Strategic behaviour, equilibria and economic optimality in settings with competing interests

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Strategic Behaviour, Equilibria and Economic Optimality in Settings with Competing Interests

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Abstract

This thesis presents several results in game theory, optimal control theory and optimal location problems and contains some contributions to mergers and voting theory. At a glance, the topics in this thesis may seem unrelated. The short answer would be that this illustrates the power of economic theory: it does not just apply to (optimal) behaviour of firms in markets (Chapters 2 and 3) and agents in exchange economies (Chapter 4), but also in other situations where interests have to be traded off and we may ask “what is optimal?” (Chapter 5), or in voting theory, where we consider how elections should be designed to have desirable properties when aggregating individual preferences (Chapter 6). Metaphorically, one may think of the engine in cars. Different cars, big or small, black, yellow or green, automatic or manual gear etc., all need an engine to start. Albeit not in plain sight, it is what enables them to drive. We usually don’t pause to think about this, but engines, in one way or another, power all cars. Similarly, in the chapters of this thesis, some of the most important tools of economics are working behind the scenes, such as game theory (for example equilibria and strategy, see Section 1.1) and economic optimality (for example profit or utility maximisation) and this allows us to analyse a variety of settings. This shows the versatility of the machinery, and more about the sheer endless number of applications can be found in Chapter 1.

In order to acquaint the reader with some necessary machinery and a historical perspective of the concerned research areas, Chapter 1 provides an introduction to this thesis. More concretely, it sets out some indispensable definitions and results from game theory and (stochastic) optimal control theory.

Market clearing is folklore in economics: as long as prices can adjust freely, demand and supply will adjust until an equilibrium price and quantity is established and

markets equilibrate. What we see in practice, however, is often different: demand in a market may exceed supply, or conversely, falls short of supply. Imagine a ride sharing service: at a given time, there is a certain number of customers that want to be transported. Suppliers of rides have to decide how many cars to put on the road, but competitors do the same. If in total the number of cars is less than the number of customers (assuming every car serves exactly one customer, which is in principle not always the case), supply falls short of demand and some customers are left unserved. Conversely, if there are more cars than ride requests, some cars are not serving any customers and it may be costly to have them drive around without clients on board.

This is essentially the topic of Chapter 2, where we construct Cournot games with limited demand, and show that they admit three distinct equilibrium regimes, including an intermediate regime that allows for a range of possible equilibria. When demand is modeled by a delayed diffusion process, we also show that this intermediate regime reduces to a single equilibrium while the other regimes approximate the deterministic setting as the delay tends to zero. Numerical illustrations are presented when demand is modeled by an Ornstein-Uhlenbeck process and price is an affine function of output.

Chapter 3 resembles the previous, there is limited demand, but the market structure is different. That is, there is one firm that acts as a supplier to the other and the latter then sells to consumers. A setup like this is rampant in every day economics, and in the literature such games are called newsvendor games. It is easy to see why: we could think of the first company as a wholesaler, selling not directly to the public, but rather to a retailer, who sells to consumers. This retailer faces (possibly uncertain) demand and has to choose between a) buying too much from the wholesaler, and not being able to resell all his products to consumers and b) buying too little from the wholesaler and potentially missing out some profit. We can determine the equilibrium in the market using a maximum principle in the case of uncertainty about the demand. If there is no uncertainty, we find the equilibrium by reasoning backwards, starting with the retailer, and using his decisions as input for the wholesaler.

More precisely, we extend stochastic newsvendor games with information lag by

including dynamic retail prices, and characterize their equilibria. We show that the equilibrium wholesale price is a nonincreasing function of the demand, while the retailer's output increases with demand until we recover the usual equilibrium. In particular, it is then optimal for retailer and wholesaler to have demand at least equal to some threshold level, beyond which the retailer's output tends to an upper bound which is absent in fixed retail price models. When demand is given by an Ornstein-Uhlenbeck process and price is an affine function of output we numerically compute the equilibrium output. We also show that the lagged case can be seen as a smoothing of the no lag case.

In the contemporary internet age, markets are more and more found online. It is progressively easier to trade, buy and sell using the internet and networks. Here, but also in traditional markets, we must ask: is it possible to exploit the mechanism that underpins the market? In other words: can users, or groups of users, manipulate markets for their own gain? Especially online, it is rather easy to lie about personal characteristics that serve as inputs to the market mechanism. For example, one could easily create an identity stating to have very little money, and in fact be a millionaire. Such manipulations may potentially influence the allocation of goods. This is the topic of Chapter 4, in which we try to understand how much an agent may gain by lying about his characteristics when he knows how the exchange mechanism in the economy works.

First, we will introduce the reader to the incentive ratio, a concept to measure the extent to which an equilibrium mechanism may be manipulated. If this ratio is close to 1, the opportunities for manipulation by means of misreporting one's personal parameters (i.e. utility function) are small and can be expected not to outweigh (computational) costs. The main contributions are that, contrary to what was shown by previous research on Fisher markets, the benefits from manipulation can be arbitrarily large in a more general model. Zooming in on the case of Cobb-Douglas markets, we show that we can break the upper bound from Fisher models already in markets with only three commodities; nonetheless, in such markets, the incentive ratio is still bounded by a reasonably small constant (3). More generally, we show

that the incentive ratio is bounded by the number of commodities.

It is easy to take decisions when everyone agrees what needs to be done. When there is some degree of disagreement, a compromise that is good for everyone can usually (but not always) be found. But when interests are opposed, it is far from obvious that any good solution exists. This is the situation we explore in Chapter 5, where two groups have opposing interests. We use a mathematical result that says there exists at least one location on a sphere (such as the Earth) that both groups consider preferable to its antipode. Then, we try to select economically optimal points among such locations. It is shown that such points always exist.

Concretely, we consider the facility location on a sphere. There have been few contributions so far considering the sphere as target location, and the most recent one has been restrictive in the sense that it presumes that every agent is alike in their taste for the proximity of the facility. But in fact, it is much more natural to assume a heterogeneous agent population, containing both agents that like and dislike the facility, so that interests are opposed: some agents will want to be near to the facility, whereas others will want to be far away. If such agents are located near each other on the sphere, this poses an interesting problem. We show that there are points so that neither group of agents (both those who consider the facility desirable as well as the one who consider it undesirable) prefers its antipodal. We use this as the starting point for our analysis, and then select a Pareto optimal location (which is guaranteed to exist under mild continuity assumptions) as the plausible destination for the facility. This result encompasses the previous solution offered in the literature when the population is homogeneous.

Chapter 6 contains some work in progress and partial results. It is concerned with three things: the consequences of mergers, the range of the Borda rule and Condorcet winning sets. More precisely, Section 6.1 on mergers describes markets in which a leader emerges: pre-merger, firms simultaneously choose their outputs and we determine the Cournot equilibrium. We then suppose that a merger among the firms takes place, and the merging firms become market leader doing so, and the market now has a leader-follower structure. Again we determine the equilibrium and compare the

post-merger equilibrium to the pre-merger equilibrium. This section is inspired by a recent publication in this area and extends it to the case of asymmetric firms, but only when the inverse demand takes the (familiar) form $P(Q) = a - bQ$. Section 6.2 about the Borda rule asks which outcomes can be obtained by an appropriate collection of individual preferences and specifically targets the open question in which the number of alternatives/candidates is a multiple of 4, and the number of voters is odd. We provide code in Julia meant to search for counterexamples of outcomes that are *not* in the range of Borda's rule in Appendix A. Section 6.3 on Condorcet winning sets presents an improved lower bound on the number of candidates needed for Condorcet winning sets of cardinality four or higher. Python code that can be used to search for such sets is given in Appendix B.

Finally, Chapter 7 concludes the material presented in this thesis and presents some directions for future research.

Chapter 1

Introduction

This chapter describes some necessary background material for the subsequent chapters. We will cover some basics of game theory, strategic behaviour and optimal control theory, that can serve as an embedding of the new results of this thesis in an appropriate context. We amplify that most material in this chapter is not an original contribution made in this thesis, but taken from other sources, with the aim to equip the reader with a basis so as to be able to understand the results presented later on. For a much more elaborate and detailed treatment, the reader is referred to one of the many references below.

1.1 Game theory

Game theory is the mathematical study of strategic interaction between “agents” or “players”. Two branches can be distinguished: a cooperative branch and a noncooperative branch. The former deals with problems in which (coalitions of) players have to divide a certain amount of revenue or profit in a way that satisfies certain requirements. For example, it may be natural to require “group rationality”, that is, no collection of agents has an incentive to suspend negotiations, and instead divide whatever revenue they can generate by themselves. In other words, what they get out of the negotiations should not be less than what they can already attain by themselves. In particular, if the “group” contains just a single agent, we have “individual rationality”, which requires that no single agent is better off on his own than

under the coalitional scheme. Such requirements lead to a host of different solution concepts, containing all admissible ways of splitting the proceeds of the labour among the agents, of which the *core* is probably the best known one. It was first considered in [24] but its modern day definition is credited to [31]. One well known application of this theory is a bankruptcy problem from the Talmud in [3]. In noncooperative game theory there is no formation of coalitions, but players try to maximise their own payoffs while considering the choices of others. This basically applies to any situation where there is interaction between (economic) agents, which may often be thought of as firms, labour unions, private individuals, governments etc. depending on the context and the problem at hand. Particularly important are the notions of *strategy* and *solution concept* i.e. equilibrium, which we will make precise in what follows. We will treat strategic behaviour in production settings (Chapters 2 and 3) and in exchange economies (Chapter 4). No treatment of strategic interaction can be carried out without some necessary machinery. Although the reader may have some image in mind of a “game” that is potentially not far off of what is actually studied in game theory, e.g. chess, rock–paper–scissors, tic–tac–toe or something along the lines of [8], it is important to define this rigorously and in an abstract way the vital notion of a game.¹

Definition 1.1.1. *A normal form game is a tuple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where N is a set of agents or players, S_i is a set of (pure) strategies for player $i \in N$ and $u_i : \prod_{i \in N} S_i \rightarrow \mathbb{R}$ is i ’s payoff function that maps an tuple of strategies as chosen by the players to the reals.*

We assume from the outset that players are rational, that is, they act to maximise u_i given that they know “everything” about the game: the (number of) players, their payoff functions, their strategies etc.. Also, all players know that all players know this, and all players know that all players know that all players know, *ad infinitum*. Moreover, the players have no problems making the computations involved in the analysis of the game (e.g. finding equilibria or determining the best response corre-

¹We could consider something more rudimentary, called a *game form*, in which a tuple of strategies is mapped to an outcome and players have preferences over the outcomes that are complete and transitive.

spondence). All these assumptions can be relaxed and indeed this is the objective of much contemporary research, for example bounded rationality which as discussed in [73], [2] refer to Simon's contributions [79] and [78] that were later picked up by economists. We immediately see why game theory can be useful: agent i seeks to maximise u_i , but doing so, he only controls $s_i \in S_i$. Thus, the best strategy may not be obvious, and indeed, may depend on the strategies chosen by the other players. Therefore, all agents need to be able to reason how other agents may react to their strategies, in order to find the best choice.

The set N may, at least in principle, be of infinite size (countable or not) although here we will focus on the case where N is finite. Sometimes there are only two players (e.g. Section 1.1.1) and as we will see there is a wealth of literature covering just this case. A priori, S_i may be any kind of set or space, e.g. $S_i = \{H, T\}$, $S_i = (0, 1)$ or $S_i = \{f : [0, T] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$. In principle S_i could even be empty, reducing i to a dummy player, but we ignore such possibilities in the remainder.

What we ultimately want to achieve in game theory is to “predict” the “outcome” of a game. Of course we do not really predict anything: under the assumption that agents behave in a very precisely described rational sense, the “outcome” is the logical consequences of their strategies. Such an outcome is determined by a solution concept, formally a function that associates to a game the strategy profiles that satisfy certain properties. For example, if all agents are rational decision makers that wish to maximise their own payoff function, knowing that all other agents do the same, we may look for (pure) Nash equilibria. In other words, Nash equilibrium is one of the many possible solution concepts, and indeed it is the most popular one. Nevertheless it may fail to exist, although there is a large class of games (the finite ones) that always possess at least one equilibrium; see Theorem 1.1.12. On the other hand, sometimes there may be many Nash equilibria, and we may need a more stringent solution concept i.e. we want to rule out certain Nash equilibria as the likely outcomes of the game. In this case we may resort to equilibrium refinements such as subgame perfect equilibria [76]; see also [2] for an enumeration of such refinements and their relation with “superrationality”.

Definition 1.1.2 (Strategy profile). *We call an element of $\prod_{i \in N} S_i$ a strategy profile. In case $|N| = n < \infty$, we may write it as $(s_1, \dots, s_n)$. When considering the strategy of agent i , it will be convenient to sometimes write this as (s_i, s_{-i}) , where s_{-i} is the tuple of strategies of every agent except i i.e. $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.*

Strategies may be pure, as mentioned above, or “mixed” in the following sense. The modern definition dates back to Borel [61], [12], [11]. The following definition comes from [46] (and applies when S_i is finite or countable).

Definition 1.1.3 (Mixed strategy). *Let $\sigma : S_i \rightarrow [0, 1]$ such that $\sigma(s_i) \geq 0$ for all $s_i \in S_i$ and $\sum_{s_i \in S_i} \sigma(s_i) = 1$. Then we call σ a mixed strategy for Player i .*

In other words, a mixed strategy is a probability distribution over Player i ’s pure strategies. When this set is finite, we can identify such a mixed strategy to a point in the simplex. We denote the set of all mixed strategies by $\Delta(S_i)$, $i \in N$.

We call the game $\Gamma = (N, (\Delta(S_i))_{i \in N}, (u_i)_{i \in N})$ the mixed extension of G . We can extend u_i in a multilinear way accordingly for games with finitely many players, see [43] for a more general treatment. For ease of exposition, we demonstrate here the familiar case $|N| = 2$. $u_i : \Delta(S_1) \times \Delta(S_2) \rightarrow \mathbb{R}$ is then defined as

$$u_i(s_1, s_2) = \sum_{(j,k) \in S_1 \times S_2} s_{1j} s_{2k} u_i(j, k), \quad i = 1, 2.$$

Equipped with the basics, let us turn now to two player zero-sum games. Indeed, they were regarded as a crucial milestone in the development of game theory by John von Neumann, who is credited saying that: “As far as I can see, there could be no theory of games on these bases without that theorem... Throughout the period in question I felt that there was nothing worth publishing until the “minimax theorem” was proved” [35], [74]. It should however be stressed that there were other significant contributions already, as pointed out in [52] by Zermelo (1913, [91], in German, see [75] for a discussion and translation of the original paper in English) and Borel (1921, 1924, in French) (see [12], [11] respectively for English translations)

1.1.1 Two player zero-sum games

Much of this section is based on [43] (in French). Another reference in English is [62], or the reader may refer to one of the many references at the end of this section. We start off with the definition of a two-player zero-sum game.

Definition 1.1.4 (Two player zero-sum game). *A zero-sum game in strategic form is defined as a tuple $G = (S_1, S_2, u)$ where S_1 (S_2) is a nonempty set of strategies for Player 1 (2) and $u: S_1 \times S_2 \rightarrow \mathbb{R}$ is the payoff function of Player 1.*

Notice how, in this case, we abuse notation by omitting the set of players. In this (default) notation, it is immediately understood there are only two players. When Player 1 chooses $j \in S_1$ and Player 2 chooses $k \in S_2$, the payoff is $u(j, k)$ and this can be seen as the amount that Player 2 pays to Player 1 i.e. Player 2's payoff is $-u(j, k)$. In this sense, interests are diametrically opposed and the name “zero-sum” is appropriate. Notice that we can describe the game in this case using just one payoff function. If the game is finite i.e. if S_1 and S_2 are both finite, we may represent the game using a matrix. The following game called Matching Pennies is folklore:

	H	T
H	1	-1
T	-1	1

Table 1.1: Matching pennies

Here, $S_1 = S_2 = \{H, T\}$ and it is clear that the objective of Player 1 (the row player) is to choose the same action as Player 2, while the latter tries to choose a different strategy compared to Player 1. The payoffs in the matrix above are Player 1's payoffs.

Value of a game

In this section we want to find equilibrium strategies (provided that they exist) and study the consequences for the payment that the players make to each other. To do so, we start off with a definition of what it means to “guarantee” a payoff.

Definition 1.1.5. *Let $G = (S_1, S_2, u)$ be a two-player zero-sum game.*

Player 1 can guarantee or assure $w \in \mathbb{R} \cup \{-\infty\}$ in G if he has a strategy that always gives him a payoff of at least w , i.e.

$$\exists j \in S_1 \quad \forall k \in S_2 \quad u(j, k) \geq w.$$

Similarly, Player 2 can guarantee or assure $w \in \mathbb{R} \cup \{+\infty\}$ in G if he has a strategy that makes him have to pay at most w , i.e.

$$\exists k \in S_2 \quad \forall j \in S_1 \quad u(j, k) \leq w.$$

It is clear from the above definition that for all $j \in S_1$ Player 1 can guarantee $\inf_{k \in S_2} u(j, k)$ and for all $k \in S_2$ Player 2 can guarantee $\sup_{j \in S_1} u(j, k)$. We are now ready to introduce the following vital notions about payoffs.

Definition 1.1.6. *The maxmin value of G , denoted by $\underline{v}$ is the quantity*

$$\sup_{j \in S_1} \inf_{k \in S_2} u(j, k) \in \mathbb{R} \cup \{\pm\infty\}.$$

The minmax value of G , denoted by $\bar{v}$ is the quantity

$$\inf_{k \in S_2} \sup_{j \in S_1} u(j, k) \in \mathbb{R} \cup \{\pm\infty\}.$$

Thus the maxmin of a game is the supremum of all the payoffs that can be guaranteed by Player 1; similarly, the minmax of a game is the infimum of all the payoffs that can be guaranteed by Player 2. Consider a situation in which Player 1 would have to choose $j \in S_1$ first and Player 2 would observe his choice. In such case, Player 2 would choose a strategy $k \in S_2$ that minimizes $u(j, k)$: it is exactly this that the maxmin value captures since in such scenario, Player 1 would want to play a strategy that gives him the best worst-case payoff i.e. the maxmin value. A similar story applies to the minmax value. It is easily verified that for any game we always have $\underline{v} \leq \bar{v}$.

Definition 1.1.7 (Value of a game). *The game $G = (S_1, S_2, u)$ has a value if $\underline{v} = \bar{v}$*

and in this case we denote this value by $\text{val}(G) = v = \underline{v} = \bar{v}$.

In Matching Pennies, see Table (1.1.1) above, there is no value for the game in pure strategies since we have $-1 = \underline{v} < \bar{v} = 1$. If the value of a game exists, it may be seen as the “price” corresponding to the natural outcome of the game: this is what, given the rationality assumptions (Player 1 tries to maximise u , Player 2 tries to minimise it), Player 1 can be expected to receive from or pay to Player 2. It may also be verified that if there exists w that can be guaranteed by both players, then this value is unique and $\text{val}(G) = w$.

We may consider the mixed extension of a game G , Γ , and define the maxmin and minmax accordingly. Let us now present the celebrated Minimax theorem [86].

Theorem 1.1.8 (Minimax). *Let $G = (S_1, S_2, u)$ be a finite two-player zero-sum game and let M be a real matrix of dimensions $|S_1| \times |S_2|$, where $M_{jk} = u(j, k)$. Then there exist (j^*, k^*, v) in $\Delta(S_1) \times \Delta(S_2) \times \mathbb{R}$ such that:*

$$\forall k \in \Delta(S_2), j^* M k \geq v$$

and

$$\forall j \in \Delta(S_1), j M k^* \leq v.$$

That is, G has a value in mixed strategies and the players have optimal strategies (in the sense that $u(j^*, k) \geq v = u(j^*, k^*) \geq u(j, k^*)$ for all $(j, k) \in \Delta(S_1) \times \Delta(S_2)$).

Various different proofs of the theorem can be found, e.g. based on linear programming.

Remark 1.1.9. *Notice that it is indeed crucial that the strategy sets of the players are finite. Else, consider the trivial two player zero-sum game where $S_1 = (0, 1)$, $S_2 = \{0\}$ and $u(j, k) = j$. This game has a value (1) but clearly Player 1 has no optimal strategy.*

Many other important results for zero-sum games were established subsequently, the most well known probably being the extensions by Fan [26], [27] and Sion’s minmax theorem [80].

1.1.2 Nonzero-sum games

Gradually, interest shifted from zero-sum games to nonzero-sum games. The result we are about to introduce next has become one of the cornerstones of contemporary economics, but has also seen applications for outside this realm: for example in biology [87], [56], [34], computer science [55], auction theory/mechanism design [85], inspection games [6] and even sports [63]. The history of Nash (equilibrium) is rife with amusing anecdotes. First, when John F. Nash presented his result [54], then still called “equilibrium points” to von Neumann, the latter said: “That’s trivial, you know. That’s just a fixed point theorem.” [53]. The fixed point theorem that this statement alluded to, is Kakutani’s fixed point theorem. But as Binmore recounts in his book “Playing for Real” [9], Kakutani was rather puzzled by the presence of numerous economists during one of his talks. He asked Binmore why there had been so many economists attending and when Binmore told him that it was due to Kakutani’s fixed point theorem, Kakutani replied “What is the Kakutani fixed point theorem?”. The aforementioned paper by Nash is hardly one page long, but nonetheless highly influential, helping Nash win the 1994 Nobel Prize (together with Harsanyi and Selten).

We shall present here the results without proof; for completeness we mention Kakutani’s fixed point theorem [37].

Theorem 1.1.10 (Kakutani fixed point theorem). *Let S be a non-empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $\phi: S \rightarrow 2^S$ be a correspondence on S with a closed graph and non-empty, convex values for all $s \in S$. Then ϕ has a fixed point i.e. there exists $s^* \in S$ such that $s^* \in \phi(s^*)$.*

It is often encountered in equivalent form, e.g. in [46], using upper hemicontinuity of ϕ instead of the closed graph property. It is this theorem, essentially the generalisation to correspondences of Brouwer’s fixed point theorem (Let C be a nonempty, convex, compact subset of a Euclidean space and $f: C \rightarrow C$ and continuous, then f has a fixed point) [15], that is the vital ingredient for Nash’s celebrated result.

Nash Equilibrium

We wish to generalise the notion of equilibrium to nonzero-sum games and kickoff this section right away with the concept of Nash equilibrium.

Definition 1.1.11 (Nash equilibrium). *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game. We say that a strategy profile $(s_i^*)_{i \in N}$ is a Nash equilibrium if*

$$\forall i \in N \quad \forall s_i \in S_i \quad u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

In other words, a strategy profile is a Nash equilibrium if there do not exist unilaterally profitable deviations for any player, *given the strategies chosen by the other players*. We are now ready to state Nash's famous theorem [54]:

Theorem 1.1.12 (Nash). *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a finite game i.e. N contains finitely many players, each of whom possesses a finite number of pure strategies. Then there exists an equilibrium in mixed strategies.*

In the case of G is a zero-sum game, the solution coincides with von Neumann's minimax theorem. Of course, not every game has a finite number of players, each of whom possess only finitely many pure strategies; indeed, Remark 1.1.9 already pointed this out. Many extensions and variations to the Nash equilibrium have been considered to deal with the limitations of the equilibrium concept, such as the one by Glicksberg [32] and Debreu–Glicksberg–Fan (see e.g. [29]), evolutionary stable strategies [48] or trembling hand perfect equilibrium [77]. It is important to stress that Nash equilibrium is a very widely applied concept and it may even be used for nonsequential games, as we will now point out.

Extensive Form Games

We will encounter some basic extensive form game in Chapter 3. First, we will need the definition of an extensive form game with perfect information. We take this definition, and much of what follows in this section, from [61]. We amplify that the setting is the one of complete and perfect information, where everything about the

game is known: preferences/payoff functions, strategies, past events etc.. Indeed, it is possible to speak about games of incomplete or imperfect information, in which there is uncertainty about the *type*, strategies, payoff functions of other players or past events, see also [61] for more on (im)perfect information and the remarks at the end of this section.

Definition 1.1.13 (Extensive form game). *An extensive form game with perfect information is a tuple $(N, H, P, (u_i)_{i \in N})$ where N is the set of players, H is a set of sequences that satisfies:*

- *The empty sequence $\emptyset$ is a member of H .*
- *If $(a^k)_{k=1, \dots, K} \in H$ (K may be infinite) and $L < K$ then $(a^k)_{k=1, \dots, L} \in H$.*
- *If an infinite sequence $(a_k)_{k=1}^\infty$ satisfies $(a^k)_{k=1, \dots, L} \in H$ for every positive integer L then $(a_k)_{k=1}^\infty \in H$.*

We call H the set of histories of the game and each component of a history a^k is an action taken by a player. A history $(a^k)_{k=1, \dots, K} \in H$ is called terminal if it is infinite or if there is no a^{K+1} such that $(a^k)_{k=1, \dots, K+1} \in H$. We denote the set of such terminal histories by Z . $P : H \setminus Z \rightarrow N$ is the player function, indicating who's the player to choose an action. Finally, u_i is a utility function representing a preference relation $\succsim_i$ (i.e. a complete and transitive binary relation) on Z for Player $i \in N$.²

Given a nonterminal history $h \in H \setminus Z$, Player $P(h)$ chooses an action from

$$A(h) = \{a \mid (h, a) \in H\}.$$

If action $a^0 \in A(h)$ has been chosen, Player $P(h, a^0)$ chooses the next action (provided the history (h, a^0) is not terminal). We say that a game is finite if H is finite.

Next, we define the notion of strategy for an extensive form game. It may be thought of as a comprehensive plan of how to play the game: for every history where

²Some sources, such as [61], use relations, others directly assume (von Neumann-Morgenstern) payoffs function, e.g. [60] and [36]. Indeed, it is possible to build a large share of game theory based on preference relations instead of payoff functions, and arguing that the preference relation may under certain conditions be represented by a function.

Player i may be called on to move, the strategy specifies which action to play. Generally, given a strategy profile, some nodes are never reached, yet a strategy should specify an action also for such nodes, in particular even if a previous action by the same player assures play will not reach the designated node. A strategy may also be thought of as an instruction to another agent on how to play the game: this agent does not need to reason about the consequences of the actions by him-/herself, but simply implements whatever it is told before the game starts.

Definition 1.1.14 (Strategy in an extensive form game). *A strategy of Player $i \in N$ in an extensive form game with perfect information is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \setminus Z$ for which $P(h) = i$.*

A strategy profile induces an outcome $O(s)$, in the sense that when the actions are played according to the strategies in s , we get a terminal history as a result. This history $(a^1, \dots, a^K) \in Z$ (possibly $K = \infty$) satisfies $s_{P(a^1, \dots, a^k)}(a^1, \dots, a^k) = a^{k+1}$ for all $0 \leq k \leq K$.

We are now in a position to define the Nash equilibrium for extensive form games.

Definition 1.1.15 (Nash equilibrium of an extensive form game). *A Nash equilibrium for the game $(N, H, P, (u_i)_{i \in N})$ is a strategy profile s^* such that*

$$\forall i \in N \quad \forall s_i \in S_i \quad u_i(O(s_i^*, s_{-i}^*)) \geq u_i(O(s_i, s_{-i}^*)),$$

It is common to represent an extensive form game using a tree, which may be a more compact representation of a game than a lengthy definition. By now, it is probably a good idea to discuss an example extensive form game and illustrate the above concepts based on the example.

In Figure 1.1, Player 1 has two actions (hence two strategies): A and B . Player 2, however, can choose from two actions at two different nodes, for a total of four pairs of actions i.e. four strategies. We usually denote them as a tuple, in the order in which they appear from left to right in the game tree, that is, as (C, C) , (C, D) , (D, C) or (D, D) . Therefore, the strategy (C, D) means that Player 2 will choose C if he has to move on the left node (after Player 1 played A), and D when is called for action on

Figure 1.1: Finite extensive form game, the payoffs are of the form (payoff Player 1, payoff Player 2) and labels on edges corresponds to players' actions.

the right node (after player 1 played B), even though only one of the two will happen when the game is actually played, of course depending on the strategy of Player 1.

We may convert this form to a game in normal form to analyse it and find its Nash equilibria. The corresponding normal form game is given by:

	(C,C)	(C,D)	(D,C)	(D,D)
A	(3,0)	(3,0)	(1,-1)	(1,-1)
B	(0,0)	(0,0)	(2,1)	(2,1)

Table 1.2: Strategic form game corresponding to extensive form game 1.1. Player 1 is the row player, Player 2 is the column player.

Therefore, the Nash equilibria are:

$$(A, (C,C)), (A, (C,D)), (B,(D,C)) \text{ and } (B,(D,D)).$$

Notice however, that there is something odd about the last two equilibria: when Player 1 actually chooses A, Player 2 replies by choosing D, while choosing C would give him/her a higher payoff ($0 > -1$). However, the game will never reach this branch according to the given strategy profiles, since Player 1 will start the game picking B as his action. We say that $(B,(D,C))$ and $(B,(D,D))$ are strategy profiles containing an *incredible threat*: if Player 2 would have been called for action after Player 1's move A, he would, in his own interest, not choose D. Therefore, such equilibria are unlikely to be established, and we wish to rule them out. This is the purpose of the subgame perfect equilibrium (SPE) concept, which we introduce next.

Subgame Perfect Equilibrium

First of all, we need a precise definition of “subgame” of a game.

Definition 1.1.16. *The subgame of the extensive game $G = (N, H, P, (u_i)_{i \in N})$ that follows the history h is the extensive game $G(h) = (N, H|_h, P|_h, (u_i|_h)_{i \in N})$, where $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$, $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and $u_i|_h$ is defined by $u_i|_h(h') \geq u_i|_h(h'')$ if and only if $u_i(h, h') \geq u_i(h, h'')$.*

Notice that from the definition, in particular, the whole game is a subgame of itself. Before stating the definition of a subgame perfect equilibrium, we need a little bit more notation. Given a strategy s_i for Player i and a history h , let $s_i|_h$ be the strategy that s_i induces in the subgame $G(h)$: $s_i|_h(h') = s_i(h, h')$ for all $h' \in H|_h$ and let O_h be the outcome function of $G(h)$.

Definition 1.1.17 (Subgame perfect equilibrium). *A subgame perfect equilibrium of an extensive form game $G = (N, H, P, (u_i)_{i \in N})$ is a strategy profile s^* such that*

$$\forall i \in N, \forall h \in H \setminus Z \text{ for which } P(h) = i \quad u_i(O_h(s_i^*|_h, s_{-i}^*|_h)) \geq u_i(O_h(s_i, s_{-i}^*)),$$

for every strategy s_i of Player i in the subgame $G(h)$.

In other words, a strategy profile s^* is a subgame perfect equilibrium if it induces a Nash equilibrium in every subgame of the game. Since the whole game is a subgame of itself, it is immediate that for a strategy profile to be a subgame perfect equilibrium, it has to satisfy stricter requirements than Nash equilibrium.

We state the following important result without a proof; the interested reader is referred to [61]. Subgame perfect equilibria can be found using a procedure called backward induction, a technique which we shall use in Section 3.4.

Theorem 1.1.18 (Kuhn). *Every finite extensive form game possesses a subgame perfect equilibrium.*

Many settings in economics naturally arise as extensive form games and we will

consider a wholesaler who sells to a retailer in Chapter 3 and a classical Stackelberg setup in Section 6.1. Here, the player making the first move knows that his follower(s) will react to his decision in an optimal way and this can be used to determine his own economically optimal decision.

Further reading

As mentioned in the beginning of this section, it is possible to consider games of incomplete or imperfect information, and indeed there are many interesting results in this area. Perhaps the most relevant question is how to generalise the notion of (Nash) equilibrium to the setting of incomplete/imperfect information. The interested reader may explore topics such as Nash equilibrium in Bayesian games or beliefs and sequential equilibrium. Some references with varying amounts of mathematical rigour and proofs and applications (in economics) include [29], [36], [60], [61], [46], [9], [62], [47], [43] (in French). Much special attention within the game theory community has been, and still is, devoted to the study of repeated games and their applications. See [45], [4], [51] or [81].

1.2 Optimal control

1.2.1 Deterministic control

We present here two different approaches to solve optimal control problems, namely maximum principles and Hamilton–Jacobi–Bellman (HJB) equations. The two were developed separately, but exhibit an interesting connection, see e.g. [90]. After that, we will extend the deterministic maximum principle to the stochastic case, while only providing a sketch and rough directions for the stochastic HJB approach. This is because Chapters 2 and 3 are in fact based on stochastic maximum principles and do not use the HJB equations. References are provided for much more details and context.

Pontryagin's maximum principle

We shall introduce this section with some (nonstochastic) optimal control theory. We start with Pontryagin's maximum principle. The following is based on a more general statement in [22], which can also be consulted for more details, in particular for weaker assumptions, and an introduction to the discrete time case.

Let a time interval $[0, T]$, $0 < T \leq \infty$ be given. The goal is to maximise some function f . on this interval. f . depends on three things: x_t , the state of the system at time $t \in [0, T]$; u_t , the control at time $t \in [0, T]$ and on t itself. We assume the controls may be chosen from the set $\mathcal{U}[0, T] = \{u : [0, T] \rightarrow \mathbb{R}^m \mid u. \text{ is measurable}\}$. The change in state of the system is governed by a differential equation (referred to as “law of motion” or “trajectory” of x_t)

$$dx_t = g_t(x_t, u_t)dt, \quad (1.1)$$

and we are given some initial state x so that we may formulate the problem as

$$\begin{cases} \max_{u \in \mathcal{U}[0, T]} \int_0^T f_t(x_t, u_t)dt, \\ dx_t = g_t(x_t, u_t)dt, \\ x_0 = x \in \mathbb{R}^n. \end{cases} \quad (1.2)$$

In order to find the optimal control $u^* \in \mathcal{U}$, we set up the Hamiltonian:

$$\mathcal{H}_t(x_t, u_t) = f_t(x_t, u_t) + \lambda_t g_t(x_t, u_t), \quad (1.3)$$

we refer to λ_t as the adjoint variable.

We have the following cornerstone result:

Theorem 1.2.1 (Pontryagin's maximum principle). *Let f . and g . be continuously differentiable functions with respect to x_t, u_t and t . Let $(u_t^*)_{t \in [0, T]}$, be a solution to problem (1.2). Then there exist continuous functions of time λ_t such that $\forall t \in [0, T]$,*

$$1. \quad u_t^* = \arg \max_{u_t} \mathcal{H}_t(x_t, u_t) = \arg \max_{u_t} \{f_t(x_t, u_t) + \lambda_t g_t(x_t, u_t)\}$$

$$2. \quad dx_t^* = g_t(x_t^*, u_t^*)dt$$

$$3. \quad -\frac{\partial \mathcal{H}_t(x_t^*, u_t^*)}{\partial x_t} = \frac{\partial \lambda_t}{\partial t}$$

I.e. Theorem 1.2.1 provides necessary conditions for the optimality of the controls.

Hamilton–Jacobi–Bellman

Let us now turn to the HJB approach. This section is entirely taken from [90] and many more details and results may be found there.

We will consider the following control system:

$$\begin{cases} dx_t = \mu_t(x_t, u_t)dt, & \text{a.e. } t \in [0, T], \\ x_0 = x \in \mathbb{R}^n. \end{cases} \quad (1.4)$$

Here, again, u_t is the control, which is an element of

$$\mathcal{U}[0, T] = \{u.: [0, T] \rightarrow U \mid u. \text{ is measurable}\}, \quad (1.5)$$

where U is a metric space, $0 < T \leq \infty$, $\mu.: [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, a given function.

We associate a cost functional to (1.4):

$$J(u.) = \int_0^T f_t(x_t, u_t)dt + h(x_T), \quad (1.6)$$

where $f.$ and $h.$ are given functions.

Now what we want to do is find a control that minimises (1.6) subject to the constraint imposed by the stochastic differential equation for the state variable; i.e. minimise (1.6) over (1.5) subject to (1.4).

However, without further restrictions, (1.6) may be not well-defined: it is possible that (1.4) admits multiple solutions for given $u. \in \mathcal{U}[0, T]$; we want $x_t: [0, T] \rightarrow \mathbb{R}^n$ to be continuous and unique. We will take care of this later.

Next, we will take a seemingly peculiar step: we will consider a generalisation of our given problem, and attempt to solve it. Let $(s, y) \in [0, T] \times \mathbb{R}^n$ and consider the

following control system over $[s, T]$ (cf. (1.4)):

$$\begin{cases} dx_t = \mu_t(x_t, u_t)dt, & \text{a.e. } t \in [s, T], \\ x_s = y \in \mathbb{R}^n. \end{cases} \quad (1.7)$$

and analogous to (1.5), u is an element of

$$\mathcal{U}[s, T] = \{u : [s, T] \rightarrow U \mid u \text{ is measurable}\}, \quad (1.8)$$

and the cost function is given by (cf. (1.6))

$$J(s, y; u) = \int_s^T f_t(x_t, u_t)dt + h(x_T), \quad (1.9)$$

our optimisation problem now becomes, for given $(s, y) \in [0, T] \times \mathbb{R}^n$:

Minimise (1.9) over (1.8) subject to (1.7).

Note that, if we choose $s = 0$ and $y = x$, we get back exactly the formulation of the initial problem. We now provide the assumptions under which (1.7) admits a unique solution which we will denote by $x := x(s, y, u)$.

(A1) (U, d) is a separable metric space.

(A2) $\mu_t : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $f_t : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are uniformly continuous and there exists a constant $L > 0$ such that for $\phi_t(x, u) = \mu_t(x, u)$, $f_t(x, u)$, $h(x)$,

$$\begin{cases} |\phi_t(x, u) - \phi_t(\hat{x}, u)| \leq L |x - \hat{x}|, \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U, \\ |\phi_t(0, u)| \leq L, \quad \forall (t, u) \in [0, T] \times U. \end{cases} \quad (1.10)$$

Next, we introduce the value function for our (initial) problem as

$$\begin{cases} V(s, y) = \inf_{u \in \mathcal{U}[s, T]} J(s, y, u), \quad \forall (s, y) \in [0, T] \times \mathbb{R}^n, \\ V(T, y) = h(y), \quad \forall y \in \mathbb{R}^n. \end{cases} \quad (1.11)$$

We are now in the position to state the following result:

Theorem 1.2.2 (Bellman's principle of optimality). *Assume (A1) and (A2) hold. Then for all $(s, y) \in [0, T) \times \mathbb{R}^n$,*

$$V(s, y) = \inf_{u \in \mathcal{U}[s, T]} \left\{ \int_s^{\hat{s}} f_t(x_t(s, y, u), u_t) dt + V(\hat{s}, x_{\hat{s}}(s, y, u)) \right\}, \forall 0 \leq s \leq \hat{s} \leq T. \quad (1.12)$$

It is relatively straightforward to verify that as a consequence of (1.12) that an optimal control that is globally optimal, is also locally optimal. One problem however is that (1.12) is complicated and difficult to handle. We would like to have an expression for V that is easier to digest. We have the following:

Proposition 1.2.1. *Assume (A1) and (A2) hold. Suppose $V \in \mathcal{C}^1([0, T] \times \mathbb{R}^n)$. Then V solves the following terminal value problem of a first-order partial differential equation (notice how the term over which we take the sup is actually a Hamiltonian):*

$$\begin{cases} -\frac{\partial V}{\partial t} + \sup_{u \in \mathcal{U}} -f_t(x, u) - \frac{\partial V}{\partial x} \mu_t(x, u) = 0, \\ V|_{t=T} = h(x), \quad x \in \mathbb{R}^n. \end{cases} \quad (1.13)$$

(1.13) is called the HJB equation associated with the initial problem. If we can solve it, from the solution, we are able to find an optimal pair (x^*, u^*) via the so called verification theorem. However, even this result may be too restrictive, for example because V may not be of class $\mathcal{C}^1$ or the HJB equation does not have a unique solution: we can then resort to *viscosity subsolutions* of (1.13). Again, for much more details, the interested reader is referred to [90].

1.2.2 Stochastic control

Stochastic maximum principle

Chapters 2 and 3 involve stochastic maximum principles. It can be instructive to think of a control as a strategy in the sense described in Section 1.1, but since we will model demand by a stochastic process, it is incumbent to go through some preliminaries.

Most of the following is taken from [66] (Section 6.4.2); alternatively see also [83] (Section 9.1.1) for explanation about the stochastic Pontryagin maximum principle.

Suppose X_t at time $t \in [0, T]$ is described by a stochastic differential equation of the form

$$\begin{cases} dX_t = \mu_t(X_t, u_t)dt + \sigma_t(X_t, u_t)dB_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \quad (1.14)$$

where $(B_t)_{t \in [0, T]}$ is a standard Brownian motion that generates the $\mathbb{P}$ -augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. Furthermore, $u_t \in \mathcal{U}[0, T]$, the set of progressively measurable control processes with values in $A \subset \mathbb{R}^d$.

What does it mean to solve (1.14)? We can distinguish strong and weak solutions and here highlight the former, see [90]. A strong solution to (1.14) is an $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted continuous stochastic process $(X_t)_{t \in [0, T]}$ that satisfies:

- $X_0 = x_0, \quad \mathbb{P} - a.s.,$
- $\int_0^t \{|\mu_s(X)| + |\sigma_s(X)|^2\}ds < \infty, \quad \forall t \geq 0 \quad \mathbb{P} - a.s.,$
- $X_t = X_0 + \int_0^t \mu_s(X)ds + \int_0^t \sigma_s(X)dB_s, \quad t \geq 0, \quad \mathbb{P} - a.s.$

Here, the first integral is a Lebesgue integral called the drift term; the second integral is an Itô integral called the diffusion term.

In case two strong solutions X and Y of (1.14) defined on any $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ and any standard $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion satisfy

$$\mathbb{P}(X_t = Y_t \mid 0 \leq t < \infty) = 1,$$

then we say that the strong solution is unique.

The performance or gain functional, the functional that u_t seeks to maximise, is given by

$$J(u) = \mathbb{E} \left[\int_0^T f_t(X_t, u_t)dt + g(X_T) \right],$$

where $f: [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}$ is continuous in (t, x) for all $a \in A$, $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is a concave $\mathcal{C}^1$ function and f, g satisfy a quadratic growth condition in x , i.e. there

exists positive constants C_1, C_2 and a function $\kappa : A \rightarrow \mathbb{R}_+$ such that:

$$|f_t(x, a)| \leq C_1(1 + |x|^2) + \kappa(a), \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^n \times A,$$

and

$$|g(x)| \leq C_2(1 + |x|^2), \quad \forall x \in \mathbb{R}^n.$$

Now we introduce the Hamiltonian $\mathcal{H} : [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}$ and define it by:

$$\mathcal{H}_t(x, u, y, z) = f_t(x, u) + \mu_t(x, u)y + \sigma_t(x, u)z,$$

and we assume that $\mathcal{H}$ is differentiable with respect to x . For each $u \in \mathcal{U}[0, T]$ we have a Backward Stochastic Differential Equation (BSDE)³ called the adjoint equation as follows:

$$\begin{cases} dY_t = -\frac{\partial \mathcal{H}_t}{\partial x}(X_t, u_t, Y_t, Z_t)dt + Z_t dB_t, \\ Y_T = \frac{\partial g}{\partial x}(X_T), \end{cases} \quad (1.15)$$

in particular when $g \equiv 0$, a case that we will encounter in Chapters 2 and 3,

$$\begin{cases} dY_t = -\frac{\partial H}{\partial x}(t, X_t, u_t, Y_t, Z_t)dt + Z_t dB_t, \\ Y_T = 0. \end{cases} \quad (1.16)$$

We then have the following result:

Theorem 1.2.3 (Stochastic maximum principle). *Let $u^* \in \mathcal{U}[0, T]$ and let X^* be the associated controlled diffusion. Suppose that (1.15) admits a solution (Y^*, Z^*) such that*

$$\mathcal{H}_t(X_t^*, u_t^*, Y_t^*, Z_t^*) = \max_{u \in \mathcal{U}} \mathcal{H}_t(X_t^*, u, Y_t^*, Z_t^*), \quad 0 \leq t \leq T, \quad a.s.$$

and

$$(x, u) \rightarrow \mathcal{H}_t(x, u, Y_t^*, Z_t^*) \text{ is a concave function,}$$

³This body of research was pioneered in [64] although BSDEs first appeared in [10], where it was attempted to extend the Pontryagin maximum principle to the stochastic setup, see [83].

for all $t \in [0, T]$. Then u^* is an optimal control, i.e.

$$J(u^*) = \sup_{u \in \mathcal{U}[0, T]} J(u).$$

We will encounter such stochastic maximum principles in Section 2.3 and Section 3.3.

We say that X_t is *uncontrolled* if μ_t and σ_t do not depend on the control u . i.e. if we may write (1.14) as

$$\begin{cases} dX_t = \mu_t(X_t)dt + \sigma_t(X_t)dB_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \quad (1.17)$$

and in particular, when $\mu_t(X_t) = \alpha(\beta - X_t)$ and $\sigma(X_t) = \sigma$, $\alpha, \beta, \sigma > 0$, we call this stochastic process an Ornstein–Uhlenbeck process [84], a stationary process which is both Gaussian and Markovian and is the only that satisfies these three conditions, up to linear transformations of the space and time variables [23]. Moreover, it is also mean-reverting.

We have:

$$\begin{cases} dX_t = \alpha(\beta - X_t)dt + \sigma dB_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \quad (1.18)$$

with solution, see e.g. [57]:

$$X_t = X_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t}) + \int_0^t \sigma e^{\alpha(s-t)} dB_s.$$

Idea for stochastic Hamilton-Jacobi-Bellman

Since we will be working with maximum principles for the stochastic case in Chapters 2 and 3, we will not go deeply into a stochastic extension of the HJB approach. Roughly, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, (1.4) is “extended” with a random component, i.e. a Brownian motion B_t , in the following way; notice that we use X_t now instead of x_t to denote a stochastic process:

$$\begin{cases} dX_t = \mu_t(X_t, u_t)dt + \sigma_t(X_t, u_t)dB_t, & \text{a.e. } t \in [0, T], \\ X_0 = x \in \mathbb{R}^n. \end{cases} \quad (1.19)$$

We recall that given such a filtered probability space, an $(\mathcal{F}_t)_{t \geq 0}$ -adapted $\mathbb{R}^n$ -valued process $X(\cdot)$ is called an n -dimensional $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion over $[0, \infty)$ if for all $0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and covariance $(t - s)I$. That is, for any $0 \leq s \leq t$,

$$\begin{cases} \mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0, & \mathbb{P} - a.s., \\ \mathbb{E}[(X_t - X_s)(X_t - X_s)^T \mid \mathcal{F}_s] = (t - s)I, & \mathbb{P} - a.s.. \end{cases} \quad (1.20)$$

And if $\mathbb{P}(X_0 = 0) = 1$ then X is called an n -dimensional standard $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion over $[0, \infty)$. A Brownian motion is an example of a diffusion process, i.e. a Markov process that has continuous sample paths or trajectories, i.e. without jumps.

Because of the randomness, the cost functional (1.6) now needs an expectation operator:

$$J(u) = \mathbb{E} \left[\int_0^T f_t(X_t, u_t)dt + h(X_T) \right]. \quad (1.21)$$

The set of controls $\mathcal{U}[0, T]$, is also modified to account for the introduction of the Brownian motion:

$$\mathcal{U}[0, T] = \{u : [0, T] \rightarrow \mathbb{R}^n \mid u \text{ is measurable and } (\mathcal{F}_t)_{t \geq 0} \text{ adapted}\}.$$

What we want now is to minimise (1.21) subject to (1.19) over $\mathcal{U}[0, T]$ as above and a new expression for the value function may be derived.

Chapter 2

Cournot games with limited demand: from multiple equilibria to stochastic equilibrium

This chapter is the result of research carried out together with Nicolas Privault.

2.1 Introduction

Cournot games with multiple equilibria under uncertain demand have attracted significant attention in recent years, see for example [42], [41], [16]. On the other hand, the stochastic control of diffusions with delayed filtrations has recently been applied to newsvendor games, cf. [58], [59] in order to deal with partial information. Under model uncertainty, these results have been extended in [50] to the case of delayed coefficients in the equation driving the state process.

In this chapter, we construct Cournot games in which limited demand can induce multiple equilibria in absence of lag. This range of equilibria collapses to a single equilibrium when demand is delayed and modeled by a random process. We choose to model limited demand by sharing sales proportionally to production in case total output exceeds demand. This deviates from standard assumptions stating that all production is sold as long as prices can adjust to their market clearing levels.

In Section 2.2 we consider one-shot Cournot games with limited demand. In this

setting, Proposition 2.2.1 presents a general formula for the individual equilibrium production shares, and we derive an equation satisfied by the total production. When the market price $P(q)$ is of the familiar affine form $P(q) = a - bq$ for $a, b > 0$, we show the existence of three different regimes of equilibria, including an intermediate regime that includes multiple equilibria, cf. Proposition 2.2.2 and Figures 2.1 and 2.2.

Second, in Section 2.3 we consider a setup in which demand is modeled by a stochastic diffusion process $(D_t)_{t \in [0, T]}$ and information about demand is lagged by a time amount $\delta > 0$ in such a way that firms have to decide on their strategies for time t based on the delayed value $D_{t-\delta}$ of a random demand process. Using the maximum principle for the stochastic control of diffusions with delayed filtrations [58], [59], we derive necessary and sufficient conditions for optimality of controls in this delayed setting, see also [21] in the case of delayed stochastic differential equations. In Proposition 2.3.2 we present a natural generalization of Proposition 2.2.1 to the case of delayed controls, based on the derivation of an equation for the total production.

A more explicit formulation is then obtained in Proposition 2.3.3 when the stochastic demand $(D_t)_{t \in [0, T]}$ is modeled exogenously according to an Ornstein-Uhlenbeck process. Finally, in the setting of affine prices we show in Corollary 2.3.4 that the ranges of equilibria obtained in Proposition 2.2.2 reduce to a unique equilibrium at every time, no matter how small the lag. In addition to leading to a unique equilibrium in the intermediate regime, the lagged case can be seen as a smoothing of the no lag case, as the solution of the deterministic no lag problem can be obtained as the limit of the lagged setup as δ tends to 0.

It turns out that when demand is low, the equilibrium is unique in both the lag and no lag cases, and total output exceeds demand (overproduction). When demand is high enough, individual equilibrium outputs are also unique and in line with the usual Cournot equilibrium, and total output falls short of demand (underproduction) in both setups. Finally, when demand is within an intermediate range, the output exactly meeting demand belongs to the range of possible equilibria in the no lag case, while output falls short of demand in the lag case.

2.2 Cournot game with limited demand

We consider two competitors facing a demand $d \geq 0$ for goods priced $P(q) \geq 0$ where $q := q^{(1)} + q^{(2)}$ is the sum of the production outputs $q^{(1)} \geq 0, q^{(2)} \geq 0$ of Players 1 and 2. We choose to model limited demand by assuming that the profit $f^{(i)}(d, q^{(1)}, q^{(2)})$ of player $i = 1, 2$ is given by

$$\begin{aligned} f^{(i)}(d, q^{(1)}, q^{(2)}) &:= P(q^{(1)} + q^{(2)}) \min \left(q^{(i)}, \frac{q^{(i)} d}{q^{(1)} + q^{(2)}} \right) - c^{(i)} q^{(i)} \\ &= P(q^{(1)} + q^{(2)}) \left(q^{(i)} \mathbf{1}_{\{q^{(1)} + q^{(2)} \leq d\}} + \frac{q^{(i)} d}{q^{(1)} + q^{(2)}} \mathbf{1}_{\{d < q^{(1)} + q^{(2)}\}} \right) - c^{(i)} q^{(i)}, \end{aligned} \quad (2.1)$$

where $c^{(i)} > 0$ is firm i 's per unit production cost, $i = 1, 2$. In other words, when the total output $q^{(1)} + q^{(2)}$ becomes larger than d , the demand is shared between Players 1 and 2 according to their respective shares $q^{(1)}/(q^{(1)} + q^{(2)})$ and $q^{(2)}/(q^{(1)} + q^{(2)})$ of total production. The intuition is that a firm that produces more generates more exposure, so if a buyer would randomly buy the product, the probability that firm i makes a sale is $q^{(i)}/(q^{(1)} + q^{(2)})$, $i = 1, 2$.

A Nash equilibrium for the game (2.1) is a pair $(q^{(1)*}, q^{(2)*})$ such that

$$f^{(1)}(d, q^{(1)}, q^{(2)*}) \leq f^{(1)}(d, q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} \geq 0, \quad (2.2)$$

and

$$f^{(2)}(d, q^{(1)*}, q^{(2)}) \leq f^{(2)}(d, q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} \geq 0, \quad (2.3)$$

so that neither player has an incentive to deviate given the control of the other.

In Proposition 2.2.1 we present a general characterization of equilibrium for the game (2.1) under limited demand. This allows for negative values of the equilibrium production outputs $q^{(1)*}$ and $q^{(2)*}$ and it does not treat the case where production $q^{(1)*}$ and $q^{(2)*}$ meets the demand d , as the derivatives of the profit functions $f^{(i)}(d, q^{(1)}, q^{(2)})$, $i = 1, 2$, are not continuous at this point and more information on the function $P(q)$ is needed. In Proposition 2.2.2 below we refine the result of Proposition 2.2.1 by constraining production to be nonnegative when the market price $P(q)$ is an affine function of the total output q .

Proposition 2.2.1. *The Cournot equilibrium action for the game (2.1) is given by*

$$q^{(i)*} = S^* \frac{\min(d, S^*)P(S^*) - c^{(i)}S^*}{2\min(d, S^*)P(S^*) - (c^{(1)} + c^{(2)})S^*}, \quad 1 \leq i \neq j \leq 2, \quad (2.4)$$

provided that $S^* = q^{(1)*} + q^{(2)*} \neq d$, where S^* solves the equation

$$S^*P'(S^*) + P(S^*) = (c^{(1)} + c^{(2)})\max(1, S^*/d) + P(S^*)\mathbf{1}_{\{S^* > d\}}.$$

Proof. Consider the partial derivatives

$$\begin{aligned} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) &= (q^{(i)}P'(q^{(1)} + q^{(2)}) + P(q^{(1)} + q^{(2)}))\mathbf{1}_{\{q^{(1)} + q^{(2)} < d\}} \\ &\quad + \left(\frac{q^{(i)}P'(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}}d + \frac{q^{(j)}P(q^{(1)} + q^{(2)})}{(q^{(1)} + q^{(2)})^2}d \right) \mathbf{1}_{\{d < q^{(1)} + q^{(2)}\}} - c^{(i)} \\ &= \left(\frac{q^{(i)}P'(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}} + \frac{q^{(j)}P(q^{(1)} + q^{(2)})}{(q^{(1)} + q^{(2)})^2} \right) \min(q^{(1)} + q^{(2)}, d) \\ &\quad + \frac{q^{(i)}P(q^{(1)} + q^{(2)})}{q^{(1)} + q^{(2)}}\mathbf{1}_{\{q^{(1)} + q^{(2)} \leq d\}} - c^{(i)}, \quad 1 \leq i \neq j \leq 2. \end{aligned}$$

a) When $q^{(1)*} + q^{(2)*} < d$, the first order conditions become

$$q^{(i)*}P'(q^{(1)*} + q^{(2)*}) + P(q^{(1)*} + q^{(2)*}) - c^{(i)} = 0, \quad i = 1, 2,$$

and the equilibrium agrees with the “usual” Cournot equilibrium. Namely, we have

$$\begin{aligned} q^{(i)*} &= \frac{c^{(i)} - P(q^{(1)*} + q^{(2)*})}{P'(q^{(1)*} + q^{(2)*})} \\ &= (q^{(1)*} + q^{(2)*}) \frac{(q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2(q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})}, \end{aligned}$$

$i = 1, 2$, and by summation we find

$$(q^{(1)*} + q^{(2)*})P'(q^{(1)*} + q^{(2)*}) + 2P(q^{(1)*} + q^{(2)*}) = c^{(1)} + c^{(2)}.$$

b) On the other hand, if $d < q^{(1)*} + q^{(2)*}$ we have

$$q^{(i)*}P'(q^{(1)*} + q^{(2)*}) + \frac{q^{(j)*}}{q^{(1)*} + q^{(2)*}}P(q^{(1)*} + q^{(2)*}) = \frac{c^{(i)}}{d}(q^{(1)*} + q^{(2)*}), \quad 1 \leq i \neq j \leq 2,$$

hence by summation we have

$$(q^{(1)*} + q^{(2)*})P'(q^{(1)*} + q^{(2)*}) + P(q^{(1)*} + q^{(2)*}) = \frac{c^{(1)} + c^{(2)}}{d}(q^{(1)*} + q^{(2)*}),$$

and

$$q^{(i)*} = (q^{(1)*} + q^{(2)*}) \frac{dP(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2dP(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})}, \quad 1 \leq i \neq j \leq 2.$$

In general, we conclude that

$$S^*P'(S^*) + 2P(S^*) = (c^{(1)} + c^{(2)}) \max\left(1, \frac{S^*}{d}\right) + P(S^*)\mathbf{1}_{\{S^* > d\}},$$

and

$$q^{(i)*} = (q^{(1)*} + q^{(2)*}) \frac{\min(d, q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - c^{(i)}(q^{(1)*} + q^{(2)*})}{2\min(d, q^{(1)*} + q^{(2)*})P(q^{(1)*} + q^{(2)*}) - (c^{(1)} + c^{(2)})(q^{(1)*} + q^{(2)*})},$$

$$1 \leq i \neq j \leq 2. \quad \square$$

Affine prices

Until the end of this section we focus on the case

$$P(q^{(1)} + q^{(2)}) = a - b(q^{(1)} + q^{(2)}). \quad (2.5)$$

We set $q^{(i)} = 0$ if the production cost $c^{(i)}$ is larger than $a > 0$, henceforth we focus on the case where $0 < c^{(i)} < a$ for $i = 1, 2$. The next proposition shows the existence of three equilibrium regimes, depending on the level of the demand.

Proposition 2.2.2. *Assume that $P(q)$ is given by (2.5), and that*

$$a - 2c^{(1)} + c^{(2)} \geq 0 \quad \text{and} \quad a + c^{(1)} - 2c^{(2)} \geq 0,$$

and let

$$\underline{d} := \frac{a - c^{(1)} - c^{(2)}}{2b} \quad \text{and} \quad \bar{d} := \frac{2a - c^{(1)} - c^{(2)}}{3b}.$$

a) In case $d > \bar{d}$, we have the equilibrium

$$(q^{(1)*}, q^{(2)*}) = \left(\frac{a - 2c^{(1)} + c^{(2)}}{3b}, \frac{a + c^{(1)} - 2c^{(2)}}{3b} \right) = \left(2(\bar{d} - \underline{d}) - \frac{c^{(1)}}{b}, 2(\bar{d} - \underline{d}) - \frac{c^{(2)}}{b} \right), \quad (2.6)$$

with $q^{(1)*} + q^{(2)*} = \bar{d} < d$.

b) In case $d < \underline{d}$, we have the equilibrium

$$(q^{(1)*}(d), q^{(2)*}(d)) = \left(\frac{ad(c^{(2)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2}, \frac{ad(c^{(1)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2} \right), \quad (2.7)$$

with $d < q^{(1)*}(d) + q^{(2)*}(d) = \frac{ad}{c^{(1)} + c^{(2)} + 2bd}$.

c) In case $\underline{d} \leq d \leq \bar{d}$, a point $(q^{(1)*}(d), q^{(2)*}(d))$ is an equilibrium if and only if

$$\max \left(\left(2(d - \underline{d}) - \frac{c^{(i)}}{b} \right)^+, \frac{(a - c^{(i)} - bd)d}{a} \right) \leq q^{(i)*}(d) \leq \min \left(\frac{(c^{(j)} + bd)d}{a}, 2\underline{d} - d + \frac{c^{(j)}}{b} \right), \quad (2.8)$$

with $q^{(1)*}(d) + q^{(2)*}(d) = d$, $1 \leq i \neq j \leq 2$.

The proof of Proposition 2.2.2 is deferred to the appendix.

Remarks.

1) We check that when $d = \bar{d}$, the lower bound in (2.8) reads

$$\frac{a - 2c^{(i)} + c^{(j)}}{3b} \leq q^{(i)*},$$

which is consistent with the equality (2.6).

2) When $d = \underline{d}$, the upper bound in (2.8) is also consistent with the solution

$$(q^{*1}(\underline{d}), q^{*2}(\underline{d})) = \left(\frac{(c^{(2)} + b\underline{d})\underline{d}}{a}, \frac{(c^{(1)} + b\underline{d})\underline{d}}{a} \right)$$

given in (2.7).

3) When $\underline{d} < d < \bar{d}$, the condition $q^{(1)*}(d) + q^{(2)*}(d) = d$, $1 \leq i \neq j \leq 2$, shows that only one of the two controls $q^{(1)*}(d)$, $q^{(2)*}(d)$ can be chosen independently.

The following Figures 2.1 and 2.2 illustrate the three different regimes (a)-(b)-(c) of Proposition 2.2.2, including the existence of a range of equilibria that result from (2.8) in the regime (c) when $\underline{d} \leq d \leq \bar{d}$.

Figure 2.1: Equilibrium output $q^{(1)*}(d)$ as a function of d_t .

Figure 2.2: Equilibrium output $q^{(2)*}(d)$ as a function of d .

Figure 2.3 shows that we move from overproduction to underproduction as the value of d increases, while demand is exactly met in the regime (c) when $\underline{d} \leq d \leq \bar{d}$.

Figure 2.3: Total output $d \mapsto q^{(1)*}(d) + q^{(2)*}(d)$.

The parametric plot of Figure 2.4 shows the possible equilibrium quantities $(q^{(1)*}(d), q^{(2)*}(d))$ depending on the level of demand $d > 0$. We check that every value of $d \in [0, \underline{d}] \cup [\bar{d}, \infty)$ correspond to a unique choice of $(q^{(1)*}(d), q^{(2)*}(d))$, whereas the values $d \in (\underline{d}, \bar{d})$ yield a continuum of equilibria located on the isolines $q^{(1)*}(d) + q^{(2)*}(d) = d$.

Figure 2.4: Parametric plot of $(q^{(1)*}(d), q^{(2)*}(d), d)$ with its projection onto the plane $d = 0$.

The next three figures present the corresponding profits $f^{(i)}(d, q^{(1)}, q^{(2)})$ of players $i = 1, 2$ as given from (2.1), with $a = 7$, $b = 2$, $c^{(1)} = 1$, $c^{(2)} = 2$.

Figure 2.5: Profit of Player 1.

Figure 2.6: Profit of Player 2.

Figure 2.7: Total profit.

The setting of this section can be made time-dependent with demand $d_t \geq 0$ at time $t > 0$, where $q_t^{(1)} \geq 0$, $q_t^{(2)} \geq 0$ denote the production outputs of Players 1 and 2. In this case,

$$J^{(i)}(q^{(1)}, q^{(2)}) := \int_0^T f^{(i)}(d_t, q_t^{(1)}, q_t^{(2)}) dt \quad (2.9)$$

with $(q^{(1)}, q^{(2)}) := (q_t^{(1)}, q_t^{(2)})_{t \in [0, T]}$, denotes the total profit of Player $i = 1, 2$, and a Nash equilibrium for the game (2.9) is a pair $(q^{(1)*}, q^{(2)*}) = (q_t^{(1)*}, q_t^{(2)*})_{t \in [0, T]}$ such that

$$J^{(1)}(q^{(1)}, q^{(2)*}) \leq J^{(1)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} = (q_t^{(1)})_{t \in [0, T]}, \quad (2.10)$$

and

$$J^{(2)}(q^{(1)*}, q^{(2)}) \leq J^{(2)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} = (q_t^{(2)})_{t \in [0, T]}. \quad (2.11)$$

In the next section we will consider such a time-dependent setting in which competitors face uncertain stochastic demand and make optimal decisions based on historical data.

2.3 Cournot game with delayed stochastic demand

Let $(B_t)_{t \in [0, T]}$ denote a standard Brownian motion that generates the $\mathbb{P}$ -augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. The information flow available to the players is represented by the delayed filtration $(\mathcal{E}_t)_{t \in [\delta, T]}$ defined by

$$\mathcal{E}_t := \mathcal{F}_{t-\delta}, \quad t \in [\delta, T], \quad (2.12)$$

where $\delta > 0$ denotes the information delay or lag.

We let $\mathcal{A}^{(i)}$ denote a given set of admissible control processes for Player i , which are made of real-valued $(\mathcal{E}_t)_{t \in [\delta, T]}$ -predictable processes, $i = 1, 2$. We endow $\mathcal{A}^{(i)}$ with the supremum norm and denote $\mathcal{A} := \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$.

In this section we introduce uncertainty in the demand, which is now modeled as the solution $(D_t)_{t \in [0, T]}$ of a controlled stochastic differential equation of the form

$$\begin{cases} dD_t = \mu_t(D_t, q_t^{(1)}, q_t^{(2)})dt + \sigma_t(D_t, q_t^{(1)}, q_t^{(2)})dB_t, \\ D_0 = d_0 \in \mathbb{R}, \end{cases} \quad (2.13)$$

where $\mu.(d, q^{(1)}, q^{(2)})$, $\sigma.(d, q^{(1)}, q^{(2)})$ are given predictable processes for each $d \in \mathbb{R}$, such that (2.13) admits a unique strong solution.

Given $f_t^{(i)} : \mathbb{R}^3 \rightarrow \mathbb{R}$, the profit function of Player i defined in (2.1), $i = 1, 2$, $t \in [0, T]$, let

$$J^{(i)}(q^{(1)}, q^{(2)}) := \mathbb{E} \left[\int_{\delta}^T f_t^{(i)}(D_t, q_t^{(1)}, q_t^{(2)}) dt \right], \quad (2.14)$$

denote the expected profits of Players 1 and 2, where D_t denotes the solution of (2.13), provided the integrals and expectations exist.

A Nash equilibrium for the Forward-Backward Stochastic Differential Equation (FBSDE) game (2.14) is a pair $(q^{(1)*}, q^{(2)*}) \in \mathcal{A}$ such that

$$J^{(1)}(q^{(1)}, q^{(2)*}) \leq J^{(1)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(1)} \in \mathcal{A}^{(1)}, \quad (2.15)$$

and

$$J^{(2)}(q^{(1)*}, q^{(2)}) \leq J^{(2)}(q^{(1)*}, q^{(2)*}) \quad \text{for all } q^{(2)} \in \mathcal{A}^{(2)}, \quad (2.16)$$

so that neither player has an incentive to deviate given the control of the other.

Maximum principle

We rely on the framework developed in [59], under the following conditions on the set $\mathcal{A} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$ of control processes.

(A1) For all $u^{(i)} \in \mathcal{A}^{(i)}$ and all bounded $\beta^{(i)} \in \mathcal{A}^{(i)}$ there exists $\varepsilon > 0$ such that

$$u^{(i)} + s\beta^{(i)} \in \mathcal{A}^{(i)} \quad \text{for all } s \in (-\varepsilon, \varepsilon); \quad i = 1, 2.$$

(A2) For all $t_0 \in [\delta, T]$ and all bounded $\mathcal{E}_{t_0}$ -measurable random variables $\alpha^{(i)}$, the control process $\beta_t^{(i)}$ defined by

$$\beta_t^{(i)} = \mathbf{1}_{[t_0, T]}(t) \alpha^{(i)}(\omega), \quad t \in [\delta, T],$$

belongs to $\mathcal{A}^{(i)}$; $i = 1, 2$.

Define the Hamiltonians $H_t^{(i)} : \mathbb{R}^5 \rightarrow \mathbb{R}$ of this game by

$$H_t^{(i)}(d, q^{(1)}, q^{(2)}, y^{(i)}, z^{(i)}) := f_t^{(i)}(d, q^{(1)}, q^{(2)}) + y^{(i)} \mu_t(d, q^{(1)}, q^{(2)}) + z^{(i)} \sigma_t(d, q^{(1)}, q^{(2)}), \quad (2.17)$$

$i = 1, 2$. To these Hamiltonians we associate the backward SDEs

$$\begin{cases} dY_t^{(i)} = -\frac{\partial H_t^{(i)}}{\partial d}(D_t, q_t^{(1)}, q_t^{(2)}, Y_t^{(i)}, Z_t^{(i)}) dt + Z_t^{(i)} dB_t, & t \in [\delta, T], \\ Y_T^{(i)} = 0 \end{cases} \quad (2.18)$$

in the adjoint processes $Y_t^{(i)}$ and $Z_t^{(i)}$, $i = 1, 2$, which can be solved under the assumptions (A1)-(A2) above. We will rely on the following maximum principle, cf. Theorem 3.1 in [7] and Theorem 2.2 in [59].

Theorem 2.3.1. *The following are equivalent:*

(1) For all bounded $\beta^{(1)} \in \mathcal{A}^{(1)}$, $\beta^{(2)} \in \mathcal{A}^{(2)}$ we have

$$\frac{d}{ds} J^{(1)}(q^{(1)} + s\beta^{(1)}, q^{(2)})|_{s=0} = \frac{d}{ds} J^{(2)}(q^{(1)}, q^{(2)} + s\beta^{(2)})|_{s=0} = 0.$$

(2) We have

$$\mathbb{E} \left[\frac{\partial H_t^{(i)}}{\partial u^{(i)}}(D_t, u^{(1)}, u^{(2)}, Y_t^{(i)}, Z_t^{(i)}) \middle| \mathcal{E}_t \right]_{u^{(1)}=q_t^{(1)}, u^{(2)}=q_t^{(2)}} = 0, \quad i = 1, 2, \quad t \in [\delta, T]. \quad (2.19)$$

In fact, when demand is controlled, as in the next section, we actually need a more advanced maximum principle, requiring more technical assumptions. The interested reader is referred to [59]. We will relax this assumption in the section “Exogenous demand”.

Cournot equilibrium

In particular, when $f^{(i)}$ is given by

$$f^{(i)}(d_t, q_t^{(1)}, q_t^{(2)}) = P(q_t^{(1)} + q_t^{(2)}) \left(q_t^{(i)} \mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq d_t\}} + \frac{q_t^{(i)} d_t^+}{q_t^{(1)} + q_t^{(2)}} \mathbf{1}_{\{d_t < q_t^{(1)} + q_t^{(2)}\}} \right) - c^{(i)} q_t^{(i)},$$

$i = 1, 2$, by (2.17) we find that (2.19) reads

$$\begin{aligned} & \mathbb{E} \left[\frac{\partial H_t^{(i)}}{\partial r^{(i)}}(D_t, r^{(i)}, q_t^{(j)}, Y_t^{(i)}, Z_t^{(i)}) \middle| \mathcal{E}_t \right]_{r^{(i)}=q_t^{(i)}} \\ &= \mathbb{E} \left[\frac{\partial f^{(i)}}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) + y^{(i)} \frac{\partial \mu_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) + z^{(i)} \frac{\partial \sigma_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} \\ &= \mathbb{E} \left[P'(r^{(1)} + r^{(2)}) \left(r^{(i)} \mathbf{1}_{\{D_t \geq r^{(1)} + r^{(2)}\}} + \frac{r^{(i)}}{r^{(1)} + r^{(2)}} D_t^+ \mathbf{1}_{\{D_t < r^{(1)} + r^{(2)}\}} \right) \right. \\ & \quad \left. + P(r^{(1)} + r^{(2)}) \left(\mathbf{1}_{\{D_t \geq r^{(1)} + r^{(2)}\}} + \frac{r^{(j)}}{(r^{(1)} + r^{(2)})^2} D_t^+ \mathbf{1}_{\{D_t < r^{(1)} + r^{(2)}\}} \right) \right. \\ & \quad \left. + y^{(i)} \frac{\partial \mu_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) + z^{(i)} \frac{\partial \sigma_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} - c^{(i)} \\ &= 0, \quad 1 \leq i \neq j \leq 2. \end{aligned}$$

Hence from the maximum principle we must have

$$\begin{aligned}
& \left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(i)} P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E} \left[\mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq D_t\}} \middle| \mathcal{E}_t \right] \\
& + \left(q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(i)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right] \\
& + \mathbb{E} \left[y^{(i)} \frac{\partial \mu_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) + z^{(i)} \frac{\partial \sigma_t}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} = c^{(i)},
\end{aligned} \tag{2.20}$$

$$1 \leq i \neq j \leq 2.$$

Exogenous demand

In order to reach a more explicit solution to (2.20), in the remainder of this section we consider the case where demand is modeled exogenously as

$$\begin{cases} dD_t = \mu_t(D_t)dt + \sigma_t(D_t)dB_t, \\ D_0 = d_0 \in \mathbb{R}, \end{cases} \tag{2.21}$$

where the coefficients $\mu_t(d)$ and $\sigma_t(d)$ do not depend on the control processes $(q^{(1)}, q^{(2)})$. In this way we obtain the next proposition, which is the counterpart of Proposition 2.2.1 in the delayed setting. In the following we focus on nonnegative equilibrium outputs, $(q_t^{(1)}, q_t^{(2)})$ for all $t \in [\delta, T]$, which are economically relevant, although $\mathcal{A}^{(i)}$ contains functions that may take negative values.

Proposition 2.3.2. *The Cournot equilibrium action is given at any $t \in [\delta, T]$ by*

$$q_t^{(i)*} = S_t^* \frac{\mathbb{E} [P(S_t^*) \min(D_t, S_t^*) - c^{(i)} S_t^* | \mathcal{E}_t]}{\mathbb{E} [2P(S_t^*) \min(D_t, S_t^*) - (c^{(1)} + c^{(2)}) S_t^* | \mathcal{E}_t]}, \quad i = 1, 2, \tag{2.22}$$

where $S_t^* = q_t^{(1)*} + q_t^{(2)*}$ solves the equation

$$\begin{aligned}
& S_t^* P'(S_t^*) \mathbb{E} [\min(D_t, S_t^*) | \mathcal{E}_t] + 2P(S_t^*) \mathbb{E} [\min(D_t, S_t^*) | \mathcal{E}_t] \\
& = (c^{(1)} + c^{(2)}) S_t^* + P(S_t^*) \mathbb{E} [D_t \mathbf{1}_{\{D_t \leq S_t^*\}} | \mathcal{E}_t].
\end{aligned}$$

Proof. We apply the maximum principle Theorem 2.3.1 to the functional $J^{(i)}(q^{(1)}, q^{(2)})$

given in (2.14). Namely, taking $f^{(i)}$ in (2.14) given by

$$f^{(i)}(d_t, q_t^{(1)}, q_t^{(2)}) = P(q_t^{(1)} + q_t^{(2)}) \left(q_t^{(i)} \mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq d_t\}} + \frac{q_t^{(i)} d_t^+}{q_t^{(1)} + q_t^{(2)}} \mathbf{1}_{\{d_t < q_t^{(1)} + q_t^{(2)}\}} \right) - c^{(i)} q_t^{(i)},$$

$i = 1, 2$, by (2.17) we find that (2.20) reads

$$\begin{aligned} & \mathbb{E} \left[\frac{\partial H_t^{(i)}}{\partial r^{(i)}}(D_t, r^{(i)}, q_t^{(j)}, Y_t^{(i)}, Z_t^{(i)}) \middle| \mathcal{E}_t \right]_{r^{(i)}=q_t^{(i)}} \\ &= \mathbb{E} \left[\frac{\partial f^{(i)}}{\partial r^{(i)}}(D_t, r^{(i)}, r^{(j)}) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} \\ &= \mathbb{E} \left[P'(r^{(1)} + r^{(2)}) \left(r^{(i)} \mathbf{1}_{\{D_t \geq r^{(1)} + r^{(2)}\}} + \frac{r^{(i)}}{r^{(1)} + r^{(2)}} D_t^+ \mathbf{1}_{\{D_t < r^{(1)} + r^{(2)}\}} \right) \right. \\ & \quad \left. + P(r^{(1)} + r^{(2)}) \left(\mathbf{1}_{\{D_t \geq r^{(1)} + r^{(2)}\}} + \frac{r^{(j)}}{(r^{(1)} + r^{(2)})^2} D_t^+ \mathbf{1}_{\{D_t < r^{(1)} + r^{(2)}\}} \right) \right]_{r^{(1)}=q_t^{(1)}, r^{(2)}=q_t^{(2)}} \\ & \quad - c^{(i)} \\ &= 0, \quad 1 \leq i \neq j \leq 2. \end{aligned}$$

Hence from the maximum principle we must have

$$\begin{aligned} & \left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(i)} P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E} \left[\mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq D_t\}} \middle| \mathcal{E}_t \right] \\ & + \left(q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(i)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right] = c^{(i)}, \end{aligned} \quad (2.23)$$

$1 \leq i \neq j \leq 2$. By summing the above two equations we find that the sum $S_t^* := q_t^{(1)*} + q_t^{(2)*}$ satisfies the equation

$$\begin{aligned} & \left(2P(q_t^{(1)} + q_t^{(2)}) + (q_t^{(1)} + q_t^{(2)}) P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E} \left[\mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq D_t\}} \middle| \mathcal{E}_t \right] \\ & + \left(\frac{P(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} + P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right] = c^{(1)} + c^{(2)}. \end{aligned} \quad (2.24)$$

or equivalently,

$$\begin{aligned} & 2P(S_t^*) \mathbb{E} [\min(D_t, S_t^*) | \mathcal{E}_t] - (c^{(1)} + c^{(2)}) S_t^* \\ &= P(S_t^*) \mathbb{E} [D_t \mathbf{1}_{\{D_t \leq S_t^*\}} | \mathcal{E}_t] - S_t^* P'(S_t^*) \mathbb{E} [\min(D_t, S_t^*) | \mathcal{E}_t]. \end{aligned} \quad (2.25)$$

By (2.23) and (2.24) we get

$$\begin{aligned}
& \left(-P(q_t^{(1)} + q_t^{(2)}) - q_t^{(j)} P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E} \left[\mathbf{1}_{\{q_t^{(1)} + q_t^{(2)} \leq D_t\}} \middle| \mathcal{E}_t \right] \\
& + \left(q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} - \frac{P(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} - q_t^{(j)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right] \\
& = -c^{(j)},
\end{aligned} \tag{2.26}$$

hence

$$\begin{aligned}
& q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right] - q_t^{(j)} P'(q_t^{(1)} + q_t^{(2)}) \mathbb{E} \left[\min \left(1, \frac{D_t}{q_t^{(1)} + q_t^{(2)}} \right) \middle| \mathcal{E}_t \right] \\
& = -c^{(j)} + P(q_t^{(1)} + q_t^{(2)}) \mathbb{E} \left[\min \left(1, \frac{D_t}{q_t^{(1)} + q_t^{(2)}} \right) \middle| \mathcal{E}_t \right],
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
q_t^{(j)} = & \frac{c^{(j)} - P(q_t^{(1)} + q_t^{(2)}) \mathbb{E} \left[\min \left(1, \frac{D_t}{q_t^{(1)} + q_t^{(2)}} \right) \middle| \mathcal{E}_t \right]}{P'(q_t^{(1)} + q_t^{(2)}) \mathbb{E} \left[\min \left(1, \frac{D_t}{q_t^{(1)} + q_t^{(2)}} \right) \middle| \mathcal{E}_t \right] - \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} \mathbb{E} \left[D_t \mathbf{1}_{\{D_t < q_t^{(1)} + q_t^{(2)}\}} \middle| \mathcal{E}_t \right]},
\end{aligned} \tag{2.28}$$

$1 \leq i \neq j \leq 2$. I.e. using (2.25),

$$q_t^{(i)*} = S_t^* \frac{\mathbb{E} [P(S_t^*) \min(D_t, S_t^*) - c^{(i)} S_t^* | \mathcal{E}_t]}{\mathbb{E} [2P(S_t^*) \min(D_t, S_t^*) - (c^{(1)} + c^{(2)}) S_t^* | \mathcal{E}_t]}, \tag{2.29}$$

$i = 1, 2$. □

Ornstein-Uhlenbeck setting

In order to derive explicit solutions we henceforth assume that $\mu_t(D_t, q_t) = \alpha(\beta - D_t)$ where $\alpha, \beta \in \mathbb{R}$, and $\sigma_t(D_t, q_t) = \sigma(t)$, where σ is a deterministic function of $t \in \mathbb{R}$.

Therefore the dynamics of D_t is given by

$$\begin{cases} dD_t = \alpha(\beta - D_t)dt + \sigma(t)dB_t \\ D_0 = d_0 \in \mathbb{R}, \end{cases}$$

whose solution $(D_t)_{t \in [\delta, T]}$ satisfies

$$D_t = D_{t-\delta}e^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + \int_{t-\delta}^t e^{\alpha(s-t)}\sigma(s)dB_s, \quad t \in [\delta, T]. \quad (2.30)$$

In the sequel we denote by $\Phi : \mathbb{R} \rightarrow [0, 1]$ the standard Gaussian cumulative distribution function.

Proposition 2.3.3. *In the Ornstein-Uhlenbeck setting, the Cournot equilibrium action at any $t \in [\delta, T]$ is given by*

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) + (c^{(j)} - c^{(i)})P(S_t^*) - c^{(j)}(P(S_t^*) - S_t^*P'(S_t^*))}{2P^2(S_t^*)\Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) - (P(S_t^*) - S_t^*P'(S_t^*))(c^{(1)} + c^{(2)})},$$

$1 \leq i \neq j \leq 2$, where $S_t^* = q_t^{(1)*} + q_t^{(2)*}$ solves the equation

$$\begin{aligned} & (P(S_t^*) + S_t^*P'(S_t^*)) \left(\varphi(D_{t-\delta})\Phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) - \varepsilon(t)\phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) \right) \\ & + S_t^* (2P(S_t^*) + S_t^*P'(S_t^*)) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) = (c^{(1)} + c^{(2)})S_t^*, \quad 1 \leq i \neq j \leq 2, \end{aligned}$$

with

$$\varphi(d) := de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}),$$

and

$$\varepsilon^2(t) := \int_{t-\delta}^t e^{2\alpha(s-t)}\sigma^2(s)ds, \quad t \in [\delta, T].$$

In contrast with Proposition 2.2.1, this result covers the case where production $q_t^{(1)*}$ and $q_t^{(2)*}$ meets the demand d_t , as the derivatives of the profit functions $f^{(i)}(d_t, q_t^{(1)}, q_t^{(2)})$, $i = 1, 2$, are smoothened by the introduction of a conditional expectation due to the stochastic delay.

The proof of Proposition 2.3.3 is deferred to the appendix.

Affine prices

In the sequel we consider the affine price model $P(q) = a - bq$, which satisfies the equation $P(q) - qP'(q) = a$. If $c^{(i)} \geq a$ then $P(q^{(i)}) - c^{(i)} = a - bq^{(i)} - c^{(i)} \leq 0$, in which case we have $q^{(i)*} = 0$, henceforth we focus on the case where $0 < c^{(i)} < a$, $i = 1, 2$. Proposition 2.3.3 yields the following corollary. When δ tends to zero, the equation (2.31) below converges to (2.4) in Proposition 2.2.1, and it also recovers the result of Proposition 2.2.2, since $\lim_{\varepsilon \rightarrow 0} \Phi(x/\varepsilon) = \mathbf{1}_{[0, \infty)}(x)$, $x \in \mathbb{R}$.

Corollary 2.3.4. *In the affine price model the Cournot equilibrium action at any $t \in [\delta, T]$ is given by*

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) + (c^{(j)} - c^{(i)})P(S_t^*) - ac^{(j)}}{2P^2(S_t^*) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) - a(c^{(1)} + c^{(2)})}, \quad (2.31)$$

$1 \leq i \neq j \leq 2$, where $S_t^* = q_t^{(1)*} + q_t^{(2)*}$ solves the equation

$$(a - 2bS_t^*) \left(\varphi(D_{t-\delta}) \Phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) - \varepsilon(t) \phi\left(\frac{S_t^* - \varphi(D_{t-\delta})}{\varepsilon(t)}\right) \right) + S_t^* (2a - 3bS_t^*) \Phi\left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)}\right) = (c^{(1)} + c^{(2)})S_t^*. \quad (2.32)$$

Figure 2.8 presents a numerical solution of (2.32) showing the values of $(q_t^{(1)}, q_t^{(2)})$ as functions of the demand d_t .

Figure 2.8: Graphs of $d_t \mapsto q^1(d), q^2(d)$.

As the delay $\delta > 0$ tends to zero, the lagged equilibrium converge to a limiting

equilibrium which belongs to the range of equilibria of Proposition 2.2.2, as illustrated in the following Figures 2.9-2.11.

Figure 2.9: Equilibrium output $q_t^{(1)*}(d_t)$ as a function of d_t .

Figure 2.10: Equilibrium output $q_t^{(2)*}(d_t)$ as a function of d_t .

Figure 2.11: Total output $d_t \mapsto q_t^{(1)*}(d_t) + q_t^{(2)*}(d_t)$.

2.4 Appendix: proof of propositions

Proof of Proposition 2.2.2

We have

$$\begin{aligned} & \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) \\ &= (a - 2bq^{(i)} - bq^{(j)}) \mathbf{1}_{\{q^{(1)}+q^{(2)} < d\}} + \left(\frac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d \mathbf{1}_{\{d < q^{(1)}+q^{(2)}\}} - c^{(i)}, \end{aligned} \quad (2.33)$$

$1 \leq i \neq j \leq 2$.

a) If $q^{(1)} + q^{(2)} < d$, (2.33) reads

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)} \quad 1 \leq i \neq j \leq 2,$$

We distinguish two cases:

1. $q^{(i)*}, q^{(j)*} > 0$. In this case we must have

$$\begin{cases} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)} = 0 \\ \frac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(j)} - bq^{(i)} - c^{(j)} = 0, \end{cases}$$

hence

$$q^{(i)*} = \frac{a - 2c^{(i)} + c^{(j)}}{3b} \quad \text{and} \quad q^{(j)*} = \frac{a + c^{(i)} - 2c^{(j)}}{3b},$$

and

$$q^{(1)*} + q^{(2)*} = \frac{2a - c^{(1)} - c^{(2)}}{3b} < d.$$

2. $q^{(i)*} < d$ and $q^{(j)*} = 0$. In this case we may have an inequality

$$\begin{cases} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(i)} - bq^{(j)} - c^{(i)} = 0 \\ \frac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = a - 2bq^{(j)} - bq^{(i)} - c^{(j)} \leq 0, \end{cases}$$

i.e.

$$\begin{cases} a - 2bq^{(i)} - c^{(i)} = 0 \\ a - bq^{(i)} - c^{(j)} \leq 0, \end{cases}$$

or $(a - c^{(j)})/b \leq q^{(i)*} = (a - c^{(i)})/(2b)$, which together with the condition $a + c^{(i)} - 2c^{(j)} \geq 0$, implies $a + c^{(i)} - 2c^{(j)} = 0$ and in this case, the equilibrium is already taken care of in point 1 above.

b) On the other hand, if $d < q^{(1)} + q^{(2)}$, (2.33) reads

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)} \quad 1 \leq i \neq j \leq 2,$$

There are two kinds of candidate equilibria:

1. $q^{(i)*}, q^{(j)*} > 0$ and we must have:

$$\begin{cases} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)} = 0 \\ \frac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(i)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(j)} = 0, \end{cases}$$

In this case,

$$\begin{cases} q^{(1)}a - b(q^{(1)} + q^{(2)})^2 = \frac{c^{(2)}}{d}(q^{(1)} + q^{(2)})^2, \\ q^{(2)}a - b(q^{(1)} + q^{(2)})^2 = \frac{c^{(1)}}{d}(q^{(1)} + q^{(2)})^2, \end{cases}$$

hence

$$q^{(1)*}(d) = \frac{ad(c^{(2)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2}, \quad \text{and} \quad q^{(2)*}(d) = \frac{ad(c^{(1)} + bd)}{(c^{(1)} + c^{(2)} + 2bd)^2},$$

and

$$d < q^{(1)*}(d) + q^{(2)*}(d) = \frac{ad}{c^{(1)} + c^{(2)} + 2bd}.$$

2. $q^{(i)*} > d$ and $q^{(j)*} = 0$ and instead we may have an inequality:

$$\begin{cases} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(j)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(i)} = 0 \\ \frac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \left(\frac{aq^{(i)}}{(q^{(1)} + q^{(2)})^2} - b \right) d - c^{(j)} \leq 0, \end{cases}$$

And therefore,

$$\begin{cases} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) = -bd - c^{(i)} = 0 \\ \frac{\partial f^{(j)}}{\partial q^{(j)}}(d, q^{(1)}, q^{(2)}) = \frac{ad}{q^{(i)}} - bd - c^{(j)} \leq 0, \end{cases}$$

which is impossible.

c) Finally we consider what happens when $\underline{d} \leq d \leq \bar{d}$, i.e. d is between the above boundaries. The sum of derivatives

$$\begin{aligned} \frac{\partial f^{(1)}}{\partial q^{(1)}}(d, q^{(1)}, q^{(2)}) + \frac{\partial f^{(2)}}{\partial q^{(2)}}(d, q^{(1)}, q^{(2)}) &= (a - 3b(q^{(1)} + q^{(2)})) \mathbf{1}_{\{q^{(1)} + q^{(2)} < d\}} + \\ &\quad \left(\frac{a}{q^{(1)} + q^{(2)}} - b \right) d \mathbf{1}_{\{d < q^{(1)} + q^{(2)}\}} - c^{(1)} - c^{(2)} \end{aligned} \quad (2.34)$$

is a decreasing function of $q^{(1)} + q^{(2)}$ which vanishes at $q^{(1)} + q^{(2)} = \bar{d}$, therefore it is strictly positive when $q^{(1)} + q^{(2)} < d \leq \bar{d}$, hence at least one of the individual derivatives must be strictly positive at such points. Therefore, $q^{(1)} + q^{(2)} < d$ cannot be an equilibrium.

In the case $q^{(1)} + q^{(2)} > d$ the summation of derivatives vanishes at $q^{(1)} + q^{(2)} = \underline{d}$ hence it is strictly negative when $q^{(1)} + q^{(2)} > d \geq \underline{d}$ and at least one of the derivatives must be negative. Therefore $q^{(1)} + q^{(2)} > d$ cannot be an equilibrium either.

Consider a point $(q^{(1)*}, q^{(2)*}) \in [0, d] \times [0, d]$ on the line $q^{(1)} + q^{(2)} = d$. This is an equilibrium if there exists no profitable deviations. We need to show that the value

function

$$\begin{aligned} f^{(i)}(d, q^{(1)}, q^{(2)}) \\ = (a - b(q^{(i)} + q^{(j)*})) \left(q^{(i)} \mathbf{1}_{\{q^{(i)} + q^{(j)*} \leq d\}} + \frac{q^{(i)} d}{q^{(i)} + q^{(j)*}} \mathbf{1}_{\{d < q^{(i)} + q^{(j)*}\}} \right) - c^{(i)} q^{(i)}, \end{aligned}$$

$1 \leq i \neq j \leq 2$, is increasing in $q^{(i)} \in [0, q^{(i)*}]$ and decreasing in $q^{(i)} \in (q^{(i)*}, \infty)$. Given that the partial derivative

$$\begin{aligned} \frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(1)}, q^{(2)}) \\ = (a - 2bq^{(i)} - bq^{(j)*}) \mathbf{1}_{\{q^{(i)} + q^{(j)*} < d\}} + \left(\frac{aq^{(j)*}}{(q^{(i)} + q^{(j)*})^2} - b \right) d \mathbf{1}_{\{d < q^{(i)} + q^{(j)*}\}} - c^{(i)} \end{aligned}$$

is a decreasing function of $q^{(i)}$, it is necessary and sufficient to have the left derivative

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(i)*-}, q^{(j)*}) = a - bq^{(i)*} - bd - c^{(i)} = a + bq^{(j)*} - 2bd - c^{(i)} \geq 0, \quad (2.35)$$

and the right derivative satisfies

$$\frac{\partial f^{(i)}}{\partial q^{(i)}}(d, q^{(i)*+}, q^{(j)*}) = \frac{aq^{(j)*}}{d} - bd - c^{(i)} \leq 0. \quad (2.36)$$

Therefore $(q^{(1)*}, q^{(2)*}) \in (0, d) \times (0, d)$ on the line $q^{(1)} + q^{(2)} = d$ is an equilibrium if and only if

$$\left(\frac{-a + c^{(i)} + 2bd}{b} \right)^+ \leq q^{(j)*} \leq \frac{(c^{(i)} + bd)d}{a}, \quad 1 \leq i \neq j \leq 2, \quad (2.37)$$

or equivalently

$$\left(\frac{-a + c^{(j)} + 2bd}{b} \right)^+ \leq d - q^{(j)*} \leq \frac{(c^{(j)} + bd)d}{a}, \quad j = 1, 2, \quad (2.38)$$

which yields (2.8). Notice that from (2.8), if there exists $q^{(i)*}, q^{(j)*}$ that satisfy the inequalities, then $\underline{d} \leq d \leq \bar{d}$, or in other words, if d is outside these bounds, we do not have equilibria satisfying $q^{(i)*} + q^{(j)*} = d$. Finally, consider $(q^{(1)*}, q^{(2)*})$ on the line

$q^{(1)} + q^{(2)} = d$ with $q^{(i)*} = d$ and $q^{(j)*} = 0$. In this case, the value function for i

$$f^{(i)}(d, q^{(1)}, q^{(2)}) = (a - bq^{(i)}) (q^{(i)} \mathbf{1}_{\{q^{(i)} \leq d\}} + d \mathbf{1}_{\{d < q^{(i)}\}}) - c^{(i)} q^{(i)},$$

is clearly decreasing in $q^{(i)} \in (d, \infty)$, and it suffices to show that it is increasing in $q^{(i)} \in [0, d]$, and from (2.35) for this we need

$$d \leq \frac{a - c^{(i)}}{2b}.$$

On the other hand, we check that the right derivative for j , (2.36) is negative if

$$\frac{a - c^{(j)}}{b} \leq d,$$

hence we need

$$\frac{a - c^{(j)}}{b} \leq d \leq \frac{a - c^{(i)}}{2b}.$$

However, $(q^{(i)*}, q^{(j)*}) = (d, 0)$ cannot be an equilibrium unless

$$d = \frac{a - c^{(j)}}{b} = \frac{a - c^{(i)}}{2b},$$

as we assume that $a + c^{(i)} - 2c^{(j)} \geq 0$. In the latter case we have $a + c^{(i)} - 2c^{(j)} = 0$ and the inequalities (2.8) read

$$d = \frac{-a + c^{(j)} + 2bd}{b} \leq q^{(i)*} \leq d, \text{ i.e. } q^{(i)*} = d, \quad \text{and} \quad 0 \leq q^{(j)*} \leq 0, \text{ i.e. } q^{(j)*} = 0,$$

which is compatible with the equilibrium $(q^{(i)*}, q^{(j)*}) = (d, 0)$. Notice that $(d, 0)$ is an equilibrium if and only if d is i 's monopoly quantity.

Proof of Proposition 2.3.3

From (2.20) we have

$$\begin{aligned} & \left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(i)} P'(q_t^{(1)} + q_t^{(2)}) \right) \mathbb{E}[\mathbf{1}_{\{q \leq D_t\}}]_{q=q_t^{(1)}+q_t^{(2)}} \\ & + \left(q_t^{(j)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(i)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \mathbb{E}[D_t \mathbf{1}_{\{q > D_t\}}]_{q=q_t^{(1)}+q_t^{(2)}} = c^{(i)}, \end{aligned} \quad (2.39)$$

$1 \leq i \neq j \leq 2$. From (2.30) we write $D_t = d_{t-\delta}e^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X$ where $X \simeq \mathcal{N}(0, \varepsilon^2(t))$ and

$$\varepsilon^2(t) = \mathbb{E} \left[\left(\int_{t-\delta}^t e^{\alpha(s-t)} \sigma(s) dB_s \right)^2 \right] = \int_{t-\delta}^t e^{2\alpha(s-t)} \sigma^2(s) ds = \sigma^2 \frac{1 - e^{-2\alpha\delta}}{2\alpha},$$

we have

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{q \leq D_t\}}]_{q=q_t^{(1)}+q_t^{(2)}} &= \mathbb{P}(q \leq de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)_{d=d_{t-\delta}, q=q_t^{(1)}+q_t^{(2)}} \\ &= \int_{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})}^{\infty} e^{-x^2/(2\varepsilon^2(t))} \frac{dx}{\sqrt{2\pi\varepsilon^2(t)}} \\ &= \Phi \left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right). \end{aligned}$$

For the second expectation we have

$$\begin{aligned} \mathbb{E}[D_t \mathbf{1}_{\{q > D_t\}}]_{q=q_t^{(1)}+q_t^{(2)}} &= \mathbb{E}[(de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X) \mathbf{1}_{\{q > de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X\}}]_{d=d_{t-\delta}, q=q_t^{(1)}+q_t^{(2)}} \\ &= \varphi(D_{t-\delta}) \mathbb{E}[\mathbf{1}_{\{q > \varphi(D_{t-\delta}) + X\}}]_{q=q_t^{(1)}+q_t^{(2)}} + \mathbb{E}[X \mathbf{1}_{\{q > \varphi(D_{t-\delta}) + X\}}]_{q=q_t^{(1)}+q_t^{(2)}} \\ &= \varphi(D_{t-\delta}) \mathbb{P}(X < q - \varphi(D_{t-\delta}))_{q=q_t^{(1)}+q_t^{(2)}} + \int_{-\infty}^{q_t^{(1)}+q_t^{(2)}-\varphi(D_{t-\delta})} x e^{-x^2/(2\varepsilon^2(t))} \frac{dx}{\sqrt{2\pi\varepsilon^2(t)}} \\ &= \varphi(D_{t-\delta}) \Phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) - \varepsilon(t) \phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right). \end{aligned}$$

Hence (2.39) reads

$$\begin{aligned} &\left(q_t^{(i)} \frac{P(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)})^2} + q_t^{(j)} \frac{P'(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} \right) \\ &\times \left(\varepsilon(t) \phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) - \varphi(D_{t-\delta}) \Phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) \right) \\ &- \left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(j)} P'(q_t^{(1)} + q_t^{(2)}) \right) \Phi \left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right) \\ &= -c^{(j)}, \quad 1 \leq i \neq j \leq 2. \end{aligned} \tag{2.40}$$

By summing the above two equations we find that the sum $S_t^* := q_t^{(1)} + q_t^{(2)}$ satisfies

the equation

$$\begin{aligned}
& \left(\frac{P(q_t^{(1)} + q_t^{(2)})}{q_t^{(1)} + q_t^{(2)}} + P'(q_t^{(1)} + q_t^{(2)}) \right) \\
& \times \left(\varepsilon(t) \phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) - \varphi(D_{t-\delta}) \Phi \left(\frac{q_t^{(1)} + q_t^{(2)} - \varphi(D_{t-\delta})}{\varepsilon(t)} \right) \right) \quad (2.41) \\
& - \left(2P(q_t^{(1)} + q_t^{(2)}) + (q_t^{(1)} + q_t^{(2)})P'(q_t^{(1)} + q_t^{(2)}) \right) \Phi \left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right) \\
& = -c^{(1)} - c^{(2)}, \quad 1 \leq i \neq j \leq 2.
\end{aligned}$$

Combining (2.40) with (2.41), we get

$$\begin{aligned}
& \left(\frac{q_t^{(i)} P(q_t^{(1)} + q_t^{(2)}) + q_t^{(j)} (q_t^{(1)} + q_t^{(2)}) P'(q_t^{(1)} + q_t^{(2)})}{(q_t^{(1)} + q_t^{(2)}) P(q_t^{(1)} + q_t^{(2)}) + (q_t^{(1)} + q_t^{(2)})^2 P'(q_t^{(1)} + q_t^{(2)})} \right) \quad (2.42) \\
& \times \left(c^{(1)} + c^{(2)} - \left(2P(q_t^{(1)} + q_t^{(2)}) + (q_t^{(1)} + q_t^{(2)})P'(q_t^{(1)} + q_t^{(2)}) \right) \Phi \left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right) \right) \\
& + \left(P(q_t^{(1)} + q_t^{(2)}) + q_t^{(j)} P'(q_t^{(1)} + q_t^{(2)}) \right) \Phi \left(\frac{\varphi(D_{t-\delta}) - q_t^{(1)} - q_t^{(2)}}{\varepsilon(t)} \right) \\
& = c^{(j)}, \quad 1 \leq i \neq j \leq 2,
\end{aligned}$$

or

$$\begin{aligned}
& -q_t^{(j)} (P(S_t^*) - S_t^* P'(S_t^*)) \left(c^{(1)} + c^{(2)} - (2P(S_t^*) + S_t^* P'(S_t^*)) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) \right) \\
& + q_t^{(j)} P'(S_t^*) (S_t^* P(S_t^*) + S_t^{*2} P'(S_t^*)) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) \\
& = c^{(j)} (S_t^* P(S_t^*) + S_t^{*2} P'(S_t^*)) - P(S_t^*) (S_t^* P(S_t^*) + S_t^{*2} P'(S_t^*)) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) \\
& - S_t^* P(S_t^*) \left(c^{(1)} + c^{(2)} - (2P(S_t^*) + S_t^* P'(S_t^*)) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) \right), \quad 1 \leq i \neq j \leq 2,
\end{aligned}$$

which shows that

$$q_t^{(j)*} = S_t^* \frac{P^2(S_t^*) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) + (c^{(j)} - c^{(i)}) P(S_t^*) - c^{(j)} (P(S_t^*) - S_t^* P'(S_t^*))}{2P^2(S_t^*) \Phi \left(\frac{\varphi(D_{t-\delta}) - S_t^*}{\varepsilon(t)} \right) - (P(S_t^*) - S_t^* P'(S_t^*)) (c^{(1)} + c^{(2)})}.$$

In the limit when d_t is large, (2.42) yields the system of equations

$$\begin{cases} 2bq_t^{(1)} + bq_t^{(2)} + c^{(1)} = a \\ bq_t^{(1)} + 2bq_t^{(2)} + c^{(2)} = a, \end{cases}$$

with solution

$$\begin{cases} q_t^{(1)} = \frac{a - 2c^{(1)} + c^{(2)}}{3b} \\ q_t^{(2)} = \frac{a + c^{(1)} - 2c^{(2)}}{3b}, \end{cases}$$

and sum

$$q_t^{(1)} + q_t^{(2)} = \frac{2a - c^{(1)} - c^{(2)}}{3b}, \quad t \in [\delta, T],$$

which recover the no lag setting of Proposition 2.2.2.

2.5 Conclusion

We have shown that the inclusion of limited demand in our model results in the existence of three different regimes, depending on the level of the demand, including an intermediate regime that shows a range of possible equilibria.

When demand is low, the market is inefficient as there will be overproduction, whereas when demand is high, there is unfulfilled demand since the usual Cournot equilibrium will materialise. When demand is at an intermediate level, it is exactly met by the combined production of the firms when there is no lag, but output falls short of demand in case of lag.

We have also shown that the introduction of a delayed stochastic demand reduces the range of equilibria to a single equilibrium in the intermediate regime. In addition, equilibria in other regimes converge to the equilibrium obtained in the non delayed setting. In this sense, the lagged case can be seen as a smoothing of the no lag case.

In order to treat the case of n -player oligopolies, we would replace (2.1) with

$$f^{(i)}(d, q^{(1)}, \dots, q^{(n)}) \\ = P(q^{(1)} + \dots + q^{(n)}) \left(q^{(i)} \mathbf{1}_{\{q^{(1)} + \dots + q^{(n)} \leq d\}} + \frac{q^{(i)} d}{q^{(1)} + \dots + q^{(n)}} \mathbf{1}_{\{d < q^{(1)} + \dots + q^{(n)}\}} \right) - c^{(i)} q^{(i)}.$$

In this case, the optimal $q^{(i)*}$ depends on the sum $\sum_{\substack{j=1 \\ j \neq i}}^n q^{(j)*}$ and this will result into more first order conditions in the proof of Propositions 2.2.1 and 2.3.2. This requires different solution techniques and will be considered for future research.

Chapter 3

A stochastic newsvendor game with dynamic retail prices

This chapter is the result of research carried out together with Nicolas Privault.

3.1 Introduction

The newsvendor problem is a two-staged game in which a wholesaler sells products to a retailer, and the retailer sells to consumers in a market according to a demand constraint. The retailer's optimal quantity is a function of the wholesaler's quoted per-unit price. Such problems have yielded a rich literature, see [72] for an overview of known results and possible extensions, such as demand that is a function of retail prices or quantities and variable wholesale prices. In [65] an extended model in which both retail price and quantity are under the control of the retailer and demand depends on the retail price, is presented.

Stochastic differential Stackelberg games have been studied via the maximum principle, and applied to continuous time newsvendor problems with exogenously fixed retail prices in [58], where demand is modeled by an Itô-Lévy process, and information about the demand is lagged. In this framework, the retailer can only sell a quantity limited to a level $D_t \geq 0$ of products at time $t \in [\delta, T]$, with $\delta > 0$. In this setup, information about demand is lagged, that is to say, $\delta > 0$ represents the magnitude of the information lag in the following way: at time $t - \delta$ firms have to decide on their

strategies for time t , unknowing of D_t i.e. demand at time t . This approach relies on the vanishing of derivatives of Hamiltonians at any time t in order to determine the equilibrium values of the controls.

This chapter extends the framework of [58], in which retail prices were exogenously fixed, to newsvendor games in which the maximum price that can be charged in the market when selling quantity q_t at time t , $0 \leq q_t \leq D_t$, is given as a function $P(q_t)$. In our framework, selling more than demanded is costly to the retailer, not only because he pays for buying an excess supply, but also because it forces down the price that he can charge to sell off d units. Although we present our model in its most general form, we will explicitly treat the case where the market price $P(q_t)$ is of the familiar affine form $P(q_t) = a - bq_t$ for $a, b > 0$, with numerical illustrations. We also treat the limiting case $\delta = 0$ in which everything about the games is deterministic as there is no uncertainty about the demand d_t . We rely on optimal stochastic control of diffusions, cf. [7] and [59], where necessary and sufficient maximum principles are derived for both the zero sum as well as the nonzero sum cases using forward-backward stochastic differential equations, with applications to newsvendor models.

In Theorem 3.3.1 we present necessary and sufficient conditions for optimality of controls based on the stochastic maximum principle in the delayed case with $\delta > 0$, by allowing retail prices to depend on the quantity of products sold between wholesaler and retailer. When demand D_t is modeled by an Ornstein-Uhlenbeck process we characterise the optimal control of Player 2 (the retailer) at time t in Proposition 3.3.2, and we also find a characterisation of the optimal control for Player 1 (the wholesaler) at time t .

When the market price is an affine function of the output we derive an equation for the equilibrium output based on the conditional distribution function of the demand D_t given the delayed information, which parallels the equation (3) in [72], see (3.29) below. In this setting we compute numerically the equilibrium output as a function of wholesale prices, cf. Figures 3.1-3.3. We note that the retail output remains capped at the level $a/(4b)$ as a function of demand, whereas it remains strictly increasing in the case of fixed prices, cf. Figure 3.1 below and Figure 3 in [58].

In the no lag case with $\delta = 0$, we are able to obtain closed form solutions based on the parameters of the economic environment in the affine case. We show in Proposition 3.4.1 that equilibrium output is always below demand, and both the wholesaler and retailer would prefer demand at least equal to a given threshold, after which profits remain constant. In general, if the demand is affected by only a short delay, decisions based on the delayed rule will be very close to the optimal decisions taken in the absence of delay. This is illustrated by the convergence of (3.30) to (3.39) below.

We also note from Figures 3.1-3.3 that the lagged case can be seen as a smoothing of the deterministic case, and that the solution of the deterministic no lag problem can be obtained as the limit of the lagged setup as δ tends to 0. However, unlike in the Cournot setup [69], we do not observe a collapse from multiple equilibria to a single equilibrium.

3.2 Notation and setup

We consider a sequential game in which Player 1 is selling a quantity $q_t \geq 0$ of products priced w_t to Player 2 who in turn sells to customers according to the demand rate D_t .

Let $t \in [\delta, T]$, where $\delta \geq 0$ is the information delay. In the first stage, Player 1 chooses a price w_t per unit to charge to Player 2, who in the second period orders the quantity $q_t \in \mathbb{R}_+$ and sells to customers according to a demand D_t . Precisely, Player 1 tries to maximise the wholesaler profit function

$$f^{(1)}(d_t, w_t, q_t) = w_t q_t \quad (3.1)$$

and Player 2 tries to maximise the retailer profit function

$$f^{(2)}(d_t, w_t, q_t) = P(q_t) \min(d_t^+, q_t) - w_t q_t, \quad (3.2)$$

where $P(q_t) \geq 0$ denotes the price function, depending on output, thereby finding the optimal value q_t^* based on the knowledge of d_t and w_t . This optimum is known to Player 1, who attempts in turn to maximize the quantity $w_t q_t^*$ using his control

variable w_t in the second stage. We assume $w_t \geq 0$, this is without loss of generality since $w_t < 0$ implies $q_t > 0$ and therefore $w_t q_t < 0$, which cannot be an equilibrium.

We assert there exists a map $\psi^* : w = (w_t)_{t \in [\delta, T]} \mapsto q = (q_t)_{t \in [\delta, T]}$ such that the function

$$q_t \mapsto f^{(2)}(d_t, w_t, q_t) \quad (3.3)$$

is maximal for $q_t = q_t^* = \psi_t^*(w_t)$. This is the case if the profit function (3.2) is strictly concave in q_t for given d_t, w_t .

Since Player 1 knows that Player 2 will act in this rational way, he will choose w_t such that the profit function

$$w_t \mapsto f^{(1)}(d_t, w_t, q_t) = w_t \psi_t^*(w) \quad (3.4)$$

is maximal for $w_t = w_t^*$. Given two performance functionals

$$J^{(1)}(w, \psi^*(w)) = \mathbb{E} \left[\int_{\delta}^T f^{(1)}(d_t, w_t, \psi_t^*(w)) dt \right]$$

and

$$J^{(2)}(w, q) = \mathbb{E} \left[\int_{\delta}^T f^{(2)}(D_t, w_t, q_t) dt \right]$$

that measure the (expected) profit over the time period $\delta \leq t \leq T$, an equilibrium of the game is defined as a pair (w^*, ψ^*) , where $w^* = (w_t^*)_{t \in [\delta, T]}$, such that

$$J^{(2)}(w^*, \psi(w^*)) \leq J^{(2)}(w^*, \psi^*(w^*)) \quad \text{for all } \psi : w \mapsto q \quad (3.5)$$

and

$$J^{(1)}(w, \psi^*(w)) \leq J^{(1)}(w^*, \psi^*(w^*)) \quad \text{for all } (w_t)_{t \in [\delta, T]} \geq 0. \quad (3.6)$$

3.3 Delayed stochastic demand

We consider a demand rate D_t at time $t \in [0, T]$, which is described by a controlled stochastic differential equation (SDE) of the form

$$\begin{cases} dD_t = \mu_t(D_t, w_t, q_t)dt + \sigma_t(D_t, w_t, q_t)dB_t, \\ D_0 = d_0 \in \mathbb{R}, \end{cases} \quad (3.7)$$

where $\mu.(d, q, w)$, $\sigma.(d, q, w)$ are given predictable processes for each $d, q, w \in \mathbb{R}$ and $(B_t)_{t \in [0, T]}$ is a standard Brownian motion that generates the $\mathbb{P}$ -augmented filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. We assume that (3.7) has a unique solution.

The information flow available to the players is represented by the filtration $(\mathcal{E}_t)_{t \in [0, T]}$ defined by

$$\mathcal{E}_t := \mathcal{F}_{t-\delta}, \quad t \in [\delta, T], \quad (3.8)$$

where $\delta > 0$ denotes the information delay or lag. In this lagged scenario, Player 2 maximizes the functional

$$J^{(2)}(w, q) = \mathbb{E} \left[\int_{\delta}^T (P(q_t) \min(D_t^+, q_t) - w_t q_t) dt \right],$$

where $P(q_t)$ is the market price at time $t \in [\delta, T]$ when output equals q_t , in order to find the optimal value $\psi_t^*(w)$ based on the knowledge of the demand d_t and the wholesale price w_t . This optimum is known to Player 1, who attempts in turn to maximize the functional

$$J^{(1)}(w, \psi^*(w)) = \mathbb{E} \left[\int_{\delta}^T w_t \psi_t^*(w) dt \right],$$

using Player 1's control variable w_t , namely the per unit price he charges to Player 2.

Maximum principle

We base our analysis on the stochastic control framework developed in [7], [59]. The control processes $q = (q_t)_{t \in [\delta, T]} = \psi(w)$ and $w = (w_t)_{t \in [\delta, T]}$ belong to the family

$\mathcal{A} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$, where the set $\mathcal{A}^{(i)}$ of admissible control processes for Player i is made of real-valued $(\mathcal{E}_t)_{t \in [0, T]}$ -predictable processes, $i = 1, 2$. We endow $\mathcal{A}^{(i)}$ with the supremum norm, $i = 1, 2$. We note that $\mathcal{A}^{(i)}$ contains functions that may take negative values, however in the sequel we will only focus on nonnegative equilibrium outputs, which are economically relevant.

In the sequel we work under the following conditions on the set $\mathcal{A}$ of control processes.

(A1) For all $u^{(i)} \in \mathcal{A}^{(i)}$ and all bounded $\beta^{(i)} \in \mathcal{A}^{(i)}$ there exists $\varepsilon > 0$ such that

$$u^{(i)} + s\beta^{(i)} \in \mathcal{A}^{(i)} \text{ for all } s \in (-\varepsilon, \varepsilon); \quad i = 1, 2.$$

(A2) For all $t_0 \in [\delta, T]$ and all bounded $\mathcal{E}_{t_0}$ -measurable random variables $\alpha^{(i)}$, the control process $\beta_t^{(i)}$ defined by

$$\beta_t^{(i)} = \mathbf{1}_{[t_0, T]}(t) \alpha^{(i)}(\omega), \quad t \in [\delta, T],$$

belongs to $\mathcal{A}^{(i)}$; $i = 1, 2$.

The profit functions $J^{(1)}, J^{(2)} : \mathcal{A} \rightarrow \mathbb{R}$, of Players 1 and 2 are defined by

$$J^{(1)}(w, q) := \mathbb{E} \left[\int_{\delta}^T f^{(1)}(D_t, w_t, q_t) dt \right] \text{ and } J^{(2)}(w, q) := \mathbb{E} \left[\int_{\delta}^T f^{(2)}(D_t, w_t, q_t) dt \right].$$

The Hamiltonians

$$H_t^{(1)} : \mathbb{R} \times \mathcal{A}^{(1)} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

and

$$H_t^{(2)} : \mathbb{R} \times \mathcal{A} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

for Players 1 and 2 are defined by

$$H_t^{(1)}(d, w, y, z) = f^{(1)}(d, w, \psi(w)) + y\mu_t(d, w, \psi(w)) + z\sigma_t(d, w, \psi(w)), \quad (3.9)$$

and

$$H_t^{(2)}(d, w, q, y, z) := f^{(2)}(d, w, q) + y\mu_t(d, w, q) + z\sigma_t(d, w, q), \quad (3.10)$$

$t \in [\delta, T]$. To these Hamiltonians we associate the backward SDEs

$$\begin{cases} dY_t^{(i)} = -\frac{\partial H_t^{(i)}}{\partial d}(D_t, w_t, q_t, Y_t^{(i)}, Z_t^{(i)})dt + Z_t^{(i)}dB_t, & t \in [\delta, T], \\ Y_T^{(i)} = 0, \end{cases} \quad (3.11)$$

in the adjoint processes $Y_t^{(i)}$ and $Z_t^{(i)}$, $i = 1, 2$, which can be solved under the assumptions (A1)-(A2) above. We will rely on the following maximum principle, cf. Theorem 3.1 in [7] and Theorem 2.2 in [59].

We now have the following necessary maximum principle for equilibria, cf. Theorem 1 in [58], where $(Y_t^{(1)})_{t \in [\delta, T]}$ and $(Y_t^{(2)})_{t \in [\delta, T]}$ are the solutions of (3.11).

Theorem 3.3.1. *The following are equivalent.*

(i) *For all bounded $\beta^{(1)} \in \mathcal{A}^{(1)}$, $\beta^{(2)} \in \mathcal{A}^{(2)}$ we have*

$$\frac{d}{ds}(J^{(1)}(w^* + s\beta^{(1)}, \psi^*(w^* + s\beta^{(1)})))|_{s=0} = \frac{d}{ds}(J^{(2)}(w^*, \psi^*(w^*) + s\beta^{(2)}))|_{s=0} = 0. \quad (3.12)$$

(ii) *We have*

$$\mathbb{E} \left[\frac{\partial H_t^{(1)}}{\partial u^{(1)}}(D_t, u^{(1)}, Y_t^{(1)}, Z_t^{(1)}) \middle| \mathcal{E}_t \right]_{|u^{(1)}=w_t^*} = 0 \quad (3.13)$$

and

$$\mathbb{E} \left[\frac{\partial H_t^{(2)}}{\partial u^{(2)}}(D_t, w_t^*, u^{(2)}, Y_t^{(2)}, Z_t^{(2)}) \middle| \mathcal{E}_t \right]_{|u^{(2)}=q_t^*=\psi_t^*(w^*)} = 0, \quad (3.14)$$

for $t \in [\delta, T]$.

Wholesaler / retailer equilibrium

In the sequel the profit functions $f^{(1)} : \mathbb{R} \times \mathcal{A} \rightarrow \mathbb{R}$ and $f^{(2)} : \mathbb{R} \times \mathcal{A} \rightarrow \mathbb{R}$ will be given by (3.1) and (3.2), i.e.,

$$f^{(1)}(d_t, w_t, q_t) = w_t q_t \quad \text{and} \quad f^{(2)}(d_t, w_t, q_t) = P(q_t) \min(d_t^+, q_t) - w_t q_t.$$

In this case, during the time period $\delta \leq t \leq T$, Player 1 will get the expected profit

$$J^{(1)}(w, q) = \mathbb{E} \left[\int_{\delta}^T w_t q_t dt \right],$$

and Player 2 will get the expected profit

$$J^{(2)}(w, q) = \mathbb{E} \left[\int_{\delta}^T (P(q_t) \min(D_t^+, q_t) - w_t q_t) dt \right].$$

To find an equilibrium, we use the maximum principle of Theorem 3.3.1. The Hamiltonian are given by

$$H_t^{(1)}(d, w, a^{(1)}, b^{(1)}) = w\psi_t(w) + a^{(1)}\mu_t(d, w, \psi_t(w)) + b^{(1)}\sigma_t(d, w, \psi_t(w)), \quad (3.15)$$

cf. (3.9), and

$$H_t^{(2)}(d, w, q, a^{(2)}, b^{(2)}) = P(q) \min(d^+, q) - wq + a^{(2)}\mu_t(d, w, \psi_t(w)) + b^{(2)}\sigma_t(d, w, \psi_t(w)), \quad (3.16)$$

with $q = \psi_t(w) = q_t(w)$, cf. (3.10).

Optimal retailer quantity

By (3.16), the first order condition (3.14) becomes

$$\begin{aligned} \mathbb{E} \left[P'(q_t) \min(D_t^+, q_t) + P(q_t) \mathbf{1}_{(-\infty, D_t]}(q_t) - w_t \right. \\ \left. + a^{(1)} \frac{\partial \mu_t}{\partial u^{(2)}}(d, w, \psi_t(w)) + b^{(1)} \frac{\partial \sigma_t}{\partial u^{(2)}}(d, w, \psi_t(w)) \mid \mathcal{E}_t \right]_{u^{(2)}=q_t} = 0 \end{aligned} \quad (3.17)$$

for the optimal values q_t^* . In fact, when demand is controlled, as in this case, we actually need a more advanced maximum principle, requiring more technical assumptions. The interested reader is referred to [58]. We will relax this assumption in the sequel, and we focus on the case where the equation (3.7) for the demand D_t reduces to an

exogenous SDE of the form

$$\begin{cases} dD_t = \mu_t(D_t)dt + \sigma_t(D_t)dB_t, \\ D_0 = d_0 \in \mathbb{R}, \end{cases}$$

where the coefficients $\mu.(d)$ and $\sigma.(d)$ do not depend on the controls q and w . In this case, since $\mu_t(d)$ and $\sigma_t(d)$ do not depend on q , (3.17) simplifies to

$$\mathbb{E} [P'(q_t) \min(D_t^+, q_t) + P(q_t) \mathbf{1}_{(-\infty, D_t]}(q_t) - w_t \mid \mathcal{E}_t]_{u^{(2)}=q_t} = 0 \quad (3.18)$$

To see why (3.18) has a unique solution, we note that since w_t is $\mathcal{E}_t$ -measurable, it is equivalent to

$$\mathbb{E}[P'(q_t) \min(D_t^+, q_t) + P(q_t) \mathbf{1}_{(-\infty, D_t]}(q_t) \mid \mathcal{E}_t] = w_t. \quad (3.19)$$

Existence and uniqueness of q_t then follows from the monotonicity of the conditional expectation. In the sequel we denote by $q_t(w_t) = \psi_t^*(w)$ the solution of (3.19). Assuming that for every $t \in [0, T]$, the function

$$w \longmapsto wq_t(w) \quad (3.20)$$

has a unique maximum at $w = w_t^*$, then the equilibrium is (w^*, ψ^*) and we have

$$q_t^*(w_t^*) = \psi_t^*(w^*), \quad t \in [\delta, T].$$

The maximization problem (3.20) can be solved for the optimal value w_t^* from the first order condition

$$w_t^* q_t'(w_t^*) + q_t(w_t^*) = 0 \quad (3.21)$$

that follows from (3.13).

Ornstein-Uhlenbeck setting

Henceforth we assume that

$$\mu_t(D_t) = \alpha(\beta - D_t) \quad \text{and} \quad \sigma_t(D_t) = \sigma,$$

with $\alpha, \beta, \sigma > 0$, therefore the dynamics (3.7) of $(D_t)_{t \in \mathbb{R}_+}$ becomes

$$\begin{cases} dD_t = \alpha(\beta - D_t)dt + \sigma dB_t \\ D_0 = d_0 \in \mathbb{R}, \end{cases}$$

with the Ornstein-Uhlenbeck solution

$$D_t = D_{t-\delta}e^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + \int_{t-\delta}^t \sigma e^{\alpha(s-t)} dB_s.$$

We note that given $\mathcal{F}_{t-\delta}$, the (unknown) demand D_t is Gaussian

$$\mathcal{N}\left(\varphi(D_{t-\delta}), \sigma^2 \frac{1 - e^{-2\alpha\delta}}{2\alpha}\right)$$

distributed, where

$$\varphi(d) := de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}), \quad d > 0, \quad (3.22)$$

and we let $\Phi : \mathbb{R} \rightarrow [0, 1]$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ respectively denote the standard Gaussian cumulative distribution and probability density functions.

Proposition 3.3.2. *The equilibrium action $q_t := \psi(w_t)$ for Player 2 at any $t \in [\delta, T]$ is the solution of the equation*

$$\begin{aligned} (P(q_t) + P'(q_t)q_t)\Phi\left(\frac{\varphi(D_{t-\delta}) - q_t}{\varepsilon}\right) \\ + P'(q_t)\left(\varphi(D_{t-\delta})\Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) - \varepsilon\phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right)\right) = w_t, \end{aligned} \quad (3.23)$$

with $\varepsilon^2 = \sigma^2(1 - e^{-2\alpha\delta})/(2\alpha)$.

Proof. The equation (3.19) can be written as

$$\begin{aligned} & P'(q_t) \mathbb{E}[\min((de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)^+, q)]|_{d=D_{t-\delta}, q=q_t} \\ & + P(q_t) \mathbb{E}[\mathbf{1}_{(-\infty, de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)}(q)]|_{d=D_{t-\delta}, q=q_t} = w_t, \end{aligned} \quad (3.24)$$

where

$$X := \mathcal{N}(0, \varepsilon^2), \text{ with } \varepsilon^2 = \sigma^2 \int_{t-\delta}^t e^{2\alpha(s-t)} ds = \sigma^2 \frac{1 - e^{-2\alpha\delta}}{2\alpha},$$

is a Gaussian random variable independent of $\mathcal{F}_{t-\delta}$. Focusing on the first expectation in (3.24) we have

$$\begin{aligned} & P'(q_t) \mathbb{E}[\min((de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)^+, q)]|_{d=D_{t-\delta}, q=q_t} \\ & = P'(q_t) \left(q_t \int_{q_t - \varphi(D_{t-\delta})}^{\infty} \frac{e^{-x^2/(2\varepsilon^2)}}{\sqrt{2\pi\varepsilon^2}} dx + \int_{-\infty}^{q_t - \varphi(D_{t-\delta})} (\varphi(D_{t-\delta}) + x) e^{-x^2/(2\varepsilon^2)} \frac{dx}{\sqrt{2\pi\varepsilon^2}} \right) \\ & = P'(q_t) \left(q_t \left(1 - \Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) \right) + \varphi(D_{t-\delta}) \Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) - \varepsilon \phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) \right) \\ & = P'(q_t) \left(q_t \Phi\left(\frac{\varphi(D_{t-\delta}) - q_t}{\varepsilon}\right) + \varphi(D_{t-\delta}) \Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) - \varepsilon \phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) \right). \end{aligned}$$

Considering the second expectation in (3.24), due to the relation (3.22) we have

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_{(-\infty, de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)}(q)]|_{d=D_{t-\delta}, q=q_t} = \mathbb{P}(q \leq de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X)|_{d=D_{t-\delta}, q=q_t} \\ & = \mathbb{P}(de^{-\alpha\delta} + \beta(1 - e^{-\alpha\delta}) + X \geq q)|_{d=D_{t-\delta}, q=q_t} \\ & = \int_{q_t - \varphi(D_{t-\delta})}^{\infty} \frac{e^{-x^2/(2\varepsilon^2)}}{\sqrt{2\pi\varepsilon^2}} dx \\ & = \Phi\left(\frac{\varphi(D_{t-\delta}) - q_t}{\varepsilon}\right), \end{aligned}$$

hence (3.19) boils down to (3.23). $\square$

By abuse of notation we will write the solution to (3.23) as

$$q_t = \psi_{D_{t-\delta}}(w_t), \quad (3.25)$$

where $\psi_{D_{t-\delta}} : \mathbb{R} \rightarrow \mathbb{R}$ is the function used in Figures 3.2 and 3.3. Next, we can solve

$$\max_{w_t \in \mathbb{R}_+} w_t \psi_{D_{t-\delta}}(w_t) \quad (3.26)$$

for the optimal w_t^* using the first order condition (3.21), which will yield

$$q_t^* = \psi_{D_{t-\delta}}(w_t^*). \quad (3.27)$$

Affine prices

In case $P(q_t) = a - bq_t$ with $a, b > 0$, i.e. retail prices are affine functions of output, which is a common assumption, (3.23) becomes

$$(a - 2bq_t)\Phi\left(\frac{\varphi(D_{t-\delta}) - q_t}{\varepsilon}\right) - b\varphi(D_{t-\delta})\Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) + \varepsilon b\phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) = w_t. \quad (3.28)$$

In particular, when $b = 0$ and retail price are fixed at a , independent of q_t , we have

$$\Phi\left(\frac{q_t - \varphi(D_{t-\delta})}{\varepsilon}\right) = \frac{a - w_t}{a}, \quad (3.29)$$

where

$$x \mapsto \Phi\left(\frac{x - \varphi(D_{t-\delta})}{\varepsilon}\right)$$

is the conditional distribution function of the demand D_t given $\mathcal{F}_{t-\delta}$, which can be seen as an analog of the equation

$$F(q_t) = \frac{a - w_t}{a}$$

in the framework of fixed retail prices, where $F(\cdot)$ is the cumulative distribution function of a random demand, see e.g. the equation (3) in [72].

When δ tends to zero, the equation (3.28) converges to

$$(a - 2bq_t)\mathbf{1}_{\{q_t \leq D_t\}} - bD_t\mathbf{1}_{\{q_t > D_t\}} = w_t, \quad (3.30)$$

which recovers (3.39) below, since $\lim_{\varepsilon \rightarrow 0} \Phi(x/\varepsilon) = \mathbf{1}_{[0, \infty)}(x)$, $x \in \mathbb{R}$.

3.4 Deterministic demand with no lag

In this section the respective performance functionals of Players 1 and 2 are given from (3.1) and (3.2) by

$$J^{(1)}(w, q(w)) = \int_0^T f^{(1)}(d_t, w_t, q_t) dt = \int_0^T w_t q_t(w_t) dt, \quad (3.31)$$

and

$$J^{(2)}(w, q) = \int_0^T f^{(2)}(d_t, w_t, q_t) dt = \int_0^T (P(q_t) \min(D_t^+, q_t) - w_t q_t) dt. \quad (3.32)$$

From (3.2) it is clear that $q_t^*(d_t, w_t) \leq d_t$ and so q_t^* exists since the functional (3.2) is continuous on a compact set. A sufficient but not necessary condition for $q_t^*(d_t, w_t)$ is given by the first order condition

$$P'(q_t) \min(d_t, q_t) + P(q_t) \mathbf{1}_{[0, d_t]}(q_t) - w_t = 0 \quad (3.33)$$

We can rewrite (3.33) as

$$P'(q_t) d_t \mathbf{1}_{\{q_t > d_t\}} + (P'(q_t) q_t + P(q_t)) \mathbf{1}_{\{q_t \leq d_t\}} - w_t = 0. \quad (3.34)$$

Therefore, when $q_t \leq d_t$ we have that q_t solves

$$P'(q_t) q_t + P(q_t) - w_t = 0.$$

If on the other hand $q_t > d_t$ then we need q_t to satisfy

$$P'(q_t) = \frac{w_t}{d_t}. \quad (3.35)$$

Assuming that the function

$$w_t \longmapsto w_t q_t^*(d_t, w_t) \quad (3.36)$$

has a unique maximum at $w_t = w_t^* > 0$, we get the first order condition

$$w_t \frac{\partial q_t^*}{\partial w_t}(d_t, w_t) + q_t^*(d_t, w_t) = 0 \quad (3.37)$$

for the optimal value w_t^* , which yields $q_t^*(d_t, w_t^*)$.

Affine prices

We now work out the case when $P(q_t) = a - bq_t$, i.e. retail prices are affine functions of output, which is a common assumption.

Proposition 3.4.1. *Assume that $P(q_t) = a - bq_t$, $a, b > 0$. Then the equilibrium is given by*

$$w_t^* = \max(a - 2bd_t, a/2)$$

and

$$q_t^*(d_t, w_t) = \min(d_t, (a - w_t)^+ / (2b)) = \begin{cases} d_t & \text{for } 0 \leq w_t \leq a - 2bd_t \\ \frac{a - w_t}{2b} & \text{for } a - 2bd_t < w_t \leq a \\ 0 & \text{for } w_t > a. \end{cases} \quad (3.38)$$

Therefore, the actions played in equilibrium are given by

$$w_t^* = \max(a - 2bd_t, a/2) \quad \text{and} \quad q_t^*(d_t, w_t^*) = \min(d_t, a/(4b)).$$

Proof. From (3.34) we have

$$-bd_t \mathbf{1}_{\{q_t > d_t\}} + (-bq_t + a - bq_t) \mathbf{1}_{\{q_t \leq d_t\}} - w_t = 0. \quad (3.39)$$

or

$$(a - bq_t) \mathbf{1}_{\{q_t \leq d_t\}} = w_t + b \min(d_t, q_t). \quad (3.40)$$

For $d_t > q_t$ we have

$$a - 2bq_t - w_t = 0, \quad (3.41)$$

so that

$$q_t^*(d_t, w_t) = \frac{a - w_t}{2b} < d_t, \quad (3.42)$$

provided that $w_t \leq a$. If however

$$\frac{a - w_t}{2b} \geq d_t \geq 0,$$

then we take

$$q_t^*(d_t, w_t) = d_t, \quad (3.43)$$

to see this, note that the derivative of the profit function, (3.41), is positive on $[0, d_t]$.

Finally if $w_t > a$, then $q_t^*(d_t, w_t) = 0$. This is optimal since profit is given by

$$(a - bq_t) \min(d_t, q_t) - wq_t$$

so clearly it is optimal to choose $q_t^*(d_t, w_t) = 0$ in this case. Hence we find

$$q_t^*(d_t, w_t) = \min(d_t, (a - w_t)^+ / (2b)) = \begin{cases} d_t & \text{for } 0 \leq w_t \leq a - 2bd_t \\ \frac{a - w_t}{2b} & \text{for } a - 2bd_t < w_t \leq a \\ 0 & \text{for } w_t > a, \end{cases} \quad (3.44)$$

and

$$w_t q_t^*(d_t, w_t) = w_t \min(d_t, (a - w_t)^+ / (2b)) = \begin{cases} w_t d_t & \text{for } 0 \leq w_t \leq a - 2bd_t, \\ \frac{aw_t - w_t^2}{2b} & \text{for } a - 2bd_t < w_t \leq a, \\ 0 & \text{for } w_t > a. \end{cases} \quad (3.45)$$

The function $w_t \mapsto (aw_t - w_t^2)/(2b)$ reaches its maximum $a^2/(8b)$ at $\bar{w} = a/2$. In order to maximize (3.45) as a function of w_t we need to consider two cases:

- i) In case we have $d \geq a/(4b)$, i.e. $a - 2bd \leq a/2$, the maximum is reached at $w_t^* = \bar{w} = a/2$ and it equals $(a\bar{w} - \bar{w}^2)/(2b) = a^2/(8b)$.
- ii) In case we have $d_t < a/(4b)$, i.e. $a - 2bd_t > a/2$, the maximum is reached at $w_t^* = a - 2bd_t$ which solves $w_t^*d_t = (aw_t^* - (w_t^*)^2)/(2b)$, and it is equal to $w_t^*d_t = d_t(a - 2bd_t)$.

Therefore we have

$$w_t^* = \max \left(a - 2bd_t, \frac{a}{2} \right),$$

and consequently

$$\begin{aligned} q_t^*(d_t, w_t^*) &= \frac{a - \max(a - 2bd_t, a/2)}{2b} \\ &= \frac{a + \min(2bd_t - a, -a/2)}{2b} \\ &= \frac{\min(2bd_t, a/2)}{2b} = \min \left(d_t, \frac{a}{4b} \right). \end{aligned}$$

□

The plots in Figures 3.1-3.4 have been made for $P(q_t) = a - bq_t$ with $a, b > 0$.

Figure 3.1: Retailer output $d \mapsto q_t^*(d, w) = \psi_d(w)$ in (3.38) with $w = 0.5$.

From Figure 3.1 we note that, due to dynamic retail prices, the retail output remains capped at the level $a/(4b)$ as a function of demand, whereas it remains strictly increasing in the case of fixed prices, cf. also Figure 3 in [58].

Figure 3.2: Wholesaler profit $w \mapsto wq_t^*(d, w) = w\psi_d(w)$ with $d = 0.2 < a/(4b)$.

Figure 3.3: Wholesaler profit $w \mapsto wq_t^*(d, w) = w\psi_d(w)$ with $d = 0.35 > a/(4b)$.

Figure 3.4: Wholesaler *vs* retailer profits (3.46) and (3.47) as functions of d_t with $\delta = 0$.

Figure 3.4 illustrates the equilibrium wholesaler and retailer profits respectively given by (3.38),

$$w_t^* q_t^*(d_t, w_t^*) = \max \left(a - 2bd_t, \frac{a}{2} \right) \min \left(d_t, \frac{a}{4b} \right) \quad (3.46)$$

and

$$\begin{aligned} & P(q_t^*(d_t, w_t^*)) q_t^*(d_t, w_t^*) - w_t^* q_t^*(d_t, w_t^*) \\ &= \left(\left(a - b \min \left(d_t, \frac{a}{4b} \right) \right)^+ - \max \left(a - 2bd_t, \frac{a}{2} \right) \right) \min \left(d_t, \frac{a}{4b} \right). \end{aligned} \quad (3.47)$$

3.5 Conclusion

We have studied stochastic newsvendor games in which dynamic retail prices have been added to delayed demand. We have shown that the presence of dynamic retail prices forces the retail output to remain capped at the level $a/(4b)$ as a function of demand, cf. Figure 3.1 above, whereas it remained strictly increasing in the case of fixed prices, cf. Figure 3 in [58]. Indeed, here the wholesaler and retailer have aligned interests as they would both want demand to equal at least $a/(4b)$, in which case the equilibrium output tends to $a/(4b)$. Below this point, the equilibrium output increases with d , less output is produced (namely $q_t^* = d_t$, cf. (3.38)), and profits are lower for both wholesaler and retailer. The lagged case is a smoothing of the no-lag case, and unlike in the Cournot setup [69], there is no transition from single to multiple equilibria.

Chapter 4

The incentive ratio in exchange economies

An abridged version of this chapter was presented at the 10th Annual International Conference on Combinatorial Optimization and Applications (COCOA'16) in Hong Kong, China.

4.1 Introduction

General equilibrium theory and (noncooperative) game theory are among the most successful and well-studied areas in economic theory. The former seeks to explain the existence of equilibria in multiple markets at the same time. The latter serves as the primary tool for predicting, analysing and describing the behaviour of rational agents' actions both in and out of equilibrium.

Both branches have yielded an abundance of literature and served as the basis of much fruitful research. In this chapter, we try to combine the two approaches for exchange economies. Specifically, we ask how much any individual agent can gain from strategically misreporting some of his/her utility function. The primary tool for this will be the *incentive ratio* as introduced by [19], [18]. In a nutshell, when all agents report truthfully, an equilibrium will materialise, assigning all agents an equilibrium bundle. But given the endowments and utility functions of the other agents, a strategic agent may be better off by misreporting his/her own characteristics, thereby enforcing

another equilibrium and, therefore, potentially obtaining a better equilibrium bundle. The incentive ratio tries to capture the maximal magnitude of such gains over all envisionable setups. A formal definition will follow below. The results presented here suggest that, contrary to previous findings in Fisher markets, the gains from strategic behaviour may be significant, even allowing an agent to improve his equilibrium utility without bound. If we impose the (common) restriction that all agents possess at least a little bit of every commodity and the market is strongly competitive, the utility gain in Cobb–Douglas markets is bounded by the number of commodities, but it may exceed the upper bound from Fisher markets, which we show by means of an example. The results obtained show a sharp contrast with the findings in [19], [18]: in the Fisher market setup, incentive ratios are bounded by the small constants 2, 2 and $e^{1/e} \approx 1.44$ for linear, Leontief and Cobb–Douglas markets, respectively.

Related work

In [14], price of anarchy bounds are computed for linear, Leontief and Cobb–Douglas markets in the Fisher model. That is, the ratio between (utilitarian) welfare of all agents in the worst possible pure Nash equilibrium (for which existence is proven) and maximum welfare. It primarily differs from the analysis presented here in that it focuses on welfare of all agents rather than measuring the benefits of strategic behaviour for one specific agent. As in this chapter, the strategic variable of interest is the utility function. Arguably, misreporting the utility function i.e. one’s preferences, is easier than misreporting (physical) endowments, which can, at least in theory, be inspected and whose withholding may be difficult and costly (e.g. due to storage costs). It is known that the Walrasian mechanism is susceptible to manipulation via endowments: via withholding endowments and recovering it fully [70], recovering part of it [82] (see also [89]) and even destroying part of one’s initial endowment [5]. However, the aforementioned studies are qualitative, that is, they show that manipulation of the equilibrium mechanism is possible, but do not quantify it. The incentive ratio is a first step to quantifying the possible gains of misreporting in exchange economies. It is not so difficult to see from Proposition 4.3.1 below that

even when destroying part of one's initial endowment (in the case e_i is the strategic variable of interest), the incentive ratio may still tend to infinity.

The idea that an agent may act strategically in a (Fisher) market by misreporting his/her utility function in order to get a better equilibrium bundle, compared to the scenario where everyone is truthful, was already considered in [1] for the case of linear utility functions. In this paper, Nash equilibria and the relation with conflict-freeness (meaning that every agent can get an optimal equilibrium bundle given the strategy profile) is studied. It is shown that being conflict-free is a necessary condition for a Nash equilibrium, and for a symmetric strategy profile it is also sufficient. The incentive ratio was first coined in [19], where the strategic variable of interest is the (Leontief) utility function of a player and bidding the true budget is a dominant strategy. In [18], [17], a slightly more sophisticated version of the incentive ratio is presented, in which players may also strategise on their endowments. The “exchange market game” is introduced in [49], and agents have linear utility functions. They may lie about their utility function to manipulate the outcome of the exchange process. It is consequently shown that a symmetric strategy profile is a Nash equilibrium if and only if it is conflict-free. Several properties of the symmetric Nash equilibria (e.g. Pareto-optimal payoffs, a characterisation for its uniqueness) are subsequently derived.

Organisation

The rest of this chapter is organised as follows. Section 4.2 discusses the necessary machinery, definitions and introduces some notation. Next, Section 4.3 presents the results for incentive ratios in Linear, Leontief and Cobb–Douglas exchange economies. The latter receives most attention. It is shown that, even if the strategy space is restricted to the utility function only, all three incentive ratios are unbounded. Under some specific, common assumptions, the $e^{1/e}$ bound is recovered for Cobb–Douglas markets with 2 commodities. For the case where there are at least 3 commodities, the incentive ratio is shown to be greater than $e^{1/e}$ but bounded by the number of commodities. Finally, 4.4 concludes and provides some directions for future research.

4.2 Preliminaries, exchange economies and competitive equilibrium

We use the following notation. Suppose $x, y \in \mathbb{R}^n$. Then $x \cdot y = \sum_{k=1}^n x_k y_k$ denotes the dot product of x and y . $x \leq y$ means $x_k \leq y_k$ for $k = 1, \dots, n$. For a vector $u = (u_1, \dots, u_n)$, by u_{-i} we mean the vector $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$ (i.e. all entries except the i -th). For positive integer n , we use $[n]$ as shorthand notation for the set $\{1, \dots, n\}$. I_m is the $m \times m$ identity matrix. The transpose of a matrix M is denoted by M^T and its determinant by $|M|$. If $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, then $Df(x)$ represents the Jacobian matrix of f at x .

Depending on agents' endowments we speak either of Fisher markets [13] (in case of monetary endowments) or, more generally, in case of (a vector of) commodity endowments, of exchange economies. In the former case, all agents bring a certain amount of cash to the market and budgets are exogeneous to the model, whereas in the latter the value of the endowment is endogeneous, determined by market prices. That is, in exchange economies, each agent i is endowed with a bundle of commodities $e_i \in \mathbb{R}_+^m$, whereas in the case of Fisher markets, he simply possesses an amount of wealth $w_i \in \mathbb{R}_+$. We will focus on the more general model of exchange economies with n agents and m commodities.

Definition 4.2.1 (Exchange economy). *An exchange economy (henceforth also simply referred to as economy) is a tuple $\xi = ((u_i)_{i=1}^n, (e_i)_{i=1}^n)$, where $u_i : \mathbb{R}_+^m \rightarrow \mathbb{R}$ is the utility function of agent $i \in [n]$ and $e_i \in \mathbb{R}_+^m$ is a vector where e_{ij} indicates how much agent $i \in [n]$ possesses of commodity $j \in [m]$.*

In an economy, agents interact with each other to obtain a bundle $x_i \in \mathbb{R}_+^m$ by trading commodities given a price vector $p \in \mathbb{R}^m$. If p is such a price vector, then every agent solves the following consumer problem ($\mathcal{CP}$).

Definition 4.2.2 (Demand).

$$\begin{aligned}
& \text{maximize} && u_i(x_i) \\
& \text{subject to} && p \cdot x_i \leq p \cdot e_i \\
& && x_i \geq 0
\end{aligned} \tag{CP}$$

We call the set of solutions to problem (CP) the demand of agent i (at prices p).

We can write $x_i(p, p \cdot e_i)$ to show explicitly that demand depends on endowments and prices. Since prices are in turn determined by endowments and utility functions, we may also write $x_i(u_i, u_{-i}, e)$ or, when it is understood that u_{-i} and e are fixed, simply as $x_i(u_i)$. The central topic in this study is the notion of equilibrium, specifically, that of Walrasian or competitive equilibrium.

Definition 4.2.3 (Competitive/Walrasian equilibrium). *A competitive equilibrium is a pair $(p, x) \in \mathbb{R}^m \times (\mathbb{R}_+^m)^n$ such that:*

1. *For all $j \in [m]$, $\sum_{i=1}^n x_{ij} = \sum_{i=1}^n e_{ij}$ i.e. markets clear*
2. *For all $i \in [n]$, x_i is a solution to (CP), i.e. x_i is the best bundle among the possible choices in the budget set.*

4.2.1 Definition of the incentive ratio and illustration in linear markets

Every agent is characterized by two parameters, his endowment e_i and his utility function u_i . Generally, different endowments and different utility functions will lead to different equilibria. What if an agent purposely misreports his utility function, thereby trying to get a better equilibrium allocation?

The incentive ratio is a concept introduced in [19]. It attempts to measure the (maximum) benefits of manipulating the equilibrium mechanism by strategically misreporting personal parameters. Formally, we define it as follows (adapted for exchange economies, the original definition was given for Fisher markets in [17], [19], [18]):

Definition 4.2.4 (Incentive ratio). *The incentive ratio of agent i in a market M (e.g. linear, Cobb–Douglas or Leontief), denoted ζ_i^M , is defined as:*

$$\zeta_i^M = \max_{u_{-i} \in U_{-i}, e_{-i} \in (\mathbb{R}_+^m)^{n-1}} \max_{u'_i \in U_i} \frac{\max_{x' \in \mathcal{E}(u'_i)} u_i(x'_i(u'_i, u_{-i}, e))}{\min_{x \in \mathcal{E}(u_i)} u_i(x_i(u_i, u_{-i}, e))}$$

The incentive ratio of the market M is subsequently defined as

$$\zeta^M = \max_{i \in [n]} \zeta_i^M$$

Remark 4.2.5. *In this definition:*

- *Variables with a prime ($'$) refer to the scenario in which agent i misreports his parameters (and all other agents report truthfully). That is, he reports u'_i and as a result, obtains a bundle $x'_i(u'_i)$. Notice that this bundle is evaluated by the true utility function.*
- *Given that player i reports $\tilde{u}_i$ (i.e. truthful or not) as his utility function (and the other players u_{-i}), we denote by $\mathcal{E}(\tilde{u}_i)$ the set of equilibrium allocations, that is,*

$$\mathcal{E}(\tilde{u}_i) = \{x \in (\mathbb{R}_+^m)^n \mid \exists p \in \mathbb{R}_+^m \text{ } (p, x) \text{ is a Walras equilibrium}\}.$$

under some (mild) assumptions this set is nonempty, but it could contain multiple equilibrium allocations.

- *U_i contains the admissible strategies/utility functions for player i , including the one that agent i chooses when he misreports his utility function. We denote $U_{-i} = \prod_{k \neq i} U_k$. We will only consider the case where all U_k 's are equal to a common U , thus $U_{-i} = U^{n-1}$.*
- *From the preceding arguments, we may restrict attention to agent i , since all agents can be treated symmetrically, thus we may rewrite the incentive ratio for the market M as*

$$\zeta^M = \max_{u_{-i} \in U^{n-1}, e_{-i} \in (\mathbb{R}_+^m)^{n-1}} \max_{u'_i \in U} \frac{\max_{x' \in \mathcal{E}(u'_i)} u_i(x'_i(u'_i, u_{-i}, e))}{\min_{x \in \mathcal{E}(u_i)} u_i(x_i(u_i, u_{-i}, e))}$$

Example 4.2.6. *The following example for linear markets shows that the consequences for the incentive ratio stemming from the nonuniqueness of equilibrium can be large. Here, the strategic influence of an agent is perhaps less relevant than the equilibrium selection problem that is embodied in the definition of the incentive ratio: the truthful equilibrium could be very bad for an agent (yielding low utility) while the nontruthful equilibrium could be very good for him (yielding high utility).*

Setup: $e_1 = (\epsilon, 1 - \epsilon)$, $e_2 = (1 - \epsilon, \epsilon)$, $u_2(x_2) \equiv 0$, $\epsilon > 0$ and small

Truthful

$$u_1(x_1) = x_{11}$$

$$p = (1, 0)$$

$$x_1 = (\epsilon, 1 - \epsilon), x_2 = (1 - \epsilon, \epsilon)$$

Nontruthful

$$u'_1(x'_1) = x'_{11}$$

$$p' = (1, 1)$$

$$x'_1 = (1, 0), x'_2 = (0, 1)$$

We have

$$\frac{u_1(x'_1)}{u_1(x_1)} = \frac{1}{\epsilon}$$

Letting ϵ tend to 0, the incentive ratio tends to ∞ .

Although arguably the utility function for agent 2 is not very realistic, he doesn't care about any of the products on the market, it does not seem that such a scenario was ruled out by the authors in [18]. In fact, in [19], [18], the following markets are considered, with arbitrarily many agents and commodities:

1. Linear, i.e. U is the set of utility functions of the form $u(x) = \alpha \cdot x$ where $\alpha \in \mathbb{R}_+^m$.
2. Leontief, i.e. U is the set of utility functions of the form $u(x) = \min_{j \in [m]} \{x_j / \alpha_j\}$ where $\alpha \in \mathbb{R}_{++}^m$
3. Cobb–Douglas, i.e. U is the set of utility functions of the form $u(x) = \prod_{j=1}^m x_j^{\alpha_j}$ where $0 \leq \alpha_j \leq 1$ for all $j \in [m]$ and $\sum_{j=1}^m \alpha_j = 1$, so that U is the set of Cobb–Douglas functions that are homogeneous of degree 1.

The (tight) bounds proven in [19], [18] for Fisher markets are 2, 2 and $e^{1/e}$ for linear, Leontief and Cobb–Douglas respectively.

4.3 The incentive ratio in Leontief and Cobb–Douglas markets

We will, without loss of generality, restrict ourselves to scenarios where $e_i \in [0, 1]^m$ for all $i \in [n]$ and $\sum_{i=1}^n e_{ij} = 1$ for all $j \in [m]$. The results indicate that, without any further restrictions on the setup presented in [19], [18] incentive ratios in Leontief and Cobb–Douglas exchange economies are unbounded.

Proposition 4.3.1. *The incentive ratio for Leontief and Cobb–Douglas exchange economies equals $+\infty$ (i.e. $\forall n \in \mathbb{N}$ there exists a market such that the incentive ratio for player i is at least n).*

Proof. (Leontief). $e_1 = (1 - \epsilon, \epsilon)$, $e_2 = (\epsilon, 1 - \epsilon)$, $u_2(x_2) = \min\{x_{21}, x_{22}\}$, $\epsilon > 0$ and small

Truthful

$$u_1(x_1) = \min\{x_{11}, x_{12}\}$$

then

$$p = (\delta, 1)$$

$$x_1 = ((\epsilon + \delta - \delta\epsilon)/(1 + \delta), (\epsilon + \delta - \delta\epsilon)/(1 + \delta))$$

$$x_2 = (1 - \epsilon + \delta\epsilon)/(1 + \delta), (1 - \epsilon + \delta\epsilon)/(1 + \delta))$$

Nontruthful

$$u'_1(x'_1) = \min\{x'_{11}, x'_{12}\}$$

then

$$p' = (1, 1)$$

$$x'_1 = (1/2, 1/2)$$

$$x'_2 = (1/2, 1/2)$$

We have

$$\frac{u_1(x'_1)}{u_1(x_1)} = \frac{1 + \delta}{2(\epsilon + \delta - \delta\epsilon)}$$

Letting δ, ϵ tend to 0, the incentive ratio tends to ∞ . □

Proof. (Cobb–Douglas). $e_1 = (1, 0)$, $e_2 = (0, 1)$, $u_2(x_2) = x_{21}^\epsilon x_{22}^{(1-\epsilon)}$, $\epsilon > 0$ and small

Truthful

$$u_1(x_1) = x_{11}^{.5} x_{12}^{.5}$$

then

$$p = (1, 1/(2\epsilon))$$

$$x_1 = (1/2, \epsilon)$$

$$x_2 = (1/2, 1 - \epsilon)$$

Nontruthful

$$u'_1(x'_1) = x'_{11}$$

then

$$p' = (1, 0)$$

$$x'_1 = (1, 1)$$

$$x'_2 = (0, 0)$$

We have

$$\frac{u_1(x'_1)}{u_1(x_1)} = \sqrt{2\epsilon^{-1}}$$

Letting ϵ tend to 0, the incentive ratio tends to ∞ . □

The intuition behind the proof for Cobb–Douglas markets is that it is possible for player 1 to completely annihilate the equilibrium value of player 2’s endowment. Since $u_2(e_2) = 0$, the bundle $x_2 = (0, 0)$ solves the consumer problem $(\mathcal{CP})$, since the indifference curve $u_2(x_2) = 0$ is odd-shaped compared to the other indifference curves.

We make the following assumption to ensure all equilibrium prices are positive; this is rather standard in algorithmic game theory.¹

Assumption 4.3.2. • *(Positivity of endowments)* Every agent possesses a strictly positive amount of every commodity: $\forall i \in [n], \forall j \in [m] \quad e_{ij} > 0$;

• *(Strong competitiveness (see e.g. [14]))* Every commodity is demanded by at least one agent: $\forall j \in [m] \exists i \in [n] \alpha_{ij} > 0, \forall j \in [m] \exists i \in [n] \alpha'_{ij} > 0$

This entails that the economy excess demand function

$$z(p) := \sum_{i=1}^n (x_i(p, p \cdot e_i) - e_i) = \sum_{i=1}^n x_i(p, p \cdot e_i) - 1$$

has the gross substitute property and this implies that the equilibrium price is

¹Alternatively, we could assume the existence of a nonmanipulating agent who possesses at least a little bit of all commodities and who desires every commodity.

unique (see e.g. [46]).

For Cobb–Douglas markets with 2 commodities, the incentive ratio is $e^{1/e}$ i.e. as in the Fisher market scenario. Here, we rule out a setup in which there exists an agent that owns all of a certain commodity (as was the case in the above example), an assumption that is not unusual. The result is easily extended to an arbitrary number of agents. We present here the case $n = 2$.

Proposition 4.3.3. *Consider a Cobb–Douglas economy with $n = 2$ players and $m = 2$ commodities, in which both players hold a strictly positive amount of both commodities. The incentive ratio is $e^{1/e}$ and this bound is tight.*

Proof. First consider such an exchange economy in its most general form:

- Endowments: $e_1 = (e_{11}, e_{12}), e_2 = (1 - e_{11}, 1 - e_{12}), (e_{11}, e_{12}) \in (0, 1)^2$
- Utility functions: $u_1(x_{11}, x_{12}) = x_{11}^\alpha x_{12}^{1-\alpha}, u'_1(x'_{11}, x'_{12}) = x'_{11}^{\alpha'} x'_{12}^{1-\alpha'}$ and $u_2(x_{21}, x_{22}) = x_{21}^\beta x_{22}^{1-\beta}; \alpha, \alpha', \beta \in (0, 1)$.

Notice that, here both agents have a little bit of both commodities so we can not be in the situation of the example above. Normalize $p_1 = 1$. Then the value of agent 1's endowment is $p \cdot e_1 = e_{11} + e_{12}p_2$ and the value of agent 2's endowment is $p \cdot e_2 = 1 - e_{11} + (1 - e_{12})p_2$. Demands follow from the Karush-Kuhn-Tucker conditions as per usual:

$$\begin{cases} x_{11} = \alpha e_{11} + \alpha e_{12} p_2 \\ x_{12} = (1 - \alpha) e_{11} / p_2 + (1 - \alpha) e_{12} \\ x_{21} = \beta (1 - e_{11}) + \beta (1 - e_{12}) p_2 \\ x_{22} = (1 - \beta) (1 - e_{11}) / p_2 + (1 - \beta) (1 - e_{12}) \end{cases}$$

We can solve for p_2 and we find that:

$$p_2 = \frac{1 - \alpha e_{11} - \beta (1 - e_{11})}{\alpha e_{12} + \beta (1 - e_{12})}$$

Therefore

$$x_{11} = \frac{\alpha[(1-\beta)e_{12} + \beta e_{11}]}{\alpha e_{12} + \beta e_{22}} \text{ and } x_{12} = \frac{(1-\alpha)[(1-\beta)e_{12} + \beta e_{11}]}{1 - \alpha e_{11} - \beta e_{21}}$$

Now we can find a closed form expression for the incentive ratio as follows:

$$\frac{u_1(x'_1)}{u_1(x_1)} = \left(\frac{\alpha'[\alpha e_{12} + \beta e_{22}]}{\alpha[\alpha' e_{12} + \beta e_{22}]} \right)^\alpha \left(\frac{(1-\alpha')[1 - \alpha e_{11} - \beta e_{21}]}{(1-\alpha)[1 - \alpha' e_{11} - \beta e_{21}]} \right)^{1-\alpha} =: T_1^\alpha T_2^{1-\alpha}$$

The following facts are easily verified:

1. $\alpha' \geq \alpha \Rightarrow T_1 \leq \alpha'/\alpha$
2. $\alpha' < \alpha \Rightarrow T_1 < 1$
3. $\alpha' \geq \alpha \Rightarrow T_2 \leq 1$
4. $\alpha' < \alpha \Rightarrow T_2 < (1 - \alpha')/(1 - \alpha)$

These facts and the inequality from [18] that $\forall x, y \geq 0, x^y \leq e^{xy/e}$ show that the incentive ratio is bounded by $e^{1/e}$. The example in [18] for Fisher markets shows the bound is tight. $\square$

This bound can be exceeded, even when $m = 3$, as the following example shows.

Example 4.3.4 (Incentive ratio $> e^{1/e}$). *Suppose the market is as follows:*

$$\left\{ \begin{array}{l} e_1 = (.99, .01, .01) \\ e_2 = (.01, .99, .99) \\ u_1(x_1) = x_{11}^2 x_{12}^3 x_{13}^5 \\ u_2(x_2) = x_{21}^4 x_{12}^6 \\ u'_1(x'_1) = x'_{11}{}^{.85} x'_{12}{}^{.1} x'_{13}{}^{.05} \end{array} \right. \text{ then } \left\{ \begin{array}{l} p = (.398, .597, .201) \\ x_1 \approx (.202, .202, 1) \\ u_1(x_1) \approx .4495 \\ p' = (.4045, .1344, .0201) \\ x'_1 \approx (.845, .299, 1) \\ u_1(x'_1) \approx .6731 \end{array} \right.$$

Therefore the incentive ratio is approximately 1.50; a tight bound remains an open question.

The remainder of this section is devoted to the proof that the incentive ratio for Cobb–Douglas markets is bounded. We will use that, by the AM–GM inequality,

$$\prod_{j=1}^m \left(\frac{\alpha'_{ij} p' \cdot e_i p_j}{\alpha_{ij} p \cdot e_i p'_j} \right)^{\alpha_{ij}} \leq \sum_{j=1}^m \frac{p' \cdot e_i}{p \cdot e_i} \max_{\alpha'_i} \frac{\alpha'_{ij}}{p'_j} \max_{\alpha_i} p_j$$

and then try to bound the j – th term in this sum by choosing a particular normalisation for p, p' . The following proposition contains three parts: first, it shows that p_j attains its maximum at $\alpha_i = (0, \dots, 1, \dots, 0)$ where the 1 is the j – th element in the vector; second α'_{ij}/p'_j also attains its maximum at $\alpha'_i = \alpha_i = (0, \dots, 1, \dots, 0)$. Finally, the budgets $p \cdot e_i$ and $p' \cdot e_i$ are equal for any choice of α_i, α'_i .

Proposition 4.3.5. *i) Normalise $p_j = 1$, then $p_j(\alpha_i)$ reaches its maximum when*

$$\alpha_{ij} = 1 \text{ and } \alpha_{ik} = 0 \text{ for all } k \neq j.$$

ii) Normalise $p'_j = 1$, then $\alpha'_{ij}/p'_j(\alpha'_i)$ reaches its maximum when $\alpha'_{ij} = 1$ and $\alpha'_{ik} = 0$ for all $k \neq j$.

iii) Let A be the “exponents matrix” i.e. it contains the strategies for each player in such a way that column $i \in [n]$ contains α_i and so A is an $m \times n$ matrix. Similarly, E is the “endowment matrix”, where columns are indexed by agents and rows by commodities, so that it is $m \times n$. Consider the matrices $EA^T - I_m$ and $E(A')^T - I_m$ and their adjugates, $\text{Adj}(EA^T - I_m)$ and $\text{Adj}(E(A')^T - I_m)$ respectively. Then the first row of the adjugate matrices contains the equilibrium price vectors p and p' (upto a nonzero constant) and moreover, $p \cdot e_i = p' \cdot e_i$.

Proof. For the first two points in the proof, without loss of generality we consider the case where $j = m$ i.e. $p_m = 1$.

i) Let $p(\alpha)$ be the equilibrium price when players report strategies according to $\alpha = (\alpha_1, \dots, \alpha_n)$. We can see a strategy as follows: $\alpha_{ij} = \alpha_{ij} / \sum_{k=1}^m \alpha_{ik}$ for all $i \in [n], j \in [m]$ i.e. we transform the explicit constraint to an implicit one. Next, let $\hat{p}$ and $\hat{\alpha}_i$ be vectors with (the first) $m - 1$ components and $\hat{z}$ is the excess demand function for the

first $m - 1$ commodities. By the implicit function theorem

$$D_{\hat{\alpha}_{im}} \hat{p}(\alpha) = - [D_{\hat{p}} \hat{z}(p(\alpha); \alpha)]^{-1} D_{\hat{\alpha}_{im}} \hat{z}(p(\alpha); \alpha)$$

By Proposition 17.G.3 in [46], $[D_{\hat{p}} \hat{z}(p(s); \alpha)]^{-1}$ has all its entries negative.

$D_{\hat{\alpha}_{im}} \hat{z}(p(\alpha); \alpha)$ is a negative vector, this completes the proof.

ii) Since $p_m = 1$, trivially we need $\alpha_{im} = 1$.

iii) We now choose a particular normalisation for p and show that we have $p \cdot e_i = p' \cdot e_i$. We apply an argument along the lines of [71]. First, notice that an equilibrium price vector p satisfies (see [20])

$$p^T (EA^T - I_m) = 0$$

We have, where $e_j \alpha_k$ is a shorthand notation for $\sum_{i=1}^n e_{ij} \alpha_{ik}$,

$$EA^T - I_m = \begin{bmatrix} e_1 \alpha_1 - 1 & e_1 \alpha_2 & \dots & e_1 \alpha_m \\ e_2 \alpha_1 & e_2 \alpha_2 - 1 & \dots & e_2 \alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_m \alpha_1 & e_m \alpha_2 & \dots & e_m \alpha_m - 1 \end{bmatrix}$$

As argued in [71], the rows of $\text{Adj}(EA^T - I_m)$ are proportional to p . We only need to make sure that the “proportionality factor” c is not 0. The $(1, j) - th$ entry of $\text{Adj}(EA^T - I_m)$ is what we get when we compute the determinant of the matrix that we get when removing the first column and $j - th$ row from $EA^T - I_m$ if j is odd and -1 times this determinant when j is even. For example, for p_1 , because of the assumption,

$$\begin{vmatrix} e_2 \alpha_2 - 1 & e_2 \alpha_3 & \dots & e_2 \alpha_m \\ e_3 \alpha_2 & e_3 \alpha_3 - 1 & \dots & e_3 \alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_m \alpha_2 & e_m \alpha_3 & \dots & e_m \alpha_m - 1 \end{vmatrix}$$

is a (strictly) diagonal matrix and therefore, nonsingular by the Levy–Desplanques theorem. This shows that the above determinant is nonzero and so, up to the sign,

the equilibrium price of commodity 1, p_1 . Therefore $c \neq 0$ and we conclude that up to a (possibly negative) scalar, we have the equilibrium price p in the first row of $\text{Adj}(EA^T - I_m)$.

We note that, instead of computing equilibrium prices, we could just compute $p \cdot e_i$ directly by replacing the first column in $EA^T - I_m$ by e_i and computing the determinant (with respect to the first column) of the resulting matrix:

$$p \cdot e_i = \frac{1}{c} \begin{vmatrix} e_{i1} & e_1\alpha_2 & \dots & e_1\alpha_m \\ e_{i2} & e_2\alpha_2 - 1 & \dots & e_2\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_{im} & e_m\alpha_2 & \dots & e_m\alpha_m - 1 \end{vmatrix}$$

We now prove the claim by noting that $p \cdot e_i = p' \cdot e_i$ if and only if this property is true when we change just two entries in α_i and leave the others unchanged i.e. $\alpha'_j = \alpha_j + \delta$ and $\alpha'_k = \alpha_k - \delta$ for $0 \leq \delta \leq \alpha_k$ for $j, k \in [m], j \neq k$ and $\alpha'_r = \alpha_r$ for all $r \in [m] \setminus \{j, k\}$.

We distinguish two cases depending on whether $1 \in \{j, k\}$. First suppose that $j = 1$ and without loss of generality assume $k = 2$. Let δ be the vector with zeroes except at position i where it equals δ . Then we have

$$\begin{aligned} p' \cdot e_i &= \frac{1}{c} \begin{vmatrix} e_{i1} & e_1(\alpha_2 - \delta) & \dots & e_1\alpha_m \\ e_{i2} & e_2(\alpha_2 - \delta) - 1 & \dots & e_2\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_{im} & e_m(\alpha_2 - \delta) & \dots & e_m\alpha_m - 1 \end{vmatrix} \\ &= \frac{1}{c} \left(\begin{vmatrix} e_{i1} & e_1\alpha_2 & \dots & e_1\alpha_m \\ e_{i2} & e_2\alpha_2 - 1 & \dots & e_2\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_{im} & e_m\alpha_2 & \dots & e_m\alpha_m - 1 \end{vmatrix} + \begin{vmatrix} e_{i1} & -\delta e_{i1} & \dots & e_1\alpha_m \\ e_{i2} & -\delta e_{i2} & \dots & e_2\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ e_{im} & -\delta e_{im} & \dots & e_m\alpha_m - 1 \end{vmatrix} \right) \\ &= p \cdot e_i \end{aligned}$$

and the conclusion follows; the case where $j, k \neq 1$ is similar: adding up columns j, k in the expression for the determinant does not change the value of $p \cdot e_i$ and $p' \cdot e_i$ and

after that we can compute the determinant by splitting it as above. $\square$

Theorem 4.3.6. *The incentive ratio for Cobb–Douglas markets is at most m .*

Proof. By the results presented in Proposition 4.3.5,

$$\begin{aligned} \frac{u_i(x'_i)}{u_i(x_i)} &= \prod_{j=1}^m \left(\frac{\alpha'_{ij} p' \cdot e_i p_j}{\alpha_{ij} p \cdot e_i p'_j} \right)^{\alpha_{ij}} \leq \prod_{j=1}^m \left(\frac{1}{\alpha_{ij}} \frac{p' \cdot e_i}{p \cdot e_i} \max_{\alpha'_i} \frac{\alpha'_{ij}}{p'_j} \max_{\alpha_i} p_j \right)^{\alpha_{ij}} \\ &\leq \prod_{j=1}^m \left(\frac{1}{\alpha_{ij}} \right)^{\alpha_{ij}} \leq m, \end{aligned}$$

where the last step follows from the weighted AM–GM inequality. $\square$

Before concluding, we summarize the results presented here in Table 4.1.

Table 4.1: Upper bounds on the incentive ratio in $n \times m$ exchange economies. The bounds for Fisher markets come from [19], [18].

Market	Fisher	Exchange
Leontief	2	∞
Linear	2	∞
Cobb–Douglas (Without Assumption 4.3.2, $n = 2$, $m = 2$)	$e^{1/e}$	∞
Cobb–Douglas (With Assumption 4.3.2, any n , $m = 2$)	$e^{1/e}$	$e^{1/e}$
Cobb–Douglas (With Assumption 4.3.2, any n , $m \geq 3$)	$e^{1/e}$	m

4.4 Conclusion

This chapter surveyed the concept of incentive ratio in the much more general model of exchange economies as compared to Fisher markets. Results in Fisher markets were encouraging: the maximum gains from strategic behaviour were bounded by reasonably small constants and therefore equilibrium mechanisms could be expected to work rather well, meaning that the profits from (computationally challenging) strategic behaviour were small relative to the costs, and thus, not worthwhile on most occasions. In other words, the equilibrium mechanism in Fisher markets is quite robust against strategic behaviour. However, the results here indicate that in the more general setup of exchange economies, results are diametrically different and

without further restrictions, ratios in linear, Leontief and Cobb–Douglas markets are unbounded.

The Cobb–Douglas case demonstrates that, when equilibrium prices (and hence allocations and utilities) are unique, the incentive ratio is bounded by the number of commodities. Therefore it may be argued that, unlike in Fisher markets, gains from strategic behaviour can be significant and manipulation could be worthwhile. The fact that the bound m is unlikely to be tight (as the case $m = 2$ demonstrates) suggests that a better upper bound may be found. This, the incentive ratio in linear markets when there exist no agent with utility function identically zero, and the question of the incentive ratio when agents are also allowed to misreport their endowments in Cobb–Douglas markets, are left as directions for future research.

Chapter 5

The facility location problem on a sphere

5.1 Introduction

We consider a variation to the facility location problem considered in [88]. In our setup, there is heterogeneity among agents: some agents may find a particular facility desirable, whereas others may find it undesirable. The results presented here allow for a different scoring system between the two groups. One could think of a factory: for some agents, it is convenient to live close to work, whereas others view it as a source of noise or pollution.

As in [88], we consider a sphere as the facility's destination. There, the method used to score locations is the Borda count, extended to the case with infinitely many alternatives, where the score of a location is based on the surface area of the locations at a certain geodesic distance from a given point. Subsequently it is shown that, if the facility is desirable (undesirable), the optimal location for the facility is the (antipode of the) center of population of the voters if the voters' average coordinate is not at the centre of the sphere, and any location is optimal, otherwise. Such an optimal location is Pareto optimal, but generally not Benthamite or Rawlsian optimal. Another contribution where the facility has to be placed on a sphere is [38], where the optimal location is the one where a weighted sum of distances to a set of given points is minimised. Three different such distances are considered and it is shown that

minimisers may not be unique. Moreover, an algorithm for computing minimisers is presented and its convergence is studied. Other notable papers where the space of interest is not a sphere include [39], where agents bargain about the optimal location on a tree and use the unanimity rule to choose the best destination. In [33], the optimal location found under voting (by using the Condorcet criterion, which stipulates that no other location is preferred by a majority of agents) is contrasted with the Weber and Rawls optimal solution. The inverse Borda count is used in [28], so that a Borda winner is well-defined if the space under consideration is unbounded. Next, it is shown that Condorcet and Borda winners coincide in one dimension, but generally diverge in higher dimensions.

The results presented here are different in the sense that we allow for heterogeneity among agents' preferences. That is, we have two sets of agents, namely those who regard the facility as desirable, and those who view it as undesirable. In the literature it has so far been assumed that a facility is either desirable or undesirable for everyone. Moreover, we do not even presume that sets of agents apply the same scoring system e.g. the agents who find the facility desirable may use scoring based on the geodesic distance whereas the other group of agents may use the Euclidean distance. This, then, introduces a challenge for deciding an optimal location. We solve this by arguing that, since interests are opposed, an optimal location is a point where both groups of players do not prefer the antipode.

The remainder is organised as follows. Section 5.2 discusses the setup, notation and terminology and recalls some previous results from [88]. Section 5.3 presents the results and Section 5.4 concludes.

5.2 Setup: the sphere, locations and scoring functions

We use notation and definitions as used in [88], we recall them here for completeness. There are $n \geq 1$ agents. $\langle \cdot, \cdot \rangle: \mathbb{R}^3 \rightarrow \mathbb{R}$ is the Euclidean inner product. $d: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+$ is the induced distance on $\mathbb{R}^3$: $\forall x, x' \in \mathbb{R}^3, d(x, x') = \sqrt{\langle x - x', x - x' \rangle}$. The set

$X = \{x \in \mathbb{R}^3 \mid d(0, x) = 1\}$ represents the sphere centered at 0 with radius 1. An element $x \in X$ is called a location and the location of agent i ($i = 1, \dots, n$) is denoted by y^i . $\rho: X \times X \rightarrow \mathbb{R}_+$ is a distance function, e.g. the geodesic distance: $\forall x, x' \in X, \rho^G(x, x') = \arccos(\langle x, x' \rangle)$. Then $S: [0, \pi] \rightarrow \mathbb{R}$, defined as $\rho^G \mapsto 2\pi(1 - \cos \rho^G)$, calculates the surface area of locations whose geodesic distance from a given location are at most ρ^G . For all i , we set $\rho^i(x) = \rho(y^i, x)$. A facility is said to be desirable for agent i if location x' is at least as good as x if and only if $\rho^i(x') \leq \rho^i(x)$. The inequality is reversed for undesirable facilities. A represents the set of k agents that consider the facility desirable; B represents the set of $n - k$ agents that consider the facility undesirable. Given k ($n - k$) locations $(y^i)_{i \in A}$ ($(y^i)_{i \in B}$), we denote $\bar{y}_A = \sum_{i \in A} \frac{y^i}{k}$ ($\bar{y}_B = \sum_{i \in B} \frac{y^i}{n-k}$) the average coordinate of A (B). Assuming $\bar{y}_A(\bar{y}_B) \neq 0$, $y_A^* = \frac{\bar{y}_A}{d(0, \bar{y}_A)}$ ($y_B^* = \frac{\bar{y}_B}{d(0, \bar{y}_B)}$) is A 's (B 's) center of population. $\leq$ is used for the usual vector inequalities: if $x, x' \in \mathbb{R}^2$ then $x \leq x'$ if and only if $x_1 \leq x'_1$ and $x_2 \leq x'_2$.

We let

$$f: X \rightarrow \mathbb{R}^2$$

$$x \mapsto (f_A(x; (y^i)_{i \in A}), f_B(x; (y^i)_{i \in B}))$$

be a continuous function, associating to each point on the sphere a pair of scores for A and B . The higher the score, the more preferred the location e.g. if $f_A(x) \geq f_A(x')$ then the group of agents that considers the facility to be desirable prefers location x to location x' . It is important to amplify that we do not require f_A and f_B to use the same scoring or underlying notion of distance. For example, it may be that the agents in A compute a Borda count based on their scoring on geodesic distance and the agents in B use the sum of the individual Euclidean distances i.e. $f_B(x) = \sum_{i \in B} \|y^i - x\|$. In case $B = \emptyset$, we may think of f as

$$f: X \rightarrow \mathbb{R}^2$$

$$x \mapsto (f_A(x), 0)$$

or simply as $f: X \rightarrow \mathbb{R}$, and similarly when $A = \emptyset$. In [88] the location with the highest Borda (based on geodesic distance) count is computed for the desirable and undesirable facility problem:

$$\max_{x \in X} 2n\pi + 2k\pi \sum_{i \in A} \langle y^i, x \rangle - 2(n-k)\pi \sum_{i \in B} \langle y^i, x \rangle \quad (5.1)$$

which can be solved by setting up the Lagrangian:

$$x_A^* = \frac{n\pi \bar{y}_A}{d(0, n\pi \bar{y}_A)} \quad (k = n) \quad (5.2)$$

$$x_B^* = \frac{-n\pi \bar{y}_B}{d(0, -n\pi \bar{y}_B)} \quad (k = 0) \quad (5.3)$$

assuming the above expression are not of the form 0/0. If it is, any point is a maximiser. This shows that A and B would prefer antipodal optimal locations if their average coordinates coincide, and generally, that their optimal choices are “nearly antipodal” if their average coordinates are almost the same. On the other hand, if the average coordinates are themselves antipodal, there is unanimity about where to put the facility. In case A or B is empty, the problem is just an optimisation problem.

5.3 Results for the case of heterogeneous agents

Let

$$T = \{x \in X \mid f(x) \geq f(-x)\}.$$

T contains the locations that have the desirable property that neither group of agents prefers its antipode.

Lemma 5.3.1. *T is nonempty; in particular when f is the Borda count based on the geodesic distance, there exists $\tilde{x} \in T$ orthogonal to $\bar{y}_A$ and $\bar{y}_B$.*

Proof. $f: X \rightarrow \mathbb{R}^2$ is a continuous function from the sphere to $\mathbb{R}^2$ and therefore by the Borsuk–Ulam theorem, there exists $\tilde{x} \in X$ such that $f(\tilde{x}) = f(-\tilde{x})$. If f is given

by

$$x \mapsto (2k\pi + \sum_{i \in A} \langle y^i, x \rangle, 2(n-k)\pi - \sum_{i \in B} \langle y^i, x \rangle)$$

then

$$\begin{aligned} 2k\pi + 2k\pi \sum_{i \in A} \langle y^i, \tilde{x} \rangle &= 2k\pi + 2k\pi \sum_{i \in A} \langle y^i, -\tilde{x} \rangle && \Longleftrightarrow \\ \sum_{i \in A} \langle y^i, \tilde{x} \rangle &= 0 && \Longleftrightarrow \\ \langle \tilde{y}_A, \tilde{x} \rangle &= 0 \end{aligned}$$

and similarly for B . □

Furthermore T is closed and bounded, hence compact.

We let

$$T' = \{x \in T \mid f(x) \text{ is maximal for } \leq \text{ in } f(T)\}$$

be the set of Pareto efficient locations: they satisfy the mild, standard requirement that we cannot make at least one group better off while not making the other group worse off.

Proposition 5.3.2. *T' is nonempty.*

Proof. The images $f(T)$ form a partial order for $\leq$ and so since every chain contains an upper bound in $f(T)$, there exists at least one maximal element i.e. T' contains at least one Pareto optimal location from T . □

Scoring locations using the Borda count

In particular, when it is meaningful to add up $f_A(x)$ and $f_B(x)$ e.g. if they are both the Borda count of location x , we define

$$T'' = \left\{ x \in T \mid f_A(x) + f_B(x) = \max_{s \in T} f_A(s) + f_B(s) \right\}.$$

Hence T'' is nonempty as well and we get $T'' \subseteq T' \subseteq T$. Of course to find a location in T'' , a setup with heterogeneous agents using the Borda count can be

readily transformed into an equivalent problem with homogeneous agents.¹

Proposition 5.3.3. *If $A = \emptyset$ ($B = \emptyset$), then the solution in T'' coincides with (5.2) ((5.3)) from [88].*

Proof. We consider the desirable facility problem, the undesirable case is similar. In this case x^* is given by (5.2) and clearly $x^* \in T$. Since this point is a maximiser of $f_A(s)$, $x^* \in T' = T''$. □

Now consider the case $k = 1, n - k = 1$, $\bar{y}_A = \bar{y}_B = (1, 0, 0)$. Here interests are opposed and even though any point is a Borda maximiser, there seems no reason to favour A over B or vice versa. Hence it seems more plausible that the selected location is a “compromise” i.e. a location from the set $T = T' = \{x \in X \mid x_1 = 0\}$, for which neither agent prefers the antipodal and $f(x)$ is maximal.

This example also shows that the optimal location in general is not Benthamite or Rawlsian optimal for either group, as we may expect when interests are opposed.

5.4 Conclusion

We have provided a new solution for facility location problems on a sphere with a heterogeneous population of agents. A Pareto optimal location that is at least as good as its antipode has been shown to exist, making it a plausible candidate for groups with opposed interests, which would otherwise generally prefer locations that are far apart. Moreover, in case the agent population is homogeneous, the proposed set of solutions contains the optimal location proposed in previous research and can thus still be selected as the desired location.

¹As pointed out by referees.

Chapter 6

Mergers and social choice

This chapter contains work in progress including some minor results.

6.1 Leading merger in a Stackelberg oligopoly: the case of asymmetric firms

6.1.1 Introduction and initial setup: classical Cournot

We extend the model of [44] to allow for firms with asymmetric unit production costs. We obtain their results as a special case to the model. There are $m + n$ asymmetric firms, m of them have the option to merge. The m merging firms are referred to as “insiders”, the others n as “outsiders”, and we assume that the outsiders are numbered $1, \dots, n$. Firm i can produce a nondifferentiable good at a per unit cost of $c_i \geq 0$ and chooses its desired output $q_i \geq 0$ to maximise its profits, given the decision of the other firms. Let $Q = \sum_{i=1}^{m+n} q_i$ represent total output (or industry output) under Cournot competition, and $P(Q)$ the inverse demand function satisfying $P'(Q) < 0$ for all $Q \geq 0$. Each firm simultaneously solves:

$$\max_{q_i \geq 0} (P(Q) - c_i)q_i \tag{6.1}$$

We let C and S superscripts on variables refer to Cournot and Stackelberg equilibrium quantities, respectively. In the absence of mergers, all firms choose q_i simul-

taneously, and if (6.1) holds for all firms, the vector

$(q_1^C, \dots, q_n^C, \dots, q_{m+n}^C) \in \mathbb{R}_+^{m+n}$ is said to be a Cournot equilibrium.

We will contrast the Cournot equilibrium with the equilibrium in a merger-Stackelberg game. That is, we suppose that a merger of m firms will take place, and that these m firms subsequently become the market leader: they choose their output $\bar{q}$ and the remaining n firms react to this. Industry output then equals $Q = \bar{q} + \sum_{i=1}^n q_i$. Without loss of generality, we shall assume $c_1 \leq c_2 \leq \dots \leq c_n$. The Cournot equilibrium price is $p^C = P(Q^C)$. For convenience, we will use the notation $\hat{q}^C = \sum_{i=n+1}^{m+n} q_i^C$, i.e. the aggregate production of the m insiders in Cournot equilibrium prior to the merger. Finally, $\Pi_m^C = \sum_{i=n+1}^{m+n} (p^C - c_i) q_i^C$ is the total profit of the merging insiders in the Cournot model. Let $\bar{c} \geq 0$ be the Stackelberg leader's unit production costs.

Cournot equilibrium

The Cournot equilibrium solves $k \leq m + n$ first order conditions simultaneously

$$P(Q) - c_i + q_i P'(Q) = a - b \sum_{i=1}^{m+n} q_i - c_i - b q_i = 0 \quad (6.2)$$

for $q_i \geq 0$, and for the other $m + n - k$ firms we must have

$$P(Q) - c_i + q_i P'(Q) = a - b \sum_{i=1}^{m+n} q_i - c_i - b q_i \leq 0$$

and $q_i = 0$. As before, we have that if $c_i < c_j$ and $q_i = 0$ then $q_j = 0$.

We may write the solution for the k firms as a matrix equation.

$$\begin{bmatrix} 2b & b & \dots & b \\ b & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b & \dots & \dots & 2b \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} = \begin{bmatrix} a - c_1 \\ a - c_2 \\ \vdots \\ a - c_k \end{bmatrix}$$

And therefore

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} = \begin{bmatrix} \frac{a + \sum_{j=1}^k c_j - (k+1)c_1}{b(k+1)} \\ \frac{a + \sum_{j=1}^k c_j - (k+1)c_2}{b(k+1)} \\ \vdots \\ \frac{a + \sum_{j=1}^k c_j - (k+1)c_k}{b(k+1)} \end{bmatrix}$$

6.1.2 Merger to Stackelberg leader

In the Stackelberg game, given $\bar{q} > 0$, every outsider $i = 1, \dots, n$ tries to solve

$$P(Q) - c_i + q_i P'(Q) = 0 \quad (6.3)$$

for $q_i \geq 0$, $i = 1, \dots, n$. The above expression gives the derivative of firm i 's profit function. It may be negative for any $q_i \geq 0$, and in that case the optimal choice is $q_i = 0$.

Proposition 6.1.1. *If $(\bar{q}, q_1^S, \dots, q_n^S)$ is a Stackelberg equilibrium then it is the solution to the following problem. If $\bar{q} > 0$, the converse is true as well, a solution exists and is unique.*

$$\begin{aligned} & \underset{\bar{q} \geq 0}{\text{maximize}} \quad (P(Q) - \bar{c})\bar{q} \\ & \text{subject to} \quad P(Q) + q_i P'(Q) - c_i \leq 0, \quad i = 1, \dots, n. \\ & \quad \quad \quad q_i \geq 0, \quad i = 1, \dots, n. \end{aligned}$$

Proof. ($\Rightarrow$): immediate

($\Leftarrow$): A consequence of the fact that P is strictly decreasing: if $P(Q) - c_i + q_i P'(Q) < 0$ then $q_i = 0$, since $P'(Q) < 0$ for all $Q \geq 0$.

A solution exists if the constraint set is compact since we may assume $\bar{q}, q_i \leq a/b$ for all $i = 1, \dots, n$ and is unique due to the concavity of the objective function. $\square$

Henceforth, we focus on the case where $P(Q) = a - bQ$. We assume without loss of generality that $c_i < a$ for $i = 1, \dots, m + n$, since else $q_i = 0$ in equilibrium (for both Cournot and Stackelberg). The Karush–Kuhn–Tucker (KKT) conditions for the above optimisation problem are both necessary and sufficient. The associated KKT

conditions are ($i = 1, \dots, n$):

$$\left\{ \begin{array}{l} a - b \sum_{i=1}^n q_i - 2b\bar{q} - \bar{c} + \bar{\lambda} + b \sum_{i=1}^n \lambda_i = 0 \\ -b\bar{q} + 2b\lambda_i + b \sum_{j=1, j \neq i}^n \lambda_j + \lambda_{n+i} = 0 \\ \bar{\lambda}\bar{q} = 0 \\ \lambda_i(-a + b \sum_{i=1}^n q_i + b\bar{q} + c_i + bq_i) = 0 \\ \lambda_{n+i}q_i = 0 \\ -a + b \sum_{i=1}^n q_i + b\bar{q} + c_i + bq_i \geq 0 \\ \bar{q}, q_i \geq 0 \\ \bar{\lambda}, \lambda_i, \lambda_{n+i} \geq 0 \end{array} \right.$$

Suppose there are $0 \leq k \leq n$ followers who produce positive output. Notice that, in such scenario this must be the first k followers (i.e. the the k firms with lowest costs). To see this, suppose $c_i < c_j$ and suppose that, in equilibrium, $q_j > q_i = 0$. From the first order conditions, $P(Q) - c_i \leq 0$ and so $P(Q) \leq c_i < c_j$, but then $P(Q) - c_j < 0$, so that $q_j = 0$ would have been optimal for follower j .

It may be verified that the following values satisfy the KKT conditions if $\bar{q} > 0$:

$$\left\{ \begin{array}{ll} \bar{q} = \frac{a + \sum_{i=1}^k c_i - (k+1)\bar{c}}{2b} & \\ q_i = \frac{a - (2k+1)c_i + \sum_{j=1, j \neq i}^k c_j + (k+1)\bar{c}}{2b(k+1)} & (i = 1, \dots, k) \\ q_i = 0 & (i = k+1, \dots, n) \\ \lambda_1 = \dots = \lambda_k = \frac{a + \sum_{i=1}^k c_i - (k+1)\bar{c}}{2b(k+1)} & \\ \lambda_{k+1} = \dots = \lambda_n = 0 & \\ \lambda_{n+1} = \dots = \lambda_{n+k} = 0 & \\ \lambda_{n+k+1} = \dots = \lambda_{2n} = \frac{a + \sum_{i=1}^k c_i - (k+1)\bar{c}}{2(k+1)} & \\ \bar{\lambda} = 0 & \end{array} \right.$$

Provided that the solution satisfies:

- $a + \sum_{i=1}^k c_i - (k+1)\bar{c} \geq 0$

- $a - (2k + 1)c_i + \sum_{j=1, j \neq i}^k c_j + (k + 1)\bar{c} \geq 0$
- $P(Q) + q_i P'(Q) - c_i \leq 0, \quad i = 1, \dots, n.$

If no such k exists for $0 \leq k \leq n$, then $\bar{q} = 0$.

In this case industry output is

$$\bar{q} + \sum_{i=1}^k q_i = \frac{(2k + 1)a - \sum_{i=1}^k c_i - (k + 1)\bar{c}}{2b(k + 1)}$$

and the equilibrium price is

$$P\left(\bar{q} + \sum_{i=1}^k q_i\right) = \frac{a + \sum_{i=1}^k c_i + (k + 1)\bar{c}}{2(k + 1)}$$

Using this we can compute an upper bound on $\bar{c}$ such that the leader's profit $(P(Q) - \bar{c})\bar{q} \geq \Pi_m^C$. As an example, consider the case $\bar{c} \leq c_1 \leq \dots \leq c_n$ and the leader can produce its monopoly quantity $(a - \bar{c})/(2b)$ and $q_i = 0$ for $i = 1, \dots, n$. Producing the equilibrium quantity at cost $\bar{c}$ is profitable for the merging insiders when $(P(\bar{q}) - \bar{c})\bar{q} \geq \Pi_m^C \iff (a - \frac{a - \bar{c}}{2b} - \bar{c})\frac{a - \bar{c}}{2b} \geq \Pi_m^C \iff \frac{(a - \bar{c})^2}{4b} \geq \Pi_m^C$ so that being the leader is profitable if $0 \leq \bar{c} \leq a - 2\sqrt{b\Pi_m^C}$. This is intuitive since the higher the pre-merger profits for the insiders, the lower $\bar{c}$ needs to be in order to guarantee post-merger profitability.

From Cournot to Stackelberg

We can now recover the result from [44] for the case where $\bar{c} = c_1 = \dots = c_{m+n} = c > 0$, when initially all firms produce positive Cournot quantities and all n followers produce positive Stackelberg quantities:

Proposition 6.1.2. *Suppose $\bar{c} = c_1 = \dots = c_n = \dots = c_{m+n} = c > 0$. The leading merger would increase industry output and benefit consumers if and only if $m < n + 1$*

Proof. The Cournot industry output is $\frac{(m+n)(a-c)}{(m+n+1)b}$, while the Stackelberg industry output is $\frac{(2n+1)(a-c)}{2b(n+1)}$. Therefore, industry output grows and consumers benefit if and

only if

$$\frac{(2n+1)(a-c)}{2b(n+1)} > \frac{(m+n)(a-c)}{(m+n+1)b} \iff n+1 > m \quad \square$$

6.2 On the range of the Borda count

This section is the result of research carried out together with Matthew Lou-Magnuson; in particular, the Julia code in Appendix A is all written by him.

The inspiration for this part comes from [40], and the central question is whether any ordering over alternatives can be attained as the image by the Borda rule of a collection of individual preference orders. The authors settle all cases except the case where the number of alternatives/candidates, m , is a multiple of 4 and the number of voters, n , is odd. They conjecture that any ordering can be attained by an appropriate preference profile. We will now make this question more formal and use the notation as used in [40].

$N = \{1, \dots, n\}$ is a finite set of voters, $n \geq 2$ and X is a set of alternatives or candidates, $|X| = m \geq 2$. An ordering R (on X) is called *strong* if it is reflexive, complete, transitive and asymmetric. The set of all such strong orderings on X is denoted by $L(X)$. A *profile* is an ordered tuple $u = (u(1), u(2), \dots, u(n)) \in L(X)^n$ of strong orderings. If a voter ranks an alternative at the k -th ($1 \leq k \leq m$) position of his ordering (in descending order of preference), he awards this alternative k points in the Borda ranking; the Borda score of an alternative x , $S(u, x)$, is the sum of the scores by all individuals. The Borda ranking at u is a binary relation $\succeq$ on X such that $x \succeq y$ if and only if $S(u, x) \geq S(u, y)$. Borda's rule f_B is a mapping from $L(X)^n$ to the set of all weak orderings on X . The outcome of Borda's rule can be depicted also weak ordering $\succeq$ on subsets of X :

$$X_1 \succ X_2 \succ \dots \succ X_T$$

As usual, $\succ$ is the asymmetric part of $\succeq$. Here, the X_i form a partition X and

two alternatives are in the same X_i if and only if they have the same Borda score. The alternatives in X_i have Borda scores strictly greater than those in X_j if and only if $i > j$. $\mathcal{L}$ is used to denote a weak ordering $X_1 \succ X_2 \succ \dots \succ X_T$, and each X_i is referred to as a level of $\mathcal{L}$. Since the Borda rule is neutral with respect to candidates, we may ask questions about $\mathcal{L}$ by just considering the cardinalities $(m_1, m_2, \dots, m_T)$ of the X_i 's, for example, $(1, 2, 3, 4)$ refers to a profile that generates a single winner, followed by two candidates with the same score, then 3 candidates with equal score and finally 4 candidates with the same Borda count i.e. $X_1 = 1$, $X_2 = 2$, $X_3 = 3$ and $X_4 = 4$. For simplicity, we will just say $\mathcal{L} = (m_1, m_2, \dots, m_T)$. Finally, the index of $\mathcal{L}$, denoted by $I(\mathcal{L})$, is defined as follows: let 2^k be the largest power of 2 dividing all the m_i ($k \in \mathbb{N}$) i.e. $\mathcal{L} = (m_1, m_2, \dots, m_T) = (2^k s_1, 2^k s_2, \dots, 2^k s_T)$. Then $I(\mathcal{L}) = s_1 + s_2 + \dots + s_T = m/2^k$.

The question that we wish to answer is essentially about the range of the Borda rule. The authors settle all cases, except when n is odd, m is even and $I(\mathcal{L})$ is even (hence m is a multiple of 4). The only result they present is when all the s_i are odd and in this case for every odd $n \geq 3$, there exists $u \in L(X)^n$ such that $f_B(u) = \mathcal{L}$. Furthermore, the authors claim they have many numerical examples for all $m < 24$.

We attempt to find a counterexample using the Julia code in Appendix A, but to run it on larger examples ($m \geq 24$), we would either need to be able to speed up the search or use much more powerful hardware.

6.3 On Condorcet winning sets of size at least four

This section is based on an open question in [25] about Condorcet winning sets. In this paper, the classical concept of a Condorcet winner (a candidate that beats any other candidate in a pairwise contest) is extended to sets, called Condorcet winning sets, which have the property that for any other alternative outside the set, a majority of the voters prefers at least one candidate inside the set.

6.3.1 Introduction

In elections, Condorcet winners are candidates that can beat any other candidate in a pairwise contest. Condorcet winners do not always exist as even small examples illustrate. An attractive property of Condorcet winning sets, introduced in [25], is that they overcome this difficulty: a Condorcet winning set exists in any election. A formal definition follows after first introducing the setup, which is taken from [25].

Given a set of alternatives or candidates $X = \{x_1, \dots, x_m\}$ and a set of voters $N = \{1, \dots, n\}$, we can associate a linear order $\succ_i$ to every voter $i \in N$, which is referred to as his preference order. A vector $(\succ_1, \dots, \succ_n)$ of voters' preference orders is called a preference profile and is denoted by P .

A candidate $x \in X$ beats another candidate $y \in X$ in a pairwise election if a majority of the voters prefers x to y . If they exist, Condorcet winners are unique. Next, we state the definition of a Condorcet winning set.

Definition 6.3.1 (Condorcet winning set). *Given an election over X with a preference profile $P = (\succ_1, \dots, \succ_n)$, a set $Y \subseteq X$ is said to cover an alternative $z \in X \setminus Y$ if*

$$\#\{i \in N \mid \exists y \in Y \text{ such that } y \succ_i z\} > \frac{n}{2}$$

Y is a Condorcet winning set if Y covers each alternative in $X \setminus Y$.

From the definition it is clear that X is a Condorcet winning set and that any superset of a Condorcet winning set is itself also a Condorcet winning set i.e. Condorcet winning sets are “upwards closed”. Therefore we are interested in minimal Condorcet winning sets. The Condorcet dimension of a preference profile P , the minimum size of Condorcet winning sets, is denoted by $\dim_C(P)$.

Condorcet winning sets of size 1 are simply Condorcet winners; it is known that Condorcet winning sets of size 2 and 3 exist. A profile of Condorcet dimension 3 has been provided in [25]. In [30] it is shown that there exists a profile of Condorcet dimension 3 with 6 voters and 6 candidates, and that this is in fact the smallest possible example.

6.3.2 An improved lower bound for profiles of dimension at least four

A “pressing” open question posed by the authors in [25] is whether profiles of dimension 4 or higher exists. We present an easy proof of one of their results that does not resort to tournaments and dominating sets; cf. Corollary 1 in [25]. Henceforth assume n is odd, so that every pairwise contest has a winner.

Proof. If there is no Condorcet winning set of size 3, then each candidate x loses in a pairwise competition against *at least* seven other candidates. The reason is that else we can partition these other candidates into two groups of size at most 3, and include two candidates u, v that beat the two groups and then $\{x, u, v\}$ forms a Condorcet winning set of size at most 3. Notice that there are $m(m-1)/2$ losses (and wins) resulting from the pairwise contests. Now consider the candidate that loses the least pairwise contests. Clearly, he loses at most $(m-1)/2$ times. Therefore we must have $(m-1)/2 \geq 7$ and the conclusion follows. $\square$

We can use this last observation as follows. This candidate with the least number of pairwise losses, say u , wins at least $m - (m-1)/2 = (m+1)/2$ pairwise contests. Remove u , plus all candidates that he beats, and consider a new contest with the $(m-3)/2$ remaining candidates. Again, there must be a candidate, say v , who wins at least $((m-3)/2 + 1)/2 = (m-1)/4$ pairwise contests; remove this candidate and everyone he beats. Then, there are $(m-9)/4$ candidates remaining and so there must exist at least one, say w , with $((m-9)/4 + 1)/2 = (m-5)/8$ wins against the candidates that are by now remaining. Then $\{u, v, w\}$ attains a total of at least $(7m-3)/8$ pairwise wins against other candidates, but this ignores synergetic effects from adding u, v, w together into one set, so the total number of wins is possibly higher.

From this, and the fact that the pairwise wins between u, v and w do not matter if they are a candidate Condorcet winning set, we have the following result.

Proposition 6.3.2. *If n is odd and $m \leq 21$, then a Condorcet winning set has size at most 3.*

Proof. As outlined above, there exists a set $\{u, v, w\}$ that defeats at least $(7m - 3)/8$ other candidates pairwise. If there are m candidates, then such a set of size 3 must obtain $m - 3$ pairwise wins to be a Condorcet winning set, i.e. if we have $(7m - 3)/8 \geq m - 3$ or when $m \leq 21$, this is the case. $\square$

Code written in Python used to search for Condorcet winning sets in an m candidates, n voters election can be found in Appendix B. Each of the n voters is given a random preference order and the programme then verifies if there exists a Condorcet winning set of size three.

Chapter 7

Conclusion

This thesis presented several results in the realms of strategic behaviour. The variety of topics treated here indicates the multitude of directions in which considerations of strategy, equilibrium and optimality, in very precise senses, play a role. This clearly shows the enormous versatility of game theory; it is ubiquitous and easily transcends the borders of research areas.

Chapters 2 and 3 have shown how firms need to behave strategically in their decisions when facing uncertainty. Building on maximum principles, we have exposed a sharp contrast between the lagged and the no lag settings in Cournot games. When demand is between certain bounds and there is no lag, we get a whole range of equilibria that reduces to single equilibrium even at the smallest delay. This discontinuity suggests a very strong difference for firms facing potentially uncertain demand: when there are many equilibria, it is not as easy to determine output compared to the case where there is a single equilibrium. The sequential setup on the other hand, demonstrated a certain smoothness between the lagged and no lag scenarios. It may therefore be argued that all the details in a market matter: the structure (leader-follower with endogeneous retailer costs vis-à-vis direct competition with exogeneous costs), delay vis-à-vis no delay in information flow and access to information as given by the filtration, the actual level of demand et cetera.

The results on the incentive ratio in Chapter 4 have indicated that the encouraging results from Fisher markets do not extend to exchange economies. This is important, because it showed that strategic behaviour is worthwhile even if it's difficult to compute

an optimal manipulating strategy. There was also good news: for Cobb–Douglas economies, under some mild assumptions, incentive ratios are bounded by the number of commodities in the market. For any given economy this is a constant, and based on the two commodities case, there is reason to believe that even better bounds may be found.

Chapter 5 has touched upon another area of interest: optimal location problems. It is clear that, when agents’ preferences are diverse and unanimity cannot be obtained, some compromise needs to be found. But given the choice of antipodal locations chosen by groups in isolation, this is a daunting task. This motivates the desire to have “stability by antipodality” i.e. to find a location such that none of the groups would rather move the facility to the antipode of the location. Moreover, it seems natural to desire that this compromise at least satisfies Pareto optimality, so that no location can be found that is at least as good for both groups and even strictly so for at least one group. We have shown that on a sphere, such a location always exists and this would be a plausible choice for the facility.

Future Research

There is ample reason to believe that the topics discussed in this thesis are part of a thriving field of research: game theory has many applications and some call for new theoretical results e.g. regarding existence of equilibria, or new kinds of equilibria. As a result, there are many possibilities for extensions. To name but a few, for the incentive ratio, we could study exchange economies in which it is possible for groups of agents (rather than just individual agents) to behave strategically and measure the consequences of such behaviour. Another good direction would be to see if the incentive ratio is bounded for other markets where the excess demand function satisfies the weak gross substitution property.¹ The maximum principles could be used in much more general settings e.g. where the demand is controlled, or where agents possess different information as reflected by their filtrations. Another obvious road would be to consider more than two agents, perhaps even a continuum of agents and evaluate

¹I am grateful to a referee for pointing out these possibilities.

the consequences of this new framework. For the facility location problems, we could switch direction to different spaces e.g. nonconvex ones (such as a torus) or unbounded ones e.g. cones. Certainly for the undesirable facility problem such unbounded spaces are of interest. Also, we could think of introducing a smoothing of the preferences of agents. Now, agents consider the facility either desirable or undesirable, with no position in between. This would make the whole problem a lot more realistic. Another opportunity would be to introduce multiple facilities, with or without constraints e.g. the requirement that they must be placed in each other's vicinity. Finally, Chapter 6 provides many opportunities in the areas of mergers and voting theory.

List of publications related to this thesis

1. I. Polak. The Incentive Ratio in Exchange Economies. *Combinatorial Optimization and Applications, 10th International Conference, COCOA 2016, Hong Kong, China, December 16-18, 2016, Proceedings*. [67].
See also [68] for the full length version.
2. I. Polak & N. Privault. Cournot games with limited demand: from multiple equilibria to stochastic equilibrium. *Submitted to Journal of Economic Dynamics & Control*
3. I. Polak & N. Privault. A stochastic newsvendor game with dynamic retail prices. *Submitted to Journal of Industrial and Management Optimization*.

Appendices

Appendix A

Julia code pertaining to “On the range of the Borda count”

This code was written by Matthew Lou-Magnuson.

```
####FACTORIAL NUMS
function inc_fac(fac)
    rev = reverse(fac)
    rev[1] += 1
    stop_i = length(rev)
    for i in 1:stop_i
        if rev[i] > i-1 && i < stop_i
            rev[i] = 0
            rev[i+1] += 1
        end
        if rev[i] > i-1 && i == stop_i
            rev = push!(rev, 1)
        end
    end
    return reverse(rev)
end
```

```

function fac_to_i(fac)
    eyes = copy(fac)
    end_i = length(eyes)

    for i in 0:end_i - 1
        value = eyes[end - i]
        position = end_i - i
        for j in position:end_i
            if eyes[j] >= value && j != position
                eyes[j] += 1
            end
        end
    end

    return eyes
end

```

```

function dec_to_fac(dec)
    places = []
    quotient = dec
    remainder = 0
    place = 1

    while quotient != 0
        remainder = quotient % place
        quotient = floor(quotient / place)
        push!(places, round(Int, remainder))
        place += 1
    end

    return reverse(places)
end

```

```

function i_val(eyes , base)
    return eyes
end

```

```

type State
    elements :: Array{Int8,1}
    stop :: Int128
    count :: Int128
    factorial :: Array{Int64,1}
end

```

```

function int_zeros(n)
    zeros = []
    for i in 1:n
        push!(zeros , 0::Int)
    end
    return zeros
end

```

####FACTORIAL STATE OBJECTS

```

function matt_fac(n)
    result = 1
    result = convert(Int128 , result)
    for i in 1:n
        result *= i
    end
    return result
end

```

```

function new_state(elements)

    return State(elements, matt_fac(length(elements)), 1,
        int_zeros(length(elements)))
end


function next_state(state)

    n_fac = inc_fac(state.factorial)
    n_count = state.count + 1
    if n_count > state.stop
        state.count = false
        return false
    end

    state.factorial = n_fac
    state.count = n_count
return state
end


function state_val(state)

    if state.count == false
        return false
    end

    return i_val(fac_to_i(state.factorial) , state.elements)
end


### ANALYSIS FUNCTIONS

function ballot_sum(bs)

    sum = bs[1]
    for b in bs[2:end]
        sum += b
    end

```

```

    end
    return sum
end

function bsum_outcome(bsum)
    sort!(bsum, rev=true)
    outcome = []
    count = 1
    i = 1
    while i < length(bsum)
        if bsum[i] == bsum[i+1]
            count += 1
            i += 1
        elseif bsum[i] != bsum[i+1]
            push!(outcome, count)
            count = 1
            i += 1
        end
    end
    push!(outcome, count)
    return outcome
end

function all_even(outcome)
    for level in outcome
        if level % 2 != 0
            return false
        end
    end
    return true
end

```



```

end

function k_p2(outcome, count = 1)
    step = outcome./2
    if all_even(step)
        return k_p2(step, count + 1)
    else
        return count
    end
end

function index(outcome)
    k = k_p2(outcome)
    return sum(outcome) / (2 ^ k)
end

function interesting(outcome)
    if all_even(outcome)
        if index(outcome) % 2 == 0
            return true
        end
    end
    return false
end

#####TEST CODE#####
init = [1,2,3,4,5,6,8]
v1 = new_state(copy(init))
v2 = new_state(copy(init))
v3 = new_state(copy(init))

```

```

outs = Dict()
while v1.count != false
    b1 = state_val(v1)
    v2.count = v1.count
    while v2.count != false
        b2 = state_val(v2)
        v3.count = v2.count
        if v2.count % 1000 == 0
            println("$ (v1.count) $ (v2.count) $ (v3.count)")
        end

        while v3.count != false
            b3 = state_val(v3)

            outcome = bsum_outcome(ballot_sum([b1,b2,b3]))
            if interesting(outcome)
                outs[outcome] = 1
            end

            next_state(v3)
            if v3.count == false
                v3 = new_state(copy(init))
                break
            end
        end
    end

    next_state(v2)
    if v2.count == false
        v2 = new_state(copy(init))
    end

```

```
        break
    end
end

next_state(v1)
if v1.count == false
    break
end
end

println(outs)
```

Appendix B

Python code pertaining to “On Condorcet winning sets of cardinality at least four”

```
import numpy as np
import itertools
from random import randint

def column_max(winning, k, c):
    M = pref_to_matrix(k)
    for i in winning:
        if M[i, c] == 1:
            return 1
    return 0

def win(winning):
    for j in range(0, nr_candidates, 1):
        result_new = 0
        if j not in winning:
            for k in range(0, nr_voters, 1):
```

```

        result_new += column_max(winning, k, j)
    if result_new < nr_voters / 2 + 1:
        return 0
    return 1

def pref_to_matrix(k):
    V = np.zeros((nr_candidates, nr_candidates))
    for i in range(0, nr_candidates, 1):
        for j in range(i+1, nr_candidates, 1):
            V[voters[k][i], voters[k][j]] = 1
    return V

for i in range(0, 1000, 1):
    print i
    nr_voters = 15
    nr_candidates = 23
    candidates = range(0, nr_candidates)

    result = np.zeros((nr_candidates, nr_candidates))
    voters = []

    tuples = list(itertools.combinations(candidates, 3))

    for k in range(0, nr_voters, 1):
        voters.append(np.random.permutation(candidates))

    for winning in tuples:
        x = win(winning)
        if (x == 1):

```

```
        break
if (x != 1):
    print 'm: ', nr_candidates
    print 'n: ', nr_voters
    for k in range(0, nr_voters, 1):
        print voters[k]
raw_input()
```

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