see Section 11.2.3).

10 mV. With the sensitivity of the sensors (see Table 11.1), this offset corresponds to a value that is around 0.5 G, much larger than the milligauss sensitivity of the atomic transition. Hence, it is important to remove the bridge offset from the sensor measurements.

Shunt resistors with appropriate values (shown as a dotted line in the right figure of Figure 11.4) is added on certain arms of the Wheatstone bridge to reduce the bridge offset at zero magnetic field. For the example in Figure 11.7 the shunt resistors have been added to the sensor to reduce the bridge offset. While it is difficult to exactly balance the bridge by adding the shunt resistors, we can still significantly reduce the bridge offset values. To completely remove the bridge offset, the following method is eventually used.

With the measurements in the set and reset states, the magnetic field, V_B , can be obtained directly by computing,

$$V_B = \frac{1}{2} [V_{o,\text{set}} - V_{o,\text{reset}}]. \quad (11.3)$$

Using the above expression, the value of V_B is measured directly, by eliminating the bridge offset V_b ⁹. In the actual implementation of the magnetic field measurement, the set and reset currents are applied alternatingly one after another. Two consecutive outputs of sensor in the set and the reset states are used to compute the magnetic field. An example of this measurement procedure is shown in the inset of Figure 11.7.

Next, we discuss the amplification and digitalization of the readings, before they are sent to the PC.

11.2.4 Amplification and digitalization of the signal

The typical sensitivity of the HMC sensors is listed in the data sheet to be 3.2 mV/V/G, under the condition of a 3 A set-reset current. In the HMC sensors, a supply voltage of $V_w = 5$ V is applied to the Wheatstone bridges. For a 1 mG magnetic field along the sensitive direction, the output voltage across the **OUT+** and **OUT-** pins is 16 μ V. This output value is too small to be detected and has to be amplified. An ultra-low offset and drift op-amp (MAX4238) is used to amplify the signal for each component of the magnetic field. The amplification is set to be 200 for all the three components.

⁹A reduction of the bridge offset value by the shunt resistors is still useful as it can increase the operational range of the sensor, by avoiding a saturation of the sensor readings in one of the set/reset states.

Once the signal has been amplified, it is digitalized by a four-channel ADC located in the PCB of the three-axis sensor. Three of the input channels are used for the x , y and z components of the magnetic field. The digital conversions are timed by USB-CAN adapter to perform simultaneously for all sensors. Thus, the digitalized sensor readouts for all the sensors correspond to the same point in time, ensuring a correct interpolation of the magnetic field.

The ADC operates in a unipolar mode; it linearly maps the amplified voltage input, in a range of 0-4.096 V, to a 16-bit digital signal from 0x0000 to 0xFFFF (in the hexadecimal representation). A voltage reference (REF3120) of 2.048 V is applied so that the actual zero magnetic field is shifted to a voltage of 2.048 V. With a gain of 200, the magnetic field sensor is expected to operate in a range of ± 0.64 G. A magnetic field component that is larger than this range leads to a saturation of the ADC output to either 0x0000 or 0xFFFF, depending on the sign of the magnetic field component.

From the values given in the data sheets, we expect that each digital level in the output of the three-axis sensor should correspond to a magnetic field of $20 \mu\text{G}$. However, we measure a lower sensitivity of $40 \mu\text{G}$ per digital level, for our sensors. With this sensitivity, the operational range of the three-axis sensors is actually larger, i.e., ± 1.3 G.

Since the sensitivity measured is two times smaller compared to the expected value, a further investigation is required to determine the discrepancy in the sensitivity. One possible reason is due to a set-reset current that is different from 3 A. It is important to note that this discrepancy does not affect the performance of our control system, since a $40 \mu\text{G}$ sensitivity is already sufficient for the intended accuracy of the magnetic field control system.

11.2.5 Characterization of the individual sensors

A good characterization of the magnetic field sensors is required to understand the performance of the magnetic field system. In this section, we report the important characteristics of all of the three-axis sensors that are used in the sensor network.

11.2.5.1 Measurement of the sensor PCB magnetic field

One important quantity to measure is the magnetic field due to the three-axis sensor itself. Some of the electronics components mounted on the PCB can become magnetized when a strong magnetic field is applied. These magnetized components, being close to the HMC sensors, cause a bias in the readings of the sensor. It is crucial to measure this magnetic field value, and remove it from the readings of the three-axis sensor.

Sensor	Channel	Sensitivity ($\mu\text{G}/\text{level}$)	RMS Noise 0.01 – 367 Hz (digital level)	Noise Density 0.01 – 100 Hz (level/ $\sqrt{\text{Hz}}$)	Magnetic Field (digital level)
1	x	39.5	1.8	0.11	974
	y	36.4	1.7	0.11	-800
	z	38.6	1.7	0.11	4
2	x	39.5	1.7	0.11	689
	y	33.4	1.8	0.11	574
	z	38.3	1.8	0.11	401
3	x	39.0	1.7	0.11	557
	y	34.6	1.9	0.11	265
	z	35.8	1.8	0.11	-564
4	x	36.9	1.8	0.11	660
	y	35.6	1.8	0.11	377
	z	34.6	1.9	0.11	797
5	x	39.6	1.8	0.12	666
	y	35.6	1.8	0.13	284
	z	36.7	1.7	0.10	-724
6	x	37.9	1.8	0.11	1781
	y	35.2	1.8	0.11	1010
	z	36.6	1.7	0.11	1621
7	x	37.2	1.8	0.11	348
	y	36.2	1.7	0.11	121
	z	36.1	1.5	0.10	-396
8	x	39.0	1.7	0.10	3051
	y	37.7	1.8	0.11	768
	z	36.9	1.8	0.11	1932

Table 11.1: The specifications of the three-axis sensors used in the cubic network. The labels of the sensors are labeled as depicted in Figure 11.11.

The magnetic field contributed by the sensor PCB is measured inside the ZGC, and tabulated in Table 11.1. From our measurements, the magnetic field of each sensor typically has a magnetic field a few tens of milligauss, due to the close proximity of the magnetized components to the HMC sensors. These measured values are then subtracted from the subsequent measurement outcomes of the three-axis sensors.

11.2.5.2 Noise of the sensor

The noise characteristics of the three-axis sensor is another important parameter that we have measured. The readings of the sensor, when placed inside the ZG, is used to compute the noise spectrum of the sensor. The time dependent signal measured by the sensor is divided into 14 partitions, with each partition having a

Figure 11.8: Plots of the noise spectral density for all the eight three-axis sensors of the sensor network.

duration of typically three minutes. Fast Fourier transform (FFT) is then performed on each partitions. All the spectra are then averaged to give the noise power spectral density (PSD) plots shown in Figure 11.8. We observe a similar noise spectrum for each channel of the sensors.

Since we expect that the fluctuations of the magnetic field is small in the ZGC, the measured noise can be attributed to the electrical noise of the three-axis sensor circuit. This measurement noise is a limiting factor for the resolution of the three-axis sensor. Thus, the sensor noise level has to be low enough to meet the targeted

submilligauss resolution. We look into the following two quantities to have an idea of the sensor noise level.

The first quantity is the root-mean-square (rms) noise, which is computed over the full bandwidth of sensor, i.e., from 0.01 Hz to the Nyquist frequency 365.5 Hz¹⁰. The rms noise for all channels of the sensors are listed in Table 11.1. Typically, the rms value of the noise corresponds to roughly 2 digital levels of the sensor output. Hence, assuming that the noise between the different sensors in the network are not correlated, the sensor network can have a resolution of 0.3 mG.

The second quantity is the white noise level in the range of 0.1 – 100 Hz, as can be seen in Figure 11.8. It has a value of typically 0.11 level/ $\sqrt{\text{Hz}}$ (see Table 11.1). For a 100 Hz bandwidth around a central frequency of 50 Hz, this noise level leads to an rms noise contribution that is just one digital level of the sensor output.

From the results, we see that the overall noise of the three-axis sensor is low enough to give a good resolution of the magnetic field in the targeted submilligauss regime.

11.2.5.3 Sensitivity of the sensor

Finally, we need to measure accurately the sensitivity of the sensors, so that the digital readings of to calibrate The measurement of the sensor sensitivity is also performed using the solenoid in the ZGC. A sinusoidal magnetic field is applied in the ZGC (see also an example in Figure 11.7), with a peak-to-peak current of 70-80 mA in the solenoid. This generates a peak-to-peak magnetic field of typically 1 G, at the center of the solenoid. An example of the recorded signal is shown in Figure 11.9. Using the peak-to-peak value of the sensor reading, the sensitivity of the three-axis sensor can be measured.

To check the linearity in the response of the three-axis sensors to the magnetic field, we perform an FFT on the measurement signal. For an ideal linear response to the sinusoidal applied field, there should only be a single peak in the FFT spectrum, which occurs at the modulation frequency. In practice, we still observe some peaks associated with the higher harmonics of the modulation, but they are more than four orders of magnitude weaker in terms of the power density. Thus, we can consider that the sensor readings are linearly proportional to the external magnetic field, up to 1 G.

When placing the sensors into the ZGC, care has been taken to ensure that the y - and z -components are as small as possible. Nevertheless, we still observe a sinusoidal modulation in the readings of the y - and z -channels in Figure 11.9. This could be

¹⁰Used individually, each sensor can have a sampling time that is as short as 1.368 ms. However, a lower sampling frequency is compulsory when we use the sensor network (see Section 11.4.1)

Figure 11.9: A typical sensor output signal in the x channel, as measured in the ZGC. This is the measurement for sensor 5 of the sensor network (see Figure 11.11 for the position of sensor 5 in the cuboid network).

due to some imperfections of the solenoid. Furthermore, the relative orientation of the sensitive axes of the two HMC sensors might not be perfectly perpendicular when mounted on the PCB, leading to a small magnetic field component that is picked up by the other two measurement channels. Despite this imperfection, the peak-to-peak values of 2050 and 1190 measured on the y - and z -components of the magnetic field, lead only to a small error (around 5%) in the calibration of the sensor sensitivity along the x -axis. The results of the sensitivity calibration for the x -channel are tabulated in Table 11.1.

The method to measure the sensitivity of the sensor, as described above, only works for the sensitivity in the x -channel. Due to the space constrain inside the ZGC, the three-axis sensor cannot be oriented in a way that allows the sensitivities along the y - and z -directions to be calibrated. To measure the sensitivity in the y - and z -channels, we perform a relative measurement using a pair of Helmholtz coil. The three-axis sensors are placed at the center of the Helmholtz coil pair. For each sensor, it is first arranged such that the x -direction of the sensor is aligned along the axis of the Helmholtz coil. With a sinusoidal magnetic field, the peak-to-peak value of the sensor reading along the x -direction is first recorded. The sensor is then reoriented to have the y - and z -channels measuring the same magnetic field modulation. The sensitivities of the y - and z -channels are then determined relative to the sensitivity in the x -channel (see Table 11.1).

Figure 11.10: The Faraday cages (shown in inset) covers the three-axis sensors, which are mounted on the MOT coil cooling discs.

Once we have characterized the three-axis sensors, they are then assembled in a cuboid configuration to form the sensor network. The next section is dedicated to the description of the sensor network.

11.2.6 Sensor network

A direct measurement of the magnetic field at the positions of the atomic cloud is not possible, since the atoms are trapped inside the science chamber under a high vacuum condition. Instead, eight three-axis sensors are arranged in a cuboid configuration around the science chamber (see Figure 11.11), with the atomic cloud trapped at the geometric center of this cuboid configuration. The sensors are mounted directly on the surface of the cooling discs for the MOT coils (see Figure 11.10).

The cuboid configuration has sides S_x , S_y and S_z along the x , y and z directions, with $S_x = 11.4$ cm, $S_y = 8.4$ cm and $S_z = 7.2$ cm, respectively. The x -direction is aligned along the direction of the atomic beam. The positive z -direction is chosen to point vertically up, and the y -direction is chosen to be consistent with the right-hand convention.

Each sensor is labeled by a number ranging from 1 to 8. With the arrangement of the sensors as shown in Figure 11.11, the signs for each magnetic field component has to be chosen consistently (see Table 11.2). These signs are important when we want to interpolate for the magnetic field at the center of the sensor network, using Equation (11.9).

Figure 11.11: The sensors are arranged in a cuboid geometry around the science chamber. The orientations of the sensors and the convention used for the positive directions are shown in the figure. Each sensor is labeled by a number ranging from 1 to 8.

$s_{i,\nu}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
$\nu = x$	+1	+1	-1	-1	-1	-1	+1	+1
$\nu = y$	+1	+1	-1	-1	+1	+1	-1	-1
$\nu = z$	+1	+1	+1	+1	-1	-1	-1	-1

Table 11.2: The signs of the magnetic field readings for the three channels of the sensors in the cuboid network.

11.2.7 Faraday cages

Besides stray magnetic field, there is also stray electrical field in the lab. When a stray electric field is picked up by the three-axis sensor, it may be misinterpreted as a stray magnetic field. In our setup, we enclose each three-axis sensor PCB by a faraday cage (see Figure 11.10), which reduces the electrical field noise in the low frequency range. An example of the Faraday cage is shown in the inset of Figure 11.10. It is made from a copper sheet, that is shaped into an enclosure. Each enclosure covers one three-axis sensor in the sensor network.

Figure 11.12 shows the effect of the Faraday cage on the measurement noise of sensor 1 in the network. In absence of the Faraday cage, the power spectral density (PSD) curves of the sensor network measurement is shown in blue for all three channels. With the Faraday cage, the PSD curves are shown in green. We only see a slight reduction in the sensor noise, within the range of 0.1 – 10 Hz. There are still low frequency noise that is picked up by the sensor even with the Faraday cage mounted. We attribute this noise as magnetic in origin, since it is not present when the sensor is inside the ZGC. One of the main aim of our system is to compensate for this low frequency magnetic field noise (see Section 11.4.2.1).

Figure 11.12: The spectra of the magnetic field measured in the laboratory, for each channel of sensor 1 (with and without the Faraday cage). The Faraday cage is seen to reduce slightly the low frequency noise within a bandwidth of 0.1 Hz to 10 Hz.

Even if the effect of the Faraday cage is weak on the sensor noise, it still has an added advantage. The enclosed environment of the sensor leads to a better temperature stability for the magnetic field sensors, and therefore a better sensitivity stability for the sensors.

So far, we have described the cuboid network of sensors used in the control system. In the following, we describe the compensation coils that are used to cancel the magnetic field noise.

Figure 11.13: The Helmholtz and anti-Helmholtz configurations of the compensation coils. The position $\mathbf{O}$, which is the geometric center of the coil pair, is where the atomic cloud is trapped. The sides of the coil pair is denoted by a , while the separation between the coil pair is denoted by d .

	a (cm)	d (cm)
x	78	43
y	81	44
z	83	45

Table 11.3: A summary of the sizes and the separations of the coil pairs used to compensate the magnetic field.

11.3 Compensation coils

The compensation coil system consists of a total of 24 coils. They are wound on six aluminum square frames, forming three pairs with different sizes. Each frame contains four independent coils. Each coil consists of 12 turns of copper wires. The wires are enameled copper wires with a rectangular cross section ($0.66 \text{ mm} \times 1.98 \text{ mm}$).

Let us denote the length of the sides of the square coils by a , and the separation of the coil pair by d (see Figure 11.13) (these values are summarized in Table 11.3). The smallest coils are used for the magnetic field compensation along the x -direction. The medium sized coils are used for the compensation along y -direction. The largest coils are used for the vertical direction, which is denoted by z . We follow the convention defined in Figure 11.11 for the x -, y - and z -directions.

There are four pairs of coils available for each component of the magnetic field. Among these four pairs, one is connected in the Helmholtz configuration (see the left figure of Figure 11.13). This pair is responsible for the control of the magnetic field at the position of the cold atomic cloud. Another pair of coils is connected

in the anti-Helmholtz configuration to control the spatial gradient of the magnetic field (see the right figure in Figure 11.13). These coils control the $\partial B_x/\partial x$, $\partial B_y/\partial y$ and $\partial B_z/\partial z$ components of the magnetic field gradient. The remaining two unused pairs for each direction can be connected for other purposes¹¹.

In the following, we focus only on the Helmholtz coils, and do not discuss the anti-Helmholtz coils, since the control feature for the field gradient is still not ready for use.

11.3.1 Helmholtz condition

The separation between the coils has to be set correctly to obtain the Helmholtz configuration. Let us denote the distance along the central axis of the coil pair by s . The origin is assumed to be located at $\mathbf{O}$ (see Figure 11.13), the center of the coil pair. From the symmetry of the coil pair, the magnetic field generated at a point on the axis is ideally directed along this axis. We further see that the odd derivatives of the magnetic field with respect to s are all zero. This leaves us with only the derivatives of even orders. For the best uniformity of the magnetic field, the following Helmholtz condition has to be satisfied,

$$\boxed{\left. \frac{\partial^2 B_s}{\partial s^2} \right|_{\text{at } \mathbf{O}} = 0,} \quad (11.4)$$

Thus, the lowest order nonzero derivative of the field at $\mathbf{O}$ is $\partial^4 B_s/\partial s^4$. The magnetic field along the central axis is given by

$$B_s = \frac{4\mu_0 a^2 N_t I}{\pi} \times \left[\frac{1}{[a^2 + (2s + d)^2] \sqrt{2a^2 + (2s + d)^2}} + \frac{1}{[a^2 + (2s - d)^2] \sqrt{2a^2 + (2s - d)^2}} \right]. \quad (11.5)$$

The number of turns of the coil is denoted by N_t . The current that flows through the coil pair is denoted by I . The permeability of free space is denoted by μ_0 . The Helmholtz condition allows us to determine the value of d/a , using the following

¹¹For instance, along the x -direction, an additional pair of coils connected in Helmholtz configuration is used to provide a bias magnetic field to isolate the $m = 0$ substate of $^3\text{P}_1$. This pair is used in the experiments reported in Part I.

equation,

$$\left. \frac{\partial^2 B_s}{\partial s^2} \right|_{\text{at } \mathbf{O}} = \frac{64\mu_0 N_t I}{\pi a^3} \frac{6(d/a)^6 + 18(d/a)^4 + 11(d/a)^2 - 5}{[1 + (d/a)^2]^3 [2 + (d/a)^2]^{5/2}} = 0. \quad (11.6)$$

Its only valid solution gives

$$d = 0.5445a. \quad (11.7)$$

Using the above relation, the values of d for the coil pairs are calculated and summarized in Table 11.3. The magnetic field at $\mathbf{O}$ of a pair of Helmholtz coil, is then given by

$$\begin{aligned} B_s &= 195.4 \times 10^{-7} \frac{I}{a}, & (\text{T}) \\ &= 195.4 \frac{I}{a}, & (\text{mG}) \end{aligned} \quad (11.8)$$

by letting $N_t = 12$.

The typical maximum separation between any two sensors in the sensor network is around 10 cm. This is smaller compared to the typical size of the Helmholtz coils. Under this arrangement, we expect that the value of B_s , as measured by the sensor network, is only about 0.05% higher than the value at $\mathbf{O}$. Thus, a good level of magnetic field uniformity can be applied by the Helmholtz coils, within the sensor network.

Furthermore, since the sensors are not located on the axis of the coil pair, they measure a nonzero value in the orthogonal components. Nevertheless, these components have typical values that is $< 0.1\%$ of the value of B_s at $\mathbf{O}$.

11.3.2 Calibration of the Helmholtz coils

To calibrate the coil pairs, several measurements are carried out to determine the magnetic field generated by the coil pairs. We summarize in Table 11.4, the the magnetic field generated by coil pair per unit current. We use several different methods to calibrate the magnetic field generated by the compensation coils. They are compared to the theoretical prediction given by Equation (11.8).

11.3.2.1 Calibration with HMR2300

The largest Helmholtz coil, employed for the compensation along the z direction, is calibrated by measuring the magnetic field using a commercial Honeywell magne-

B-field per unit current	Theory (mG/A)	HMR2300 magnetometer (mG/A)	Sensor network (mG/A)	Spectroscopy (mG/A)	Modulation coefficient (A/V)
x	251	-	280	-	0.407
y	241	-	260	228	0.412
z	235	239	249	-	0.412

Table 11.4: A table summarizing the calibration measurements of the compensation coils. The different calibration methods are described in Sections 11.3.2.1, 11.3.2.2 and 11.3.2.3. A voltage modulation is applied to the current controller unit to change the coil current. The modulation coefficient for the current controller units is measured and tabulated in the last column.

tometer (HMR2300¹²). This calibration is carried out before the coils are mounted on the experimental setup, allowing the magnetometer to be placed at the center of the coil pair.

We vary manually the magnetic field generated by the coil pair and plot the readings of the magnetometer in Figure 11.14. A good agreement is obtained between the theoretical values of the magnetic field (in green), and the experimentally measured value (blue open circles). A linear fit of the experiment data, shown as the red curve, allows us to determine the gradient of the line (see Table 11.4 for the results).

11.3.2.2 Calibration with sensor network

The calibration procedure described in previous section is carried out before the compensation coils are mounted around the experimental setup. After the mounting of the coils, the HMR2300 magnetometer cannot be placed at the center to perform the calibration of the coils. The sensor network is used instead to perform the calibration measurements.

In contrast to the previous case where the current is manually changed, a voltage modulation is applied to the current controller unit to vary the current¹³. We find a typical deviation of 6-12% from the theoretical value, as summarized in Table 11.4. Since the Helmholtz coils lead to a good uniformity of the magnetic field (see the discussion in Section 11.3.1), this cannot be attributed to the off-center position of the three-axis sensors in the network. On the other hand, the theoretical prediction in Equation (11.8) assumes ideally a thin wire. It is possible that this discrepancy is related to the fact that the actual coils are made of many turns of wires with

¹²The HMR2300 magnetometer is commercial a three-axis magnetometer that is also constructed from the HMC1001 and HMC1002 sensors. Each digital level in the reading of the HMR magnetometer corresponds to a magnetic field of 67 μG .

¹³The modulation coefficients for the three current controller units are summarized in Table 11.4.

Figure 11.14: Magnetic field at the center of the Helmholtz coil pair as a function of the current. The data is fitted with a linear function, shown in red. The theoretical calculation of the magnetic field based on Equation (11.8) is shown in green, showing an excellent agreement with the measured values.

nonzero thickness. Furthermore, the an error in the sensitivity of the sensor network could also cause this discrepancy.

Since this is the only calibration method that is carried out for all the three directions, we use it to calibrate the value of applied magnetic field value in the program that controls the experimental parameter and time sequence (the lab control program).

11.3.2.3 Calibration with spectroscopy on atoms

To accurately calibrate the magnetic field generated by the compensation coil, we measure the Zeeman shift due to this magnetic field experienced by the atoms. We perform this for the component of the magnetic field along the propagation direction of the probe beam, that is, the y -direction. In principle, the same calibration can also be performed for the magnetic field along x - and z -directions. Here, we perform the spectroscopy experiment only on the y -component of the magnetic field, since most of our studies are focused on the effect due to the magnetic field component along this axis.

Using the cold atomic cloud, the Zeeman shift of the $m = 1$ or $m = -1$ excited substates are measured with a transmission spectroscopy. The frequency shifts are measured relative to the resonance frequency of the $m = 0 \rightarrow m = 0$ transition of the ^{88}Sr intercombination transition. We apply three different values of magnetic field in the experiment, as listed in Table 11.5. These magnetic field values are set

Applied Field (lab control) (mG)	Expected shift (kHz)	Measured shift (kHz)	Actual field (mG)
200	420	368	175
162	340	281	134
228	480	420	200

Table 11.5: Table showing the Zeeman shifts of the $m = \pm 1$ substates of 3P_1 , when a magnetic field is applied along the propagation of the probe (y -direction). A comparison is made between the expected and the measured Zeeman shift. The actual value of the magnetic field component along y -direction, is computed from the measured Zeeman shift. We find that the calibration in the lab control program overestimates the y -component of the magnetic field by a factor of 1.14.

in the lab control program, using the calibration results of the sensor network measurements (see previous section). The absolute values of the expected Zeeman shift, $g_J\mu_B B/h$, are also tabulated. We find that the actual magnetic field experienced by the atoms is 1.14 times smaller than the values set in the lab control program. This correction is required when we study the magneto-optical rotation (MOR) effects, to have the real value of the applied magnetic field, so that a comparison with the theoretical predictions can be made. This result also allows us to calibrate the nonzero measurement of the sensor network (see Section 12.2.1.5).

With the sensor network and compensation coils, the PC program can now be used to perform the PID control. In the following section, we explain the signal processing carried out in the PC.

11.4 Signal processing in the PC

A diagram showing the digital signal processing of the signal is shown in Figure 11.15. The digital signal processing is carried out in the PC, through an in-house C++ program. In the following, we describe the different aspects of the digital signal processing.

11.4.1 Measurement of magnetic field

During each sampling cycle, the sensors in the sensor network take turn to send their digitalized 16-bit reading to the USB-CAN adapter. After one cycle has completed, all the sensor readings are sent together to the PC. The sampling cycle period is set in the PC program to be 2.048 ms¹⁴. This corresponds to a sampling frequency of 488 Hz, and a Nyquist frequency of 244 Hz. The Nyquist frequency is higher than the 50 Hz, allowing us to detect and component the AC 50 Hz magnetic field

¹⁴The sampling cycle period can be modified in the PC program.

Figure 11.15: Flow chart showing the digital signal processing and PID controllers for one component of the magnetic field. The cancellation of the magnetic field noise is performed only at the low frequency and at 50 Hz. A detailed description is given in the text.

noise. Some higher harmonics of the AC signal can even be measured, for example at 100 Hz and 150 Hz. In practice, we only compensate for the 50 Hz signal, which is the dominant AC contribution.

The sensor measurement is performed by alternating between the sensors in the set and in the reset states, as described in Section 11.2.2. We assume that magnetic field does not vary significantly between consecutive readings. This is true to a certain extent. However, since 2.048 ms is not very short compared to the 20 ms period of the AC 50 Hz signal, this would lead to a phase shift, and a slight underestimation of the 50 Hz magnetic field fluctuation. The differential

measurement with the consecutive readings are used to remove the bridge offset as described in Section 11.2.3.

The actual magnetic field measured by each sensors can be computed from their sensitivities tabulated in Table 11.1. Let us denote the magnetic field component measured by each sensor by $B_{i,\nu}$, where i refers to the label of the sensor and $\nu = x, y, z$ refers to the directions of the components. The following interpolation formula is used to find the ν -component of the magnetic field at the position of the cold atomic cloud,

$$B_\nu = \frac{1}{8} \sum_{i=1}^8 s_{i,\nu} B_{i,\nu}. \quad (11.9)$$

The values of $s_{i,\nu}$ are listed in Table 11.2. If the magnetic field is relatively homogeneous within the cuboid network of the sensors, the above expression gives a good estimate of the magnetic field at the position where the cold atomic cloud is trapped.

11.4.2 Filters and PID feedback control

A measurement of the magnetic field environment in the lab is performed over a duration of 33 minutes. The result is shown in Figure 11.16 for the x -component of the field. The green curve shows the slow variation of the field, obtained by applying a low pass filter of 10 Hz cutoff to the signal (more details in next section). The low frequency variations in the magnetic field are due to human activities in the vicinity of the lab. Additionally, the red curve in the inset of the figure is obtained by applying a band pass filter to the measurement signal with the central frequency at 50 Hz. It shows that the 50 Hz noise in the laboratory has a peak-to-peak intensity of the order of 1 mG. We further verify in Section 12.3 that the AC 50 Hz noise, as measured with MOR effect, has the same order of magnitude.

Besides the magnetic field noise, there is also the electrical noise that appears in the measurement noise of the sensors. It is important to compensate only for the noise that are magnetic in origin, since compensating the electrical noise ends up introducing additional magnetic field fluctuations. In the following, we describe the frequency filters and the PID controllers that has to be applied to cancel only the magnetic field noise.

11.4.2.1 Control of the low frequency magnetic field noise

For the low frequency noise, a first order RC low pass filter with cut-off frequency f_c is implemented digitally. Let us consider a sequence of the sensor network magnetic

Figure 11.16: The measurement of x -component magnetic field by the sensor network. The signal after a low pass filter is applied, is shown in green, while the 50 Hz frequency component extracted using a band pass filter is shown in red. The inset zooms into the results of frequency filtering being applied to the measured magnetic field signal. The filtered 50 Hz signal has a zero offset value, and is visible only in the inset.

field measured in time, which is denoted by $B_\nu(t_i)$, where $\nu = x, y, z$. The time corresponding to the measurement is denoted as t_i ($i = 0, 1, 2, \dots$). We further assume that the filtering process is initiated at t_0 .

For a sampling period of Δt , the low pass filtered signal $\tilde{B}_\nu(t_i)$ is given by,

$$\tilde{B}_\nu(t_i) = \kappa B_\nu(t_i) + (1 - \kappa) \tilde{B}_\nu(t_{i-1}), \quad (11.10)$$

where

$$\kappa = \frac{2\pi f_c \Delta t}{2\pi f_c \Delta t + 1}. \quad (11.11)$$

In Figure 11.16, we show an example of the low pass filtered signal as the green curve, with the cut-off frequency at $f_c = 10$ Hz.

The PID controller is applied using the filtered signal $\tilde{B}_\nu(t_i)$, with the set point at zero magnetic field. The error signal is given by $e_\nu(t_i) = \tilde{B}_\nu(t_i)$, which is the difference between the measured signal and the set point value. The PID controller works to bring the magnetic field to zero, by computing a control variable. To compensate the low frequency magnetic field component, this control variable is converted to a voltage value. This voltage is applied to the current controller unit to vary the current flowing in the Helmholtz coils.

A PID controller consists of three different types of controllers. The most basic

one is the proportional controller. As its name suggests, the control variable for the proportional controller is computed proportional to the error signal, i.e.,

$$V_{P,\nu}(t_i) = K_P e_\nu(t_i), \quad (11.12)$$

where K_P is the proportional gain. In addition to the proportional controller, there are the integral and derivative controllers which improves the performance of the PID loop. The control variables for these controllers, for $i > 1$, are defined by

$$V_{I,\nu}(t_i) = K_I \sum_{j=1}^i [e_\nu(t_j) + e_\nu(t_{j-1})], \quad (11.13)$$

$$V_{D,\nu}(t_i) = K_D [e_\nu(t_i) - e_\nu(t_{i-1})] \quad (11.14)$$

Here, K_I and K_D are the integral and derivative gains, respectively. The control variables of the proportional, integration and derivative controllers are added together to give a PID feedback control,

$$V_{PID,\nu} = V_{P,\nu} + V_{I,\nu} + V_{D,\nu}. \quad (11.15)$$

11.4.2.2 Control of the 50 Hz magnetic field noise

For the 50 Hz frequency component, the PID controllers act on its quadrature amplitudes. To introduce these quadrature amplitudes, we first describe a method to digitally implement a band pass filter centered on f_0 , on the measured signal.

First, the signal $B_\nu(t)$ is mixed with sinusoidal signals of frequency f_0 to have

$$B_{s,\nu}(t) = B_\nu(t) \sin(2\pi f_0 t), \quad (11.16)$$

$$B_{c,\nu}(t) = B_\nu(t) \cos(2\pi f_0 t). \quad (11.17)$$

Consider $B_{s,\nu}(t)$ and $B_{c,\nu}(t)$ in the frequency domain, we have

$$\begin{aligned} B_{s,\nu}(\omega) &= \frac{1}{2i} \int_{-\infty}^{\infty} B_\nu(t) [e^{i(\omega+2\pi f_0)t} - e^{i(\omega-2\pi f_0)t}] dt \\ &= \frac{1}{2i} [B_\nu(\omega + 2\pi f_0) - B_\nu(\omega - 2\pi f_0)], \end{aligned} \quad (11.18)$$

$$\begin{aligned} B_{c,\nu}(\omega) &= \frac{1}{2} \int_{-\infty}^{\infty} B_\nu(t) [e^{i(\omega+2\pi f_0)t} + e^{i(\omega-2\pi f_0)t}] dt \\ &= \frac{1}{2} [B_\nu(\omega + 2\pi f_0) + B_\nu(\omega - 2\pi f_0)]. \end{aligned} \quad (11.19)$$

We see that $B_{s,\nu}(\omega)$ and $B_{c,\nu}(\omega)$ are basically constructed from $B_\nu(\omega)$, with $\pm 2\pi f_0$ displacements in the frequency. The $B_\nu(\omega - 2\pi f_0)$ term takes the angular frequency component of B_ν at $2\pi f_0$, and bring it to DC component. The $B_\nu(\omega + 2\pi f_0)$ term takes the angular frequency component of $-2\pi f_0$, instead, and bring it to DC component. The contribution at frequency f_0 , with a bandwidth of $2f_c$, can be found by applying a low pass filter on $B_{s,\nu}$ and $B_{c,\nu}$, with a cut-off frequency of f_c . We find

$$\tilde{B}_{s,\nu}(t) = \frac{1}{4\pi i} \int_{-2\pi f_c}^{2\pi f_c} [B_\nu(\omega + 2\pi f_0) - B_\nu(\omega - 2\pi f_0)] e^{-i\omega t} d\omega, \quad (11.20)$$

$$\tilde{B}_{c,\nu}(t) = \frac{1}{4\pi} \int_{-2\pi f_c}^{2\pi f_c} [B_\nu(\omega + 2\pi f_0) + B_\nu(\omega - 2\pi f_0)] e^{-i\omega t} d\omega. \quad (11.21)$$

They can be interpreted as the quadrature amplitudes of the filtered signal around f_0 . The filtered signal is obtained by mixing again the quadrature amplitudes with the sinusoidal signals. We further allow an additional freedom to shift the phase of the filtered signal by Φ , i.e.,

$$\tilde{B}_\nu(t) = \tilde{B}_{s,\nu}(t) \sin(2\pi f_0 t + \Phi) + \tilde{B}_{c,\nu}(t) \cos(2\pi f_0 t + \Phi). \quad (11.22)$$

It takes time to communicate the readings of the signal to the PC for processing. This leads to a time delay, and therefore, a phase shift on the signal. This phase shift can be significant if the time delay is comparable to the 20 ms period of the 50 Hz frequency component. Furthermore, since the sampling time of 2.048 ms is just one order of magnitude smaller than the 20 ms period, the set/reset differential measurement of the magnetic field also contributes to an additional phase shift. A freedom to choose Φ allows us to bring the filtered signal in phase with the actual 50 Hz frequency component of the magnetic field.

In the inset of Figure 11.16, we show the action of band pass filter ($f_0 = 50$ Hz) on the x -component of the magnetic field. The filtered signal, with a width of $2f_c = 10$ Hz, and a phase shift of $\Phi = -1.5\pi$, is shown in red.

As mentioned earlier, the PID controllers are applied to the quadrature amplitudes $\tilde{B}_{s,\nu}(t)$ and $\tilde{B}_{c,\nu}(t)$ [Equations (11.20) and (11.21)], and not on the filtered signal. From Equation (11.22), the 50 Hz compensation can be successfully suppressed as long as the quadrature amplitudes are suppressed. The PID control variables for the two quadrature amplitudes at $f_0 = 50$ Hz in the magnetic field

component along ν , are given by

$$\begin{aligned}
V_{PID,s,\nu}(t_i) &= K_{P,50} \tilde{B}_{s,\nu}(t_i) \\
&+ K_{I,50} \sum_{j=1}^i \left[\tilde{B}_{s,\nu}(t_j) + \tilde{B}_{s,\nu}(t_{j-1}) \right] + K_{D,50} \left[\tilde{B}_{s,\nu}(t_i) - \tilde{B}_{s,\nu}(t_{i-1}) \right],
\end{aligned} \tag{11.23}$$

$$\begin{aligned}
V_{PID,c,\nu}(t_i) &= K_{P,50} \tilde{B}_{c,\nu}(t_i) \\
&+ K_{I,50} \sum_{j=1}^i \left[\tilde{B}_{c,\nu}(t_j) + \tilde{B}_{c,\nu}(t_{j-1}) \right] + K_{D,50} \left[\tilde{B}_{c,\nu}(t_i) - \tilde{B}_{c,\nu}(t_{i-1}) \right],
\end{aligned} \tag{11.24}$$

for $i > 1$. Here, the error signals are $\tilde{B}_s(t_i)$ and $\tilde{B}_c(t_i)$. The PID gain values for both the quadrature amplitudes are the same. They are denoted by $K_{P,50}$, $K_{I,50}$ and $K_{D,50}$, respectively, for the proportional, integration and derivative gains.

The control variable for the 50 Hz frequency component, is then given by

$$V_{50,\nu}(t_i) = V_{PID,s,\nu}(t_i) \sin(2\pi f_0 t_i + \Phi_\nu) + V_{PID,c,\nu}(t_i) \cos(2\pi f_0 t_i + \Phi_\nu).$$

(11.25)

In Figure 11.17, the PSD of magnetic field signal, with and without the magnetic field compensation, is shown. The PSD of the measurement without any compensation of the field is plotted in blue. This is compared to the PSD shown in green, when the compensation of the low frequency and 50 Hz magnetic field noise is active. For the low frequency component, the PID gain values are $K_P = 3 \times 10^{-4}$, $K_I = 2.7 \times 10^{-4}$ and $K_D = 0$. For the 50 Hz component, the PID gain values are $K_{P,50} = 8 \times 10^{-3}$, $K_{I,50} = 3.3 \times 10^{-4}$ and $K_{D,50} = 0$. In fact, we only perform a PI control of the magnetic field, as the derivative gains are set to zero for both frequency components. The applied values of the phase shifts for the compensation at 50 Hz, are $\Phi_x = -1.5\pi$, $\Phi_y = -1.8\pi$ and $\Phi_z = -1.2\pi$, respectively.

11.4.2.3 Remarks on the performance of the PID controller

For the frequency components up to 1 Hz of the magnetic field, the PID feedback control manages to reduce the low frequency magnetic field noise. Importantly, we observe a reduction of the DC noise power density by typically five orders of magnitude. For the cancellation of the 50 Hz noise, the noise level is reduced down to the level of the sensor network white noise, as shown in the inset of Figure 11.17.

Figure 11.17: Comparison between the PSD of the stray magnetic field (blue) and the PSD of the sensor measurement under PID control (green). We consider only the x -component of the magnetic field. The inset shows the PSD for both situations around 50 Hz. The gain values for the PID are set to be $K_P = 3 \times 10^{-4}$, $K_I = 2.7 \times 10^{-4}$ and $K_D = 0$ for the low frequency component, and $K_{P,50} = 8 \times 10^{-3}$, $K_{I,50} = 3.3 \times 10^{-4}$ and $K_{D,50} = 0$ at 50 Hz. The value of phase shift Φ is set to be -1.5π . While the PID controller removes the magnetic field noise at low frequency and at 50 Hz, a slight increase of measurement noise is observed for the other frequency components above 1 Hz.

11.4.2.4 Tracking of PID action during an open loop

So far, we have discussed the action of the PID controller when the loop is closed. When the PID controller loop is closed, the sensor network reading is required to compute the error signals for magnetic field compensation. However, during the MOT stage when a magnetic field gradient is applied by the MOT coils, the sensor network readings are saturated. As discussed in the next chapter, the sensors are switched off during this preparation stage. Thus, the PID loop has to be opened during this preparation stage¹⁵.

However, after the MOT stage, there is not enough time to restore the PID action. Typically, we need more than 0.5 s for the PID controller to achieve its steady state in compensating the magnetic field. Furthermore, we have observed that sensor 6 stops working after the magnetic field of the MOT is turned on. This sensor can be recovered only by switching the sensor network off and on several times. Therefore, it is not possible to operate the PID controller correctly and quickly enough, before an experiment is performed on the cold atoms.

¹⁵The opening and closing of the PID loop is controlled by a digital trigger. The switching of this trigger is performed in the lab control program

Figure 11.18: Figure showing tests of PID tracking with three different lengths of duration when the PID is paused, namely 2 s, 1 s and 0.5 s. Only the data for the magnetic field component along y direction is shown. The PID feedback control is periodically applied and paused. We apply the same PID settings, as used in Figure 11.17.

Nevertheless, the compensation of the magnetic field is still required to carry out certain experiments. In view of this, we implement a feature that allows us to track the action of the PID controller, even though the PID loop is opened. The stray magnetic field is compensated approximately based on the values of the control variables, just before the PID loop is opened. The success of this tracking feature requires that the behavior of the stray magnetic field does not change significantly

over the duration when the PID loop is opened.

For the low frequency component of the magnetic field noise, the action of the PID is tracked by simply retaining the previous control variable, just before the loop is open. For the 50 Hz magnetic field noise, we retain the values of $V_{PID,s,\nu}(t_i)$ and $V_{PID,c,\nu}(t_i)$ just before the loop is opened. With these two values, Equation (11.25) can be used to compute the new voltage values to be applied to the current controllers units. Hence, if the amplitude and phase of the 50 Hz magnetic field noise remains unchanged when the PID loop is opened, a good cancellation of the 50 Hz noise can be achieved.

We test the performances of this PID tracking feature over different duration where the PID loop is opened. The results are shown in Figure 11.18 for the y -component of the magnetic field. The period reported for each figure corresponds to the time for one full cycle of opening and closing the PID loop.

At the shortest period of 1 s, there is no significant increase in the noise of the sensor network measurements when PID action is tracked. The fluctuations of the magnetic field remain under 1 mG. As the period increases, the noise of the magnetic field component along y also increases during the time when the loop is opened. At the largest period of 4 s, the opening of the PID loop results an increased noise, with the peak-to-peak magnetic field fluctuations going up to 2 mG. From the results shown in Figure 11.18, we can open the PID loop for as long as 1 s, while maintaining a fluctuation of the magnetic field that is within ± 1 mG. This is taken into consideration in the experimental time sequence, as shown in Figure 12.3 of the following chapter.

11.4.2.5 Applying a magnetic field offset

In addition to canceling the stray magnetic field, the control system can also apply an offset magnetic field at the position of the atoms. The magnetic field value to be applied is set in the lab control program. We can incorporate directly the magnetic field control into the experiment time sequence. The value of the magnetic field to be applied is sent to the sensor program through an asynchronous local area network. A digital trigger is then used to indicate when the magnetic field offset is applied. When the trigger to apply the magnetic field offset is high, the offset magnetic field is applied. Conversely, when the trigger is low, the offset magnetic field is not applied.

The offset magnetic field is applied by the same pair of compensation coils that performs the magnetic field compensation. Using the calibration of the Helmholtz coils by the sensor network (see Table 11.4), the magnetic field offset values are converted to a change in modulation voltage, $V_{B_{\text{off}},\nu}$. When the magnetic field

offset is applied, we have to work with the PID action in tracked mode. When the PID gains are nonzero, the applied voltage to the current controller $V_{cc,\nu}$, is given by

$$V_{cc,\nu} = V_{B_{\text{off}},\nu} + V_{PID,\nu} + V_{50,\nu}. \quad (11.26)$$

Here, $V_{PID,\nu}$ and $V_{50,\nu}$ are the tracked values. Otherwise, when the trigger to apply the offset field is low, only the PID control variables contribute. We have,

$$V_{cc,\nu} = V_{PID,\nu} + V_{50,\nu}. \quad (11.27)$$

So far, we have described our active control system for the magnetic field, which can be operated to control the magnetic field in a standalone manner. In the next chapter, we consider the performance of this system, when it is integrated into the cold atomic experimental setup. In particular, the high sensitivity of the atoms to the magnetic field is used to calibrate this control system, and to study its performance.

Chapter 12

Magneto-Optical Rotation Experiments with Cold Atoms

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In this chapter, we describe the implementation of magneto-optical rotation experiments on cold atoms, making use of the active magnetic field control system. An important objective of this chapter is to report the performance of the active magnetic field control system, tested using the cold atomic cloud as a sensitive in-situ detector of the magnetic field.

In particular, the magneto-optical rotation (MOR) of light transmitted through the atomic cloud is used to calibrate the sensor network measurement of the magnetic field. This calibration is necessary because the inhomogeneity of the magnetic field could lead to an error in the interpolation of the magnetic field at the location of the cold atoms. In Section 12.2.1, we use the linear magneto-optical rotation (LMOR) of a weak probe beam to compensate for an offset in the sensor network reading.

We also investigate the long term stability of the magnetic field cancellation using the control system. We check that over the duration of one month, the PID controllers consistently cancels the magnetic field noise (see Section 12.2.2) to a level that is lower than 1 mG. This confirms also that in the long run, we do not lose the calibration of the sensor network.

The compensation of the stray magnetic field is performed only for the low frequency and 50 Hz frequency components. While we have mentioned that the 50 Hz noise is dominated by magnetic field noise, this fact remains to be verified. In Section 12.3, we use the LMOR effect on the atoms to show that the 50 Hz noise measured by the sensor network is, indeed, dominated by the magnetic field noise.

Before we proceed to describe the various MOR experiments that are used to do the calibration and performance study of the control system, we first comment on the properties of cold atomic clouds that are used in the MOR studies. We also describe the balanced polarimeter setup [164, 170] that is used to measure the MOR effect. The time sequence of the experiments is also presented. An important point, here, is the integration of active magnetic field control system into the cold atomic experimental procedure. The description here is not only relevant for the experiments to study the performance of the control system. The study of the nonlinear magneto-optical rotation (NMOR), as reported in Chapter 13, is also performed using the same experimental setup and time sequence.

12.1 Details of the MOR experiments

The preparation of the ^{88}Sr cold clouds for the MOR experiments is described in Appendix C and Ref. [46]. The MOR effects are studied on the narrow ^{88}Sr intercombination transition at 689 nm

The same probe beam described in Section 4.1, is used to perform the MOR experiments. Usually, it is linearly polarized along the horizontal x -direction. The probe beam propagates along the y -direction (see Figure 11.11 for the convention for the x -, y - and z -directions). This probe beam, along with an electron multiplying charge-coupled device (EMCCD) camera, allows us to record the absorption images of the cold atomic cloud. Through the balanced polarimeter setup, the absorption images in the two polarization channels a and b (these channels are defined in Figure 10.2) are captured simultaneously. This is discussed in further details in Section 12.1.2.

Finally, the experimental time sequence for both the experiments reported in this chapter and the following chapter, is presented in Section 12.1.3.

12.1.1 Cold atomic sample

An important parameter of the cold atomic cloud is its optical thickness. For the LMOR effect, the rotation of polarization is proportional to the optical thickness of the atomic cloud¹. One may then think that a larger optical thickness is desired for the MOR experiments. However, the measurement of the rotation relies also on the transmission of light through the atomic cloud (see Section 12.1.2). A large optical thickness leads to a strong attenuation of the light, making the transmission signal more susceptible to bias. A trade-off between the two effects mentioned above, is to prepare the cold atoms with a resonant optical thickness that is typically $b_{\bar{v}}(0) \simeq 1$.

Another important property of the cloud is its temperature. Since the cloud has less atoms, it can now reach a temperature lower than the 3 μK reported for an optically thick cloud in Part I². In the MOR experiments, the cloud has a temperature of typically 1 μK . Compared to the transition linewidth Γ , where have $k\bar{v}/\Gamma = 2$, where $k\bar{v}$ is the root mean square Doppler shift experienced by the atoms. Importantly, due to this Doppler broadening, the LMOR sensitivity is reduced, as discussed in Chapter 10.

12.1.2 Experimental setup to measure MOR

The balanced polarimeter setup [164,170] as shown in Figure 12.1 is used to study the MOR of the probe beam polarization.

We use a half wave plate and a polarizing beam splitter (PBS), mounted on rotating mounts, to set the polarization of the incident beam. The polarization

¹The rotation angle of nonlinear MOR (NMOR) effects, in general, does not depend linearly on the optical thickness. An example of this is shown in Figure 13.5.

²The cloud temperature is affected by the density, and therefore the number of atoms, as shown in Figure C.5

Figure 12.1: The experimental setup to measure the MOR effect. A half wave plate and a PBS, are placed before the probe beam transmits through the cold atomic cloud. Usually, the incident beam is horizontally polarized. In certain cases, the polarization can be changed by rotating both the half wave plate and the PBS. After being transmitted through the atomic cloud, a half wave plate is placed at an angle of 22.5° with the horizontal plane, to rotate the polarization of the transmitted field. The PBS after the half wave plate separates the two polarization channels a and b into the transmitted and reflected path. The two polarization channels are 45° away from the horizontal and vertical polarization basis (see Figure 10.2). On the reflection path, another PBS is placed to clean the polarization on this path. These two paths are recombined using a non-polarizing beam splitter (NPBS). The two transmission images in the two polarization channels are measured simultaneously by an EMCCD camera.

of the 689 nm incident probe beam is usually set to the horizontal direction (x -direction). For certain experiments, they can be rotated to set a different incident polarization.

The polarization of the transmitted light is then analyzed at 45° from the horizontal axis, by placing a half wave plate at an angle of 22.5° after the transmission

Figure 12.2: An example of the false color absorption images of the cold cloud in the two polarization channels. The images are taken at $\delta = 0$, with an applied magnetic field $B_y = 23.5$ mG along the probe beam propagation direction. The cloud has an peak optical thickness at resonance of $b_{\bar{v}}(0) = 2$ and a temperature of $1 \mu\text{K}$.

from the cloud. A PBS is placed just after this half-wave plate, to split the light into the transmitted and the reflected paths. On the reflected path, another PBS, that is rotated by 90° , is placed to clean the polarization on this path. The two beams are finally brought together by an NPBS. The transmission images of the cloud in the a and b polarization channels, are captured side-by-side on an EMCCD camera. An example of this measurement is shown in Figure 12.2, with an applied magnetic field of $B_y = 23.5$ mG along the probe propagation direction (y -direction). The cold atomic cloud has a peak resonant optical thickness of $b_{\bar{v}}(0) = 2$ and a temperature of $T = 1 \mu\text{K}$. We set the probe frequency to be at resonance with the $m = 0 \rightarrow m = 0$ transition of the intercombination line, which does not experience any Zeeman shift.

The cold atomic cloud has a Gaussian density profile in the magneto-optical trap (MOT). The two transmission images in both polarization channels are fitted using Equation (4.1) from Chapter 4, to find the transmitted intensity at the center of the cloud. We denote the peak transmitted intensity at the center of the cloud³, by I_a and I_b respectively, for the polarization channels a and b . This allows us to

³Our studies here focuses only on the polarization rotation experienced by the part of the probe beam that passes through the center of the cloud.

compute b_a and b_b , which are defined in Section 10.2.3 to be

$$b_a = -\log(2I_a/I_0), \quad (12.1)$$

$$b_b = -\log(2I_b/I_0). \quad (12.2)$$

In the above expressions, I_0 is the intensity of the peak incident probe beam.

Due to the MOR effect, the transmitted intensity in one of the channel can be larger than $I_0/2$. This is seen in the channel b of Figure 12.2, leading to a b_b value that is negative.

The measurement of the transmitted intensities I_a and I_b at the center of the cloud, is used to compute the rotation angle θ of the light polarization,

$$\theta = \frac{1}{2} \sin^{-1} \frac{I_{a,0}e^{-b_a} - I_{b,0}e^{-b_b}}{I_{a,0}e^{-b_a} + I_{a,0}e^{-b_b}}. \quad (12.3)$$

$I_{a,0}$ and $I_{b,0}$ are the peak intensity values of the incident beam in the a and b channels respectively. Ideally, we have $I_{a,0} = I_{b,0} = I_0/2$. Due to imperfections in the setup, there is a slight imbalance in the intensities of both channels. This leads to a spurious rotation of θ_0 , even if the atomic cloud is absent. This spurious rotation angle is given by

$$\theta_0 = \frac{1}{2} \sin^{-1} \frac{I_{a,0} - I_{b,0}}{I_0}. \quad (12.4)$$

In our experiments, θ_0 is typically measured to be in the range of 0.03-0.06 rad. We assume that this imbalance is solely caused by a slightly rotated incident polarization. In general, a slightly rotated incident polarization will offset the rotation angle θ differently for different probe beam frequency. However, in our case where the intensity imbalance is small, it is a good approximation to take θ_0 as a constant offset to the rotation angle. The actual rotation angle experienced by the light polarization then becomes $\theta - \theta_0$.

12.1.3 Experimental time sequence

The time sequence for a typical MOR experiment is shown in Figure 12.3.

At the start of the time sequence, the PID controller is applied for 1 s to cancel the stray magnetic field, before the cold atomic cloud is prepared. The default settings of the filters and the PID controllers are summarized in Table 12.1.

During the preparation stage of the atomic cloud, a large magnetic field generated by the MOT coils saturates the reading of the sensor. Furthermore, when

Figure 12.3: The time sequence of a typical experiment to study the MOR effect using the cold ^{88}Sr cloud. The PID controller is active for a duration of 1 s, before the blue and red MOTs are performed. During the preparation of the atoms, the PID loop is opened, and the cancellation of the magnetic field noise is tracked. After that, the atomic cloud is held in an optical molasses created by the red MOT beams for 20 ms, to allow for the MOT magnetic field to be turned off completely. An offset magnetic field is applied after this stage. A further 20 ms is waited before the measurement of the MOR effect is performed. After the measurement is completed, the sensors are restarted. The whole experimental time sequence for one run of the experiment takes about 3.5 s to complete.

Gains	Low Frequency	50 Hz
K_P	3×10^{-4}	2×10^{-3}
K_I	2.7×10^{-4}	3.3×10^{-4}
K_D	0	0
Φ_ν	Low Frequency	50 Hz
$\nu = x$	-	-1.8π
$\nu = y$	-	-1.8π
$\nu = z$	-	-1.5π

Table 12.1: Table summarizing the default settings for the PID gains and phase shift (50 Hz).

the current of the coil is changed, the sensors are likely to stop working due to the large induced current. To avoid this, the sensor network is switched off. Hence, the PID loop is also opened for a time that is typically 1 s. Using the tracking feature described in Section 11.4.2.4, we can continue to cancel the magnetic field noise during this stage.

After the red MOT stage, the magnetic field for MOT is switched off. The atoms are then held in a molasses for 20 ms, to make sure that the magnetic field generated by the MOT coils is fully turned off before running the experiments. The molasses is performed using the red MOT beams, with a detuning of -40 kHz. The optical power of the red MOT beams are set at the value corresponding to the final stage of red MOT. This is maintained for 18 ms. The final 2 ms of the molasses

stage is performed with a reduced optical power, to avoid a heating of the atomic cloud. At the end of the molasses stage, we check with a Hall effect sensor that the MOT field has completely turned off.

An offset magnetic field is then applied after the red MOT stage. As described in Section 11.4.2.5, this offset magnetic field is applied by an additional current flowing in the Helmholtz coils. This is on top of what is necessary to track the PID action and cancel the magnetic field noise. The magnetic field offset is used to study the dependence of the MOR effects on the applied magnetic field.

The measurement of the polarization rotation is performed after a further time-of-flight of 20 ms. This delay allows for the offset magnetic field to reach its intended value. A check of the magnetic field value after this time, shows that 98% of the steady state magnetic field is achieved when the measurement is performed on the MOR effect. Furthermore, the optical thickness of the cloud is reduced within this duration, due the ballistic expansion of the cloud.

To measure the transmission images of the cold atomic cloud in the two channels (see an example in Figure 12.2), the 689 nm probe beam is turned on for a total duration of 120 μ s. This time includes also the time to flush the atoms, and to record the background and the reference images. The probe duration is short to make sure that the steady state of coherent transmission is achieved. At the same time, the probe duration is short enough to avoid any significant mechanical effects on the atomic cloud. We verify that there is, indeed, no mechanical effect due to the probe beam.

After the experiment is completed, the sensors are switched on and off repetitively for 1 s to reboot the sensors. The next run of the experiment begins after the sensors are rebooted.

12.2 Performance of the control system

We turn our attention now to a study of the performance of the control system. As a first step, we calibrate the sensor network at zero magnetic field, making use of the LMOR effects on the atomic cloud (see Section 12.2.1). This calibration is required for the PID controller to correct for the magnetic field noise below 1 mG. In addition, we also comment briefly on the calibration of the sensor network for the nonzero readings.

In Section 12.2.2, we further discuss the long term stability of the magnetic field cancellation by the PID controller. The results show a good long term stability of the PID controller loop, over the duration of one month. This means that, the calibration of the sensor network measurement is robust over this period.

12.2.1 Calibration of the sensor network measurement

We perform the calibration of the sensor network measurement by first bringing to zero the DC component of the magnetic field at the center of the cold atomic cloud. To avoid a possible bias in the reading of the sensor network, the actual zero magnetic field condition is found by looking at the MOR effects on the atoms.

In principle, to have a good calibration accuracy on all three components, we need to send the probe beam from all three directions. Along each probe propagation direction, the Faraday effect with the linear dependence on the magnetic field, is most suited for the calibration of the sensor network reading. In practice, due to a geometrical limitation in the direction where the probe beam may propagate, we can only send it along the y -direction. Thus, along the x - and z -directions, the sensitivity is lower because the MOR effects depend only quadratically on the transverse component of the small stray magnetic field.

In the following, we investigate first the frequency dependence of the LMOR effect experienced by the atomic cloud. This study is useful to reduce the magnitude of the transverse magnetic field components.

12.2.1.1 Frequency dependence of LMOR

In Figure 12.4, we plot the curves of b_a (blue open circle) and b_b (green dots) against the probe frequency⁴. In this experiment, the peak resonant optical thickness of the cloud is typically measured to be $b_{\bar{v}}(0) = 1.1$. This measurement is done without any compensation of the magnetic field noise, where all the PID gains are set to zero. An external magnetic field of 50 mG is applied by a pair of compensation coils along the propagation direction of the probe beam. We use the calibration of the compensation coil based on the sensor network measurement (see Section 11.3.2.2), to determine the compensation coil current necessary to apply this magnetic field. This applied magnetic field is larger than the stray magnetic field, to put us in the condition close to Faraday rotation. The applied field shifts the $m = \pm 1$ substates by typically 100 kHz.

The frequency of the red probe beam is then scanned in a range of ± 200 kHz around the transition resonance to perform the experiment. The transmission images in both the polarization channels of a and b are recorded for each frequency value of the probe beam. They are then used to calculate the rotation angle θ .

We compare between the experimental data, and the theoretical predictions given in Equations (10.53) and (10.54) for the case of Faraday rotation. Following the experimental conditions, we set $b_{\bar{v}}(0) = 1.1$ and $k\bar{v}/\Gamma = 2$ in the theoretical

⁴The polarization channels a and b forms a polarization basis that is rotated by 45° from the horizontal-vertical polarization basis.

Figure 12.4: The values of b_a and b_b for the cold atomic clouds are plotted in the figure. The solid curves are the theoretical predictions computed at $b_v(0) = 1.1$ and $k\bar{v}/\Gamma = 2$. The magnetic field in the theoretical calculation is set to closely match the general feature of the experimental curves. In this case, we find $g_J\mu_B B_y/\hbar = 13.2\Gamma$, i.e., $B_y = 47$ mG. Note that the negative values of b_a and b_b does not imply the existence of a gain (see Section 10.2.3). This negative value is due to the rotation of the polarization that may lead to a higher transmitted intensity in one of the polarization channel, as compared to its lower incident intensity in that particular channel.

calculations. The magnetic field along the y -direction is then chosen to match the experimentally measured b_a and b_b curves. We find a value of $B_y = 47$ mG. A small bump close to $\delta = 0$ is observed for the two curves in Figure 12.4. These bumps are absent in the theoretical predictions. In the ideal Faraday rotation, only the $\sigma_{\pm}$ transitions contributes to the transmission of light. However, in reality, there exist a component of the small stray magnetic field that is parallel to the polarization of the light. This component is coupled to the π transition in the atoms. The additional attenuation of light by this π transition causes the bumps around $\delta = 0$, in the curves of b_a and b_b , as seen in Figure 12.4.

We plot the corresponding measurements of the rotation angle θ in Figure 12.5. The green open circles are θ_0 , as defined in Equation (12.4). These offset values are removed from the experimentally measured values of θ . We find a good agreement of $\theta - \theta_0$ with the theoretical predictions. As expected, the shape of the rotation angle consist of two dispersive curves, each centered around ± 100 kHz. These two dispersive curves originate from the Zeeman shifted $\sigma_{\pm}$ transitions. The signatures due to a transverse stray field component parallel to the light polarization is not

Figure 12.5: The rotation angle θ measured under the same experimental condition as in Figure 12.4. The experimental results are represented by the blue stars. The green open circles indicate the offset in the rotation angle that are due to the imbalance in the incident intensity of the two polarization channels, a and b . The blue curve is the theoretical prediction of the rotation angle, calculated using Equation (10.25) with $b_{\bar{v}}(0) = 1.1$, $k\bar{v}/\Gamma = 2$ and $g\mu_B B/\hbar/\Gamma = 13.2$. It shows a good agreement with the experimentally measured value.

clearly seen here for the rotation angle. Indeed, the large magnetic field along y -direction masks any signature from the component of the stray field parallel to the polarization.

Nevertheless, in the absence of a strong bias field, the transverse components of the stray field still affect the Faraday rotation angle θ (see an example Figure 12.9).

12.2.1.2 Effect of the transverse magnetic field components on Faraday rotation

Let us consider theoretically the effect due to a small transverse stray magnetic field, B_t , around the science chamber. The stray field is assumed to make an angle of $\pi/4$ with the polarization of a probe beam. In the section, we study theoretically how this stray magnetic field affects the Faraday rotation experiment. The applied magnetic field along the propagation direction is denoted as B_y .

In the absence of a stray field, we find the usual dispersive-like curve of Faraday rotation (see Figure 10.4). In particular, we see that θ is zero when $B_y = 0$. The point where $\theta = 0$ is the zero crossing point of the curve. In the presence of a stray

Figure 12.6: The dependence of the zero crossing value on the stray magnetic field B_t along the transverse direction. The curve is plotted for a transverse stray field that makes an angle of $\pi/4$ with the light polarization. The point of zero crossing in Faraday rotation, corresponds to the value of applied B_y to achieve a zero rotation of $\theta = 0$.

field along the transverse direction, the zero crossing point occurs at a different value of B_y . This leads to a bias in the value of the crossing, as shown in Figure 12.6. We find that the value of the zero crossing for a Faraday rotation experiment does not depend sensitively on the value of B_t . Even with a 4 mG transverse component of the stray field, the changes in the zero crossing value is still less than 1 mG. This puts a less demanding requirement on putting the transverse stray field components to zero, when calibrating the sensor network measurement along the y -direction.

12.2.1.3 Setting stray magnetic field to zero

In this section, we describe the methods used to put the actual magnetic field experienced at the center of the atomic cloud, to a level below 1 mG⁵. We do this by minimizing the values of b_a or b_b at resonance, to reduce the transverse stray magnetic field component. Finally, we use the zero crossing point of the Faraday rotation angle to bring the magnitude of the y -component of the stray magnetic field down to a level less than 1 mG.

To manually compensate the x -component of the stray magnetic field, a hori-

⁵Note that we have typically 0.5–1 mG variation of the magnetic field across the size of the cold atomic cloud. Since the magnetic field gradient is not compensated, the sensor network measurement is only correctly calibrated at the center of the cold cloud.

Figure 12.7: Figure showing b_b as a function of B_x , when bias a magnetic field of $B_y = 50$ mG is applied. The left panel shows a plot of the measurement data before the manual compensation of B_x . The right panel shows the measurement data after the manual compensation. A quadratic function is fitted to the experimental data, to locate the minimum point. The manual compensation of the the x -component stray field is carried out based on the location of the minimum point (see text for more details). Before the compensation, we find the minimum of b_b to occur when $B_x = 3.3$ mG is applied. After the manual compensation, the minimum point of b_b occurs when $B_x = 0.1$ mG.

zontally polarized probe beam at $\delta = 0$ is used. An offset magnetic field of 50 mG is applied along the y -direction by a pair of Helmholtz coil. Another pair of Helmholtz coil along the x -component varies the applied magnetic field B_x . As discussed in the previous section, when B_x is varied, we expect a minimum for b_b when the component of the magnetic field along x -direction is zero. As seen in the left panel of Figure 12.7, a minimum point of b_b corresponds to $B_x = 3.3$ mG. This value is obtained by fitting a quadratic function to the experimental curve close to zero magnetic field⁶. To remove the average stray field component along x , the current of the corresponding Helmholtz coil pair is changed by an amount corresponding to 3.3 mG. After this, we repeat the experiment, with the results shown in the right panel of Figure 12.7. The minimum point of b_b is now located at 0.1 mG, indicating that we have successfully reduced the x -component of the stray magnetic field.

⁶Each data point results from an average over three runs of the experiment. The 50 Hz noise is not canceled during the experiment. Nevertheless, the repeated runs, and the fit over several data points around zero magnetic field, leads to an average over the effect of the AC magnetic noise.

Figure 12.8: The rotation angle (blue open circles), measured by varying the B_y values. The offset value of the rotation angle, θ_0 , is shown as the red crosses. The intersection between the the blue open circles and the red crosses, gives the magnetic field that has to be applied along the y -direction, in order to the atom experience an actual zero magnetic field component along y -direction. In this case, we find it to be $B_y = 1.9$ mG

The same procedure is then repeated for the stray magnetic field component along the z -direction. The polarization of the incident beam is first changed to align along the vertical direction. The Helmholtz coil pair along the z -direction is now used to vary B_z along the z -direction. Initially, the minimum of b_b occurs at $B_z = 1.4$ mG. After the manual compensation, the minimum of b_b is now occurring at B_z less than 0.1 mG. This shows again that the manual compensation of the stray field succeeds in reducing the magnitude of the stray field. We find a residual magnetic field component that is lower than 1 mG.

From Figure 12.6, a transverse magnetic field component of less than 1 mG has no significant effect on the calibration of the y -component stray field, at the milli-gauss level of accuracy. To do the final calibration of the magnetic field component along y -direction, we plot the rotation angle θ against B_y as the blue open circles in Figure 12.8. This experiment is also performed at $\delta = 0$. We see now the dispersive profile of the Faraday rotation angle. The red crosses in Figure 12.8, refers to the offset of the rotation angle, θ_0 . The intersection between the curves of the red crosses and the blue open circles gives the required B_y value to put the component along the y -direction to zero. We find $B_y = 1.9$ mG through a linear fit of θ around zero magnetic field.

Figure 12.9: The rotation angle θ as a function of the probe beam frequency, before and after we manually compensate the magnetic field. The blue open circles are the rotation angles with a nonzero stray magnetic field. Here, the LMOR effect due to the stray field is clearly observed. The rotation angles are measured again after the stray magnetic field is brought close to zero (the black open squares). Now, they are distributed randomly around the red dots, which are the measured θ_0 values. This is consistent with the fact that the stray field has been compensated. The optical thickness of the cloud in this experiment is around 1.4. The cloud temperature is 1 μ K.

12.2.1.4 Calibration at zero magnetic field

By carrying out the procedures in the previous section, each component of the magnetic field has a time-averaged value of less than 1 mG. We further verify this by performing a measurement of θ , as a function of the probe beam frequency. In this study, we do not apply any bias field. In Figure 12.9, we compare between the frequency dependence of θ , before (blue open circles) and after (black open squares) the manual compensation of the stray field. The black open squares representing θ is seen to be randomly distributed around θ_0 (red dots), without any clear dependence on the frequency. This is an indication that the magnetic field experienced by the atoms are indeed close to zero after the manual compensation procedure is performed. In contrast, the blue open circles show a clear dependence on the probe frequency, indicating a LMOR effect that originates from the uncompensated stray magnetic field.

So far, the manual compensation procedure only reduces the average value of the stray magnetic field at the center of the cold atomic cloud. They are still 50 Hz fluctuations of the magnetic field that can only be removed by applying an

active compensation. Nevertheless, the magnetic field fluctuations should now be centered around the zero magnetic field value. However, the sensor network still gives a nonzero time-averaged reading of $\mathbf{B} = (0.0, 2.2, 5.0)$ mG. This offset value is removed from the readings of the sensor network. After removing this offset value, the readings of the sensor network at zero magnetic field is well calibrated to a level below 1 mG.

12.2.1.5 Correction of the sensor network reading at nonzero value

In addition to knowing that the sensor network gives a reliable zero magnetic field value, it is also important to know if the non-zero values of the sensor network are accurate. Using the results in Table 11.5, we can give an answer to this question. In That table, we summarize the Zeeman shifts of the $m = \pm 1$ substates, when a magnetic field is applied along the y -direction. The magnetic field applied by the Helmholtz coils is based on the calibration by the sensor network measurements (the first column of the table). The actual Zeeman shift measured from a spectroscopy experiment is recorded in the last column of the table. From this measurement, the actual magnetic field component along the y -direction is found to be typically 1.14 times smaller than the sensor network reading. Eventually, this correction factor has to be applied to the sensor network measurement output, in order to calibrate its reading.

Since only three measurements are made, the calibration here depends critically on the assumed linearity of the sensor measurement output. We did not check this directly in our studies. Nevertheless, we can have an idea on the linearity of a single three-axis sensor by analyzing its frequency spectrum, using the results shown in Figure 11.7. The power in the second harmonic of the modulation is typically four orders of magnitude lower than the power at the modulation frequency. This means that there is only a small nonlinearity in the response of a single three-axis sensor. Since the sensor network linearly combines the measurement outcomes of the eight three-axis sensors, we may deduce that the sensor network measurement is also linear.

12.2.2 PID control of magnetic field and its long term behavior

With a proper calibration of the sensor network, we expect that the PID controller can correctly compensate the stray magnetic field. This is checked directly on the atoms. Furthermore, it is also important to check the long term stability of the sensor network reading. A sensor network reading that has good long term stability,

will ensure a long terms stability of the stray magnetic field compensation.

To study the performance of the PID control of the magnetic field, we scan the magnetic field along the y -direction, B_y , within a small range of ± 5 mG around zero. Since the transition is Doppler broadened, this range is sufficiently small for us to treat θ is a linear function of B_y . For each B_y value, the experiment under the same condition is typically repeated four times to improve the signal-to-noise ratio⁷. A linear fit of θ against the applied magnetic field B_y then allows the zero crossing of the rotation angle to be determined. The magnetic field at the center of the cloud can be determined from the zero crossing point, as described in the previous section.

Two types of experiments are performed carried to study the performance of the PID controller. The first type of experiment is performed with the stray magnetic field locked to zero using the PID controller. We recall that the cancellation of the magnetic field is performed only before the preparation of the cold atomic cloud, and the tracking of this magnetic field cancellation is applied during the experiment. The default settings for the filters and PID controllers are listed in Table 12.1. The second type of experiment is performed with all the PID gain values set to zero. There is no cancellation of the stray field during the experiment. These two types of experiments are performed over the duration of more than one month, to investigate the long term stability of magnetic field compensation. In both experiments, the readings of the sensor network are recorded and compared to the values measured by locating the zero crossing point.

In Figure 12.10, we summarized the results of the experiments. We plot as the blue dots, the measurements results from the zero crossing points of the y -component magnetic field when the cancellation of the magnetic field is carried out. The corresponding sensor network reading values are found to be within the gray shaded area. A good agreement, with an accuracy of less than 1 mG, is achieved between the two types of measurement methods. The results here show that the control of the magnetic field fluctuations, at a level below 1 mG, is achieved in our setup.

In the case where the stray magnetic field value is not locked by the PID controllers, the results are plotted as the red open circles in Figure 12.10. Since the stray magnetic field is not canceled, the y -component of the magnetic field fluctuates within a range of several milligauss. We find, generally, good qualitative agreement between this result and the readings of the sensor network (green triangles) under the same condition. Nevertheless, the agreement is not quantitative. This is because the measurement of the magnetic field through the zero crossing of

⁷Since each run of the experiment lasts about 3.5 s, this limits the range and the number of repeated experimental runs that can be performed within a reasonable duration.

Figure 12.10: The results of various experiments performed over a duration of one month to test the performance of the PID controller on the y -component of the magnetic field. The measurement results from the zero crossing method, when the magnetic field values are locked at zero value, are shown as the blue dots. The red open circles are the zero crossing measurements, when the PID gain values are zero. These values are compared to the sensor network readings. In the case where the PID controller is applied, the typical range of the sensor network reading is shaded in gray. When PID controller is not applied, the sensor network readings are shown as the green open triangle.

the Faraday rotation experiment is accurate only when the x - and z -components of the stray magnetic field are small (see Section 12.2.1.2). Since there is no cancellation of the stray magnetic field, this transverse magnetic field may be larger and could lead to an offset in the values obtained from zero crossing.

In Figure 12.11, we look into the daily 24-hour variation of the stray magnetic field. This is done by overlapping the measurements from different days, over the same 24-hour duration. In the cases where the stray magnetic field value is locked at zero, the control system compensates correctly the magnetic field noise throughout the day. However, when the PID controller is not applied, the measurement of the magnetic field shows a variation in the order of a few milligauss over the 24-hour period. In particular, a larger variation is observed between 9 am to 7 pm, suggesting an origin of the magnetic field noise that is related to human activities around the laboratory.

Our results as reported in this section, show a good stability of the PID controller in compensating the stray magnetic field, over a duration that is typically a month. This also means that the calibration for the sensor network at zero magnetic field,

Figure 12.11: The measurement results in Figure 12.10 are overlapped in a 24-hour duration, showing the daily variation of the magnetic field. For the case where the PID feedback is applied, the magnetic field remains close to zero throughout the day. However, if the PID feedback is not applied, the value of B_y fluctuates during the day. In particular, due to human activities, particularly between 9 am to 7 pm, additional noise on the magnetic field can be seen.

is also retained over the duration of one month. No significant drift of the offset in the sensor network readings is observed.

12.3 50 Hz magnetic field noise

In this section, we complement the discussion of the cancellation of the magnetic field noise, by investigating the origin of the 50 Hz measurement noise. To perform this study, the PID gains for the control of the 50 Hz frequency component are all set to zero. We keep only the low frequency PID gains to compensate the low frequency component of the stray magnetic field. Many runs of the experiment are performed. Each time an experiment is carried out, we also record a stepped-down voltage of the 50 Hz AC line power supply. This is done using a transformer inside a DC power supply. We finally have a peak-to-peak value of 2 V. At the same time, the rotation angle of the transmitted light through the atomic cloud is measured. This rotation angle is plotted against the corresponding output of the transformer in Figure 12.12. The raw data of the measurement is shown as the blue open circles. To reduce the fluctuations, the experimental data is collected into 25 bins, where

Figure 12.12: A plot showing the correlation between the rotation angle in the MOR effect, and the stepped-down voltage of the AC line.

the average and standard deviation of the voltage and rotation angle in each bin are calculated. This is shown as the red open circles with the associated error bars. A correlation is found between θ and the value of the AC voltage at the moment when the experiment is carried out.

The output from the transformer is also recorded using an oscilloscope. By fitting the measured signal with a sinusoidal function, the phase of the AC line power supply can be measured. The angle of rotation can be plotted against this phase value as shown in Figure 12.13. The raw data is plotted as the blue open circles. The experimental data is again partitioned into 25 bins to compute the average values as shown by the red open circles. The error bars reflect the standard deviation associated with the bin-averaged values.

The averaged data is fitted with a sinusoidal function (black solid curve in Figure 12.13). We find a peak-to-peak value of 0.0857 rad for the rotation angle θ experienced by the probe beam. The cloud optical thickness is $b_{\bar{v}}(0) = 0.9$, with a temperature of $T = 1.3 \mu\text{K}$. Assuming that the magnetic field components along x - and z -directions have negligible impact here, a use of Equation (10.44) results in a peak-to-peak value of 2 mG. This falls within the range of peak-to-peak values of 50 Hz noise that has been measured by the sensor network, i.e., 1-2 mG along each component. Thus, this is consistent with the assumption that the canceled 50 Hz noise is magnetic in origin.

In this chapter, we have described the calibration of the sensor network measure-

Figure 12.13: A reconstruction of the rotation angle θ experienced by a probe beam, as a function of the phase of the AC line voltage. The blue open circles are the raw data, while the red open circles with the error bars are the average values computed for the 25 bins of data. The black solid curve shows a fit of the averaged values to a sinusoidal function. The function for the fit is given in the plot.

ment, which is a prerequisite for a correct operation of the active control system. We then study the performance and stability of the magnetic field compensation. We find that the compensation of the magnetic field component along y -direction is achieved at the level below 1 mG, with a good stability that does not drift even after one month. In the next chapter, we describe the use of this system to study the NMOR effects.

Chapter 13

Study of NMOR Effects

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In the previous chapter, the magneto-optical rotation (MOR) experiments are performed with weak probe beam resonant to the $^{88}\text{Sr } ^1\text{S}_0 \rightarrow ^3\text{P}_1$ intercombination transition, with intensity restricted to be below the saturation intensity $I_{\text{sat}} = 3 \mu\text{W cm}^{-2}$ of the atom. In this chapter, we remove the restriction on the intensity of the probe beam, allowing it to be larger than the saturation intensity. The nonlinear MOR (NMOR) effects can now be observed on the transmitted light through the cold atomic cloud.

We focus our studies only on the special cases of Faraday and Voigt rotations. In the case of nonlinear Faraday effect, the two polarization modes for the coherent transmission of light, are still the $\sigma_{\pm}$ polarizations related to the transitions from the $^1\text{S}_0$ ground state to the $m = \pm 1$ substates of the excited $^3\text{P}_1$ state. In the case of the nonlinear Voigt effect, the two polarization modes are aligned parallel and perpendicular to the direction of the magnetic field. The component of light that is parallel to the magnetic field, probes the magnetically insensitive π transition of ^{88}Sr , while the other component probes both the $\sigma_{\pm}$ transitions.

Recall that, for the linear MOR (LMOR) effect, the π and $\sigma_{\pm}$ transitions are independent. They also propagate independently through the atomic medium. This is not true for the NMOR effects, since there is a significant saturation effect, which causes a population hole burning in the ground state. Furthermore, the coherences between the different excited substates also become important for NMOR. Additional nonlinear effect also comes from the absorption experienced by light propagating through the atomic medium. The complications here mean that there is no analytical solution for the NMOR effect, unlike the simple case of LMOR. A numerical calculation is performed here, for comparison with the experimental results. The governing equations used to numerically study the NMOR effect are presented in Section 13.1. The experimental results are reported later in Section 13.2.

13.1 NMOR on the $J = 0 \rightarrow J = 1$ transition

The intercombination line of ^{88}Sr is one of the simplest systems to study the NMOR effect. There is no hyperfine structure in this transition. Thus, we have just to consider a four-level system. In this section, we extend the Maxwell-Bloch equations [117, 173] to this $J = 0 \rightarrow J = 1$ transition. The $m = 0 \rightarrow m = 0$ transition of the intercombination line is insensitive to Zeeman shift. Its resonance frequency is denoted by ω_0 , and the probe beam detuning δ is measured with respect to this frequency.

In Section 13.1.1, we first introduce the dipole operator for the $J = 0 \rightarrow J = 1$ transition, which is used to define the Rabi frequency. The Rabi frequency is a

measure of the strength of light-atom interaction. Furthermore, the dipole moment density is the source term in the Helmholtz equation, that is used to model the coherent propagation of light inside the atomic medium.

The dipole moment of the atom is induced by light-atom interaction, and can be computed from the optical Bloch equations (see Section 13.1.2)¹. The Helmholtz equations for the nonlinear Faraday and Voigt rotations, are then described in Sections 13.1.3 and 13.1.4, respectively.

13.1.1 Dipole operator

We denote the $J = 0$ ground state by $|g\rangle$. The $J = 1$ excited state has three substates. The $m = \pm 1$ excited states are denoted by $|\pm 1\rangle$, while the $m = 0$ excited substate is denoted by $|0\rangle$.

The magnetic field is directed along the unit vector $\boldsymbol{\epsilon}_0$, giving the π -transition. The other two circular unit vectors $\boldsymbol{\epsilon}_{\pm}$ are perpendicular to $\boldsymbol{\epsilon}_0$. We use these three unit vectors to be our basis. The dipole matrix elements are,

$$\mathbf{d}_{0g} = \mathbf{d}_{g0} = \mathcal{D}\boldsymbol{\epsilon}_0, \quad (13.1)$$

$$\mathbf{d}_{\pm 1g} = \mathbf{d}_{g\pm} = \mathcal{D}\boldsymbol{\epsilon}_{\pm}. \quad (13.2)$$

The phase of the states are chosen so that the reduced dipole moment $\mathcal{D}$ is real valued. $\mathcal{D}$ is related to the linewidth of the excited state through,

$$\Gamma = \frac{\omega_0^3 |\mathcal{D}|^2}{3\pi\epsilon_0 \hbar c_0^3}, \quad (13.3)$$

where c_0 is the speed of light in vacuum, ϵ_0 is the permittivity of free space, and $\hbar$ is the reduced Planck's constant.

In the electric dipole approximation, the interaction between the light field $\mathbf{E}$ (frequency ω_l) and the atomic dipole, is given by the term $-\mathbf{E} \cdot \hat{\mathbf{d}}$ in the Hamiltonian. The spatial variation of the light is ignored here. For an atom located at $\mathbf{r} = \mathbf{0}$, the light field experienced by the atom is

$$\mathbf{E} = E\boldsymbol{\epsilon} = \text{Re} \{ E_0 e^{-i\omega_l t} \} \boldsymbol{\epsilon}. \quad (13.4)$$

To quantify the strength of light-atom interaction, we need to define the Rabi

¹The optical Bloch equation is extended from the usual two-level-atom case, to account for the four sublevels of the $J = 0 \rightarrow J = 1$ transition.

frequencies. For the $\sigma_{\pm}$ transitions, we have

$$\Omega_{\pm 1} = -\frac{E_0}{\hbar} \boldsymbol{\epsilon} \cdot \mathbf{d}_{0g} = \Omega \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}_{\pm}, \quad (13.5)$$

and for the π transition,

$$\Omega_0 = \Omega \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}_0. \quad (13.6)$$

In both expressions, $\Omega = -\frac{E_0 \mathcal{D}}{\hbar}$.

In the following, we introduce the optical Bloch equations, for the four-level $J = 0 \rightarrow J = 1$ transition.

13.1.2 Optical Bloch equations

Let us denote the density matrix for an atom by $\hat{\rho}$. We define a new operator $\hat{\sigma}$ in the interaction picture, which is related to $\hat{\rho}$ by the following relations between the matrix elements,

$$\sigma_{\pm 1g} = \rho_{\pm 1g} e^{-i\omega_l t}, \quad (13.7)$$

$$\sigma_{0g} = \rho_{0g} e^{-i\omega_l t}. \quad (13.8)$$

The other elements of $\hat{\sigma}$ are identical to the corresponding elements of $\hat{\rho}$.

The time dependence of $\hat{\sigma}$ is governed by the following equation,

$$\left. \frac{d\hat{\sigma}}{dt} \right|_{\text{relax}} = -\frac{i}{\hbar} [\hat{H}, \hat{\sigma}] + \left. \frac{d\hat{\sigma}}{dt} \right|_{\text{relax}}, \quad (13.9)$$

where $\left. \frac{d\hat{\sigma}}{dt} \right|_{\text{relax}}$ accounts for the relaxation of the operator $\hat{\sigma}$. In the matrix form, it is

$$\left. \frac{d\hat{\sigma}}{dt} \right|_{\text{relax}} = - \begin{pmatrix} \Gamma \sigma_{-1-1} & \Gamma \sigma_{-10} & \Gamma \sigma_{-1+1} & \frac{\Gamma}{2} \sigma_{-1g} \\ \Gamma \sigma_{0-1} & \Gamma \sigma_{00} & \Gamma \sigma_{0+1} & \frac{\Gamma}{2} \sigma_{0g} \\ \Gamma \sigma_{+1-1} & \Gamma \sigma_{+10} & \Gamma \sigma_{+1+1} & \frac{\Gamma}{2} \sigma_{+1g} \\ \frac{\Gamma}{2} \sigma_{g-1} & \frac{\Gamma}{2} \sigma_{g0} & \frac{\Gamma}{2} \sigma_{g+1} & -\Gamma \sum_{q=-1}^1 \sigma_{qq} \end{pmatrix}. \quad (13.10)$$

In the above, the first line and column correspond to the $m = -1$ substate of the excited state. The second line and column correspond to the $m = 0$ substate, and the third line and column corresponds to the $m = +1$ substate. The last line and column correspond to the ground state.

The cold ^{88}Sr atomic cloud in our experiment still experiences significant Doppler effect, since we have $k_l \bar{v} / \Gamma \approx 2$. Here, k_l is the probe beam wave vector, $\bar{v}$ is

the root-mean-square velocity and Γ is the linewidth of the intercombination line transition. Therefore, it is important to include the motion of the atoms in our model. For an atom moving with a velocity component v along the propagation direction of the probe beam, a Doppler shift of $-k_l v$ has to be taken in account in the expression of the Hamiltonian $\hat{H}$. Thus, in the rotating wave approximation (RWA), the Hamiltonian is,

$$\hat{H} = \begin{pmatrix} -\delta_{-1} + k_l v & 0 & 0 & \frac{\Omega_{-1}}{2} \\ 0 & -\delta + k_l v & 0 & \frac{\Omega_0}{2} \\ 0 & 0 & -\delta_{+1} + k_l v & \frac{\Omega_{+1}}{2} \\ \frac{\Omega_{-1}}{2} & \frac{\Omega_0}{2} & \frac{\Omega_{+1}}{2} & 0 \end{pmatrix}, \quad (13.11)$$

where B is the magnitude of the applied field. The detuning of the $m = \pm 1$ substates, are given by $\delta_{\pm 1} = \delta \mp \delta_B$. The quantity $\delta_B = g_J \mu_B B / \hbar$ is the absolute value of the Zeeman shift experienced by both the $m = \pm 1$ substates. The Landé g factor is $g_J = 1.5$, and the Bohr's magneton is denoted by μ_B .

Next, we consider specifically the cases of nonlinear Faraday and Voigt rotations. In each case, the optical Bloch equations described here are coupled to a Helmholtz equation to describe the propagation of light through the atomic medium.

13.1.3 Nonlinear Faraday rotation

For the Faraday rotation, only the $\sigma_{\pm}$ transitions are driven by the probe beam. This means that $\Omega_0 = 0$. Essentially, Equation (13.9) reduces to an equation for a 3×3 matrix. For the incident probe beam, the Rabi frequencies of the two circular polarization modes are $|\Omega_{\pm}| = |\Omega|/\sqrt{2}$, with Ω being the Rabi frequency associated with the total intensity of the beam.

We reduce the system to a 1D problem, by modeling the cold atomic cloud as a slab of thickness L with a uniform density ρ_0 . The propagation of the light is directed along the y -axis. The expectation value of the dipole operator is given by $\langle \hat{\mathbf{d}} \rangle = \text{Tr}\{\hat{\rho} \hat{\mathbf{d}}\}$. For an atom moving with a velocity component v along the y -direction, the dipole moment is found to be

$$\langle \hat{\mathbf{d}}(y, v, t) \rangle = 2\text{Re}\{\mathcal{D}\sigma_{-1g}(y, v)e^{-i\omega_l t}\}\epsilon_- + 2\text{Re}\{\mathcal{D}\sigma_{+1g}(y, v)e^{-i\omega_l t}\}\epsilon_+. \quad (13.12)$$

In general, this formalism can treat the MOR effect for an arbitrary time dependence. Here, we focus only on the steady state regime of coherent transmission, similar to what has been done in the experiment. The steady state solutions of Equation (13.9) for the coherences σ_{+1g} and σ_{-1g} are applied to the equation above.

The position dependence appears, since $\sigma_{\pm g}$ depends on the electric fields that are attenuated as the light propagates through the medium.

We assume that the medium is dilute and the atoms scatter light independently. It is then possible to average over the velocity distribution of the atoms. We find the velocity-averaged polarization density $\mathbf{P}$ of the atomic medium to be given by,

$$\mathbf{P}(y, t) = \rho_0 \int_{-\infty}^{\infty} \langle \hat{\mathbf{d}}(y, v, t) \rangle f(v) dv, \quad (13.13)$$

where v is distributed according to the thermal Gaussian distribution,

$$f(v) = \frac{1}{\sqrt{2\pi\bar{v}}} e^{-v^2/2\bar{v}^2}. \quad (13.14)$$

Using Equation (13.12), we can rewrite

$$\mathbf{P}(y, t) = \text{Re} \{ [P_-(y) \boldsymbol{\epsilon}_- + P_+(y) \boldsymbol{\epsilon}_+] e^{-i\omega_l t} \}, \quad (13.15)$$

with

$$P_{\pm}(y) = 2\rho_0 \mathcal{D} \int_{-\infty}^{\infty} \sigma_{\pm 1g}(y, v) f(v) dv. \quad (13.16)$$

Let the steady state field that propagates through the uniform slab, be denoted by

$$\mathbf{E}(y, t) = \text{Re} \{ (E_-(y) \boldsymbol{\epsilon}_- + E_+(y) \boldsymbol{\epsilon}_+) e^{-i\omega_l t} \}. \quad (13.17)$$

The components of the polarization density, $P_{\pm}$, are the source terms for amplitudes $E_{\pm}$ of the two circular polarization components. The Helmholtz equations for the two components are given by,

$$\frac{d^2 E_{\pm}(y)}{dy^2} + k_l^2 E_{\pm}(y) = -2\mu_0 \omega_l^2 \rho_0 \mathcal{D} \int_{-\infty}^{\infty} \sigma_{\pm 1g}(y, v) f(v) dv, \quad (13.18)$$

where μ_0 is the permeability of free space.

Equations (13.18) and (13.9) are coupled, and can be solved numerically to describe the nonlinear Faraday rotation in the cold atomic cloud. In the experiments, we study the NMOR effects on the light passing through the center of the cloud. Thus, the peak optical thickness of the cold cloud in the experiment is used as the optical thickness of the uniform slab. The Helmholtz equations are then discretized into a number of thin slabs. Each slab element has an optical thickness that is

small, ensuring that the light field does not get attenuated significantly when propagating through each element. θ is then computed from the transmitted intensity and compared with the experimental results.

13.1.4 Nonlinear Voigt rotation

For the case of Voigt rotation, all the three Rabi frequencies, Ω_0 and $\Omega_{\pm}$, are in general nonzero. Similar to the case of Faraday rotation, the expectation value of the dipole moment for an atom with a velocity component v along the y -direction, has three components,

$$\langle \hat{\mathbf{d}} \rangle = 2\text{Re} \{ \mathcal{D}\sigma_{-1g}e^{-i\omega_l t} \} \boldsymbol{\epsilon}_- + 2\text{Re} \{ \mathcal{D}\sigma_{0g}e^{-i\omega_l t} \} \boldsymbol{\epsilon}_0 + 2\text{Re} \{ \mathcal{D}\sigma_{+1g}e^{-i\omega_l t} \} \boldsymbol{\epsilon}_+. \quad (13.19)$$

Using the steady state coherence terms solved from the optical Bloch equations, the average dipole moment density for a thermal cloud is given by

$$\mathbf{P}(y, t) = \text{Re} \{ [P_-(y) \boldsymbol{\epsilon}_- + P_0(y) \boldsymbol{\epsilon}_0 + P_+(y) \boldsymbol{\epsilon}_+] e^{-i\omega_l t} \}, \quad (13.20)$$

with

$$P_-(y) = 2\rho_0 \mathcal{D} \int_{-\infty}^{\infty} \sigma_{-1g}(y, v) f(v) dv, \quad (13.21)$$

$$P_0(y) = 2\rho_0 \mathcal{D} \int_{-\infty}^{\infty} \sigma_{0g}(y, v) f(v) dv, \quad (13.22)$$

$$P_+(y) = 2\rho_0 \mathcal{D} \int_{-\infty}^{\infty} \sigma_{+1g}(y, v) f(v) dv. \quad (13.23)$$

The polarization density is also averaged over the thermal velocity distribution $f(v)$. For convenience, we rewrite $\mathbf{P}(y, t)$ in terms of its components along x - and z -directions, i.e.,

$$P_x(y) = \mathbf{e}_x \cdot \mathbf{P}(y, t), \quad (13.24)$$

$$P_z(y) = \mathbf{e}_z \cdot \mathbf{P}(y, t). \quad (13.25)$$

These are the two propagation eigenmodes for the Voigt rotation. The unit vectors along the x - and z -directions are denoted by $\mathbf{e}_x$ and $\mathbf{e}_z$, respectively. This allows us to define

$$\sigma_x(y, v) = \mathbf{e}_x \cdot [\sigma_{-1g}(y, v) \boldsymbol{\epsilon}_- + \sigma_{0g}(y, v) \boldsymbol{\epsilon}_0 + \sigma_{+1g}(y, v) \boldsymbol{\epsilon}_+] \quad (13.26)$$

$$\sigma_z(y, v) = \mathbf{e}_z \cdot [\sigma_{-1g}(y, v) \boldsymbol{\epsilon}_- + \sigma_{0g}(y, v) \boldsymbol{\epsilon}_0 + \sigma_{+1g}(y, v) \boldsymbol{\epsilon}_+] \quad (13.27)$$

Using the above definitions, the Helmholtz equations are given by,

$$\frac{d^2 E_x(y)}{dy^2} + k_l^2 E_x(y) = -2\mu_0\omega_l^2\rho_0\mathcal{D} \int_{-\infty}^{\infty} \sigma_x(y, v)f(v) dv, \quad (13.28)$$

$$\frac{d^2 E_z(y)}{dy^2} + k_l^2 E_z(y) = -2\mu_0\omega_l^2\rho_0\mathcal{D} \int_{-\infty}^{\infty} \sigma_z(y, v)f(v) dv, \quad (13.29)$$

for the steady state propagation of light, where the optical field assumed to take the following form,

$$\mathbf{E}(y, t) = \text{Re} \left\{ [E_x(y)\mathbf{e}_x + E_z(y)\mathbf{e}_z]e^{-i\omega_l t} \right\}. \quad (13.30)$$

The numerical solution of the nonlinear Voigt rotation can be performed in a similar manner as described in the previous section for the case of nonlinear Faraday rotation.

In the following section, we turn our attention to the experimental studies of NMOR.

13.2 Experimental studies

The experimental studies of the NMOR are performed using the time sequence presented in Figure 12.3. The active control system for the magnetic field is applied during the experiment. The experiment is performed with the probe beam detuning set at $\delta = 0$. The applied magnetic field is scanned by applying an additional static magnetic field on top of the field required to cancel the stray magnetic field (see Section 11.4.2.5 for a description of how this field is applied). We present the experimental results in this section.

Similar to the studies of the previous chapter, θ is measured only at the center of the atomic cloud, where the optical thickness of the cloud is the largest. The calculation of the rotation angle is performed using also Equation (12.3).

The typical curves of the rotation angle θ measured in the cases of Faraday and Voigt rotations, are shown in Sections 13.2.1 and 13.2.2, respectively. An interesting narrow feature, appears around zero magnetic field, is observed when the probe beam intensity is high. Our study focuses on this narrow central feature, which is attributed to the nonlinear effect.

Figure 13.1: Faraday rotation at different values of the incident intensity ($\delta = 0$). The experimental results for $I_0 = 33.6I_{\text{sat}}$ (nonlinear Faraday rotation) and $I_0 = 0.1I_{\text{sat}}$ (linear Faraday rotation) are fitted with the appropriate profiles (more details given in Section 13.2.3). This fitting procedure measures the width, Γ_{NMOR} , of the central feature associated with the NMOR effect. The experiments are performed typically with cold clouds of optical thickness 1.5, and cloud temperature in the range of 1-1.5 μK .

13.2.1 Faraday rotation configuration

For the nonlinear Faraday effect, the magnetic field along the propagation direction, B_y , is varied. The rotation angle, θ , is measured using the balanced polarimeter method (see Section 12.1). Typical results are shown in Figure 13.1 for several values of the incident intensity I_0 . The peak optical thickness (at the center of the cloud) of the cold atomic cloud is typically 1.5. The temperature of the cold atomic cloud is typically within a range of 1-1.5 μK . This leads to a Doppler broadening of the transitions. At low incident intensity, this increases the width of the dispersive profile, and reduces the sensitivity of the transition to magnetic field. When $I_0 > I_{\text{sat}}$, instead of a dispersive-like curve seen for the one at low incident intensity (black open circles), an anomalous dispersive-like behavior is observed for θ as a function of B_y . It shows a narrower central feature. This nonlinear contribution is, interestingly, unaffected by Doppler broadening. This is discussed further in Section 13.2.3.

In addition to the width of the Faraday rotation, we investigate the difference in the behavior of $d\theta/dB_y$ between the linear and nonlinear Faraday rotations. In Figure 13.1, the linear Faraday rotation curve has a negative slope around $B_y = 0$. On the other hand, the nonlinear Faraday rotation curves have positive slopes close

Figure 13.2: Rotation angle θ as a function of B for the linear and nonlinear Voigt rotations. The experimental results are compared to the numerical calculation results at $\delta = 0$, using the equations in Section 13.1.4. The optical thickness of the cloud in the experiment is 1.4. The temperature of the cloud is $1 \mu\text{K}$. At small magnetic field values, the nonlinear Voigt rotation displays a curvature that is opposite in sign to the one shown by the linear Voigt rotation. The numerical calculations are performed at the same experimental condition. The results show excellent agreement with the experimental data.

to $B_y = 0$. Further studies of the slope, which quantifies the sensitivity of the nonlinear Faraday rotation, are reported in Section 13.2.4

13.2.2 Voigt rotation configuration

A similar study of the Voigt effect is carried out. The applied magnetic field component along the x -direction is varied. The polarization of the incident field is rotated to make a 45° angle with the x -direction. In our case this gives the strongest rotation signal. We use a cloud of optical thickness $b_{\bar{v}}(0) = 1.4$, and temperature $1 \mu\text{K}$. The rotation angle θ is shown in Figure 13.2. For the linear case, the experimental data is plotted as the blue open circles, while, for the nonlinear case, the experimental data is plotted as the red dots. They are compared to the numerically simulated results (using the equations discussed in Section 13.1.4), computed at the experimental conditions. The results for the linear and the nonlinear cases are plotted as the blue and the red solid curves, respectively.

At small values of the magnetic field, the rotation angle θ depends quadratically

on B . This is observed for both the linear and the nonlinear cases. At small magnetic field, the linear Voigt rotation shows a negative curvature, while the curvature is positive in the nonlinear case.

In the following, we do not study in detail the properties of the nonlinear Voigt rotation, choosing to focus our study only on the nonlinear Faraday rotation.

13.2.3 Width of nonlinear Faraday effect

In this section, we define the width of the central feature for the Faraday effect, through a fitting procedure implemented on θ . We start by first considering the linear Faraday rotation, which displays a dispersive-like profile for θ . This is expected from Equation (10.42). It comes from the mismatch in the refractive indices of the $\sigma_{\pm}$ modes, due to the difference Zeeman shift of the two $m = \pm 1$ substates. At $\delta = 0$, this mismatch has the same form as the real dispersive part of the complex refractive index, with its imaginary counterpart which is the optical thickness profile. This optical thickness profile (imaginary part) is related to the dispersive profile through the Kramers-Kronig relation. This relation is expressed by the Hilbert transform [174, 175]. In the linear regime, at low optical thickness, we have the Voigt profile as the absorption profile for both the $\sigma_{\pm}$ transitions². This Hilbert transform can be computed numerically, and fitted to the experimental data points. This is shown as the black curve in Figure 13.1, which fits well to the experimental data. The width of the Gaussian velocity distribution is found to be 8.2 mG, consistent with the width of the velocity distribution at a temperature of 1.5 μ K.

We consider now the nonlinear Faraday rotation. As mentioned before, the incident probe beam consists of the $\sigma_{\pm}$ components that are co-propagating through the medium. Under this configuration, the two photon detuning between these two components are unaffected by the Doppler effect. The central feature of the anomalous dispersive curve appears due to the coherence between these two $m = \pm 1$ substates.

We also apply the fitting procedure with the Hilbert transform to the nonlinear case. We do this as a heuristic method to extract the width of this central anomalous feature. We first postulate an absorption profile for the nonlinear case. We note that the situation here is similar to the case of an electromagnetically induced transparency (EIT) scheme, except that both the $\sigma_{\pm}$ components have equal incident intensity here. Following this idea, we postulate that the absorption profile is modified from the low intensity Voigt profile, by a Lorentzian dip

²The Voigt profile for the absorption curve appears from the convolution between a Lorentzian absorption profile of a single atom, and the Gaussian distribution of the atomic velocities. This profile is given by Equation (3.65).

at $B = 0$ [176]. This Lorentzian dip takes into account of the effects due to the coherence between the $m = \pm 1$ substates, and the population hole burning in the ground state. This dip has a Lorentzian profile because the photon redistribution between the $\sigma_{\pm}$ transitions is independent of the motion of the atoms. Similar effect involving a narrowing of the linewidth has also been observed for a hot vapor of rubidium [177]. In that work, a four-level atomic system is studied. It is composed of a Λ EIT system, with an additional control beam that coupled to another excited state. The frequency of this control beam is tunable. At the line center of the probe beam, the authors reported a width of typically 100 kHz, even though the probe beam in their setup is tuned on a transition with 6 MHz linewidth. Remarkably, they showed that this reduction in the linewidth happens only at high temperature, thanks to the contribution from off-resonant velocities in the distribution.

We then compute Hilbert transform of the nonlinear absorption profile, and fit it to the experimental result. An example is shown for the case of $I_0 = 33.6I_{\text{sat}}$, as the red curve in Figure 13.1. Our postulate leads to an excellent fit of the experimental data. A more solid theoretical foundation is required to understand why this fit function works well. This is left for future investigations. This method is then applied to all the curves of the nonlinear Faraday rotation to extract the Lorentzian width. This Lorentzian width is then equated to the width of the central feature for θ .

We denote the width of the central Lorentzian shape by Γ_{NMOR} . The values of Γ_{NMOR}^2 calculated from the fit is plotted against I_0/I_{sat} in Figure 13.3. The result is consistent with notion that the central feature is affected only by the power broadening effect. Indeed, in both the $\sigma_{\pm}$ polarization channels, we have an incident beam of intensity $I_0/2$. An extrapolation of the Γ_{NMOR} at zero intensity gives a value of 3.7 mG. This magnetic field value is consistent with the magnetic field required for a Zeeman shift corresponding to one linewidth. With this result, we validate the following empirical relation [compare the following expression with Equation (A.22) in Appendix A]

$$\Gamma_{\text{NMOR}} = \frac{\hbar\Gamma}{g_J\mu_b} \sqrt{1 + \frac{I_0}{2I_{\text{sat}}}}. \quad (13.31)$$

In the next section, we focus on the studies relevant to the slope $d\theta/dB$, of the narrow central feature. This slope is related to the magnetic field sensitivity of the Faraday rotation. In particular, we analyze this sensitivity as a function of various parameters such as the probe intensity I_0 , the resonant optical thickness $b_{\bar{v}}(0)$, and cloud temperature T .

Figure 13.3: The width of the nonlinear contribution, Γ_{NMOR} is plotted as a function of I_0/I_{sat} . The width is well explained, considering only the power broadening of the $\sigma_{\pm}$ transition.

13.2.4 Sensitivity of the Faraday rotation

In computing the slope $d\theta/dB$, we have to correct the value of B_y applied by the Helmholtz coils. We use the results of the calibration of the Helmholtz coils using the spectroscopy experiment, which is discussed in Section 11.3.2.3.

13.2.4.1 Dependence on intensity

In this section, we present first the dependence of $d\theta/dB$ on I_0 . In Figure 13.4, the results are plotted for the experiments that are performed on a cloud of peak optical thickness, $b_{\bar{v}}(0) = 1.6$ and temperature, $T = 1 \mu\text{K}$. For each value of I_0/I_{sat} , the magnetic field B is scanned within a range of $\pm 3.5 \text{ mG}$. The slope $d\theta/dB$ is then measured within a small range of magnetic field values around zero. The error bars represent the standard deviations of several measurement results under the same experimental conditions.

Using the governing equations in Section 13.1, the theoretical calculation results are compared to the experimental data (see the green curve in Figure 13.4). While the theory captures well the qualitative behavior of the experimental results, it overestimates the absolute value of the slope by a factor of 0.8 (see the black dash-dotted line in Figure 13.4). We believe that imperfections in the experiment systemati-

Figure 13.4: The slope, $d\theta/dB$, plotted as a function of I_0/I_{sat} for different values of the cloud temperature. The experiments are carried out at $T = 1 \mu\text{K}$, with the peak optical thickness of the cloud taking a value of $b_{\bar{v}}(0) = 1.6$. The theoretical prediction at this temperature, as outlined in Section 13.1, is shown as the green solid curve. It differs from the experimentally measured results, by a factor of 0.8. The black dash-dotted line takes the 0.8 factor into account, which agrees better with the experimental results. The theoretical predictions are also shown for other temperature values, i.e., at $T = 0 \mu\text{K}$ and at $10 \mu\text{K}$.

cally reduce the experimentally measured sensitivity. This systematic reduction could be due to the finite linewidth of the probe laser, as discussed in Ref. [40], which is not taken into account in our modeling. Its could also be due to an error in balancing the polarization channels of the incident probe beam. The optical thickness of the cloud could also be systematically lower than the value measured experimentally, leading to an overestimation of the rotation angle in the theory. A slight misalignment of the probe beam with the magnetic field also leads to an error in determining the magnetic field component along the propagation direction. We checked that each factor by itself does not account fully for the discrepancy. However, a combination of the several factors mentioned above could add up to account for the lower slope values measured in the experiment. A further investigation is required to fully explain this discrepancy.

At $T = 1 \mu\text{K}$, the slope $d\theta/dB$ around $I_0 = 2I_{\text{sat}}$ is zero. Above this value, the sign of the slope $d\theta/dB$ switches sign from negative to positive. This is when the nonlinear effects starts to dominate. At large intensity values, the magnitude of the slope decreases due to the larger power broadening effect.

In the same figure, two other theoretical curves are plotted with a different

cloud temperature. This gives an idea of the effect of cloud temperature on $d\theta/dB$. The red dotted line is computed for a zero temperature atomic cloud. At zero temperature, as expected, the sensitivity in the linear regime is higher since there is no Doppler broadening.

The other calculation is performed at a higher temperature of $10 \mu\text{K}$, and shown as the magenta dashed curve in Figure 13.4. We note that $|d\theta/dB|(I_0 = 5I_{\text{sat}}) > |d\theta/dB|(I_0 = 0)$. Indeed, a large Doppler broadening reduces significantly the value of $d\theta/dB$ in the linear regime. Thus, at high probe intensity, the nonlinear Faraday rotation can have a better sensitivity, as long as the power broadening effect is not larger than the Doppler broadening effect.

13.2.4.2 Dependence on optical thickness

We consider now the dependence of $d\theta/dB$ on the cloud optical thickness. For the case of linear Faraday rotation, the rotation angle for the Faraday effect at small values of magnetic field B is given by [see Equation (10.44)],

$$\theta = -\frac{b_{\bar{v}}(0)\Gamma}{4(k\bar{v})^2} \left[\frac{1 - g\left(\frac{k\bar{v}}{\Gamma}\right)}{g\left(\frac{k\bar{v}}{\Gamma}\right)} \right] \delta_B. \quad (13.32)$$

At a given temperature, we see that the value of $d\theta/dB$ is proportional to $b_{\bar{v}}(0)$. This is observed experimentally, as shown by the red open circles in Figure 13.5, which indeed display a linear behavior. Consistent with the result of the previous section, the theoretical prediction (red solid line) overestimates by a factor of 0.8, compared to the actual experimental curve. This factor is corrected for in the right panel.

In general, $d\theta/dB$ for the nonlinear Faraday rotation, does not depend linearly on $b_{\bar{v}}(0)$. This is because of the absorption of light during its propagation through the atomic medium [see for example Equation (13.18)]. Such a behavior is observed for the case of $I_0 = 4.8I_{\text{sat}}$, where the theoretical curve (green dashed line) shows a deviation from a linear behavior. For higher values of the probe intensity, the deviation from the linear dependence on $b_{\bar{v}}(0)$ starts to become significant only at a larger value of optical thickness.

The comparison between the theoretical and experimental results for the nonlinear Faraday rotation, also show good agreement. When the factor of 0.8 is multiplied with the theoretical calculation results, an even better agreement is achieved (see the right panel).

As seen from Figure 13.5, $d\theta/dB$ values for both the linear and nonlinear Faraday effects depend linearly on $b_{\bar{v}}(0)$ for optical thickness values up to $b_{\bar{v}}(0) = 1.5$.

Figure 13.5: The dependence of the slope $d\theta/dB$ on the optical thickness $b_{\bar{v}}(0)$ for the linear and nonlinear Faraday effects. The left panel shows the theoretical predictions at the various incident intensity. The right panel rescales the theoretical curves by 0.8, resulting in a better agreement with the experimental result. The experiments are performed with the cloud temperature fixed around $1 \mu\text{K}$. The loading rate of the magneto-optical trap (MOT) is modified to vary the number of atoms in the cloud. This changes the optical thickness of the cloud. The power of the red MOT beams are then varied accordingly to maintain the cloud temperature around $1 \mu\text{K}$.

Keeping the optical thickness below this value, the slope $d\theta/dB$ can be normalized by $b_{\bar{v}}(0)$, to avoid possible fluctuations in the values of $b_{\bar{v}}(0)$ over different runs of the experiment. This normalization is used in the next section, when we consider the temperature dependence of $d\theta/dB$.

13.2.4.3 Dependence on temperature

The dependence of $d\theta/dB$ on the cloud temperature is studied in this section. For the linear Faraday rotation, θ is measured with $I_0/I_{\text{sat}} = 0.04$, while for the case of nonlinear Faraday rotation, it is measured with $I_0/I_{\text{sat}} = 15$. The cloud has an optical thickness in the range of 0.5-1.5, which allows us to normalize the slope by $b_{\bar{v}}(0)$ (see Figure 13.5 and discussion in the previous section). The absolute values of $|(d\theta/dB)|b_{\bar{v}}(0)^{-1}$ are plotted in Figure 13.6, allowing a direct comparison between the sensitivities of linear and nonlinear Faraday rotations. The error bars represent the standard deviation of several measurements at the same experimental

Figure 13.6: The dependence of the normalized slope $|(\frac{d\theta}{dB})|b_{\bar{v}}(0)^{-1}$ on the cloud temperature, for the linear and nonlinear Faraday effects. In both cases, the theoretical curves over-predicts the absolute value of the slope, by a factor of 0.8. The linear and nonlinear effects are studied experimentally at $I_0/I_{\text{sat}} = 0.04$ and $I_0/I_{\text{sat}} = 15$ respectively. The atomic clouds used in the experiments have $b_{\bar{v}}(0)$ values in the range of 0.5-1.5. The high temperature behavior of the normalized slope for linear Faraday rotation is shown as the black dashed line. For the nonlinear Faraday rotation, the normalized slope value saturates to 0.051 in the limit of large cloud temperature, as indicated in the figure.

conditions.

The theoretical curves rescaled by 0.8 in Figure 13.6 agrees well with the experimentally measured points. At low temperature, the slope of the linear Faraday effect has a large absolute value, in agreement with the discussion in Section 13.2.4.1. At high cloud temperature, the sensitivity reduces as shown in Equation (10.48),

$$|\theta| = \frac{b_{\bar{v}}(0)}{\sqrt{2\pi}} \frac{1}{k\bar{v}} |\delta_B|, \quad (13.33)$$

which suggests that the slope is proportional to $1/\sqrt{T}$. This behavior is indicated in Figure 13.6 by black dashed curve.

For the nonlinear Faraday rotation, the normalized slope increases as the cloud temperature increases. We note that above 2.5 μK , the sensitivity of the nonlinear Faraday rotation is better than the sensitivity for the linear Faraday rotation. At a sufficiently high temperature, the Doppler broadening reduces significantly the sensitivity of the linear Faraday rotation. The nonlinear case is, however, only

affected by the power broadening. Thus, as long as the intensity is not too high, nonlinear Faraday rotation could have a better sensitivity at small magnetic field. In fact, the value of $|(d\theta/dB)|b_v(0)^{-1}$ is expected to saturate to a value of 0.051 at high temperature. This fact is promising to extend our study of NMOR to a hot vapor of atoms.

In this chapter, we have reported our experiments that study the magnetic field sensitivities of linear and nonlinear MOR effects. These studies achieved generally good agreement with the theoretical modeling. In the process, we have used the active control system for the magnetic field, thus, demonstrating its usefulness for cold atomic experiments. In the next chapter, we conclude our studies reported in Part II. We also give some prospective applications for the studies that are performed.

Chapter 14

Conclusion to Part II

In Part II, we have described the effects of magnetic field on the coherent transmission of light through a cloud of cold ^{88}Sr atoms, using the narrow $^1\text{S}_0 \rightarrow ^3\text{P}_1$ intercombination transition. It is among the narrowest single-photon transition used for the study of the magneto-optical rotation (MOR) effects in a cold atomic setup¹. Here, the atoms are sensitive to a magnetic field lower than a milligauss. Furthermore, the ^{88}Sr atoms offer a simple way to model and understand the MOR effects, since the transition is purely a $J = 0 \rightarrow J = 1$ transition.

We studied the Faraday and Voigt rotations of light polarization in the linear and nonlinear regime. To perform these studies and other magnetic field sensitive experiments, an active magnetic field control system is implemented in the experimental setup. It cancels the stray magnetic field. This system was calibrated on the atoms, and shown to achieve its target of canceling the magnetic field below the level of 1 mG. We further checked, by performing a long term experiment over the period of one month, that the calibration and performance of this system are robust.

Our studies then focused mainly on the sensitivity of the MOR effects to a static magnetic field. For the case of the linear MOR (LMOR), we observed that the sensitivity reduces as the cloud temperature increases. This is due to the Doppler broadening of the transitions. Nevertheless, the magnetic field sensitivity is still higher than the LMOR sensitivity for alkali-metal atoms such as the rubidium atoms. Here, we measured a maximum Verdet constant value of 3×10^8 rad/Tm. Even with Doppler broadening, it is still higher than the corresponding LMOR sensitivity value of 2×10^6 rad/Tm measured for the rubidium cold atoms in Ref. [40].

The LMOR sensitivity is then compared to the sensitivity in the nonlinear MOR (NMOR) case. The NMOR effects display a qualitatively different behavior. It has a narrow central feature around the zero magnetic field, which is shown to be due to the coherences between the excited substates of $^3\text{P}_1$. Interestingly, the NMOR sensitivity is only affected by the power broadening of the transitions, and not by the Doppler broadening. As a consequence, for a given cloud temperature, we observed experimentally that the NMOR effect can have a higher sensitivity than LMOR at the same cloud temperature. Moreover, unlike the LMOR sensitivity which decays following $1/\sqrt{T}$ at high temperature, this NMOR sensitivity saturates to a constant value.

In terms of practical application, even though the sensitivity of the intercombination transition is still not competitive with many commercially available magnetometers, the strontium atoms can still be useful as magnetometers. We note that

¹With hot rubidium atoms, the 421.6 nm blue transition with a linewidth of 240 kHz has also been used to study the nonlinear MOR effects [150]. This transition is more complicated as it involves multiple decay paths from the excited state.

the results of our studies can be extended to a hot vapor gas. Since the sensitivity saturates to a constant value at large temperature, a hot strontium gas, with a lower saturated vapor pressure as compared to the rubidium gas, can be then useful for magnetic field measurements at a temperature well above room temperature.

Chapter 15

Outlook

In this thesis, we have reported theoretical and experimental studies on the coherent transmission of light through a cloud of cold ^{88}Sr atoms.

In Part I, we have exploited the long lifetime of the excited state to perform experimental studies on the coherent transmission of light in the transient regime. Our studies reveal the superflash effect, along with the underlying cooperative emission of light. The superflash is an emission in the forward direction, with a peak intensity that is larger than the incident intensity. Theoretically, we have predicted a peak intensity that is 9 times the incident intensity, if the steady state regime is achieved before a phase jump is applied to the probe beam. In a more general setting, we have also shown numerically that a peak of 12 times the incident intensity can be reached. One may then wonder if a higher maximum peak intensity is physically achievable for the cooperative emission. For the case of superradiance, a dense cloud with N_a atoms leads to an emission with a peak that is amplified by N_a^2 . In the extended optically thick media studied here, the attenuation of the coherent incident light in the medium means that only the atomic dipoles, located within a small layer of thickness comparable to the light mean free path, are excited by the laser light. The atoms which are located deeper in the medium are excited by the incoherent diffuse scattered light. Any attempt to achieve a higher peak intensity has to overcome this absorption in the medium. In Ref. [79], large flashes are also generated by timing appropriately the switching of the probe field. This suggests that one can explore with a more general modulation of the incident field, to generate a flash with higher peak intensity. A future study of this particular aspect of cooperative emission, is interesting for the generation of high intensity pulse trains using the mechanism described in the thesis.

Besides, the cooperative emission of the flashes can also be used to probe a medium which is opaque in the steady state regime. For instance, since the decay time of the flash is shortened by the optical thickness at resonance and zero temperature $b_0(0)$, the time scale of the flash carries information on the number of atoms in the medium. Such an information is hard to obtain in the steady state due to the low transmission signal.

For the cooperative emission, we have been working with atomic clouds of low density. However, it is well-known that the coherent transmission of light is affected by the density of atomic cloud [34]. Since the atomic clouds consists of atoms which are randomly distributed and moving, the theory of wave propagation in disorder medium is applied in our studies. Using a diagrammatic approach, we have computed the lowest order correction terms, which come from recurrent scatterings between pairs of atoms, and the pair correlation function of the atoms. Due to the modifications to the light scattering behavior by these phenomena, a superflash is now expected to be emitted by a dense medium, even if the probe beam is at

resonance. Thus, the cooperative emission could be an additional tool to study the light atom interaction in a dense medium.

Furthermore, since quantum gases have peculiar pair correlation function in the position of the atoms, the coherent transmission of light is also modified differently from a classical gas. Thus, the coherent transmission of light may also be used to probe the quantum gases. This was recently shown for a BEC [93]. It is then possible to consider, in the future, the transmission of light through a quantum gas as a method to study its pair correlation function.

Another interesting aspect of coherent transmission is the effect of cloud temperature. When the atomic cloud is dilute, the temperature effect on the coherent transmission of light can be easily understood and modeled. For a dense atomic cloud, however, the inclusion of the Doppler effect to the coherent transmission of light, due to the motion of the atoms is not straightforward. Several recent works have been performed by various groups to experimentally and theoretically understand how the temperature of the dense atomic medium affects the scattering of light [26, 99, 106]. In these studies, the observation of a frequency shift, or a lack of it, in an atomic cloud is studied by modeling the atoms as coupled dipoles. Our work could be extended to complement these studies, at least in terms of the lowest order correction to the ISA term. Following the method in Ref. [109], the theoretical framework used to study the recurrent scattering, can be extended to include the effects of atomic motion. One can then study how the contribution of recurrent scattering is affected by the dephasing due to Doppler effect.

Besides the study of the cooperative emission in the transient regime, we also investigated the effects of magnetic field on the coherent transmission. The results are reported in Part II. In presence of a weak magnetic field, the internal structure of the atoms plays a role. We have observed both the linear and nonlinear magneto-optical rotation of light. Importantly, we have also implemented an active magnetic field control system to control the stray magnetic field down to a level lower than 1 mG. Thus, the subtle effects of magnetic field on the coherent transmission light can be further explored using our experimental setup. For instance, the study in Ref. [123] has revealed a difference in the scattering of light, when the degeneracy of the sublevels is lifted by a static magnetic field. In that work, it has been proposed that the magnetic field can be used to control the trapping of light inside the medium. Recently, the Faraday rotation of light polarization has also been proposed as a method to isolate the optical precursors [178]. We see that the presence of magnetic field can potentially lead to nontrivial results. Thus, it is worthwhile to pursue further studies on the effect of magnetic field on a dense atomic cloud.

Appendices

Appendix A

Maxwell Bloch Equations

In this appendix, we review the Maxwell-Bloch equations (MBE) that are useful to study the general case of a propagating signal through an atomic medium [117,173]. The MBE consists of two parts that are coupled together. There is the set of optical Bloch equations (OBE) which describes the interaction of light with a single atom, and a wave equation which describes the propagation of the light inside the atomic medium.

The MBE allows us to study the general case of coherent transmission of light, even beyond the linear response regime. Here, we focus on the case where the atom is modeled by a two-level system. We further assume in this appendix that the atomic medium is dilute and the independent scattering approximation (ISA) holds. We apply the electric dipole approximation in our study. The rotating wave approximation (RWA) is also applied in the following.

A.1 Optical Bloch equation

A.1.1 The governing equations for stationary atoms

The OBE is well-known and widely studied in the literature [117, 179]. Let us denote $|a\rangle$ as the ground state and $|b\rangle$ as the excited state of a two-level atom. The resonant frequency of the transition is denoted by ω_0 . The excited state has a linewidth of Γ , which is also the decay rate of the excited state population. It is related to the dipole matrix element $\mathbf{d}_{ab} = \langle a|\hat{\mathbf{d}}|b\rangle$ of the transition, by

$$\Gamma = \frac{|\mathbf{d}_{ab}|^2 \omega_0^3}{3\pi\epsilon_0 \hbar c_0^3}, \quad (\text{A.1})$$

where ε_0 is the permittivity of vacuum, c_0 is the speed of light in vacuum, and $\hbar$ is the reduced Planck's constant. The dipole moment operator is denoted by $\hat{\mathbf{d}}$. The internal state of the atom is described by its density matrix $\hat{\rho}$.

The light that interacts with the atom is assumed to be a plane wave of frequency ω_l . Its amplitude is denoted by $\mathbf{E}_0$. The interaction strength between light and atom is quantified by the Rabi frequency,

$$\Omega_0 = -\frac{\mathbf{d}_{ab} \cdot \mathbf{E}_0}{\hbar}. \quad (\text{A.2})$$

Since the atom is considered as a point particle with dimensions much shorter than the wavelength of the light (Rayleigh scatterer), we can apply the electric dipole approximation.

We assume that the light has a detuning $\delta = \omega_l - \omega_0$, and satisfies $|\delta| \ll \omega_0$. In the interaction picture, the OBE in the RWA is given by [117],

$$\dot{\sigma}_{aa} = -\frac{i\Omega_0}{2}(\sigma_{ab}^* - \sigma_{ab}) + \Gamma\sigma_{bb}, \quad (\text{A.3})$$

$$\dot{\sigma}_{bb} = \frac{i\Omega_0}{2}(\sigma_{ab}^* - \sigma_{ab}) - \Gamma\sigma_{bb}, \quad (\text{A.4})$$

$$\dot{\sigma}_{ab} = -\left[\frac{\Gamma}{2} + i\delta\right]\sigma_{ab} - \frac{i\Omega_0}{2}(\sigma_{bb} - \sigma_{aa}), \quad (\text{A.5})$$

$$\sigma_{ba} = \sigma_{ab}^*. \quad (\text{A.6})$$

The density matrix in this interaction picture, $\hat{\sigma}$, is related to $\hat{\rho}$ by the following relations between their elements,

$$\sigma_{aa} = \rho_{aa}, \quad (\text{A.7})$$

$$\sigma_{bb} = \rho_{bb}, \quad (\text{A.8})$$

$$\sigma_{ab} = \rho_{ab}e^{-i\omega_l t}, \quad (\text{A.9})$$

$$\sigma_{ba} = \rho_{ba}e^{i\omega_l t}. \quad (\text{A.10})$$

A further change of variables, using

$$\sigma_{ab} = s_1 + is_2, \quad (\text{A.11})$$

$$\sigma_{aa} - \sigma_{bb} = s_3, \quad (\text{A.12})$$

which leads us to the following OBE for the Bloch vector $\mathbf{s} = (s_1, s_2, s_3)$,

$$\dot{s}_1 = -\frac{\Gamma}{2}s_1 + \delta s_2, \quad (\text{A.13})$$

$$\dot{s}_2 = -\delta s_1 - \frac{\Gamma}{2}s_2 + \frac{\Omega_0}{2}s_3, \quad (\text{A.14})$$

$$\dot{s}_3 = -\Gamma(s_3 - 1) - 2\Omega_0 s_2. \quad (\text{A.15})$$

The components s_1 and s_2 are related to the coherence terms, while the s_3 is the population difference between the ground and excited states. The Bloch vector $\mathbf{s}$ allows an interpretation of the atomic state geometrically in the Bloch sphere (also known as the Poincaré sphere).

A.1.2 Solution of the OBE in the steady state regime

The solution of the OBE in the steady state regime is obtained by setting $\dot{\mathbf{s}} = 0$ in Equations (A.13) to (A.15). The resonant saturation parameter of the incident probe is defined as $s_0 = 2|\Omega_0|^2/\Gamma^2$. The intensity of this probe beam can be high, but still having $\Omega_0 \ll \omega_0$. The intensity of the probe beam, I_0 , is related to the saturation intensity, I_{sat} , through $I_0 = s_0 I_{\text{sat}}$. The saturation intensity is given by $I_{\text{sat}} = \pi \hbar c_0 \Gamma / 3 \lambda_0^3$, where $\lambda_0 = 2\pi c_0 / \omega_0$ is the resonant wavelength of the transition. For the narrow intercombination transition, I_{sat} takes a small value of $3 \mu\text{W}/\text{cm}^2$. The effective saturation parameter for a probe detuned by δ , is

$$s(\delta) = \frac{s_0}{1 + (2\delta/\Gamma)^2}. \quad (\text{A.16})$$

In terms of $s(\delta)$, the steady state solutions are found to be

$$s_1 = \frac{\Omega_0}{2} \frac{\delta}{\delta^2 + \Gamma^2/4} \frac{1}{s(\delta) + 1}, \quad (\text{A.17})$$

$$s_2 = \frac{\Omega_0}{2} \frac{\Gamma/2}{\delta^2 + \Gamma^2/4} \frac{1}{s(\delta) + 1}, \quad (\text{A.18})$$

$$s_3 = \frac{1}{s(\delta) + 1}. \quad (\text{A.19})$$

In terms of the population and coherence of the density matrix, we have

$$\sigma_{bb} = \frac{1 - s_3}{2} = \frac{s(\delta)}{2[1 + s(\delta)]}, \quad (\text{A.20})$$

$$\sigma_{ab} = s_1 + i s_2 = \frac{i\Omega_0}{2} \frac{1}{\Gamma/2 + i\delta} \frac{1}{1 + s(\delta)}. \quad (\text{A.21})$$

At zero temperature, the power broadening from the intensity of the probe beam leads to the following effective linewidth, Γ' ,

$$\Gamma' = \Gamma\sqrt{1 + s_0}. \quad (\text{A.22})$$

We encounter this in the study of the width of the nonlinear magnetic-optical rotation, in Chapter 13.

For a weak probe illumination, we can set $s_0 \approx 0$, leading to $\sigma_{bb} = 0$, that is, the excited state population is almost zero. The coherence element in this linear optical regime is

$$\sigma_{ab} = \frac{i\Omega_0}{2} \frac{1}{\Gamma/2 + i\delta}. \quad (\text{A.23})$$

A.1.3 Two-level atom polarizability

The expectation value of the dipole operator is defined as $\text{Tr}\{\hat{\rho}\hat{\mathbf{d}}\} = \rho_{ab}\mathbf{d}_{ba} + \rho_{ba}\mathbf{d}_{ab}$. This may also be rewritten as $\langle\mathbf{d}\rangle = 2\text{Re}\{\sigma_{ba}e^{-i\omega_L t}\mathbf{d}_{ab}\}$.

The polarizability $\alpha(\delta)$ of an atom, is defined by

$$\langle\hat{\mathbf{d}}(z, t)\rangle = \varepsilon_0\alpha(\delta)\mathbf{E}(z, t), \quad (\text{A.24})$$

where $\mathbf{E}(z, t)$ is the optical field that induces the dipole moment. Using the expression for the coherence element σ_{ab} , we find a polarizability of the atoms, in the RWA, to be

$$\alpha(\delta) = -\frac{6\pi c_0^3}{\omega_0^3} \frac{\Gamma/2}{\delta + i\Gamma/2} \frac{1}{1 + s(\delta)}. \quad (\text{A.25})$$

In general the polarizability is dependent on the intensity of the field, through the saturation parameter $s(\delta)$.

For a weak probe beam (linear response regime), the polarizability of the two level atom take the following Lorentzian form,

$$\alpha(\delta) = -\frac{\alpha_0\Gamma/2}{\delta + i\Gamma/2}. \quad (\text{A.26})$$

In the equation above, α_0 is the magnitude of the resonant atomic polarizability, which has a value of $6\pi c_0^3/\omega_0^3$. Note that the atomic polarizability, in the linear response regime, is not dependent on the intensity of the light.

A.1.4 Moving atoms

In the cold ^{88}Sr cloud, the motion of the atoms still causes to a significant Doppler shift to the transition frequency of the atoms. For an atom moving with a velocity component of v along the probe propagation direction, the detuning δ is modified by $\delta - k_l v$ due to the Doppler effect. Here, $k_l = \omega_l/c_0$ is the wave vector of the probe beam. In the linear regime, the averaged polarizability of an atom in the medium, is

$$\alpha_{\bar{v}}(\delta) = -\alpha_0 \frac{\Gamma}{2} \int_{-\infty}^{\infty} \frac{f(v)}{\delta - kv + i\Gamma/2} dv_z. \quad (\text{A.27})$$

The integral above can be evaluated using the Faddeeva function (see Appendix B).

In the next section, we describe the propagation of light through the atomic medium, by coupling the solution of the OBE with a wave equation. This results in the MBE.

A.2 Light propagation through an atomic medium

A.2.1 1-D wave equation

For a theoretical consideration of the transmission of light, we suppose that the atomic medium has a slab geometry with thickness L , and a position dependent density of $\rho(z)$. The light is assumed to propagate along the z direction in the slab.

The propagation of light through the medium is described by the 1-D wave equation shown below,

$$\frac{\partial^2 \mathbf{E}(z, t)}{\partial z^2} - \frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}(z, t)}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}(z, t)}{\partial t^2}. \quad (\text{A.28})$$

The equation above, together with Equations (A.13) to (A.15) are known as the Maxwell-Bloch equation (MBE)¹ [117].

The field of the incident light is given by,

$$\mathbf{E}_0(z, t) = \text{Re} \{ E_0(z, t) e^{-i\omega_l t} \boldsymbol{\epsilon} \}. \quad (\text{A.29})$$

Here, we assume that the typical time scale for the field amplitude $E_0(z, t)$ is much

¹A recent derivation, by A. A. Svidzinsky and coworkers, of the MBE from a full quantum treatment leads to a slightly different form and interpretation of the MBE [180]. The difference between the two forms become significant when the amplitude of the light varies rapidly. In our case, the amplitude of the light is assumed to be varying slowly and thus the semiclassical expression is considered to be valid.

longer compared to the time scale of the optical oscillation at ω_l . The incident light is polarized along the direction of the unit vector ϵ . Since we assume that there is no external magnetic field, the transmitted field has the same polarization with the incident field.

The dipole moment density $\mathbf{P}(z, t)$ of the medium is given by,

$$\mathbf{P}(z, t) = \text{Re} \left\{ 2\rho(z) \left[\int_{-\infty}^{\infty} \sigma_{ba}(z, v_z, t) f(v_z) dv_z \right] e^{-i\omega_l t} \epsilon \right\}. \quad (\text{A.30})$$

where the averaging over the thermal velocity distribution of the atoms is performed for the component along the propagation direction. By this, we have assumed that all the atoms scatter light independently putting us in the ISA regime, which holds true for a dilute atomic cloud. The dipole moment in the equation above, is time dependent, since the amplitude of the field that induces it is also time dependent.

In general, this set of coupled equations do not have an analytical solution. Nevertheless, it can be solved numerically, to give the transmission signal through the atomic medium. In the special case of a weak probe beam, an analytical solution can be found for the steady state coherent transmission. This is discussed in the following section.

A.2.2 Steady state solution with weak probe beam

For simplicity, we assume that the density is uniform for the atomic medium. To be in the linear response regime, the incident light is assumed to have an intensity lower than the saturation intensity. The following ansatz is used for the steady state solution of the optical field,

$$\mathbf{E}(z, t) = \text{Re} \{ E(z) \exp(-i\omega_l t) \epsilon \}. \quad (\text{A.31})$$

In the absence of the magnetic field, the induced atomic dipole is parallel to ϵ . Thus, the vector notation is not necessary and the wave equation reduces to the following Helmholtz equation,

$$\frac{d^2 E_t(z)}{dz^2} + k_l^2 E_t(z) = \rho_0 k_l^2 E_t(z) \alpha_0 \frac{\Gamma}{2} \int_{-\infty}^{\infty} \frac{f(v_z)}{\delta - kv_z + i\Gamma/2} dv_z. \quad (\text{A.32})$$

Upon rearranging the terms, we find

$$\boxed{\frac{d^2 E_t(z)}{dz^2} + k_l^2 [1 + \rho \alpha_v(\delta)] E_t(z) = 0,} \quad (\text{A.33})$$

which is a 1-D Helmholtz equation. The light is seen to be propagating in an effective homogeneous medium, with a wave vector k . It is related to k_l through,

$$k = k_l \sqrt{1 + \rho \alpha_{\bar{v}}(\delta)}. \quad (\text{A.34})$$

The effective relative permittivity, defined as $\varepsilon_r = k/k_l$, is then found to be

$$\varepsilon_r(\delta) = 1 + \rho \alpha_{\bar{v}}(\delta). \quad (\text{A.35})$$

This is consistent with the results obtained in Section 3.4.3 in the ISA regime of steady state coherent transmission.

Finally, we see that the corresponding Beer-Lambert law for the optical field, as shown below, is a solution of Equation (A.33).

$$E(L) = E_0(0) \exp(ik_l \sqrt{\varepsilon_r(\delta)} L). \quad (\text{A.36})$$

Appendix B

The Faddeeva Function

This appendix lists some expressions involving the Faddeeva function that are useful to consider the effect of Doppler broadening for a dilute medium. These results and many others related identities can be found in Ref. [122]. The Faddeeva function is defined by the following integral

$$w(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-s^2)}{z - s} ds. \quad (\text{B.1})$$

It is related to the complex error function

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz). \quad (\text{B.2})$$

It obeys the following property

$$w(z^*) = [w(-z)]^*. \quad (\text{B.3})$$

The first derivative of $w(z)$ is

$$\frac{dw(z)}{dz} = -2zw(z) + \frac{2i}{\sqrt{\pi}}. \quad (\text{B.4})$$

Higher order derivatives can be obtained using the following recurrence relation,

$$w^{(n+2)}(z) + 2zw^{(n+1)}(z) + 2(n+1)w^{(n)}(z) = 0, \quad \text{for } n = 0, 1, 2, \dots, \quad (\text{B.5})$$

where we denote the n -th order derivative of $w(z)$ by $w^{(n)}(z)$. Using this, the expansion of $w(z)$ around z_0 up to second order is found to be

$$w(z) \approx w(z_0) + \left[\frac{2i}{\sqrt{\pi}} - 2z_0 w(z_0) \right] (z - z_0) + \left[(2z_0^2 - 1)w(z_0) - \frac{2iz_0}{\sqrt{\pi}} \right] (z - z_0)^2. \quad (\text{B.6})$$

A useful and related function is $g(x)$. It is defined as

$$g(x) = \sqrt{\frac{\pi}{8}} \exp\left[\frac{1}{8x^2}\right] \operatorname{erfc}\left[\frac{1}{\sqrt{8}x}\right] \frac{1}{x}. \quad (\text{B.7})$$

This function appears when the incident beam is at resonance with the atomic medium [see Equation (3.67) and the discussion around it]. At large x , the limiting behavior is

$$g(x) \simeq \sqrt{\frac{\pi}{8}} \frac{1}{x}. \quad (\text{B.8})$$

At $x \approx 0$, the following expansion holds,

$$g(x) \simeq 1 - 4x^2 + 48x^4 - \dots \quad (\text{B.9})$$

This leads to the following limits at $x \rightarrow 0$, i.e.,

$$\lim_{x \rightarrow 0} \frac{1 - g(x)}{4x^2} = 1, \quad (\text{B.10})$$

$$\lim_{x \rightarrow 0} \frac{1}{4x^2} \left[\frac{1 - g(x)}{4x^2} - g(x) \right] = -2. \quad (\text{B.11})$$

Appendix C

Experimental Setup for Laser Cooling and Trapping of ^{88}Sr

In this appendix, the experimental setup of the cold atomic cloud is briefly described. The full description of the experimental setup is found in Ref. [46]. A lot of the techniques applied in the preparation of the cold strontium cloud have been developed by many different groups. More details on these techniques can be found in Refs. [45, 181–185].

The experimental setup is shown in Figure C.1. This setup is used to prepare both the cold atomic clouds of ^{88}Sr and ^{87}Sr atoms. Here, we focus only on the preparation of ^{88}Sr , since the studies carried out and reported in this thesis involves only this isotope.

The strontium atom source is loaded into an oven. When heated, strontium atoms leaves this oven in the form of an atomic beam. The atomic beam is first

Figure C.1: The experimental setup used to prepare the cold strontium atomic cloud.

Figure C.2: The transitions of the strontium atom used for the various cooling and trapping stages of the cold atomic cloud.

cooled along the transverse direction by a 2D optical molasses (OM), before the atomic beam velocity is slowed down by a Zeeman slower. The longitudinal velocity of the atomic beam is reduced to 60 ms^{-1} . At this velocity, the atoms can be captured by a 2D magneto-optical trap (MOT). The 2D MOT is aligned at an angle of 30° from the axis of the Zeeman slower. It deflects the atomic beam by 30° into the science cell. The 2D OM, Zeeman slower and the 2D MOT stages are performed on the blue $^1\text{S}_0 \rightarrow ^1\text{P}_1$ (linewidth $\Gamma_b = 32 \text{ MHz}$) transition. The wavelength of this transition is $\lambda_b = 461 \text{ nm}$.

Finally, in the science cell, the ^{88}Sr atoms are trapped and cooled in the 3D MOT. This 3D MOT is divided into two stages. The first 3D MOT stage also uses the blue 461 nm transition. It is later referred to as the blue MOT. The atoms are then transferred into another 3D MOT working on the $\lambda_r = 689 \text{ nm}$ red transition of $^1\text{S}_0 \rightarrow ^3\text{P}_1$ (linewidth $\Gamma_r = 7.5 \text{ kHz}$). We call it the red MOT. The two cooling and trapping transitions are shown in Figure C.2, along with the transitions targeted by the repumper beams to improve the performance of the blue MOT. These various cooling and trapping stages are described further in the following sections.

Up until the blue MOT stage, the performances of the preparation stages are measured by the loading rate, L_0 , of the atoms into the blue MOT. This loading rate is calculated from a fluorescence measurement of the blue MOT, 5 ms after the blue MOT beams are turned on. The loading of atoms is considered to be linear

within this short duration of time, allowing the loading rate to be computed directly from the number of atoms in the MOT.

Besides describing the cooling and trapping of the cold atoms, the blue and red laser systems are also described briefly in this appendix. Finally, we give also a brief remark on the two repumper beams.

C.1 Atomic beam emitted from the oven

The strontium source for the experiment is loaded in a stainless steel oven. The oven is heated up to 550 °C. The oven temperature is controlled using a feedback loop, so that a constant flux of strontium atomic beam is emitted from the oven through the nozzle. The oven nozzle contains 900 Monel400 micro tubes that are 1 cm in length. They are arranged in a 1 cm $\times$ 1 cm square area. Each tube has a inner and outer diameters of 200 μ m and 300 μ m, respectively. The nozzle is heated at a temperature that is roughly 50 °C higher than the oven to avoid the clogging of the tubes by the atoms.

The 2D OM is located at 15 cm after the nozzle. Two pairs of retroreflected 461 nm laser beams (along the horizontal and the vertical directions), with a total power of 120 mW, are used in the 2D OM. We operate the 2D OM at a detuning of -18 MHz. Due to the reduced transverse velocities of the atoms, the loading rate L_0 of the atoms into the blue MOT is improved by a factor of three.

After passing through the 2D OM, the atomic beam travels through a 10 cm long tube, with an inner diameter of 1 cm. This tube filters out the atoms with a large transverse velocity components before the atomic beam enters into the Zeeman slower (discussed in the next section). The small diameter of this tube also creates a differential pressure between the oven and the rest of the experimental apparatus. This provides a better vacuum environment for the subsequent stages of cooling.

C.2 The Zeeman slower

The Zeeman slower makes use of a non-homogeneous magnetic field, along with a counter propagating laser beam, to reduce the longitudinal velocity component of the atomic beam. The Zeeman slower in our setup has a length of 32 cm. Flat cooper wires are wound in a tapered manner along its axis, to generate the inhomogeneous magnetic field required for an optimal deceleration of the atoms. The magnetic field varies from 500 G to -100 G, between the entrance and exit of the Zeeman slower, respectively. A 461 nm laser beam (optical power up to 110 mW) at a detuning of $-13\Gamma_b$ is sent into the Zeeman slower through a sapphire viewport. To avoid the

deposition of Sr atoms on this viewport, it is heated up to 375 °C.

With these settings, the Zeeman decelerate the strontium atoms that have longitudinal velocity up to 520 m/s. After the Zeeman slower, the peak velocity occurs at a value of typically 100 m/s. This fraction of atoms with lower values of longitudinal velocity is increased (see Fig. 5a in Ref [46]). A larger number of atoms can then be captured by the MOT. The usage of this Zeeman slower increases L_0 by typically a factor of 300.

In the setup, the Zeeman slower is placed close to the 2D MOT. Thus, the large magnetic field from the Zeeman slower can affect the operation of the 2D MOT. To avoid this, three layers of μ -metal sheets are used to enclose the Zeeman slower (see Figure C.1).

C.3 2D MOT

The 2D MOT is performed using two pairs of coils, which produce an equal magnetic field gradient along the two cooling and trapping directions (see Figure C.3). Each coil has a square geometry, formed by 22 turns $\times$ 20 layers of 1 mm diameter copper wires. Each coil pair is separated by a distance of typically 20 cm. These pair of coils generated a magnetic field gradient at the maximum level of 7.5 G/cm. The current of the coils are set to a typical value of 1.5 A.

The total optical power in the 2D MOT beams is 220 mW. The detuning of the laser used in the experiment is set at $-0.9\Gamma_b$. 60 % of the power is sent on the horizontal pair, to provide the power for the deflection of the atomic beam, in addition to the usual transversal cooling and focusing of the atomic beam. The atomic beam is deflected by 30°. This deflection has several advantages as summarized below [46].

- The Zeeman optical beam, which enters through the viewport, do not cross the region in the science cell where the experiment is carried out.
- The uncooled atoms in the atomic beam, including the unwanted isotopes of strontium are unaffected by the 2D MOT. These atoms do not enter the science cell, leading to a better vacuum environment in the science cell.
- This frees up an additional optical axis opposite to the direction of the atomic beam inside the science chamber.
- The 2D MOT also acts as a switch to turn the atomic beam on and off.

After leaving from the 2D MOT, the atom enters the science cell through another differential pumping tube, to have a better lifetime of the atoms inside the science cell.

Figure C.3: A schematic plot of the 2D MOT. Two pairs of coils generate the magnetic field gradient at an optimal value of 7.5 G/cm along the two cooling directions. The blue 461 nm lasers form two pairs of counter propagating beams. The 2D MOT is aligned at an angle of 30° with the initial atomic beam, to deflect the atomic beam. A differential pumping tube is placed after the 2D MOT, which leads into the science cell.

C.4 3D MOT

The atoms entering the science cell have a typical longitudinal velocity of 50 m/s. This is within the capture velocity range of the blue MOT, but not for the red MOT which has a lower capture velocity of 1 cm/s¹. Thus, the blue MOT is performed first in the science cell for initial cooling and trapping of the atoms. They are then transferred into the red MOT where the cloud temperature will be lower.

The same pair of coils is used for both stages of the MOT. The MOT coils are aligned along the vertical direction to counter the effect of gravity. There are three pairs of MOT beams for both the 461 nm and 689 nm lasers. Each pair of the MOT beam is retroreflected. The MOT beams for the two wavelengths are overlapped, to ensure that the atoms in the blue MOT can be transferred effectively to the red MOT. The time sequence of the ⁸⁸Sr 3D MOT is shown in Figure C.4.

¹With a broadband modulation of the red laser frequency, it is possible to extend the capture velocity up to 1m/s.

Figure C.4: The time sequence for the preparation of the atoms in the MOT. This time sequence starts from the blue MOT stage. After typically several hundred milliseconds of loading time in the blue MOT, the atoms are transferred to a broadband red MOT. This is followed by two stages of single band red MOT with different optical power. The frequency bandwidth, optical power and the magnetic field gradient are indicated in the figure.

C.4.1 3D blue MOT

The 3D blue MOT is performed on the $^1S_0 \rightarrow ^3P_1$ transition at 461 nm. The maximum total intensity of the MOT beams is 76 mW cm^{-2} . The MOT beams are set at a detuning of -58 MHz. A magnetic field gradient of 45 G/cm is applied to trap the atoms. At this stage, a loading rate of $L_0 = 6 \times 10^9 \text{ s}^{-1}$ into the blue MOT is measured.

Importantly, the cooling transition for the blue MOT is not closed. As shown in Figure C.2, the atoms in the 1P_1 may decay to the 1D_2 state, and subsequently to the long-lived 3P_2 triplet state. This is undesirable, because these atoms are lost from the blue MOT. To overcome this loss, a repumper laser at 707 nm is applied to repump the atoms from this long-lived state to the higher 3S_1 state. The atoms subsequently decay into all of the 3P substates. The atoms that decay to the 3P_1 substate can decay back to the ground state with a lifetime of $21 \mu\text{s}$. Besides going back to the 3P_2 substate, the atoms can also go into the 3P_0 substate. For these atoms, another repumper laser at 679 nm is used to repump them. With these two lasers, the transition is closed, thereby improving the number of atoms that are cooled and trapped inside the MOT.

As shown in Figure C.4, the blue MOT stage is further divided into two phases. The first phase of the blue MOT is performed using the full optical power of the MOT beam. This is followed by the second phase of the blue MOT, which is

performed at a lower optical power (see Figure C.4). This reduces the temperature of the atomic cloud, to optimize the transfer of the atoms into the red MOT stage. The optical power is gradually reduced to 5 mW over a duration of 50 ms. The low power MOT stage with the final optical power is performed for a further 50 ms.

In the blue MOT, we have a maximum of $N_a = 10^9$ ^{88}Sr atoms. The cloud typically have a size of the order of 1 cm. This leads to a maximum peak density of typically $\rho_0 = 3 \times 10^9 \text{ cm}^{-3}$. At this stage, the cloud has a temperature of typically 2 mK.

C.4.2 3D red MOT

To further reduce the cloud temperature, the atoms are transferred into the red MOT. At 2 mK, the root-mean-square (rms) Doppler shift is roughly 600 kHz. For the intercombination transition, this is 80 times the linewidth. Thus, to optimize the transfer between the low power blue MOT and the red MOT, the frequency of the MOT beam is first broadened to a bandwidth of 2 MHz, to capture atoms from a wider velocity class. The broadband red MOT stage starts when the blue MOT is turned off². During this broadband stage, the magnetic field ramped up to 0.75 G/cm. Towards the end of the broadband red MOT stage, the total power of the red MOT beam is halved to further reduce the cloud temperature.

After this broadband stage, the modulation on the red MOT beam is removed. It is then set to a detuning of -200 kHz. For the intercombination line, this detuning corresponds to $-26\Gamma_r$. For the single band operation, the total power of the red MOT beams is reduced to 3 mW, and eventually to 2 mW. The magnetic field gradient is ramped up from 0.75 G/cm to 1 G/cm, over a duration of 100 ms. This ramping process is responsible to compress the atomic cloud in the trap.

At the end of the red MOT stage, the atomic cloud has a typical volume of 0.15 mm^3 , with a maximum number of 3.5×10^8 atoms. A peak number density of $\rho_0 = 2.5 \times 10^{12} \text{ cm}^{-3}$ is achievable in the red MOT.

The temperature of the cold cloud in the red MOT is typically around 1-3 μK . The cloud temperature can be varied by changing the total number of atoms, or the density of the cloud. An example of how the cloud temperature changes, as a function of ρ_0 , is shown in Figure C.5.

²The red MOT beams are turned at a maximum optical power value of 30 mW, even during the blue MOT phase, as it does not affect the blue MOT.

Figure C.5: A plot of the cloud temperature versus the peak cloud density ρ_0 . With a higher density, the cloud temperature is typically found to be higher.

C.5 Laser systems

Several lasers are required for the cooling and trapping of the ^{88}Sr . They are the blue laser at 461 nm and the red laser at 689 nm, which performs the MOT. In addition, there are the two repumper lasers that improve the numbers of atoms in the blue MOT. A brief description of these lasers is given in this section.

C.5.1 Blue laser system

There are two sources for the blue laser at 461 nm. There is a commercial source (Toptica TA-SHG Pro) with 400 mW optical power output at 461 nm. This laser is composed of a 922 nm diode laser as a master laser. The output is amplified by an in-built 1.5 W infra-red (IR) tapered amplifier (TA). The 461 nm laser is then generated by a SHG resonant cavity. It gives a total power of 400 mW. The laser here is used for the 2D OM, 2D MOT and the 3D blue MOT. A small fraction of the laser power at 461 nm is also sent to a reference cell. The frequency of the master laser is locked by performing an transmission spectroscopy on an atomic beam inside this reference cell. An AOM in the double pass configuration is used to vary the frequency of the laser.

A 3 mW beam is extracted from the 922 nm master laser, and sent to a home-made SHG cavity. It is used to inject a 30 mW 922 nm slave laser. A TA amplifies this 922 nm beam, before it is coupled into a SHG cavity composed of a bow type

Figure C.6: A schematic drawing of the red laser system for the cooling and trapping of ^{88}Sr .

ring cavity and a periodically poled potassium titanyl phosphate (PPKTP) crystal. The 461 nm output from the SHG cavity has a detuning of -408 MHz from the $^1\text{S}_0 \rightarrow ^3\text{P}_1$ transition. We can obtain a maximum optical power of up to 200 mW from this SHG cavity. This laser output is sent through a fiber to be used in the Zeeman slower.

C.5.2 Red laser system

The red laser system is shown in Figure C.6. Besides trapping and cooling the atoms, it is also used as a probe beam for the intercombination transition. The discussion here focuses only on the part of the red laser system which is used in the preparation of the ^{88}Sr cold atomic cloud.

The master laser is an external cavity diode laser (ECDL). Its frequency is locked using a ultra low expansion (ULE) high finesse Fabry-Pérot (FP). The FP cavity experiences a long term drift with a rate of less than 10 Hz/s. This affects the accuracy of the laser beam frequency, and is corrected by performing a saturated fluorescence spectroscopy on a strontium atomic beam in the reference cell. The alignment of the retroreflected probe beam is achieved using a cat's eye configuration. The probe beam frequency for the spectroscopy is varied using an AOM (AOM1 in Figure C.6) in double pass configuration. By applying a large bias magnetic field, the $^1S_0, m = 0 \rightarrow ^3P_1, m = 0$ transition is isolated due to large Zeeman shifts of the other sublevels³. As discussed in Section 4.1.3, this locking system achieves an accuracy of ± 5 kHz the frequency of the red laser.

The master laser seeds a slave laser (slave1 in Figure C.6). The laser from slave 1 is then used to seed other diode lasers, to generate the relevant cooling and trapping laser beams for the preparation of the ^{88}Sr and ^{87}Sr cold atomic clouds. The frequency of the seeding beam is tunable using another AOM in double pass configuration (AOM2). The output of slave1 is typically 80 MHz away from the resonance frequency.

A part of the slave1 output injects the slave2 laser. To obtain the laser beams for the red MOT of ^{88}Sr , we take the -1 order of the output from AOM3. This AOM is driven at 80 MHz to bring the detuning close to the resonance of the intercombination transition. The red MOT beam can be switched off by turning off the radio frequency (rf) input to the AOM.

As discussed in Section 4.1, the 689 nm probe beam is also derived from slave2. When the red MOT beam is turned off at AOM3, the zero order output passes through another AOM driven at 80 MHz. The -1 order of the output is then used as the probe beam.

C.5.3 Repumper lasers

The repumper lasers are at 707 nm and 679 nm respectively. They are homemade ECDL lasers. Their frequencies are locked on a wavelength meter (HighFinesse, WS/7). An accuracy of 40 MHz is achievable with this locking. The repumper beams can be turned off to reduce L_0 , when a smaller number of atoms is required for the experiment. Together with the 2D OM, and the power and detuning of the cooling beams, the density and the temperature of the atomic cloud can be varied.

³The $m = 0$ sublevel is not shifted by the magnetic field.

Appendix D

Detailed Calculations for the Temporal Behavior of the Flashes

In this appendix, we supplement the details for the derivation of the various expressions for the temporal behavior of the flash. The final expression has already been summarized in Section 5.2. We recall here that the transition frequency is given by $\omega_0 = k_0 c_0$, where k_0 is the resonant wave vector and c_0 is the speed of light in vacuum. The transition linewidth is denoted by Γ . The probe beam has a frequency of ω_l , with detuning $\delta = \omega_l - \omega_0$.

We assume in the general case that, the atoms are moving in the cloud following a Gaussian velocity distribution, with a root-mean-square (rms) velocity of $\bar{v}$.

D.1 Temporal Behavior of the Flashes

D.1.1 Small optical thickness

In this section, we provide the details of the derivation for Equation (5.22). We start from Equation (5.21), which describes the transmitted field through a slab of thickness L . It is repeated in the following,

$$E_t(t, L) = E_0(t, L) - \frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) w \left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \exp(-i\omega t) d\omega. \quad (\text{D.1})$$

It is assumed here that the optical thickness at resonance and at zero temperature is small, i.e., $b_0(0) \ll 1$. The incident field $E_0(t, L)$ is defined at the out-going surface of the slab. The second term on the right hand side is an expansion in first order of $b_0(0)$, consistent with the small optical thickness assumption. It is an integral over the various frequency component ω of the incident field $E_0(\omega, L)$.

The Faddeeva function, w , as described in Appendix B, is a convolution product between the atomic Lorentzian profile and the Gaussian distribution of the atomic velocities.

Using the integral definition of the Faddeeva function in Equation (B.1),

$$w\left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2k\bar{v}}}\right) = \frac{\sqrt{2ik\bar{v}}}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-s^2)}{\omega - \omega_0 - \sqrt{2k\bar{v}}s + i\Gamma/2} ds, \quad (\text{D.2})$$

we find

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) w\left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2k\bar{v}}}\right) \exp(-i\omega t) d\omega \\ = \frac{1}{2\pi} \frac{2ik\bar{v}}{\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds E_0(\omega, L) \frac{\exp(-s^2) \exp(-i\omega t)}{\omega - \omega_0 - \sqrt{2k\bar{v}}s + i\Gamma/2}. \end{aligned} \quad (\text{D.3})$$

This integration is separated two parts, corresponding to the two terms of $E_0(\omega, L)$, as written in Equation (5.20). The expression of $E_0(\omega, L)$ is expressed as,

$$E_0(\omega, L) = i\xi E_0 \text{PV} \frac{1}{\omega - \omega_l} + \eta \pi E_0 \delta(\omega - \omega_l). \quad (\text{D.4})$$

The values of the factors ξ and η for the various cases are summarized in Table 5.1.

The simpler integration over ω for the term containing the Dirac delta term gives

$$\begin{aligned} \frac{\eta E_0}{2} \frac{\sqrt{2ik\bar{v}}}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-s^2) \exp(-i\omega t)}{\omega - \omega_0 - \sqrt{2k\bar{v}}s + i\Gamma/2} \delta(\omega - \omega_l) d\omega \\ = \frac{\eta E_0 \exp(-i\omega_l t)}{2} \frac{\sqrt{2ik\bar{v}}}{\pi} \frac{\exp(-s^2)}{\omega_l - \omega_0 - \sqrt{2k\bar{v}}s + i\Gamma/2}. \end{aligned} \quad (\text{D.5})$$

This is followed by the integration over s for the same term, giving

$$\begin{aligned} \frac{\eta E_0}{2} \frac{\sqrt{2ik\bar{v}}}{\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds \frac{\exp(-s^2) \exp(-i\omega t)}{\omega - \omega_0 - \sqrt{2k\bar{v}}s + i\Gamma/2} \delta(\omega - \omega_l) \\ = \frac{\eta E_0 \exp(-i\omega_l t)}{2} w \left[\frac{\delta + i\Gamma/2}{\sqrt{2k\bar{v}}} \right]. \end{aligned} \quad (\text{D.6})$$

The other term in the calculation of the transmitted field using Equation (D.1), involves a principal value integration. It originates from the first term of Equa-

tion (5.20). Again the ω integration is carried out first,

$$\begin{aligned} & \frac{i\xi E_0}{2\pi} \frac{\sqrt{2}ik\bar{v}}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\exp(-s^2) \exp(-i\omega t)}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} \frac{1}{\omega - \omega_l} d\omega \\ &= \frac{\xi E_0 \exp(-i\omega_l t)}{2} \frac{\sqrt{2}ik\bar{v}}{\pi} \begin{cases} -\frac{\exp(-s^2)}{\delta - \sqrt{2}k\bar{v}s + i\Gamma/2}, & t < 0, \\ \frac{\exp(-s^2)[1 - 2\exp(i\delta t - \sqrt{2}ik\bar{v}st)\exp(-\Gamma t/2)]}{\delta - \sqrt{2}k\bar{v}s + i\Gamma/2}, & t \geq 0. \end{cases} \end{aligned} \quad (\text{D.7})$$

Next, the integration over s is carried out, leading to

$$\begin{aligned} & \frac{i\xi E_0}{2\pi} \frac{\sqrt{2}ik\bar{v}}{\pi} \text{PV} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt \frac{\exp(-s^2) \exp(-i\omega t)}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} \frac{1}{\omega - \omega_l} d\omega \\ &= \frac{\xi E_0 \exp(-i\omega_l t)}{2} \begin{cases} -w \left[\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right], & t < 0, \\ w \left[\frac{\delta + i\Gamma/2}{\sqrt{2}k\bar{v}} \right] - 2e^{i\delta t - \Gamma t/2} e^{-(k\bar{v}t)^2/2} w \left[\frac{\delta + i(\Gamma/2 + k^2\bar{v}^2 t)}{\sqrt{2}k\bar{v}} \right], & t \geq 0. \end{cases} \end{aligned} \quad (\text{D.8})$$

Putting the results above together, we obtain Equation (5.22) for the transmitted field in the case of small optical thickness.

D.1.2 Large optical thickness at zero temperature

For the case of large optical thickness, the calculation of the transmitted field is performed analytically only at zero temperature. We start from Equation (5.42), i.e.,

$$E_t(t, L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) \exp\left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\omega - \omega_0 + i\Gamma/2}\right] \exp(-i\omega t) d\omega. \quad (\text{D.9})$$

The integration over the Dirac delta term in $E_0(\omega, L)$ leads to

$$\begin{aligned} & \frac{\eta E_0}{2} \int_{-\infty}^{\infty} \delta(\omega - \omega_l) \exp\left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\omega - \omega_0 + i\Gamma/2}\right] \exp(-i\omega t) d\omega \\ &= \frac{\eta E_0}{2} \exp\left[-\frac{b_0(0)}{2} \frac{i\Gamma/2}{\delta + i\Gamma/2}\right] \exp(-i\omega_l t). \end{aligned} \quad (\text{D.10})$$

The ω integration for the other term, which involves the principal value integration, can be as a summation in increasing orders of $b_0(0)$. We first interchange the integration and summation order. The p -th order term of the series is given by the

following integration,

$$\begin{aligned} & \frac{i\xi E_0}{2\pi} \text{PV} \int_{-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{1}{p!} \left(-\frac{ib_0(0)\Gamma}{4} \right)^p \frac{1}{(\omega - \omega_0 + i\Gamma/2)^p} \frac{\exp(-i\omega t)}{\omega - \omega_l} d\omega \\ &= \frac{i\xi E_0}{2\pi} \sum_{p=0}^{\infty} \frac{1}{p!} \left(-\frac{ib_0(0)\Gamma}{4} \right)^p \text{PV} \int_{-\infty}^{\infty} \frac{1}{(\omega - \omega_0 + i\Gamma/2)^p} \frac{\exp(-i\omega t)}{\omega - \omega_l} d\omega. \end{aligned} \quad (\text{D.11})$$

We denote

$$\mathcal{I}_p = \text{PV} \int_{-\infty}^{\infty} \frac{1}{(\omega - \omega_0 + i\Gamma/2)^p} \frac{\exp(-i\omega t)}{\omega - \omega_l} d\omega. \quad (\text{D.12})$$

The value of $\mathcal{I}_p$ for $p = 0$ can be computed easily to be,

$$\mathcal{I}_0 = \begin{cases} i\pi \exp(-i\omega_l t), & t < 0, \\ -i\pi \exp(-i\omega_l t), & t \geq 0. \end{cases} \quad (\text{D.13})$$

For $p \geq 1$, $\mathcal{I}_p$ has two poles. One of it is a simple pole at $\omega = \omega_l$, which is located on the real axis. The other is a p -th order pole at $\omega = \omega_0 - i\Gamma/2$, located on the negative half plane. For $t < 0$, only the simple pole contributes to the integral. For $t \geq 0$, both poles contribute to the integral.

At $t < 0$,

$$\mathcal{I}_p(t < 0) = i\pi \frac{e^{-i\omega_l t}}{(\delta + i\Gamma/2)^p}. \quad (\text{D.14})$$

At $t \geq 0$,

$$\mathcal{I}_p(t \geq 0) = -i\pi \frac{e^{-i\omega_l t}}{(\delta + i\Gamma/2)^p} - 2i\pi \sum_{q=0}^{p-1} \frac{(-it)^q}{q!} \frac{e^{-i\omega_0 t - \Gamma t/2}}{(\delta + i\Gamma/2)^{p-q}}. \quad (\text{D.15})$$

Putting all the terms together, we find Equation (5.43) as reported in Chapter 5.

D.2 Calculation of the initial decay time

As discussed in Chapter 5, the exact form of the coherently transmitted field through the optically thick atomic cloud is rather complicated. We introduced the initial decay time $\tau_{\bar{v}}$ in Section 5.2.5, to compute analytically a time scale for the flashes that are emitted in our experiments. Here, we present the details of its derivation.

The initial decay time is defined in Equation (5.46) in terms of the transmitted intensities. However, the analytical expression of the intensity is typically more complicated than the expression for the field. It is possible to rewrite the definition

of the initial decay time in terms of the optical field,

$$\tau_{\bar{v}}(\delta) = \left| \frac{1 - |E_t(t = \infty)|^2 / |E_t(t = 0^+)|^2}{2 \operatorname{Re} \left\{ \frac{dE_t/dt(t=0^+)}{E_t(t=0^+)} \right\}} \right|. \quad (\text{D.16})$$

In the above expression and the following equations, we will write $E(t)$ in place of $E(t, L)$ for all the fields, dropping from the notation, the explicit reference to L . The expressions for $E_t(t = 0^+)$ and $E_t(t = \infty)$ are summarized in Table 5.2 for the all three cases studied in this thesis.

Let us remove first the oscillation at the optical frequency, ω_l , from the forward scattered field. We have,

$$\begin{aligned} E_s(t) e^{i\omega_l t} \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega, L) \left\{ \exp \left[-\frac{b_0(0)}{2} \sqrt{\frac{\pi}{8}} \frac{\Gamma}{k\bar{v}} w \left(\frac{\omega - \omega_0 + i\Gamma/2}{\sqrt{2}k\bar{v}} \right) \right] - 1 \right\} e^{-i(\omega - \omega_l)t} d\omega. \end{aligned} \quad (\text{D.17})$$

The above expression can be differentiated with respect to time t , which results in a factor $\omega - \omega_l$ that appears inside the integral. The singularity due to $E_0(\omega, L)$ is removed by this differentiation, since $(\omega - \omega_l)E_0(\omega, L) = i\xi E_0$.

After expanding the exponential function that contains $b_0(0)$, and interchange the order of integration, we find

$$\begin{aligned} \frac{d}{dt} [E_s(t) e^{i\omega_l t}] &= \frac{\xi E_0}{2\pi} \\ &\times \sum_{p=1}^{\infty} \frac{1}{p!} \left[-\frac{b_0(0)}{2} \frac{i\Gamma}{2} \right]^p \int_{-\infty}^{\infty} d\omega e^{-i(\omega - \omega_l)t} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \frac{\exp(-s^2)}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} \right]^p. \end{aligned} \quad (\text{D.18})$$

We denote

$$\mathcal{J}_p = \int_{-\infty}^{\infty} d\omega e^{-i(\omega - \omega_l)t} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \frac{\exp(-s^2)}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} \right]^p. \quad (\text{D.19})$$

In the definition of $\tau_{\bar{v}}$, only the fields at $t = 0^+$ is needed. Thus, we only compute for the expressions at $t > 0$. The $p = 1$ term can be calculated exactly to be,

$$\mathcal{J}_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\omega \frac{e^{-s^2} e^{-i(\omega - \omega_l)t}}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} = -2\pi i e^{i\delta t} e^{-\Gamma t/2} e^{-(k\bar{v}t)^2/2}. \quad (\text{D.20})$$

For $p > 1$, an exact calculation of $\mathcal{J}_p$ is difficult. Nevertheless, at $t = 0^+$, the values of $\mathcal{J}_p$ for $p > 1$ can be shown to be zero. For example, at $p = 2$,

$$\mathcal{J}_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \frac{e^{-(s^2+s'^2)} e^{-i(\omega-\omega_l)t}}{(\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2)(\omega - \omega_0 - \sqrt{2}k\bar{v}s' + i\Gamma/2)}. \quad (\text{D.21})$$

The above integrand for $s \neq s'$ can be rewritten as

$$\frac{e^{-(s^2+s'^2)} e^{-i(\omega-\omega_l)t}}{(\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2)(\omega - \omega_0 - \sqrt{2}k\bar{v}s' + i\Gamma/2)} = \frac{e^{-(s^2+s'^2)} e^{-i(\omega-\omega_l)t}}{\sqrt{2}k\bar{v}(s-s')} \left[\frac{1}{\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2} - \frac{1}{\omega - \omega_0 - \sqrt{2}k\bar{v}s' + i\Gamma/2} \right]. \quad (\text{D.22})$$

For $s \neq s'$,

$$\mathcal{J}_2(t > 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \frac{e^{-(s^2+s'^2)} e^{i\delta t} e^{-\Gamma t/2}}{\sqrt{2}k\bar{v}(s-s')} \left[e^{-\sqrt{2}ik\bar{v}s t} - e^{-\sqrt{2}ik\bar{v}s' t} \right]. \quad (\text{D.23})$$

When $s = s'$, the singularity in the integrand is now a second order pole. We find

$$\int_{-\infty}^{\infty} d\omega \frac{e^{-i(\omega-\omega_l)t}}{(\omega - \omega_0 - \sqrt{2}k\bar{v}s + i\Gamma/2)^2} = -2\pi t e^{i\delta t} e^{-\sqrt{2}ik\bar{v}s t} e^{-\Gamma t/2}. \quad (\text{D.24})$$

In both cases considered, the value of the integration at $t = 0^+$ is zero. Thus, we have,

$$\mathcal{J}_2(t = 0^+) = 0. \quad (\text{D.25})$$

For the other terms where $p > 2$, a similar argument can be made to show that $\mathcal{J}_p(t = 0^+) = 0$. We first expand the integrand in terms of its partial fractions,

$$\prod_{k=1}^p \frac{1}{\omega - \omega_0 - \sqrt{2}k\bar{v}s_k + i\Gamma/2} = \sum_{k=1}^p \frac{a_k}{\omega - \omega_0 - \sqrt{2}k\bar{v}s_k + i\Gamma/2}. \quad (\text{D.26})$$

The weights a_k obey

$$\sum_{k=1}^p a_k = 0. \quad (\text{D.27})$$

This can be show by writing the right hand side of Equation (D.26) into the form of a fraction, and compare the numerator on both sides. The leading power of ω^{p-1} has a coefficient that is equal to $\sum_{k=1}^p a_k$, which is zero on the left hand side. This validates the relation in Equation (D.27). We can now integrate individually each

term on the right hand side. For each term, we calculate the following integration,

$$\int_{-\infty}^{\infty} \frac{a_k e^{-i(\omega - \omega_l)t}}{\omega - \omega_0 - \sqrt{2}k\bar{v}s_k + i\Gamma/2} d\omega, \quad (\text{D.28})$$

which gives a value of $-2\pi i a_k$ when $t = 0^+$. It is important to note also that the isolated points where some of the values s_k are identical, do not lead to any singularities. Thus, they do not affect the integration results here at $t = 0^+$. Together with Equation (D.27), we find $\mathcal{J}_p(t = 0^+) = 0$ for $p > 1$.

Finally, we have

$$\frac{d}{dt} [E_s e^{i\omega_l t}] (t = 0^+) = \frac{dE_s}{dt}(t = 0^+) + i\omega E_s(t = 0^+) = -\xi E_0 \frac{b_0(0)\Gamma}{4}. \quad (\text{D.29})$$

We can relate the forward scattered field to the transmitted field easily by,

$$E_s(t = 0^+) = E_t(t = 0^+) - E_0(t = 0^+). \quad (\text{D.30})$$

This results in

$$\frac{dE_t}{dt}(t = 0^+) = -\xi E_0 \frac{b_0(0)\Gamma}{4} - i\omega E_t(t = 0^+). \quad (\text{D.31})$$

The denominator of Equation (D.16), apart from a factor of two, is

$$\text{Re} \left\{ \frac{1}{E_t(t = 0^+)} \frac{dE_t}{dt}(t = 0^+) \right\} = -\frac{b_0(0)\Gamma}{4|E_t(t = 0^+)|^2} \text{Re} \{ \xi E_0 E_t^*(t = 0^+) \}. \quad (\text{D.32})$$

Putting the expression above back in Equation (D.16), we obtain Equation (5.47).

Appendix E

Detailed Calculations of second order terms

In this appendix, the details of the derivation for the two-body positional correlation and recurrent scattering contributions to the mass operator are presented. We recall that for the vector waves, the mass operator $\overleftrightarrow{\mathbf{M}}$ up to the terms proportional to ρ_0^2 , can be split into three terms. Here, ρ_0 is the atomic density. This classification is based on their different origins, i.e.,

$$\overleftrightarrow{\mathbf{M}} = \overleftrightarrow{\mathbf{M}}_{\text{ISA}} + \overleftrightarrow{\mathbf{M}}_{\text{cor}} + \overleftrightarrow{\mathbf{M}}_{\text{rec}}, \quad (\text{E.1})$$

where $\overleftrightarrow{\mathbf{M}}_{\text{ISA}}$ reflects the ISA contribution, $\overleftrightarrow{\mathbf{M}}_{\text{cor}}$ reflects the contribution from the pair correlation function, and $\overleftrightarrow{\mathbf{M}}_{\text{rec}}$ reflects the recurrent scattering between a pair of atoms. From Equation (3.39), we see that only the transverse component of the dyadic mass operator is relevant for the effective relative permittivity, ϵ_r . The transverse component of the mass operator M_t can be written as,

$$M_t = M_{t,\text{ISA}} + M_{t,\text{cor}} + M_{t,\text{rec}}, \quad (\text{E.2})$$

with the corresponding transverse components of the three contributing terms. Similarly, for scalar waves, we have

$$M = M_{\text{ISA}} + M_{\text{cor}} + M_{\text{rec}}. \quad (\text{E.3})$$

The ISA mass operator terms, $\overleftrightarrow{\mathbf{M}}_{\text{ISA}}$ for vector waves, and M_{ISA} for scalar waves, have been computed in Chapter 3. In this appendix, we only focus on the derivations of the terms $M_{t,\text{cor}}$, $M_{t,\text{rec}}$, M_{cor} , and M_{rec} . We start by discussing the calculation of $M_{t,\text{cor}}$ and $M_{t,\text{rec}}$ in the case of vector EM waves.

E.1 Vector waves

We recall that the free space dyadic Green function, $\overleftrightarrow{\mathbf{G}}_0$, for the case of vector waves (see Section 3.2.1), is given by

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) = \frac{1}{4\pi} \frac{\omega}{c_0} \left[\beta(\omega R/c_0) \overleftrightarrow{\mathbf{P}} + \gamma(\omega R/c_0) \overleftrightarrow{\mathbf{U}} \right] - \frac{\overleftrightarrow{\mathbf{I}}}{3(\omega/c_0)^2} \delta(\mathbf{R}), \quad (\text{E.4})$$

for a light of frequency ω . The speed of light in the vacuum is denoted by c_0 . The functions β and γ are defined in the following,

$$\beta(x) = \frac{e^{ix}}{x} \left[1 - \frac{1}{ix} - \frac{1}{x^2} \right], \quad (\text{E.5})$$

$$\gamma(x) = \frac{2e^{ix}}{x} \left[\frac{1}{ix} + \frac{1}{x^2} \right]. \quad (\text{E.6})$$

The dyadics $\overleftrightarrow{\mathbf{U}}$ and $\overleftrightarrow{\mathbf{P}}$ are defined by,

$$\overleftrightarrow{\mathbf{U}} = \mathbf{u} \otimes \mathbf{u}, \quad (\text{E.7})$$

$$\overleftrightarrow{\mathbf{P}} = \overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\mathbf{U}}, \quad (\text{E.8})$$

where $\mathbf{u} = \mathbf{R}/R$, i.e., the unit vector along $\mathbf{R}$.

We have also denoted the resonance frequency of the transition by ω_0 , and the linewidth by Γ . The polarizability is given by

$$\alpha(\omega) = \alpha_0 \tilde{\alpha}(\omega). \quad (\text{E.9})$$

The modulus of the resonant polarizability, α_0 , has a value of $6\pi/k_0^3$ for the vector waves. Here, $k_0 = \omega_0/c_0$. In the linear response regime, which is the focus of our study,

$$\tilde{\alpha}(\omega) = -\frac{\Gamma/2}{\omega - \omega_0 + i\Gamma/2}. \quad (\text{E.10})$$

This expression holds when $|\omega - \omega_0| \ll \omega_0$, which is the case in this thesis.

E.1.1 Mass operator for two-body correlation

The detailed derivation of Equation (7.16) in Section 7.2.1 is presented here. We start from Equation (7.15), which reads,

$$\overleftrightarrow{\mathbf{M}}_{\text{cor}}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^4 \int \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) g_c(R) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R}. \quad (\text{E.11})$$

Only the following form of the pair correlation function, $g_c(R)$, is considered for our studies, i.e.,

$$g_c(R) = \begin{cases} -1, & R < d_{\min}, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{E.12})$$

In fact, for a typical atomic scatterer, $d_{\min} \rightarrow 0$. Using Equation (E.12), the two-body correlation term becomes an integration over the exclusion volume V_{ex} , i.e.,

$$\overleftrightarrow{\mathbf{M}}_{\text{cor}}(\mathbf{k}, \omega) = -\rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^4 \int_{V_{\text{ex}}} \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R} \quad (\text{E.13})$$

We further define a non-dimensionalized radius of the exclusion volume, $\tilde{d}_{\min} = \omega d_{\min} / c_0$.

The integration over the Dirac delta function term in $\overleftrightarrow{\mathbf{G}}_0$ is performed first, giving simply

$$\frac{1}{3} \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^2 \int_{V_{\text{ex}}} \delta(\mathbf{R}) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R} = \frac{1}{3} \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^2. \quad (\text{E.14})$$

Even in the limit of $\tilde{d}_{\min} \rightarrow 0$, this contribution still remains.

For the other $\overleftrightarrow{\mathbf{G}}_0$ terms, the integration is performed first over the solid angle Ω . We make use of the following angular integrals, in the computation of $\overleftrightarrow{\mathbf{M}}_{\text{cor}}$ and $\overleftrightarrow{\mathbf{M}}_{\text{rec}}$,

$$\int \overleftrightarrow{\mathbf{U}} d\Omega = \frac{4\pi}{3} \overleftrightarrow{\mathbf{I}}, \quad (\text{E.15})$$

$$\int \overleftrightarrow{\mathbf{P}} d\Omega = \frac{8\pi}{3} \overleftrightarrow{\mathbf{I}}, \quad (\text{E.16})$$

$$\int \overleftrightarrow{\mathbf{U}} e^{-i\mathbf{k} \cdot \mathbf{R}} d\Omega = 4\pi \frac{j_1(kR)}{kR} \overleftrightarrow{\mathbf{I}} + \left[4\pi j_0(kR) - 12\pi \frac{j_1(kR)}{kR} \right] \overleftrightarrow{\mathbf{K}}, \quad (\text{E.17})$$

$$\int \overleftrightarrow{\mathbf{P}} e^{-i\mathbf{k} \cdot \mathbf{R}} d\Omega = \left[4\pi j_0(kR) - 4\pi \frac{j_1(kR)}{kR} \right] \overleftrightarrow{\mathbf{I}} - \left[4\pi j_0(kR) - 12\pi \frac{j_1(kR)}{kR} \right] \overleftrightarrow{\mathbf{K}}, \quad (\text{E.18})$$

where $\overleftrightarrow{\mathbf{K}} = \mathbf{k} \otimes \mathbf{k} / k^2$, with $k = |\mathbf{k}|$. The functions $j_0(x)$ and $j_1(x)$ are the zeroth and first order spherical Bessel functions, respectively. They are defined by

$$j_0(x) = \frac{\sin x}{x}, \quad (\text{E.19})$$

$$j_1(x) = \frac{\sin x - x \cos x}{x^2}. \quad (\text{E.20})$$

As mentioned before, only the transverse component of the mass operator plays

a role in the coherent transmission of light. For the two-body positional correlation, we find the following contribution in the transverse direction,

$$M_{t,\text{cor}}(\mathbf{k}, \omega) = \frac{1}{3} \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^2 - \rho_0^2 \alpha_0^2 \tilde{\alpha}(\omega)^2 \left(\frac{\omega}{c_0} \right)^5 \int_0^{d_{\min}} \gamma(\omega R/c_0) \frac{j_1(kR)}{kR} R^2 dR \\ - \rho_0^2 \alpha_0^2 \tilde{\alpha}(\omega)^2 \left(\frac{\omega}{c_0} \right)^5 \int_0^{d_{\min}} \beta(\omega R/c_0) \frac{kR j_0(kR) - j_1(kR)}{kR} R^2 dR. \quad (\text{E.21})$$

Using the *on-shell* approximation, i.e., replacing k by ω/c_0 , the integration over R is found to be

$$M_{t,\text{cor}}(\omega) = (\rho_0 \alpha_0)^2 \tilde{\alpha}(\omega)^2 \left(\frac{\omega}{c_0} \right)^2 \\ \times \frac{2i + 2i\tilde{d}_{\min}^2 - \tilde{d}_{\min}^3 - 2i\tilde{d}_{\min}^4 - \left(2i + 4\tilde{d}_{\min} - 2i\tilde{d}_{\min}^2 - \tilde{d}_{\min}^3 \right) e^{2i\tilde{d}_{\min}}}{4\tilde{d}_{\min}^3}, \quad (\text{E.22})$$

as reported in Section 7.2.1.2 [see Equations (7.16) and (7.17)].

E.1.2 Mass operator for recurrent scattering contribution

We now consider the mass operator term originating from the recurrent scattering contribution. The recurrent scattering process consists of different orders, as indicated below,

$$\overleftrightarrow{\mathbf{M}}_{\text{rec}}(\omega) = \sum_{p=3}^{\infty} \overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(p)}(\omega). \quad (\text{E.23})$$

Here, $\overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(p)}(\omega)$ reflects to the contribution from a two-body recurrent scattering with p scattering events.

E.1.2.1 Loop diagrams

The simplest contribution comes from recurrent scattering involving three scattering events, i.e., $p = 3$. According to the classification introduced in Ref. [121], which is summarized in Figure 7.1, this is the simplest example of the loop diagrams. The contribution to the mass operator is given by

$$\overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(3)}(\mathbf{r} - \mathbf{r}', \omega) = \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \int \cdots \int \mathcal{T}_{\mathbf{r}_1}(\mathbf{r}, \mathbf{r}_3, \omega) \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_3 - \mathbf{r}_4, \omega) \mathcal{T}_{\mathbf{r}_2}(\mathbf{r}_4, \mathbf{r}_5, \omega) \\ \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}_5 - \mathbf{r}_6, \omega) \mathcal{T}_{\mathbf{r}_1}(\mathbf{r}_6, \mathbf{r}', \omega) P_2(\mathbf{r}_1, \mathbf{r}_2) d^3\mathbf{r}_1 \dots d^3\mathbf{r}_6, \quad (\text{E.24})$$

with the scattering operator of an atom located at $\mathbf{r}_0$ given by

$$\mathcal{T}_{\mathbf{r}_0}(\mathbf{r}, \mathbf{r}', \omega) = \alpha(\omega) \left(\frac{\omega}{c_0} \right)^2 \delta(\mathbf{r} - \mathbf{r}_0) \delta(\mathbf{r}' - \mathbf{r}_0). \quad (\text{E.25})$$

This leads to

$$\begin{aligned} & \vec{\mathbf{M}}_{\text{rec}}^{(3)}(\mathbf{r} - \mathbf{r}', \omega) \\ &= \rho_0^2 \alpha(\omega)^3 \left(\frac{\omega}{c_0} \right)^6 \delta(\mathbf{r} - \mathbf{r}') \int \vec{\mathbf{G}}_0(\mathbf{r}' - \mathbf{r}_4, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{r}_4 - \mathbf{r}', \omega) [1 + g_c(|\mathbf{r}' - \mathbf{r}_4|)] d^3 \mathbf{r}_4. \end{aligned} \quad (\text{E.26})$$

In the $\mathbf{k}$ -space, we find

$$\vec{\mathbf{M}}_{\text{rec}}^{(3)}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^3 \left(\frac{\omega}{c_0} \right)^6 \int \vec{\mathbf{G}}_0(\mathbf{R}, \omega) \cdot \vec{\mathbf{G}}_0(-\mathbf{R}, \omega) [1 + g_c(R)] d^3 \mathbf{R}. \quad (\text{E.27})$$

The free space Green function obeys, $\vec{\mathbf{G}}_0(\mathbf{R}, \omega) = \vec{\mathbf{G}}_0(-\mathbf{R}, \omega)$. Hence, writing $\vec{\mathbf{G}}_0(\mathbf{R}, \omega)^2 = \vec{\mathbf{G}}_0(\mathbf{R}, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{R}, \omega)$, we find

$$\vec{\mathbf{M}}_{\text{rec}}^{(3)}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^3 \left(\frac{\omega}{c_0} \right)^6 \int \vec{\mathbf{G}}_0(\mathbf{R}, \omega)^2 [1 + g_c(R)] d^3 \mathbf{R}. \quad (\text{E.28})$$

In fact, this can be generalized for all the loop diagrams to give,

$$\vec{\mathbf{M}}_{\text{rec}}^{(p)}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^p \left(\frac{\omega}{c_0} \right)^{2p} \int \vec{\mathbf{G}}_0(\mathbf{R}, \omega)^{p-1} [1 + g_c(R)] d^3 \mathbf{R}. \quad (\text{E.29})$$

E.1.2.2 Boomerang diagrams

We consider now the case of even values of p , i.e., the boomerang diagrams. The simplest case is at $p = 4$, which can be written as

$$\begin{aligned} \vec{\mathbf{M}}_{\text{rec}}^{(4)}(\mathbf{r} - \mathbf{r}', \omega) &= \sum_{i=1}^{N_a} \sum_{\substack{j=1 \\ j \neq i}}^{N_a} \int \cdots \int \mathcal{T}_{\mathbf{r}_2}(\mathbf{r}, \mathbf{r}_3, \omega) \vec{\mathbf{G}}_0(\mathbf{r}_3 - \mathbf{r}_4, \omega) \mathcal{T}_{\mathbf{r}_1}(\mathbf{r}_4, \mathbf{r}_5, \omega) \\ &\cdot \vec{\mathbf{G}}_0(\mathbf{r}_5 - \mathbf{r}_6, \omega) \mathcal{T}_{\mathbf{r}_2}(\mathbf{r}_6, \mathbf{r}_7, \omega) \cdot \vec{\mathbf{G}}_0(\mathbf{r}_7 - \mathbf{r}_8, \omega) \mathcal{T}_{\mathbf{r}_1}(\mathbf{r}_8, \mathbf{r}', \omega) P_2(\mathbf{r}_1, \mathbf{r}_2) d^3 \mathbf{r}_1 \dots d^3 \mathbf{r}_8. \end{aligned} \quad (\text{E.30})$$

After some calculations, we have

$$\begin{aligned} \overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(4)}(\mathbf{r} - \mathbf{r}', \omega) \\ = \rho_0^2 \alpha(\omega)^4 \left(\frac{\omega}{c_0} \right)^8 \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r}' - \mathbf{r}, \omega) \cdot \overleftrightarrow{\mathbf{G}}_0(\mathbf{r} - \mathbf{r}', \omega) [1 + g_c(|\mathbf{r} - \mathbf{r}'|)] . \end{aligned} \quad (\text{E.31})$$

In the $\mathbf{k}$ -space, this is

$$\overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(4)}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^4 \left(\frac{\omega}{c_0} \right)^8 \int \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega)^3 [1 + g_c(R)] e^{-i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{R}. \quad (\text{E.32})$$

In general, the boomerang diagrams are calculated by,

$$\overleftrightarrow{\mathbf{M}}_{\text{rec}}^{(p)}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^p \left(\frac{\omega}{c_0} \right)^{2p} \int \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega)^{p-1} [1 + g_c(R)] e^{-i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{R}. \quad (\text{E.33})$$

The inclusion of the factor $1 + g_c(R)$ in the integrand means that the integration is carried out over the volume $V - V_{\text{ex}}$, where V is the volume of the scattering medium.

E.1.2.3 Total recurrent scattering mass operator

Summing up all orders (both loop and boomerang diagrams), we have

$$\begin{aligned} \overleftrightarrow{\mathbf{M}}_{\text{rec}}(\mathbf{k}, \omega) &= \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^4 \sum_{l=0}^{\infty} \left(\frac{\alpha_0 \omega^2}{c_0^2} \right)^{2l+1} \tilde{\alpha}(\omega)^{2l+3} \\ &\times \left[\int_{V-V_{\text{ex}}} \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega)^{2l+2} d^3 \mathbf{R} + \frac{\alpha_0 \omega^2}{c_0^2} \tilde{\alpha}(\omega) \int_{V-V_{\text{ex}}} \overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega)^{2l+3} e^{-i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{R} \right]. \end{aligned} \quad (\text{E.34})$$

Since the $\overleftrightarrow{\mathbf{U}}$ and $\overleftrightarrow{\mathbf{P}}$ are projectors, they satisfy the relations $\overleftrightarrow{\mathbf{U}}^2 = \overleftrightarrow{\mathbf{U}}$, $\overleftrightarrow{\mathbf{P}}^2 = \overleftrightarrow{\mathbf{P}}$ and $\overleftrightarrow{\mathbf{U}} \cdot \overleftrightarrow{\mathbf{P}} = \overleftrightarrow{\mathbf{P}} \cdot \overleftrightarrow{\mathbf{U}} = \overleftrightarrow{\mathbf{0}}$. Thus,

$$\overleftrightarrow{\mathbf{G}}_0(\mathbf{R}, \omega)^p = \left(\frac{\omega}{4\pi c_0} \right)^p \left[\beta(\omega R/c_0)^p \overleftrightarrow{\mathbf{P}} + \gamma(\omega R/c_0)^p \overleftrightarrow{\mathbf{U}} \right]. \quad (\text{E.35})$$

The Dirac delta term in $\overleftrightarrow{\mathbf{G}}_0$ can be neglected, since its integration is over the volume $V - V_{\text{ex}}$ is zero.

Only the transverse component of the recurrent mass operator is relevant here,

$$\begin{aligned}
M_{t,\text{rec}}(\mathbf{k}, \omega) = & \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^5 \sum_{l=0}^{\infty} \left(\frac{\alpha_0 \omega^3}{4\pi c_0^3} \right)^{2l+1} \tilde{\alpha}(\omega)^{2l+3} \\
& \times \left\{ \frac{1}{3} \int_{d_{\min}}^{\infty} [2\beta(\omega R/c_0)^{2l+2} + \gamma(\omega R/c_0)^{2l+2}] R^2 dR \right. \\
& + \frac{\alpha_0 \omega^3}{4\pi c_0^3} \tilde{\alpha}(\omega) \int_{d_{\min}}^{\infty} \beta(\omega R/c_0)^{2l+3} \frac{kR j_0(kR) - j_1(kR)}{kR} R^2 dR \\
& \left. + \frac{\alpha_0 \omega^3}{4\pi c_0^3} \tilde{\alpha}(\omega) \int_{d_{\min}}^{\infty} \gamma(\omega R/c_0)^{2l+3} \frac{j_1(kR)}{kR} R^2 dR \right\}. \quad (\text{E.36})
\end{aligned}$$

The *on-shell* approximation is applied to set $k = \omega/c_0$. Furthermore, we have approximated $\alpha_0 \omega^3/c_0^3 \approx 6\pi$. Defining $\tilde{R} = \omega R/c$,

$$\begin{aligned}
M_{t,\text{rec}}(\omega) = & \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^2 \sum_{l=0}^{\infty} \left(\frac{3}{2} \right)^{2l+1} \tilde{\alpha}(\omega)^{2l+3} \left\{ \frac{1}{3} \int_{\tilde{d}_{\min}}^{\infty} [2\beta(\tilde{R})^{2l+2} + \gamma(\tilde{R})^{2l+2}] \tilde{R}^2 d\tilde{R} \right. \\
& \left. + \frac{3}{2} \tilde{\alpha}(\omega) \int_{\tilde{d}_{\min}}^{\infty} \beta(\tilde{R})^{2l+3} \tilde{R} [\tilde{R} j_0(\tilde{R}) - j_1(\tilde{R})] + \gamma(\tilde{R})^{2l+3} \tilde{R} j_1(\tilde{R}) d\tilde{R} \right\}. \quad (\text{E.37})
\end{aligned}$$

The order of the integration and summation in the equation above, is interchanged to find Equations (7.27) and (7.28). These two equations are repeated here for completeness,

$$M_{t,\text{rec}} = (\rho_0 \alpha_0)^2 \left(\frac{\omega}{c_0} \right)^2 \frac{\tilde{\alpha}(\omega)^3}{2} \mathcal{J}_v(\omega), \quad (\text{E.38})$$

with

$$\begin{aligned}
\mathcal{J}_v(\omega) = & \int_{\tilde{d}_{\min}}^{\infty} \frac{2x^2 [\beta(x)^2 + \frac{9}{4} \tilde{\alpha}(\omega) \beta(x)^3 \{j_0(x) - j_1(x)/x\}]}{1 - \frac{9}{4} \tilde{\alpha}(\omega)^2 \beta(x)^2} \\
& + x^2 \frac{\gamma(x)^2 + \frac{9}{2} \tilde{\alpha}(\omega) \gamma(x)^3 j_1(x)/x}{1 - \frac{9}{4} \tilde{\alpha}(\omega)^2 \gamma(x)^2} dx. \quad (\text{E.39})
\end{aligned}$$

E.2 Scalar waves

In the case of scalar EM waves, the free space Green function, G_0 , is given by the following simple expression (see Section 3.2.2),

$$G_0(\mathbf{R}, \omega) = \frac{1}{4\pi R} e^{i\omega R/c_0}, \quad (\text{E.40})$$

Equation (E.9) also holds for the atomic polarizability, but with $\alpha_0 = 4\pi/k_0^3$.

In the following, the mass operators M_{cor} and M_{rec} are calculated for the scalar waves.

E.2.1 Mass operator for two-body correlation

From Equation (7.21), which is repeated below,

$$M_{\text{cor}}(\mathbf{k}, \omega) = \rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^4 \int G_0(\mathbf{R}, \omega) g_c(R) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R}, \quad (\text{E.41})$$

the pair correlation function as given in Equation (E.12) leads to,

$$M_{\text{cor}}(\mathbf{k}, \omega) = -\rho_0^2 \alpha(\omega)^2 \int_{V_{\text{ex}}} G_0(\mathbf{R}, \omega) e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R}. \quad (\text{E.42})$$

The angular integration over the solid angle is carried out first to find

$$M_{\text{cor}}(\mathbf{k}, \omega) = -\rho_0^2 \alpha_0^2 \tilde{\alpha}(\omega)^2 \left(\frac{\omega}{c} \right)^4 \int_0^{d_{\text{min}}} e^{-i\omega R/c} \frac{\sin kR}{kR} R dR. \quad (\text{E.43})$$

The *on-shell* approximation is again applied. This finally leads to Equation (7.23), i.e.,

$$M_{\text{cor}}(\omega) = -\rho_0^2 \alpha(\omega)^2 \left(\frac{\omega}{c_0} \right)^2 \left[\frac{1}{4} \left(1 - e^{2i\tilde{d}_{\text{min}}} \right) + \frac{i}{2} \tilde{d}_{\text{min}} \right]. \quad (\text{E.44})$$

E.2.2 Mass operator for recurrent scattering contribution

In analogy to the vector waves, the recurrent scattering contribution to the mass operator, for scalar waves, are given by the following integrations,

$$M_{\text{rec}}^{(p)}(\mathbf{k}, \omega) = \rho_0^2 \alpha_0^2 \tilde{\alpha}(\omega)^p \left(\frac{\omega}{c_0} \right)^{2p} \begin{cases} \int G_0(\mathbf{R})^{p-1} [1 + g_c(R)] d^3\mathbf{R}, & \text{loops,} \\ \int G_0(\mathbf{R})^{p-1} [1 + g_c(R)] e^{-i\mathbf{k} \cdot \mathbf{R}} d^3\mathbf{R}. & \text{boomerangs,} \end{cases} \quad (\text{E.45})$$

for $p \geq 3$. Similar to the case of vector waves, the expressions for the loop and boomerang diagrams have different forms.

Summing up these contributions, the recurrent scattering mass operator in the

$\mathbf{k}$ -space is found to be,

$$M_{\text{rec}}(\mathbf{k}, \omega) = \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^4 \sum_{l=0}^{\infty} \left(\frac{\alpha_0 \omega^2}{c_0^2} \right)^{2l+1} \tilde{\alpha}(\omega)^{2l+3} \\ \times \left[\int_{V-V_{\text{ex}}} G_0(\mathbf{R}, \omega)^{2l+2} d^3 \mathbf{R} + \left(\frac{\alpha_0 \omega^2}{c_0^2} \right) \tilde{\alpha}(\omega) \int_{V-V_{\text{ex}}} G_0(\mathbf{R}, \omega)^{2l+3} e^{-i\mathbf{k} \cdot \mathbf{R}} d^3 \mathbf{R} \right]. \quad (\text{E.46})$$

Again, the angular integration is performed first,

$$M_{\text{rec}}(\mathbf{k}, \omega) = 4\pi \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^4 \sum_{l=0}^{\infty} \left(\frac{\alpha_0 \omega^2}{c_0^2} \right)^{2l+1} \\ \times \left[\int_{d_{\min}}^{\infty} G_0(R, \omega)^{2l+2} R^2 dR + \left(\frac{\alpha_0 \omega^2}{c_0^2} \right) \tilde{\alpha}(\omega) \int_{d_{\min}}^{\infty} G_0^{2l+3}(R, \omega)^{2l+3} \frac{\sin kR}{kR} R^2 dR \right]. \quad (\text{E.47})$$

The *on-shell* approximation is then applied. We also note that $\alpha_0 \omega^3 / c_0^3 \approx 4\pi$. Thus, in terms of $\tilde{R} = \omega R / c_0$, we find

$$M_{\text{rec}}(\omega) = \rho_0^2 \alpha_0^2 \left(\frac{\omega}{c_0} \right)^2 \left[\sum_{l=0}^{\infty} \tilde{\alpha}(\omega)^{2l+3} \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{i(2l+2)\tilde{R}}}{\tilde{R}^{2l}} d\tilde{R} \right. \\ \left. + \sum_{l=0}^{\infty} \tilde{\alpha}(\omega)^{2l+4} \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{i(2l+3)\tilde{R}}}{\tilde{R}^{2l+2}} \sin \tilde{R} d\tilde{R} \right].$$

The lowest order term of the integral can actually be evaluated. We have

$$\int_{\tilde{d}_{\min}}^{\infty} e^{2i\tilde{R}} d\tilde{R} = \lim_{\eta \rightarrow 0^+} \int_{\tilde{d}_{\min}}^{\infty} e^{(2i-\eta)\tilde{R}} d\tilde{R} = \frac{i}{2} e^{2i\tilde{d}_{\min}}, \quad (\text{E.48})$$

where η is a positive number introduced as a dissipation at large distance. The recurrent scattering mass operator for scalar waves is rewritten as

$$M_{\text{rec}}(\omega) = \rho_0^2 \alpha_0^2 \frac{i\tilde{\alpha}(\omega)^3}{2} \left(\frac{\omega}{c_0} \right)^2 \left[e^{2i\tilde{d}_{\min}} + \sum_{l=0}^{\infty} \tilde{\alpha}(\omega)^{2l+1} \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{i(2l+2)\tilde{R}}}{\tilde{R}^{2l+2}} d\tilde{R} \right. \\ \left. - (1 + 2i\tilde{\alpha}) \sum_{l=0}^{\infty} \tilde{\alpha}(\omega)^{2l+1} \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{i(2l+4)\tilde{R}}}{\tilde{R}^{2l+2}} \sin \tilde{R} d\tilde{R} \right].$$

The integration and the summation order can be interchanged. This finally

leads to the result expressed by Equations (7.31) and (7.31), which is

$$M_{\text{rec}}(\omega) = (\rho_0 \alpha_0)^2 \left(\frac{\omega}{c} \right)^2 \frac{i \tilde{\alpha}(\omega)^3}{2} \left[e^{2i\tilde{d}_{\min}} + \tilde{\alpha}(\omega) \mathcal{I}_s(\omega) \right], \quad (\text{E.49})$$

with

$$\mathcal{I}_s(\omega) = \int_{\tilde{d}_{\min}}^{\infty} \frac{e^{2ix} [1 - (1 + 2i\tilde{\alpha}(\omega)) e^{2ix}]}{x^2 - \tilde{\alpha}(\omega)^2 e^{2ix}} dx. \quad (\text{E.50})$$

Appendix F

List of Publication

1. [C. C. Kwong](#), T. Yang, M. S. Pramod, K. Pandey, D. Delande, R. Pierrat and D. Wilkowski, “Cooperative emission of a coherent superflash of light”, Phys. Rev. Lett. **113**, 223601 (2014)
2. [C. C. Kwong](#), T. Yang, D. Delande, R. Pierrat and D. Wilkowski, “Cooperative emission of a pulse train in an optically thick scattering medium”, Phys. Rev. Lett. **115**, 223601 (2015)
3. T. Yang, K. Pandey, M. S. Pramod, F. Leroux, [C. C. Kwong](#), E. Hajiyev, Z. Y. Chia, B. Fang and D. Wilkowski, “A high flux source of cold strontium atoms”, Eur. Phys. J. D **69**, 226 (2015)
4. K. Pandey, [C. C. Kwong](#), M. S. Pramod and D. Wilkowski, “Linear and non-linear magneto-optical rotation on the narrow strontium intercombination line”, Phys. Rev. A **93**, 053428 (2016)

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