Output regulation of linear heterogeneous multi-agent systems

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OUTPUT REGULATION OF LINEAR HETEROGENEOUS MULTI-AGENT SYSTEMS

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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

Date

FARNAZ ADIB YAGHMAIE

To my family and my husband, Meysam, with endless love.

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Abstract

In this thesis, we address output regulation of linear heterogeneous multi-agent systems and use a general model which accounts for heterogeneity of the agents and an arbitrarily finite number of inputs and outputs. We suggest distributed controller design methods for different scenarios.

In the first part of this thesis, we study output regulation of a group of linear heterogeneous agents communicating over an acyclic graph. We first show that the output regulation can be achieved through local controller design, then we formulate the output regulation in a graphical game framework. We show that the graphical formulation is robust to multiplicative uncertainties satisfying an upper bound with an infinite gain margin.

In the second part of this thesis, we address output regulation of a group of linear heterogeneous agents communicating over a general graph. We derive global and local sufficient conditions for the existence of a distributed controller. We show that our suggested controllers have an H_∞ criterion as a part of their sufficient conditions. We design the controllers based on a set of Linear Matrix Inequalities (LMIs).

In the third part of this thesis, we introduce unmodeled disturbances to the multi-agent system and investigate H_∞ output regulation of linear heterogeneous agents communicating over a general cyclic graph. We suggest zero/nonzero-sum multiplayer games and LMI designs for H_∞ output regulation.

In the fourth part of this thesis, we generalize the idea of classical output regula-

tion to bipartite output regulation. In the bipartite output regulation, the agents communicate over a signed structurally balanced graph and it is desired that the outputs of all agents synchronize in moduli but with different signs. For a structurally balanced signed graph, we prove that the bipartite output regulation is equivalent to the classical output regulation over an unsigned graph whose adjacency matrix is obtained by taking the absolute value of each entry in the adjacency matrix of the signed graph. We obtain a new H_∞ criterion which is sufficient for both the classical output regulation and the bipartite output regulation.

In the last part of this thesis, we define multi-party output regulation of linear heterogeneous agents over a general graph with complex weights. In the multi-party output regulation, the outputs of all agents in the same party synchronize to each other but the synchronized trajectory of a party is rotated with respect to another party. As one application, we discuss how the developed theory is applied in formation control of linear heterogeneous multi-agent systems.

Chapter 1

Introduction

Distributed control of multi-agent systems is an active research area with the applications in consensus [1–5] and formation control [6–8]. The vast interest to this field is due to the successful application of distributed cooperative control in Unmanned Air Vehicles (UAVs) [9], sensor networks [10], formation control of a group of mobile robots [6–8], and so on. There are many advantages for distributed control such as less computational complexity, high reliability and scalability due to the lack of a centralized decision-making center, which make this field challenging and interesting.

In the study of distributed control of multi-agent systems, one may think of two possible scenarios: 1- One can consider a group of *homogeneous agents*, *i.e.* agents with identical dynamics. In this case, state synchronization of the agents is of interest [1,11–20] where a common approach is to stabilize the state synchronization error. The study of state synchronization of homogeneous agents is important because its results are widely used in formation control and output regulation problems. 2- It is also possible to consider a group of *uncertain homogeneous agents*, *i.e.* uncertain agents with identical nominal dynamics or a group of *heterogeneous agents*, *i.e.* agents with different dynamics in terms of different definitions for the

state variables and a different number of the state variables. In the case of heterogeneous agents, it is not meaningful to study state synchronization anymore and instead one needs to consider output synchronization problem. Such a problem is called *output regulation problem of multi-agent systems* in the literature [21–24].

The term *output regulation* stands for asymptotic tracking a reference trajectory and disturbance rejection when the model of disturbance is known [25]. If the reference trajectory is a zero signal or converges to zero, the problem is called *regulation to zero*. In this thesis, we are interested in studying output regulation problem of multi-agent systems where it is desired that the outputs of all agents synchronize to a nonzero reference trajectory or a nonzero leader (*leader–follower output regulation*). Tracking a reference trajectory which converges to zero is trivial; because in this case, all agents need only to stabilize their dynamics and there is no need for communication over a network; *i.e.* the multi-agent structure is not needed anymore.

In the sequel, we bring a literature review on the output regulation problem and the related concepts.

1.1 Literature Review

Consider a group of heterogeneous agents containing N followers and one leader whose models are described as follows

$$\dot{x}_0 = A_0 x_0, \tag{1.1}$$

$$y_0 = C_0 x_0, \tag{1.2}$$

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{1.3}$$

$$y_i = C_i x_i + D_i u_i, \tag{1.4}$$

$$e_i = y_i - y_0 = C_i x_i - C_0 x_0, \tag{1.5}$$

in which $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^p$ denote the state, the control signal and the output of agent $i = 1, \dots, N$ respectively, and $x_0 \in \mathbb{R}^{n_0}$ and $y_0 \in \mathbb{R}^p$ denote the state and the output of the leader. e_i is the output regulation error and it is desired to design the controllers u_i distributively such that $|e_i| \rightarrow 0$ as $|t| \rightarrow +\infty$.

According to [26], there are two general approaches toward the output regulation problem in multi-agent systems. Each approach makes use of one of the following explicit assumptions.

Assumption 1.1.1. *The regulator equation assumption:* *There exists a solution (Φ_i, Γ_i) to the regulator equation*

$$\begin{aligned}\Phi_i A_0 &= A_i \Phi_i + B_i \Gamma_i, \\ 0 &= C_i \Phi_i + D_i \Gamma_i + C_0.\end{aligned}\tag{1.6}$$

Assumption 1.1.2. *The transmission zero condition assumption:*

$$\text{rank} \left(\begin{bmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{bmatrix} \right) = n_i + p, \quad \lambda \in \text{spec}(A_0).\tag{1.7}$$

It is shown in Theorem 1.9 of [25] that the satisfaction of Assumption 1.1.2 implies the satisfaction of Assumption 1.1.1 but not vice versa. For uncertain systems Assumption 1.1.1 is not enough and Assumption 1.1.2 must be adopted.

1.1.1 Output regulation problem with regulator equation assumption

One can use Assumption 1.1.1 to solve the output regulation problem of linear heterogeneous multi-agent systems [22,23,27,28]. For this purpose, each follower

employs a distributed observer and a local controller. The distributed observer is designed to estimate the state of the leader and to accomplish this task, the followers need to communicate the observers' states distributively. The controller consists of a state-feedback, which is used to stabilize the system's dynamics, and a feed-forward term, which is used to perform tracking of a reference signal. The reference signal for each follower is the output of the distributed observer, which converges to the output of the leader. The feed-forward gain is derived from a solution to the regulator equation (1.6) [22,23]. Since a solution to (1.6) is required, if the followers have uncertainty in their parameters, one needs to solve (1.6) for every sample of the uncertain system. As a result, this method is not commonly used for uncertain systems. An adaptive algorithm to calculate a solution to the regulator equation is given in [29,30] which results in an adaptive distributed control law for the output regulation problem.

A *leaderless output synchronization problem* of linear multi-agent systems is analyzed in [27] where the controllers are designed such that the outputs of all agents synchronize to a common trajectory namely a virtual leader. The problem in [27] is completely different from [23] because in [23] a leader does exist while in [27], the agents synchronize to a virtual leader.

In [31], pre-compensators are used to make the agents *almost identical*. The pre-compensators are used to synchronize the identical parts of the agents to each other and to stabilize the remaining parts. Recently, the output consensus of heterogeneous *non-introspective* agents subject to actuator saturation is considered in [32]. The agents are non-introspective if they do not have an access to their states and outputs and instead, they know the local output synchronization error and the local observer error. The controller design consists of three steps. First, a state transformation is performed to remove the redundant modes that have no effect on the regulated output. In the second step, a low-gain feedback controller is designed to

ensure that the saturation level is not violated. In the third step, an observer is constructed and used in the controller.

In [33,34], Assumption 1.1.1 is used to solve the output regulation of uncertain multi-agent systems. The authors use a robust regulator and an observer to accomplish the task. The observer estimates the state of the leader and the robust regulator compensates for the uncertainty in the system. In [35], a local dynamic controller is suggested which uses two local errors. The first error is an output local neighborhood error and the second error is a weighted local dynamic compensator error. Such a problem formulation needs the stability of uncontrolled agents which is unnecessarily strong.

1.1.2 Output regulation problem with transmission zero assumption

Assuming that the pair (A_i, B_i) is controllable (A_i, C_i) is observable, the transmission zeros of system (A_i, B_i, C_i, D_i) is defined as follows

$$\left\{ \lambda \mid \text{rank} \left(\begin{bmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{bmatrix} \right) < n_i + p \right\}.$$

To solve the output regulation problem *via* the transmission zero assumption, one needs to incorporate a p -copy (Definition 1.4.1) of the internal model of the leader in the dynamic controller of each follower [25]. Assumption 1.1.2 ensures that there is no (unstable) zero-pole cancellation between the agent's dynamics and the p -copy of the leader. Since a solution to (1.6) is not required anymore, the solution based on Assumption 1.1.2 can be used for uncertain systems as long as this assumption holds for all uncertain samples of the agents.

There is rich literature that uses Assumption 1.1.2 as one of the basic assumptions

[21,24,26,36–42]. For example, [24,36] solve the leader-follower output regulation problem of linear heterogeneous multi-agent systems over acyclic graphs (*i.e.*, graphs with no loop). In [36], it is proved that the extended agents (augmenting the agent dynamics with the controller dynamics), need to share an invariant subspace to achieve output regulation. Under a set of standard assumptions, it is proven that the stability of the global closed-loop system matrix is necessary and sufficient for the output regulation, which reduces to a single-agent stability criterion by the acyclic assumption.

Considering an acyclic graph eliminates many possible scenarios for the communication topology. In [21], this assumption is removed and instead it is assumed that the nominal dynamics of the uncertain agents are identical. The communication graph in [21] is assumed to contain a spanning tree and loops are allowed. Both distributed dynamic state-feedback and distributed dynamic output-feedback are suggested for the output regulation problem. The leader-follower output regulation of homogeneous agents is also considered in [37]. Even though in the original paper, the authors assume neither Assumption 1.1.1 nor Assumption 1.1.2, the paper's assumptions have been modified in [26] and it is shown that by Assumption 1.1.2, there exists a distributed controller that solves the output regulation problem. Output regulation of homogeneous linear multi-agent systems over switching networks is considered in [38].

A leaderless output regulation of linear heterogeneous SISO (Single Input Single Output) multi-agent systems is considered in [39]. It is assumed that the agents communicate their outputs and hence a pure output-feedback is used. The distributed controller in [39] consists of a type generator, a stable compensator, and a robust regulator. Assuming that the network topology contains a directed spanning tree, the type generators are synchronized to each other *via* the stable compensators. Then, the robust regulator takes care of convergence to the type generator

and handling the uncertainty. The controller has a simple structure when the relative degree is one and becomes more complicated and needs a high-gain feedback for higher relative degrees.

Recently, the leader-follower output regulation of linear heterogeneous multi-agent systems over a general graph is analyzed in [40]. The authors of [40] use a p -copy of the leader and design a dynamic state-feedback controller to achieve output regulation. The output regulation problem in [40] is formulated as an H_∞ control problem and a version of the small gain theory is used to derive the stability condition. In [43], the properties of Positive Real (PR) system are used for output regulation of linear heterogeneous multi-agent systems with an equal number of inputs and outputs.

The output regulation of a class of non-introspective agents is considered in [44]. The controller design contains three main steps. In the first step, a similarity transformation is used to reduce the dimension of the problem; for example to remove the unobservable modes. In the second step, a state-feedback is constructed to regulate the tracking error. Since not all of the states are available for the feedback, a high-gain observer is constructed in the third step. In [45], a new form of the internal model is introduced and combined with some robust techniques to achieve output regulation of heterogeneous uncertain multi-agent systems. The suggested methodology uses a distributed high-gain state-feedback and a distributed high-gain observer to achieve output regulation for a class of minimum phase systems. Similarly, [46] uses the same internal model for the output regulation of switching heterogeneous multi-agent systems with uncertainty.

1.1.3 H_∞ output regulation

In the leader-follower output regulation problem, if the agents are subject to modeled disturbances, it is common to stack both the state of the leader and the distur-

bance model together and name the resulting system an *exo-system*. The Internal Model Principle (IMP) is used to address such a problem [27]. When the agents are subject to external unmodeled disturbances, the problem is called an H_∞ *leader-follower output regulation problem*, which cannot be solved by the IMP alone and H_∞ methods are required [47–51].

A significant part of the research on H_∞ control of multi-agent systems refers to H_∞ state synchronization of homogeneous agents [52–55]. A common approach there is to design either a cooperative state-feedback [52,54–56] or an output-feedback [53], satisfying a set of LMIs for fixed or dynamic communication graphs [53], possibly with a time-delay [52].

It is also possible to design an optimal controller for systems subject to additive disturbances *via* the concept of zero-sum games [57–60]. The theory of zero-sum games is well developed for single-agent systems [58,59] and it covers both state-feedback [59], and output-feedback [61]. Zero-sum games are also used for H_∞ leader-follower state synchronization of homogeneous agents [49,50]. Considering a local state synchronization error as the state variable for each player, the H_∞ state synchronization problem is formulated as a zero-sum graphical game in [49]. A coupled local cost function J_i is defined for each player where J_i depends on the control signal and the disturbance of player i and the control signals and the disturbances of the players in the neighborhood. An online algorithm is suggested in [49] which guarantees convergence to a Nash Equilibrium (NE) and attenuation of the disturbance with a selected $\mathcal{L}_2$ gain. In comparison, [50] considers a global cost function and aims to design a distributed output-feedback to attenuate the disturbance with a pre-specified $\mathcal{L}_2$ gain.

The H_∞ output regulation of heterogeneous agents is studied in [51]. The design procedure requires a solution to the regulator equation (1.6) (Assumption 1.1.1) and the agents need to communicate the internal state of their dynamic controller

to achieve H_∞ output regulation. The controller is designed *via* a solution to a Riccati equation. The H_∞ output regulation of heterogeneous introspective agents (who have access to their internal states and outputs) under external disturbances is also considered in [48]. The agents need to communicate their outputs and the internal states of their dynamic controllers for the H_∞ output regulation. It is shown in [48] that the suggested controller attains a perfect regulation when there is no disturbance and a bounded $\mathcal{L}_2$ gain from the disturbance to output synchronization error in its presence.

1.1.4 Bipartite output regulation

For decades, the problem of reaching consensus among cooperative agents has been studied in the literature [1–5]. In all mentioned references, the agents communicate cooperatively with each other and a graph with positive edge weights is used to represent their cooperative behavior.

However, in a number of contexts, the interaction among agents is not necessarily cooperative. In contexts like social networks, mutual relationships between pairs of individuals can be antagonistic or cooperative [62]. Another context is the marketing or game where two rivals compete over a shared resource [63]. As an engineering context, one can name the opinion forming over coopetition networks [64]. Usually, a signed graph is used to represent the interaction of the agents in those scenarios where an edge weight is positive if the interacting nodes are cooperative and negative if they are competitive. Such a representation leads to a polarization of the agents into two subgroups, where the agents reach an agreement over a variable within each group but the agreed variables are different from one group to another. Such a phenomenon appears in synchronization between two pendulum clocks, salt water oscillator experiments and chaotic systems [64]. Besides, many engineering problems fall within this framework, *e.g.*, analyzing trustworthiness of the nodes in a network

[65,66] and anticipating unanimity of the opinions in a decision/voting process in the presence of stubborn agents [67].

One can define different types of agreement over signed graphs like *modulus consensus* [68], *bipartite consensus* [62,63,69,70], and *sign consensus* [67]. Modulus consensus and bipartite consensus are conceptually close to each other; in both, the agents reach an agreement about the modulus of a variable and the variable's sign could be different among the agents. While the modulus consensus allows convergence to a zero trajectory, the bipartite consensus is meant for convergence to a non-zero trajectory. In the sign consensus, the agreement over the sign of a variable is of interest [67].

The pioneering work of [62] studies the bipartite state consensus of first-order linear and nonlinear systems where the main limiting factors are considering 1- state synchronization and 2- first-order systems. The recent work [71] extends the results of [62] for time-varying communication graphs. In [70,72] and [63], the bipartite state synchronization of a general linear, yet homogeneous multi-agent system is analyzed over directed and undirected graphs respectively. The bipartite output regulation of homogeneous and heterogeneous linear multi-agent systems is studied in [73]. It is required that the agents communicate their states or the state of their dynamic compensators to achieve bipartite regulation.

1.1.5 Multi-party output regulation

In cooperative and bipartite regulation, the agents synchronize to a single or two trajectories respectively. It is natural to consider the feasibility of synchronization to multiple trajectories. Generally speaking, synchronization of the agents to multiple trajectories or *multi-party regulation* can be seen as a generalization of cooperative (single-party) and bipartite (two-party) regulation and it can be found in a number of contexts such as social networks and politics. For example, the users in Facebook

can express their emotions in six different ways or as a political example, the United State has three political parties [74]. Such observations reveal that the interaction among different members in a system is more complicated than only cooperative or friend-foe.

The concept of multi-party regulation is completely new and it is introduced in this thesis for the first time. A relevant approach which contains some aspects of this concept is cluster synchronization [75–78]. This field studies synchronization of homogeneous coupled systems within clusters *via* a protocol which is designed to magnify the effect of the agents within the same cluster and suppress the effect of others. As a result, the agents within a cluster synchronize to a trajectory which is either determined by their initial conditions or is given by a leader node in the cluster. This can be achieved only if the inter-cluster couplings have either a constant sum [76] or a zero sum [75].

1.2 Motivation

We are motivated by a practical example of formation control of linear heterogeneous multi-agent systems. Assume that we are given a mix of mobile robots and UAVs and our task is to design a distributed controller for them such that they form a predefined time-varying pattern and follow a reference trajectory. Throughout this thesis, we show how such an example is formulated as an output regulation problem of linear heterogeneous multi-agent systems and how the developed theories are used to provide a solution to this problem.

Studying output regulation of heterogeneous agents is more challenging than studying state synchronization of homogeneous agents because of the following reasons. 1- Because of the heterogeneity, it is not possible to obtain the state synchronization error and deriving the output synchronization error does not help so much

in solving the problem. 2- The output regulation problem of heterogeneous multi-agent systems requires the internal model principle which is inherently satisfied in the case of homogeneous agents.

This thesis addresses the following questions

1. What are the sufficient conditions for the output regulation of a general linear heterogeneous multi-agent system and how to design the controller?
2. What are the sufficient conditions for the H_∞ output regulation of a general linear heterogeneous multi-agent system subject to additive disturbance? Can we attenuate the disturbance optimally?
3. Is it possible to achieve output synchronization into two trajectories where one of them is the mirror of the another one (bipartite output regulation)? What are the sufficient conditions for it?
4. Is it possible to achieve synchronization into multi-trajectories (multi-party output regulation)? What are the sufficient conditions for it?

1.3 Overview of the Thesis

In this thesis, we investigate different aspects of output regulation of linear heterogeneous multi-agent systems. In each technical chapter, we remove some assumptions that we have made in the previous chapters to give a very general result pertaining to the output regulation problem of linear heterogeneous agents. This step-by-step development helps us to provide a solution to the problem of simultaneous time-varying formation control and reference tracking at the end.

This chapter gives an introduction to output regulation in multi-agent systems and reviews existing results. The rest of this thesis is organized as follows. In Chapter 2, we formulate output regulation problem of linear heterogeneous multi-agent systems

over acyclic graph as a graphical game and analyze the effect of uncertainty in the system. In Chapter 3, we remove the acyclic assumption and obtain a unique sufficient condition for output regulation using state and output-feedbacks *via* Linear Matrix Inequalities (LMIs). In Chapter 4, we introduce disturbances to our system model and assume that the agents are subject to unmodeled disturbances. For such systems, we introduce the concept of H_∞ -output regulation because the IMP alone can not be used for disturbance rejection when disturbance is unmodeled. We suggest two independent approaches namely LMI and graphical game approaches to guarantee H_∞ output regulation. In Chapter 5, we generalize the idea of classical output regulation to bipartite output regulation for agents communicating over a signed graph. In Chapter 6, we introduce the concept of multi-party output regulation over a complex graph. We obtain sufficient condition for multi-party regulation and show an application of multi-party output regulation in the formation control. Conclusions and future works are drawn in Chapter 7. The proof of Theorem 2.6.1 is given in Appendix A.

1.4 Notation and Preliminaries

The following notations will be used throughout this thesis. Let $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ be the set of $n \times m$ real and complex matrices respectively. The argument and the modulus of a complex number z are denoted by $\angle z$ and $|z|$ respectively. The real and imaginary parts of a complex number z are denoted by $Re(z)$ and $Im(z)$ respectively. The imaginary unit is denoted by $\iota = \sqrt{-1}$. I_n and $\mathbf{1}_n$ denote the identity matrix of dimension $n \times n$ and an n -dimensional column vector of 1 respectively. $\mathbf{0}_{n \times m}$ denotes a matrix of zeros with of dimension $n \times m$. We drop the subscript in I_n , $\mathbf{1}_n$, and $\mathbf{0}_{n \times m}$ whenever there is no confusion.

Let $A = [a_{ij}] \in \mathbb{C}^{n \times m}$. We define $|A| = [|a_{ij}|]$. Given $A = [a_{ij}] \in \mathbb{R}^{n \times m}$, let

$B := [A]_{n_1:n_2 \times m_1:m_2} \in \mathbb{R}^{(n_2-n_1+1) \times (m_2-m_1+1)}$ be a matrix formed by rows $n_1, \dots, n_2$ and columns $m_1, \dots, m_2$ of A . The Kronecker product of two matrices A and B is denoted by $A \otimes B$. Let $A_i \in \mathbb{C}^{n_i \times m_i}$ for $i = 1, \dots, N$. The operator $\underset{1:N}{Diag}\{A_i\}$ builds a block diagonal matrix with N diagonal blocks whose i th diagonal block is A_i . The notation $Diag\{A_1, A_2, \dots, A_N\}$ is also used to refer to the same concept when there is no ambiguity. The (semi) positive definiteness constraint on the matrix $P \in \mathbb{R}^{n \times n}$ is formulated as $(P \geq 0) \ P > 0$. For $P > 0$, there exists a unique matrix $N > 0$ such that $P = N^2$. We write $N = \sqrt{P}$ for simplicity. The vectorization of a matrix $A = [a_1, \dots, a_m] \in \mathbb{C}^{n \times m}$, $a_j \in \mathbb{C}^{n \times 1}$ is denoted by $vec(A) = [a_1^T, \dots, a_m^T]^T$. The spectrum of matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $spec(A)$ which is the multiset of its eigenvalues λ_i . The spectral radius of A is denoted as $\rho(A) = \max_{\lambda_i \in spec(A)} |\lambda_i|$. Let $\sigma_i(A)$ be the i th singular value of a matrix $A \in \mathbb{R}^{n \times m}$. Then, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ represent the maximum and minimum singular values of matrix A respectively and we have

$$\bar{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})}. \quad (1.8)$$

The Euclidean norm of a vector a is denoted by $\|a\|_2$. Let $\mathcal{L}_2^n[0, +\infty)$ denote the space of square integrable vectors $x \in \mathbb{R}^n$ over $[0, +\infty)$. For $a \in \mathcal{L}_2^n[0, +\infty)$, the $\mathcal{L}_2$ norm is denoted by the notation $\|a\|_{\mathcal{L}_2}$. We will drop the subscript n in $\mathcal{L}_2[0, +\infty)$ when there is no confusion.

The cardinality of a set V is denoted by $|V|$. The disjoint union of two sets V_1 and V_2 is denoted by $V_1 \dot{\cup} V_2$ where $V_1 \cap V_2 = \emptyset$.

Definition 1.4.1 ([25]). A pair of matrices $M_1 = I_p \otimes \beta$, $M_2 = I_p \otimes \tau$ incorporate a *p-copy* internal model of a square matrix A if (β, τ) is controllable and the minimal polynomial of A divides the characteristic polynomial of β .

1.4.1 Preliminaries on graph theory

Consider graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}})$ with a finite set of $N + 1$ nodes $\bar{V} = \{v_0, v_1, \dots, v_N\}$ and a set of edges $\bar{\mathcal{E}} \subseteq \bar{V} \times \bar{V}$. Note that node v_0 is associated with the leader and the remaining N nodes are associated with the N followers. $\bar{E} = [\bar{\alpha}_{ij}]_{(N+1) \times (N+1)}$ is the *adjacency matrix* with $\bar{\alpha}_{ij} = 1$ if $(v_j, v_i) \in \bar{\mathcal{E}}$ and $\bar{\alpha}_{ij} = 0$ otherwise. $\bar{\alpha}_{0j} = 0, \forall j$; i.e. there is no incoming edge to the leader.

Consider graph $\mathcal{G} = (V, \mathcal{E})$ which is obtained from $\bar{\mathcal{G}}$ by removing node v_0 and its relating edges. This graph represents the interaction among the followers and it has a finite set of N nodes $V = \{v_1, \dots, v_N\}$ and a set of edges $\mathcal{E} \subseteq V \times V$. The adjacency matrix $E = [\alpha_{ij}]_{N \times N}$ is obtained by removing the row and the column of $\bar{E}$ relating to node v_0 . The set of neighbors of node v_i is $N_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. For graph $\mathcal{G}$, define the *in-Degree matrix* D as a diagonal matrix $D = \underset{1:N}{\text{Diag}}\{d_i\}$ with $d_i = \sum_{j \in N_i} \alpha_{ij}$, the weighted in-degree of node i . The *laplacian* matrix for graph $\mathcal{G}$ is defined as $L = D - E$. Matrix $G = \underset{1:N}{\text{Diag}}\{g_i\}$ shows the pinning from the leader to the followers; $g_i = \bar{\alpha}_{i0} = 1$ if there is a link between the leader and follower i and $g_i = \bar{\alpha}_{i0} = 0$ otherwise. We say the graph $\bar{\mathcal{G}}$ is normalized if the matrix $\Lambda = L + G$ has all the diagonal elements equal to one.

A *weak path* $i_1 i_2 \dots i_k i_{k+1}$ is an alternating sequence of nodes $i_r \in V, r = 1, \dots, k$ such that either $(i_r, i_{r+1}) \in \mathcal{E}$ or $(i_{r+1}, i_r) \in \mathcal{E}$. If $\forall r = 1, \dots, k, (i_r, i_{r+1}) \in \mathcal{E}$, then the weak path is a *directed path*. If $i_1 = i_{k+1}$, then the weak path is named a *weak cycle*. A *directed cycle* is defined similarly. If there exists a directed path from node v_i to node v_j then it is said that node v_i is reachable from v_j . A graph $\mathcal{G}$ is *strongly connected* if there exists a path from v_i to v_j for $\forall v_i, v_j \in V, v_i \neq v_j$. A directed graph is a *connected directed tree* where every node except one node, called the root, has in-degree equal to one. The root node has in-degree equal to zero. A graph has a spanning tree if a subset of its edges forms a directed tree over all nodes.

Chapter 2

Output Regulation of Linear Heterogeneous Agents over Acyclic Graphs

This chapter addresses output regulation of linear heterogeneous multi-agent systems communicating over an acyclic graph in a differential graphical game sense. To be more specific, consider a group of $N + 1$ agents whose acyclic communication graph is fixed and directed. The group consists of N followers and one leader as the exo-system. We use the internal model principle and incorporate the unstable dynamic of the leader into the agents' dynamics. We show that output regulation of heterogeneous linear multi-agent system is equivalent to state stabilization of a local system. The notation of the graphical game is derived for the output regulation problem and we show that the response of the graphical game is obtained through a solution to the coupled Hamilton-Jacobi equations. Both offline and online algorithms are suggested for the solution to the coupled Hamilton-Jacobi equations. It is proved that the online solution converges to a Nash Equilibrium (NE). Besides, we analyze the robustness of the graphical game to the multiplicative uncertainty and

obtain an upper bound for the uncertainty such that the uncertain system remains stable under the graphical control.

This chapter has three main contributions. Firstly, we formulate the output regulation problem in the graphical game framework. It is shown that this formulation results in an NE profile. Secondly, we present the robustness analysis of the graphical game and derive an upper bound for the multiplicative uncertainty. Our results show that the graphical formulation has an infinite gain margin. More specifically, the robust property and infinite gain margin of Linear Quadratic Regulator (LQR) method which are studied in [79] and [80], are satisfied in the graphical formulation as well. Finally, we provide a Policy Iteration (PI) algorithm as an offline solution to the coupled Hamilton-Jacobi equations resulting from the graphical game formulation. It is shown that this algorithm converges to an NE when all agents update their control policies. The PI algorithm provides the basis for the online solution which converges to an NE. The results of this chapter have been published in [81,82].

This chapter is organized as follows. Section 2.1 defines the output regulation problem. It is shown in Section 2.2 that the output regulation problem for a group of heterogeneous agents over an acyclic graph is solved through a local controller design. Section 2.3 discusses optimal output regulation in the game theoretical approach. Section 2.4 is devoted to the robustness analysis of the graphical games. Sections 2.5 and 2.6 address the offline and online solutions to output regulation of heterogeneous linear agents in the graphical game framework. The simulation results are represented in Section 2.7 that show the effectiveness of the proposed method.

2.1 Problem Formulation

In this section, we give the definition of the output regulation problem of linear heterogeneous multi-agent systems over acyclic graph and bring the necessary assumptions for the existence of a solution to this problem. We use a setup similar to [24] and define a new concept of output regulation games in section 2.3.

Consider a group of $N + 1$ heterogeneous agents containing N followers and one leader. The agents communicate over $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}})$ defined in Subsection 1.4.1 and their models are described as follows

$$\dot{x}_0 = A_0 x_0, \tag{2.1}$$

$$y_0 = C_0 x_0, \tag{2.2}$$

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{2.3}$$

$$y_i = C_i x_i, \tag{2.4}$$

$$e_i = y_i - y_0 = C_i x_i - C_0 x_0, \tag{2.5}$$

$$\tag{2.6}$$

in which $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^p$, $i = 1, \dots, N$ denote the state, control signal and output for agent i , $x_0 \in \mathbb{R}^{n_0}$ and $y_0 \in \mathbb{R}^p$ are the state and output of the leader and e_i is the regulated error that is desired to be zero. Define an auxiliary system for each follower as

$$\dot{z}_i = G_1 z_i + G_2 e_{iv}, \tag{2.7}$$

in which, $z_i \in \mathbb{R}^{n_z}$ and the pair (G_1, G_2) is a p -copy of the leader. Since the leader's output is not measurable for all agents, the virtual regulating error is introduced and used in (2.7)

$$e_{iv} = \frac{1}{d_i + g_i} \left[\sum_{j=1}^N \alpha_{ij}(y_i - y_j) + g_i(y_i - y_0) \right] = y_i - r_i, \quad (2.8)$$

in which r_i is the reference signal for agent i and is defined as follows

$$r_i = \frac{1}{d_i + g_i} \left[\sum_{j=1}^N \alpha_{ij}y_j + g_i y_0 \right]. \quad (2.9)$$

From (2.5) and (2.8), the global regulated error e and the global virtual regulating error e_v are given by

$$e = [e_1^T, e_2^T, \dots, e_i^T, \dots, e_N^T]^T, \quad (2.10)$$

$$e_v = \left[\text{Diag}\left\{\frac{1}{d_i + g_i}\right\}_{1:N} (L + G) \right] \otimes I_p e. \quad (2.11)$$

We make the following assumptions through this chapter [24].

Assumption 2.1.1. *The leader is marginally stable; i.e. $\text{Re}(\lambda_0) = 0$, $\forall \lambda_0 \in \text{spec}(A_0)$ and no repeated pole on the imaginary axis is allowed.*

Assumption 2.1.2. *The pair (A_i, B_i) is stabilizable and the pair (A_i, C_i) is detectable for all $i = 1, \dots, N$.*

Assumption 2.1.3.

$$\text{rank} \begin{bmatrix} A_i - \lambda_0 I & B_i \\ C_i & 0 \end{bmatrix} = n_i + p, \quad \lambda_0 \in \text{spec}(A_0). \quad (2.12)$$

Assumption 2.1.4. *The directed graph $\bar{\mathcal{G}}$ is acyclic and it contains a spanning tree with the leader as its root node.*

Based on Assumption 2.1.1, we only consider a leader which is marginally stable and its output is bounded. Formulating the problem with a strictly stable leader ($Re(\lambda_0) < 0, \forall \lambda_0 \in spec(A_0)$) is equivalent to the state stabilization of all agents and is trivial. Hence, a marginally stable leader is considered. Assumptions 2.1.2-2.1.3 are some standard assumptions for the output regulation. Because of Assumption 2.1.4, all eigenvalues of $(L + G)$ have positive real parts (Lemma 4 of [83]), hence

$$\lim_{t \rightarrow \infty} e_i(t) = 0 \Leftrightarrow \lim_{t \rightarrow \infty} e_{iv}(t) = 0, \quad i = 1, \dots, N. \quad (2.13)$$

It means for a graph satisfying Assumption 2.1.4, the regulated output error goes to zero if and only if the virtual regulated error goes. Now, we define the output regulation problem.

Problem 2.1.1. (*The Output Regulation Problem-*) Consider a group of N heterogeneous agents and their auxiliary systems defined in (2.1)-(2.7). Design

$$u_i = K_{x_i} x_i + K_{z_i} z_i, \quad (2.14)$$

such that $e_i \rightarrow 0$ for $i = 1, \dots, N$ or equivalently the output regulation is achieved.

2.2 Local Controller Design

In this section, a distributed solution to the output regulation problem is derived. Note that Lemma 2.2.1 is very similar to Theorem 3.1 of [24], with a small difference in the definition of the local error. Since the proof of this lemma is constructive for

our approach, we mention it explicitly here. Define

$$\begin{aligned}\tilde{Y} &:= \underset{1:N}{Diag}\{Y_i\} \{Y_i = A_i, B_i, C_i, K_{x_i}, K_{z_i}\}, i = 1, \dots, N, \\ \tilde{\chi} &:= I_N \otimes \chi, \{\chi = A_0, G_2\}, \tilde{H} := \left(\underset{1:N}{Diag}\left\{\frac{1}{d_i + g_i}\right\}(L + G) \right) \otimes I_p.\end{aligned}$$

Using the control signal (2.14), the overall closed-loop system containing all agents (2.3) and their auxiliary systems (2.7) is

$$\begin{aligned}\dot{\nu} &= A_c \nu + B_c \underline{x}_0, \quad \nu = \begin{bmatrix} x_1^T, x_2^T, \dots, x_N^T, z_1^T, z_2^T, \dots, z_N^T \end{bmatrix}^T, \\ \dot{\underline{x}}_0 &= \tilde{A}_0 \underline{x}_0, \\ e &= \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} \nu - I_N \otimes C_0 \underline{x}_0,\end{aligned}\tag{2.15}$$

in which $\underline{x}_0 = \underbrace{[x_0^T, x_0^T, \dots, x_0^T]^T}_N$ and

$$A_c = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_x & \tilde{B}\tilde{K}_z \\ \tilde{G}_2\tilde{H}\tilde{C} & I_N \otimes G_1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ -\tilde{G}_2\tilde{H}(I_N \otimes C_0) \end{bmatrix}.\tag{2.16}$$

Note that A_c can be decomposed as

$$A_c = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{G}_2\tilde{H}\tilde{C} & I_N \otimes G_1 \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_x & \tilde{K}_z \end{bmatrix} = A_{op} + B_{op}K.\tag{2.17}$$

The aforementioned form is useful in the following results.

Theorem 2.2.1 ([24]). *Consider Assumptions 2.1.1-2.1.4. Suppose A_c is Hurwitz. Then the dynamic controller (2.14), solves Problem 2.1.1.*

According to Theorem 2.2.1, stability of A_c is a globally sufficient condition for the existence of a solution to Problem 2.1.1. The next lemma shows that by considering

an acyclic graph, A_c is transformed to a lower or upper triangular matrix and a local stability criterion is obtained.

Lemma 2.2.1 ([24]). *Stabilizing the pair (A_{op}, B_{op}) is equivalent to stabilizing the pair $(\bar{A}_i, \bar{B}_i)$ for $i = 1, \dots, N$*

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ G_2 C_i & G_1 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}. \quad (2.18)$$

Proof. Similar to [24], since Assumption 2.1.4 holds, rename the agents in (2.3) and (2.7) such that $i > j$ if $(v_j, v_i) \in \mathcal{E}$. Select $T = [T_1; T_2; \dots; T_N]$, in which T_k is a matrix defined as follows:

$$T_k = \begin{bmatrix} i_{(\sum_{q=1}^{k-1} n_q + 1)} \\ \vdots \\ i_{(\sum_{q=1}^k n_q)} \\ i_{(\sum_{q=1}^N n_q + (k-1)n_z + 1)} \\ \vdots \\ i_{(\sum_{q=1}^N n_q + kn_z)} \end{bmatrix}, \quad (2.19)$$

in which i_r denotes r th row of $I_{(\sum_{q=1}^N n_q + Nn_z)}$. Using T as a similarity transformation, we have $\tilde{A}_{op} = T A_{op} T^{-1}$, which is a lower block triangular matrix with diagonal block $\bar{A}_i$. Also, $\tilde{B}_{op} = T B_{op}$ is a block diagonal matrix with diagonal block $\bar{B}_i$ and this completes the proof. $\square$

Remark 2.2.1. *Lemma 2.2.1 gives a distributed criterion for the output regulation problem.*

Remark 2.2.2. *Denoting the new state variable w_i , then the state space represen-*

tation of the pair $(\bar{A}_i, \bar{B}_i)$ is

$$\begin{aligned} \dot{w}_i &= \begin{bmatrix} A_i & 0 \\ G_2 C_i & G_1 \end{bmatrix} w_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_{ps_i} := \bar{A}_i w_i + \bar{B}_i u_{ps_i}, \\ y_{w_i} &= [C_i, 0] w_i := \bar{C}_i w_i, \end{aligned} \quad (2.20)$$

which is the same as the system defined in (2.3) and (2.7) when the reference signal r_i (2.9) is disconnected. We call (2.20) as the pseudo system for agent i . One immediate result is that the pair $(\bar{A}_i, \bar{B}_i)$ can be controlled through u_{ps_i} such that the closed-loop system becomes stable. Once $u_{ps_i} = K_i w_i$ is designed optimally, $u_i = K_i \begin{bmatrix} x_i \\ z_i \end{bmatrix}$ will be used to control the original system in (2.3) and (2.7) to solve Problem 2.1.1.

2.3 Distributed Output Regulation Games

In the output regulation of heterogeneous multi-agent games, we define the cost function J_i for each pseudo system associating with player i as

$$\begin{aligned} J_i(w_i(0), u_{ps_i}, u_{ps_{-i}}) &= \int_0^{+\infty} L_i dt \\ &= \int_0^{+\infty} \{y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j}\} dt \end{aligned} \quad (2.21)$$

in which $Q_{ii} > 0$, $R_{ii} > 0$ and $R_{ij} > 0$ for $i = 1, \dots, N$ and $j \in N_i$. Let $u_{ps_{-i}}$ be the vector of control inputs of the neighbors of node i ; i.e., $u_{ps_{-i}} = \{u_{ps_j}, j \in N_i\}$. Given policies u_{ps_i} and $u_{ps_{-i}}$ the value function for each agent i is

$$V_i(w_i(t)) = \int_t^{+\infty} y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} dt. \quad (2.22)$$

Each agent selects its policy assuming that other agents have already selected their desired policies and those are fixed. Hence V_i is selected to be a function of w_i only. Now we define an admissible control policy.

Definition 2.3.1. (Admissible Control Policy [84]): A control policy u_{ps_i} is admissible if it is continuous, $u_{ps_i}(0) = 0$, it stabilizes the pseudo system (2.20) and the value function (2.22) is finite.

Problem 2.3.1. (The Optimal Output Regulation Problem-) Consider a group of N heterogeneous agents defined in (2.3) -(2.7) and their pseudo systems defined in (2.20). Assume the cost function J_i associated with each pseudo system as (2.21). Achieve the output regulation through the game theoretical optimal design of u_{ps_i} from the set of admissible controls for stabilizing (2.20).

According to Remark 2.2.2, once the control $u_{ps_i} = K_i w_i$ stabilizes (2.20), the output regulation is achieved by using the control signal $u_i = K_i[x_i^T, z_i^T]^T$ in the original system in (2.3) and (2.7). Introducing the Hamiltonian for the constrained optimization problem as

$$H_i = y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} + \frac{\partial V_i}{\partial w_i} (\bar{A}_i w_i + \bar{B}_i u_{ps_i}). \quad (2.23)$$

Employing the stationary condition, one obtains the optimal control policy [79]

$$\frac{\partial H_i}{\partial u_{ps_i}} = 0 \rightarrow u_{ps_i}^* = -\frac{1}{2} R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i}{\partial w_i}. \quad (2.24)$$

When the value function is finite, using the Bellman equation, a differential equiv-

alent to the value integral is given in terms of the Hamiltonian function

$$H_i = y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} + \frac{\partial V_i^{*T}}{\partial w_i} (\bar{A}_i w_i + \bar{B}_i u_{ps_i}) = 0. \quad (2.25)$$

where V_i^* denotes the optimal value. Replacing the optimal control policy (2.24) in (2.25) yields to the coupled Hamilton-Jacobi Equation

$$\begin{aligned} y_{w_i}^T Q_{ii} y_{w_i} + \frac{1}{4} \left(\frac{\partial V_i^*}{\partial w_i} \right)^T \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial w_i} + \frac{1}{4} \sum_{j \in N_i} \left(\frac{\partial V_j^*}{\partial w_j} \right)^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T \left(\frac{\partial V_j^*}{\partial w_j} \right) + \\ + \frac{\partial V_i^{*T}}{\partial w_i} (\bar{A}_i w_i - \frac{1}{2} \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial w_i}) = 0. \end{aligned} \quad (2.26)$$

In the graphical game, we are interested in the convergence to Nash Equilibrium (NE). To define NE, first we need to defined the *best response*.

Definition 2.3.2. (*Best Response*)-For agent i , $u_{ps_i}^*$ is the best response to the fixed policies $u_{ps_{-i}}$ if

$$J_i(u_{ps_i}^*, u_{ps_{-i}}) \leq J_i(u_{ps_i}, u_{ps_{-i}}), \quad (2.27)$$

for all admissible policies u_{ps_i} for agent i [85].

Definition 2.3.3. (*Nash Equilibrium*)-A global policy profile $\{u_{ps_1}^*, u_{ps_2}^*, \dots, u_{ps_N}^*\}$ constitutes a Nash Equilibrium if $u_{ps_i}^*$ is the best response to $u_{ps_{-i}}^*$ for $i = 1, \dots, N$ [85].

Now, we show that the solution to the Hamilton-Jacobi-Bellman equation provides the optimal control policy and the convergence to an NE. To show this, we need the following Lemma.

Lemma 2.3.1. *Suppose that V_i is the corresponding solution to (2.25). Consider $u_{ps_i}^*$ as (2.24). Then for any admissible control policy*

$$\begin{aligned} H_i\left(\frac{\partial V_i}{\partial w_i}, u_i, u_{-i}\right) &= H_i\left(\frac{\partial V_i}{\partial w_i}, u_i^*, u_{-i}^*\right) + (u_{ps_i} - u_{ps_i}^*)^T R_{ii}(u_{ps_i} - u_{ps_i}^*) \\ &\quad + \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij}(u_{ps_j} - u_{ps_j}^*) + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij}(u_{ps_j} - u_{ps_j}^*). \end{aligned} \quad (2.28)$$

Proof. By adding and subtracting the optimal control u_i^* for agent i and its neighbors

$$\begin{aligned} H_i\left(\frac{\partial V_i}{\partial w_i}, u_i, u_{-i}\right) &= H_i\left(\frac{\partial V_i}{\partial w_i}, u_i^*, u_{-i}^*\right) + u_{ps_i}^T R_{ii} u_{ps_i} - u_{ps_i}^{*T} R_{ii} u_{ps_i}^* + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} \\ &\quad - \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j}^* + \frac{\partial V_i}{\partial w_i}^T \bar{B}_i u_{ps_i} - \frac{\partial V_i}{\partial w_i}^T \bar{B}_i u_{ps_i}^* \\ &= H_i\left(\frac{\partial V_i}{\partial w_i}, u_i^*, u_{-i}^*\right) + (u_{ps_i} - u_{ps_i}^*)^T R_{ii}(u_{ps_i} - u_{ps_i}^*) + 2u_{ps_i}^{*T} R_{ii} u_{ps_i} \\ &\quad - 2u_{ps_i}^{*T} R_{ii} u_{ps_i}^* + \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij}(u_{ps_j} - u_{ps_j}^*) \\ &\quad + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j} - 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j}^* + \frac{\partial V_i}{\partial w_i}^T \bar{B}_i (u_{ps_i} - u_{ps_i}^*). \end{aligned}$$

Given the optimal control in (2.24), the above equation is simplified to

$$\begin{aligned} H_i\left(\frac{\partial V_i}{\partial w_i}, u_i, u_{-i}\right) &= H_i\left(\frac{\partial V_i}{\partial w_i}, u_i^*, u_{-i}^*\right) + (u_{ps_i} - u_{ps_i}^*)^T R_{ii}(u_{ps_i} - u_{ps_i}^*) + 2u_{ps_i}^{*T} R_{ii} u_{ps_i} \\ &\quad - 2u_{ps_i}^{*T} R_{ii} u_{ps_i}^* + \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij}(u_{ps_j} - u_{ps_j}^*) \\ &\quad + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j} - 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j}^* - 2u_{ps_i}^{*T} (u_{ps_i} - u_{ps_i}^*) \\ &= H_i\left(\frac{\partial V_i}{\partial w_i}, u_i^*, u_{-i}^*\right) + (u_{ps_i} - u_{ps_i}^*)^T R_{ii}(u_{ps_i} - u_{ps_i}^*) \\ &\quad + \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij}(u_{ps_j} - u_{ps_j}^*) + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij}(u_{ps_j} - u_{ps_j}^*). \end{aligned}$$

□

Theorem 2.3.1. *Consider Problem 2.3.1 and Assumptions 2.1.1-2.1.4. Let V_i^* be a smooth solution to (2.26) and $u_{ps_i}^*$ be given by (2.24) in terms of this solution. Then*

1. *the closed-loop system of (2.20) is stable.*
2. *$[u_{ps_1}^*, u_{ps_2}^*, \dots, u_{ps_N}^*]$ constitute an NE and the optimal value of the cost function is*

$$J_i^* = V_i^*(w_i(0)). \quad (2.29)$$

Proof. 1. Because V_i^* is a smooth solution to (2.26), it satisfies (2.25). Rewrite (2.25) as $H_i = L_i + \dot{V}_i^* = 0$. Then, $\dot{V}_i^* = -L_i \leq 0$. Since L_i is a quadratic positive function, $\dot{V}_i^*$ is always negative and hence V_i^* is the candidate Lyapunov function for w_i and (2.20) is stable.

2. Based on the proof of part 1, (2.20) is stable. For any smooth function $V_i(w_i)$ such that $V_i(0) = 0$ and $V_i(w_i(+\infty)) = 0$, one can write the performance index as

$$J_i = \int_0^{+\infty} L_i dt + \int_0^{+\infty} \dot{V}_i dt - V_i(w_i(+\infty)) + V_i(w_i(0)).$$

Using the result of Lemma 2.3.1

$$\begin{aligned} J_i &= \int_0^{+\infty} (L_i + \dot{V}_i) dt + V_i(w_i(0)) \\ &= \int_0^{+\infty} \{H_i^* + (u_{ps_i} - u_{ps_i}^*)^T R_{ii}(u_{ps_i} - u_{ps_i}^*) + \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij}(u_{ps_j} - u_{ps_j}^*) \\ &\quad + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij}(u_{ps_j} - u_{ps_j}^*)\} dt + V_i(w_i(0)). \end{aligned}$$

Now, suppose $V_i = V_i^*$ is a smooth solution to (2.26) then, $H_i^* = 0$. Assume

agent i uses the best response strategy to the fixed strategies u_{ps-i} of its neighbors

$$J_i(u_{ps_i}^*, u_{ps-i}) = \int_0^{+\infty} \left\{ \sum_{j \in N_i} (u_{ps_j} - u_{ps_j}^*)^T R_{ij} (u_{ps_j} - u_{ps_j}^*) \right. \\ \left. + 2 \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} (u_{ps_j} - u_{ps_j}^*) \right\} dt + V_i^*(w_i(0)).$$

Clearly, $J_i(u_{ps_i}^*, u_{ps-i}) \leq J_i(u_{ps_i}, u_{ps-i})$. Because every agent is in the best response to its neighbors, the system is in an NE. The optimal value of the cost function is obtained by using the optimal control for all agents, $u_{ps_i} = u_{ps_i}^*, \forall i = 1, \dots, N \rightarrow J_i(u_{ps_i}^*, u_{ps-i}^*) = V_i^*(w_i(0))$.

□

Corollary 2.3.1. *Suppose there exists a solution to (2.26) that is quadratic in the state w_i . Then the conclusions of Theorem 2.3.1 still hold. Moreover, the optimal control (2.24) is of the form (2.14).*

Proof. The proof follows from Theorem 2.3.1.

□

2.4 Robustness Analysis of the Graphical Games

In this section, the robustness of the graphical game formulation is analyzed. Suppose that the nominal system (2.20) is used for optimal design, however, the original system is exposed to an unstructured uncertainty

$$\dot{w}_i = (\bar{A}_i + \Delta \bar{A}_i)w_i + (\bar{B}_i + \Delta \bar{B}_i)u_{ps_i}. \quad (2.30)$$

Let $g_i(s) = (sI - \bar{A}_i)^{-1}\bar{B}_i$ show the transfer function from input u_{ps_i} to the state for the nominal plant. Then, the actual transfer function due to the uncertainty in matrix parameters is

$$\begin{aligned}
g'_i(s) &= (sI - \bar{A}_i - \Delta \bar{A}_i)^{-1} (\bar{B}_i + \Delta \bar{B}_i) \\
&= [(sI - \bar{A}_i) (I - (sI - \bar{A}_i)^{-1} \Delta \bar{A}_i)]^{-1} (\bar{B}_i + \Delta \bar{B}_i) \\
&= \left[(I - (sI - \bar{A}_i)^{-1} \Delta \bar{A}_i)^{-1} (sI - \bar{A}_i)^{-1} \right] (\bar{B}_i + \Delta \bar{B}_i) \\
&\approx [(I + (sI - \bar{A}_i)^{-1} \Delta \bar{A}_i) (sI - \bar{A}_i)^{-1}] (\bar{B}_i + \Delta \bar{B}_i) \\
&\approx (sI - \bar{A}_i)^{-1} \bar{B}_i + (sI - \bar{A}_i)^{-1} \Delta \bar{B}_i + (sI - \bar{A}_i)^{-1} \Delta \bar{A}_i (sI - \bar{A}_i)^{-1} \bar{B}_i,
\end{aligned}$$

which is obtained by neglecting the second order variations. Then, $g'_i(s)$ reads

$$g'_i(s) = g_i(s) + \Delta g_i(s), \quad (2.31)$$

where,

$$\Delta g_i(s) = (sI - \bar{A}_i)^{-1} \Delta \bar{B}_i + (sI - \bar{A}_i)^{-1} \Delta \bar{A}_i (sI - \bar{A}_i)^{-1} \bar{B}_i. \quad (2.32)$$

Observe that the additive uncertainty (2.31) can be written in the form of multiplicative uncertainty as well

$$g'_i(s) = (I + \Delta g_i(s) g_i^+) g_i(s) = (I + M_i(s)) g_i(s), \quad (2.33)$$

where g_i^+ is Moore - Penrose pseudoinverse of $g_i(s)$. Hence, we only consider the multiplicative uncertainty. Now, assume that the uncertainty satisfies an upper known bound

$$\bar{\sigma}(M_i(\iota w)) \leq l_i(w) \quad (2.34)$$

with $l_i(\omega)$ known for all frequencies $0 \leq \omega < +\infty$. Suppose that the control signal u_{ps_i} is designed for the nominal plant g_i based on the solution to the coupled Hamilton Jacobi (2.26) in the graphical game framework. We want to derive a stability condition for the uncertain plant g'_i under the same controller. To accomplish this, first we need to show that the Hamilton-Jacobi equations (2.26) are equivalent to the coupled Riccati equations. Let

$$\left[\frac{\partial V_1}{\partial w_1}^T, \dots, \frac{\partial V_N}{\partial w_N}^T \right]^T = Pw, \quad (2.35)$$

where $w = [w_1^T, w_2^T, \dots, w_i^T, \dots, w_N^T]^T$ is the global state vector of w_i and $P = \text{Diag}\{P_i\}_{1:N} > 0$ is a positive definite matrix whose diagonal blocks are derived from $\frac{\partial V_i}{\partial w_i} = P_i w_i$.

The global control vector is

$$u_{ps} = [u_{ps_1}^T, \dots, u_{ps_i}^T, \dots, u_{ps_N}^T]^T = -\frac{1}{2} \text{Diag}\{R_{ii}^{-1} \bar{B}_i^T\}_{1:N} Pw. \quad (2.36)$$

We take the Hamiltonian equation in (2.25) and write it with respect to the global variables. The last term in (2.25) reads

$$\begin{aligned} \frac{\partial V_i}{\partial w_i}^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}) &= w^T P \left(\begin{bmatrix} 0 & \dots & 0 \\ 0 & \bar{A}_i & 0 \\ \vdots & \dots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ 0 & \frac{-1}{2} \bar{B}_i R_{ii}^{-1} \bar{B}_i^T & 0 \\ \vdots & \dots & \ddots \end{bmatrix} P \right) w \\ &= w^T P \bar{A}_{ic} w. \end{aligned} \quad (2.37)$$

Besides, the second and third terms in (2.25) can be written as

$$u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} = u_{ps}^T \begin{bmatrix} \alpha_{i1} R_{i1} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & R_{ii} \\ & & & \ddots \\ & & & & \alpha_{iN} R_{iN} \end{bmatrix} u_{ps}. \quad (2.38)$$

Defining the following matrices,

$$\begin{aligned} \bar{Q}_{ii} &:= \begin{bmatrix} 0 & \dots & 0 \\ 0 & 2\bar{C}_i^T Q_{ii} \bar{C}_i & 0 \\ \vdots & \dots & \ddots \end{bmatrix}, \quad \hat{A}_i := \begin{bmatrix} 0 & \dots & 0 \\ 0 & \bar{A}_i & 0 \\ \vdots & \dots & \ddots \end{bmatrix}, \\ \bar{R}_i &:= \frac{1}{2} \text{Diag}_{1:N} \{ \bar{B}_i R_{ii}^{-1} \} \begin{bmatrix} \alpha_{i1} R_{i1} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & R_{ii} \\ & & & \ddots \\ & & & & \alpha_{iN} R_{iN} \end{bmatrix} \text{Diag}_{1:N} \{ R_{ii}^{-1} \bar{B}_i^T \}, \\ \hat{R}_i &:= \frac{1}{2} \text{Diag}_{1:N} \{ \bar{B}_i R_{ii}^{-1} \} \begin{bmatrix} -\alpha_{i1} R_{i1} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & R_{ii} \\ & & & \ddots \\ & & & & -\alpha_{iN} R_{iN} \end{bmatrix} \text{Diag}_{1:N} \{ R_{ii}^{-1} \bar{B}_i^T \}. \end{aligned} \quad (2.39)$$

Then we have

$$P^T \bar{A}_{ic} + \bar{A}_{ic}^T P + \bar{Q}_{ii} + P^T \bar{R}_i P = 0,$$

$$P^T \hat{A}_i + \hat{A}_i^T P + \bar{Q}_{ii} - P^T \hat{R}_i P = 0.$$

We summarize the aforementioned results in the following proposition.

Proposition 2.4.1. *Hamilton-Jacobi equations (2.26) are equivalent to the coupled Riccati equation in closed-loop form*

$$P^T \bar{A}_{ic} + \bar{A}_{ic}^T P + \bar{Q}_{ii} + P^T \bar{R}_i P = 0, \quad (2.40)$$

or the open-loop form

$$P^T \hat{A}_i + \hat{A}_i^T P + \bar{Q}_{ii} - P^T \hat{R}_i P = 0. \quad (2.41)$$

Remark 2.4.1. *Equations (2.40) and (2.41) provide a compact representation of the Hamilton-Jacobi equation (2.26). Even though matrix P is a block diagonal matrix of order N , (2.40) and (2.41) depend only on $|N_i| + 1$ diagonal blocks of it. To clarify this point, note that the only part that interconnects agent i to other agents is through $\bar{R}_i$ and $\hat{R}_i$ which are both block diagonal matrices and they have a nonzero diagonal block for i and $j \in N_i$. Hence the suggested solution has a distributed structure.*

Motivated by Corollary 2.3.1, we will show that the solution to the Riccati equation has the robustness property.

Theorem 2.4.1. *Consider the system defined in (2.20). Suppose $u_{ps_i} = -\frac{1}{2} R_{ii}^{-1} \bar{B}_i^T P_i w_i = -K_i w_i$ and u_{ps_i} is the solution to the coupled Riccati equation (2.41). Then the return difference $D_i(s) = I + K_i(sI - \bar{A}_i)^{-1} \bar{B}_i$ satisfies Kalman inequality*

$$D_i^T(-i\omega) \Gamma_i D_i(i\omega) \geq \Gamma_i, \quad (2.42)$$

for all real ω , where $\Gamma_i = 2R_{ii}$.

Proof. Rewriting (2.41)

$$P_i \bar{A}_i + \bar{A}_i^T P_i - \frac{1}{2} P_i \bar{B}_i R_{ii}^{-1} \bar{B}_i^T P_i + 2 \bar{C}_i^T Q_{ii} \bar{C}_i + \frac{1}{2} \sum_{j \in N_i} P_j^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T P_j = 0.$$

Now, add and subtract sP_i

$$\begin{aligned} & 2 \bar{C}_i^T Q_{ii} \bar{C}_i - P_i(sI - \bar{A}_i) - (-sI - \bar{A}_i^T) P_i - \frac{1}{2} P_i \bar{B}_i R_{ii}^{-1} \bar{B}_i^T P_i \\ & + \frac{1}{2} \sum_{j \in N_i} P_j^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T P_j = 0. \end{aligned}$$

Pre-multiplying the aforementioned equation by $\bar{B}_i^T(-sI - \bar{A}_i^T)^{-1}$, post-multiplying it by $(sI - \bar{A}_i)^{-1} \bar{B}_i$ and adding $2R_{ii}$ to its both sides lead to

$$\begin{aligned} & \bar{B}_i^T(-sI - \bar{A}_i^T)^{-1} \left(2 \bar{C}_i^T Q_{ii} \bar{C}_i + \frac{1}{2} \sum_{j \in N_i} P_j^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T P_j \right) (sI - \bar{A}_i)^{-1} \bar{B}_i + 2R_{ii} \\ & = \left[I + \frac{1}{2} \bar{B}_i^T(-sI - \bar{A}_i^T)^{-1} P_i \bar{B}_i R_{ii}^{-1} \right] 2R_{ii} \left[I + \frac{1}{2} R_{ii}^{-1} \bar{B}_i^T P_i (sI - \bar{A}_i)^{-1} \bar{B}_i \right]. \end{aligned}$$

Noting that $K_i = \frac{1}{2} R_{ii}^{-1} \bar{B}_i^T P_i$, the aforementioned equation reads

$$\begin{aligned} & \bar{B}_i^T(-sI - \bar{A}_i^T)^{-1} \left(2 \bar{C}_i^T Q_{ii} \bar{C}_i + \frac{1}{2} \sum_{j \in N_i} P_j^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T P_j \right) (sI - \bar{A}_i)^{-1} \bar{B}_i + 2R_{ii} \\ & = [I + \bar{B}_i^T(-sI - \bar{A}_i^T)^{-1} K_i^T] 2R_{ii} [I + K_i(sI - \bar{A}_i)^{-1} \bar{B}_i]. \end{aligned}$$

Using the definition of the return difference, we have

$$\begin{aligned} & \bar{B}_i^T(-sI - \bar{A}_i^T)^{-1} \left(2 \bar{C}_i^T Q_{ii} \bar{C}_i + \frac{1}{2} \sum_{j \in N_i} P_j^T \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T P_j \right) \\ & \times (sI - \bar{A}_i)^{-1} \bar{B}_i + 2R_{ii} = D_i^T(2R_{ii})D_i. \end{aligned}$$

The first term in LHS is nonnegative-definite Hermitian. Then (2.42) is concluded. $\square$

Remark 2.4.2. *Based on Theorem 2.1 of [80], one important conclusion of Theorem 2.4.1 is that increase in the steady state mean square error due to the plant disturbance is less or equal to that for open-loop steady state equivalent.*

Lemma 2.4.1. *Consider the system defined in (2.20). Assume that one of the following conditions is satisfied*

1. $m_i = 1$ (single input agents), or

2. $2R_{ii} = I_{m_i}$,

for $i = 1, \dots, N$. Using the control signal $u_{ps_i} = -\frac{1}{2}R_{ii}^{-1}\bar{B}_i^T P_i w_i = -K_i w_i$ which is the solution to the coupled Riccati equation (2.41) results in a decreased sensitivity. The sensitivity function is defined as

$$S_i = D_i^{-1} = (I + K_i(sI - \bar{A}_i)^{-1}\bar{B}_i)^{-1}. \quad (2.43)$$

Proof. We use (2.42) in our proof.

1. If $m_i = 1$, then, R_{ii} is a scalar and we have

$$2R_{ii}\sigma^2(D_i) \geq 2R_{ii} \rightarrow \underline{\sigma}(D_i) \geq 1 \rightarrow \bar{\sigma}(S_i) \leq 1.$$

2. If $2R_{ii} = I$

$$D_i^T(-i\omega)D_i(i\omega) \geq I \rightarrow \underline{\sigma}(D_i) \geq 1 \rightarrow \bar{\sigma}(S_i) \leq 1.$$

□

Remark 2.4.3. *The inequality $\underline{\sigma}(D_i(s)) = \underline{\sigma}(I + K_i(sI - \bar{A}_i)^{-1}\bar{B}_i) \geq 1$ means that the Nyquist plot of the loop gain $K_i(sI - \bar{A}_i)^{-1}\bar{B}_i$ does not enter the unit circle*

around -1. As a result, if the loop gain is multiplied by a constant $k' > 1$, then the Nyquist plot expands and the stability of the system does not change. Hence, similar to the LQR design, the graphical control of the system in (2.20) provides an infinite gain margin.

Remark 2.4.4. *It is common to select R_{ii} diagonal. Hence, the case 2) in Lemma 2.4.1 can be satisfied without loss of generality in the case of diagonal R_{ii} . In the other words, by setting $u_{new_i} = (\sqrt{2R_{ii}})u_{ps_i}$, $B_{new_i} = \bar{B}_i(\sqrt{2R_{ii}})^{-1}$ and $R_{new_{ij}} = (\sqrt{2R_{ii}})^{-T}R_{ij}\sqrt{2R_{ii}}^{-1}$ and using these variables in the system (2.20) and its cost function (2.21), the condition in Lemma 2.4.1 is satisfied.*

According to [79], the stability of the uncertain system is guaranteed if the following condition holds

$$\bar{\sigma}(K_i g_i (I + K_i g_i)^{-1}) \leq \frac{1}{l_i(\omega)}. \quad (2.44)$$

In the next theorem, we obtain the required condition for the stability of the uncertain system.

Theorem 2.4.2. *The uncertain system (2.30) remains stable under the graphical control, for all multiplicative uncertainties satisfying*

$$l_i(\omega) < \frac{1}{2}. \quad (2.45)$$

Proof. To simplify our notation, let $\tau_i = K_i g_i$. From (2.44)

$$l_i < \frac{1}{\bar{\sigma}(\tau_i(I + \tau_i)^{-1})} = \underline{\sigma}[(\tau_i(I + \tau_i)^{-1})^{-1}] = \underline{\sigma}(I + \tau_i^{-1}). \quad (2.46)$$

Using $(I + \tau_i^{-1})^{-1} = I - (I + \tau_i)^{-1}$ we have

$$\bar{\sigma} [(I + \tau_i^{-1})^{-1}] = \bar{\sigma} [I - (I + \tau_i)^{-1}] \leq \bar{\sigma}(I) + \bar{\sigma} [(I + \tau_i)^{-1}] = 1 + \frac{1}{\underline{\sigma}(I + \tau_i)} \leq 2,$$

where we used the result of Lemma 2.4.1. According to the above inequality

$$\underline{\sigma}(I + \tau_i^{-1}) = \frac{1}{\bar{\sigma} [(I + \tau_i)^{-1}]} \geq \frac{1}{2}. \quad (2.47)$$

Then, from, (2.46) and (2.47), the inequality (2.45) is concluded. $\square$

Remark 2.4.5. *Theorem 2.4.2 gives an upper bound for the multiplicative uncertainty such that the system remains stable under the controller design in the graphical game framework using the nominal value. comparing this result to LQR, one can find that both of them result in the same upper bound for the uncertainty.*

2.5 Offline Solution to Output Regulation Games

Reinforcement learning techniques are widely used to solve the single player [86] and the multiple players optimal control [84,87,88]. These methods are based on the adaptive learning laws to estimate the value function and usually grouped into two categories: 1- Policy Iteration (PI) algorithms, and 2- Value iteration algorithms. In what follows, a PI algorithm is presented for the output regulation graphical game (Algorithm 2.1). While PI provides an offline solution to the output regulation graphical games, its structure will be used in the next section to derive an online solution.

The following theorems show the convergence of the PI algorithm for two cases: (1) only agent i updates its policy and (2) all agents update their policies.

Theorem 2.5.1. *Consider the fixed policies u_{ps-i} of the neighbors of agent i . Assume that there exists a stabilizing control policy u_{ps_i} and agent i performs Algo-*

Algorithm 2.1 Policy Iteration algorithm for output regulation of an N -heterogeneous player graphical game

Require: Initialize with an admissible control policy for each pseudo system.

repeat

Step 1. Obtain V_i^k based on u_i^k such that (2.25) holds.

Step 2. Update the control policy by

$$u_{ps_i}^{k+1} = \arg \min_{u_{ps_i}} H_i(w_i, \frac{\partial V_i^k}{\partial w_i}, u_{ps_i}, u_{ps_{-i}}^k), \quad (2.48)$$

which yields to

$$u_{ps_i}^{k+1} = -\frac{1}{2} R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i^k}{\partial w_i}. \quad (2.49)$$

until convergence.

rithm 2.1 while the neighbors do not update their control policies. Then, Algorithm 2.1 converges to the best response u_{ps_i} to $u_{ps_{-i}}$ and the value function converges to the optimal game value $V_i^k \rightarrow V_i^$, where V_i^* is the solution to HJB in (2.26).*

Proof. Consider iteration k of Algorithm 2.1. Based on step 2, $u_{ps_i}^{k+1}$ is obtained from V_i^k . In the next round of the algorithm, V_i^{k+1} is obtained from $u_{ps_i}^{k+1}$ in step 1. We prove the convergence of the algorithm by integrating $\dot{V}_i^k - \dot{V}_i^{k+1}$ along the system trajectory. We know that $a(t) = a(+\infty) - \int_t^{+\infty} \dot{a} dt$, hence

$$\begin{aligned} V_i^k - V_i^{k+1} &= \int_t^{+\infty} \dot{V}_i^{k+1} - \dot{V}_i^k dt \\ &= \int_t^{+\infty} \left(\frac{\partial V_i^{k+1}}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^{k+1}) - \left(\frac{\partial V_i^k}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^{k+1}) dt. \end{aligned} \quad (2.50)$$

Based on (2.25)

$$\begin{aligned} \left(\frac{\partial V_i^{k+1}}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^{k+1}) &= -(y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^{k+1 T} R_{ii} u_{ps_i}^{k+1} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j}), \\ \left(\frac{\partial V_i^k}{\partial w_i} \right)^T \bar{A}_i w_i &= -(y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^k{}^T R_{ii} u_{ps_i}^k + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} + \frac{\partial V_i^k}{\partial w_i} \bar{B}_i u_{ps_i}^k). \end{aligned}$$

Using the above equations in (2.50),

$$\begin{aligned}
V_i^k - V_i^{k+1} &= \int_t^{+\infty} u_{ps_i}^k{}^T R_{ii} u_{ps_i}^k - u_{ps_i}^{k+1}{}^T R_{ii} u_{ps_i}^{k+1} + \left(\frac{\partial V_i^k}{\partial w_i} \right)^T \bar{B}_i (u_{ps_i}^k - u_{ps_i}^{k+1}) dt \\
&= \int_t^{+\infty} u_{ps_i}^k{}^T R_{ii} u_{ps_i}^k - u_{ps_i}^{k+1}{}^T R_{ii} u_{ps_i}^{k+1} - u_{ps_i}^{k+1}{}^T R_{ii} (u_{ps_i}^k - u_{ps_i}^{k+1}) dt \\
&= \int_t^{+\infty} (u_{ps_i}^{k+1} - u_{ps_i}^k)^T R_{ii} (u_{ps_i}^{k+1} - u_{ps_i}^k) dt \geq 0.
\end{aligned}$$

Hence we have $V_i^{k+1} \leq V_i^k$. As a result, V_i^k is a non-increasing function bounded below by zero. Hence V_i^k is convergent as $k \rightarrow \infty$. Let $V_i^{(\infty)} = \lim_{k \rightarrow \infty} V_i^k$. According to principle of optimality

$$\begin{aligned}
V_i^k &\geq \min_{u_{ps_i}} \int_t^{+\infty} y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^T R_{ii} u_{ps_i} + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} dt \\
&= \int_t^{+\infty} y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^{*T} R_{ii} u_{ps_i}^* + \sum_{j \in N_i} u_{ps_j}^T R_{ij} u_{ps_j} dt = V_i^*
\end{aligned}$$

where $u_{ps_i}^*$ is the optimal control and V_i^* is the best response solution to (2.26). Let $k \rightarrow \infty$. Then, $V_i^{(\infty)} \geq V_i^*$ and the algorithm converges to the best response of agent i to the fixed policies $u_{ps_{-i}}$. $\square$

Theorem 2.5.2. *Assume that there exists a stabilizing control policy u_{ps_i} for each agent and all agents update their control policies during each iteration of Algorithm 2.1. Then for small $\rho_{ij} = \bar{\sigma}(R_{jj}^{-1} R_{ij})$, u_{ps_i} converges to an NE and the value function converges to the optimal value function $V_i^k \rightarrow V_i^*$ where V_i^* is the solution to HJB in (2.26).*

Proof. Consider iteration k of the Algorithm 2.1. Based on step 2, $u_{ps_i}^{k+1}$ is obtained from V_i^k . In the next round of the algorithm, V_i^{k+1} is obtained from $u_{ps_i}^{k+1}$ in step 1. We prove the convergence of the algorithm by integrating $\dot{V}_i^k - \dot{V}_i^{k+1}$ along the

system trajectory

$$\begin{aligned}
V_i^k - V_i^{k+1} &= \int_t^{+\infty} \dot{V}_i^{k+1} - \dot{V}_i^k dt \\
&= \int_t^{+\infty} \left(\frac{\partial V_i^{k+1}}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^{k+1}) - \left(\frac{\partial V_i^k}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^k) dt.
\end{aligned} \tag{2.51}$$

based on (2.25)

$$\begin{aligned}
\left(\frac{\partial V_i^{k+1}}{\partial w_i} \right)^T (\bar{A}_i w_i + \bar{B}_i u_{ps_i}^{k+1}) &= -(y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^{k+1 T} R_{ii} u_{ps_i}^{k+1} + \sum_{j \in N_i} (u_{ps_j}^{k+1})^T R_{ij} u_{ps_j}^{k+1}), \\
\left(\frac{\partial V_i^k}{\partial w_i} \right)^T \bar{A}_i w_i &= -(y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^k R_{ii} u_{ps_i}^k + \sum_{j \in N_i} (u_{ps_j}^k)^T R_{ij} u_{ps_j}^k + \frac{\partial V_i^k}{\partial w_i} \bar{B}_i u_{ps_i}^k).
\end{aligned}$$

Using the above equations in (2.51),

$$\begin{aligned}
V_i^k - V_i^{k+1} &= \int_t^{+\infty} u_{ps_i}^k R_{ii} u_{ps_i}^k - u_{ps_i}^{k+1 T} R_{ii} u_{ps_i}^{k+1} + \left(\frac{\partial V_i^k}{\partial w_i} \right)^T \bar{B}_i (u_{ps_i}^k - u_{ps_i}^{k+1}) \\
&\quad + \sum_{j \in N_i} (u_{ps_j}^k)^T R_{ij} u_{ps_j}^k - \sum_{j \in N_i} (u_{ps_j}^{k+1})^T R_{ij} u_{ps_j}^{k+1} dt \\
&= \int_t^{+\infty} u_{ps_i}^k R_{ii} u_{ps_i}^k - u_{ps_i}^{k+1 T} R_{ii} u_{ps_i}^{k+1} - u_{ps_i}^{k+1 T} R_{ii} (u_{ps_i}^k - u_{ps_i}^{k+1}) \\
&\quad + \sum_{j \in N_i} (u_{ps_j}^{k+1} - u_{ps_j}^k)^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) - 2 \sum_{j \in N_i} (u_{ps_j}^{k+1})^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) dt \\
&= \int_t^{+\infty} (u_{ps_i}^{k+1} - u_{ps_i}^k)^T R_{ii} (u_{ps_i}^{k+1} - u_{ps_i}^k) \\
&\quad + \sum_{j \in N_i} (u_{ps_j}^{k+1} - u_{ps_j}^k)^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) - 2 \sum_{j \in N_i} (u_{ps_j}^{k+1})^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) dt \geq 0.
\end{aligned}$$

In order to have $V_i^{k+1} \leq V_i^k$, it is sufficient that the following statement is positive

$$\sum_{j \in N_i} (u_{ps_j}^{k+1} - u_{ps_j}^k)^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) - 2 \sum_{j \in N_i} (u_{ps_j}^{k+1})^T R_{ij} (u_{ps_j}^{k+1} - u_{ps_j}^k) \geq 0.$$

Let $\Delta u_j = u_{ps_j}^{k+1} - u_{ps_j}^k$. Then the above equation is equivalent to

$$(\Delta u_j)^T R_{ij}(\Delta u_j) > -\left(\frac{\partial V_j^k}{\partial w_j}\right)^T \bar{B}_j R_{jj}^{-1} R_{ij}(v) \Delta u_j,$$

and hence the sufficient condition is $\|\Delta u_j\| \underline{\sigma}(R_{ij}) > \left\| \frac{\partial V_j^k}{\partial w_j} \right\| \|\bar{B}_j\| \rho_{ij}$. The mentioned inequality is valid for $\rho_{ij} = 0$. By the continuity of the solution, it holds for small value of ρ_{ij} as well. As a result, V_i^k is a non-increasing function bounded below by zero. Hence V_i^k is convergent as $k \rightarrow \infty$. Let $V_i^{(\infty)} = \lim_{k \rightarrow \infty} V_i^k$. According to principle of optimality

$$V_i^k \geq \int_t^{+\infty} y_{w_i}^T Q_{ii} y_{w_i} + u_{ps_i}^{*T} R_{ii} u_{ps_i}^* + \sum_{j \in N_i} u_{ps_j}^{*T} R_{ij} u_{ps_j}^* dt = V_i^*$$

where $u_{ps_i}^*$ and $u_{ps_{-i}}^*$ are the optimal control and V_i^* is the solution to (2.26). Let $k \rightarrow \infty$. Then, $V_i^{(\infty)} \geq V_i^*$ and the algorithm converges to an NE. $\square$

2.6 Online Solution to Output Regulation Games

In Section 2.3, the optimal solution to Problem 2.3.1 is obtained in the form of (2.24). Calculating (2.24) for all agents needs to solve the cooperative Hamilton-Jacobi equations (2.26) for $i = 1, \dots, N$ simultaneously. To solve these equations, we use the structure of the PI algorithm to develop a critic-actor adaptive control architecture. The algorithm uses two adaptive networks for each agent. One is used to approximate the value function and is named the critic network and the second one approximates the control policy and is named the actor network. The following assumption is required to prove that the approximated solution converges to the optimal solution to the Bellman equation (2.25).

Assumption 2.6.1. *The solution to the Bellman equation is smooth and nonnegative.*

It is possible that the global solution V_i to (2.25) is not smooth. It is shown in [89] that under certain reachability and observability assumptions, it has a local smooth solution V_i . Since the reachability and observability hold for our system setup, the local smooth solution V_i always exists.

According to the Weierstrass higher order approximation theorem [86], assume that the solution $V_i(w_i)$ to the Bellman equation is uniformly approximated by (2.52). Note that using w_i in (2.52) is motivated by Corollary 2.3.1

$$V_i(w_i) = W_i^T \Phi_i(w_i) + \epsilon_i. \quad (2.52)$$

In (2.52), Φ_i is the activation function vector with h neurons and W_i is the optimal weight for the value function. According to Theorem 3 of [86], the approximation error converges to zero as $h \rightarrow \infty$. Denoting the current weights as $\hat{W}_i$, the estimated value function is

$$\hat{V}_i(w_i) = \hat{W}_i^T \Phi_i(w_i). \quad (2.53)$$

Hence, the approximated error of the Bellman equation in each iteration is

$$H_i = L_i + \hat{W}_i^T \frac{\partial \Phi_i}{\partial w_i} \dot{w}_i = \epsilon_{H_i}. \quad (2.54)$$

Defining the total error as

$$E_{critic_i} = \frac{1}{2} \epsilon_{H_i}^T \epsilon_{H_i}. \quad (2.55)$$

One can use the normalized gradient descent to obtain the weights in each iteration.

The actor network is defined to estimate the optimal policy as

$$\hat{u}_{ps_i} = \hat{u}_{ps_{i+N}} = -\frac{1}{2}R_{ii}^{-1}\bar{B}_i^T \left[\frac{\partial \Phi_i}{\partial w_i} \right]^T \hat{W}_{i+N}. \quad (2.56)$$

The next theorem shows how to tune the critic and actor weights such that (1) the set of Hamilton-Jacobi equations are solved simultaneously, (2) the state w_i and the estimation error of the critic ($\tilde{W}_i = W_i - \hat{W}_i$) and the actor ($\tilde{W}_{i+N} = W_i - \hat{W}_{i+N}$) networks are Uniformly Ultimately Bounded.

Definition 2.6.1. (Uniformly Ultimately Bounded-) A timed signal $v(t) \in \mathbb{R}^n$ is said to be Uniformly Ultimately Bounded (UUB), if $\exists S \subset \mathbb{R}^n$ such that for all $v(0) \in S$, there exists a bound b and a time $T(v(0), b)$ such that $\|v(t)\| \leq b, \forall t \geq T$ [90].

Theorem 2.6.1. Consider Problem 2.3.1 and Assumptions 2.1.1-2.6.1. Suppose that the value function and the control policy are estimated by (2.53) and (2.56). Assume that $\bar{\sigma}_{i+N}$ is Persistently Exciting (PE). Tune the weights of the critic network as

$$\dot{\hat{W}}_i = -a_i \frac{\sigma_{i+N}}{(1 + \sigma_{i+N}^T \sigma_{i+N})^2} \varepsilon_{H_i}^{opt}, \quad (2.57)$$

$$\sigma_{i+N} = \frac{\partial \Phi_i}{\partial w_i} (\bar{A}_i w_i + \bar{B}_i \hat{u}_{ps_{i+N}}), \quad (2.58)$$

where $\varepsilon_{H_i}^{opt}$ is obtained by replacing (2.56) in (2.54) and $a_i > 0$ is a tuning parameter.

Tune the weights of the actor network as

$$\begin{aligned} \dot{\hat{W}}_{i+N} = & -a_{i+N} \left\{ \left(S_i \hat{W}_{i+N} - F_i \bar{\sigma}_{i+N}^T \hat{W}_i \right) - \frac{1}{4} \bar{D}_i \hat{W}_{i+N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i \right. \\ & \left. - \frac{1}{4} \sum_{j \in N_i} \left[\frac{\partial \Phi_j}{\partial w_j} \right] \bar{B}_j R_{jj}^{-T} R_{ij} R_{jj}^{-1} \bar{B}_j^T \left[\frac{\partial \Phi_j}{\partial w_j} \right]^T \hat{W}_j \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_{i+N} \right\}, \end{aligned} \quad (2.59)$$

where $a_{i+N} > 0$, $F_i > 0$ and $S_i >$ are tuning parameters, $\bar{\sigma}_{i+N} = \frac{\sigma_{i+N}}{1 + \sigma_{i+N}^T \sigma_{i+N}}$,

$m_{s_i} = 1 + \sigma_{i+N}^T \sigma_{i+N}$ and $\bar{D}_i = \left[\frac{\partial \Phi_i}{\partial w_i} \right] \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \left[\frac{\partial \Phi_i}{\partial w_i} \right]^T$. Then

1. the states w_i , $\tilde{W}_i = W_i - \hat{W}_i$ and $\tilde{W}_{i+N} = W_i - \hat{W}_{i+N}$ are UUB.
2. H_i is UUB and $\hat{W}_i$ converges to the approximated solution to the coupled Hamilton-Jacobi equation.
3. $\hat{u}_{i+N}$ converges to an approximated NE.

Proof. 1. The proof is similar to Theorem 2 of [91] and it is brought in Appendix A.

2. Assume that $\|W_i\| \leq W_{i \max}$, Then $\|\hat{W}_i\| \leq \|\tilde{W}_i\| + W_{i \max}$. Substituting (2.56) in the Hamilton-Jacobi equation (2.26) yields to the approximated Hamilton-Jacobi equation. According to Theorem 2.3.1, the system (2.20) is stable. Hence, w_i and u_{ps_i} , $i = 1, \dots, N$ remain arbitrary small. Taking norm of the approximated Hamilton-Jacobi equation, one can see all signals in the approximated Hamilton-Jacobi equations are UUB which results in UUB for the approximated Hamiltonian.

3. According to Theorem 2.6.1, $\tilde{W}_{i+N} = W_i - \hat{W}_{i+N}$ is UUB. Hence $\hat{u}_{i+N}$ converges to an approximated NE.

□

2.7 Simulation Results

Consider a group of five followers and one leader with the communication graph depicted in Fig 2.1. We analyze two cases: 1) When the agents' dynamics are certain and 2) when there is a multiplicative uncertainty in the system. In both cases, the

Figure 2.1: The communication graph

following dynamics is considered for the leader

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_0, \quad y_0 = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} x_0. \quad (2.60)$$

Select the 1-copy for each follower as

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (2.61)$$

2.7.1 Output regulation of five certain agents

Consider the followers' dynamics as

$$\begin{aligned} \dot{x}_i &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \quad y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i, \quad i = 1, 3, \\ \dot{x}_i &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \quad y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i, \quad i = 2, 4, 5. \end{aligned}$$

Note that the agents' dynamics are certain in this example. Select the activation function Φ_i as

$$\begin{aligned} \Phi_i(w_i) &= [w_{i1}^2, w_{i1}w_{i2}, w_{i1}w_{i3}, w_{i1}w_{i4}, w_{i1}w_{i5}, w_{i2}^2, w_{i2}w_{i3}, w_{i2}w_{i4}, w_{i2}w_{i5}, \\ &\quad w_{i3}^2, w_{i3}w_{i4}, w_{i3}w_{i5}, w_{i4}^2, w_{i4}w_{i5}, w_{i5}^2]^T. \end{aligned} \quad (2.62)$$

Select the weight matrices as $Q_{ii} = 10I$, $R_{ii} = 1$ for $i = 1, \dots, N$ and $R_{32} = 0.1$, $R_{41} = 0.1$, $R_{43} = 0.1$ and $R_{54} = 0.1$ and use $a_i = 10$, $a_{i+N} = 1$, $S_i = I$ and $F_i = I$. To assure that $\bar{\sigma}_{i+N}$ is PE, a decreasing probing noise is added to the control signal. Figure 2.2 shows the evolution of the output synchronization errors and their convergences to zero after 1000th second. The 3D plot of the outputs and their time derivatives for $i = 0, 1, \dots, 5$ is shown in Fig. 2.3. These figures illustrate both temporal and spatial convergence of the outputs of all followers to the output of the leader.

According to our discussion in Section 2.6, we employ the critic and actor networks to approximate the value function and the control policy online. we use this online learning algorithm without any previous learning or training which results to the

Figure 2.2: Output synchronization errors for the certain followers

oscillation in the first 100 seconds. One can obtain fewer oscillations and smaller control efforts by selecting bigger weighting matrices in the cost function to increase the penalty of the error in this period. This can be done by selecting bigger values for Q_{ii} , R_{ii} and R_{ij} . After 100th second, the agents remain in a closed vicinity of each other; the regulating errors for all agents are small and the algorithm performs fine-tuning of the weights to reach their optimal value. The duration of the fine tuning depends on the preset accuracy threshold for estimating the Hamiltonian (2.54), *e.g.*, in our case it is 10^{-8} . Also note that this algorithm scales up well because the whole convergence time (both the first 100 seconds and the fine-tuning) is related to the maximum in-degree, rather than the number of agents, as the agents perform the computations in parallel and their learning algorithm is only affected by their neighbors.

Figure 2.3: The outputs and the time derivative of the outputs for the certain followers and the leader

Figure 2.4: The magnitude plot of l_1 and l_2 and the authorized upper bound

2.7.2 Output regulation of five uncertain agents

Consider the same system setup as Subsection 2.7.1. Now, assume that the agents are subject to the uncertainty in their dynamics as follows

$$\begin{aligned} A_i^u &= A_i + \begin{bmatrix} 0 & 0 \\ -0.25 & -0.5 \end{bmatrix}, \quad i = 1, 3, \\ A_i^u &= A_i + \begin{bmatrix} 0 & 0.45 \\ 0 & 0 \end{bmatrix} \quad i = 2, 4, 5. \end{aligned} \tag{2.63}$$

Figure 2.4 shows the bode diagram of l_i for $i = 1, 2$ and the authorized upper bound. From Fig. 2.4, it can be understood that the inequality (2.45) in Theorem 2.4.2 is satisfied.

While the state w_i for $i = 1, \dots, 5$ evolves based on the uncertain system in (2.63), the nominal system models are used in the online algorithm. Since the inequality (2.45) is satisfied for $i = 1, \dots, N$, it is expected that the uncertain systems remain stable when the nominal systems are used to obtain the control signal. Figure 2.5

Figure 2.5: Output synchronization errors for the uncertain followers

shows the evolution of the output synchronization errors for the uncertain followers. The 3D plot of the outputs and their time derivatives is given in Fig. 2.6. Generally speaking, a system of uncertain followers may or may not need more time for the convergence. This issue highly depends on the uncertainty. In the current setup, the output of the followers converges to the output of the leader within 1000 s.

Figure 2.6: The outputs and the time derivative of the outputs for the uncertain followers and the leader

Chapter 3

Output Regulation of Linear Heterogeneous Agents over General Graphs

This chapter addresses output regulation of heterogeneous linear multi-agent systems communicating over a general graph *via* output-feedback and state-feedback controllers, respectively. We use a general model for the output regulation problem of linear multi-agent systems which accounts for a general directed and fixed communication graph, heterogeneity of the agents, an arbitrarily finite number of inputs and outputs and both state and output-feedbacks. First, we derive global and local sufficient conditions for the existence of local output-feedback and local state-feedback controllers. We show that both controllers have an H_∞ criterion as a part of their sufficient conditions. Second, we design these controllers locally to ensure those sufficient conditions. Third, for the case of output-feedback, we show that the design procedure requires a feasible solution to the set of two LMIs and one non-convex algebraic constraint. We provide an existing algorithm to find a feasible solution to this problem. Fourth, for the case of state-feedback, we prove

that the H_∞ criterion is satisfied if and only if there exists a feasible solution to an LMI. Moreover, we define an output regulation region such that selecting the state-feedback coupling gain from this region ensures that the sufficient condition is always satisfied.

This chapter has four main contributions. Firstly, we derive the global and local sufficient conditions for the existence of a local output-feedback that solves the output regulation problem and design the output-feedback locally. Other papers based on the transmission zero condition generally consider state-feedback [40] or distributed dynamic output-feedback [21]. Secondly, we propose a local state-feedback synthesis based on a feasible solution to an LMI. Thirdly, we define a novel concept of output regulation region for heterogeneous multi-agent systems using state-feedback such that selecting the controller coupling gain from this region guarantees that the sufficient condition is always satisfied. Finally, we show connections among output regulation problem, output regulation of the system without tracking a reference input, and H_∞ control. The results of this chapter have been published in [92,93].

The rest of this chapter is organized as follows: Section 3.1 defines the output regulation problem. Section 3.2 addresses the global and local sufficient conditions for the existence of an output-feedback controller and the local design procedure. Section 3.3 discusses the global and local sufficient conditions for the existence of a state-feedback controller. In addition, it shows how to design the controller locally and defines an output regulation region. The simulation results are presented in Section 3.4.

3.1 Problem Formulation

Consider a set of heterogeneous agents with N followers and one leader communicating over $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}})$ defined in Subsection 1.4.1 whose models are described

bellow

$$\dot{x}_0 = A_0 x_0, \quad (3.1)$$

$$y_0 = C_0 x_0, \quad (3.2)$$

$$\dot{x}_i = A_i x_i + B_i u_i + R_i x_0, \quad (3.3)$$

$$y_i = C_i x_i, \quad (3.4)$$

$$e_i = y_i - y_0 = C_i x_i - C_0 x_0, \quad (3.5)$$

in which $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ denote respectively the state and control signal for agent i , $x_0 \in \mathbb{R}^{n_0}$ is the state of the leader, $y_i \in \mathbb{R}^p$, $i = 0, 1, \dots, N$ is the measured output by agent i (0 denotes the leader) and e_i is the regulated error that is desired to be zero. The agents are heterogeneous in that they may have different dynamics. Define the auxiliary systems for each follower as

$$\dot{z}_i = G_{1i} z_i + G_{2i} e_{iv}, \quad (3.6)$$

where (G_{1i}, G_{2i}) is the p -copy of the leader's dynamic and $z_i \in \mathbb{R}^{n_{z_i}}$, $i = 1, \dots, N$. Since the leader's output is not measurable for all agents, we introduce the virtual regulating error e_{iv}

$$e_{iv} = \frac{1}{d_i + g_i} \left[\sum_{j \in N_i} \alpha_{ij} (y_i - y_j) + g_i (y_i - y_0) \right]. \quad (3.7)$$

From (3.5) and (3.7), the global regulated error e and the global virtual regulating error e_v are given by

$$e = [e_1^T, e_2^T, \dots, e_i^T, \dots, e_N^T]^T, \quad (3.8)$$

$$e_v = \left[\text{Diag} \left\{ \frac{1}{d_i + g_i} \right\}_{1:N} (L + G) \right] \otimes I_p e. \quad (3.9)$$

We also would like to introduce two terms: local reference signal r_i and the global reference signal r_G as

$$r_i = \frac{1}{d_i + g_i} \left[\sum_{j \in N_i} \alpha_{ij} y_j + g_i y_0 \right], \quad (3.10)$$

$$r_G = \underbrace{[x_0^T, x_0^T, \dots, x_0^T]^T}_N. \quad (3.11)$$

Then, $e_{iv} = y_i - r_i$. We make the following assumptions.

Assumption 3.1.1. $Re(\lambda_0) \geq 0, \forall \lambda_0 \in spec(A_0)$.

Assumption 3.1.2. *The pair (A_i, B_i) is stabilizable and the pair (A_i, C_i) is detectable for all $i = 1, \dots, N$. Matrix B_i is full column rank.*

Assumption 3.1.3.

$$rank \begin{bmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{bmatrix} = n_i + p, \lambda_0 \in spec(A_0). \quad (3.12)$$

Assumption 3.1.4. *The pair (G_{1i}, G_{2i}) contains a p -copy of the leader introduced in Definition 1.4.1.*

Assumption 3.1.5. *The directed graph $\bar{\mathcal{G}}$ has a spanning tree and the leader is its root node.*

Assumption 3.1.6. *The triple $(\begin{bmatrix} A_i & 0 \\ G_{2i}C_i & G_{1i} \end{bmatrix}, \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \begin{bmatrix} C_i & 0 \\ 0 & I \end{bmatrix})$ is output-feedback stabilizable.*

Assumption 3.1.1 can be made without loss of generality (Remark 1.3 of [25]); the modes associated with the eigenvalues with negative real parts decay to zero

exponentially and has no permanent effect on the asymptotic behavior of the leader. In this thesis, we consider a leader which generates a nonzero trajectory. Tracking a leader which generates a trajectory which converges to zero is trivial. Because in this case, all followers need only to stabilize their dynamics and there is no need for communication over a network; *i.e.* the multi-agent structure is not needed anymore.

Assumption 3.1.2 is the standard assumption of stabilizability of (A_i, B_i) and detectability of (A_i, C_i) . Stabilizability of (A_i, B_i) means that the uncontrollable poles of A_i are strictly stable and detectability of (A_i, C_i) is defined similarly. Assumption 3.1.6 means that those poles of $\begin{bmatrix} A_i & 0 \\ G_{2i}C_i & G_{1i} \end{bmatrix}$ which are not controllable via the output-feedback, are stable. Assumptions 3.1.2 and 3.1.6 are used interchangeably based on whether we use the state-feedback or the output-feedback. We select (G_{1i}, G_{2i}) to contain a p -copy of the leader's dynamics in Assumption 3.1.4 to solve the problem *via* the internal model principle. Hence, we put Assumption 3.1.3 to make sure that there is no unstable zero-pole cancellation between the follower's dynamics and the p -copy and our system is internally stable. Assumption 3.1.5 ensures that the reference signal (the leader's output) is reachable from all followers and it guarantees that all eigenvalues of $(L + G)$ have positive real parts. Hence,

$$\lim_{t \rightarrow \infty} e_i(t) = 0 \Leftrightarrow \lim_{t \rightarrow \infty} e_{iv}(t) = 0, \quad i = 1, \dots, N. \quad (3.13)$$

Remark 3.1.1. *To select (G_{1i}, G_{2i}) to contain a p -copy of A_0 , the leader's dynamics is needed. One can use estimation theories or reinforcement learning techniques to estimate A_0 [29,30].*

Problem 3.1.1. *Output Regulation Problem via Output-Feedback (ORPOF): Con-*

sider a group of $N + 1$ heterogeneous agents and their auxiliary systems defined in (3.1-3.7). Design

$$u_i = K_{y_i} y_i + K_{z_i} z_i \quad (3.14)$$

such that $e_i \rightarrow 0$, $i = 1, \dots, N$.

Problem 3.1.2. *Output Regulation Problem via State-Feedback (ORPSF): Consider a group of $N + 1$ heterogeneous agents and their auxiliary systems defined in (3.1-3.7). Design*

$$u_i = K_{x_i} x_i + K_{z_i} z_i \quad (3.15)$$

such that $e_i \rightarrow 0$, $i = 1, \dots, N$.

To implement the controllers in (3.14) and (3.15), each follower needs to measure its outputs or its states in a reference coordinate. Note that the reference coordinates do not need to be the same between the followers. Each follower only uses the local output synchronization error e_{iv} and an absolute measurement of the outputs of the neighbors is not required.

3.2 Output-Feedback Design for Output Regulation

In this section, we derive global and local sufficient conditions for the existence of a solution to Problem 3.1.1 and then, we design the controller.

3.2.1 Global sufficient condition for the existence of a solution to problem 3.1.1

Let

$$\begin{aligned}\tilde{A}_0 &:= I_N \otimes A_0, \quad \tilde{C}_0 = I_N \otimes C_0, \\ \tilde{Y} &:= \text{Diag}_{1:N}\{Y_i\}, \text{ where, } \{Y_i = A_i, B_i, C_i, K_{y_i}, K_{z_i}, G_{1i}, G_{2i}, R_i\}, \\ H &:= \text{Diag}_{1:N}\left\{\frac{1}{d_i + g_i}\right\}(L + G), \quad \tilde{H} := H \otimes I_p.\end{aligned}$$

Using the control signal (3.14), the closed loop system of all agents (3.3) and their auxiliary systems (3.6) is

$$\dot{\varrho}_g = A_{c_1} \varrho_g + B_c r_G, \quad (3.16)$$

$$\dot{r}_G = \tilde{A}_0 r_G,$$

$$\begin{aligned}e &= \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} \varrho_g - \tilde{C}_0 r_G, \quad \varrho_g = \begin{bmatrix} x_1^T, \dots, x_N^T, z_1^T, \dots, z_N^T \end{bmatrix}^T, \\ A_{c_1} &= \begin{bmatrix} \tilde{A} + \tilde{B} \tilde{K}_y \tilde{C} & \tilde{B} \tilde{K}_z \\ \tilde{G}_2 \tilde{H} \tilde{C} & \tilde{G}_1 \end{bmatrix}, \quad B_c = \begin{bmatrix} \tilde{R} \\ -\tilde{G}_2 \tilde{H} \tilde{C}_0 \end{bmatrix}. \quad (3.17)\end{aligned}$$

Theorem 3.2.1. *Consider Assumptions 3.1.1, 3.1.3- 3.1.6. If A_{c_1} is Hurwitz, then the output-feedback controller in (3.14) solves Problem 3.1.1.*

Proof. Assume, A_{c_1} is Hurwitz. Then, by Lemma 1.6 of [25], Problem 3.1.1 has a solution in the form (3.14) if the following matrix equation has a solution

$$\begin{aligned}\Pi\tilde{A}_0 &= (\tilde{A} + \tilde{B}\tilde{K}_y\tilde{C})\Pi + \tilde{B}\tilde{K}_z\Omega + \tilde{R}, \\ \Omega\tilde{A}_0 &= \tilde{G}_2\tilde{H}\tilde{C}\Pi + \tilde{G}_1\Omega - \tilde{G}_2\tilde{H}\tilde{C}_0,\end{aligned}\tag{3.18}$$

$$\tilde{C}\Pi - \tilde{C}_0 = 0.\tag{3.19}$$

Using (3.19), we can simplify (3.18) and then the problem reduces to a solution to the following system

$$\begin{aligned}\Pi_i A_0 &= (A_i + B_i K_{y_i} C_i) \Pi_i + B_i K_{z_i} \Omega_i + R_i, \\ \Omega_i A_0 &= G_{1i} \Omega_i, \\ C_i \Pi_i - C_0 &= 0.\end{aligned}\tag{3.20}$$

Because of Assumptions 3.1.1, 3.1.3- 3.1.6 and incorporating the p -copy of dynamic of the leader in the system, (3.20) has a solution (Π_i, Ω_i) (Remark 3.8 of [94]). Then, $\Pi = \underset{1:N}{Diag}\{\Pi_i\}$ and $\Omega = \underset{1:N}{Diag}\{\Omega_i\}$ form a solution to (3.18) and (3.19). $\square$

Remark 3.2.1. While [21] considers output regulation of homogeneous agents, we analyze output regulation of heterogeneous agents. Besides, the controller in [21] is different from our controllers. To be more specific, the controllers in [21] are $K_x(\sum \alpha_{ij}(x_i - x_j) + \alpha_{i0}x_i) + K_z z_i$ and $K_1(\sum \alpha_{ij}(\xi_i - \xi_j) + \alpha_{i0}\xi_i) + K_2 z_i$, where x_i is the state of agent i and ξ_i is its observer. Due to the homogeneity, these controllers make use of the local synchronization error in the states or observers which is not possible here (because of heterogeneity). Instead, we propose explicit dynamic output-feedback $K_{y_i}y_i + K_{z_i}z_i$ and dynamic state-feedback $K_{x_i}x_i + K_{z_i}z_i$ where we use the output or the state of agent i and the internal state only.

3.2.2 Local sufficient condition and local controller design for problem 3.1.1

In this subsection, we address the design of the output-feedback controller gains locally such that it provides stability for A_{c_1} . Define

$$\underline{A}_i := \begin{bmatrix} A_i + B_i K_{y_i} C_i & B_i K_{z_i} \\ G_{2i} C_i & G_{1i} \end{bmatrix}, \quad \hat{B}_i := \begin{bmatrix} 0 \\ G_{2i} \end{bmatrix}, \quad (3.21)$$

$$\hat{C}_i := \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad \varpi_1^i(s) := -\hat{C}_i(sI - \underline{A}_i)^{-1} \hat{B}_i. \quad (3.22)$$

Before proceeding to one of our main results in this section, we bring two technical lemmas.

Lemma 3.2.1 ([95]). *Let $Q = [Q_{ij}]$ with $Q_{ij} \in \mathbb{C}^{n_i \times n_j}$ and $\tilde{Q} = [\|Q_{ij}\|]$ be its block-norm matrix. Then $\rho(Q) \leq \rho(\tilde{Q})$.*

Lemma 3.2.2 ([40]). *Let Q be a nonnegative matrix and D a positive definite diagonal matrix $D = \text{Diag}\{d_i\}_{i=1:N}$. Then, $\rho(DQ) \leq \max_i d_i \rho(Q)$.*

Theorem 3.2.2. *Consider Assumptions 3.1.1, 3.1.3- 3.1.6 and assume that K_{y_i} and K_{z_i} are selected such that*

$$\max \|\varpi_1^i\|_\infty < \frac{1}{\rho(Z)} = \gamma, \quad i = 1, \dots, N, \quad (3.23)$$

in which $Z = \tilde{H} - I$ and $\rho(Z)$ is the spectral radius of Z . Then, the global system matrix A_{c_1} in (3.17) is stable and the output-feedback controller (3.14) solves Problem 3.1.1.

Proof. Because of Assumption 3.1.5, we can write $\tilde{H} = Z + I$. Denote the charac-

teristic equation of the matrix A_{c_1} by Δ

$$\Delta = \det(sI - A_{c_1}) = \det\left(X + \begin{bmatrix} 0 \\ -\tilde{G}_2 Z \end{bmatrix} \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}\right)$$

where

$$X = \begin{bmatrix} sI - (\tilde{A} + \tilde{B}\tilde{K}_y\tilde{C}) & -\tilde{B}\tilde{K}_z \\ -\tilde{G}_2\tilde{C} & sI - \tilde{G}_1 \end{bmatrix}.$$

Assume that matrix X is invertible for $\forall s > 0$. Then,

$$\Delta = \det(X)\det\left(I + \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} X^{-1} \begin{bmatrix} 0 \\ -\tilde{G}_2 \end{bmatrix} Z\right).$$

Observe that X is a 2×2 block matrix and each block entry is a diagonal matrix.

Hence, one can use a permutation matrix T to bring it to a block diagonal form

$$\begin{aligned} \Delta &= \det(TXT^T)\det\left(I + \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} T^T T X^{-1} T^T T \begin{bmatrix} 0 \\ -\tilde{G}_2 \end{bmatrix} Z\right) \\ &= \det(\text{Diag}\{sI - \underline{A}_i\}_{1:N}) \times \det(I + \varpi_1(s)Z) \end{aligned}$$

where $\varpi_1(s) = \text{Diag}\{\varpi_1^i\}_{1:N}$. By the small gain theorem, a sufficient condition for stability of the interconnected system $(I + \varpi_1 Z)$ is $\rho(\varpi_1 Z) < 1$. Since $\rho(\varpi_1 Z) \leq \rho(\text{Diag}\{\|\varpi_1^i\|_\infty\}_{1:N} Z) \leq \max_{1:N} \|\varpi_1^i\|_\infty Z < 1$. Hence, (3.23) guarantees stability of A_{c_1} . According to Theorem 3.2.1, the controller (3.14) solves Problem 3.1.1. $\square$

Remark 3.2.2. *Theorem 3.2.2 links output regulation problem and H_∞ control together. Consider the system defined in (3.3, 3.6-3.7) with the reference signal r_i . Assume that the output-feedback gain $[K_{y_i}, K_{z_i}]$ is designed such that the H_∞ norm*

in Theorem 3.2.2 is satisfied. Then, the output-feedback controller $u_i = K_{y_i}y_i + K_{z_i}z_i$ solves Problem 3.1.1.

The following theorem shows how to design the gain $\bar{K}_i = \begin{bmatrix} K_{y_i} & K_{z_i} \end{bmatrix}$ such that the norm condition in Theorem 3.2.2 is satisfied. Note that matrix $\underline{A}_i$ in (3.21) can be decomposed as

$$\begin{aligned} \underline{A}_i &= \begin{bmatrix} A_i & 0 \\ G_{2i}C_i & G_{1i} \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \begin{bmatrix} K_{y_i} & K_{z_i} \end{bmatrix} \begin{bmatrix} C_i & 0 \\ 0 & I \end{bmatrix} \\ &= \bar{A}_i + \bar{B}_i \bar{K}_i \bar{C}_i. \end{aligned} \quad (3.24)$$

Define

$$\Psi_i := \begin{bmatrix} \bar{A}_i^T P_i + P_i \bar{A}_i & P_i \hat{B}_i & \hat{C}_i^T \\ \hat{B}_i^T P_i & -\gamma^2 I & 0 \\ \hat{C}_i & 0 & -I \end{bmatrix}, \quad \Psi_{Y_i} := \begin{bmatrix} X_i \bar{A}_i^T + \bar{A}_i X_i & \hat{B}_i & X_i \hat{C}_i^T \\ \hat{B}_i^T & -\gamma^2 I & 0 \\ \hat{C}_i X_i & 0 & -I \end{bmatrix}, \quad (3.25)$$

$$U_i := \begin{bmatrix} \bar{C}_i & 0 & 0 \end{bmatrix}, \quad V_i := \begin{bmatrix} \bar{B}_i^T P_i & 0 & 0 \end{bmatrix} = Y_i E_i, \quad (3.26)$$

$$Y_i := \begin{bmatrix} \bar{B}_i^T & 0 & 0 \end{bmatrix}, \quad E_i := \begin{bmatrix} P_i & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \quad (3.27)$$

The matrices $P_i > 0$ and $X_i > 0$ will be introduced shortly.

Theorem 3.2.3. *Let W_{U_i} , W_{V_i} and W_{Y_i} be the orthogonal complements of U_i , V_i and Y_i respectively. Then, the norm condition in Theorem 3.2.2 is satisfied if and only if there exist matrices $P_i, X_i > 0 \in \mathbb{R}^{n_i+n_{z_i}}$ such that*

$$W_{Y_i}^T \Psi_{Y_i} W_{Y_i} < 0, \quad (3.28)$$

$$W_{U_i}^T \Psi_i W_{U_i} < 0, \quad (3.29)$$

$$X_i P_i = I. \quad (3.30)$$

Then, the controller gain is obtained from

$$\Psi_i + V_i^T \bar{K}_i U_i + U_i^T \bar{K}_i^T V_i < 0. \quad (3.31)$$

Proof. According to [96], the condition in (3.23) is satisfied if and only if there exists a matrix $P_i > 0$ satisfying

$$\begin{bmatrix} \underline{A}_i^T P_i + P_i \underline{A}_i & P_i \hat{B}_i & \hat{C}_i^T \\ \hat{B}_i^T P_i & -\gamma^2 I & 0 \\ \hat{C}_i & 0 & -I \end{bmatrix} < 0. \quad (3.32)$$

Using (3.24), the LMI condition in (3.32) is equivalent to (3.31). Using the elimination of matrix variable Lemma [96], (3.31) is feasible if and only if

$$\begin{aligned} W_{V_i}^T \Psi_i W_{V_i} &< 0, \\ W_{U_i}^T \Psi_i W_{U_i} &< 0. \end{aligned} \quad (3.33)$$

As the columns of W_{Y_i} spans $\text{Ker}(Y_i)$: ($Y_i W_{Y_i} = 0$), the columns of $E_i^{-1} W_{Y_i}$ spans $\text{Ker}(V_i) = \text{Ker}(Y_i E_i)$: ($Y_i E_i E_i^{-1} W_{Y_i} = 0$) and we can conclude that $W_{V_i} = E_i^{-1} W_{Y_i}$.

Then (3.33) becomes

$$\begin{aligned} W_{Y_i}^T E_i^{-1} \Psi_i E_i^{-1} W_{Y_i} &< 0, \\ W_{U_i}^T \Psi_i W_{U_i} &< 0. \end{aligned} \tag{3.34}$$

Introducing $\Psi_{Y_i} = E_i^{-1} \Psi_i E_i^{-1}$

$$\Psi_{Y_i} = \begin{bmatrix} P_i^{-1} \bar{A}_i^T + \bar{A}_i P_i^{-1} & \hat{B}_i & P_i^{-1} \hat{C}_i^T \\ \hat{B}_i^T & -\gamma^2 I & 0 \\ \hat{C}_i P_i^{-1} & 0 & -I \end{bmatrix}.$$

Let $X_i = P_i^{-1}$, then (3.30). It completes the proof. $\square$

Remark 3.2.3. According to Theorems 3.2.2 and 3.2.3, a set of sufficient conditions for stability of A_{c_1} is:

1. The existence of matrices $P_i > 0$ and $X_i > 0$ satisfying (3.28-3.30) and
2. The existence of a spanning tree for the graph $\bar{\mathcal{G}}$.

The first condition examines the output-feedback stabilizability of the extended agent such that an H_∞ norm is satisfied. While the output-feedback stabilizability of the extended agent is a local condition, the upper bound for the H_∞ norm is obtained from the graph structure. Similarly, the second condition is related to the graph. Hence, we have a mix of local and global conditions. The local condition is related to each agent and the global conditions are related to the graph.

Before proceeding to finding a solution to (3.28-3.30), we review some materials

from [97]. There exists a feasible solution $S_i, E_i > 0 \in \mathbb{R}^{h_i}$ to

$$\begin{aligned} W_{Y_i}^T(S_i A_{h_i} + A_{h_i}^T S_i) W_{Y_i} &< 0, \\ W_{U_i}^T(A_{h_i} E_i + E_i A_{h_i}^T) W_{U_i} &< 0, \\ S_i E_i &= I_{h_i}. \end{aligned} \tag{3.35}$$

if and only if the global minimum of the cost function in the following optimization problem is h_i

$$\begin{aligned} \min \quad & \mathbf{Tr}(S_i E_i), \\ \text{w.r.t} \quad & \begin{cases} W_{Y_i}^T(S_i A_{h_i} + A_{h_i}^T S_i) W_{Y_i} < 0, \\ W_{U_i}^T(A_{h_i} E_i + E_i A_{h_i}^T) W_{U_i} < 0, \\ \begin{bmatrix} S_i & I_{h_i} \\ I_{h_i} & E_i \end{bmatrix} \geq 0. \end{cases} \end{aligned} \tag{3.36}$$

According to [97], such a problem is named Cone Complementarity Problem (CCP). Now, we claim that there exists a feasible solution $X_i, P_i > 0 \in \mathbb{R}^{n_i+n_{z_i}}$ to (3.28-3.30) if and only if the global minimum of the cost function in the following optimization problem is $n_i + n_{z_i}$

$$\begin{aligned} \min \quad & \mathbf{Tr}(X_i P_i), \\ \text{w.r.t} \quad & \begin{cases} W_{Y_i}^T \Psi_{Y_i} W_{Y_i} < 0, \\ W_{U_i}^T \Psi_i W_{U_i} < 0, \\ \begin{bmatrix} X_i & I_{n_i+n_{z_i}} \\ I_{n_i+n_{z_i}} & P_i \end{bmatrix} \geq 0. \end{cases} \end{aligned} \tag{3.37}$$

Note that the matrices Ψ_i and Ψ_{Y_i} are not in a form corresponding to (3.35) or

(3.36). Let

$$A_{h_i} = \begin{bmatrix} \bar{A}_i^T & 0 & \hat{C}_i^T \\ \hat{B}_i^T & -0.5I_p & 0 \\ 0 & 0 & -0.5I_p \end{bmatrix} \quad (3.38)$$

$$E_i = \begin{bmatrix} P_i & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}, \quad S_i = \begin{bmatrix} X_i & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix}. \quad (3.39)$$

Then the set of (3.28-3.30) is equivalent to (3.35) and $h_i = n_i + n_{z_i} + 2p$.

Sufficiency: If (X_i, P_i) is a feasible solution to (3.28-3.30), then (3.39) is a solution to (3.35) and based on [97], (3.39) is a solution to the optimization problem in (3.36) with the optimal value of the cost function equal to $n_i + n_{z_i} + 2p$ and hence the optimal value of the cost function in (3.37) is $n_i + n_{z_i}$.

Necessity: If there exists a pair (X_i, P_i) as a solution to (3.37) with the optimal value of the cost function $n_i + n_{z_i}$, then (E_i, S_i) in (3.39) is a solution to (3.36) with the optimal value of the cost function equal to $n_i + n_{z_i} + 2p$. According to [97], the pair (E_i, S_i) in (3.39) is a feasible solution to (3.35) and hence, (X_i, P_i) is a feasible solution to (3.28-3.30).

The following algorithm is given in [97] and provides a solution to CCPs.

Algorithm 3.1 iterates over a linear programming problem. Note that the objective function in the optimization problem in Algorithm 3.1 is $\mathbf{Tr}(X_i^{(k+1)} \Xi_i^{(k)} + \Phi_i^{(k)} P_i^{(k+1)})$. Hence, there exists a feasible solution to (3.28-3.30) if and only if the global solution to the optimization problem in Algorithm 3.1 is $2(n_i + n_{z_i})$. The proof of convergence is given in Theorem 2.1 of [97].

Remark 3.2.4. According to Theorem 3.2.3, the norm condition in (3.23) is satisfied if and only if there exists a solution to the set of two LMIs and one nonconvex constraint in (3.28-3.30), and based on [97], there exists a solution to (3.28-3.30) if

Algorithm 3.1 Iterative solution to (3.28-3.30)

Require: Find a feasible solution $X_i^{(0)}, P_i^{(0)}$ to (3.28-3.29) if any. Set $k=0$.

repeat

Set $\Phi_i^{(k)} = X_i^{(k)}$ and $\Xi_i^{(k)} = P_i^{(k)}$. Find $X_i^{(k+1)}$ and $P_i^{(k+1)}$ that solve

$$\min \quad \text{Tr}(X_i^{(k+1)}\Xi_i^{(k)} + \Phi_i^{(k)}P_i^{(k+1)}), \quad (3.40)$$

$$\text{w.r.t} \quad \begin{cases} W_{Y_i}^T \Psi_{Y_i}^{k+1} W_{Y_i} < 0, \\ W_{U_i}^T \Psi_{U_i}^{k+1} W_{U_i} < 0, \\ \begin{bmatrix} X_i^{(k+1)} & I \\ I & P_i^{(k+1)} \end{bmatrix} \geq 0. \end{cases} \quad (3.41)$$

until $|\text{Tr}(X_i^{(k+1)}\Xi_i^{(k)} + \Phi_i^{(k)}P_i^{(k+1)} - X_i^{(k)}\Xi_i^{(k-1)} - \Phi_i^{(k-1)}P_i^{(k)})| < \varepsilon$

and only if the solution to Algorithm 3.1 is $2(n_i + n_{z_i})$. Hence, the solution $2(n_i + n_{z_i})$ for Algorithm 3.1 is both necessary and sufficient for the existence of a gain matrix $\bar{K}_i$ to satisfy (3.23).

Remark 3.2.5. In [97], a dynamic output-feedback is designed assuming that $m \leq n$, where m and n are the orders of the controller and the system respectively. As in Theorem 3.2.3, we are designing a **static output-feedback gain** for the system $(\bar{A}_i, \bar{B}_i, \bar{C}_i, 0)$ in (3.24), the controller order is equal to zero and the condition $m \leq n$ is satisfied in our framework. Also, according to [97], Y_i^T and U_i should be of full column and full row ranks respectively. As the matrices B_i and C_i are of full column and full row ranks, so are Y_i^T and U_i .

Remark 3.2.6. If the matrix C_i is invertible, then $W_{U_i}^T = \begin{bmatrix} 0_{(n_i+n_{z_i}) \times 2p} & I_{2p} \end{bmatrix}$ and as a result, (3.29) is generally satisfied and one can use a feasible solution to (3.28) to design the controller gain. In this case, a feasible solution to an LMI is required only.

3.3 State-feedback Design for Output Regulation

Designing output-feedback requires the output-feedback stabilizability condition (Assumption 3.1.6). In the case that the states are measurable or they can be estimated, we can use state-feedback which is simpler and requires a more relaxed condition like stabilizability of the agents.

3.3.1 Global sufficient conditions for the existence of a solution to Problem 3.1.2

Let $\tilde{K}_x := \text{Diag}_{1:N}\{K_{x_i}\}$. Using the control signal (3.15), the overall closed loop system containing all agents (3.3) and their auxiliary systems (3.6) is

$$\dot{\xi}_g = A_{c_2}\xi_g + B_c r_G, \xi_g = \begin{bmatrix} x_1^T, \dots, x_N^T, z_1^T, \dots, z_N^T \end{bmatrix}^T \quad (3.42)$$

$$A_{c_2} = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_x & \tilde{B}\tilde{K}_z \\ \tilde{G}_2\tilde{H}\tilde{C} & \tilde{G}_1 \end{bmatrix}. \quad (3.43)$$

The next result shows that the stability of A_{c_2} is a sufficient condition for the existence of a solution to Problem 3.1.2.

Theorem 3.3.1 ([40]). *Consider Assumptions 3.1.1-3.1.5. If A_{c_2} is Hurwitz, then the state-feedback controller in (3.15) solves Problem 3.1.2.*

Remark 3.3.1. *The global system matrices A_{c_1} and A_{c_2} are obtained by incorporating a p -copy of the internal model of the leader in each agent. The stability of A_{c_1} and A_{c_2} is a sufficient condition for Problems 3.1.1 and 3.1.2 respectively and it can be recast as the state regulation of the global system ϱ_g in (3.16) and ξ_g in (3.42) without the reference signal ($r_G = 0$).*

3.3.2 Local sufficient condition and local controller design for Problem 3.1.2

Define

$$\hat{A}_i := \begin{bmatrix} A_i + B_i K_{x_i} & B_i K_{z_i} \\ G_{2i} C_i & G_{1i} \end{bmatrix}, \quad \varpi_2^i(s) := -\hat{C}_i (sI - \hat{A}_i)^{-1} \hat{B}_i. \quad (3.44)$$

Theorem 3.3.2 ([40]). *Consider Assumptions 3.1.1-3.1.5 and assume that K_{x_i} and K_{z_i} are selected such that*

$$\max \|\varpi_2^i\|_\infty < \frac{1}{\rho(Z)} = \gamma. \quad (3.45)$$

Then, the global system matrix A_{c_2} in (3.43) is stable and the state-feedback controller (3.15) solves Problem 3.1.2.

Decompose matrix $\hat{A}_i$ in (3.44) as

$$\hat{A}_i = \begin{bmatrix} A_i & 0 \\ G_{2i} C_i & G_{1i} \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \begin{bmatrix} K_{x_i} & K_{z_i} \end{bmatrix} = \bar{A}_i + \bar{B}_i K_i.$$

Theorem 3.3.3. *The norm condition in Theorem 3.3.2 is satisfied if and only if there exist a matrix $Q_i > 0$ and a scalar $\sigma_1 \in \mathbb{R}$ solving the following LMI*

$$\begin{bmatrix} \bar{A}_i Q_i + Q_i \bar{A}_i^T - \sigma_1 \bar{B}_i \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 & Q_i \hat{C}_i^T \\ \hat{C}_i Q_i & -I \end{bmatrix} < 0 \quad (3.46)$$

or equivalently,

$$\bar{A}_i Q_i + Q_i \bar{A}_i^T - \sigma_1 \bar{B}_i \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 + Q_i \hat{C}_i^T \hat{C}_i Q_i < 0. \quad (3.47)$$

The corresponding controller gain is $K_i = -\frac{\sigma_1}{2} \bar{B}_i^T Q_i^{-1}$.

Proof. According to [96], the condition in (3.45) is satisfied if and only if there exists a matrix $Q_i > 0$ satisfying

$$\begin{bmatrix} \bar{A}_i Q_i + Q_i \bar{A}_i^T + \bar{B}_i K_i Q_i + Q_i K_i^T \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 & Q_i \hat{C}_i^T \\ \hat{C}_i Q_i & -I \end{bmatrix} < 0. \quad (3.48)$$

By renaming $\Omega_i = K_i Q_i$, (3.48) is satisfied if and only if

$$\begin{bmatrix} \bar{A}_i Q_i + Q_i \bar{A}_i^T + \bar{B}_i \Omega_i + \Omega_i^T \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 & Q_i \hat{C}_i^T \\ \hat{C}_i Q_i & -I \end{bmatrix} < 0. \quad (3.49)$$

Using Finsler's Lemma [96], the above inequality is equivalent to (3.46) for some $\sigma_1 \in \mathbb{R}$. Then, $K_i = -\frac{\sigma_1}{2} \bar{B}_i^T Q_i^{-1}$. Using Schur complement, (3.46) is equivalent to (3.47). $\square$

Remark 3.3.2. According to Theorems 3.3.2 and 3.3.3, a set of sufficient conditions for stability of A_{c_2} is:

1. The existence of matrix $Q_i > 0$ and a scalar $\sigma_1 \in \mathbb{R}$ satisfying (3.46) and
2. The existence of a spanning tree for the graph $\bar{\mathcal{G}}$.

The first condition examines the state-feedback stabilizability of the extended agent such that an H_∞ norm is satisfied. Note that the stabilizability of the extended agent is a local condition and it is equivalent to the stabilizability of (A_i, B_i) . The upper bound for the H_∞ norm as well as the existence of a spanning tree for the graph are

global conditions. Totally, they make a set of sufficient conditions that guarantees stability of A_{c_2} .

3.3.3 Output regulation region

Based on our discussion in Subsections 3.3.1 and 3.3.2, the solution to ORPSF can be recast into designing a controller of the form of $u_i = -\frac{\sigma_1}{2}\bar{B}_i^T Q_i^{-1}[x_i^T, z_i^T]^T$ such that the norm condition in (3.45) is satisfied. Observe that the parameters σ_1 and γ are related to each other through (3.46). Motivated by the definition of the synchronization region [98], we define an output regulation region for heterogeneous multi-agent system as follows.

Definition 3.3.1. Given matrices $\bar{B}_i$ and $Q_i > 0$, a region $\mathcal{J}_\gamma \subset \mathbb{R}$ is called an *output regulation region* with index $\gamma > 1$, if for any $\sigma \in \mathcal{J}_\gamma$, the controller gain $[K_{x_i}, K_{z_i}] = -\frac{\sigma}{2}\bar{B}_i^T Q_i^{-1}$ leads to $\|\varpi_2^i\|_\infty < \gamma$.

Lemma 3.3.1. Suppose matrix $Q_i > 0$ and scalar $\sigma_1 \in \mathbb{R}$ form a solution to (3.47). Then, for any $\sigma > \sigma_1$, selecting the controller gain as $K_i = [K_{x_i}, K_{z_i}] = -\frac{\sigma}{2}\bar{B}_i^T Q_i^{-1}$ yields to $\|\varpi_2^i\|_\infty < \gamma$ and an output regulation region is $\mathcal{J}_\gamma = [\sigma_1, +\infty)$.

Proof. Adding and subtracting $\sigma\bar{B}_i\bar{B}_i^T$ from (3.47)

$$\bar{A}_i Q_i + Q_i \bar{A}_i^T - \sigma \bar{B}_i \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 + Q_i \hat{C}_i^T \hat{C}_i Q_i < (\sigma_1 - \sigma) \bar{B}_i \bar{B}_i^T. \quad (3.50)$$

Then for any $\sigma > \sigma_1$ we have

$$\bar{A}_i Q_i + Q_i \bar{A}_i^T - \sigma \bar{B}_i \bar{B}_i^T + \hat{B}_i \hat{B}_i^T / \gamma^2 + Q_i \hat{C}_i^T \hat{C}_i Q_i < 0$$

and hence based on Theorem 3.3.3, the controller gain $K_i = -\frac{\sigma}{2}\bar{B}_i^T Q_i^{-1}$ leads to (3.45). Based on the definition of the output regulation region, $\mathcal{J}_\gamma = [\sigma_1, +\infty)$. $\square$

The following Theorem provides a distributed controller design procedure for Problem 3.1.2.

Theorem 3.3.4. *Consider Problem 3.1.2 and Assumptions 3.1.1-3.1.5. Let the set of $\sigma_1 \in \mathbb{R}$ and $Q_i > 0$ be a solution to (3.47). Then, for any $\sigma \in [\sigma_1, +\infty)$, the controller $u_i = -\frac{\sigma}{2}\bar{B}_i^T Q_i^{-1}[x_i^T, z_i^T]^T$ solves Problem 3.1.2.*

Proof. The proof follows from Theorems 3.3.2, 3.3.3 and the definition of the output regulation region. $\square$

3.4 Simulation Results

Consider a group of five followers and one leader with the dynamics given below and the communication graph depicted in Fig 3.1 with $\rho(Z) = 0.8285 = \frac{1}{\gamma}$

$$\left\{ \begin{array}{l} \dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0, \quad y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0, \end{array} \right. \quad (3.51)$$

$$\left\{ \begin{array}{l} \dot{x}_i = \begin{bmatrix} 0.3 & -2 \\ 0.1 & -0.2 \end{bmatrix} x_i + \begin{bmatrix} 1.8 & -0.8 \\ 0.9 & 1.6 \end{bmatrix} u_i, \\ y_i = \begin{bmatrix} -0.1 & 1.2 \\ 0.4 & 1.4 \end{bmatrix} x_i, \quad i = 1, 3, 5, \end{array} \right. \quad (3.52)$$

$$\left\{ \begin{array}{l} \dot{x}_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} x_i + \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_i, \\ y_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_i, \quad i = 2, 4. \end{array} \right. \quad (3.53)$$

We design the state-feedback for followers 1,3,5 and output-feedback for followers 2,4. The solutions to the LMI in (3.46) for followers 1,3,5 are $f_{\gamma}^{1,3,5} = [0.9, +\infty)$, and

$$Q_{1,3,5} = \begin{bmatrix} 4.57 & -0.12 & 0.73 & 1.55 & 2.08 & -2.53 \\ -0.12 & 0.64 & -0.01 & -0.58 & -0.27 & -0.02 \\ 0.73 & -0.01 & 31278 & -356.4 & -625.52 & 53.94 \\ 1.55 & -0.58 & -356.41 & 8.63 & 64.09 & -2.93 \\ 2.08 & -0.27 & -625.52 & 64.09 & 30419 & -295.2 \\ -2.53 & -0.02 & 53.94 & -2.93 & -295.19 & 6.89 \end{bmatrix}.$$

Let us select $\sigma = 1 \in f_{\gamma}^{1,3,5}$. Then the controller is

$$K_{1,3,5} = \begin{bmatrix} -0.33 & -0.86 & -0.00 & -0.10 & -0.00 & -0.26 \\ 0.10 & -1.45 & -0.00 & -0.24 & -0.00 & -0.05 \end{bmatrix}.$$

Algorithm 3.1 admits the following solution for followers 2,4 in 10 iterations

Figure 3.1: The communication graph

$$\begin{aligned}
P_{2,4} &= \begin{bmatrix} 0.53 & 0.02 & -0.58 & 0.16 & 0.74 & 0.02 & 0.11 \\ 0.02 & 0.37 & 0.12 & 0.00 & 0.02 & 0.12 & 0.52 \\ -0.58 & 0.12 & 3.20 & -0.14 & -0.82 & -0.04 & -0.35 \\ 0.16 & 0.00 & -0.14 & 0.38 & 0.48 & -0.00 & 0.01 \\ 0.74 & 0.02 & -0.82 & 0.48 & 3.65 & 0.01 & 0.17 \\ 0.02 & 0.12 & -0.04 & -0.00 & 0.01 & 0.39 & 0.46 \\ 0.11 & 0.52 & -0.35 & 0.01 & 0.17 & 0.46 & 3.33 \end{bmatrix}, \\
K_{2,4} &= \begin{bmatrix} 7.33 & 14.20 & -2.39 & -11.18 & 4.84 & 20.40 \\ 99.65 & -218.16 & 33.70 & 144.17 & -72.98 & -314.13 \end{bmatrix}.
\end{aligned}$$

The output tracking errors of the followers are shown in Fig. 3.2. The trajectories of all followers and the leader are depicted in Fig. 3.3.

Figure 3.2: The tracking error e_i for followers 1-5

Figure 3.3: The trajectories of followers 1-5

Chapter 4

H_∞ Output Regulation of Linear Heterogeneous Agents

This chapter investigates H_∞ output regulation of Linear Time Invariant (LTI) heterogeneous agents subject to unmodeled disturbances *via* a linear distributed output-feedback. We assume that the agents communicate a set of outputs in a distributed manner and aim to design a static distributed output-feedback controller to synchronize on another set of outputs with an H_∞ bound. We use a necessary geometrical structure of single-agents' drift dynamics for synchronization in a disturbance-free case and propose two independent sufficient conditions through the graphical game and LMI approaches that guarantee H_∞ output regulation and remove the acyclic condition on the graph topology [24,36,81]. As those sufficient conditions are independent, whenever the required condition for one method is not satisfied, one can check the condition for the other method.

Our problem is different from [27,36] in the sense that our agents are subject to unmodeled disturbances and we aim for H_∞ output regulation. We emphasize that we do not design any observers in contrast to [27,99], and do not rely on passivity properties in contrast to [100].

This chapter has four main contributions. (1) We obtain a sufficient condition for H_∞ output regulation of heterogeneous LTI agents subject to unmodeled disturbances based on zero/nonzero-sum graphical games. (2) We use reinforcement learning to obtain a solution to coupled Hamilton-Jacobi equations resulted from this novel zero/nonzero-sum graphical game formulation, which to our best knowledge has not appeared in the literature. (3) We obtain another sufficient condition to guarantee H_∞ output regulation based on an upper bound of a single-agent H_∞ norm. We further design the controller gains based on a set of LMIs. (4) We also obtain upper bounds for the $\mathcal{L}_2$ -gain of output synchronization errors with respect to unmodeled disturbances in both the graphical game and LMI approaches in contrast to [49], which does not calculate the upper bound but only contends its existence. The results of this chapter have been published in [101–103].

The rest of this chapter is organized as follows. Section 4.1 defines the H_∞ output regulation problem. Section 4.2 presents a necessary geometrical condition for the output regulation problem. Also, it introduces two coordinate transformations that are used in Sections 4.3 and 4.4. Section 4.3 suggests a graphical game controller design for H_∞ output regulation and computes the overall $\mathcal{L}_2$ -gain of output synchronization error with respect to disturbances. Section 4.4 introduces a single-agent H_∞ criterion for H_∞ output regulation and presents a way of computing the resulting overall $\mathcal{L}_2$ -gain of the output synchronization error with respect to disturbances. Also, it proposes the controller design *via* a set of LMIs. Simulations are provided in Section 4.5.

4.1 Problem Formulation

Consider a set of $N + 1$ heterogeneous agents communicating over $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}})$ defined in Subsection 1.4.1. The set contains N followers given as LTI systems

$$\dot{x}_i = A_i x_i + B_i u_i + P_i \omega_i, \quad (4.1)$$

$$y_i = C_i x_i, \quad (4.2)$$

$$z_i = D_i x_i, \quad (4.3)$$

and a leader given by

$$\dot{x}_0 = A_0 x_0, \quad (4.4)$$

$$y_0 = C_0 x_0, \quad (4.5)$$

$$z_0 = D_0 x_0, \quad (4.6)$$

in which $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and the control, $y_i \in \mathbb{R}^p$ is an output that is desired to be synchronized and $z_i \in \mathbb{R}^q$ is an output that is used in the control signal. The motivation behind introducing two outputs is to achieve synchronization in one set of outputs, y_i s, by communicating z_i s. All followers are subjected to external unmodeled disturbances $\omega_i \in \mathcal{L}_2^{r_i}[0, +\infty)$. The local neighborhood error in the z -outputs is

$$e_{zi} = \sum_{j=1}^N \alpha_{ij} (z_j - z_i) + g_i (z_0 - z_i). \quad (4.7)$$

Define the following global states $\rho = [\rho_1^T, \rho_2^T, \dots, \rho_N^T]^T$ for $\rho_i \in \{x_i, z_i, y_i, u_i, e_{zi}, \omega_i\}$.

The total local neighborhood error in the z -outputs reads

$$e_z = -(L + G) \otimes I_q(z - \bar{z}_0), \quad (4.8)$$

where $\bar{z}_0 = \mathbf{1}_N \otimes z_0$. The y -synchronization error is defined as

$$\begin{aligned} e_i &= y_i - y_0, \\ e &= [e_1^T, e_2^T, \dots, e_N^T]^T. \end{aligned} \quad (4.9)$$

The linear distributed output-feedback is formed from (4.7) as

$$u_i = K_i e_{zi}, \quad (4.10)$$

where the feedback gains K_i will be designed later.

Definition 4.1.1. H_∞ Output Regulation: Consider a group of $N + 1$ heterogeneous LTI systems defined by (4.1-4.6). The H_∞ output regulation problem is to achieve y -output regulation for the disturbance-free system asymptotically, i.e. $\|e_i\| \rightarrow 0$, $i = 1, \dots, N$, as $t \rightarrow \infty$ and a bounded $\mathcal{L}_2$ -gain from the global disturbance ω to the global y -synchronization error e .

Our problem formulation in Definition 4.1.1, contains as special instances the state synchronization of homogeneous agents, $A_i = A$, $B_i = B$, $K_i = K$, communicating their states, $D_i = I_n$, [12,104], or their outputs, $D_i = C$, [50], and the output regulation of heterogeneous agents with outputs communicated $D_i = C_i$, [24,27,40, 81,92]. The H_∞ output regulation problem can be viewed as the classical cooperative output regulation in the absence of the disturbances plus output regulation with an H_∞ -bound in their presence.

4.2 Necessary Conditions for Output Regulation

Make Assumption 3.1.5. For a group of $N + 1$ heterogeneous LTI systems (4.1-4.6) to achieve y -output regulation in the absences of disturbances, there must exist an invariant subspace in the closed-loop system of (4.1-4.6). We call such a subspace *the target invariant subspace* [27]. Due to linearity, $x = 0$ is always an invariant subspace of the closed-loop system and it implies *trivial y -synchronization subspace*. In this thesis, we are interested in a non-trivial y -synchronization subspace. Let a nontrivial target invariant subspace be spanned by the columns of some matrix $\Pi = [\Pi_1^T, \dots, \Pi_N^T, \Pi_0^T]^T$, where $\Pi_i \in \mathbb{R}^{n_i \times l}$. Then, Π satisfies the invariance and y -output synchronization conditions

$$\begin{aligned} A^* \Pi &= \Pi X, \\ C^* \Pi &= \bar{R}_1, \end{aligned} \tag{4.11}$$

where $\bar{R}_1 = \mathbf{1}_{N+1} \otimes R_1$, for some matrix R_1 [27] and

$$\begin{aligned} A^* &= \text{Diag}\{A_1, \dots, A_N, A_0\} - \text{Diag}\{B_1 K_1, \dots, B_N K_N, 0_{n_0 \times q}\} (\bar{L} \otimes I_q) D^*, \\ C^* &= \text{Diag}\{C_1, \dots, C_N, C_0\}, D^* = \text{Diag}\{D_1, \dots, D_N, D_0\}, \end{aligned}$$

where $\bar{L}$ is the augmented graph Laplacian. The system matrix $X \in \mathbb{R}^{l \times l}$ in (4.11) describes the dynamics on the target invariant subspace, does not have any stable poles. We make the following assumption.

Assumption 4.2.1. *Distributed control (4.10) vanishes on the target invariant sub-*

space

$$K_i \left(\sum_{j=1}^N \alpha_{ij}(z_j - z_i) + g_i(z_0 - z_i) \right) = 0. \quad (4.12)$$

We can certainly make Assumption 4.2.1 holds by using z -synchronization on the target invariant subspace

$$D_i \Pi_i = R_2, \quad i = 0, \dots, N. \quad (4.13)$$

The following lemma is an immediate result of Assumption 4.2.1 stated in the literature [27].

Lemma 4.2.1. *Under Assumption 4.2.1, there exist an invariant subspaces for single-agent drift dynamics $A_i, i = 0, \dots, N$*

$$A_i \Pi_i = \Pi_i X, \quad (4.14)$$

$$C_i \Pi_i = R_1, \quad i = 0, \dots, N. \quad (4.15)$$

Proof. Equation (4.11) under Assumption 4.2.1, reduces to (4.14-4.15). $\square$

Remark 4.2.1. *In summary, Assumption 3.1.5 and (4.13-4.15) form a set of necessary assumptions for nontrivial y -output regulation. One can satisfy (4.13-4.15) by incorporating an internal model of the leader in the dynamics of the controllers, as described in [23,40,92]. In this case, the dynamics of the controller and the original agent can be considered together as the augmented single-agent in this chapter. Hence, from now on, we consider systems that satisfy (4.13-4.15).*

Remark 4.2.2. Equation (4.14) implies that all single-agent drift dynamics, including the leader, contains a copy of the same shared invariant subspace and on all those subspaces, the drift dynamics is given by matrix X .

4.2.1 Coordinate transformations

In this subsection, we introduce two coordinate transformations, which reveal the structure of the agents' dynamics, and motivate the subsequent graphical game and the LMI design.

According to the necessary conditions in (4.14), all single-agent drift dynamics A_i , $i = 0, \dots, N$ share one invariant subspace, which is spanned by columns of Π_i satisfying (4.13-4.15). We supplement the column of Π_i by Ψ_i to form a basis of the single-agent state space $\mathbb{R}^{n_i}$. Then, with such basis, we have the following transformed state $[\xi_i^T, \nu_i^T]^T$

$$x_i = \begin{bmatrix} \Pi_i & \Psi_i \end{bmatrix} \begin{bmatrix} \xi_i \\ \nu_i \end{bmatrix} = T_i \begin{bmatrix} \xi_i \\ \nu_i \end{bmatrix}, \quad (4.16)$$

with (4.14) leads to

$$\begin{bmatrix} \dot{\xi}_i \\ \dot{\nu}_i \end{bmatrix} = T_i^{-1} A_i T_i \begin{bmatrix} \xi_i \\ \nu_i \end{bmatrix} = \begin{bmatrix} X & F_i \\ 0 & \tilde{A}_i \end{bmatrix} \begin{bmatrix} \xi_i \\ \nu_i \end{bmatrix}, \quad (4.17)$$

describing an uncontrolled, disturbance-free system. In the above system, ξ_i s parameterize the subspace spanned by the columns of Π_i , and ν_i s represent the remaining

coordinates in the single-agent state-space. Define the following matrices

$$\begin{aligned}\hat{A}_i &:= T_i^{-1} A_i T_i = \begin{bmatrix} X & F_i \\ 0 & \tilde{A}_i \end{bmatrix}, \quad \hat{B}_i := T_i^{-1} B_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix}, \quad \hat{P}_i := T_i^{-1} P_i = \begin{bmatrix} \tilde{P}_{1i} \\ \tilde{P}_{2i} \end{bmatrix}, \\ \tilde{C}_i &:= C_i T_i = \begin{bmatrix} R_1 & C_i \Psi_i \end{bmatrix}, \quad \tilde{D}_i := D_i T_i = \begin{bmatrix} R_2 & D_i \Psi_i \end{bmatrix}.\end{aligned}\tag{4.18}$$

We make Assumption 3.1.1 and describe the leader's dynamics by $\dot{\xi}_0 = X\xi_0$ and $\nu_0 \equiv 0$. This assumption can be made without loss of generality; the strictly stable poles on the target invariant subspace do not have any permanent (lasting) effect on the outputs, and their pertaining effects will go to zero exponentially in time.

Define the neighborhood synchronization error in ξ_i s

$$e_{\xi_i} = \sum_{j=1}^N \alpha_{ij}(\xi_i - \xi_j) + g_i(\xi_i - \xi_0).\tag{4.19}$$

Let $\epsilon_i = [e_{\xi_i}^T, \nu_i^T]^T$. Then, the closed-loop dynamics of system (4.1) in the transformed coordinates ϵ_i equals

$$\begin{aligned}\dot{\epsilon}_i &= \bar{A}_i \epsilon_i + \bar{B}_i u_i + \bar{P}_i \omega_i - \sum_{j=1}^N \alpha_{ij}(\bar{B}_j u_j + \bar{P}_j \omega_j + \bar{F}_j \nu_j), \\ u_i &= K_i \left(\sum_{j=1}^N \alpha_{ij}(D_j x_j - D_i x_i) + g_i(D_0 x_0 - D_i x_i) \right) \\ &= K_i \left(\sum_{j=1}^N \alpha_{ij}(R_2 \xi_j - R_2 \xi_i + D_j \Psi_j \nu_j - D_i \Psi_i \nu_i) \right) \\ &\quad + g_i K_i (R_2 \xi_0 - R_2 \xi_i - D_i \Psi_i \nu_i) \\ &= -K_i R_2 e_{\xi_i} - (d_i + g_i) K_i D_i \Psi_i \nu_i + K_i \sum_{j=1}^N \alpha_{ij} D_j \Psi_j \nu_j,\end{aligned}\tag{4.20}$$

where

$$\begin{aligned} \bar{A}_i &:= \begin{bmatrix} X & (d_i + g_i)F_i \\ 0 & \tilde{A}_i \end{bmatrix}, \quad \bar{B}_i := \begin{bmatrix} (d_i + g_i)\tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix}, \quad \bar{P}_i := \begin{bmatrix} (d_i + g_i)\tilde{P}_{1i} \\ \tilde{P}_{2i} \end{bmatrix}, \\ \underline{B}_j &:= \begin{bmatrix} \tilde{B}_{1j} \\ 0 \end{bmatrix}, \quad \underline{P}_j := \begin{bmatrix} \tilde{P}_{1j} \\ 0 \end{bmatrix}, \quad \underline{F}_j := \begin{bmatrix} F_j \\ 0 \end{bmatrix}. \end{aligned} \quad (4.21)$$

The transformed system (4.20) is in a standard form for the graphical/ $2N$ -player games [49,84] and will be used in Section 4.3.

Define $\bar{x}_i = [\delta_{\xi_i}^T, \nu_i^T]^T$, where $\delta_{\xi_i} = \xi_i - \xi_0$. Then, the closed-loop dynamics of system (4.1) in the transformed coordinates equals

$$\begin{aligned} \dot{\bar{x}}_i &= \hat{A}_i \bar{x}_i + \hat{B}_i u_i + \hat{P} \omega_i, \\ e_i &= y_i - y_0 = \tilde{C}_i \bar{x}_i, \\ e_{z_i} &= z_i - z_0 = \tilde{D}_i \bar{x}_i, \\ u_i &= K_i \left(\sum_{j=1}^N \alpha_{ij} (\tilde{D}_j \bar{x}_j - \tilde{D}_i \bar{x}_i) - g_i \tilde{D}_i \bar{x}_i \right). \end{aligned} \quad (4.22)$$

The transformed system (4.22) is used in Section 4.4.

Let $e_\xi, \delta_\xi, \nu, \epsilon$ and $\bar{x}$ be the total vectors of $e_{\xi_i}, \delta_{\xi_i}, \nu_i, \epsilon_i$ and $\bar{x}_i$. Then,

$$\begin{bmatrix} e_\xi \\ \nu \end{bmatrix} = \begin{bmatrix} (L + G) \otimes I_l & 0 \\ 0 & I_{\sum_{i=1}^N n_i - Nl} \end{bmatrix} \begin{bmatrix} \delta_\xi \\ \nu \end{bmatrix} = T_g \begin{bmatrix} \delta_\xi \\ \nu \end{bmatrix}. \quad (4.23)$$

Based on the above equation, we can derive

$$\|\epsilon\|_2 \leq \bar{\sigma}(T_g) \|\bar{x}\|_2, \quad \|\bar{x}\|_2 \leq \frac{1}{\underline{\sigma}(T_g)} \|\epsilon\|_2. \quad (4.24)$$

Remark 4.2.3. In (4.17), we can always select Ψ_i such that $F_i = 0$. From (4.17),

we have $A_i \Psi_i - \Psi_i \tilde{A}_i = \Pi_i F_i$, which is a Sylvester equation. Using the Kronecker product's property, $(I_{\tilde{n}_i} \otimes A_i - \tilde{A}_i^T \otimes I_{n_i}) \text{vec}(\Psi_i) = \text{vec}(\Pi_i F_i)$, where $\tilde{n}_i = n_i - l$. If $F_i = 0$, then we should select

$$\text{vec}(\Psi_i) \in \text{null}(I_{\tilde{n}_i} \otimes A_i - \tilde{A}_i^T \otimes I_{n_i}). \quad (4.25)$$

Let $\lambda_k \in \text{spec}(A_i)$, $\tilde{\lambda}_k \in \text{spec}(\tilde{A}_i)$. Then, $(\lambda_k - \tilde{\lambda}_k) \in \text{spec}(I_{\tilde{n}_i} \otimes A_i - \tilde{A}_i^T \otimes I_{n_i})$. Since $\text{spec}(\tilde{A}_i) \subset \text{spec}(A_i)$, we have $0 \in \text{spec}(I_{\tilde{n}_i} \otimes A_i - \tilde{A}_i^T \otimes I_{n_i})$ and it is always possible to satisfy (4.25).

4.3 Graphical Game H_∞ Controller Design

In this section we formulate the H_∞ output regulation problem as a graphical game and propose how to design the controller gains K_i s. We formulate the H_∞ output regulation in the graphical game framework in Subsection 4.3.1 and derive the overall $\mathcal{L}_2$ -gain of the y -synchronization error with respect to the disturbance in Subsection 4.3.2. Also in Subsection 4.3.3, we propose a numerical solution to the graphical game formulation of the H_∞ output regulation problem which yields the final design.

Let $F_i = 0$, by the appropriate choice of Ψ_i according to Remark 4.2.3. Then, the dynamics in (4.20) is in a standard form for the graphical game, *i.e.*, it depends on the control signals and the disturbances of other agents in the neighborhood. Note however that, in contrast to the homogeneous players in [49,84], the dynamics of the players in (4.20) are generally heterogeneous. The closed-loop system of (4.20)

Figure 4.1: The representation of the nominal and the interconnected systems

reads

$$\begin{aligned}
 \dot{e}_{\xi_i} &= X e_{\xi_i} - (d_i + g_i) \tilde{B}_{1i} K_i R_2 e_{\xi_i} - (d_i + g_i)^2 \tilde{B}_{1i} K_i D_i \Psi_i \nu_i \\
 &\quad + (d_i + g_i) \tilde{B}_{1i} K_i \sum_{j=1}^N \alpha_{ij} D_j \Psi_j \nu_j + (d_i + g_i) \tilde{P}_{1i} \omega_i \\
 &\quad + \sum_{j=1}^N \alpha_{ij} \tilde{B}_{1j} K_j R_2 e_{\xi_j} + \sum_{j=1}^N \alpha_{ij} (d_j + g_j) \tilde{B}_{1j} K_j D_j \Psi_j \nu_j \\
 &\quad - \sum_{j=1}^N \alpha_{ij} \tilde{B}_{1j} K_j \sum_{l=1}^N \alpha_{jl} D_l \Psi_l \nu_l - \sum_{j=1}^N \alpha_{ij} \tilde{P}_{1j} \omega_j, \\
 \dot{\nu}_i &= \tilde{A}_i \nu_i - \tilde{B}_{2i} K_i R_2 e_{\xi_i} - (d_i + g_i) \tilde{B}_{2i} K_i D_i \Psi_i \nu_i + \tilde{P}_{2i} \omega_i \\
 &\quad + \tilde{B}_{2i} K_i \sum_{j=1}^N \alpha_{ij} D_j \Psi_j \nu_j
 \end{aligned} \tag{4.26}$$

One can represent system (4.26) as an interconnection of two subsystems as shown in

Fig. 4.1, which allows the use of an appropriate version of the small gain theorem to ensure $\mathcal{L}_2$ -gain stability of y -synchronization errors with respect to the disturbances. We consider the upper block system in Fig. 4.1 as *the nominal system*, whose equation is given by

$$\begin{aligned}\dot{e}_{\xi_i} = & X e_{\xi_i} - (d_i + g_i) \tilde{B}_{1i} K_i R_2 e_{\xi_i} - (d_i + g_i)^2 \tilde{B}_{1i} \bar{K}_i \nu_i \\ & + (d_i + g_i) \tilde{B}_{1i} \beta_i + (d_i + g_i) \tilde{P}_{1i} \omega_i \\ & + \sum_{j=1}^N \alpha_{ij} (\tilde{B}_{1j} K_j R_2 e_{\xi_j} + (d_j + g_j) \tilde{B}_{1j} \bar{K}_j - \tilde{B}_{1j} \beta_j - \tilde{P}_{1j} \omega_j), \\ \dot{\nu}_i = & \tilde{A}_i \nu_i - \tilde{B}_{2i} K_i R_2 e_{\xi_i} - (d_i + g_i) \tilde{B}_{2i} \bar{K}_i \nu_i + \tilde{P}_{2i} \omega_i + \tilde{B}_{2i} \beta_i,\end{aligned}\tag{4.27}$$

where $\bar{K}_i \in \mathbb{R}^{m_i \times (n_i - l)}$ is a gain matrix to be specified later. The lower block is named *the interconnected system*, which is defined by

$$\begin{aligned}\beta_i = & K_i \sum_{j=1}^N \alpha_{ij} D_j \Psi_j \nu_j - (d_i + g_i) (K_i D_i \Psi_i - \bar{K}_i) \nu_i, \\ \beta = & \begin{bmatrix} \beta_1^T, & \dots, & \beta_N^T \end{bmatrix}^T.\end{aligned}\tag{4.28}$$

The interconnected system (4.28) is resulted from the control signal. Hence, the implementable control signal u_i in (4.20) is decomposed into two terms

$$\begin{aligned}u_i = & u_{op_i} + \beta_i, \\ u_{op_i} = & -K_i R_2 e_{\xi_i} - (d_i + g_i) \bar{K}_i \nu_i.\end{aligned}\tag{4.29}$$

According to (4.29), an open-loop representation of the system in (4.27) is

$$\dot{e}_i = \bar{A}_i e_i + \bar{B}_i u_{op_i} + \bar{P}_i \omega_i + \bar{B}_i \beta_i - \sum_{j=1}^N \alpha_{ij} (\underline{B}_j u_{op_j} + \underline{P}_j \omega_j + \underline{B}_j \beta_j).\tag{4.30}$$

Remark 4.3.1. We name u_{op_i} the optimal control signal (since it will be designed

optimally) and consider β_i as both the interconnected system and the control mismatch, since it comes from the mismatch between the implementable control u_i (4.20) and the optimal control u_{op_i} . The control mismatch β_i contains the term $(d_i + g_i)\bar{K}_i\nu_i$ and u_{op_i} contains $-(d_i + g_i)\bar{K}_i\nu_i$, as shown in (4.28-4.29). Clearly, $(d_i + g_i)\bar{K}_i\nu_i$ has no effect on u_i but we introduce it to facilitate the theoretical developments of the optimal design.

Remark 4.3.2. In the ideal case, it is desired to have $\bar{K}_i = K_i D_i \Psi_i$, then the interconnected system of β_i is simplified to $\beta_i = K_i \sum_{j=1}^N \alpha_{ij} D_j \Psi_j \nu_j$ and the optimal control would be $u_{op_i} = -K_i R_2 e_{\xi_i} - (d_i + g_i) K_i D_i \Psi_i \nu_i$. Requiring $\bar{K}_i = K_i D_i \Psi_i$, however, leads to a constrained optimization problem in which there is no guarantee for the existence of a solution. Hence, $\bar{K}_i = K_i D_i \Psi_i$ is not strictly required in the following development. It is worth to point out that, for the homogeneous agents, i.e., $\nu_i \equiv 0$, $\beta_i \equiv 0$, our problem formulation reduces to the existing one [49].

4.3.1 Controller design *via* zero/nonzero-sum game

In this subsection, the H_∞ y -output regulation is formulated in a zero/nonzero-sum graphical game. Define the following $\mathcal{L}_2$ condition for all $\omega_i, \beta_i \neq 0$, $\cdot \in \mathcal{L}_2[0, +\infty)$

$$\begin{aligned} \int_0^T \{ \epsilon_i^T Q_i \epsilon_i + u_{op_i}^T R_{ii} u_{op_i} + \sum_{j=1}^N \alpha_{ij} u_{op_j}^T R_{ij} u_{op_j} \} dt \leq \\ \gamma_{\xi\nu_i}^2 \int_0^T \{ \omega_i^T S_{1ii} \omega_i + \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j \} dt \\ + \gamma_{\beta_i}^2 \int_0^T \{ \beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j \} dt, \end{aligned} \quad (4.31)$$

where $T > 0$, $Q_i > 0$, $R_{ii} > 0$, $R_{ij} \geq 0$, $S_{1ii} > 0$, $S_{1ij} \geq 0$, $S_{2ii} > 0$, $S_{2ij} \geq 0$, $\gamma_{\xi\nu_i} > 0$, $\gamma_{\beta_i} > 0$. Let $u_{op-i} = \{u_{op_j}, j \in N_i\}$, $\omega_{-i} = \{\omega_j, j \in N_i\}$ and $\beta_{-i} = \{\beta_j, j \in N_i\}$

be the vector of the control and disturbance inputs of the neighbours of agent i . Let $\tau_i(t)$ be the performance output such that

$$\|\tau_i(t)\|_2^2 = \epsilon_i^T Q_i \epsilon_i + u_{op_i}^T R_{ii} u_{op_i} + \sum_{j=1}^N \alpha_{ij} u_{op_j}^T R_{ij} u_{op_j}, \quad (4.32)$$

and defined the disturbances of the system (4.30) as

$$\begin{aligned} \underline{\omega}_i &= \text{Diag}\{\sqrt{\alpha_{i1} S_{1i1}}, \dots, \sqrt{S_{1ii}}, \dots, \sqrt{\alpha_{iN} S_{1iN}}\}, \\ \underline{\beta}_i &= \text{Diag}\{\sqrt{\alpha_{i1} S_{2i1}}, \dots, \sqrt{S_{2ii}}, \dots, \sqrt{\alpha_{iN} S_{2iN}}\}. \end{aligned} \quad (4.33)$$

The $\mathcal{L}_2$ -gain bound in (4.31) is equivalent to the optimization of the following quadratic performance index

$$\begin{aligned} &J_i(\epsilon_i(0), u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) \\ &= \int_0^{+\infty} L_i(\epsilon_i, u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) dt \\ &= \int_0^{+\infty} \left\{ \epsilon_i^T Q_i \epsilon_i + u_{op_i}^T R_{ii} u_{op_i} + \sum_{j=1}^N \alpha_{ij} u_{op_j}^T R_{ij} u_{op_j} - \gamma_{\xi\nu_i}^2 [\omega_i^T S_{1ii} \omega_i \right. \\ &\quad \left. + \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j] - \gamma_{\beta_i}^2 [\beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j] \right\} dt. \end{aligned} \quad (4.34)$$

According to the performance index, the quadratic value function for this $2N$ -player zero/nozero sum graphical game is defined as

$$\begin{aligned} V_i &= \min_{u_{op_i}} \max_{\omega_i} \max_{\beta_i} \int_0^{+\infty} \left\{ \epsilon_i^T Q_i \epsilon_i + u_{op_i}^T R_{ii} u_{op_i} + \sum_{j=1}^N \alpha_{ij} u_{op_j}^T R_{ij} u_{op_j} \right. \\ &\quad \left. - \gamma_{\xi\nu_i}^2 [\omega_i^T S_{1ii} \omega_i + \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j] - \gamma_{\beta_i}^2 [\beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j] \right\} dt. \end{aligned} \quad (4.35)$$

The value function for agent i contains other agent's control actions and disturbances. We consider two types of disturbances: $\underline{\omega}_i$, which accounts for the actual disturbances and $\underline{\beta}_i$, which stands for the control mismatches.

Definition 4.3.1. *A control policy u_{op_i} is admissible if (a) it is continuous, (b) $u_{op_i}(0) = 0$, (c) it stabilizes (4.30), and (d) the value function (4.35) is finite [84].*

Problem 4.3.1. (H_∞ Output Regulation via Zero/Nonzero-sum Graphical Games): *Consider a group of $N + 1$ heterogeneous agents defined by (4.1-4.6). Achieve H_∞ y -output regulation through the optimal design of u_{op_i} from the set of admissible control policies, which minimizes the quadratic value function (4.35) with respect to (4.30), while satisfying (4.31).*

Remark 4.3.3. *The zero-sum qualification relates the gains of $[u_{op_i}, u_{op_{-i}}]$ and the opposing joint gains of $[\omega_i, \omega_{-i}]$, and $[\beta_i, \beta_{-i}]$. The nonzero-sum qualification comes from the cooperation of all agents' controllers to minimize the value functions, and from the cooperation of disturbances to maximize them.*

Define the Hamiltonian as

$$H_i = L_i(\epsilon_i, u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) + \frac{\partial V_i^T}{\partial \epsilon_i} \dot{\epsilon}_i. \quad (4.36)$$

At the equilibrium, one has the stationary conditions

$$\frac{\partial H_i}{\partial u_{op_i}} = 0, \rightarrow u_{op_i}^* = -\frac{1}{2} R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i}{\partial \epsilon_i}, \quad (4.37)$$

$$\frac{\partial H_i}{\partial \omega_i} = 0, \rightarrow \omega_i^* = \frac{1}{2} \gamma_{\xi \nu_i}^{-2} S_{1ii}^{-1} \bar{P}_i^T \frac{\partial V_i}{\partial \epsilon_i}, \quad (4.38)$$

$$\frac{\partial H_i}{\partial \beta_i} = 0, \rightarrow \beta_i^* = \frac{1}{2} \gamma_{\beta_i}^{-2} S_{2ii}^{-1} \bar{B}_i^T \frac{\partial V_i}{\partial \epsilon_i}. \quad (4.39)$$

When the value function (4.35) is finite, using the Bellman equation, a differential

equivalent to the value integral is given in terms of the Hamiltonian function

$$H_i = L_i(\epsilon_i, u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) + \frac{\partial V_i^{*T}}{\partial \epsilon_i} \dot{\epsilon}_i = 0. \quad (4.40)$$

Substituting the control policy (4.37) and the disturbances (4.38-4.39) into (4.40) yields the Hamilton-Jacobi equation

$$\begin{aligned} & \epsilon_i^T Q_i \epsilon_i + \frac{1}{4} \frac{\partial V_i^{*T}}{\partial \epsilon_i} \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial \epsilon_i} + \frac{1}{4} \sum_{j=1}^N \alpha_{ij} \frac{\partial V_j^{*T}}{\partial \epsilon_j} \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T \frac{\partial V_j^*}{\partial \epsilon_j} \\ & - \gamma_{\xi \nu_i}^2 \frac{1}{4} [\gamma_{\xi \nu_i}^{-4} \frac{\partial V_i^{*T}}{\partial \epsilon_i} \bar{P}_i S_{1ii}^{-1} \bar{P}_i^T \frac{\partial V_i^*}{\partial \epsilon_i} + \sum_{j=1}^N \alpha_{ij} \gamma_{\xi \nu_j}^{-4} \frac{\partial V_j^{*T}}{\partial \epsilon_j} \bar{P}_j S_{1jj}^{-1} S_{1ij} S_{1jj}^{-1} \bar{P}_j^T \frac{\partial V_j^*}{\partial \epsilon_j}] \\ & - \gamma_{\beta_i}^2 \frac{1}{4} [\gamma_{\beta_i}^{-4} \frac{\partial V_i^{*T}}{\partial \epsilon_i} \bar{B}_i S_{2ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial \epsilon_i} + \sum_{j=1}^N \alpha_{ij} \gamma_{\beta_j}^{-4} \frac{\partial V_j^{*T}}{\partial \epsilon_j} \bar{B}_j S_{2jj}^{-1} S_{2ij} S_{2jj}^{-1} \bar{B}_j^T \frac{\partial V_j^*}{\partial \epsilon_j}] \\ & + \frac{\partial V_i^{*T}}{\partial \epsilon_i} [\bar{A}_i \epsilon_i - \frac{1}{2} \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial \epsilon_i} + \frac{1}{2} \gamma_{\xi \nu_i}^{-2} \bar{P}_i S_{1ii}^{-1} \bar{P}_i^T \frac{\partial V_i^*}{\partial \epsilon_i} + \frac{1}{2} \gamma_{\beta_i}^{-2} \bar{B}_i S_{2ii}^{-1} \bar{B}_i^T \frac{\partial V_i^*}{\partial \epsilon_i}] \\ & - \frac{\partial V_i^{*T}}{\partial \epsilon_i} [\sum_{j=1}^N \alpha_{ij} \{ -\frac{1}{2} \bar{B}_j R_{jj}^{-1} \bar{B}_j^T \frac{\partial V_j^*}{\partial \epsilon_j} + \frac{1}{2} \gamma_{\xi \nu_j}^{-2} \bar{P}_j S_{1jj}^{-1} \bar{P}_j^T \frac{\partial V_j^*}{\partial \epsilon_j} + \frac{1}{2} \gamma_{\beta_j}^{-2} \bar{B}_j S_{2jj}^{-1} \bar{B}_j^T \frac{\partial V_j^*}{\partial \epsilon_j} \}] = 0. \end{aligned} \quad (4.41)$$

Let V_i^* be the quadratic optimal solution to (4.41) and let $u_{op_i}^*(V_i^*)$, $\omega_i^*(V_i^*)$ and $\beta_i^*(V_i^*)$ be the optimal control policy and the worst-case disturbances in terms of V_i^* . The following theorem shows that such a solution provides stability for the disturbance-free system (4.30) ($\underline{\omega}_i \equiv 0$ and $\underline{\beta}_i \equiv 0$) and bounded $\mathcal{L}_2$ -gains with respect to disturbances $\underline{\omega}_i, \underline{\beta}_i \in \mathcal{L}_2[0, +\infty)$.

Remark 4.3.4. *The solvability of (4.41) via a numerical approach will be discussed in Subsection 4.3.3.*

Theorem 4.3.1. *Suppose V_i^* is a smooth quadratic positive definite solution to (4.41). Using the optimal control $u_{op_i}^*(V_i^*)$ given by (4.37), the disturbance-free closed-loop system of (4.30) is asymptotically stable. Moreover, for all disturbances*

$\omega_i, \omega_{-i}, \beta_i, \beta_{-i} \in \mathcal{L}_2[0, +\infty)$, (4.31) is satisfied.

Proof. First we show the stability of the disturbance-free system, and then we guarantee the $\mathcal{L}_2$ -gain bound. For any smooth value function V_i , the Hamiltonian is defined as

$$\begin{aligned} H_i(\epsilon_i, \nabla V_i, u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) = \\ \epsilon_i^T Q_i \epsilon_i + u_{op_i}^T R_{ii} u_{op_i} + \sum_{j=1}^N \alpha_{ij} u_{op_j}^T R_{ij} u_{op_j} - \gamma_{\xi_{\nu_i}}^2 [\omega_i^T S_{1ii} \omega_i \\ + \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j] - \gamma_{\beta_i}^2 [\beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j] + \frac{dV_i}{dt}. \end{aligned} \quad (4.42)$$

Let V_i^* be a smooth positive semi-definite solution to (4.41). Completing the squares leads to

$$\begin{aligned} H_i(\epsilon_i, \nabla V_i^*, u_{op_i}, u_{op_{-i}}^*, \omega_i, \omega_{-i}^*, \beta_i, \beta_{-i}^*) = \\ H_i(\epsilon_i, \nabla V_i^*, u_{op_i}^*, u_{op_{-i}}^*, \omega_i^*, \omega_{-i}^*, \beta_i^*, \beta_{-i}^*) + (u_{op_i} - u_{op_i}^*)^T R_{ii} (u_{op_i} - u_{op_i}^*) \\ - \gamma_{\xi_{\nu_i}}^2 (\omega_i - \omega_i^*)^T S_{1ii} (\omega_i - \omega_i^*) - \gamma_{\beta_i}^2 (\beta_i - \beta_i^*)^T S_{2ii} (\beta_i - \beta_i^*). \end{aligned}$$

Selecting $u_{op_i} = u_{op_i}^*(V_i^*)$, one has

$$\begin{aligned} \epsilon_i^T Q_i \epsilon_i + u_{op_i}^{*T} R_{ii} u_{op_i}^* + \sum_{j=1}^N \alpha_{ij} u_{op_j}^{*T} R_{ij} u_{op_j}^* - \gamma_{\xi_{\nu_i}}^2 [\omega_i^T S_{1ii} \omega_i \\ + \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j] - \gamma_{\beta_i}^2 [\beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j] + \frac{dV_i^*}{dt} \leq 0. \end{aligned} \quad (4.43)$$

Set $\omega_i = 0, \omega_{-i} = 0, \beta_i = 0, \beta_{-i} = 0$. According to (4.43)

$$\frac{dV_i^*}{dt} \leq -(\epsilon_i^T Q_i \epsilon_i + u_{op_i}^{*T} R_{ii} u_{op_i}^* + \sum_{j=1}^N \alpha_{ij} u_{op_j}^{*T} R_{ij} u_{op_j}^*) \leq 0.$$

Hence, the disturbance-free system (4.30) is asymptotically stable. Next, integrate

(4.43)

$$\begin{aligned}
V_i^*(\epsilon_i(t)) - V_i^*(\epsilon_i(0)) &+ \int_0^t \{ \epsilon_i^T Q_i \epsilon_i + u_{op_i}^{*T} R_{ii} u_{op_i}^* + \sum_{j=1}^N \alpha_{ij} u_{op_j}^{*T} R_{ij} u_{op_j}^* - \gamma_{\xi \nu_i}^2 [\omega_i^T S_{1ii} \omega_i \\
&+ \sum_{j=1}^N \alpha_{ij} \omega_j^T S_{1ij} \omega_j] - \gamma_{\beta_i}^2 [\beta_i^T S_{2ii} \beta_i + \sum_{j=1}^N \alpha_{ij} \beta_j^T S_{2ij} \beta_j] \} dt \leq 0.
\end{aligned}$$

Select $\epsilon_i(0) = 0$. Since $V_i^*(\epsilon_i(0)) = 0$ and $V_i^*(\epsilon_i(t)) > 0$, we have (4.31) guaranteeing the $\mathcal{L}_2$ -gain bound and this completes the proof. $\square$

Remark 4.3.5. *Theorem 4.3.1 uses Hamilton-Jacobi equation (4.41) to prove the stability of the nominal system (4.30) under optimal control (4.37) in the absence of ω , β , disturbances. While ω_i is an external disturbance and could be zero, β_i is not, because β_i is resulted from the implementable control signal and is considered as the interconnected system (4.28). Later, in Theorem 4.3.4 we will present the proof of stability in the presence of β_i s, which analyzes the stability of the overall system containing both the nominal and interconnected systems via the small gain theorem. Although the standard Hamilton-Jacobi approach is used here, we apply it to heterogeneous agents in multi-player games, which did not appear before.*

Remark 4.3.6. *Assuming a quadratic structure for the value function, the control signal (4.37) is*

$$u_{op_i} = K_{i1} e_{\xi_i} + K_{i2} \nu_i,$$

where K_{i1} and K_{i2} are gains obtained from the optimization problem and u_{op_i} (4.29) has the following structure

$$u_{op_i} = -K_i R_2 e_{\xi_i} - (d_i + g_i) \bar{K}_i \nu_i.$$

We can always satisfy $K_{i2} = -(d_i + g_i) \bar{K}_i$ as $\bar{K}_i$ is an arbitrary parameter. In order

to calculate K_i from $K_{i1} = -K_i R_2$, the following condition is required

$$\text{Rank}(R_2) = \text{Rank} \left(\begin{bmatrix} R_2 \\ -K_{i1} \end{bmatrix} \right). \quad (4.44)$$

One can satisfy (4.44) by selecting any invertible R_2 similar to [99,105].

Assumption 4.3.1. R_2 is invertible.

In a zero/nonzero-sum game, we are interested in convergence to a Nash equilibrium (NE). The game has a well-defined NE if

$$\begin{aligned} J_i(u_{op_i}^*, u_{op_{-i}}^*, \omega_i, \omega_{-i}^*, \beta_i, \beta_{-i}^*) &\leq J_i(u_{op_i}^*, u_{op_{-i}}^*, \omega_i^*, \omega_{-i}^*, \beta_i^*, \beta_{-i}^*) \\ &\leq J_i(u_{op_i}, u_{op_{-i}}^*, \omega_i^*, \omega_{-i}^*, \beta_i^*, \beta_{-i}^*). \end{aligned} \quad (4.45)$$

The following theorem shows that a smooth solution to (4.41) guarantees convergence to an NE. For the policies (4.37-4.39) to form an NE, we need the reachability condition for a system with $(\bar{A}_i, \bar{B}_i)$ as its system and input matrices respectively [84]. As the reachability and controllability conditions coincide for continuous-time systems, we make the following assumption hereafter.

Assumption 4.3.2. The pair $(\bar{A}_i, \bar{B}_i)$ is controllable.

Theorem 4.3.2. Suppose V_i^* is a smooth quadratic positive definite solution to (4.41) such that the closed-loop system of

$$\dot{\epsilon}_i = \bar{A}_i \epsilon_i + \bar{B}_i u_{op_i}^* + \bar{P}_i \omega_i^* + \bar{B}_i \beta_i^* - \sum_{j=1}^N \alpha_{ij} (\underline{B}_j u_{op_j}^* + \underline{P}_j \omega_{op_j}^* + \underline{B}_j \beta_j^*) \quad (4.46)$$

is asymptotically stable. Then, the control signal and disturbances (4.37-4.39) form an NE and the optimal value of the game is $V_i^*(\epsilon_i(0))$.

Proof. First, we show (4.37-4.39) form an NE. Rewriting (4.34) and adding zero

$$\begin{aligned} J_i &= \int_0^{+\infty} L_i dt + \int_0^{+\infty} \dot{V}_i dt + V_i(\epsilon_i(0)) - V_i(\epsilon_i(+\infty)) \\ &= \int_0^{+\infty} H_i(\epsilon_i, \nabla V_i, u_{op_i}, u_{op_{-i}}, \omega_i, \omega_{-i}, \beta_i, \beta_{-i}) dt + V_i(\epsilon_i(0)) - V_i(\epsilon_i(+\infty)). \end{aligned}$$

Let V_i^* be a smooth positive semi-definite solution to (4.41). By completing the squares one has

$$\begin{aligned} J_i &= V_i(\epsilon_i(0)) - V_i(\epsilon_i(+\infty)) + \int_0^{+\infty} \{ H_i(\epsilon_i, \nabla V_i^*, u_{op_i}^*, u_{op_{-i}}^*, \omega_i^*, \omega_{-i}^*, \beta_i^*, \beta_{-i}^*) \\ &\quad + (u_{op_i} - u_{op_i}^*)^T R_{ii} (u_{op_i} - u_{op_i}^*) + \sum_{j=1}^N \alpha_{ij} (u_{op_j} - u_{op_j}^*)^T R_{ij} (u_{op_j} - u_{op_j}^*) \\ &\quad + 2 \sum_{j=1}^N \alpha_{ij} u_{op_j}^{*T} R_{ij} (u_{op_j} - u_{op_j}^*) - \sum_{j=1}^N \alpha_{ij} \frac{\partial V_i^T}{\partial \epsilon_i} \underline{B}_j (u_{op_j} - u_{op_j}^*) \\ &\quad - \gamma_{\xi\nu_i}^2 (\omega_i - \omega_i^*)^T S_{1ii} (\omega_i - \omega_i^*) - \gamma_{\xi\nu_i}^2 \sum_{j=1}^N \alpha_{ij} (\omega_j - \omega_j^*)^T S_{1ij} (\omega_j - \omega_j^*) \\ &\quad - 2\gamma_{\xi\nu_i}^2 \sum_{j=1}^N \alpha_{ij} \omega_j^{*T} S_{1ij} (\omega_j - \omega_j^*) - \sum_{j=1}^N \alpha_{ij} \frac{\partial V_i^T}{\partial \epsilon_i} \underline{P}_j (\omega_j - \omega_j^*) \\ &\quad - \gamma_{\beta_i}^2 (\beta_i - \beta_i^*)^T S_{2ii} (\beta_i - \beta_i^*) - \gamma_{\beta_i}^2 \sum_{j=1}^N \alpha_{ij} (\beta_j - \beta_j^*)^T S_{2ij} (\beta_j - \beta_j^*) \\ &\quad - \gamma_{\beta_i}^2 \sum_{j=1}^N \alpha_{ij} \beta_j^{*T} S_{2ij} (\beta_j - \beta_j^*) - \sum_{j=1}^N \alpha_{ij} \frac{\partial V_i^T}{\partial \epsilon_i} \underline{B}_j (\beta_j - \beta_j^*) \} dt. \end{aligned}$$

Since (4.46) is asymptotically stable, $V_i^*(\epsilon_i(+\infty)) = 0$. Set $u_{op_j} = u_{op_j}^*$, $\omega_j = \omega_j^*$, $\beta_j = \beta_j^*$, $\forall j \in N_i$ in the above equation. Then (4.45) follows easily. Next, we obtain the optimal value of the game. Set $u_{op_i} = u_{op_i}^*$, $\omega_i = \omega_i^*$, $\beta_i = \beta_i^*$; then, $J_i^* = V_i^*(\epsilon_i(0))$. $\square$

Figure 4.2: The interconnected system

4.3.2 Small gain theorem and the overall $\mathcal{L}_2$ -gain bound

In this subsection, we derive the overall $\mathcal{L}_2$ -gain of the y -output synchronization error with respect to the total disturbance. For this purpose, we review a well-known version of the small gain theorem.

Theorem 4.3.3. *Let H_1 and H_2 be two systems, whose inputs are u_1, ω_1 and u_2, ω_2 and outputs are y_1, y_2 , respectively. Assume that H_1 and H_2 are interconnected as depicted in Fig. 4.2. Let $\omega_1 = \omega_2$. Furthermore, assume that*

$$\|u_1\|_2 \leq M_1 \|y_2\|_2, \quad \|u_2\|_2 \leq M_2 \|y_1\|_2, \quad (4.47)$$

$$\|y_1\|_{\mathcal{L}_2} \leq L_{11} \|\omega_1\|_{\mathcal{L}_2} + L_{12} \|u_1\|_{\mathcal{L}_2}, \quad (4.48)$$

$$\|y_2\|_{\mathcal{L}_2} \leq L_{22} \|\omega_2\|_{\mathcal{L}_2} + L_{21} \|u_2\|_{\mathcal{L}_2}. \quad (4.49)$$

Then,

$$\|y_1\|_{\mathcal{L}_2} \leq (1 - L_{12} L_{21} M_1 M_2)^{-1} (L_{11} + L_{12} L_{22} M_1) \|\omega\|_{\mathcal{L}_2}, \quad (4.50)$$

and the bounded $\mathcal{L}_2$ -gain of the interconnected system is guaranteed if $L_{12} L_{21} M_1 M_2 <$

1.

Proof. According to the outputs $\mathcal{L}_2$ bounds and the instantaneous norms of the interconnections (4.47-4.49)

$$\begin{aligned}\|y_1\|_{\mathcal{L}_2} &\leq L_{11}\|\omega_1\|_{\mathcal{L}_2} + L_{12}\|u_1\|_{\mathcal{L}_2} \leq L_{11}\|\omega_1\|_{\mathcal{L}_2} + L_{12}M_1\|y_2\|_{\mathcal{L}_2} \\ &\leq L_{11}\|\omega_1\|_{\mathcal{L}_2} + L_{12}M_1(L_{22}\|\omega_2\|_{\mathcal{L}_2} + L_{21}M_2\|y_1\|_{\mathcal{L}_2}).\end{aligned}$$

Take $\omega_1 = \omega_2$. According to the small gain theorem, the finite-gain $\mathcal{L}_2$ stability is guaranteed if $L_{12}L_{21}M_1M_2 < 1$ and (4.50) follows. $\square$

Now, we calculate the $\mathcal{L}_2$ -gains of the performance output τ_i with respect to ω and β which are useful in deriving the overall $\mathcal{L}_2$ -gains of the y -output synchronization error with respect to disturbances. According to (4.32), the performance output τ_i satisfies the following bounded $\mathcal{L}_2$ condition

$$\begin{aligned}\int_0^T \|\tau_i(t)\|_2^2 dt &\leq \gamma_{\xi\nu_i}^2 \int_0^T \|\underline{\omega}_i\|_2^2 dt + \gamma_{\beta_i}^2 \int_0^T \|\underline{\beta}_i\|_2^2 dt \\ &\leq \int_0^T \gamma_{\xi\nu_i}^2 S_{m_i}^2 \|\omega\|_2^2 dt + \int_0^T \gamma_{\beta_i}^2 \beta_{m_i}^2 \|\beta\|_2^2 dt,\end{aligned}\tag{4.51}$$

where

$$\begin{aligned}S_{m_i} &= \max_{i,j \in N_i} \{\bar{\sigma}\{\sqrt{\alpha_{ij}S_{1ij}}\}, \bar{\sigma}\{\sqrt{S_{1ii}}\}\}, \\ \beta_{m_i} &= \max_{i,j \in N_i} \{\bar{\sigma}\{\sqrt{\alpha_{ij}S_{2ij}}\}, \bar{\sigma}\{\sqrt{S_{2ii}}\}\},\end{aligned}\tag{4.52}$$

The following theorem presents the main result of this section, which guarantees the $\mathcal{L}_2$ -gain stability of the total y -output synchronization error e with respect to ω . It is worth to point out that the following result does not need the acyclic condition in contrast to [24,36,81], since we use the small gain theorem to prove the closed-loop stability.

Theorem 4.3.4. *Consider Problem 4.3.1 and let Assumptions 3.1.1, 3.1.5 4.2.1-4.3.2 hold. Let V_i^* be a smooth quadratic positive definite solution to (4.41). Let the*

control signal in (4.37) be the optimal control policy in terms of V_i^* and the controller gain K_i is obtained according to Remark 4.3.6. If $\bar{\sigma}(D_i\Psi_i)$ and $\bar{\sigma}(\bar{K}_i - K_i D_i\Psi_i)$ are sufficiently small, then distributed control (4.10) leads to H_∞ y -output regulation. Furthermore, an upper bound for the $\mathcal{L}_2$ -gain of the y -output synchronization error e with respect to ω is given by

$$\|e\|_{\mathcal{L}_2} \leq L_y(1 - L_{12}M_1)^{-1}L_{11}\|\omega\|_{\mathcal{L}_2}, \quad (4.53)$$

where

$$L_y = \max_i \{\bar{\sigma}(\tilde{C}_i)\} \frac{1}{\underline{\sigma}(T_g)}, \quad (4.54)$$

$$L_{11} = \sqrt{\sum_{i=1}^N \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\xi\nu_i}^2 S_{m_i}^2}, \quad L_{12} = \sqrt{\sum_{i=1}^N \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\beta_i}^2 \beta_{m_i}^2}, \quad (4.55)$$

$$M_1 = \max_i \bar{\sigma}(K_i) \bar{\sigma}(L + G) \max_i \bar{\sigma}(D_i\Psi_i) + \max_i \bar{\sigma}((d_i + g_i)\bar{K}_i). \quad (4.56)$$

Proof. We use Theorem 4.3.3 in this proof. For the performance output τ_i we have

$$\int_0^T \underline{\sigma}(\sqrt{Q_i})^2 \|\epsilon_i\|_2^2 dt \leq \int_0^T \epsilon_i^T Q_i \epsilon_i dt \leq \int_0^T \|\tau_i(t)\|_2^2 dt.$$

Consider system H_1 in Theorem 4.3.3 as the overall systems of (4.30), having output ϵ and two inputs ω and β where ω is the disturbance input and β is related to the control mismatch. Let H_2 be an identity system with $L_{22} = 0$, $L_{21} = 1$ and $M_2 = 1$.

For system H_1 we have

$$\begin{aligned}
\int_0^T \|\epsilon\|_2^2 dt &= \int_0^T \sum_{i=1}^N \|\epsilon_i\|_2^2 dt \\
&\leq \sum_{i=1}^N \left(\int_0^T \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\xi\nu_i}^2 S_{m_i}^2 \|\omega\|_2^2 dt + \int_0^T \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\beta_i}^2 \beta_{m_i}^2 \|\beta\|_2^2 dt \right) \\
&= \sum_{i=1}^N \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\xi\nu_i}^2 S_{m_i}^2 \int_0^T \|\omega\|_2^2 dt + \sum_{i=1}^N \underline{\sigma}(\sqrt{Q_i})^{-2} \gamma_{\beta_i}^2 \beta_{m_i}^2 \int_0^T \|\beta\|_2^2 dt.
\end{aligned}$$

According to the above, (4.55) follows. The total vector β is written as

$$\beta = \left(\text{Diag}\{(d_i + g_i)\bar{K}_i\}_{1:N} - \text{Diag}\{K_i\}_{1:N} (L + G) \otimes I_q \text{Diag}\{D_i \Psi_i\}_{1:N} \right) \begin{bmatrix} 0 & I \end{bmatrix} \epsilon$$

which motivates the following definition for M_1

$$\begin{aligned}
M_1 &:= \max_i \bar{\sigma}(K_i) \bar{\sigma}(L + G) \max_i \bar{\sigma}(D_i \Psi_i) + \max_i \bar{\sigma}((d_i + g_i)\bar{K}_i) \\
&\geq \bar{\sigma} \left(\left(\text{Diag}\{(d_i + g_i)\bar{K}_i\}_{1:N} - \text{Diag}\{K_i\}_{1:N} (L + G) \otimes I_q \text{Diag}\{D_i \Psi_i\}_{1:N} \right) \begin{bmatrix} 0 & I \end{bmatrix} \right).
\end{aligned}$$

Then, (4.50) reads

$$\|\epsilon\|_{\mathcal{L}_2} \leq (1 - L_{12} M_1)^{-1} L_{11} \|\omega\|_{\mathcal{L}_2},$$

and with (4.24) results in (4.53). According to the small gain theorem the finite-gain $\mathcal{L}_2$ stability is guaranteed if $L_{12} M_1 < 1$ which is satisfied in general by keeping M_1 sufficiently small. Hence, sufficiently small $\bar{\sigma}(D_i \Psi_i)$ and $\bar{\sigma}(\bar{K}_i)$ guarantee stability. This is achieved by sufficiently small $\bar{\sigma}(D_i \Psi_i)$ and $\bar{\sigma}(\bar{K}_i - K_i D_i \Psi_i)$, which together guarantee sufficiently small $\bar{\sigma}(\bar{K}_i)$. Hence, the closed-loop systems of (4.1-4.6) achieve H_∞ y -output regulation. $\square$

Remark 4.3.7. *If one selects K_i to be equal to its ideal value $K_i D_i \Psi_i$ (Remark 4.3.2), then having $\bar{\sigma}(D_i \Psi_i)$ sufficiently small is the only requirement for the finite-*

gain $\mathcal{L}_2$ stability in Theorem 4.3.4.

Remark 4.3.8. Theorem 4.3.4 provides a conservative criterion for H_∞ y -output regulation, which is $L_{12}M_1 < 1$. Because of such conservatism, there may exist some choices of K_i , $\bar{K}_i$, ensuring H_∞ y -output regulation but not satisfying the criterion of Theorem 4.3.4. Note that Theorem 4.3.4 implies both an upper bound and a lower bound for the feedback gain. On one hand, the feedback gain should be large enough to guarantee the cooperative stability of ξ_i s. To see this point, considering an extreme case when $K_i = \mathbf{0}$, the system of ξ_i s cannot be stabilized and hence K_i should exceed some lower bound. On the other hand, K_i should be small enough to guarantee the stability of the interconnection of the nominal (4.30) and interconnected (4.28) systems in a small gain sense.

4.3.3 Online policy iteration solution to zero/nonzero-sum graphical game

In this subsection we present a numerical procedure to solve (4.41) which makes the aforementioned design of K_i s feasible.

The Policy Iteration algorithm is widely used to provide an offline solution to the coupled Hamilton-Jacobi equations resulted from the graphical game formulation of state synchronization of homogeneous agents [84,91] and its structure is used to provide an online solution. In this section, we use four adaptive networks to derive an online solution to the graphical game formulation of the H_∞ output regulation of heterogeneous agents. The first network approximates the value function and is named the critic network. The second one approximates the control policy and is named the actor network. The third and fourth networks approximate the disturbance signals ω_i and β_i .

Assume that the solution $V_i(\epsilon_i(t))$ to the Bellman equation is uniformly approxi-

mated by

$$V_i(\epsilon_i) = W_i^T \Phi_i(\epsilon_i) + \varepsilon_i,$$

in which, Φ_i is the activation function vector with μ_{n_i} neurons, W_i is the optimal weight and ε_i is the approximation error. Denote the current weight of the critic network by $\hat{W}_i$, the estimated value function is

$$\hat{V}_i(\epsilon_i) = \hat{W}_i^T \Phi_i(\epsilon_i), \quad (4.57)$$

and the approximated error of the Bellman equation is

$$H_i = L_i + \hat{W}_i^T \frac{\partial \Phi_i}{\partial \epsilon_i} \dot{\epsilon}_i = \varepsilon_{H_i}. \quad (4.58)$$

The control policy is approximated by the actor network

$$\hat{u}_{op_i} = -\frac{1}{2} R_{ii}^{-1} \bar{B}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i} \hat{W}_{i+N}, \quad (4.59)$$

where $\hat{W}_{i+N}$ denotes the current weight of the actor network. Moreover, according to the worst-case disturbances in (4.38-4.39), these two signals are approximated by

$$\hat{\omega}_i = \frac{1}{2} \gamma_{\xi \nu_i}^{-2} S_{1ii}^{-1} \bar{P}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i} \hat{W}_{i+2N}, \quad (4.60)$$

$$\hat{\beta}_i = \frac{1}{2} \gamma_{\beta_i}^{-2} S_{2ii}^{-1} \bar{B}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i} \hat{W}_{i+3N}, \quad (4.61)$$

where $\hat{W}_{i+2N}$ and $\hat{W}_{i+3N}$ denote the current weights of the networks approximating ω_i and β_i respectively.

The following theorem provides the tuning laws for the adaptive networks guaranteeing the closed-loop stability of (4.30) and convergence to an approximated NE.

Theorem 4.3.5. *Consider Problem 4.3.1 and let Assumptions 3.1.1, 3.1.5 4.2.1-4.3.2 hold. Assume that the value function, the control policy and the disturbances are estimated by (4.57) and (4.59-4.61) respectively. Assume that*

$$\sigma_{i+N} = \frac{\partial \Phi_i}{\partial \epsilon_i} \{ \bar{A}_i \epsilon_i + \bar{B}_i \hat{u}_{op_i} + \bar{P}_i \hat{\omega}_i + \bar{B}_i \hat{\beta}_i - \sum_{j=1}^N \alpha_{ij} (\underline{B}_j \hat{u}_{op_j} + \underline{P}_j \hat{\omega}_j + \underline{B}_j \hat{\beta}_j) \}, \quad (4.62)$$

is Persistently Exciting (PE). Tune the weights of the critic network as

$$\dot{\hat{W}}_i = -a_i \frac{\sigma_{i+N}}{(1 + \sigma_{i+N}^T \sigma_{i+N})^2} \varepsilon_{H_i}^{op} \quad (4.63)$$

where $\varepsilon_{H_i}^{op}$ is obtained by replacing (4.59-4.61) in (4.58). Tune the weights of the actor and disturbance networks as

$$\begin{aligned} \dot{\hat{W}}_{i+N} &= -a_{i+N} \{ (\bar{S}_{1i} \hat{W}_{i+N} - \bar{T}_{1i} \bar{\sigma}_{i+N}^T \hat{W}_i) - \frac{1}{4} \bar{D}_i \hat{W}_{i+N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i \\ &\quad - \frac{1}{4} \sum_{j=1}^N \alpha_{ij} \frac{\partial \Phi_j}{\partial \epsilon_j} \bar{B}_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} \bar{B}_j^T \frac{\partial \Phi_j^T}{\partial \epsilon_j} \hat{W}_j \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_{i+N} \}, \\ \dot{\hat{W}}_{i+2N} &= -a_{i+2N} \{ (\bar{S}_{2i} \hat{W}_{i+2N} - \bar{T}_{2i} \bar{\sigma}_{i+N}^T \hat{W}_i) + \frac{1}{4} \gamma_{\xi \nu_i}^{-2} \bar{E}_{2i} \hat{W}_{i+2N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i \\ &\quad + \frac{1}{4} \gamma_{\xi \nu_i}^{-2} \sum_{j=1}^N \alpha_{ij} \frac{\partial \Phi_j}{\partial \epsilon_j} \bar{P}_j S_{1jj}^{-1} S_{1ij} S_{1jj}^{-1} \bar{P}_j^T \frac{\partial \Phi_j^T}{\partial \epsilon_j} \hat{W}_j \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_{i+2N} \}, \\ \dot{\hat{W}}_{i+3N} &= -a_{i+3N} \{ (\bar{S}_{3i} \hat{W}_{i+3N} - \bar{T}_{3i} \bar{\sigma}_{i+N}^T \hat{W}_i) + \frac{1}{4} \gamma_{\beta_i}^{-2} \bar{E}_{3i} \hat{W}_{i+3N} \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_i \\ &\quad + \frac{1}{4} \gamma_{\beta_i}^{-2} \sum_{j=1}^N \alpha_{ij} \frac{\partial \Phi_j}{\partial \epsilon_j} \bar{B}_j S_{2jj}^{-1} S_{2ij} S_{2jj}^{-1} \bar{B}_j^T \frac{\partial \Phi_j^T}{\partial \epsilon_j} \hat{W}_j \frac{\bar{\sigma}_{i+N}^T}{m_{s_i}} \hat{W}_{i+3N} \}, \end{aligned} \quad (4.64)$$

where,

$$\begin{aligned}
\bar{\sigma}_{i+N} &= \frac{\sigma_{i+N}}{1 + \sigma_{i+N}^T \sigma_{i+N}}, \\
m_{s_i} &= 1 + \sigma_{i+N}^T \sigma_{i+N}, \\
\bar{D}_i &= \frac{\partial \Phi_i}{\partial \epsilon_i} \bar{B}_i R_{ii}^{-1} \bar{B}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i}, \\
\bar{E}_{2i} &= \frac{\partial \Phi_i}{\partial \epsilon_i} \bar{P}_i S_{1ii}^{-1} \bar{P}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i}, \\
\bar{E}_{3i} &= \frac{\partial \Phi_i}{\partial \epsilon_i} \bar{B}_i S_{2ii}^{-1} \bar{B}_i^T \frac{\partial \Phi_i^T}{\partial \epsilon_i},
\end{aligned} \tag{4.65}$$

and $a_i > 0$, $a_{i+N} > 0$, $a_{i+2N} > 0$, $\bar{T}_{1i} > 0$, $\bar{S}_{1i} > 0$, $\bar{T}_{2i} > 0$, $\bar{S}_{2i} > 0$, $\bar{T}_{3i} > 0$, $\bar{S}_{3i} > 0$, $i = 1, \dots, N$ are tuning parameters. Then,

1. The states ϵ_i , $\tilde{W}_i = W_i - \hat{W}_i$, $\tilde{W}_{i+N} = W_i - \hat{W}_{i+N}$, $\tilde{W}_{i+2N} = W_i - \hat{W}_{i+2N}$ and $\tilde{W}_{i+3N} = W_i - \hat{W}_{i+3N}$ are Uniformly Ultimately Bounded (UUB).
2. ε_{H_i} is UUB and $\hat{W}_i$ converges to the approximated solution to the Hamilton-Jacobi (4.41).
3. $\hat{u}_{op_i}$, $\hat{\omega}_i$, $\hat{\beta}_i$ converge to the approximated NE.

Proof. Under Assumption 4.3.2, the proof is similar to the Proof of Theorem 2.6.1 given in Appendix A. □

Remark 4.3.9. Motivated by the optimal control signal $u_{op_i} = -K_i R_2 e_{\xi_i} - (d_i + g_i) \bar{K}_i \nu_i$ a quadratic structure for the value function is considered here. Since the state dimension of agent i is n_i , the total number of $0.5n_i(n_i + 1)$ neurons is chosen to approximate the kernel of the critic network. Let P_i^g be the kernel of the value function obtained from the weights of the value function, $\hat{W}_i$. Then, the gain $K_i = -K_{i1} R_2^{-1}$ is derived from $\begin{bmatrix} K_{i1} & K_{i2} \end{bmatrix} = -R_{ii}^{-1} \bar{B}_i^T P_i^g$.

4.4 An LMI Approach to H_∞ Control

In this section, an LMI approach for the design of the controller gains K_i s is suggested that guarantees H_∞ y -output regulation. In subsection 4.4.1, we provide a sufficient H_∞ condition to guarantee output regulation in the absence of disturbances. In subsection 4.4.2, we prove that the derived H_∞ criterion is also sufficient for the H_∞ output regulation and calculate an upper bound for the $\mathcal{L}_2$ -gain. In subsection 4.4.3 we select the controller gains to satisfy that H_∞ criterion *via* a set of LMIs. Here, we assume that the graph is normalized.

4.4.1 A single-agent sufficient condition in absence of disturbances

We first derive a sufficient condition for y -output regulation in absence of disturbances, which is useful in both revealing the appropriate structure of the matrix A_i and designing the controller gain K_i . The interest in the specific structure of A_i is due to results in distributed output regulation [23,24,40,81,92], which incorporate a p -copy model of X by some hierarchical structure and suggest a state-feedback or an output-feedback. For the developments of this section, we consider the system in (4.22)

$$\dot{\bar{x}}_i = \hat{A}_i \bar{x}_i + \hat{B}_i u_i + \hat{P} \omega_i.$$

Define

$$\begin{aligned}\tilde{\Gamma} &:= (L + G) \otimes I_q = I + Z, \\ A_i^n &:= \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} - \hat{B}_i K_i \tilde{D}_i, \quad C_i^n := \begin{bmatrix} 0 & I \end{bmatrix}.\end{aligned}\tag{4.66}$$

Since our aim is to design an output-feedback, we make the following standard assumption.

Assumption 4.4.1. *The triple $(\hat{A}_i, \hat{B}_i, \tilde{D}_i)$ is output-feedback stabilizable.*

The next theorem is based on the small gain theorem, which decouples the agents from the detailed graph topology and suggests single-agent conditions allowing for distributed design.

Theorem 4.4.1. *Consider a group of $N + 1$ heterogeneous agents defined by (4.1-4.6). Let Assumptions 3.1.1, 3.1.5, 4.2.1 and 4.4.1 hold. If*

$$\|\tilde{D}_i(sI - \hat{A}_i + \hat{B}_i K_i \tilde{D}_i)^{-1} \hat{B}_i K_i\|_\infty < \frac{1}{\rho(Z)} = \gamma_z,\tag{4.67}$$

for $i = 1, \dots, N$, then the disturbance-free system (4.1-4.6) achieves y -output regulation and there exists a $0 < \gamma_i < +\infty$ such that

$$\|\tilde{C}_i(sI - \hat{A}_i + \hat{B}_i K_i \tilde{D}_i)^{-1} \hat{P}_i\|_\infty < \gamma_i,\tag{4.68}$$

for all $i = 1, \dots, N$.

Proof. First, we show that (4.67) is a sufficient condition for y -output regulation in the disturbance-free system and then we show the existence of γ_i satisfying (4.68).

Set $\omega_i \equiv 0$ for $i = 1, \dots, N$. Let $\tilde{D} = \underset{1:N}{\text{Diag}}\{\tilde{D}_i\}$, $\tilde{K} = \underset{1:N}{\text{Diag}}\{K_i\}$, $\hat{A} = \underset{1:N}{\text{Diag}}\{\hat{A}_i\}$,

$\hat{B} = \text{Diag}\{\hat{B}_i\}_{1:N}$. Then, the total closed-loop system of ((4.22) reads

$$\dot{\bar{x}} = (\hat{A} - \hat{B}\tilde{K}\tilde{\Gamma}\tilde{D})\bar{x}. \quad (4.69)$$

According to (4.66), the characteristic equation of the above system reads

$$\begin{aligned} \Delta &= |sI - (\hat{A} - \hat{B}\tilde{K}\tilde{\Gamma}\tilde{D})| \\ &= |sI - \hat{A} + \hat{B}\tilde{K}\tilde{D} + \hat{B}\tilde{K}Z\tilde{D}| \\ &= |sI - (\hat{A} - \hat{B}\tilde{K}\tilde{D})| \times \\ &\quad |I + \tilde{D}(sI - \hat{A} + \hat{B}\tilde{K}\tilde{D})^{-1}\hat{B}\tilde{K}Z|. \end{aligned}$$

Based on the small gain theorem, if the following condition holds

$$\|\tilde{D}_i(sI - \hat{A}_i + \hat{B}_iK_i\tilde{D}_i)^{-1}\hat{B}_iK_i\|_\infty < \frac{1}{\max\{|\lambda_i(Z)|\}} = \frac{1}{\rho(Z)}$$

for $i = 1, \dots, N$, then the system in (4.69) is stable and the disturbance-free system achieves y -output regulation. According to Theorem 5.4 of [106], an LTI system with a Hurwitz system matrix, is finite-gain $\mathcal{L}_2$ stable. Since (4.67) guarantees stability of $(\hat{A}_i - \hat{B}_iK_i\tilde{D}_i)$ for $i = 1, \dots, N$, then there exists a $0 < \gamma_i < \infty$ for $i = 1, \dots, N$ such that (4.68) holds. $\square$

The next theorem suggests a structure of the drift dynamics $\hat{A}_i$ that guarantees (4.67).

Theorem 4.4.2. *Consider a group of $N + 1$ heterogeneous agents defined by (4.1-4.6). Let Assumptions 3.1.1, 3.1.5 and 4.2.1 . Given K_i for each $i = 1, \dots, N$, let*

$P_i^L > 0$ and $\sigma_i > 0$ satisfy the following LMI

$$\begin{bmatrix} (A_i^n)^T P_i^L + P_i^L A_i^n - \sigma_i (C_i^n)^T C_i^n + \tilde{D}_i^T \tilde{D}_i & P_i^L \hat{B}_i K_i \\ K_i^T \hat{B}_i^T P_i^L & -\gamma_z^2 I \end{bmatrix} < 0 \quad (4.70)$$

for each $i = 1, \dots, N$. Then, the disturbance-free system (4.1-4.6) achieves y -output regulation if the following structure is imposed on the drift dynamics of each agent

$$L_i^n := \begin{bmatrix} F_i \\ \tilde{A}_i \end{bmatrix} = -\frac{\sigma_i}{2} (P_i^L)^{-1} (C_i^n)^T. \quad (4.71)$$

Proof. The norm condition (4.67) is equivalent to

$$\begin{bmatrix} (\hat{A}_i - \hat{B}_i K_i \tilde{D}_i)^T P_i^L + P_i^L (\hat{A}_i - \hat{B}_i K_i \tilde{D}_i) + \tilde{D}_i^T \tilde{D}_i & P_i^L \hat{B}_i K_i \\ K_i^T \hat{B}_i^T P_i^L & -\gamma_z^2 I \end{bmatrix} < 0$$

or

$$\begin{bmatrix} (A_i^n + L_i^n C_i^n)^T P_i^L + P_i^L (A_i^n + L_i^n C_i^n) + \tilde{D}_i^T \tilde{D}_i & P_i^L \hat{B}_i K_i \\ K_i^T \hat{B}_i^T P_i^L & -\gamma_z^2 I \end{bmatrix} < 0.$$

Introducing $Y^i = P_i^L L_i^n$, the above equation is written as

$$\begin{bmatrix} (A_i^n)^T P_i^L + P_i^L (A_i^n) + Y^i C_i^n + (Y^i C_i^n)^T + \tilde{D}_i^T \tilde{D}_i & P_i^L \hat{B}_i K_i \\ K_i^T \hat{B}_i^T P_i^L & -\gamma_z^2 I \end{bmatrix} < 0.$$

Using Finsler's lemma and considering $\sigma_i > 0$ the above equation is equivalent to (4.70) which suggest to select L_i^n as in (4.71). $\square$

Remark 4.4.1. The LMI (4.70) is satisfied only if the pair (A_i^n, C_i^n) is detectable; otherwise, there is no L_i^n such that $A_i^n + L_i^n C_i^n$ is Hurwitz. Also, note that the matrix

F_i obtained from Theorem 4.4.2 is generally nonzero.

Remark 4.4.2. *In contrast to [24,36,81], which assume an acyclic graph in order to transform the system matrix of the total system to a triangular form, here, we drop that assumption by using the small gain theorem to prove the stability of the total system matrix. Our general result in Theorem 4.4.1 (the norm condition (4.67) contains the special case of acyclic graphs reported in [24,36,81]. Corollary 4.4.1 shows that the norm condition in (4.67) reduces to the stability of for acyclic graphs.*

Corollary 4.4.1. *Consider a group of $N + 1$ heterogeneous agents (4.1-4.6) with an acyclic normalized graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}})$. Let Assumptions 3.1.1, 4.2.1 and 4.4.1. Assume (4.68) holds for some $0 < \gamma_i < +\infty$. Then, the disturbance-free system (4.1-4.6) achieves y -output regulation.*

Proof. For an acyclic graph $\rho(Z) = 0$ [40]. Hence if (4.68) holds, $\hat{A}_i - \hat{B}_i K_i \tilde{D}_i$ is stable and the norm condition (4.67) in Theorem 4.4.1 is always satisfied. $\square$

Remark 4.4.3. *The results of Theorem 4.4.1 can be used for specifying the system matrix structure or the controller design. To find the structure of the system matrix, $\hat{A}_i$ one needs to evaluate (4.70) for the selected K_i and use (4.71) to find F_i and $\tilde{A}_i$. Later in Subsection 4.4.3 we will show that, by means of (4.67) it is possible to design K_i s by a set of LMIs.*

4.4.2 The $\mathcal{L}_2$ -gain of the total y -synchronization error with respect to disturbance

The previous subsection deals with output regulation for a disturbance-free system. We now deal with disturbances by showing that (4.67) guarantees a bounded $\mathcal{L}_2$ -gain for the output synchronization error with respect to total disturbance, and

Figure 4.3: The global system of (4.22)

derive an $\mathcal{L}_2$ -gain bound. Figure 4.3 shows the total closed-loop system of (4.22).

Define

$$\underline{A}_i := \hat{A}_i - \hat{B}_i K_i \tilde{D}_i. \quad (4.72)$$

Theorem 4.4.3. *Consider a group of $N + 1$ heterogeneous agents (4.1-4.6). Let Assumptions 3.1.1, 3.1.5, 4.2.1 and 4.4.1 hold. Assume (4.67) holds for $i = 1, \dots, N$. Then, the system of (4.1-4.6) achieves H_∞ y -output regulation and an upper bound for the $\mathcal{L}_2$ -gain of the y -synchronization error e with respect to ω is given by*

$$\mathbf{L} = \gamma_{11} + \gamma_{12} \bar{\sigma}(Z) \gamma_{22} \gamma_{21}, \quad (4.73)$$

where

$$\begin{aligned}
\gamma_{11} &= \max_i \gamma_i, \quad \gamma_{12} = \max_i \|\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\|_\infty, \\
\gamma_{21} &= \max_i \|\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\|_\infty, \\
\gamma_{22} &= \|(I - \text{Diag}_{1:N}\{\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\} Z)^{-1}\|_\infty.
\end{aligned} \tag{4.74}$$

Proof. Let

$$\begin{aligned}
F_{11} &= \text{Diag}_{1:N}\{\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\}, \\
F_{12} &= \text{Diag}_{1:N}\{\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\}, \\
F_{21} &= \text{Diag}_{1:N}\{\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\}, \\
F_{22} &= \text{Diag}_{1:N}\{\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\}.
\end{aligned}$$

Then the transfer function from input ω to e denoted by F_{tot} is

$$F_{tot} = F_{11} + F_{12}Z(I - F_{22}Z)^{-1}F_{21}.$$

According to the small gain-theorem, a sufficient condition for the stability of the F_{tot} is $\rho(F_{22}Z) < 1$. Using Lemmas 3.2.1 and 3.2.2, $\rho(F_{22}Z) \leq \max_i \|\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\|_\infty \rho(Z) < 1$. Hence, if (4.67) is satisfied, then F_{tot} is stable and the finite-gain $\mathcal{L}_2$ stability is guaranteed. Next, we obtain the overall $\mathcal{L}_2$ -gain of the y -output synchronization error e with respect to ω

$$\|F_{tot}\|_\infty \leq \|F_{11}\|_\infty + \|F_{12}\|_\infty \bar{\sigma}(Z) \|(I - F_{22}Z)^{-1}\|_\infty \|F_{21}\|_\infty.$$

Since

$$\begin{aligned}\|F_{11}\|_\infty &\leq \max_i \|\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\|_\infty, \quad \|F_{12}\|_\infty \leq \max_i \|\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{B}_i K_i\|_\infty, \\ \|F_{21}\|_\infty &\leq \max_i \|\tilde{D}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\|_\infty,\end{aligned}$$

and $\|\tilde{C}_i(sI - \underline{A}_i)^{-1} \hat{P}_i\|_\infty < \gamma_i$ (4.68), then, an upper bound for the $\mathcal{L}_2$ -gain is given by (4.73-4.74). $\square$

Now, we are ready to define the problem of H_∞ y -output regulation *via* LMIs.

Problem 4.4.1. (*Output Regulation via Single-agent LMIs*): Consider a group of $N + 1$ heterogeneous agents defined by (4.1-4.6). Achieve H_∞ y -output regulation through designing K_i via solving a set of LMIs to satisfy (4.67).

4.4.3 An LMI controller design

We now discuss how to design the controller gain K_i based on a set of LMIs to satisfy (4.67). Define

$$\Xi_i := \begin{bmatrix} \hat{A}_i^T P_i^L + P_i^L \hat{A}_i & 0 & \tilde{D}_i^T \\ 0 & -\gamma_z^2 I & 0 \\ \tilde{D}_i & 0 & -I \end{bmatrix}, \quad (4.75)$$

$$\Xi_i^y := \begin{bmatrix} \hat{A}_i X_i^L + X_i^L \hat{A}_i^T & 0 & X_i^L \tilde{D}_i^T \\ 0 & -\gamma_z^2 I & 0 \\ \tilde{D}_i X_i^L & 0 & -I \end{bmatrix}, \quad (4.76)$$

$$\mathcal{U}_i := \begin{bmatrix} -\tilde{D}_i & I & 0 \end{bmatrix}, \quad \mathcal{V}_i := \begin{bmatrix} \hat{B}_i^T P_i^L & 0 & 0 \end{bmatrix} = \mathcal{Y}_i E_i, \quad (4.77)$$

$$\mathcal{Y}_i := \begin{bmatrix} \hat{B}_i^T & 0 & 0 \end{bmatrix}, \quad E_i := \begin{bmatrix} P_i^L & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \quad (4.78)$$

The matrices P_i^L and X_i^L are decision matrices which will be introduced shortly.

Theorem 4.4.4. *Let W_{U_i} , W_{V_i} and W_{Y_i} be the orthogonal complements of $\mathcal{U}_i$, $\mathcal{V}_i$ and $\mathcal{Y}_i$ respectively. Then (4.67) is satisfied if and only if there exist matrices $X_i^L, P_i^L > 0 \in \mathbb{R}^{n_i \times n_i}$ such that*

$$W_{Y_i}^T \Xi_i^y W_{Y_i} < 0, \quad (4.79)$$

$$W_{U_i}^T \Xi_i W_{U_i} < 0, \quad (4.80)$$

$$X_i^L P_i^L = I. \quad (4.81)$$

Then, the controller gain is obtained from

$$\Xi_i + \mathcal{V}_i^T K_i \mathcal{U}_i + \mathcal{U}_i^T K_i^T \mathcal{V}_i < 0. \quad (4.82)$$

Proof. The norm condition in (4.67) is satisfied if and only if there exists a $P_i^L > 0$ such that

$$\begin{bmatrix} (\hat{A}_i - \hat{B}_i K_i \tilde{D}_i)^T P_i^L + P_i^L (\hat{A}_i - \hat{B}_i K_i \tilde{D}_i) & P_i^L \hat{B}_i K_i & \tilde{D}_i^T \\ K_i^T \hat{B}_i^T P_i^L & -\gamma_z^2 I & 0 \\ \tilde{D}_i & 0 & -I \end{bmatrix} < 0.$$

According to the definitions in (4.75-4.78), the above LMI is equivalent to (4.82).

Using the Elimination of Matrix Variable Lemma [96], (4.82) is feasible if and only if

$$\begin{aligned} W_{V_i}^T \Xi_i W_{V_i} &< 0, \\ W_{U_i}^T \Xi_i W_{U_i} &< 0. \end{aligned} \quad (4.83)$$

As the columns of W_{Y_i} span $\text{Ker}(\mathcal{Y}_i)$: $(\mathcal{Y}_i W_{Y_i} = 0)$, the columns of $E_i^{-1} W_{Y_i}$

$\text{span } \text{Ker}(\mathcal{V}_i) = \text{Ker}(\mathcal{Y}_i E_i) : \mathcal{Y}_i E_i E_i^{-1} W_{Y_i} = 0$ and we can conclude that $W_{v_i} = E_i^{-1} W_{Y_i}$. Then (4.83) becomes

$$W_{Y_i}^T E_i^{-1} \Xi_i E_i^{-1} W_{Y_i} < 0,$$

$$W_{U_i}^T \Xi_i W_{U_i} < 0.$$

Introducing $E_i^{-1} \Xi_i E_i^{-1}$

$$E_i^{-1} \Xi_i E_i^{-1} = \begin{bmatrix} \hat{A}_i (P_i^L)^{-1} + (P_i^L)^{-1} \hat{A}_i^T & 0 & (P_i^L)^{-1} \tilde{D}_i^T \\ 0 & -\gamma_z^2 I & 0 \\ \tilde{D}_i (P_i^L)^{-1} & 0 & -I \end{bmatrix}.$$

Let $X_i^L = (P_i^L)^{-1}$, then (4.81) follows. $\square$

According to Theorem 4.4.4, a feasible solution to the set of two LMIs and one non-convex algebraic constraint in (4.79-4.81) is required. According to [97], an equivalent representation for the LMIs and the non-convex algebraic constraint in (4.79-4.81) is the following optimization problem

$$\begin{aligned} \min \text{Tr}(X_i^L P_i^L) \\ \left\{ \begin{array}{l} W_{Y_i}^T \Xi_i^y W_{Y_i} < 0, \\ W_{U_i}^T \Xi_i W_{U_i} < 0, \\ \begin{bmatrix} X_i^L & I \\ I & P_i^L \end{bmatrix} \geq 0. \end{array} \right. \end{aligned} \quad (4.84)$$

Then, there exists a feasible solution to (4.79-4.81) if and only if the global minimum of the cost function in (4.84) equals n_i . According to [97], such a problem is named Cone Complementarity Problem and Algorithm 4.1 provides a solution to it. Note that ε_{est} is the estimation threshold.

Algorithm 4.1 Iterative solution to (4.79-4.81)

Require: Find a feasible solution $X_i^{L(0)}, P_i^{L(0)}$ to (4.79-4.80) if any. Set $k=0$.

repeat

Set $\Sigma_i^{(k)} = X_i^{L(k)}$ and $\Omega_i^{(k)} = P_i^{L(k)}$. Find $X_i^{L(k+1)}$ and $P_i^{L(k+1)}$ that solves

$$\min \quad \text{Tr}(X_i^{L(k+1)} \Omega_i^{(k)} + \Sigma_i^{(k)} P_i^{L(k+1)}), \quad (4.85)$$

$$\text{w.r.t} \quad \begin{cases} W_{Y_i}^T \Psi_{Y_i}^{k+1} W_{Y_i} < 0, \\ W_{U_i}^T \Psi_{U_i}^{k+1} W_{U_i} < 0, \\ \begin{bmatrix} X_i^{L(k+1)} & I \\ I & P_i^{L(k+1)} \end{bmatrix} \geq 0. \end{cases} \quad (4.86)$$

until $|\text{Tr}(X_i^{L(k+1)} \Omega_i^{(k)} + \Sigma_i^{(k)} P_i^{L(k+1)} - X_i^{L(k)} \Omega_i^{(k-1)} - \Sigma_i^{(k-1)} P_i^{L(k)})| < \varepsilon_{est}$.

Remark 4.4.4. According to Theorem 4.4.4, the norm condition (4.67) is satisfied if and only if there exists a solution to the set of two LMIs and one nonconvex constraint in (4.79-4.81), and based on [97], there exists a solution to (4.79-4.81), if and only if the optimal value of the cost function in Algorithm 4.1 equals $2n_i$. Hence, the solution $2n_i$ for the cost function in Algorithm 4.1 is both necessary and sufficient for the existence of a gain matrix K_i satisfying (4.67). Algorithm 4.1 is initiated by a feasible solution to (4.79-4.80) and it is proven in [97] that it converges. Once it converges, it gives the variables P_i^L, X_i^L and based on them, one can check if the value of the cost function in Algorithm 4.1 equals $2n_i$. If it does, one can calculate K_i as a feasible solution to the LMI (4.82) for the obtained P_i^L . It is worth to mention that Algorithm 4.1 iterates over a linear programming problem and many standard solvers are available.

4.5 Simulation Results

This section shows the effectiveness of the two independently proposed methods. Consider a group of two followers and one leader as shown in Fig. 4.4. In the graph definition, weights and pinning gains are selected as zero or unity. Consider the

Figure 4.4: The communication graph

agents' dynamics as

$$\left\{ \begin{array}{l} A_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ C_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \end{array} \right. \quad (4.87)$$

$$\left\{ \begin{array}{l} A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -7 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ -0.2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ C_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.005 \end{bmatrix}, \end{array} \right. \quad (4.88)$$

$$\left\{ \begin{array}{l} A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \\ 0.1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ C_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & -0.004 \\ 0 & 1 & 0 \end{bmatrix}. \end{array} \right. \quad (4.89)$$

Simple calculations reveal that

$$\begin{aligned} X &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Psi_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad i = 0, 1, 2, \\ R_1 &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

4.5.1 The graphical game design of the controller

We used four adaptive networks and Reinforcement Learning to solve the multi-player game as described in Subsection 4.3.3. Our algorithm uses measured data online and updates the adaptive weights to learn the optimal solution to the multi-player zero/nonzero-sum game. The overall complexity of our algorithm depends on the maximum in-degree of the graph [84]. Choose the weight matrices in (4.31) as

$$\begin{aligned} Q_i &= 10I, \quad S_{1ii} = 10I, \quad S_{2ii} = 10I, \quad R_{ii} = 1, \gamma_{\xi\nu_i} = 0.8, \quad \gamma_{\beta_i} = 0.8, \\ S_{1ij} &= 0.1I, \quad S_{2ij} = 0.1I, \quad R_{ij} = 0.1 \quad i = 1, 2, \quad j \in N_i, \end{aligned}$$

Figure 4.5: The worst-case disturbances for agents 1-2 in the graphical game

and select the tuning parameters as

$$a_i = a_{i+N} = a_{i+2N} = a_{i+3N} = 0.1,$$

$$\bar{S}_{1i} = \bar{S}_{2i} = \bar{S}_{3i} = 0.1I, \quad \bar{T}_{1i} = \bar{T}_{2i} = \bar{T}_{3i} = 0.1I, \quad i = 1, 2,$$

where I is the identity matrix of a proper dimension. The graphical game is implemented according to Theorem 4.3.5 and the gain K_i is obtained as shown in Remark 4.3.9. The worst-case disturbances in (4.60) are applied to the agents and are shown in Fig. 4.5. The y -output synchronization errors for agents 1-2 are shown in Fig. 4.6. Our simulation example illustrates feasibility and efficiency of adaptive networks for solving (4.41) - those networks perform sufficiently fast to be run online for reasonably large networks.

Remark 4.5.1. *The given multi-agent system satisfies the necessary assumptions*

Figure 4.6: y -output synchronization error in the graphical game approach

for H_∞ y -output regulation via a zero/nonzero sum game; namely Assumption 4.3.1 and Assumption 4.3.2. Moreover, the $\mathcal{L}_2$ -gain stability criterion in Theorem 4.3.4 is satisfied for this system setup.

4.5.2 The LMI design of the controller

For the graph in Fig. 4.4, $\gamma_Z = 1.4142$ and Algorithm 4.1 admits the following solutions for the followers

$$\begin{aligned}
 P_1^L &= \begin{bmatrix} 1.568 & 0.349 & -0.002 \\ 0.349 & 0.853 & 0.001 \\ -0.002 & 0.001 & 3.001 \end{bmatrix}, & X_1^L &= \begin{bmatrix} 0.702 & -0.288 & 0.001 \\ -0.288 & 1.291 & -0.001 \\ 0.001 & -0.001 & 0.333 \end{bmatrix}, \\
 P_2^L &= \begin{bmatrix} 1.282 & -0.495 & -0.001 \\ -0.495 & 1.148 & -0.001 \\ -0.001 & -0.001 & 13.09 \end{bmatrix}, & X_2^L &= \begin{bmatrix} 0.937 & 0.404 & 0.000 \\ 0.404 & 1.046 & 0.000 \\ 0.000 & 0.000 & 0.076 \end{bmatrix}, \\
 K_1 &= \begin{bmatrix} 0.8427 & 2.0008 \end{bmatrix}, & K_2 &= \begin{bmatrix} 0.9617 & 0.0651 \end{bmatrix}.
 \end{aligned}$$

Note that in this example the following disturbance is applied to each follower

$$\omega_i(t) = \begin{cases} 10 \sin t, & 100 \leq t \leq 150 \\ 0, & \text{otherwise} \end{cases}.$$

Figure 4.7 depicts the y -output synchronization errors for agents 1-2.

Figure 4.7: y -output synchronization error in the LMI approach

Chapter 5

Bipartite Output Regulation of Linear Heterogeneous Agents

In this chapter, we study Bipartite Output Regulation (BOR) of linear heterogeneous multi-agent systems, where the outputs synchronize in modulus with possibly different signs. For a structurally balanced signed graph, we prove that the BOR is equivalent to the Cooperative Output Regulation (COR) over an unsigned graph whose adjacency matrix is obtained by taking the absolute value of each entry in the adjacency matrix of the signed graph. We obtain a new H_∞ -criterion which is sufficient for both COR and BOR.

Comparing with [62,63,69,72,73], which consider a bipartite state synchronization problem, we study a bipartite output regulation problem. In contrast to other existing works which have restrictive assumptions such as homogeneity of the agents [63,72], undirected communication graphs [63] or first order dynamics [62], our framework allows heterogeneity of the agents and a general directed and time-invariant signed communication graph.

This chapter has two main contributions. Firstly, we prove that the H_∞ -criterion for the COR problems reported in [40,92] can be relaxed for some classes of com-

munication graphs to ensure the existence of solutions for a larger set of problems. Secondly, we prove that the BOR problem is equivalent to the COR problem in the sense that one can be reduced to the other one such that a control solution to one problem induces a control solution to the other problem. The significance of this contribution is that all kinds of control techniques developed for solving the COR problem can be directly applied to solve the BOR problem (and vice versa), in particular, that generalized H_∞ criterion introduced in the first contribution can also be applied to the BOR problem. The results of this chapter have been published in [107,108].

The rest of this chapter is organized as follows. In Section 5.1, we introduce some useful results on the signed graphs. In Section 5.2, we first formulate the BOR problem and the COR problem and then show that they are equivalent *via* a transformation procedure. In Section 5.3, we propose a relaxed H_∞ criterion as a sufficient condition to ensure the existence of a solution to the COR problem, which is applicable to the BOR problem as well due to the aforementioned transformation procedure. Simulation results are shown in Section 5.4.

5.1 Preliminaries on Signed Graphs

By [109], a *signed graph* is represented by a tuple $\bar{\mathcal{G}}^s = (\bar{V}, \bar{\mathcal{E}}, \bar{\theta})$, where $\bar{V} = \{v_0, \dots, v_N\}$ denotes a finite vertex set, $\bar{\mathcal{E}} \subseteq \bar{V} \times \bar{V}$ is a directed edge set, and $\bar{\theta} : \bar{\mathcal{E}} \rightarrow \{+1, -1\}$ is a partial edge labeling function, which assigns either a positive or negative sign to each edge. We call $\bar{\mathcal{G}}^u = (\bar{V}, \bar{\mathcal{E}})$ the corresponding *unsigned graph*. A (follower) subgraph of $\bar{\mathcal{G}}^u$ obtained by removing the (leader) node v_0 is represented by $\mathcal{G}^u = (V, \mathcal{E})$ and an $N \times N$ *adjacent* matrix $E^u = [\alpha_{ij}^u]$, where $\alpha_{ij}^u = 1$ if $(v_j, v_i) \in \mathcal{E}$, and $\alpha_{ij}^u = 0$, otherwise. The adjacency of node v_0 and node v_i is denoted by α_{i0}^u and it is defined similarly. The *upper stream neighbor set* of a

node $v \in V$ is defined as $N_v = \{v' \in V | (v', v) \in \mathcal{E}\}$. The in-degree matrix F of that (follower) subgraph is defined as $F = \text{Diag}\{|N_{v_i}|\}_{1:N}$. The Laplacian of that (follower) subgraph is defined as $L^s = F - E^s$, where $E^s := [\alpha_{ij}^s := \theta(v_j, v_i)\alpha_{ij}^u]$ is the signed adjacent matrix. The signed pinning gain from the node v_0 to other nodes is denoted by the matrix $G^s = \text{Diag}\{g_i^s := \theta(v_0, v_i)\alpha_{i0}^u\}_{1:N}$, and $G^u = \text{Diag}\{g_i^u := \alpha_{i0}^u\}_{1:N}$ is the unsigned pinning gain.

A subgraph $\mathcal{G}_k^s = (V_k, \mathcal{E}_k, \theta_k)$, where $V_k \subseteq V$, $\mathcal{E}_k \subseteq \mathcal{E}$ and θ_k being the restriction of θ over $\mathcal{E}_k$, is called a *strongly connected subgraph* of $\mathcal{G}^s$ if each pair of different nodes $v_{ik}, v_{jk} \in V_k$ are reachable from each other *via* respective directed paths. In particular, if the strongly connected subgraph consists of only one node, we call it a *single-node strongly connected subgraph*. $\mathcal{G}_k^s$ is *maximal* if there does not exist another strongly connected subgraph, which contains $\mathcal{G}_k^s$ as a subgraph.

Definition 5.1.1 (Structurally Balanced Graph [62]). A signed graph $\mathcal{G}^s = (V, \mathcal{E}, \theta)$ is *structurally balanced* if it admits a bipartition of the nodes, $V = V^1 \dot{\cup} V^2$, such that (i) for all $(v_i, v_j) \in \mathcal{E} \cap V^q \times V^q$, where $q = 1, 2$, $\theta(v_i, v_j) = 1$; and (ii) for all $v_i \in V^q, v_j \in V^r$ with $(v_i, v_j) \in \mathcal{E}$, $\theta(v_i, v_j) = -1$, $q, r \in \{1, 2\}$, $q \neq r$.

Let $\mathcal{D}$ be the set of gauge transformations $\mathcal{D} = \{\Sigma = \text{Diag}\{\sigma_i\}_{1:N} | \sigma_i \in \{\pm 1\}\}$. We define the following notations

$$\begin{aligned} H^s &:= \text{Diag}_{1:N} \left\{ \frac{1}{|N_{v_i}| + g_i^u} \right\} (F - E^s + G^u), \\ H^u &:= \text{Diag}_{1:N} \left\{ \frac{1}{|N_{v_i}| + g_i^u} \right\} (F - E^u + G^u). \end{aligned} \quad (5.1)$$

Lemma 5.1.1 ([62, 72]). Let $\mathcal{G}^s = (V, \mathcal{E}, \theta)$ be a signed graph which is structurally balanced with the bipartition $V = V^1 \dot{\cup} V^2$, and $\mathcal{G}^u = (V, \mathcal{E})$ be its unsigned equivalent. Then $\Sigma_1 E^s \Sigma_1 = E^u$ and $\Sigma_1 H^s \Sigma_1 = H^u$ if and only if $\Sigma_1 = \text{Diag}\{\sigma_i\}_{1:N} \in \mathcal{D}$, where for all $v_i \in V^q, v_j \in V^r$ with $q, r \in \{1, 2\}$, we have $\sigma_i = \sigma_j$ if and only if $q = r$.

Lemma 5.1.2 ([11]). *Let a graph $\mathcal{G} = (V, \mathcal{E})$ contain K maximal strongly connected subgraphs $\mathcal{G}_k = (V_k, \mathcal{E}_k)$, $k = 1, \dots, K$. One can reorder the nodes such that the adjacent matrix E of $\mathcal{G}$ is lower block triangular and its m -th diagonal blocks is $\Xi_m \in \{E_k | 1 \leq k \leq K\}$, where E_k is the adjacent matrix of $\mathcal{G}_k$.*

5.2 Bipartite and Cooperative Output Regulation Problems

In this section we define the BOR and COR problems and show that they are equivalent. Consider a group of $N + 1$ linear heterogeneous agents communicating over $\bar{\mathcal{G}}^s = (\bar{V}, \bar{\mathcal{E}}, \bar{\theta})$. The group consists of N followers whose models are described as follows

$$\dot{x}_i = A_i x_i + B_i u_i, \quad (5.2)$$

$$y_i = C_i x_i, \quad (5.3)$$

$$z_i = D_i x_i, \quad i = 1, \dots, N \quad (5.4)$$

and one leader with the following state realization

$$\dot{x}_0 = A_0 x_0, \quad (5.5)$$

$$y_0 = C_0 x_0, \quad (5.6)$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^p$, $\forall i = 0, \dots, N$ and $u_i \in \mathbb{R}^{m_i}$, $z_i \in \mathbb{R}^{q_i}$, $\forall i = 1, \dots, N$. In above, y_i , $i = 1, \dots, N$ denote the regulated outputs in the BOR and COR problems and z_i , $i = 1, \dots, N$ represent measured outputs which will be directly used in the control signal.

We make the following assumption.

Assumption 5.2.1. *The signed graph $\bar{\mathcal{G}}^s = (\bar{V}, \bar{\mathcal{E}}, \bar{\theta})$ associated to the multi-agent system is structurally balanced.*

Without loss of generality, let $\Sigma_1 = \text{Diag}\{\sigma_i\}_{i=1:N}$ be the gauge transformation introduced in Lemma 5.1.1, where $v_0 \in V^1$, $\sigma_i = 1, \forall v_i \in V^1$ and $\sigma_j = -1, \forall v_j \in V^2$.

Problem 5.2.1. *Bipartite Output Regulation (BOR) Problem:* *Consider a group of $N + 1$ linear heterogeneous agents defined by (5.2-5.6). Assume that the agents communicate their outputs, i.e., y_i 's, over a structurally balanced signed graph $\bar{\mathcal{G}}^s = (\bar{V}, \bar{\mathcal{E}}, \bar{\theta})$. The control objective is to design u_i locally*

$$\begin{aligned} u_i &= K_{1i}z_i + K_{2i}\eta_i, \\ \dot{\eta}_i &= G_{1i}\eta_i + G_{2i}\delta_i, \quad \eta_i \in \mathbb{R}^{n_{\eta_i}} \\ \delta_i &= \frac{1}{|N_{v_i}| + g_i^u} \left[\sum_{j=1}^N (\alpha_{ij}^u y_i - \alpha_{ij}^s y_j) + g_i^u y_i - g_i^s y_0 \right], \end{aligned} \tag{5.7}$$

such that $e_{bi} = y_i - \sigma_i y_0 \rightarrow \mathbf{0}$, for $i = 1, \dots, N$.

This problem has also been studied in [73], where the agents communicate their states or the states of their dynamic compensators. The present framework is different from [73] in the sense that the agents only need to communicate the outputs. In this chapter, we transform the BOR problem into another problem called cooperative output regulation problem which has been previously discussed in Chapter 3.

Problem 5.2.2. *Cooperative Output Regulation (COR) Problem:* *Consider a group of $N + 1$ linear heterogeneous agents defined by (5.2-5.6). Assume that the agents communicate their outputs y_i 's over an unsigned graph $\bar{\mathcal{G}}^u = (\bar{V}, \bar{\mathcal{E}})$. The*

control objective is to design $\bar{u}_i$ locally

$$\begin{aligned}\bar{u}_i &= K_{1i}z_i + K_{2i}\bar{\eta}_i, \\ \dot{\bar{\eta}}_i &= G_{1i}\bar{\eta}_i + G_{2i}\bar{\delta}_i, \quad \bar{\eta}_i \in \mathbb{R}^{n_{\eta_i}} \\ \bar{\delta}_i &= \frac{1}{|N_{v_i}| + g_i^u} \left[\sum_{j=1}^N \alpha_{ij}^u(y_i - y_j) + g_i^u(y_i - y_0) \right],\end{aligned}\tag{5.8}$$

such that $e_{ci} = y_i - y_0 \rightarrow \mathbf{0}$, for $i = 1, \dots, N$.

We now show the equivalence of the COR and the BOR problems by means of a similarity transformation. Let

$$\begin{aligned}\tilde{H}^s &:= H^s \otimes I_p, \quad \tilde{H}^u = H^u \otimes I_p, \quad Z = \tilde{H}^u - I_{Np}, \\ \tilde{X} &:= \text{Diag}\{X_i\}_{1:N}, \quad X_i \in \{A_i, B_i, C_i, D_i, G_{1i}, G_{2i}, K_{1i}, K_{2i}\}, \\ \tilde{A}_0 &:= I_N \otimes A_0, \quad \tilde{C}_0 = I_N \otimes C_0, \quad C_c = \begin{bmatrix} \tilde{C} & \mathbf{0} \end{bmatrix}, \\ A_c &:= \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{G}_2\tilde{H}^s\tilde{C} & \tilde{G}_1 \end{bmatrix}, \quad \bar{A}_c := \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{G}_2\tilde{H}^u\tilde{C} & \tilde{G}_1 \end{bmatrix}, \\ B_c &:= \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(G^s \otimes I_p)\tilde{C}_0 \end{bmatrix}, \quad \bar{B}_c := \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2\tilde{H}^u\tilde{C}_0 \end{bmatrix}.\end{aligned}$$

The overall closed-loop system of all agents (5.2) seeking BOR over the structurally balanced signed graph $\bar{\mathcal{G}}^s$ via controllers (5.7) is given by

$$\begin{aligned}\dot{\xi}_b &= A_c\xi_b + B_cr_G, \\ \dot{r}_G &= \tilde{A}_0r_G, \\ e_b &= C_c\xi_b - \Sigma_1 \otimes C_0r_G,\end{aligned}\tag{5.9}$$

where $\xi_b = [x_1^T, \dots, x_N^T, \eta_1^T, \dots, \eta_N^T]^T$, $r_G = \mathbf{1}_N \otimes x_0$. The closed loop system of all

agents (5.2) seeking COR over $\bar{\mathcal{G}}^u$ via the control signal (5.8) is given by

$$\begin{aligned}\dot{\xi}_c &= \bar{A}_c \xi_c + \bar{B}_c r_G, \\ \dot{r}_G &= \tilde{A}_0 r_G, \\ e_c &= C_c \xi_c - \tilde{C}_0 r_G,\end{aligned}\tag{5.10}$$

where $\xi_c = [x_1^T, \dots, x_N^T, \bar{\eta}_1^T, \dots, \bar{\eta}_N^T]^T$. The next theorem shows the equivalence between BOR and COR problems.

Theorem 5.2.1. *The control signal u_i in (5.7) solves the BOR problem over a structurally balanced signed graph $\bar{\mathcal{G}}^s = (\bar{V}, \bar{\mathcal{E}}, \bar{\theta})$ if and only if the control signal $\bar{u}_i$ in (5.8) solves the COR problem over the unsigned graph $\bar{\mathcal{G}}^u = (\bar{V}, \bar{\mathcal{E}})$.*

Proof. Denote $\Sigma_2 = \underset{1:N}{Diag}\{\sigma_i \otimes I_{n_i}\}$, $\Sigma_3 = \underset{1:N}{Diag}\{\sigma_i \otimes I_{n_{\eta_i}}\}$, $\Sigma_4 = \Sigma_1 \otimes I_p$, $\bar{\Sigma} = \begin{bmatrix} \Sigma_2 & \mathbf{0} \\ \mathbf{0} & \Sigma_3 \end{bmatrix}$. We use $\bar{\Sigma}$ as a similarity transformation for the global system in (5.9). It is trivial to see that $\bar{\Sigma}^{-1} = \bar{\Sigma}$. Let $\bar{\xi} = \bar{\Sigma} \xi_b$. Then (5.9) in the new coordinate reads

$$\begin{aligned}\dot{\bar{\xi}} &= \bar{\Sigma} A_c \bar{\Sigma} \bar{\xi} + \bar{\Sigma} B_c r_G, \\ e &= C_c \bar{\Sigma} \bar{\xi} - \Sigma_1 \otimes C_0 r_G.\end{aligned}\tag{5.11}$$

According to the definition of $\bar{\Sigma}$ we have

$$\bar{\Sigma} A_c \bar{\Sigma} = \begin{bmatrix} \tilde{A} + \tilde{B} \tilde{K}_1 \tilde{D} & \tilde{B} \tilde{K}_2 \\ \Sigma_3 (\tilde{G}_2 \tilde{H}^s \tilde{C}) \Sigma_2 & \tilde{G}_1 \end{bmatrix}.$$

It is easy to show that $\Sigma_3 (\tilde{G}_2 \tilde{H}^s \tilde{C}) \Sigma_2 = \tilde{G}_2 \Sigma_4 \tilde{H}^s \Sigma_4 \tilde{C} = \tilde{G}_2 (\Sigma_1 H^s \Sigma_1) \otimes I_p \tilde{C}$. Ac-

cording to Lemma 5.1.1, $\Sigma_1 H^s \Sigma_1 = H^u$. Hence $\bar{\Sigma} A_c \bar{\Sigma} = \bar{A}_c$. Similarly,

$$\bar{\Sigma} B_c = \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(\Sigma_1 G^s \otimes I_p) \tilde{C}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(G^u \otimes I_p) \tilde{C}_0 \end{bmatrix}.$$

Noting that $L^u \mathbf{1}_N = \mathbf{0}$, (5.11) reads

$$\dot{\bar{\xi}} = \bar{A}_c \bar{\xi} + \bar{B}_c r_G. \quad (5.12)$$

It remains to show the error (5.9) in the new coordinate

$$e_b = \begin{bmatrix} \Sigma_4 \tilde{C} & \mathbf{0} \end{bmatrix} \bar{\xi}_g - \Sigma_4 \tilde{C}_0 r_G = \Sigma_4 (C_c \bar{\xi} - \tilde{C}_0 r_G) = \Sigma_4 e_c.$$

Let $\bar{\xi} = [x_1^T, \dots, x_N^T, \bar{\eta}_1^T, \dots, \bar{\eta}_N^T]^T$ and $\Sigma_4^{-1} e_b = e_c = [e_{c1}^T, \dots, e_{cN}^T]^T$. The system in (5.12) with property $e_{ci} \rightarrow \mathbf{0}$ represents a COR problem using controller (5.8) for the multi-agent system (5.2-5.6) communicating the outputs over the unsigned graph $\bar{\mathcal{G}}^u$. Hence, the matrices $(K_{1i}, K_{2i}, G_{1i}, G_{2i})$ in (5.8) solves the COR problem if and only if, the same matrices in (5.7) solves the BOR problem. $\square$

5.3 A Relaxed H_∞ Criterion for COR and BOR Problems

In this section, we extend the results of Theorem 3.2.2 and Theorem 3.3.2 ([92]) and obtain a sufficient condition to ensure the existence of a control solution to both the BOR and the COR problems. Let graph $\mathcal{G}^u$ contain K maximal strongly connected subgraphs $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_K\}$. Order the nodes in $\mathcal{G}^u$ such that the adjacency matrix of $\mathcal{G}^u$ is lower block triangular. For each $\mathcal{G}_k$ we assume that its vertex set is

$V_k = \{v_{s_{C_k}}, \dots, v_{s_{C_k}+n_{C_k}-1}\}$, where $|V_k| = n_{C_k}$. Define

$$\begin{aligned} Z_k &:= [Z]_{(s_{C_k}:(s_{C_k}+n_{C_k}-1)) \times (s_{C_k}:(s_{C_k}+n_{C_k}-1))}, \\ \underline{A}_i &:= \begin{bmatrix} A_i + B_i K_{1i} D_i & B_i K_{2i} \\ G_{2i} C_i & G_{1i} \end{bmatrix}, \quad \hat{B}_i := \begin{bmatrix} 0 \\ G_{2i} \end{bmatrix}, \\ \hat{C}_i &:= \begin{bmatrix} C_i & 0 \end{bmatrix}, \quad h_i(s) := -\hat{C}_i (sI - \underline{A}_i)^{-1} \hat{B}_i. \end{aligned} \quad (5.13)$$

Matrix Z_k is the normalized adjacency matrix of the k th maximal strongly connected subgraph.

Theorem 5.3.1. *Consider Assumptions 3.1.1, 3.1.3-3.1.6. If K_{1i} and K_{2i} are selected such that for all $k \in \{1, \dots, K\}$,*

$$\max \|h_i\|_\infty < \frac{1}{\rho(Z_k)} = \gamma_k, \quad \forall v_i \in V_k \quad (5.14)$$

then (5.8) solves the COR problem over $\bar{\mathcal{G}}^u$.

Proof. The characteristics equation of $\bar{A}_c$ reads

$$\Delta = \det(sI - \bar{A}_c) = \det(X) \det\left(I - \begin{bmatrix} C & 0 \end{bmatrix} X^{-1} \begin{bmatrix} 0 \\ \tilde{G}_2 \end{bmatrix} Z\right),$$

where $X = \begin{bmatrix} sI - (\tilde{A} + \tilde{B}\tilde{K}_1\tilde{D}) & -\tilde{B}\tilde{K}_2 \\ -\tilde{G}_2\tilde{C} & sI - \tilde{G}_1 \end{bmatrix}$. Simple calculation shows

$$\Delta = \det(\text{Diag}_{1:N}\{sI - \underline{A}_i\}) \times \det(I + h(s)Z),$$

where $h(s) = \text{Diag}_{1:N}\{h_i\}$. Hence, the stability is guaranteed by the stability of the interconnected term $\mathcal{F}(s) = I + h(s)Z$. Since the transfer matrix $h(s)$ is diagonal and the nodes are numbered such that Z is lower block triangular (Lemma 5.1.2),

$\mathcal{F}(s)$ is lower block triangular and the stability of $\mathcal{F}(s)$ is given by the stability of its diagonal blocks. For the nodes in $\mathcal{G}_k$, the stability of $\mathcal{F}_k(s) = I + h_{n_{C_k}} Z_k$ is of interest, where

$$h_{n_{C_k}}(s) = \underset{s_{C_k}:(s_{C_k}+n_{C_k}-1)}{\text{Diag}} \{h_l(s)\}. \quad (5.15)$$

By the small gain theorem, $\mathcal{F}_k(s)$ is stable if the condition in (5.14) is satisfied. Thus, $\bar{A}_c$ is stable and the COR is achieved. $\square$

Remark 5.3.1. *Theorem 5.3.1 has two important results.*

1. *For a single-node strongly connected subgraph i , we have $Z_i = [0]$, $\rho(Z_i) = 0$. According to Theorem 5.3.1, the H_∞ -norm of $h_i(s)$ can attain any finite value, and it is only required to select (K_{1i}, K_{2i}) to ensure stability of $\underline{A}_i$.*
2. *The nodes in V_k should only know the spectral radius of their normalized adjacency matrix, $\rho(Z_k)$. Each node needs information about the strongly connected subgraph which it constitutes (rather than entire network).*

Corollary 5.3.1. *Consider Assumptions 3.1.1, 3.1.3-3.1.6 and 5.2.1. Assume that K_{1i} and K_{2i} are selected such that (5.14) holds. Then (5.7) solves the BOR problem over $\bar{\mathcal{G}}^s$.*

Proof. It follows from Theorem 5.2.1 and Theorem 5.3.1. $\square$

Remark 5.3.2. *If $D_i = I_{n_i}$, then our suggested controller (5.7) reduces to a state-feedback controller. The design of a state-feedback for the COR problem is given in Theorem 3.3.4 and it can be directly used to solve the BOR problem. Also, one can design an observer $\hat{x}_i$ to estimate x_i using y_i . By the detectability assumption of the pair (A_i, C_i) , one can consider the estimation error as one of the states of the dynamic controller and our results are still valid. Moreover, our controller covers*

the explicit output-feedback as well by $D_i = C_i$. A design procedure for an explicit output-feedback to solve the COR problem is given in Theorem 3.2.3 and Algorithm 3.1. It is worth to mention that our proposed general controller (5.7) goes beyond the mentioned special cases and in general $D_i \neq C_i, I_{n_i}$; in other words, the measured output $z_i = D_i x_i$ and the regulated output $y_i = C_i x_i$ could be completely different. In this case, Theorem 5.3.2 in the sequel provides a design criterion.

Decompose the matrix $\underline{A}_i$ as

$$\begin{aligned} \underline{A}_i &= \begin{bmatrix} A_i & \mathbf{0} \\ G_{2i}C_i & G_{1i} \end{bmatrix} + \begin{bmatrix} B_i \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} K_{1i} & K_{2i} \end{bmatrix} \begin{bmatrix} D_i & \mathbf{0} \\ \mathbf{0} & I_{n_{\eta_i}} \end{bmatrix} \\ &:= \bar{A}_i + \bar{B}_i \bar{K}_i \bar{C}_i \end{aligned} \quad (5.16)$$

and define

$$\Psi_i := \begin{bmatrix} \bar{A}_i^T P_i + P_i \bar{A}_i & P_i \hat{B}_i & \hat{C}_i^T \\ \hat{B}_i^T P_i & -\gamma_k^2 I_p & \mathbf{0} \\ \hat{C}_i & \mathbf{0} & -I_p \end{bmatrix}, \quad (5.17)$$

$$\Psi_{Y_i} := \begin{bmatrix} X_i \bar{A}_i^T + \bar{A}_i X_i & \hat{B}_i & X_i \hat{C}_i^T \\ \hat{B}_i^T & -\gamma_k^2 I_p & \mathbf{0} \\ \hat{C}_i X_i & \mathbf{0} & -I_p \end{bmatrix}, \quad (5.18)$$

$$U_i := \begin{bmatrix} \bar{C}_i & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (5.19)$$

$$V_i := \begin{bmatrix} \bar{B}_i^T P_i & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (5.20)$$

$$Y_i := \begin{bmatrix} \bar{B}_i^T & \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (5.21)$$

The matrices P_i and X_i in (5.17-5.18) and (5.20) will be introduced in the next Theorem.

Theorem 5.3.2. *Consider Assumptions 3.1.1, 3.1.3-3.1.6 and 5.2.1. Let W_{U_i}*

and W_{Y_i} be the orthogonal complements of U_i and Y_i respectively. The H_∞ -norm condition in (5.14) is satisfied if and only if there exist matrices $P_i, X_i > 0 \in \mathbb{R}^{(n_i+n_{\eta_i}) \times (n_i+n_{\eta_i})}$ such that

$$W_{Y_i}^T \Psi_{Y_i} W_{Y_i} < 0, \quad (5.22)$$

$$W_{U_i}^T \Psi_{U_i} W_{U_i} < 0, \quad (5.23)$$

$$X_i P_i = I_{n_i+n_{\eta_i}}. \quad (5.24)$$

Then, the controller gain $\bar{K}_i$ is obtained from

$$\Psi_i + V_i^T \bar{K}_i U_i + U_i^T \bar{K}_i^T V_i < 0 \quad (5.25)$$

and (5.7) solves the BOR problem over $\bar{\mathcal{G}}^s$.

Proof. The proof follows from the same lines of the proof of Theorem 3.2.3, by defining matrix $\bar{C}_i$ as (5.16). $\square$

An equivalent optimization problem to obtain the gains (K_{1i}, K_{2i}) is given in Algorithm 3.1. The next Corollary, addresses a sufficient condition for the existence of a state-feedback ($D_i = I_{n_i}$) to solve BOR.

Corollary 5.3.2. *Consider Assumptions 3.1.1, 3.1.3-3.1.6 and 5.2.1. Assume $D_i = I_{n_i}$. The controller (5.7) solves the BOR problem over $\bar{\mathcal{G}}^s$ if and only if there exist $X_i > 0 \in \mathbb{R}^{(n_i+n_{\eta_i}) \times (n_i+n_{\eta_i})}$ and $\sigma_i > 0 \in \mathbb{R}$ as a feasible solution to*

$$\begin{bmatrix} X_i \bar{A}_i^T + \bar{A}_i X_i - \sigma_i \bar{B}_i \bar{B}_i^T & \hat{B}_i & X_i \hat{C}_i^T \\ \hat{B}_i^T & -\gamma_k^2 I_p & \mathbf{0} \\ \hat{C}_i X_i & \mathbf{0} & -I_p \end{bmatrix} < 0, \quad (5.26)$$

where the controller gain is given by

$$\bar{K}_i = \begin{bmatrix} K_{1i} & K_{2i} \end{bmatrix} = -\frac{\sigma_i}{2} \bar{B}_i^T X_i^{-1}. \quad (5.27)$$

Proof. For the special case of $D_i = I_{n_i}$, the matrix W_{U_i} is of the form of

$$W_{U_i} = \begin{bmatrix} \mathbf{0}_{(n_i+n_{\eta_i}) \times p} & \mathbf{0}_{(n_i+n_{\eta_i}) \times p} \\ I_p & \mathbf{0}_{p \times p} \\ \mathbf{0}_{p \times p} & I_p \end{bmatrix}.$$

Then, (5.23) becomes $\begin{bmatrix} -\gamma_k^2 I_p & \mathbf{0} \\ \mathbf{0} & -I_p \end{bmatrix} < 0$, which is always satisfied. Equation (5.22) depends on X_i only and it is independent of P_i . Hence if $X_i > 0$ is a solution to (5.22), then $P_i = X_i^{-1}$ and (5.22-5.24) are satisfied. Hence, a feasible solution to (5.22) is required. Based on Finsler's Lemma, (5.22) is equivalent to [96]

$$\Psi_{Y_i} - \sigma_i Y_i^T Y_i < 0 \quad (5.28)$$

where $\sigma_i > 0$. The above inequality is a compact representation of (5.26). It remains to obtain the controller's gain. Based on Theorem 5.3.2, the controller gain is derived from (5.25). Rewriting (5.25) and noting that $U_i = \begin{bmatrix} I_{p+n_{\eta_i}} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ we have

$$\begin{bmatrix} \bar{A}_i^T P_i + P_i \bar{A}_i + P_i \bar{B}_i \bar{K}_i + \bar{K}_i^T \bar{B}_i^T P_i & P_i \hat{B}_i & \hat{C}_i^T \\ \hat{B}_i^T P_i & -\gamma_k^2 I_p & \mathbf{0} \\ \hat{C}_i & \mathbf{0} & -I_p \end{bmatrix} < 0. \quad (5.29)$$

Pre and post multiplying the above inequality by the positive definite matrix $\Theta =$

$\text{Diag}\{X_i, I_p, I_p\}$ result to

$$\begin{bmatrix} X_i \bar{A}_i^T + \bar{A}_i X_i + \bar{B}_i \bar{K}_i X_i + X_i \bar{K}_i^T \bar{B}_i^T & \hat{B}_i & X_i \hat{C}_i^T \\ & \hat{B}_i^T & -\gamma_k^2 I_p & \mathbf{0} \\ & \hat{C}_i X_i & \mathbf{0} & -I_p \end{bmatrix} < 0. \quad (5.30)$$

One can select $\bar{K}_i = -\frac{\sigma_i}{2} \bar{B}_i^T X_i^{-1}$, then (5.30) is equivalent to (5.26). $\square$

An important interpretation of Corollary 5.3.2 is that for the special case of $D_i = I_{n_i}$, we need to obtain a feasible solution to an LMI only. Such a result is anticipated since in this case, an output-feedback design becomes the same as a state-feedback design and it is shown in [96] that a state-feedback controller design based on a feasible solution to such an LMI, results to a bounded H_∞ -norm.

5.4 Simulation Results

Consider the signed graph $\bar{\mathcal{G}}^s$ shown in Fig. 5.1. The signed graph $\bar{\mathcal{G}}^s$ satisfies assumptions 3.1.5 and 5.2.1. Also, it has two non-single-node maximal strongly connected subgraphs containing the nodes $V_1 = \{3a, 3b, 3c\}$ and $V_2 = \{7a, 7b, 7c\}$ respectively. We have $\rho(Z_1) = 0.7937$ and $\rho(Z_2) = 0.8514$ for $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.

Consider the dynamics of the leader and the followers as

$$0 : \begin{cases} \dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0, & y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0, \end{cases}$$

$$1, 3a, 3c, 7a, 7c : \begin{cases} \dot{x}_1 = \begin{bmatrix} -0.3 & -2 \\ 0.1 & -0.2 \end{bmatrix} x_1 + \begin{bmatrix} 1.8 & -0.8 \\ 0.9 & 1.6 \end{bmatrix} u_1, \\ y_1 = \begin{bmatrix} -0.1 & 1.2 \\ 0.4 & 1.4 \end{bmatrix} x_1, \quad z_1 = x_1, \end{cases}$$

$$2, 3b, 4, 6, 7b : \begin{cases} \dot{x}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_2, \\ y_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_2, \quad z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} x_2 \end{cases}$$

$$5, 8 : \begin{cases} \dot{x}_5 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x_5 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u_5, \\ y_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_5, \quad z_5 = x_5. \end{cases}$$

Select the p -copy of the leader as $G_{1i} = I_2 \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $G_{2i} = I_2 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $i = 1, \dots, N$.

Figure 5.1: The communication graph

We select the controller gains as

$$\begin{aligned}
 K_1 &= \begin{bmatrix} -10 & -4 & 460 & 172 & -497 & -183 \\ 6 & -13 & -591 & -203 & 246 & 87 \end{bmatrix}, \\
 K_{3a,3c} &= \begin{bmatrix} -0.35 & -0.86 & 0 & -0.09 & -0.01 & -0.3 \\ 0.14 & -1.5 & -0.01 & -0.34 & 0 & -0.01 \end{bmatrix}, \\
 K_{3b} &= \begin{bmatrix} -1.93 & 1.59 & -0.47 & -2.91 & -0.09 & -2.19 \\ -10.9 & 12.45 & -2.97 & -16.12 & -0.84 & -18.36 \end{bmatrix}, \\
 K_{7a,7c} &= \begin{bmatrix} -0.37 & -0.87 & -0.003 & -0.08 & -0.02 & -0.34 \\ 0.16 & -1.55 & -0.02 & -0.43 & 0.001 & 0.011 \end{bmatrix}, \\
 K_{7b} &= \begin{bmatrix} -2.64 & 1.48 & -0.68 & -4.8 & -0.14 & -5.04 \\ -17.17 & 11.76 & -4.91 & -32.84 & -1.25 & -41.67 \end{bmatrix}, \\
 K_{2,4,6} &= \begin{bmatrix} -0.39 & 0.29 & -0.15 & -2.13 & -0.04 & -1 \\ -2.17 & 1.14 & -0.41 & -5.28 & -0.83 & -16.32 \end{bmatrix}, \\
 K_{5,8} &= \begin{bmatrix} -26.98 & -1.22 & -718 & -241 & -17.7 & -3.97 \\ 0.22 & 26 & 17.7 & 3.977 & 562 & 206 \end{bmatrix}.
 \end{aligned}$$

To have a summary, we brought $\|h_i\|_\infty$ and their authorized relaxed upper bounds in Table 5.1.

Table 5.1: $\|h_i\|_\infty$ s and their upper bounds

i	$\ h_i\ _\infty$	Upper bound by Corollary 5.3.1
1	1.3794	$< \infty$
2	5.0357	$< \infty$
3a	1.1017	1.2599
3b	1.2377	1.2599
3c	1.1017	1.2599
4	5.0357	$< \infty$
5	1.2941	$< \infty$
6	5.0357	$< \infty$
7a	1.1685	1.1746
7b	1.1559	1.1746
7c	1.1685	1.1746
8	1.2941	$< \infty$

As a comparison, according to Theorem 3.2.2, we need $\|h_i\|_\infty < \frac{1}{\rho(Z)} = 1.1746$, $i = 1 : 12$. However, the selected gains do not satisfy this requirement, hence, there is no guarantee for the existence of a solution to the COR problem. But under our generalized criterion, *i.e.*, Theorem 5.3.1, a solution does exist. Then by Corollary 5.3.1, a control solution to the BOR problem is also guaranteed.

In Fig. 5.2-5.3 the outputs of all followers, y_i , $i = 1, \dots, 12$, the output of the leader, y_0 , and its mirror, $-y_0$, are sketched. From the figure we can see that the agents achieve BOR with two subgroups $V^1 = \{0, 3c, 4, 5, 6, 7a, 7b, 8\}$ and $V^2 = \{1, 2, 3a, 3b, 7c\}$, where $y_i \rightarrow y_0, \forall i \in V^1$ and $y_j \rightarrow -y_0, \forall j \in V^2$. The initial conditions are selected randomly.

Figure 5.2: The first outputs of all followers, the leader and its mirror

Figure 5.3: The second outputs of all followers, the leader and its mirror

Chapter 6

Multi-party Output Regulation of Linear Heterogeneous Agents

In this chapter, we give a new concept of *multi-party regulation*, where the agents within each party synchronize to a reference trajectory (the leader trajectory) with an orientation. The orientation factors are captured by complex numbers and they are different among the parties. For this purpose, the edge weights of the corresponding communication graph are chosen to be complex and a consistency property should hold. It will be shown that this problem formulation contains cooperative regulation [40,81,92] and bipartite regulation [62,70,108] as its special cases. The idea of complex graphs is also used in [110] to study distributed surrounding design of a target region by a group of single-integrators. The problem in [110] is completely different from the multi-party regulation problem because firstly, [110] considers surrounding of a region while multi-party regulation considers oriented synchronization of the parties to a reference trajectory and secondly, the agents in [110] are single-integrators while in the multi-party regulation problem they are general linear heterogeneous systems.

While multi-party regulation has common values with cluster synchronization,

it introduces new directions which can only be described in this framework. To be more specific, the synchronized trajectory in multi-party regulation depends on the trajectory of a single leader and the weights of the complex graphs, whereas in cluster synchronization, it depends on the identical dynamics of the agents and their initial values [75,76,78]. Hence, one can select the leader and design the weights of the graphs to achieve the desired multi-party regulation. As an example, consider a group of linear heterogeneous agents which are supposed to achieve a time-variant formation prescribed in terms of a linear command generator and to track a reference signal. It is an open problem [111], which can be easily formulated and solved in our proposed multi-party regulation framework as described in Section 6.4 but not in cluster synchronization. Moreover, while in cluster synchronization the signals of the agents in the same cluster are magnified and the signals of the agents in other clusters are suppressed [75], in our multi-party regulation the incoming signals of all agents in the neighborhood are treated similarly. Finally, multi-party regulation deals with heterogeneous agents in general, but currently, cluster synchronization can only consider homogeneous agents [75,78].

This chapter has three main contributions. 1) We define the new concept of multi-party output regulation of linear heterogeneous agents and multi-party state regulation of linear homogeneous agents over complex graphs. 2) We provide a sufficient condition for the existence of a distributed controller to guarantee the multi-party output regulation of heterogeneous agents, which can be directly applicable to multi-party state regulation of homogeneous agents. 3) We address formation control of linear heterogeneous agents and give a new formulation for prescribing time-variant formations (rather than constant relative positions only [112,113]) in terms of linear command generators and bring sufficient conditions for simultaneous formation and reference trajectory tracking *via* the internal model principle (IMP). The result of this chapter has been published in [114].

The remainder of this chapter is organized as follows. Section 6.1 introduces some results on complex graphs. Section 6.2 contains the problem formulation of *multi-party output regulation of heterogeneous agents*. Section 6.3 brings a distributed sufficient condition to design the controllers for the mentioned problem. Section 6.4 contains an application of multi-party regulation in formation control. Section 6.5 contains the simulations.

6.1 Preliminaries on Complex Graphs

A complex graph is represented by $\mathcal{G} = (V, \mathcal{E}, E)$ where $V = \{v_1, \dots, v_N\}$ is a finite set of N nodes, $\mathcal{E} \subseteq V \times V$ is a set of directed arcs and $E = [\alpha_{i,j}] \in \mathbb{C}^{N \times N}$ is the complex adjacency matrix with $|\alpha_{i,j}| = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $\alpha_{i,j} = 0$ otherwise. The set of neighbors of node v_i is $N_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. The absolute in-degree matrix F is defined as $F = \underset{1:N}{\text{Diag}}\{f_i\}$, $f_i = \sum_{j \in N_i} |\alpha_{i,j}|$. The Laplacian is defined as $L = F - E$. The pinning gain from v_0 to $\{v_1, \dots, v_N\}$ is denoted by the matrix $G = \underset{1:N}{\text{Diag}}\{g_i\}$ with $\{g_i \in \mathbb{C}, |g_i| = 1\}$ if there is a link from v_0 to v_i and $g_i = 0$ otherwise. Graph $|\mathcal{G}| = (V, \mathcal{E}, |E|)$ denotes a modulus equivalent graph of $\mathcal{G} = (V, \mathcal{E}, E)$ where all edge weights are either zero or unity.

Definition 6.1.1 ([110]). A weak cycle $i_1 i_2 \dots i_k i_1$ is said to be consistent if

$$\prod_{r=1}^k w_r = 1 \quad (6.1)$$

where $w_r = \alpha_{i_{r+1}, i_r}$ if $(i_r, i_{r+1}) \in \mathcal{E}$ and $w_r = \alpha_{i_r, i_{r+1}}^{-1}$ if $(i_{r+1}, i_r) \in \mathcal{E}$. A graph is consistent if all its weak cycles are consistent and inconsistent otherwise.

Theorem 6.1.1. Consider a set of transformations $\mathcal{D} = \{\Gamma = \underset{1:N}{\text{Diag}}\{\gamma_i\} | \gamma_i \in \mathbb{C}, |\gamma_i| = 1\}$. Assume that the graph $\mathcal{G} = (V, \mathcal{E}, E)$ is consistent and contains a spanning tree. Then, there exists a transformation $\Gamma_m = \underset{1:N}{\text{Diag}}\{\gamma_i\} \in \mathcal{D}$ such that

Figure 6.1: Nodes r_1 and r_2 and their common ancestor k and the relevant parameters.

$\Gamma_m E \Gamma_m^{-1} = |E|$. Moreover, $L = F - E$ and $L_m = F - |E|$ are iso-spectral.

Proof. The matrix $\Gamma_m E \Gamma_m^{-1}$ reads

$$[\Gamma_m E \Gamma_m^{-1}]_{ij} = \gamma_i \alpha_{i,j} \gamma_j^{-1}. \quad (6.2)$$

Select a spanning tree of the graph and number the nodes in an increasing order such that the root node is labeled as 1, and its children are numbered 2, 3, etc. Set $\gamma_1 = 1$. For each node $i = 2, \dots, N$, define γ_i based on its ancestor. Let $\hat{i}$ denote the ancestor of i for $i = 2, \dots, N$. To have $\gamma_i \alpha_{i,\hat{i}} \gamma_{\hat{i}}^{-1} = 1$, we should select

$$\gamma_i = \gamma_{\hat{i}} \alpha_{i,\hat{i}}^{-1}. \quad (6.3)$$

This proves the result for those arcs that appear in the spanning tree. For others, the above result holds due to the presumed consistency of the graph. Consider two nodes r_1 and r_2 having a common ancestor node k in Fig. 6.1. Assume that there

exist $n_1 + 1$ nodes from k to r_1 inclusive, which are numbered $k, k + 1, \dots, k + n_1$, and $n_2 + 1$ nodes from k to r_2 inclusive, which are numbered $k, k + n_1 + 1, \dots, k + n_1 + n_2$. According to (6.3),

$$\begin{aligned}\gamma_{r_1} &= \gamma_k \prod_{i=k}^{k+n_1} \alpha_{i+1,i}^{-1}, \\ \gamma_{r_2} &= \gamma_k \alpha_{k+n_1+1,k}^{-1} \prod_{j=k+n_1+1}^{k+n_1+n_2} \alpha_{j+1,j}^{-1}.\end{aligned}$$

Assume there exists an arc from r_2 to r_1 . Then

$$\begin{aligned}\gamma_{r_1} \alpha_{r_1,r_2} \gamma_{r_2}^{-1} &= \gamma_k \prod_{i=k}^{k+n_1} (\alpha_{i+1,i}^{-1}) \alpha_{r_1,r_2} \gamma_k^{-1} \alpha_{k+n_1+1,k} \times \\ &\quad \prod_{j=k+n_1+1}^{k+n_1+n_2} (\alpha_{j+1,j}).\end{aligned}\tag{6.4}$$

Since the graph is consistence, we have

$$\prod_{i=k}^{k+n_1} (\alpha_{i+1,i}^{-1}) \alpha_{r_1,r_2} \alpha_{k+n_1+1,k} \prod_{j=k+n_1+1}^{k+n_1+n_2} (\alpha_{j+1,j}) = 1,$$

which, together with (6.4), results in $\gamma_{r_1} \alpha_{r_1,r_2} \gamma_{r_2}^{-1} = 1$. This completes the proof of the first part. Next, we prove that L and L_m are iso-spectral. The matrix Γ_m constructed above is nonsingular and it can be used as a similarity transformation

$$\Gamma_m L \Gamma_m^{-1} = \Gamma_m F \Gamma_m^{-1} - \Gamma_m E \Gamma_m^{-1} = F - |E| = L_m.$$

Hence, two matrices L and L_m are iso-spectral. □

The proof of Theorem 6.1.1 is constructive and can be used to obtain the transformation Γ_m . We call $\Gamma_m = \text{Diag}\{\gamma_i\}_{1:N}$ the *unsigned-equivalent transformation* of the consistent graph $\mathcal{G}$.

Lemma 6.1.1. (*[110]*) *0 is an eigenvalue of L if and only if the graph $\mathcal{G}$ is consistent.*

Lemma 6.1.2. *Let $\mathcal{G} = (V, \mathcal{E}, E)$ be a consistent graph with N nodes and the unsigned-equivalent transformation Γ_m . Then, $\nu_m = \Gamma_m^{-1} \mathbf{1}_N$ is an eigenvector associated with the eigenvalue 0 of L .*

Proof. According to Theorem 6.1.1, $\Gamma_m L \Gamma_m^{-1} = L_m$. Since L_m is a matrix with a zero row-sum, $\mathbf{1}_N$ is an eigenvector associated with the zero eigenvalue. Hence

$$L_m \mathbf{1}_N = \Gamma_m L \Gamma_m^{-1} \mathbf{1}_N = \mathbf{0}. \quad (6.5)$$

Hence, one can conclude that $L \Gamma_m^{-1} \mathbf{1}_N = \mathbf{0}$. □

Remark 6.1.1. *The eigenvector $\nu_m = \Gamma_m^{-1} \mathbf{1}_N = [\gamma_1^{-1}, \dots, \gamma_N^{-1}]^T$ associated with the zero eigenvalue of the Laplacian L of the consistent graph $\mathcal{G}$ gives the agreement subspace for the consensus problem $\dot{\chi} = -L\chi$, $\chi \in \mathbb{C}^N$ and denotes a possible synchronized subspace for a general multi-agent system using a Laplacian protocol as detailed in Theorem 6.2.1.*

6.2 Multi-party Regulation

In this section, we define the problem of multi-party regulation of heterogeneous agents. Consider a group of $N+1$ linear heterogeneous agents containing N followers

$$\dot{x}_i = A_i x_i + B_i u_i, \quad (6.6)$$

$$y_i = C_i x_i, \quad (6.7)$$

$$z_i = D_i x_i, \quad (6.8)$$

Figure 6.2: $N+1$ agents in three parties. The solid lines represent the inner coupling within the clusters and the dash lines represent the inter coupling between the clusters. The inter cluster coupling weights are not equal to one.

and one leader

$$\dot{x}_0 = A_0 x_0, \quad (6.9)$$

$$y_0 = C_0 x_0, \quad (6.10)$$

in which $x_i \in \mathbb{C}^{n_i}$ and $u_i \in \mathbb{C}^{m_i}$ denote the state and control signal for agent i respectively, $x_0 \in \mathbb{C}^{n_0}$ is the state of the leader, $y_i \in \mathbb{C}^p$, $i = 0, 1, \dots, N$ is the output of agent i (0 denotes the leader), and $z_i \in \mathbb{C}^{q_i}$, $i = 1, \dots, N$ is the measured output by agent i to be used in the control of the heterogeneous agents. Let $\mathcal{G} = (V, \mathcal{E}, E)$ be the communication graph of N followers and $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$ be the augmented graph obtained from $\mathcal{G}$ by attaching the leader (node v_0) and its outgoing edges to $\mathcal{G}$. The agents communicate their outputs y_i , $i = 0, 1, \dots, N$ over $\bar{\mathcal{G}}$ and constitute $\mathcal{P}$ parties. In Fig. 6.2 an example of a multi-agent system with three parties is represented. Let $\bar{i}$ denote the party which agent i belongs to. The agents in the same party communicate cooperatively, i.e., for all i, j , if $\bar{i} = \bar{j}$ then $\alpha_{i,j} \in \{1, 0\}$.

Definition 6.2.1. Multi-party Output Regulation of Heterogeneous Agents:

Consider a group of $N + 1$ heterogeneous agents defined by (6.6-6.10) with a consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$. The multi-agent system achieves multi-party output regulation if the followings hold:

1. we have output regulation within each party, i.e., $\lim_{t \rightarrow +\infty} \|y_i(t) - y_j(t)\| = \mathbf{0}$, if $\bar{i} = \bar{j}$,
2. $\lim_{t \rightarrow +\infty} \|e_i(t)\| = \lim_{t \rightarrow +\infty} \|y_i(t) - a_i y_0(t)\| = \mathbf{0}$, where $a_i \in \mathbb{C}$, $|a_i| = 1$ and $a_i \neq a_j$ if $\bar{i} \neq \bar{j}$.

In the above definition, we call a_i the orientation factor of agent i with respect to the leader. The multi-party output synchronization error is denoted by $e_i(t) = y_i(t) - a_i y_0(t)$ and it measures the convergence of y_i to y_0 with the orientation $\angle a_i$.

One can define the cooperative output regulation similar to Definition 6.2.1 by considering communication over the modulus graph $|\bar{\mathcal{G}}| = (\bar{V}, \bar{\mathcal{E}}, |\bar{E}|)$. In this case, we have $\lim_{t \rightarrow +\infty} \|y_i(t) - y_0(t)\| = \mathbf{0}$ for all $i = 1, \dots, N$.

Define the local multi-party output synchronization error as

$$\delta_i = \frac{1}{f_i + |g_i|} \sum_{j \in N_i} (y_i - \alpha_{i,j} y_j) + |g_i| (y_i - g_i y_0). \quad (6.11)$$

The following theorem proves that it is possible to achieve multi-party regulation over a consistent graph by using local multi-party synchronization error if the orientation factors are selected such that $a_i = [\nu_m]_i = \gamma_i^{-1}$.

Theorem 6.2.1. Let $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$ be a consistent graph with the unsigned-equivalent transformation $\Gamma_m = \text{Diag}\{\gamma_i\}_{0:N}$. Let $\bar{\mathcal{G}}$ contain a spanning tree with the leader as its root node. Assume that $a_i = \gamma_i^{-1}$. Then, for each $i = 1, \dots, N$, $\lim_{t \rightarrow +\infty} \|e_i(t)\| = \mathbf{0}$ if and only if $\lim_{t \rightarrow +\infty} \|\delta_i(t)\| = \mathbf{0}$.

Proof. Since the leader is the root of the spanning tree $a_0 = \gamma_0^{-1} = 1$. Choose $a_i = \gamma_i^{-1}$ for follower i , as mentioned in Theorem 6.1.1. The local neighborhood

state error reads

$$\begin{aligned} \delta_i = \frac{1}{f_i + |g_i|} & \left(\sum_{j \in N_i} [(y_i - \gamma_i^{-1}y_0) - \alpha_{i,j}(y_j - \gamma_j^{-1}y_0) + \gamma_i^{-1}y_0 - \gamma_j^{-1}\alpha_{i,j}y_0] \right. \\ & \left. + |g_i|[(y_i - \gamma_i^{-1}y_0) + \gamma_i^{-1}y_0 - g_i y_0] \right). \end{aligned} \quad (6.12)$$

For a nonzero $\alpha_{i,j}$, there exists an arc from j to i and it means j is an ancestor of i . Hence, $\gamma_i = \gamma_j \alpha_{i,j}^{-1}$. As a result,

$$(\gamma_i^{-1} - \alpha_{i,j} \gamma_j^{-1})y_0 = (\gamma_j^{-1} \alpha_{i,j} - \gamma_j^{-1} \alpha_{i,j})y_0 = \mathbf{0}. \quad (6.13)$$

Similarly, a nonzero g_i , means that there exists a path from the leader to node i and $\gamma_i = 1g_i^{-1}$. We have

$$(\gamma_i^{-1} - g_i)y_0 = (g_i - g_i)y_0 = \mathbf{0}. \quad (6.14)$$

Hence, the local neighborhood error reads

$$\delta_i = \frac{1}{f_i + |g_i|} \sum_{j \in N_i} (e_i - \alpha_{i,j} e_j) + |g_i| e_i. \quad (6.15)$$

Define the total vectors $e = [e_1^T, e_2^T, \dots, e_N^T]^T$ and $\delta = [\delta_1^T, \delta_2^T, \dots, \delta_N^T]^T$. Then we have

$$\delta = (F + |G|)^{-1} \times (L + |G|)e. \quad (6.16)$$

For a consistent graph containing a spanning tree with the leader as its root node, all eigenvalues of $L + |G|$ have positive real parts (Theorem 6.1.1). Thus, for each agent i , $\lim_{t \rightarrow +\infty} \|e_i(t)\| = \mathbf{0}$ if and only if $\lim_{t \rightarrow +\infty} \|\delta_i(t)\| = \mathbf{0}$. $\square$

For the consistent graph $\bar{\mathcal{G}}$ and its unsigned-equivalent transformation Γ_m , the parties are uniquely determined by $[a_1^T, \dots, a_N^T] = \Gamma_m^{-1} \mathbf{1}_N$; *i.e.*, $a_i = a_j$ if and only if $\bar{i} = \bar{j}$. In other words, a necessary topological condition for multi-party regulation

is consistency of the graph.

Remark 6.2.1. *In the cooperative regulation problem, graphs containing spanning trees with nonnegative edge weights are considered. It is easy to show that those graphs are always consistent and $\gamma_i = 1$, for $i = 1, \dots, N$. In this case, the local multi-party synchronization error (6.11) reduces to the cooperative output synchronization error $\frac{1}{f_i + g_i} \sum_{j \in N_i} (y_i - y_j) + g_i(y_i - g_i y_0)$ (3.7) [40, 81, 92], where all edge weights are nonnegative. In the bipartite regulation, graphs with both positive and negative edge weights are considered and a bipartition of the nodes into two subgroups comes into the picture where the nodes cooperate with each other within each subgroup but compete with the members of the other subgroup. Such a graph is called structurally balanced and it is shown in [62] that this property holds if and only if all cycles are positive. The consistency condition in Definition 6.1.1 reduces to the structurally balanced requirement for graphs with positive and negative edge weights. Also, it is easy to show that $\gamma_i = 1$, $\forall i \in V^1$ and $\gamma_j = -1$, $\forall j \in V^2$ with the bipartition $V^1 \dot{\cup} V^2 = V$. In this case, the local multi-party synchronization error (6.11) reduces to the local bipartite synchronization error (5.7) used in [62, 70, 108].*

Remark 6.2.2. *One can use Theorem 6.2.1 to design the graph weights to achieve the desired multi-party regulation behavior: Select the set of a_i , $i = 1, \dots, N$ and take a spanning tree of the graph. Set $a_0 = 1$ (the leader node). Then, select the edge weights $\alpha_{i,j}$ of the augmented graph according to (6.3), i.e., $\alpha_{i,j} = a_i a_j^{-1}$.*

Remark 6.2.3. *When all agents are homogeneous, as commonly considered in the literature, one can define multi-party state regulation of homogeneous agents similar to Definition 6.2.1 for agents defined by (6.6-6.10) but with $A_i = A$, $C_i = C$, $n_i = n$, $i = 0, 1, \dots, N$ and $B_i = B$, $m_i = m$, $i = 1, \dots, N$. The multi-party state synchronization error is defined as $\varepsilon_i = x_i - a_i x_0$, $\varepsilon = [\varepsilon_1^T, \dots, \varepsilon_N^T]^T$. Assuming that the pair (A, C) is detectable, one can show that $\lim_{t \rightarrow +\infty} \|\varepsilon(t)\| = 0$ if and only if*

$\lim_{t \rightarrow +\infty} \|\delta(t)\| = \mathbf{0}$, namely the problem of multi-party state regulation of homogeneous agents can be solved precisely by using the techniques developed for the problem of multi-party regulation of heterogeneous agents with an extra assumption that the pair (A, C) is detectable.

6.3 Sufficient Condition for Multi-party Regulation

In this section, we analyze multi-party output regulation of heterogeneous agents, and derive a sufficient condition for it. As a direct application of this result, we also derive a sufficient condition for multi-party state regulation of homogeneous agents.

Consider a Laplacian-based dynamic controller of the following form

$$\begin{aligned} u_i &= K_{1i}z_i + K_{2i}\eta_i, \quad \eta_i \in \mathbb{C}^{n_{\eta_i}} \\ \dot{\eta}_i &= G_{1i}\eta_i + G_{2i}\delta_i, \end{aligned} \tag{6.17}$$

where z_i is the measured output of agent i defined in (6.8), η_i is the internal state of the controller and $K_{1i}, K_{2i}, G_{1i}, G_{2i}$ are real controller's matrices to be designed in the sequel. When $z_i \equiv x_i$ and $z_i \equiv y_i$, the controller (6.17) simply becomes state-feedback and explicit output-feedback respectively. Define

$$\begin{aligned} H &:= (F + |G|)^{-1}(L + |G|), \\ H_m &:= (F + |G|)^{-1}(L_m + |G|), \\ \tilde{H} &:= H \otimes I_p, \quad \tilde{H}_m := H_m \otimes I_p, \quad Z := H_m - I, \end{aligned}$$

$$\begin{aligned}
\tilde{X} &:= \text{Diag}_{1:N}\{X_i\}, \quad X_i \in \{A_i, B_i, C_i, D_i, G_{1i}, G_{2i}, K_{1i}, K_{2i}\}, \\
\tilde{A}_0 &:= I_N \otimes A_0, \quad \tilde{C}_0 := I_N \otimes C_0, \quad C_c := \begin{bmatrix} \tilde{C} & \mathbf{0} \end{bmatrix}, \\
A_c &:= \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{G}_2\tilde{H}\tilde{C} & \tilde{G}_1 \end{bmatrix}, \quad \bar{A}_c := \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{G}_2\tilde{H}_m\tilde{C} & \tilde{G}_1 \end{bmatrix}, \\
B_c &:= \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(G \otimes I_p)\tilde{C}_0 \end{bmatrix}, \quad \bar{B}_c := \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2\tilde{H}_m\tilde{C}_0 \end{bmatrix}.
\end{aligned}$$

Theorem 6.3.1. *Given $N + 1$ heterogeneous agents (6.6-6.10) and the consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$, let $\Gamma_m = \text{Diag}_{0:N}\{\gamma_i\}$ be the unsigned-equivalent transformation of $\bar{\mathcal{G}}$ and $a_i = \gamma_i^{-1}$. The agents achieve multi-party output regulation over $\bar{\mathcal{G}}$ according to Definition 6.2.1, if and only if they achieve cooperative output regulation over $|\bar{\mathcal{G}}| = (\bar{V}, \bar{\mathcal{E}}, |\bar{E}|)$.*

Proof. Let

$$\begin{aligned}
\Sigma_1 &= \Gamma_m, \quad \Sigma_2 = \text{Diag}_{1:N}\{\gamma_i \otimes I_{n_i}\}, \\
\Sigma_3 &= \text{Diag}_{1:N}\{\gamma_i \otimes I_{n_{\eta_i}}\}, \quad \Sigma_4 = \Sigma_1 \otimes I_p, \quad \bar{\Sigma} = \begin{bmatrix} \Sigma_2 & \mathbf{0} \\ \mathbf{0} & \Sigma_3 \end{bmatrix}.
\end{aligned}$$

The overall closed-loop system of all agents (6.6-6.10) using (6.17) over $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$ is given by

$$\begin{aligned}
\dot{\xi} &= A_c \xi + B_c r_G, \\
\dot{r}_G &= \tilde{A}_0 r_G, \\
e &= C_c \xi - \Sigma_1^{-1} \otimes C_0 r_G,
\end{aligned} \tag{6.18}$$

where $\xi = [x_1^T, \dots, x_N^T, \eta_1^T, \dots, \eta_N^T]^T$, $r_G = \mathbf{1}_N \otimes x_0$. We treat $\bar{\Sigma}$ as a similarity

transformation for (6.18). Let $\bar{\xi} = \bar{\Sigma}\xi$. Then, (6.18) in the new coordinate reads

$$\begin{aligned}\dot{\bar{\xi}} &= \bar{\Sigma}A_c\bar{\Sigma}^{-1}\bar{\xi} + \bar{\Sigma}B_cr_G, \\ e &= C_c\bar{\Sigma}^{-1}\bar{\xi} - \Sigma_1^{-1} \otimes C_0r_G.\end{aligned}\tag{6.19}$$

According to the definition of $\bar{\Sigma}$ we have

$$\bar{\Sigma}A_c\bar{\Sigma}^{-1} = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \Sigma_3(\tilde{G}_2\tilde{H}\tilde{C})\Sigma_2^{-1} & \tilde{G}_1 \end{bmatrix}.$$

It is easy to show that $\Sigma_3(\tilde{G}_2\tilde{H}\tilde{C})\Sigma_2^{-1} = \tilde{G}_2\Sigma_4\tilde{H}\Sigma_4^{-1}\tilde{C} = \tilde{G}_2(\Sigma_1H\Sigma_1^{-1}) \otimes I_p\tilde{C}$. According to Theorem 6.1.1, $\Sigma_1H\Sigma_1^{-1} = H_m$. Hence $\bar{\Sigma}A_c\bar{\Sigma}^{-1} = \bar{A}_c$. Similarly,

$$\bar{\Sigma}B_c = \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(\Sigma_1G \otimes I_p)\tilde{C}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\tilde{G}_2(|G| \otimes I_p)\tilde{C}_0 \end{bmatrix}.$$

Noting that $(F - |E|)\mathbf{1}_N = \mathbf{0}$, (6.19) reads

$$\dot{\bar{\xi}} = \bar{A}_c\bar{\xi} + \bar{B}_cr_G.\tag{6.20}$$

It remains to show that the error (6.18) in the new coordinate is given by

$$e = \begin{bmatrix} \Sigma_4^{-1}\tilde{C} & \mathbf{0} \end{bmatrix} \bar{\xi} - \Sigma_4^{-1}\tilde{C}_0r_G = \Sigma_4^{-1}(C_c\bar{\xi} - \tilde{C}_0r_G).$$

Define the new error $\bar{e} = \Sigma_4e$. Then the closed-loop system of (6.20) with $\bar{e} \rightarrow 0$ represents a group of agents (6.6-6.10) seeking the cooperative output regulation over the modulus graph $|\bar{\mathcal{G}}| = (\bar{V}, \bar{\mathcal{E}}, |\bar{E}|)$ and it completes the proof. $\square$

Theorem 6.3.1 establishes the equivalence between the multi-party output regulation over $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$ and the cooperative output regulation over $|\bar{\mathcal{G}}| = (\bar{V}, \bar{\mathcal{E}}, |\bar{E}|)$.

Define

$$\begin{aligned} \underline{A}_i &:= \begin{bmatrix} A_i + B_i K_{1i} D_i & B_i K_{2i} \\ G_{2i} C_i & G_{1i} \end{bmatrix}, \quad \hat{B}_i := \begin{bmatrix} \mathbf{0} \\ G_{2i} \end{bmatrix}, \\ \hat{C}_i &:= \begin{bmatrix} C_i & \mathbf{0} \end{bmatrix}, \quad h_i(s) := -\hat{C}_i(sI - \underline{A}_i)^{-1} \hat{B}_i. \end{aligned} \quad (6.21)$$

The next corollary is a direct result of Theorem 6.3.1 and Theorem 3.2.2.

Corollary 6.3.1. *Given $N + 1$ heterogeneous agents (6.6-6.10) and the consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$. Let $\Gamma_m = \text{Diag}\{\gamma_i\}_{0:N}$ be the unsigned-equivalent transformation of $\bar{\mathcal{G}}$ and $a_i = \gamma_i^{-1}$. Let Assumptions 3.1.1, 3.1.3-3.1.6 hold. If K_{1i} and K_{2i} are selected such that*

$$\max_i \|h_i\|_\infty < \frac{1}{\rho(Z)}, \quad i = 1, \dots, N, \quad (6.22)$$

then (6.17) solves the multi-party output regulation defined in Definition 6.2.1.

A Linear Matrix Inequalities (LMIs) approach to design (K_{i1}, K_{i2}) satisfying (6.22) is given in Algorithm 3.1.

For the homogeneous (identical) agents mentioned in Remark 6.2.3, we suggest a Laplacian-based static controller

$$u_i = K_y \delta_i, \quad (6.23)$$

where K_y is a real gain matrix to be designed below. Define

$$m(s) := C(sI - (A + BK_y C))^{-1} BK_y. \quad (6.24)$$

Theorem 6.3.2. *Given $N + 1$ homogeneous agents (6.6-6.10) and the consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$, let $\Gamma_m = \text{Diag}\{\gamma_i\}_{0:N}$ be the unsigned-equivalent*

transformation of $\bar{\mathcal{G}}$ and $a_i = \gamma_i^{-1}$. Let Assumption 3.1.5 hold and the tuple (A, B, C) be output-feedback stabilizable. The agents achieve multi-party state regulation if the controller gain K_y is selected such that

$$\|m(s)\|_\infty < \frac{1}{\rho(Z)}. \quad (6.25)$$

Proof. In this proof, we provide a sufficient condition for the stability of the cooperative state synchronization error which is also sufficient for the stability of the multi-party state synchronization error by Theorem 6.3.1. The overall dynamics of the cooperative state synchronization error $\bar{\varepsilon}_i = x_i - x_0$ reads

$$\dot{\bar{\varepsilon}} = (I_N \otimes A + H_m \otimes BK_y C) \bar{\varepsilon}. \quad (6.26)$$

The characteristic equation of the above system reads

$$\begin{aligned} \Delta &= |sI - (I_N \otimes A + (I_N \otimes BK_y)H_m(I_N \otimes C))| \\ &= |sI - (I_N \otimes A + I_N \otimes BK_y C \\ &\quad + (I_N \otimes BK_y)Z(I_N \otimes C))| \\ &= |sI - (I_N \otimes A + I_N \otimes BK_y C)||I - \\ &\quad I_N \otimes C(sI - I_N \otimes (A + BK_y C))^{-1}(I_N \otimes BK_y)Z| \\ &= |sI - (I_N \otimes (A + BK_y C))||I - M(s)Z| \end{aligned} \quad (6.27)$$

where $M(s) = I_N \otimes m(s)$. Based on the small gain theorem, if (6.25) holds, the system in (6.26) is stable. $\square$

6.4 Formation Control using Multi-party Regulation

Formation control is one of the most active fields in the domain of multi-agent systems [111,112,115]. Majority of the existing results focuses on time-invariant pattern formation (constant relative positions) of homogeneous agents [112,115], leaving time-variant pattern formation of heterogeneous agents as an open problem [111]. The aim of this section is to give an application of multi-party regulation in formation control of heterogeneous agents *via* Internal Model Principle (IMP). More specifically, we give a new formulation to prescribe a time-variant formation (rather than constant relative positions) in terms of a linear command generator and bring a sufficient condition for simultaneous pattern formation and reference tracking for heterogeneous agents. As we analyze formation control of the agents over a plane, $p = 1$ in this section.

Let a desired formation of the agents with respect to the leader be characterized by sets of $(a_i, \psi_i(t))$, where $\angle a_i$ represents the orientation of y_i with respect to y_0 and $\psi_i(t)$ denotes the formation constraint given by the dynamic system

$$\begin{aligned}\dot{\vartheta}_i &= \Omega_i \vartheta_i, \quad \vartheta_i(0) = \vartheta_{i0}, \\ \psi_i &= \Upsilon_i \vartheta_i.\end{aligned}\tag{6.28}$$

This formation constraint describes a time-variant relative distance between y_i and y_0 . It is more general than the one defined by constant relative positions commonly used in the literature [112,113] and covers time-variant constraints like ramp, sinusoidal, *etc.* Define the formation error τ_i

$$\tau_i = y_i - \psi_i - a_i y_0.\tag{6.29}$$

As a result, each agent tracks the following reference signal

$$\begin{aligned} \dot{x}_{ai} &= \begin{bmatrix} A_0 & \mathbf{0} \\ \mathbf{0} & \Omega_i \end{bmatrix} x_{ai} = A_{ai} x_{ai} \\ y_{ai} &= \begin{bmatrix} a_i C_i & \Upsilon_i \end{bmatrix} x_{ai} \end{aligned} \quad (6.30)$$

where $x_{ai} = \begin{bmatrix} x_0^T & \vartheta_i^T \end{bmatrix}^T$ and $y_{ai} = a_i y_0 + \psi_i$. We call (6.30) *the augmented exo-system*, which should be tracked for the purpose of formation control. According to [27], an internal model of A_{ai} is necessary for this purpose. Hence, it is required that each agent select (G_{1i}, G_{2i}) as a 1-copy of A_{ai} .

According to (6.29), the controller should be designed to achieve $y_i - \psi_i \rightarrow a_i y_0$. Hence, each agent needs to communicate ψ_i in addition to y_i with its neighbors. As a result, the local formation error is defined as

$$\begin{aligned} \varsigma_i &= \delta_i - \phi_i \\ \phi_i &= \frac{1}{f_i + |g_i|} \sum_{j \in N_i} (\psi_i - \alpha_{i,j} \psi_j) + |g_i| \psi_i. \end{aligned} \quad (6.31)$$

Theorem 6.4.1. *Given $N + 1$ heterogeneous agents (6.6-6.10) and the consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$, let $\Gamma_m = \text{Diag}\{\gamma_i\}_{0:N}$ be the unsigned-equivalent transformation of $\bar{\mathcal{G}}$ and $a_i = \gamma_i^{-1}$. Let Assumptions 3.1.1, 3.1.3, 3.1.5- 3.1.6 hold. For each i , select (G_{1i}, G_{2i}) to contain a 1-copy of A_{ai} in (6.30). If K_{1i} and K_{2i} are selected such that (6.22) holds for each $i = 1, \dots, N$, then the protocol*

$$\begin{aligned} u_i &= K_{1i} z_i + K_{2i} \eta_i, \\ \dot{\eta}_i &= G_{1i} \eta_i + G_{2i} (\delta_i - \phi_i), \end{aligned} \quad (6.32)$$

will ensure that each follower i follows the leader (6.9-6.10) with the formation $(a_i, \psi_i(t))$.

Proof. The augmented exo-system for tracking a leader in (6.9-6.10) with formation constraint (6.28) is given in (6.30). The rest of proof follows from Corollary 6.3.1. $\square$

Next, we show that if $\bar{\mathcal{G}}$ is acyclic, then the H_∞ -norm condition in (6.22) is relaxed to the stability of a system matrix.

Corollary 6.4.1. *Given $N + 1$ heterogeneous agents (6.6-6.10) and the acyclic consistent communication graph $\bar{\mathcal{G}} = (\bar{V}, \bar{\mathcal{E}}, \bar{E})$ containing a spanning tree with the leader as its root node. Let $\Gamma_m = \text{Diag}\{\gamma_i\}_{0:N}$ be the unsigned-equivalent transformation of $\bar{\mathcal{G}}$ and $a_i = \gamma_i^{-1}$. Let Assumptions 3.1.1, 3.1.3, and 3.1.6 hold. For each i , select (G_{1i}, G_{2i}) to contain a 1-copy of A_{ai} in (6.30). Select K_{1i} and K_{2i} such that $\underline{A}_i$ in (6.21) is stable. Then, using the protocol*

$$\begin{aligned} u_i &= K_{1i}z_i + K_{2i}\eta_i, \\ \dot{\eta}_i &= G_{1i}\eta_i + G_{2i}(\delta_i - \phi_i), \end{aligned} \tag{6.33}$$

each follower i tracks the leader (6.9-6.10) with the formation $(a_i, \psi_i(t))$.

Proof. For an acyclic graph $\rho(Z) = 0$. Thus, (6.22) trivially holds, and the stability of $\underline{A}_i$ will be sufficient. $\square$

Remark 6.4.1. *Consider a special case where the formation constraints are constant*

$$\dot{\vartheta}_i = 0, \psi_i = \vartheta_i = \vartheta_i(0). \tag{6.34}$$

If A_0 has a single-integrator as part of its dynamics, it is easy to show that an internal model of A_0 suffices for both tracking and formation. To be specific, if A_0

has a single-integrator, one can write $A_0 = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A_r \end{bmatrix}$, and

$$\begin{bmatrix} \dot{x}_{01} \\ \dot{x}_{0r} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A_r \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{0r} \end{bmatrix}, \quad x_0(0) = \begin{bmatrix} x_{01}(0) \\ x_{02}(0) \end{bmatrix}. \quad (6.35)$$

One can combine $\dot{x}_{01} = 0$, $x_{01}(0)$ and $\dot{v}_i = 0$, $v_i(0)$ into a single-integrator again. Also, if the augmented exo-system and an agent share a dynamics, it is possible to select (G_{1i}, G_{2i}) to contain a 1-copy of the unshared dynamics of the augmented exo-system. An interested reader may refer to [116] for the technical details.

Remark 6.4.2. *Sufficient conditions for simultaneous formation and reference tracking over a general consistent graph and an acyclic consistent graph are given in Theorem 6.4.1 and Corollary 6.4.1. One can select the controller gains (K_{1i}, K_{2i}) to achieve the desired convergence behavior as long as those sufficient conditions are satisfied. Considering an acyclic consistent graph provides more freedom for selecting the controller gains for a better performance in compare with a cyclic graph. The reason is that the sufficient condition for an acyclic consistent graph is given in terms of the stability of the system matrix $\underline{A}_i$ rather than the H_∞ criterion (6.22).*

6.5 Simulation Results

In this section, we give different examples to show that how different formation scenarios can be modeled in the multi-party regulation framework.

Consider a consistent graph $\bar{\mathcal{G}}$ shown in Fig. 6.3, which contains a spanning tree with the leader as its root node and $\rho(Z) = 0.8706$. By using the procedure in

Figure 6.3: The communication graph

Theorem 6.1.1 we have

$$\Gamma_m = \text{Diag}\{1, \exp(-i\pi/6), \exp(-i\pi/2), \exp(-i0.9\pi), \exp(-i1.8\pi)\}.$$

Consider the dynamics of the followers as

$$v_{1,3,5} = \begin{cases} A_{1,3,5} = \begin{bmatrix} -0.3 & -2 \\ 0.1 & -0.2 \end{bmatrix}, B_{1,3,5} = \begin{bmatrix} 1.8 \\ 0.9 \end{bmatrix}, \\ C_{1,3,5} = \begin{bmatrix} -0.1 & 1.2 \end{bmatrix}, D_{1,3,5} = C_{1,3,5}, \end{cases}$$

$$v_{2,4} = \begin{cases} A_{2,4} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}, B_{2,4} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}, \\ C_{2,4} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D_{2,4} = C_{2,4}. \end{cases}$$

6.5.1 Time-invariant formation and tracking a double integrator

Assume that the followers are required to track a double-integrator reference signal:

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0, \quad x_0(0) = \begin{bmatrix} 1 \\ 0.3\iota \end{bmatrix}, \quad y_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_0, \quad (6.36)$$

while forming a time-invariant formation described by (a_i, ψ_i) where $a_1 = a_2 = a_3 = a_4 = a_5 = 1$ and

$$\dot{\vartheta}_i = 0, \quad \psi_i = \vartheta_i, \quad i = 1, \dots, 5, \quad (6.37)$$

with

$$\begin{aligned} \vartheta_1(0) &= 2 - 2\iota, \quad \vartheta_2(0) = 4 - 4\iota, \quad \vartheta_3(0) = 6 - 6\iota, \\ \vartheta_4(0) &= -4 - 4\iota, \quad \vartheta_5(0) = -2 - 2\iota. \end{aligned} \quad (6.38)$$

Since all orientation factors are equal to one; *i.e.*, $a_i = 1$, we assume that the agents communicate over $|\bar{\mathcal{G}}|$. The formation constraints in (6.37-6.38) describe constant relative positions with respect to the leader. Such a formation pattern is commonly used in the literature [112,113].

One can easily see that a single-integrator is a shared dynamics of the leader and the formation constraints. Hence we select (G_{1i}, G_{2i}) to contain a 1-copy the leader according to Remark 6.4.1 for $i = 1, \dots, 5$

$$G_{1i} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_{2i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Figure 6.4: Simultaneous time-invariant formation (6.38) and tracking double-integrator (6.36). The 3D trajectories of the followers and the leader.

Select the controller gains to satisfy (6.22):

$$K_{1i} = -7.09, K_{2i} = \begin{bmatrix} -0.45 & -3.87 \end{bmatrix}, i = 1, 3, 5,$$

$$K_{1i} = -5.43, K_{2i} = \begin{bmatrix} -0.98 & -5.14 \end{bmatrix}, i = 2, 4.$$

The initial conditions for the followers and their controllers are selected randomly and the whole setup is run for 100 seconds. In Fig. 6.4, the 3D trajectories of all agents are shown where the final positions are marked with $\triangle$. The followers track the leader's trajectory (a double-integrator) (6.36) with constant formation constraints given in (6.38). The evolution of the formation error (6.29) through time is shown in Fig. 6.5.

Figure 6.5: Simultaneous time-invariant formation (6.38) and tracking double-integrator (6.36). The plot of formation errors *vs.* time.

6.5.2 Time-invariant formation and tracking a circular orbit

Assume that the followers are supposed to track the following circular orbit:

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x_0, \quad x_0(0) = \begin{bmatrix} 1 \\ 1.4142t \end{bmatrix}, \quad y_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_0. \quad (6.39)$$

Consider a time-invariant formation described by (a_i, ψ_i) where $a_i, i = 1, \dots, 5$ are determined by the communication graph $\bar{\mathcal{G}}$ in Fig. 6.3; *i.e.*, $a_i = \gamma_i^{-1}$, $\angle a_1 = 0^\circ$, $\angle a_2 = 30^\circ$, $\angle a_3 = 90^\circ$, $\angle a_4 = 162^\circ$, $\angle a_5 = 324^\circ$ and

$$\dot{\vartheta}_i = 0, \quad \psi_i = \vartheta_i, \quad i = 1, \dots, 5, \quad (6.40)$$

Figure 6.6: Simultaneous time-invariant formation (6.41) and tracking circular orbit (6.39). The 3D trajectories of the followers and the leader.

with

$$\begin{aligned} \vartheta_1(0) &= 2 - 2\iota, \quad \vartheta_2(0) = 4 - 4\iota, \quad \vartheta_3(0) = 6 - 6\iota, \\ \vartheta_4(0) &= -4 - 4\iota, \quad \vartheta_5(0) = -2 - 2\iota. \end{aligned} \tag{6.41}$$

Select (G_{1i}, G_{2i}) to contain a 1-copy of the internal model of the augmented reference signal (6.30) for $i = 1, \dots, 5$

$$G_{1i} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}, \quad G_{2i} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Figure 6.7: Simultaneous time-invariant formation (6.41) and tracking circular orbit (6.39). The 2D trajectories of the followers and the leader in the last 50 seconds of the simulation.

Select the controller gains to satisfy (6.22):

$$K_{1i} = -4.57, K_{2i} = \begin{bmatrix} -0.25 & 0.02 & -0.54 \end{bmatrix}, i = 1, 3, 5,$$

$$K_{1i} = -2.96, K_{2i} = \begin{bmatrix} -0.24 & -0.06 & -0.44 \end{bmatrix}, i = 2, 4.$$

The initial conditions for the followers and their controllers are selected randomly. The whole setup is run for 700 seconds and the 3D trajectories are shown in Fig. 6.6. The 2D trajectories of the agent (excluding time) in the last 50 second of the simulation is shown in Fig. 6.7, where the final positions at $t = 700$ are marked with $\triangle$. According to Fig. 6.7, the followers track the circular orbit (6.39) and form the formation given by the set of (a_i, ψ_i) . Namely, the final position of each follower i

Figure 6.8: The plot of formation errors *vs.* time

is rotated with respect to leader by $\angle a_i$ and then it is translated by ψ_i . In other words, the formation error (6.29) converges to zero as it is shown in Fig. 6.8.

6.5.3 Time-variant formation and tracking a double integrator

Assume that the followers are required to track a double-integrator reference signal:

$$\dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0, \quad x_0(0) = \begin{bmatrix} 0.1 \\ 0.1 + 0.1t \end{bmatrix}, \quad y_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_0. \quad (6.42)$$

Consider a time-variant formation described by a set of $(a_i, \psi_i(t))$ where $a_1 = a_2 = a_3 = a_4 = a_5 = 1$ and

$$\dot{\vartheta}_i = \begin{bmatrix} 0 & 1 \\ -0.05 & 0 \end{bmatrix} \vartheta_i, \quad \psi_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \vartheta_i, \quad i = 1, \dots, 5, \quad (6.43)$$

with

$$\begin{aligned} \vartheta_1(0) &= \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \vartheta_2(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \vartheta_3(0) = \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \\ \vartheta_4(0) &= \begin{bmatrix} -5 \\ 0 \end{bmatrix}, \quad \vartheta_5(0) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}. \end{aligned} \quad (6.44)$$

Since all orientation factors are equal to one; *i.e.*, $a_i = 1$, we assume that the agents communicate over $|\bar{\mathcal{G}}|$.

Select (G_{1i}, G_{2i}) to contain a 1-copy of the internal model of the augmented reference signal (6.30) for $i = 1, \dots, 5$

$$G_{1i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.05 & 0 \end{bmatrix}, \quad G_{2i} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Select the controller gains to satisfy (6.22):

$$\begin{cases} K_{1i} = -8.07, \\ K_{2i} = \begin{bmatrix} -1.46 & -9.97 & -1.15 & -0.83 \end{bmatrix}, \quad i = 1, 3, 5, \end{cases}$$

Figure 6.9: Simultaneous time-variant formation (6.43-6.44) and tracking double-integrator (6.42). The 3D trajectories of the followers and the leader.

$$\begin{cases} K_{1i} = -5.23, \\ K_{2i} = \begin{bmatrix} -0.93 & -6.48 & -0.74 & -0.42 \end{bmatrix}, i = 2, 4. \end{cases}$$

The initial conditions for the followers and their controllers are selected randomly and the whole setup is run for 150 seconds. In Fig. 6.9, the 3D trajectories of all agents are shown where the final positions at $t = 150$ are marked with $\triangle$. According to Fig. 6.9, the followers track the leader's trajectory (a double-integrator) (6.42) with sinusoidal formation constraints given in terms of the linear command generators (6.43-6.44). The evolution of the formation error (6.29) through time is shown in Fig. 6.10.

Figure 6.10: Simultaneous time-variant formation (6.43-6.44) and tracking double-integrator (6.42). The plot of formation errors *vs.* time.

Chapter 7

Conclusions and Future Works

7.1 Conclusions

In this thesis, we have investigated different aspects of output regulation of linear heterogeneous multi-agent systems. First, we analyzed output regulation of linear heterogeneous multi-agent systems via output and state-feedbacks over a general (cyclic) graph and obtained sufficient conditions for the existence of those controllers. Second, we introduced additive unmodeled disturbances to the agents and considered H_∞ output regulation of linear heterogeneous agents, where the outputs are synchronized to each other with an $\mathcal{L}_2$ gain. Third, we extended the concept of output regulation to bipartite output regulation and multi-party output regulation where the agents communicate over a signed graph or a complex graph. We have discussed an important application of output regulation of linear heterogeneous agents and its related topics in practical applications like formation control and shown how the obtained theory can be used to formulate and solve different formation scenarios.

7.2 Future Works

Our research work could be extended in the following directions.

- In this thesis, we assumed that the communication graph is fixed. Nevertheless, considering a dynamic communication graph is more realistic. Analyzing the output regulation of multi-agent systems over dynamic communication graph would be a challenging topic and requires theories from time-variant and switching systems [22].
- The sufficient condition for output regulation of linear heterogeneous multi-agent systems is described as an upper bound on the H_∞ -norm of single agent systems. The upper bound is uniquely determined by the communication graph. One might consider a designing procedure for the weight of communication graph to obtain a desired performance in terms of a performance index. Additional constraints of the structurally balanced and the consistency of the graph need to be taken care of when a signed graph or a complex graph is considered.
- They are several practical examples where the uncontrolled agents are interconnected, *e.g.*, a group of interconnected generators in a power plant. In this case, the multi-agent system is described over two networks: 1- A physical network which interprets the physical interactions and 2- A communication network which interprets the communication topology. The physical network and the communication network might not have similar topologies. Output regulation of this system might be a challenging and interesting topic.
- One might also analyze output regulation of linear heterogeneous agents subject to attack. The attack could be in the form of a loss of a communication channel or in the form of an external signal which is added to the control signal to ruin the desired performance. One might analyze the system to see under which conditions the system can achieve output regulation and if the perfect regulation is not possible, how good is the performance.

AUTHOR'S PUBLICATIONS

1. **F. Adib Yaghmaie, F. L. Lewis, and R. Su, “Output Regulation of Heterogeneous Linear Multi-Agent Systems via Output and StateFeedback”, *Automatica*, Vol. 67, pp. 157-167, 2016.**
2. **F. Adib Yaghmaie, F. L. Lewis, and R. Su, “Output Regulation of Heterogeneous Multi-Agent Systems with Differential Graphical Game”, *International Journal of Robust and Nonlinear Control*, Vol. 26, pp. 2256-2278, 2016.**
3. **F. Adib Yaghmaie, R. Su, F. L. Lewis, and S. Olaru, “Bipartite and Cooperative Output Synchronizations of Linear Heterogeneous Agents: A Unified Framework”, *Automatica*, To appear, 2017.**
4. **F. Adib Yaghmaie, R. Su, F. L. Lewis, and L. Xie, “Multi-party Consensus of Linear Heterogeneous Multi-agent Systems”, *Automatic Control, IEEE transaction on*, To appear, 2017.**
5. **F. Adib Yaghmaie, K. Hengster Movric, F. L. Lewis, R. Su, and M. Sebek, “ H_∞ Output Regulation of Linear Heterogeneous Multi-agent Systems over Switching Networks”, Under preparation, 2017.**
6. **F. Adib Yaghmaie, K. Hengster Movric, F. L. Lewis, R. Su, and M. Sebek, “Multi-agent Differential Graphical Games for H_∞ Control of Linear Heterogeneous Players”, Under preparation, 2017.**

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Appendices

Appendix A

Proof of Theorem 2.6.1

For simplicity, we omit the overlines over $\bar{A}_i$ and $\bar{B}_i$ and write them as A_i and B_i . Also we write u_i instead of u_{ps_i} . The state variable will be shown by x_i instead of w_i to be stated clearly from the weights that are shown by W_i . In our proof, we consider only two players, $N = 2$. The result can be extended to multiple player easily. Define

$$\bar{D}_i := \left[\frac{\partial \Phi_i}{\partial x_i} \right] B_i R_{ii}^{-1} B_i^T \left[\frac{\partial \Phi_i}{\partial x_i} \right]^T, \quad \sigma_i := \nabla \Phi_i(A_i x_i - \frac{1}{2} B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T W_i), \quad (\text{A.1})$$

$$\sigma_{i+N} := \nabla \Phi_i(A_i x_i - \frac{1}{2} B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T \hat{W}_{i+N}). \quad (\text{A.2})$$

Consider the following Lyapunov function

$$V = V_1(x) + V_2(x) + \frac{1}{2} \tilde{W}_1 a_1^{-1} \tilde{W}_1 + \frac{1}{2} \tilde{W}_2 a_2^{-1} \tilde{W}_2 + \frac{1}{2} \tilde{W}_3 a_3^{-1} \tilde{W}_3 + \frac{1}{2} \tilde{W}_4 a_4^{-1} \tilde{W}_4, \quad (\text{A.3})$$

in which V_i is the approximated solution to (2.26). We derive the time derivative of the all of these terms. First we obtain the time derivative of $V_i = W_i^T \Phi_i + \epsilon_i$

$$\begin{aligned}
\dot{V}_i &= W_i^T \nabla \Phi_i \dot{x}_i + \dot{\epsilon}_i \\
&= W_i^T \nabla \Phi_i (A_i x_i - \frac{1}{2} B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T \hat{W}_{i+N}) + \dot{\epsilon}_i \\
&= W_i^T \nabla \Phi_i (A_i x_i - \frac{1}{2} B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T W_i) - \frac{1}{2} W_i^T \nabla \Phi_i B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T \hat{W}_{i+N} \quad (\text{A.4}) \\
&\quad + \frac{1}{2} W_i^T \nabla \Phi_i B_i R_{ii}^{-1} B_i^T \nabla \Phi_i^T W_i + \dot{\epsilon}_i \\
&= W_i^T \sigma_i + \frac{1}{2} W_i^T \bar{D}_i \tilde{W}_{i+N} + \dot{\epsilon}_i.
\end{aligned}$$

Based on (2.54),

$$\begin{aligned}
W_i^T \sigma_i &= -L_i + \varepsilon_{H_i} \quad (\text{A.5}) \\
&= -y_i^T Q_{ii} y_i - \frac{1}{4} W_i^T \bar{D}_i w_i - \frac{1}{4} W_j^T \nabla \Phi_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \nabla \Phi_j^T W_j + \varepsilon_{H_i}.
\end{aligned}$$

Then, $\dot{V}_1 + \dot{V}_2$ reads

$$\begin{aligned}
\dot{V}_1 + \dot{V}_2 &= W_1^T \sigma_1 + W_2^T \sigma_2 + \frac{1}{2} W_1^T \bar{D}_1 \tilde{W}_3 + \frac{1}{2} W_2^T \bar{D}_2 \tilde{W}_4 + \dot{\epsilon}_1 + \dot{\epsilon}_2 \\
&= \dot{\bar{L}}_v + \frac{1}{2} W_1^T \bar{D}_1 \tilde{W}_3 + \frac{1}{2} W_2^T \bar{D}_2 \tilde{W}_4 + \dot{\epsilon}_1 + \dot{\epsilon}_2, \quad (\text{A.6})
\end{aligned}$$

where $\dot{\bar{L}}_v = W_1^T \sigma_1 + W_2^T \sigma_2$. Now, we analyze the weight estimation errors for the critic networks

$$\dot{E}_i^c = -\frac{1}{2} \tilde{W}_i^T a_i^{-1} \dot{\hat{W}}_i. \quad (\text{A.7})$$

Using the tuning law in (2.57) for the critic weights,

$$\begin{aligned}
\dot{E}_i^c &= \tilde{W}_i^T \frac{\sigma_{i+N}}{(\sigma_{i+N}^T \sigma_{i+N} + 1)^2} \{ -\sigma_{i+N}^T \hat{W}_i + y_i^T Q_{ii} y_i + \frac{1}{4} \hat{W}_{i+N}^T \bar{D}_i \hat{W}_{i+N} \\
&\quad + \frac{1}{4} \hat{W}_{j+N}^T \nabla \Phi_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \nabla \Phi_j^T \hat{W}_{j+N} \}. \quad (\text{A.8})
\end{aligned}$$

Now, using (A.5), the above equation reads

$$\begin{aligned} \dot{E}_i^c = & \dot{\bar{E}}_i^c + \frac{1}{4} \tilde{W}_i^T \frac{\sigma_{i+N}}{(\sigma_{i+N}^T \sigma_{i+N} + 1)^2} \{ \tilde{W}_{i+N}^T \bar{D}_i \tilde{W}_{i+N} + \\ & \tilde{W}_{j+N}^T \nabla \Phi_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \nabla \Phi_j^T \tilde{W}_{j+N} \\ & - 2 W_j^T \nabla \Phi_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \nabla \Phi_j^T \tilde{W}_{j+N} \}, \end{aligned} \quad (\text{A.9})$$

where $\dot{\bar{E}}_i^c = \tilde{W}_i^T \bar{\sigma}_{i+N} (-\bar{\sigma}_{i+N}^T W_i + \frac{\varepsilon_{H_i}}{m_i})$. Next, we analyze the weight estimation errors for the actor networks

$$\dot{E}_i^a = -\frac{1}{2} \tilde{W}_{i+N}^T a_i^{-1} \dot{W}_{i+N}. \quad (\text{A.10})$$

By summing up all of the above terms, we obtain the time derivative of the global Lyapunov function

$$\begin{aligned}
\dot{V} = & \dot{\bar{L}}_v + \dot{\bar{E}}_1^c + \dot{\bar{E}}_2^c + \dot{\epsilon}_1 + \dot{\epsilon}_2 \\
& + \frac{1}{2}W_1^T \bar{D}_1 \tilde{W}_3 + \frac{1}{2}W_2^T \bar{D}_2 \tilde{W}_4 \\
& - \frac{1}{2}W_2 \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T \tilde{W}_4 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 - \frac{1}{2}W_1 \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T \tilde{W}_3 \frac{\bar{\sigma}_4}{m_2} \tilde{W}_2 \\
& + \frac{1}{4}\tilde{W}_3^T \bar{D}_1 W_1 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 - \frac{1}{4}\tilde{W}_3^T \bar{D}_1 W_1 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4}\tilde{W}_3^T \bar{D}_1 \tilde{W}_3 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4}\tilde{W}_4^T \bar{D}_2 W_2 \frac{\bar{\sigma}_4}{m_2} \tilde{W}_2 - \frac{1}{4}\tilde{W}_4^T \bar{D}_2 W_2 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& + \frac{1}{4}\tilde{W}_4^T \bar{D}_2 \tilde{W}_4 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& + \frac{1}{4}\tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 \\
& - \frac{1}{4}\tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& - \frac{1}{4}\tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T \tilde{W}_4 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4}\tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \frac{\bar{\sigma}_4}{m_2} \tilde{W}_2 \\
& - \frac{1}{4}\tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& - \frac{1}{4}\tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T \tilde{W}_3 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& - \tilde{W}_3^T \{a_3^{-1} \dot{\tilde{W}}_3 - \frac{1}{4} \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T \tilde{W}_3 \frac{\bar{\sigma}_4}{m_2} \hat{W}_2 - \frac{1}{4} \bar{D}_1 \hat{W}_3 \frac{\bar{\sigma}_3}{m_1} \hat{W}_1\} \\
& - \tilde{W}_4^T \{a_4^{-1} \dot{\tilde{W}}_4 - \frac{1}{4} \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T \tilde{W}_4 \frac{\bar{\sigma}_3}{m_1} \hat{W}_1 - \frac{1}{4} \bar{D}_2 \hat{W}_4 \frac{\bar{\sigma}_4}{m_2} \hat{W}_2\}.
\end{aligned} \tag{A.11}$$

Using the tuning law in (2.59), the last two lines in the above equation become zero. However, the following terms are added to the above equation

$$\begin{aligned}
& \tilde{W}_3^T F_2 W_1 - \tilde{W}_3^T F_2 \tilde{W}_3 - \tilde{W}_3^T F_1 \bar{\sigma}_3^T W_1 + \tilde{W}_3^T F_1 \bar{\sigma}_3^T \tilde{W}_1, \\
& \tilde{W}_4^T F_4 W_2 - \tilde{W}_4^T F_4 \tilde{W}_4 - \tilde{W}_4^T F_2 \bar{\sigma}_4^T W_2 + \tilde{W}_4^T F_2 \bar{\sigma}_4^T \tilde{W}_2.
\end{aligned} \tag{A.12}$$

Then, (A.11) reads

$$\begin{aligned}
\dot{V} = & -y_1^T Q_{11} y_1 - \frac{1}{4} W_1^T \bar{D}_1 w_1 - \frac{1}{4} W_2^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 + \varepsilon_{H_1} \\
& - y_2^T Q_{22} y_2 - \frac{1}{4} W_2^T \bar{D}_2 w_2 - \frac{1}{4} W_1^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 + \varepsilon_{H_2} \\
& + \tilde{W}_1^T \bar{\sigma}_3 (-\bar{\sigma}_3^T W_1 + \frac{\varepsilon_{H_1}}{m_1}) \\
& + \tilde{W}_2^T \bar{\sigma}_4 (-\bar{\sigma}_4^T W_2 + \frac{\varepsilon_{H_2}}{m_2}) \\
& + \dot{\epsilon}_1 + \dot{\epsilon}_2 \\
& + \frac{1}{2} W_1^T \bar{D}_1 \tilde{W}_3 + \frac{1}{2} W_2^T \bar{D}_2 \tilde{W}_4 \\
& - \frac{1}{2} W_2 \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T \tilde{W}_4 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 - \frac{1}{2} W_1 \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T \tilde{W}_3 \frac{\bar{\sigma}_4}{m_1} \tilde{W}_2 \\
& + \frac{1}{4} \tilde{W}_3^T \bar{D}_1 W_1 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 - \frac{1}{4} \tilde{W}_3^T \bar{D}_1 W_1 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4} \tilde{W}_3^T \bar{D}_1 \tilde{W}_3 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4} \tilde{W}_4^T \bar{D}_2 W_2 \frac{\bar{\sigma}_4}{m_2} \tilde{W}_2 - \frac{1}{4} \tilde{W}_4^T \bar{D}_2 W_2 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& + \frac{1}{4} \tilde{W}_4^T \bar{D}_2 \tilde{W}_4 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& + \frac{1}{4} \tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 \frac{\bar{\sigma}_3}{m_1} \tilde{W}_1 \\
& - \frac{1}{4} \tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& - \frac{1}{4} \tilde{W}_4^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T \tilde{W}_4 \frac{\bar{\sigma}_3}{m_1} W_1 \\
& + \frac{1}{4} \tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \frac{\bar{\sigma}_4}{m_2} \tilde{W}_2 \\
& - \frac{1}{4} \tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& - \frac{1}{4} \tilde{W}_3^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T \tilde{W}_3 \frac{\bar{\sigma}_4}{m_2} W_2 \\
& + \tilde{W}_3^T F_2 W_1 - \tilde{W}_3^T F_2 \tilde{W}_3 - \tilde{W}_3^T F_1 \bar{\sigma}_3^T W_1 + \tilde{W}_3^T F_1 \bar{\sigma}_3^T \tilde{W}_1 \\
& + \tilde{W}_4^T F_4 W_2 - \tilde{W}_4^T F_4 \tilde{W}_4 - \tilde{W}_4^T F_2 \bar{\sigma}_4^T W_2 + \tilde{W}_4^T F_2 \bar{\sigma}_4^T \tilde{W}_2.
\end{aligned}
\tag{A.13}$$

Assume

$$\begin{aligned} \|\epsilon_i\| &< b_{\epsilon_i}, \quad \|\nabla \epsilon_i\| < b_{\epsilon_{ix}}, \quad \|\Phi_i\| < b_{\Phi_i}, \\ \|\nabla \Phi_i\| &< b_{\Phi_{ix}}, \quad \|W_i\| < W_{i \max}, \quad \|\varepsilon_{H_i}\| < \bar{\varepsilon}_i. \end{aligned} \quad (\text{A.14})$$

Then we have a bounded norm for $\|\dot{\epsilon}_i\|$

$$\|\dot{\epsilon}_i\| < b_{\epsilon_{ix}} \|A_i\| \|x_i\| + \frac{1}{2} b_{\epsilon_{ix}} b_{\Phi_{ix}} \|B_i\|^2 \underline{\sigma}(R_{ii}) \left(\|W_i\| + \|\tilde{W}_{i+N}\| \right). \quad (\text{A.15})$$

Now, we can write $\dot{V}$ in terms of $\tilde{Z}^T = [x_1^T, x_2^T, (\bar{\sigma}_3^T \tilde{W}_1)^T, (\bar{\sigma}_4^T \tilde{W}_2)^T, \tilde{W}_3^T, \tilde{W}_4^T]$. Then, (A.13) reads

$$\dot{V} = c + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \tilde{Z}^T M \tilde{Z} + \tilde{Z} \tilde{D}, \quad (\text{A.16})$$

in which $M > 0$ and $\tilde{D}$, c are block matrices. Let m_{ij} denote the block matrix

located in the block row i and block column j .

$$\begin{aligned}
m_{11} &= \begin{bmatrix} C_1^T Q_{11} C_1 & 0 \\ 0 & C_2^T Q_{22} C_2 \end{bmatrix}, \quad m_{22} = m_{33} = I, \quad m_{53} = -\frac{1}{2}F_3 - \frac{1}{8m_2}\bar{D}_2 W_2 = m_{53}^T, \\
m_{44} &= F_2 - \frac{1}{8}(\bar{D}_1 W_1 \frac{\bar{\sigma}_3^T}{m_1} + \frac{\bar{\sigma}_3}{m_1} W_1^T \bar{D}_1 + \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_2 \frac{\bar{\sigma}_4^T}{m_2} \\
&\quad + \frac{\bar{\sigma}_4}{m_2} W_2^T \nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T), \\
m_{55} &= F_4 - \frac{1}{8}(\bar{D}_2 W_2 \frac{\bar{\sigma}_4^T}{m_2} + \frac{\bar{\sigma}_4}{m_2} W_2^T \bar{D}_2 + \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_1 \frac{\bar{\sigma}_3^T}{m_1} \\
&\quad + \frac{\bar{\sigma}_3}{m_1} W_1^T \nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T), \\
m_{42} &= -\frac{1}{2}F_1 - \frac{1}{8m_1}\bar{D}_1 W_1 = m_{24}^T, \quad m_{52} = -\frac{1}{8m_1}\Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_2 = m_{25}^T, \\
m_{43} &= +\frac{1}{4m_2}\nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \\
&\quad - \frac{1}{8m_2}\nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 = m_{34}^T,
\end{aligned} \tag{A.17}$$

and those that are not specified are zero. The elements of vector $\tilde{D}$ are

$$\begin{aligned}
d_1 &= \begin{bmatrix} b_{\epsilon_{1x}} \|A_1\| \\ b_{\epsilon_{2x}} \|A_2\| \end{bmatrix}, \quad d_2 = \frac{\varepsilon_{H_1}}{m_1}, \quad d_3 = \frac{\varepsilon_{H_2}}{m_2}, \\
d_4 &= F_2 W_1 - F_1 \bar{\sigma}_3^T W_1 + \frac{1}{2}\bar{D}_1 W_1 + \frac{1}{2}b_{\epsilon_{1x}} b_{\Phi_{1x}} \|B_1\|^2 \underline{\sigma}(R_{11}) - \frac{1}{4}\bar{D}_1 W_1 \frac{\bar{\sigma}_3^T}{m_1} W_1 \\
&\quad - \frac{1}{4}\nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T W_1 \frac{\bar{\sigma}_4^T}{m_2} W_2 \\
d_5 &= F_4 W_2 - F_3 \bar{\sigma}_4^T W_2 + \frac{1}{2}\bar{D}_2 W_2 + \frac{1}{2}b_{\epsilon_{2x}} b_{\Phi_{2x}} \|B_2\|^2 \underline{\sigma}(R_{22}) - \frac{1}{4}\bar{D}_2 W_2 \frac{\bar{\sigma}_4^T}{m_2} W_2 \\
&\quad - \frac{1}{4}\nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T W_1 \frac{\bar{\sigma}_3^T}{m_1} W_1,
\end{aligned} \tag{A.18}$$

and c is

$$\begin{aligned}
c = & \frac{1}{4} \|W_1\|^2 \|\bar{D}_1\| + \frac{1}{4} \|W_2\|^2 \|\nabla \Phi_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T \nabla \Phi_2^T\| + \bar{\varepsilon}_1 \\
& + \frac{1}{4} \|W_2\|^2 \|\bar{D}_2\| + \frac{1}{4} \|W_1\|^2 \|\nabla \Phi_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T \nabla \Phi_1^T\| + \bar{\varepsilon}_2 \\
& + \frac{1}{2} b_{\epsilon_{1x}} b_{\Phi_{1x}} \|B_1\|^2 \underline{\sigma}(R_{11}) \|W_1\| + \frac{1}{2} b_{\epsilon_{2x}} b_{\Phi_{2x}} \|B_2\|^2 \underline{\sigma}(R_{22}) \|W_2\|.
\end{aligned} \tag{A.19}$$

Using the defined bound in (A.14), c and $\tilde{D}$ are bounded by $c_{\max}$ and $\tilde{D}_{\max}$ respectively. Suppose that the parameters are selected such that matrix $M > 0$. Write M as

$$M = \begin{bmatrix} C_1^T Q_{11} C_1 & 0 & 0 & 0 \\ 0 & C_2^T Q_{22} C_2 & 0 & 0 \\ 0 & 0 & I & M_{23} \\ 0 & 0 & M_{32} & M_{33} \end{bmatrix}. \tag{A.20}$$

For matrix M to be positive definite, it is required that

1. $C_1^T Q_{11} C_1 > 0$, $C_2^T Q_{22} C_2 > 0$ which are satisfied.
2. $I > 0$.
3. $I - M_{23} M_{33}^{-1} M_{32} > 0$, which can be satisfied by proper selection of F_1 , F_2 , F_3 and F_4 .
4. $M_{33} - M_{23} I^{-1} M_{32} > 0$, which can be satisfied by proper selection of F_1 , F_2 , F_3 and F_4 .

Then (A.16) reads

$$\dot{V} < -\|\tilde{Z}\|^2 \underline{\sigma}(M) + \tilde{D}_{\max} \|\tilde{Z}\| + c_{\max} + \bar{\varepsilon}_1 + \bar{\varepsilon}_2. \tag{A.21}$$

By completing the squares, the time derivative of the Lyapunov function becomes

negative if

$$\|\tilde{Z}\| > \frac{\tilde{D}_{\max}}{2\underline{\sigma}(M)} + \sqrt{\frac{\tilde{D}_{\max}^2}{4\underline{\sigma}^2(M)} + \frac{c_{\max} + \bar{\varepsilon}_1 + \bar{\varepsilon}_2}{\underline{\sigma}(M)}}. \quad (\text{A.22})$$

Hence, if V exceeds a certain bound, then its derivative is negative and according to standard Lyapunov extension theorem (Theorem 2.5.7 of [90]), those states in $\tilde{Z}$ are UUB.