Sparse tensor product finite element method for some linear and nonlinear multiscale problems

Tan, Wee Chin

2018

Tan, W. C. (2018). Sparse tensor product finite element method for some linear and nonlinear multiscale problems. Doctoral thesis, Nanyang Technological University, Singapore.

<http://hdl.handle.net/10356/73206>

<https://doi.org/10.32657/10356/73206>

NANYANG
TECHNOLOGICAL
UNIVERSITY

**SPARSE TENSOR PRODUCT
FINITE ELEMENT METHOD FOR
SOME LINEAR AND NONLINEAR
MULTISCALE PROBLEMS**

TAN WEE CHIN

**DIVISION OF MATHEMATICAL SCIENCES
SCHOOL OF PHYSICAL AND MATHEMATICAL
SCIENCES**

2017

SPARSE TENSOR PRODUCT FINITE ELEMENT METHOD FOR SOME LINEAR AND NONLINEAR MULTISCALE PROBLEMS

TAN WEE CHIN

Division of Mathematical Sciences
School of Physical and Mathematical Sciences

A thesis submitted to Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy in Mathematical Sciences

2017

d

Acknowledgements

First and foremost, I would like to extend my heartfelt gratitude to my supervisor Associate Professor Hoang Viet Ha for the unrelenting support and guidance during my PhD studies. I am also thankful to him for providing me with the privilege of working in this interesting research area. Without his invaluable supervision, encouragement, and wisdom over these years, this thesis could not have reached its present form.

Second, I am very grateful to Agency for Science, Technology and Research for awarding me the A*STAR Graduate Scholarship (NTU/NUS). I am appreciative of the help from the IT staff, Soh Hwee Jin Melvin, for the help with the servers. I am also grateful to my colleagues and friends: She Qihao, Ian Flint, Quek Jia Hao and Xia Bingxing for their friendship and help. I am also grateful to Dr. Pham Quy Muoi of Da Nang University for his friendship and many fruitful discussions.

Lastly, I would like to thank my parents and girlfriend for their endless support during my PhD studies.

Abstract

Partial differential equations with multiple scales arise from a wide range of scientific and engineering problems, such as composite materials, oil flow and seismology. Solving multiscale partial differential equations are extremely difficult. Traditionally, numerical methods have to use a mesh size that is at most the order of the smallest scale to extract all the information in the microscopic scale. There has been extensive work to study multiscale partial differential equations and develop numerical methods to approximate the solutions. Although there is a large literature on the topic, there are still many challenges and open problems. This thesis contributes novel essentially optimal numerical methods for solving locally periodic multiscale monotone parabolic problems, multiscale linear and nonlinear equations that depend on microscopic scales in both space and time, and multiscale nonlinear monotone variational inequalities. We develop the sparse tensor product finite element (FE) method to approximate the solutions to the problems. The method approximates all the necessary microscopic and macroscopic information with a prescribed accuracy using an essentially optimal number of degrees of freedom. For locally periodic problems, this is better than other methods available in the literature.

Chapter 2 studies linear parabolic equations that depend on a microscopic time scale and a microscopic spatial scale. We consider the most interesting critical similarity case where the derivative with respect to the fast time variable plays a role in the cell problems. Using multiscale convergence, we deduce a high dimensional multiscale homogenized equation that provides the solution to the homogenized equation, which provides the macroscopic information, and the time-space corrector term, which encodes the microscopic information. We develop the sparse tensor product

FE method for solving this time-space multiscale homogenized equation. The method provides an approximation to all the necessary information with a required accuracy, using an essentially optimal number of degrees of freedom. From the FE solution of the time-space multiscale homogenized equation, we construct a numerical corrector for the solution of the original time-space multiscale equation.

Chapter 3 develops a new method for solving monotone parabolic equations that depend on n separable microscopic scales. For nonlinear problems, forming the homogenized equation is practically impossible, as for each vector in $\mathbb{R}^d$, we have to solve a nonlinear monotone cell problem. Applying multiscale convergence, we obtain a high dimensional multiscale homogenized equation which when solved, the solution will provide the necessary macroscopic and microscopic information. We develop the backward Euler and Crank-Nicholson methods for sparse tensor product FE spaces. In both approaches, the sparse tensor product FE method provides an approximation with an essentially optimal level of accuracy, but uses an essentially optimal number of degrees of freedom. For two scale problems, a new homogenization error is derived in terms of the microscopic scale. A numerical corrector is deduced with an explicit error. For general multiscale problems, a numerical corrector is derived, without an explicit error. This is because of the unavailability of a homogenization error.

Chapter 4 considers multiscale monotone parabolic equations which depend on a microscopic time scale and a microscopic spatial scale. As in Chapter 2, we consider the most interesting case when the derivative with respect to the fast time variable plays a part in the limiting equation. Applying multiscale convergence, a high dimensional multiscale homogenized equation is obtained. This chapter develops a new numerical method to

solve the multiscale monotone parabolic problems with an essentially equal level of accuracy as the full tensor product FE method. The backward Euler and Crank-Nicholson methods are developed for the sparse tensor product FE spaces. The sparse tensor product FE method uses an essentially optimal number of degree of freedom to get an approximation with the desired level of accuracy. A numerical corrector is also derived from the FE solutions.

Chapter 5 studies locally periodic multiscale variational inequalities which depend on a macroscopic and n microscopic scales. Using multiscale convergence, we deduce a multiscale homogenized variational inequality in a high dimensional tensorized domain. To the best of our knowledge, numerical method for multiscale variational inequalities have not been developed. We develop the sparse tensor product FE method for locally periodic multiscale problems which attains an essentially equal level of accuracy to that of the full tensor product FE method but requires only an essentially optimal number of degrees of freedom which is essentially equal to that for solving a problem in $\mathbb{R}^d$. In the two scale case, we deduce a new homogenization error for the nonlinear monotone variational inequalities. A numerical corrector is constructed with an explicit error in terms of the FE error and the homogenization error. In the multiscale case, we construct a numerical corrector without an explicit error.

In all the chapters, we apply the methods developed to solve some multiscale problems numerically to verify the theory.

Contents

Acknowledgements	i
Abstract	iii
1 Introduction	1
2 Time-Space Multiscale Parabolic Equations	13
2.1 Setting up of the problem	14
2.2 Multiscale homogenization of (2.2)	15
2.2.1 Multiscale convergence	15
2.2.2 Multiscale homogenized equation of problem (2.2) . .	20
2.3 FE approximations	23
2.3.1 General FE approximation	24
2.3.2 Full tensor FE	31
2.3.3 Sparse tensor FEs	34
2.4 Numerical correctors	38
2.4.1 Homogenization error	38
2.4.2 Numerical correctors	39
2.5 Regularity	44

2.6	Numerical examples	48
3	Multiscale Monotone Parabolic Equations depending on Multiple Microscales in Space	57
3.1	Problem setting and homogenization	58
3.1.1	Problem setting	58
3.1.2	Multiscale homogenization	59
3.2	FE discretization	64
3.2.1	Backward Euler method	64
3.2.1.1	Backward Euler method for general FE spaces	64
3.2.1.2	Backward Euler method for full tensor product FE spaces	71
3.2.1.3	Backward Euler method for sparse tensor product FE spaces	74
3.2.2	Crank-Nicholson method	78
3.2.2.1	Crank-Nicholson method for general FE spaces	78
3.2.2.2	Crank-Nicholson method for full tensor product FE spaces	84
3.2.2.3	Crank-Nicholson method for sparse tensor product FE spaces	86
3.3	Numerical correctors	87
3.3.1	Two scale problems	88
3.3.2	Multiscale problems	95
3.4	Numerical examples	100
4	Multiscale Monotone Parabolic Equations depending on Microscales in Space and Time	113

4.1	Problem setting and homogenization	114
4.1.1	Problem setting	114
4.1.2	Multiscale homogenization	115
4.1.3	Multiscale homogenization of (4.3)	117
4.2	FE discretization	118
4.2.1	Backward Euler method	119
4.2.1.1	Backward Euler method for general FE spaces	119
4.2.1.2	Backward Euler for full tensor product FE spaces	125
4.2.1.3	Backward Euler method for sparse tensor product FE spaces	128
4.2.2	Crank-Nicholson method	131
4.2.2.1	Crank-Nicholson method for general FE s- paces	131
4.2.2.2	Crank-Nicholson method for full tensor prod- uct FE spaces	138
4.2.2.3	Crank-Nicholson method for sparse tensor product FE spaces	140
4.3	Numerical corrector	140
4.4	Numerical examples	145
5	Multiscale Monotone Variational Inequalities	155
5.1	Multiscale variational inequalities	155
5.1.1	Homogenization problem	157
5.2	FE discretization	161
5.2.1	Hierarchical FE spaces	162
5.2.2	Full tensor product FE spaces	162

5.2.3	Sparse tensor product FE spaces	168
5.3	Numerical correctors	169
5.3.1	Two-scale problems	170
5.3.2	Numerical corrector for the multiscale problem	176
5.4	Numerical examples	180
References		185

List of Figures

2.1	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$	49
2.2	The error $\ u_1 - \hat{u}_1\ _{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$	49
2.3	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 2 dimensional problem at $t = 1$	50
2.4	The error $\ u_1 - \hat{u}_1\ _{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 2 dimensional problem at $t = 1$	51
2.5	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$	53
2.6	The error $\ u_1 - \hat{u}_1^L\ _{L^2(D \times (0,1), H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$	53
2.7	The error $\ u_2 - \hat{u}_2^L\ _{L^2(D \times (0,1) \times Y, H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$	54
2.8	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$	55

2.9	The error $\ u_1 - \hat{u}_1^L\ _{L^2(D \times (0,1), H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$	55
2.10	The error $\ u_2 - \hat{u}_2^L\ _{L^2(D \times (0,1) \times Y, H_{\#}^1(Y)/\mathbb{R})}$ for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$	56
3.1	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.	101
3.2	The error $\ u_1 - \hat{u}_1\ _{L^2(D, H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.	102
3.3	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.	102
3.4	The error $\ u_1 - \hat{u}_1\ _{L^2(D, H_{\#}^1(Y))}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for Crank-Nicholson method.	103
3.5	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for backward Euler method.	104
3.6	The error $\ u_1 - \hat{u}_1\ _{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.	104
3.7	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.	105
3.8	The error $\ u_1 - \hat{u}_1\ _{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.	105
3.9	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem at $t = 1$ for backward Euler method.	107
3.10	The error $\ u_1 - \hat{u}_1\ _{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for two dimensional problem at $t = 1$ for backward Euler method.	107
3.11	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem at $t = 1$ for Crank-Nicholson method.	108

3.12	The error $\ u_1 - \hat{u}_1\ _{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for two dimensional problem at $t = 1$ for Crank-Nicholson method. .	108
4.1	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method. . . .	147
4.2	The error $\ u_1 - \hat{u}_1\ _{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method.	147
4.3	The error $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method. . .	148
4.4	The error $\ u_1 - \hat{u}_1\ _{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method.	148
4.5	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for the backward Euler method. . . .	149
4.6	The error $\ u_1 - \hat{u}_1\ _{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method.	150
4.7	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method.	150
4.8	The error $\ u_1 - \hat{u}_1\ _{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for the Crank-Nicholson method.	151
4.9	The error $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ versus the mesh size h for 2 dimensional problem at $t = 0.5$ for the backward Euler method.	152

4.10	The error $\ u_1 - \hat{u}_1\ _{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ versus the mesh size h for 2 dimensional problem at $t = 0.5$ for the backward Euler method.	153
5.1	The $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ error versus the mesh size h	181
5.2	The $\ u_1 - \hat{u}_1\ _{L^2(D, H_{\#}^1(Y)/\mathbb{R})}$ error versus the mesh size h	181
5.3	The $\ u_0 - \hat{u}_0^L\ _{W_0^{1,4}(D)}$ error versus the mesh size h	182
5.4	The $\ u_1 - \hat{u}_1\ _{L^4(D, W_{\#}^{1,4}(Y)/\mathbb{R})}$ error versus the mesh size h	182
5.5	The $\ u_0 - \hat{u}_0^L\ _{H_0^1(D)}$ error versus the mesh size h	184
5.6	The $\ u_1 - \hat{u}_1\ _{L^2(D, H_{\#}^1(Y))}$ error versus the mesh size h	184

Chapter 1

Introduction

Partial differential equations that depend on multiple scales occur in the mathematical modelling of many real world phenomena. Composites are made of many different components that are blended together to achieve the desired toughness and lightness. Subsurface flow is another type of multiscale problems where the geometry of the underground structures, e.g. the sea bed in an oil field, is essentially modelled in multiple scales. In biology, human bones are composites. Multiscale problems are frequently encountered in seismology as the earth is intrinsically a composite. We refer to the book by Milton [61] for an extensive account of multiscale composites. We want to study how the fine scales in which the component materials are blended together affect the large scale homogeneous properties of the composites which are essential in understanding complex multiscale phenomena, in designing and manufacturing composites.

Microscopic heterogeneities occur at a much smaller scale than the macroscopic length scale of interest. The equations in consideration are usually very complex. Finding numerical approximations for the solutions, to retrieve all the information in the microscopic scale, we have to use a

mesh size that is at most the order of the smallest scale. Thus, traditional numerical methods have high complexity and are prohibitively expensive. Studying new numerical methods that can solve such problems within acceptable accuracy and computational time, and using reasonable computational resources is one of the main areas of interest in applied mathematics.

The homogenization theory studies the limit of multiscale equations as the fine scales converge to zero. The theory provides adequate theoretical framework to describe the multiscale equations macroscopically. We obtain effective homogenized problems which approximate the original multiscale problems in an average sense. The homogenized problems only depend on the macroscopic scale and can be solved by traditional numerical methods as the complexity is not as high. The subject of homogenization goes back to the work of Rayleigh [71] and Maxwell [60]. The theory of homogenization attracts a large amount of significant contributions ([16], [53], [11], [72], [10], [17], [14], [15], [4], [30], [62], [64] and [69]) in the last few decades, but there are still many unsolved problems that require theoretical study and effective numerical methods to solve: problems with multiple spatial scales, multiscale nonlinear problems and problems depending on multiple time and space scales ([36]).

For problems that depend on microscopic scales periodically, we can perform asymptotic expansion to deduce the homogenized equation (see [16], [53] and [11]). However, applying the asymptotic expansion to problems with more than two scales is highly complicated (see [16]). To compute the coefficients of the homogenized equations, we have to solve a set of cell problems. Solving the cell problems can be very expensive as the computer memory and floating point operations required are huge and this is especially if the problem depends on multiple microscopic scales or is locally

periodic (see [7]). The concept of multiscale convergence ([65], [5] and [8]) can also be employed to derive a limiting equation that contains all the necessary macroscopic and microscopic information. Solving them, we get all the desired information. For general problems, G -convergence and Γ -convergence can be employed to derive the homogenized equation (see [53], [31] and [63]) but these methods are abstract and the identification of the homogenization limits of the multiscale problems can be difficult. Furthermore, the homogenized problems only provide the macroscopic information in the solution of the multiscale problems. To extract the microscopic information, we have to find the corrector terms. Computing the correctors can be of high complexity.

There have been significant efforts to develop numerical methods that solve multiscale problems with reduced complexity and with high accuracy. The Multiscale FE method ([23], [38] and [52]) constructs FE basis functions that contain fine scale information in each macroscopic simplex. The basis functions can be computed by parallel computing. The advantage of the multiscale basis is that it can be reused for different forcing functions in the right hand side. Recently the method has been extended to “Generalized Multiscale FE method” (see [37], [26] and [42]). The Heterogeneous Multiscale Method ([35] and [1]) solves the cell problem for each macroscopic degree of freedom to account for the microscopic information. Another method that could reduce complexity is due to [70]. Though general, the complexity of these method is high. It grows super linearly with respect to the optimal complexity level when a better accuracy is required. We note also the recent numerical schemes for general multiscale problems developed in [59]. For locally periodic multiscale problems, the high dimensional FE method that was initiated by Hoang and Schwab [49] for multiscale ellip-

tic problems requires an essentially optimal number of degrees of freedom to find all the necessary microscopic and macroscopic information. The method approximates the solution of the high dimensional multiscale homogenized equation obtained from multiscale convergence (see [65] and [5]). It exploits the regularity of the corrector terms with respect to all the slow and fast variables at the same time. The complexity is essentially optimal, i.e. the complexity level defers from the optimal level by only a logarithmic multiplying factor. To achieve an error of the order $O(h)$ for a problem in $\mathbb{R}^d$, the number of degrees of freedom required is $O(h^d)$, with possibly a logarithmic multiplicative factor. For locally periodic two scale problems, Ohlberger [67] shows that the Heterogeneous Multiscale Method is equivalent to solving the high dimensional multiscale homogenized problems using full tensor product FE.

In this thesis, we consider locally periodic multiscale nonlinear monotone variational inequalities (our work thus covers also linear multiscale variational inequalities), multiscale monotone parabolic problems, multiscale linear and nonlinear equations that depend on microscopic scales in both space and time. We develop the sparse tensor product FE method to approximate their solutions. The method requires an essentially optimal number of degrees of freedom to find all the microscopic and macroscopic information.

In Chapter 2, we study the multiscale linear parabolic equation that depends on a microscopic scale in time and a microscopic scale in space. The homogenization theory of parabolic problems that depend on microscopic scales in both temporal and spatial variables is studied by Benssousan et al. in [16] (see also [6], [50] and [57]) where the second time scale, t/ε^k , is considered. This problem can be split into 3 cases with $k < 2$, $k = 2$ and

$k > 2$. The most interesting case occurs when $k = 2$, where the derivative with respect to the fast time variable plays a role in the limiting equation. When $k < 2$, this variable only plays the role of a parameter; and when $k > 2$, it is averaged out in the homogenization limit. Thus we only consider the critical case $k = 2$.

As for other multiscale problems, a direct numerical discretization method is too expensive. For the consideration of the multiple scales in the temporal and spatial variables, Efendiev and Pankov [41] employ the ideas of the multiscale FE method ([51] and [38]) to perform numerical homogenization for quasilinear parabolic equations, where a set of multiscale FE basis functions are employed which are solutions of the multiscale local problems (see also [40] and [39]). For parabolic problems that depend on multiple scales in both space and time, Owhadi and Zhang [68] construct a multiscale basis by solving a set of multiscale parabolic problems with a non-homogeneous boundary condition. Other contributions in this direction could be found in [58] and [46]. Though general, the cost of constructing these multiscale basis can still be expensive as microscopic meshes with respect to both time and space have to be used.

We restrict our consideration to the case where the coefficient is locally periodic with respect to both the temporal and spatial microscopic scales. We develop an optimal method for solving the multiscale linear parabolic equation numerically which contains all the necessary macroscopic and microscopic information. We employ the high dimensional sparse tensor product FE method to solve the multiscale homogenized equation. We derive the multiscale homogenized equation from multiscale convergence (see [65] and [5]). As the coefficient depends on the microscopic time scale as well, the concept of multiscale convergence of Nguetseng [65] and Allaire

[5] needs to be extended. This was first explored by Holmbom et al in [50]. Once the multiscale homogenized equation is solved, we obtain the solution to the homogenized equation that describes the solution of the multiscale problem macroscopically, and the corrector terms which encode the microscopic information. We construct a numerical corrector in terms of the FE solutions.

We setup the multiscale problem in Section 2.1. In Section 2.2, we recall the concept of multiscale convergence in both time and space and prove several results on time space multiscale limit of a sequence of functions, and use them to derive the multiscale homogenized equation. In Section 2.3, the numerical approximation of the multiscale homogenized equation is studied. We first consider a numerical scheme with general FE spaces and prove the convergence. We then consider the scheme using the full tensor product FE spaces and the sparse tensor product FE spaces for the corrector in Subsections 2.3.2 and 2.3.3 respectively. Assuming regularity for the solution of the multiscale homogenized equation, we derive FE error estimates in terms of the mesh size. We show that the sparse tensor product FE approximation produces essentially equal level of accuracy as the full tensor product FE approximation, but uses only an essentially optimal number of degrees of freedom. In Section 2.4, we construct a numerical corrector from the FE solution. In Section 2.5, we show that the regularity required to get the error estimates for the full and sparse tensor product FE approximations hold under some regularity conditions. In Section 2.6, we present some numerical examples in one and two dimensions to verify the FE rate of convergence for sparse tensor product FE approximations. Though we only consider the theory for the case with only one microscopic spatial scale, our method is fully capable of solving equations with more

than one microscopic spatial scales, e.g. those considered in Holmbom et al. [50]. We show this by solving some examples with one microscopic time scale and two microscopic spatial scales studied in [50], using sparse tensor product FE.

In Chapter 3 we consider multiscale monotone parabolic equations which depend on n separated microscales. The Heterogeneous Multiscale Method has been applied for the two scale parabolic monotone problems by Abdulle (see [3] and [2]). We note that the Heterogeneous Multiscale Method has been used to solve stationary two scale monotone problems in [44] and [47]. We consider locally periodic problems in this chapter. As mentioned above, the Heterogeneous Multiscale Method is not optimal in this case. Our method requires an essentially optimal number of degrees of freedom to achieve a prescribed level of accuracy. We note that for nonlinear problems, it is virtually impossible to form the homogenized equation explicitly, as for each vector in $\mathbb{R}^d$, a nonlinear monotone cell problem has to be solved. Thus, solving the high dimensional multiscale homogenized equation is an ideal method for obtaining the necessary microscopic and macroscopic information.

In Section 3.1, we setup the multiscale monotone parabolic problem and recap on the multiscale homogenization theory. We apply the multiscale homogenization theory to the monotone problem to derive the multiscale homogenized equation. The multiscale homogenization in the L^p setting of Allaire is extended to the time dependent functions ([76]). FE approximation is developed in Section 3.2. In Section 3.2.1, we consider the backward Euler method. In Section 3.2.1.1, the backward Euler for the general FE spaces is presented. We prove the convergence of the scheme for the general FE spaces. In Section 3.2.1.2, backward Euler for the full tensor product

FE spaces is developed. In Section 3.2.1.3, we develop the backward Euler for sparse tensor product FE spaces. We achieve the same level of accuracy for the sparse tensor product FE spaces but require far less number of degrees of freedom comparing to the full tensor FE spaces. In Section 3.2.2, we consider the Crank-Nicholson method. The Crank-Nicholson method for general FE spaces, for full tensor product FE spaces and for sparse tensor product FE spaces are developed in Section 3.2.2.1, 3.2.2.2 and 3.2.2.3 respectively. In Section 3.3, we derive numerical correctors from the FE solutions. For the two scale problem, we prove a new analytical homogenized error in terms of the microscopic scale, from that a numerical corrector is derived with an explicit error in terms of the homogenized error and the FE error. We do this in Section 3.3.1. In Section 3.3.2, the corrector for general multiscale problems is derived, but without the explicit error due to the unavailability of a homogenization error. Some numerical results that illustrate the theory are presented in Section 3.4.

In the field of parabolic monotone problems, even in the single macroscopic scale setting, regularity of solutions is still very much under progress (see [12]). The Crank-Nicholson method requires more regularity than the backward Euler method. We thus develop both methods for the nonlinear problems.

In Chapter 4, we consider the multiscale monotone parabolic equations which depend on a microscopic time scale, i.e. the monotone operator depends on $\frac{t}{\varepsilon^k}$. These types of monotone problems have previously been studied in [50], [74] and [66]. This problem can be split into 3 cases, $k < 2$, $k = 2$ and $k > 2$, similar to that of the linear problem in Chapter 2. The critical case occurs when $k = 2$, where the role of the fast time variable plays a role in the cell problem. When $k < 2$, this term only plays the

part of a parameter. When $k > 2$, this variable is averaged out in the homogenization limit. In this thesis, we only consider the case when $k = 2$. We develop a new numerical method to solve the multiscale parabolic monotone equations which depend on a microscopic time scale using an optimal number of degrees of freedom and achieving an essentially equal level of accuracy as the full tensor product FE method.

In Section 4.1, we setup the multiscale monotone parabolic problem which depends on a microscopic time scale and a microscopic spatial scale and derive the multiscale homogenized equation from multiscale convergence. FE discretization is developed in Section 4.2. Section 4.2.1 develops the backward Euler method for general FE spaces. We prove convergence for the general scheme. We then develop the backward Euler method for the full tensor and sparse tensor product FE method in Section 4.2.1.2 and 4.2.1.3 respectively. When the solution possesses sufficient regularity, we derive error estimates in terms of the mesh size. In Section 4.2.2.1, we consider the Crank-Nicholson method for general FE spaces. The Crank-Nicholson method for the full tensor and sparse tensor product FE spaces are developed in Sections 4.2.2.2 and 4.2.2.3 respectively. In Section 4.3, we derive the numerical corrector from the FE solutions. Several one dimensional and two dimensional numerical results are presented in Section 4.4 to illustrate the theory.

In Chapter 5, we consider multiscale variational inequalities of the monotone type. Variational inequalities occur in many applied areas, for example the obstacle problems (see [28]). Multiscale problems arise when the membranes in the obstacle problems are made of composite materials. Although there has been some works on theoretical homogenization of linear and nonlinear multiscale variational inequalities ([16], [9], [33], [73] and

[13]), not much work has been done on their numerical solutions even for linear problems. Our work on multiscale monotone variational inequalities covers multiscale linear variational inequalities.

A direct method to solve multiscale problems is prohibitively expensive. The numerical methods that have been developed to solve multiscale equations, for example the Multiscale FE method and the Heterogeneous Multiscale Method, have not been applied for multiscale variational inequalities. In this chapter, we develop the sparse tensor product FE method that uses an essentially optimal number of degrees of freedom to find all the macroscopic and microscopic information. It finds the solution of the homogenized problem and the correctors, without forming the homogenized equation.

We start by employing the multiscale convergence mentioned above to derive the multiscale homogenized variational inequality. This multiscale homogenized inequality contains all the macroscopic and microscopic information. As mentioned above, to form the homogenized equation, for each vector in $\mathbb{R}^d$, a nonlinear monotone cell problem has to be solved. This is virtually impossible. Solving the multiscale homogenized problem, we get all the necessary microscopic and macroscopic information. However, this problem is posed in a high dimensional tensorized domain. The full tensor product FE approach is very expensive. Instead we develop the sparse tensor product FE method, which produces an approximation for this high dimensional variational inequality with essentially equal accuracy as that of the full tensor product FE method, but uses an essentially optimal number of degrees of freedom. From the numerical solution of this high dimensional multiscale homogenized problem, we construct a numerical corrector for the general multiscale problem. In the two scale case, we establish a homogenization error for the nonlinear monotone variational inequalities.

To the best of our knowledge, this homogenization error is new. From this, we derive an explicit error for this corrector in terms of the FE error and the homogenization error. In the multiscale case, we construct a numerical corrector, without an explicit error.

In Section 5.1, we formulate the nonlinear multiscale monotone variational inequality and use the concept of multiscale convergence in the L^p spaces of [65] and [5] to derive the monotone multiscale homogenized variational inequalities for the general $(n + 1)$ scale problems. In Section 5.2, we construct the tensor product FE by introducing the piecewise linear FE spaces. We consider both the full tensor and sparse tensor product FE method. We then show that the sparse tensor product FE method achieves an essentially equal level of accuracy to that of the full tensor FE, but requires only an essentially equal number of degrees of freedom, when the solution satisfies some regularity conditions. In Section 5.3, we construct numerical correctors and derive a new homogenization error for two sale monotone variational inequalities. We then construct a numerical corrector with an error estimate in terms of the homogenization error and the FE error. In this section, we also construct a numerical corrector for the multiscale problems, though without an explicit convergence rate as a homogenization error for general multiscale problems is not available. We then show some numerical results in Section 5.4.

Throughout the thesis, by ∇ without indicating explicitly the variable, we denote the gradient with respect to x of a function of x , and by ∇_x we denote the partial gradient with respect to x of a function depending on x and other variables. By $\#$ we denote spaces of periodic functions. For functions depending on time t and other variables, when we only want to emphasize the time dependence, we will only indicate the time variable.

The chapters are self-contained. The same notations may denote different function spaces in different chapters.

Chapter 2

Time-Space Multiscale Parabolic Equations

Chapter 2 considers multiscale linear parabolic equations that depend a microscopic scale in time and a microscopic scale in space. This chapter considers the critical case when the derivative with respect to the fast time variable plays a role in the limiting equation. We develop an essentially optimal FE method to solve the problem. Section 2.1 sets up the problem. Multiscale homogenization is developed in Section 2.2. The FE approximation scheme for general FE spaces, full tensor and sparse tensor product FE spaces is developed in Section 2.3. In Section 2.4, we construct a numerical corrector. In Section 2.5, we show that the regularity conditions hold under some regularity conditions. We present several numerical examples in Section 2.6 to illustrate the theory.

2.1 Setting up of the problem

Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $T > 0$. Let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. We consider a symmetric matrix valued function $a : (0, T) \times D \times (0, 1) \times Y \rightarrow \mathbb{R}_{sym}^{d \times d}$ such as $a(t, x, \tau, y) \in C((0, T) \times \bar{D} \times (0, 1) \times \bar{Y})$. The function a is periodic with respect to τ and y , with the period being $(0, 1)$ and Y respectively; from now on we say that it is $(0, 1) \times Y$ periodic with respect to τ and y . We assume that a is uniformly coercive and bounded, i.e, there are positive constants c_1 and c_2 such that for all $\xi, \zeta \in \mathbb{R}^d$

$$c_1 |\xi|^2 \leq a(t, x, \tau, y) \xi \cdot \xi, \quad a(t, x, \tau, y) \xi \cdot \zeta \leq c_2 |\xi| |\zeta| \quad (2.1)$$

for all $(t, x, \tau, y) \in (0, T) \times D \times (0, 1) \times Y$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Let $\varepsilon > 0$ be a small number that represents the microscopic scale. We consider the time-space multiscale coefficient

$$a^\varepsilon(t, x) = a(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}).$$

We denote by $V = H_0^1(D)$ and $H = L^2(D)$. We have that $V \subset H \subset V'$. By $\langle \cdot, \cdot \rangle_H$, we denote the inner product in H extended to the duality pairing between V' and V . Let $T > 0$, $f \in L^2((0, T), V')$ and $g \in H$. We consider the parabolic problem

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (a^\varepsilon(t, x) \nabla u^\varepsilon) &= f(t, x), \quad x \in D, \quad t \in (0, T), \\ u^\varepsilon(0, x) &= g, \quad x \in D \end{aligned} \quad (2.2)$$

with the Dirichlet boundary condition for $u^\varepsilon(t, \cdot)$. Problem (2.2) has a unique solution $u^\varepsilon \in L^2((0, T), V) \cap H^1((0, T), V')$ which satisfies

$$\|u^\varepsilon\|_{L^2((0, T), V)} + \|u^\varepsilon\|_{H^1((0, T), V')} \leq c(\|f\|_{L^2((0, T), V')} + \|g\|_H)$$

where constant c only depends on the constants c_1, c_2 in (2.1) and T ([75] Chapter 4).

Homogenization of equation (2.2) is first studied in [16]. Indeed, Bensoussan et al. consider the general case where the multiscale coefficient $a^\varepsilon(t, x) = a(t, x, \frac{t}{\varepsilon^k}, \frac{x}{\varepsilon})$ with $k > 0$. The homogenized limit depends on the value of k with the critical value $k = 2$ giving the most interesting homogenized equation. In this thesis we consider only the case where $k = 2$. We discuss the cases where $k < 2$ and $k > 2$ in Remark 2.7. We use multiscale convergence to study this problem.

2.2 Multiscale homogenization of (2.2)

We study homogenization of (2.2) by multiscale convergence. Multiscale convergence was initiated by Nguetseng in [65], and developed further by Allaire [5]. The concept is extended to functions depending on microscopic scales with respect to both time and space in Holmbom et al.[50]. We first recall the definition of multiscale convergence in [50]. We then prove some results on time space multiscale convergence and use them to derive the multiscale homogenized equation of (2.2). As the coefficients depend on microscales in both space and time, we wish to extract microscopic properties with respect to both the space and time so we use multiscale convergence in both space and time.

2.2.1 Multiscale convergence

We first recall the definition of Holmbom et al. [50].

Definition 2.1. A sequence $\{w^\varepsilon\}_\varepsilon$ time-space multiscale converges to a function $w_0 \in L^2((0, T) \times D \times (0, 1) \times Y)$ if for all functions $\phi \in C((0, T) \times$

$\bar{D} \times (0, 1) \times \bar{Y}$) which are $(0, 1) \times Y$ periodic with respect to τ and y ,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon(t, x) \phi\left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) dx dt \\ &= \int_0^T \int_D \int_0^1 \int_Y w_0(t, x, \tau, y) \phi(t, x, \tau, y) dy d\tau dx dt. \end{aligned}$$

We can show that (see [50]):

Proposition 2.2. From a bounded sequence in $L^2((0, T) \times D)$, there is a time-space multiscale convergent subsequence.

In the following propositions, we establish the time-space multiscale convergent limits of bounded sequences in $L^2((0, T), V) \cap H^1((0, T), V')$ that will be employed to derive the multiscale homogenization limit of the solution of (2.2). These results are first derived in [50]. For completeness, we will present the proofs of these results; and the proof of Proposition 2.4 is different from that of [50].

Proposition 2.3. Let w^ε be a bounded sequence in $L^2((0, T), V) \cap H^1((0, T), V')$. Then there are functions $w_0 \in L^2((0, T), V)$ and $w_1 \in L^2((0, T) \times D \times (0, 1), H_{\#}^1(Y))$, and a subsequence (not renumbered) such that

$$\nabla w^\varepsilon \xrightarrow{\text{ts-ms}} \nabla_x w_0 + \nabla_y w_1.$$

Proof. Let $\psi(t, x, \tau, y) : (0, T) \times D \times (0, 1) \times Y \rightarrow \mathbb{R}^d$ be a smooth function that is $(0, 1) \times Y$ periodic with respect to τ and y . We consider

$$\begin{aligned} & \varepsilon \int_0^T \int_D \nabla w^\varepsilon \cdot \psi\left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) dx dt \\ &= - \int_0^T \int_D w^\varepsilon (\varepsilon \operatorname{div}_x \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) + \operatorname{div}_y \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})) dx dt. \end{aligned}$$

As $\|\nabla w^\varepsilon\|_{L^2((0, T), V)}$ is bounded, passing to the limit when $\varepsilon \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon \operatorname{div}_y \psi\left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) dx dt = 0.$$

Let $w_0 \in L^2((0, T) \times D \times (0, 1) \times Y)$ be the multiscale convergent limit of a subsequence of w^ε . We then have

$$\int_0^T \int_D \int_0^1 \int_Y w_0(t, x, \tau, y) \operatorname{div}_y \psi(t, x, \tau, y) dy dx d\tau dt = 0.$$

This implies that w_0 is independent of y . As ψ is a smooth function,

$$\left\| \varepsilon^2 \psi \left(t, \cdot, \frac{t}{\varepsilon^2}, \frac{\cdot}{\varepsilon} \right) \right\|_V \leq c\varepsilon$$

uniformly for all t . Therefore

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^T \left\langle \frac{\partial w^\varepsilon}{\partial t}, \psi \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) \right\rangle_H dt = 0,$$

i.e.

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^T \int_D w^\varepsilon \left(\frac{\partial \psi}{\partial t} \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) + \frac{1}{\varepsilon^2} \frac{\partial \psi}{\partial \tau} \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) \right) dx dt = 0.$$

As

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_0^T \int_D w^\varepsilon \frac{\partial \psi}{\partial t} \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) dx dt = 0,$$

we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon(t, x) \frac{\partial \psi}{\partial \tau} \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) dx dt \\ &= \int_0^T \int_D \int_0^1 \int_Y w_0(t, x, \tau) \frac{\partial \psi}{\partial \tau}(t, x, \tau, y) dy d\tau dx dt = 0. \end{aligned}$$

This implies that w_0 does not depend on τ . Therefore w_0 is the weak limit of w^ε in $L^2((0, T) \times D)$. Let $\chi \in L^2((0, T) \times D \times (0, 1) \times Y)$ be the multiscale limit of ∇w^ε . We choose the function

$$\psi \in C^\infty((0, T), C_0^\infty(D, C_\#^1((0, 1), C_\#^\infty(Y))))$$

so that $\operatorname{div}_y \psi = 0$. We then have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D \nabla w^\varepsilon \cdot \psi \left(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) dx dt$$

$$= \int_0^T \int_D \int_0^1 \int_Y \chi(t, x, \tau, y) \cdot \psi(t, x, \tau, y) dy d\tau dx dt.$$

Since $\operatorname{div}_y \psi = 0$, we have

$$\int_0^T \int_D \nabla w^\varepsilon \cdot \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) = - \int_0^T \int_D w^\varepsilon \operatorname{div}_x \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt.$$

Passing to the multiscale limit, we get

$$\begin{aligned} & - \int_0^T \int_D \int_0^1 \int_Y w_0(t, x) \operatorname{div}_x \psi(t, x, \tau, y) dy d\tau dx dt \\ &= \int_0^T \int_D \int_0^1 \int_Y \chi(t, x, \tau, y) \cdot \psi(t, x, \tau, y) dy d\tau dx dt. \end{aligned}$$

Thus

$$\int_0^T \int_0^1 \int_D \int_Y (\chi - \nabla_x w_0) \cdot \psi(t, x, \tau, y) dy dx d\tau dt = 0.$$

As this holds for all functions ψ with $\operatorname{div}_y \psi = 0$, there is a function $w_1 \in L^2((0, T) \times D \times (0, 1), H_\#^1(Y))$ so that

$$\chi = \nabla_x w_0(t, x) + \nabla_y w_1(t, x, \tau, y).$$

□

Proposition 2.4. Let $\{w^\varepsilon\}_\varepsilon$ be a bounded sequence in $L^2((0, T), H^1(D))$ such that

$$\nabla w^\varepsilon \xrightarrow{\text{ts-ms}} \nabla_x w_0 + \nabla_y w_1.$$

Then for all smooth functions $\psi(t, x, \tau, y)$ which are $(0, 1) \times Y$ periodic with respect to τ and y and

$$\begin{aligned} & \int_Y \psi(t, x, \tau, y) dy = 0, \\ & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D \frac{1}{\varepsilon} w^\varepsilon(t, x) \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\ &= \int_0^T \int_D \int_0^1 \int_Y w_1(t, x, \tau, y) \psi(t, x, \tau, y) dy d\tau dx dt. \end{aligned}$$

Proof. As ψ is smooth and $\int_Y \psi(t, x, \tau, y) dy = 0$, there is a smooth function $\bar{\psi}(t, x, \tau, y)$ which are $(0, 1) \times Y$ periodic with respect to τ and y such that

$$\Delta_y \bar{\psi}(t, x, \tau, \cdot) = \psi(t, x, \tau, \cdot).$$

Since

$$\begin{aligned} \Delta \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) &= \Delta_x \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) + \frac{2}{\varepsilon} \nabla_x \nabla_y \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon^2} \Delta_y \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}), \\ \Delta_y \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) &= \varepsilon^2 \Delta \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) - \varepsilon^2 \Delta_x \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}) - 2\varepsilon \nabla_x \nabla_y \bar{\psi}(t, x, \tau, \frac{x}{\varepsilon}). \end{aligned}$$

Thus

$$\begin{aligned} & \int_0^T \int_D \frac{1}{\varepsilon} w^\varepsilon(t, x) \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\ &= \frac{1}{\varepsilon} \int_0^T \int_D w^\varepsilon(t, x) \left(\varepsilon^2 \Delta \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) - \varepsilon^2 \Delta_x \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \right. \\ & \quad \left. - 2\varepsilon \nabla_x \nabla_y \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \right) dx dt. \end{aligned}$$

As $\bar{\psi}$ is a smooth function

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T \int_D w^\varepsilon \varepsilon^2 \Delta_x \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt = 0.$$

We note that

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^T \int_D w^\varepsilon \varepsilon^2 \Delta \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\ &= -\varepsilon \int_0^T \int_D \nabla w^\varepsilon(t, x) \cdot \left(\nabla_x \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_y \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \right) dx dt \\ &\rightarrow -\int_0^T \int_D \int_0^1 \int_Y (\nabla_x w_0 + \nabla_y w_1) \cdot \nabla_y \bar{\psi} dy d\tau dx dt \end{aligned}$$

as $\varepsilon \rightarrow 0$. Since w_0 is independent of y ,

$$\int_0^T \int_D \int_0^1 \int_Y \nabla_x w_0 \cdot \nabla_y \bar{\psi} dy dx d\tau dt = 0.$$

Thus

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T \int_D w^\varepsilon \varepsilon^2 \Delta \bar{\psi}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\
&= - \int_0^T \int_D \int_0^1 \int_Y \nabla_y w_1 \cdot \nabla_y \bar{\psi} dy dx d\tau dt \\
&= \int_0^T \int_D \int_0^1 \int_Y w_1 \Delta_y \bar{\psi} dy dx d\tau dt \\
&= \int_0^T \int_D \int_0^1 \int_Y w_1 \psi dy dx d\tau dt.
\end{aligned}$$

We note further that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T \int_D w^\varepsilon 2\varepsilon \nabla_x \nabla_y \bar{\psi}(t, \frac{t}{\varepsilon^2}, x, \frac{x}{\varepsilon}) dx dt \\
&= 2 \int_0^T \int_0^1 \int_D \int_Y w_0(t, x) \nabla_x \nabla_y \bar{\psi}(t, x, \tau, y) dy dx d\tau dt \\
&= 0.
\end{aligned}$$

We then get the conclusion. $\square$

2.2.2 Multiscale homogenized equation of problem (2.2)

We have the following result on the time-space multiscale limit of the solution of the multiscale problem (2.2). We denote by $V_\# = H_\#^1(Y)/\mathbb{R}$ and $H_\# = L^2(Y)/\mathbb{R}$. In the limiting equation below, $\frac{\partial u_1}{\partial \tau}$ (fixing t and x) belongs to the duality space $V'_\#$. Thus we use the Gelfand triple $V_\# \subset H_\# \subset V'_\#$; and we denote by $\langle \cdot, \cdot \rangle_{H_\#}$ the inner product in $H_\#$ extended to the duality pairing between $V_\#$ and $V'_\#$. In the proposition below, we use time-space multiscale convergence to derive a limiting equation that contains all the microscopic and macroscopic information in both space and time. The main difficulty is due to the fast time variable t/ε^2 which implies a $O(1/\varepsilon)$ term in the time derivative of the test function ψ^ε below. However, this can be overcome by using Proposition 2.4 above.

Proposition 2.5. There are functions $u_0 \in L^2((0, T), V) \cap H^1((0, T), V')$ and $u_1 \in L^2((0, T) \times D \times (0, 1), V_\#) \cap L^2((0, T) \times D, H_\#^1((0, 1), V'_\#))$ such that we can extract a subsequence from $\{u^\varepsilon\}_\varepsilon$ so that

$$\nabla u^\varepsilon \xrightarrow{\text{ts-ms}} \nabla_x u_0 + \nabla_y u_1.$$

The functions u_0 and u_1 satisfy the problem

$$\begin{aligned} & \left\langle \frac{\partial u_0}{\partial t}(t, \cdot), \phi_0(\cdot) \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial u_1}{\partial \tau}(t, x, \tau, \cdot), \phi_1(x, \tau, \cdot) \right\rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y a(t, x, \tau, y) (\nabla_x u_0(t, x) + \nabla_y u_1(t, x, \tau, y)) \\ & \quad \cdot (\nabla_x \phi_0(x) + \nabla_y \phi_1(x, \tau, y)) dy d\tau dx \\ & = \int_D f(t, x) \phi_0(x) dx \end{aligned} \quad (2.3)$$

for all $\phi_0 \in V$ and $\phi_1 \in L^2(D \times (0, 1), V_\#)$, with the initial condition $u_0(0, x) = g$.

Proof. Let $\psi_0 \in \mathcal{D}((0, T) \times D)$ and $\psi_1 \in \mathcal{D}((0, T) \times D, C_\#^\infty((0, 1), C_\#^\infty(Y)))$ be such that

$$\int_Y \psi_1(t, x, \tau, y) dy = 0$$

for all $t, x, \tau \in (0, T) \times D \times (0, 1)$. Let

$$\psi^\varepsilon(t, x) = \psi_0(t, x) + \varepsilon \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}).$$

We have from (2.2) that

$$\begin{aligned} & - \int_0^T \int_D u^\varepsilon \frac{\partial \psi^\varepsilon}{\partial t} dx dt + \int_0^T \int_D a(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \nabla u^\varepsilon(t, x) \cdot \nabla \psi^\varepsilon(t, x) dx dt \\ & = \int_0^T \int_D f(t, x) \psi^\varepsilon(t, x) dx dt. \end{aligned}$$

We note that

$$\nabla \psi^\varepsilon(t, x) = \nabla_x \psi_0(t, x) + \varepsilon \nabla_x \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) + \nabla_y \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}).$$

We then have

$$\begin{aligned}
& - \int_0^T \int_D u^\varepsilon \left(\frac{\partial \psi_0}{\partial t}(t, x) + \varepsilon \frac{\partial \psi_1}{\partial t}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \frac{\partial \psi_1}{\partial \tau}(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \right) dx dt \\
& + \int_0^T \int_D a(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \nabla u^\varepsilon(t, x) \\
& \quad \cdot \left(\nabla_x \psi_0(t, x) + \varepsilon \nabla_x \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) + \nabla_y \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \right) dx dt \\
& = \int_0^T \int_D f(t, x) (\psi_0(t, x) + \varepsilon \psi_1(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})) dx dt.
\end{aligned}$$

Passing to the multiscale convergence limit, using Propositions 2.3 and 2.4, we get

$$\begin{aligned}
& - \int_0^T \int_D u_0(t, x) \frac{\partial \psi_0}{\partial t}(t, x) dx dt \\
& - \int_0^T \int_D \int_0^1 \int_Y u_1(t, x, \tau, y) \frac{\partial \psi_1}{\partial \tau}(t, x, \tau, y) dy dx d\tau dt \\
& + \int_0^T \int_0^1 \int_D \int_Y a(t, x, \tau, y) (\nabla_x u_0(t, x) + \nabla_y u_1(t, x, \tau, y)) \\
& \quad \cdot (\nabla_x \psi_0(t, x) + \nabla_y \psi_1(t, x, \tau, y)) dy d\tau dx dt \\
& = \int_0^T \int_D f(t, x) \psi_0(t, x) dx dt. \tag{2.4}
\end{aligned}$$

By a density argument, we deduce that (2.4) hold for all $\psi_0 \in L^2((0, T), V)$ and $\psi_1 \in L^2((0, T) \times D, H_\#^1((0, 1), V_\#))$. From this, we deduce (2.3). To show the initial condition $u_0(0, \cdot) = g$, we first note that $\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_0}{\partial t}$ in $L^2((0, T), V')$. Let $\psi \in C^\infty((0, T) \times D)$ so that $\psi(T, \cdot) = 0$. We have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D u^\varepsilon \frac{\partial \psi}{\partial t} dx dt = \int_0^T \int_D u_0 \frac{\partial \psi}{\partial t} dx dt \\
& = - \int_0^T \int_D \frac{\partial u_0}{\partial t} \psi dx dt - \int_D u_0(0, x) \psi(0, x) dx.
\end{aligned}$$

On the other hand

$$\int_0^T \int_D u^\varepsilon \frac{\partial \psi}{\partial t} dx dt$$

$$\begin{aligned}
&= - \int_0^T \int_D \frac{\partial u^\varepsilon}{\partial t} \psi dx dt - \int_D u^\varepsilon(0, x) \psi(0, x) dx \\
&\rightarrow - \int_0^T \int_D \frac{\partial u_0}{\partial t} \psi dx dt - \int_D g(x) \psi(0, x) dx
\end{aligned}$$

when $\varepsilon \rightarrow 0$. These imply that $u_0(0, x) = g(x)$. $\square$

Proposition 2.6. Problem (2.3) has a unique solution. The whole sequence $\{u^\varepsilon\}_\varepsilon$ time-space multiscale converges to the solution (u_0, u_1) of (2.3).

Proof. We show that problem (2.3) has solution $u_0 = 0$ and $u_1 = 0$ when $f = 0$ and $g = 0$. We note that $\int_D \int_0^1 \langle \frac{\partial u_1}{\partial \tau}, u_1 \rangle_{H_\#} d\tau dx = 0$ due to the periodicity of u_1 with respect to τ . From (2.3), letting $\phi_0 = u_0(t, \cdot)$ and $\phi_1 = u_1(t, \cdot, \cdot, \cdot)$, and taking the integral from 0 to T with respect to t , we have

$$\frac{1}{2} \|u_0(T)\|_H^2 + \int_0^T \int_D \int_0^1 \int_Y a(\nabla_x u_0 + \nabla_y u_1) \cdot (\nabla_x u_0 + \nabla_y u_1) dy d\tau dx dt = 0.$$

From (2.1), we deduce that $u_0 = 0$ and $u_1 = 0$. $\square$

Remark 2.7. For the case where $k < 2$, u_1 depends on τ but τ only plays the role of a parameter in the multiscale homogenized equation. The derivative $\frac{\partial u_1}{\partial \tau}$ is not present. For the case $k > 2$, u_1 no longer depends on τ . The problem is equivalent to that of the multiscale coefficient $\int_Y a(t, x, \tau, y) d\tau$ which is independent of τ . Details can be found in [16].

2.3 FE approximations

In this section, we establish the convergence of the Crank-Nicolson scheme for solving equation (2.3). We first consider convergence for general FE space. We then consider the case of full tensor product and sparse tensor product FE approximations for u_1 . The main difficulty of deriving an error

estimate for the Crank-Nicolson scheme is due to the fact that the term $\int_0^1 < \frac{\partial u_1}{\partial \tau}, \phi_1 >_{H_\#} d\tau$ in the limiting equation (2.3) is not symmetric. However, we can use the scheme in [34] to show the convergence of the numerical method. To get the convergence, we need to assume regularity for $\frac{\partial u_1}{\partial \tau}$.

2.3.1 General FE approximation

We denote by V_1 the space $L^2(D \times (0, 1), V_\#)$. Let $V_0^L \subset V$ and $V_1^L \subset L^2(D \times (0, 1), V_\#) \cap L^2(D, H_\#^1((0, 1), V'_\#))$ be FE spaces. Let M be an integer. Let $\Delta t = T/M$. We consider the time sequence $0 = t_0 < t_1 < \dots < t_M = T$ where $t_m = m\Delta t$. Let $g^L \in V_0^L$ be an approximation of g . Let $t_{m+1/2} = t_m + \Delta t/2$. We consider the problem: Find $U_{0,m}^L \in V_0^L, U_{1,m}^L \in V_1^L$ for $m = 1, \dots, M$ so that

$$\begin{aligned} & \left\langle \frac{U_{0,m+1}^L - U_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial}{\partial \tau} \frac{U_{1,m+1}^L + U_{1,m}^L}{2}, \phi_1 \right\rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y a(t_{m+1/2}, x, \tau, y) \left(\nabla_x \frac{U_{0,m}^L + U_{0,m+1}^L}{2} + \nabla_y \frac{U_{1,m}^L + U_{1,m+1}^L}{2} \right) \\ & \quad \cdot (\nabla_x \phi_0 + \nabla_y \phi_1) dy d\tau dx \\ & = \int_D f(t_{m+1/2}, x) \phi_0(x) dx, \end{aligned} \quad (2.5)$$

for all $\phi_0 \in V_0^L$ and $\phi_1 \in V_1^L$. Let $Z_{0,m}^L = u_0(t_m) - U_{0,m}^L$, $Z_{1,m}^L = u_1(t_m) - U_{1,m}^L$. This is the Crank-Nicolson method for approximating (2.3). We denote by

$$Z_{0,m+1/2}^L = \frac{1}{2}(Z_{0,m}^L + Z_{0,m+1}^L), \quad Z_{1,m+1/2}^L = \frac{1}{2}(Z_{1,m}^L + Z_{1,m+1}^L).$$

We then have the following result.

Proposition 2.8. Problem (2.5) has a unique solution.

Proof. Let B be the Gram matrix of the basis functions of V_0^L in the inner product of H . Let M be the matrix describing the interaction of the basis functions of V_1^L with themselves in the bilinear form representing the second term on the left hand side of (2.5). Let A be the matrix describing the interaction of the basis functions of $V_0^L \times V_1^L$ in the bilinear form representing the third term on the left hand side of (2.5). Let $F_{m+1/2}$ be the vector representing the interaction of $f(t_{m+1/2})$ with the basis functions of V_0^L in the linear form of the right hand side in (2.5). Let $\mathbf{c}_{0,m}$ be the coordinate vector of $U_{0,m}^L$ in the linear expansion with respect to the basis functions of V_0^L . Let $\mathbf{c}_{1,m}$ be the coordinate vector of $U_{1,m}^L$ in the linear expansion with respect to the basis functions of V_1^L . Let $\mathbf{c}_m$ be the coordinate vector of the expansion of $(U_{0,m}^L, U_{1,m}^L)$ with respect to the basis functions of $V_0^L \times V_1^L$, i.e. $\mathbf{c}_m = (\mathbf{c}_{0,m}, \mathbf{c}_{1,m})$. Let $\mathbf{d}_0$ be the coordinate vector of ϕ_0 in the expansion with respect to the basis functions of V_0^L . Let $\mathbf{d}_1$ be the coordinate vector of ϕ_1 in the basis functions of V_1^L . Let $\mathbf{d}$ be the coordinate vector of (ϕ_0, ϕ_1) in the expansion with respect to the basis function of $V_0^L \times V_1^L$, i.e. $\mathbf{d} = (\mathbf{d}_0, \mathbf{d}_1)$. Problem (2.5) can be written as

$$\begin{aligned} & \frac{1}{\Delta} B \mathbf{c}_{0,m+1} \cdot \mathbf{d}_0 + \frac{1}{2} M \mathbf{c}_{1,m+1} \cdot \mathbf{d}_1 + \frac{1}{2} A \mathbf{c}_{m+1} \cdot \mathbf{d} \\ = & F_{m+1/2} \cdot \mathbf{d}_0 - \frac{1}{\Delta t} B \mathbf{c}_{0,m} \cdot \mathbf{d}_0 - \frac{1}{2} M \mathbf{c}_{1,m} \cdot \mathbf{d}_1 - \frac{1}{2} A \mathbf{c}_m \cdot \mathbf{d}. \end{aligned} \quad (2.6)$$

We denote by $\mathcal{A} : \mathbb{R}^{\dim V_0^L + \dim V_1^L} \times \mathbb{R}^{\dim V_0^L + \dim V_1^L}$ be the bilinear form

$$\mathcal{A}(\mathbf{c}, \mathbf{d}) = \frac{1}{\Delta t} B \mathbf{c}_0 \cdot \mathbf{d}_0 + \frac{1}{2} M \mathbf{c}_1 \cdot \mathbf{d}_1 + \frac{1}{2} A \mathbf{c} \cdot \mathbf{d}.$$

As M is a skew symmetric matrix, we have that

$$\mathcal{A}(\mathbf{c}, \mathbf{c}) = \frac{1}{\Delta t} B \mathbf{c}_0 \cdot \mathbf{c}_0 + \frac{1}{2} A \mathbf{c} \cdot \mathbf{c}.$$

As B is the Gram matrix, and A is a positive definite matrix, $\mathcal{A}(\mathbf{c}, \mathbf{c}) \geq c|\mathbf{c}|^2$ where $|\cdot|$ denote the Eulidean norm in $\mathbb{R}^{\dim V_0^L + \dim V_1^L}$. Therefore the bilinear

form $\mathcal{A}$ is coercive. It is also bounded. Thus equation (2.6) has a unique solution. $\square$

When the solutions of equation 2.3 is sufficiently regular with respect to t , we show the convergence of the Crank-Nicolson scheme. We follow the approach by [34].

Theorem 2.9. Assume that $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_1 \in C^2([0, T], L^2(D \times (0, 1), V_{\#}))$ and $\frac{\partial u_1}{\partial \tau} \in C^2([0, T], L^2(D \times (0, 1), V'_{\#}))$. Then

$$\begin{aligned} & \|Z_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|Z_{0,m+1/2}^L\|_V^2 + \|Z_{1,m+1/2}^L\|_{V_1}^2) \\ & \leq c\Delta t \left(\sum_{m=0}^{M-1} \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 \right. \\ & \quad + \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V'_1}^2 \\ & \quad + \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_{L^2(D)}^2 \Bigg) \\ & \quad + \max_{m=1,\dots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_{L^2(D)}^2 + \|g - g^L\|_{L^2(D)}^2 + c(\Delta t)^4 \quad (2.7) \end{aligned}$$

for all $\{\tilde{u}_{0,m}^L, m = 0, \dots, M\} \subset V_0^L$ and $\{\tilde{u}_{1,m}^L, m = 1, \dots, M\} \subset V_1^L$.

Proof. Let $\rho_{0,m} = \frac{1}{\Delta t}(u_0(t_{m+1}) - u_0(t_m)) - \frac{\partial u_0}{\partial t}(t_{m+1/2})$, $\zeta_{0,m} = \frac{1}{2}(u_0(t_{m+1}) + u_0(t_m)) - u_0(t_{m+1/2})$, $\zeta_{1,m} = \frac{1}{2}(u_1(t_{m+1}) + u_1(t_m)) - u_1(t_{m+1/2})$ and $\xi_{1,m} = \frac{1}{2}(\frac{\partial u_1}{\partial \tau}(t_{m+1}) + \frac{\partial u_1}{\partial \tau}(t_m)) - \frac{\partial u_1}{\partial \tau}(t_{m+1/2})$. Since $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_1 \in C^2([0, T], L^2(D \times (0, 1), V_{\#}))$ and $\frac{\partial u_1}{\partial \tau} \in C^2([0, T], L^2(D \times (0, 1), V'_{\#}))$, we deduce that

$$\|\rho_{0,m}\|_{L^2(D)} \leq c(\Delta t)^2,$$

$$\|\zeta_{0,m}\|_V \leq c(\Delta t)^2,$$

$$\|\zeta_{1,m}\|_{V_1} \leq c(\Delta t)^2,$$

$$\|\xi_{1,m}\|_{L^2(D \times (0,1), V'_{\#})} \leq c(\Delta t)^2$$

where the constant c does not depend on m . From (2.3) and (2.5) considered at $t = t_{m+1/2}$ we deduce that

$$\begin{aligned}
& \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H - \langle \rho_{0,m}, \phi_0 \rangle_H \\
& + \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), \phi_1 \right\rangle_{H_\#} d\tau dx + \int_D \int_0^1 \langle \xi_1, \phi_1 \rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \\
& \quad \cdot (\nabla_x \phi_0 + \nabla_y \phi_y) dy d\tau dx \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) (\nabla_x \zeta_{0,m} + \nabla_y \zeta_{1,m}) \cdot (\nabla_x \phi_0 + \nabla_y \phi_1) dy d\tau dx = 0.
\end{aligned} \tag{2.8}$$

Consider

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} \right\rangle_H \\
&+ \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right\rangle_{H_\#} \\
&+ \int_D \int_0^1 \int_Y a(t_{m+1/2}) \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \\
& \quad \cdot \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \\
&\geq \frac{1}{2\Delta t} (\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) \\
& \quad + \gamma (\|Z_{0,m+1/2}^L\|_V^2 + \|Z_{1,m+1/2}^L\|_{V_1}^2).
\end{aligned} \tag{2.9}$$

For $\{\tilde{u}_{0,m}^L, m = 0, \dots, M\} \subset V_0^L$ and $\{\tilde{u}_{1,m}^L, m = 1, \dots, M\} \subset V_1^L$, we have

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&+ \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (\tilde{u}_0 - U_0^L)_{m+1/2} \right\rangle_H
\end{aligned}$$

$$\begin{aligned}
& + \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), (u_1 - \tilde{u}_1)_{m+1/2} \right\rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), (\tilde{u}_1 - U_1^L)_{m+1/2} \right\rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \cdot \\
& \quad (\nabla_x (u_0 - \tilde{u}_0)_{m+1/2} + \nabla_y (u_1 - \tilde{u}_1)_{m+1/2}) dy d\tau dx \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \cdot \\
& \quad (\nabla_x (\tilde{u}_0 - U_0^L)_{m+1/2} + \nabla_y (\tilde{u}_1 - U_1^L)_{m+1/2}) dy d\tau dx.
\end{aligned}$$

From (2.5) we have

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& - \int_D \int_0^1 \left\langle \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2}, \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\rangle_{H_\#} \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) \cdot \\
& \quad (\nabla_x (u_0 - \tilde{u}_0)_{m+1/2} + \nabla_y (u_1 - \tilde{u}_1)_{m+1/2}) dy d\tau dx \\
& - \langle \rho_{0,m}, (\tilde{u}_0 - U_0^L)_{m+1/2} \rangle_H - \int_D \int_0^1 \langle \xi_{1,m}, (\tilde{u}_1 - U_1^L)_{m+1/2} \rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_Y a(t_{m+1/2}) (\nabla_x \zeta_{0,m} + \nabla_y \zeta_{1,m}) \cdot \\
& \quad (\nabla_x (\tilde{u}_0 - U_0^L)_{m+1/2} + \nabla_y (\tilde{u}_1 - U_1^L)_{m+1/2}) dy d\tau dx.
\end{aligned}$$

We note that $(\tilde{u}_0 - U_0^L)_{m+1/2} = (\tilde{u}_0 - u_0)_{m+1/2} + Z_{0,m+1/2}^L$ and $(\tilde{u}_1 - U_1^L)_{m+1/2} = (\tilde{u}_1 - u_1)_{m+1/2} + Z_{1,m+1/2}^L$. For a positive constant $\delta > 0$, using the Cauchy-Schwartz inequality, we have

$$I \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H$$

$$\begin{aligned}
& +\delta\|Z_{1,m+1/2}^L\|_{V_1}^2 + c\left\|\frac{\partial}{\partial\tau}(u_1 - \tilde{u}_1)_{m+1/2}\right\|_{V_1'} \\
& +\delta\|Z_{0,m+1/2}^L\|_V^2 + \delta\|Z_{1,m+1/2}^L\|_{V_1}^2 \\
& +c\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 + c\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 \\
& +c\|\rho_{0,m}\|_H^2 + c\|(\tilde{u}_0 - u_0)_{m+1/2}\|_H^2 + \delta\|Z_{0,m+1/2}^L\|_H^2 \\
& +c\|\xi_{1,m}\|_{V_1'}^2 + c\|(\tilde{u}_1 - u_1)_{m+1/2}\|_{V_1}^2 + \delta\|Z_{1,m+1/2}^L\|_{V_1}^2 \\
& +c\|\zeta_{0,m}\|_V^2 + c\|\zeta_{1,m}\|_{V_1}^2 + c\|(\tilde{u}_0 - u_0)_{m+1/2}\|_V^2 + \delta\|Z_{0,m+1/2}^L\|_V^2 \\
& +c\|(\tilde{u}_1 - u_1)_{m+1/2}\|_{V_1}^2 + \delta\|Z_{1,m+1/2}^L\|_V^2.
\end{aligned}$$

From this and (2.9), choosing δ sufficiently small, we have

$$\begin{aligned}
& \frac{1}{2\Delta t}(\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) + c(\|Z_{0,m+1/2}^L\|_V^2 + \|Z_{1,m+1/2}^L\|_{V_1}^2) \\
& \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& +c\left\|\frac{\partial}{\partial\tau}(u_1 - \tilde{u}_1)_{m+1/2}\right\|_{V_1'} \\
& +c\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 + c\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 + c(\Delta t)^4.
\end{aligned}$$

Fixing an integer $P \leq M$, taking the sum for $m = 0, \dots, P-1$, we have

$$\begin{aligned}
& \|Z_{0,P}^L\|_H^2 - \|Z_{0,0}^L\|_H^2 + c\Delta t \sum_{m=0}^{P-1} (\|Z_{0,m+1/2}^L\|_V^2 + \|Z_{1,m+1/2}^L\|_{V_1}^2) \\
& \leq c\Delta t \sum_{m=0}^{P-1} \left(\left\|\frac{\partial}{\partial\tau}(u_1 - \tilde{u}_1)_{m+1/2}\right\|_{V_1'} + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 \right. \\
& \quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 \right) + cP(\Delta t)^5 \\
& + 2\Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H. \tag{2.10}
\end{aligned}$$

We note that

$$\Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H$$

$$\begin{aligned}
&= \langle Z_{0,P}^L, (u_0 - \tilde{u}_0)_{P-1/2} \rangle_H - \langle Z_{0,0}^L, (u_0 - \tilde{u}_0)_{1/2} \rangle_H \\
&\quad + \Delta t \sum_{m=1}^{P-1} \left\langle \frac{Z_{0,m}^L}{\Delta t}, -(u_0 - \tilde{u}_0)_{m-1/2} + (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&\leq \delta \|Z_{0,P}^L\|_H^2 + c \|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + \|Z_{0,0}^L\|_H^2 + \|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
&\quad + \delta \Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 \\
&\quad + c \Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2
\end{aligned}$$

which is a consequence of the Cauchy-Schwartz inequality; δ is an arbitrary positive constant. We note that

$$\Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 \leq \Delta t P \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2 \leq T \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2.$$

From this and (2.10), choosing δ sufficiently small, we have

$$\begin{aligned}
&\|Z_{0,P}^L\|_H^2 \\
&\leq c \Delta t \sum_{m=0}^{P-1} \left(\left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^2 + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 \right. \\
&\quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 \right) \\
&\quad + c(\Delta t)^4 + c \|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + 2\|Z_{0,0}^L\|_H^2 + \|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
&\quad + c \Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \\
&\quad + \delta T \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2.
\end{aligned}$$

Choosing δ sufficiently small, we have

$$\begin{aligned}
&\max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2 \\
&\leq c \Delta t \sum_{m=0}^{M-1} \left(\left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^2 + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^2 \right. \\
&\quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 \right)
\end{aligned}$$

$$+c(\Delta t)^4 + c \max_{m=1,\dots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_H^2 + c\|Z_{0,0}^L\|_H^2$$

From this we get the conclusion. $\square$

2.3.2 Full tensor FE

To approximate $u_1(t) \in L^2(D, L^2((0, 1), V_{\#})) \cap L^2(D, H_{\#}^1((0, 1), V'_{\#})) \cong L^2(D) \otimes L^2(0, 1) \otimes V_{\#} \cap L^2(D) \otimes H_{\#}^1(0, 1) \otimes V'_{\#}$, we use tensor product FEs. Let $h_l = 2^{-l}$. Assuming that D is a polygonal domain in $\mathbb{R}^d$, we divide the domain D into a hierarchy of sets of triangular simplices $\{\mathcal{T}^l\}_{l \geq 0}$. Each simplex in the set $\mathcal{T}^l$ of mesh size $O(h_l)$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles when $d = 2$ and 8 tetrahedral when $d = 3$. For each simplex $T \in \mathcal{T}^l$, we denote by $P^1(T)$ the set of linear polynomials in T . Similarly, we divide Y into a hierarchy $\{\mathcal{T}_{\#}^l\}_{l \geq 0}$ of sets of simplices with mesh size $O(h_l)$ which are distributed periodically. For each $l = 1, 2, \dots$, the interval $(0, 1)$ for the variable τ is divided into sets $\mathcal{T}_{\#\tau}^l$ of 2^l intervals of length 2^{-l} . We define the following FE spaces:

$$\begin{aligned} V_0^l &= \{\phi \in H_0^1(D), \phi \in P^1(T) \forall T \in \mathcal{T}^l\}; \\ V^l &= \{\phi \in H^1(D), \phi \in P^1(T) \forall T \in \mathcal{T}^l\}; \\ V_{\#}^l &= \{\phi \in H_{\#}^1(Y), \phi \in P^1(T) \forall T \in \mathcal{T}_{\#}^l\}; \\ V_{\#\tau}^l &= \{\phi \in H_{\#}^1(0, 1), \phi \in P^1(T) \forall T \in \mathcal{T}_{\#\tau}^l\}. \end{aligned}$$

We then have the following approximation property

$$\begin{aligned} \inf_{w^l \in V_0^l} \|w - w^l\|_{H_0^1(D)} &\leq ch_l \|w\|_{H^2(D)}, \quad \forall w \in H_0^1(D) \cap H^2(D); \\ \inf_{w^l \in V^l} \|w - w^l\|_{L^2(D)} &\leq ch_l \|w\|_{H^1(D)}, \quad \forall w \in H^1(D); \\ \inf_{w^l \in V_{\#}^l} \|w - w^l\|_{H_{\#}^1(Y)} &\leq ch_l \|w\|_{H_{\#}^2(Y)}, \quad \forall w \in H_{\#}^2(Y); \end{aligned}$$

$$\begin{aligned}
\inf_{w^l \in V_{\#}^l} \|w - w^l\|_{L^2(Y)} &\leq ch_l \|w\|_{H_{\#}^1(Y)}, \quad \forall w \in H_{\#}^1(Y) \\
\inf_{w^l \in V_{\#\tau}^l} \|w - w^l\|_{H_{\#}^1((0,1))} &\leq ch_l \|w\|_{H_{\#}^2((0,1))}, \quad \forall w \in H_{\#}^2(0,1) \\
\inf_{w^l \in V_{\#}^l} \|w - w^l\|_{L^2((0,1))} &\leq ch_l \|w\|_{H_{\#}^1(Y)}, \quad \forall w \in H_{\#}^1((0,1)).
\end{aligned}$$

For approximating $u_1(t)$ we define the full tensor product FE space as

$$\bar{V}_1^L = V^L \otimes V_{\#\tau}^L \otimes V_{\#}^L.$$

Let $\mathcal{H}$ be the regularity space

$$\begin{aligned}
H^1(D, H_{\#}^1((0,1), H_{\#}^1(Y))) \cap L^2(D, H_{\#}^2((0,1), H_{\#}^1(Y))) \\
\cap L^2(D, H_{\#}^1((0,1), H_{\#}^2(Y)))
\end{aligned}$$

with the norm

$$\begin{aligned}
\|w\|_{\mathcal{H}} &= \|w\|_{H^1(D, H_{\#}^1((0,1), H_{\#}^1(Y)))} + \|w\|_{L^2(D, H_{\#}^2((0,1), H_{\#}^1(Y)))} \\
&\quad + \|w\|_{L^2(D, H_{\#}^1((0,1), H_{\#}^2(Y)))}.
\end{aligned}$$

The space $\mathcal{H}$ consists of functions which are more regular than $L^2(D, H^1((0,1), V_{\#}))$ with respect to each variable x , τ and y but the higher regularity needs not occur at the same time. With this regularity, we then have the following approximation property.

Proposition 2.10. For $w \in \mathcal{H}$

$$\inf_{w^L \in V_1^L} \|w - w^L\|_{L^2(D, H^1((0,1), V_{\#}))} \leq ch_L \|w\|_{\mathcal{H}}.$$

The proof of this proposition is quite standard. It is similar to the proof for similar results in Bungartz and Griebel [21] and Hoang and Schwab [49]. We refer to these references for details. We denote the solution of the Crank-Nicholson scheme (2.5) when $V_1^L = \bar{V}_1^L$ as $\bar{U}_{0,m}^L$ and $\bar{U}_{1,m}^L$ respectively, and $Z_{0,M}^L$, $Z_{0,m+1/2}^L$ and $Z_{1,m+1/2}^L$ as $\bar{Z}_{0,m}^L$, $\bar{Z}_{0,m+1/2}^L$ and $\bar{Z}_{1,m+1/2}^L$. We therefore have the following result.

Theorem 2.11. Assume that $u_0 \in C^3([0, T], H) \cap C^2([0, T], V) \cap C^1([0, T], H^2(D))$, $u_1 \in C^2([0, T], V_1) \cap C([0, T], \mathcal{H})$. If we choose the initial condition g^L such that $\|g - g^L\|_V \leq ch_L$. Then

$$\|\bar{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\bar{Z}_{0,m+1/2}^L\|_V^2 + \|\bar{Z}_{1,m+1/2}^L\|_{V_1}^2) \leq c((\Delta t)^4 + h_L^2). \quad (2.11)$$

Proof. We estimate the right hand side of (2.7). As $u_1 \in C([0, T], \mathcal{H})$, we can choose $\tilde{u}_{1,m} \in V_1^L$ for $m = 1, \dots, M$ such that

$$\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{L^2(D, H^1((0,1), V_{\#}))}^2 \leq ch_L (\|u_1(t_m)\|_{\mathcal{H}} + \|u_1(t_{m+1})\|_{\mathcal{H}}) \leq ch_L,$$

where c does not depend on t . Therefore

$$\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^2 + \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^2 \leq ch_L.$$

As $u_0 \in C([0, T], H^2(D)) \subset C([0, T], C(\bar{D}))$ (we consider $d = 2, 3$), we can define the interpolation $I^L u_0(t) \in V_0^L$ whose value at each node equals the value of $u_0(t)$. We note that

$$\|u_0(t) - I^L u_0(t)\|_V \leq ch_L \|u_0(t)\|_{H^2(D)} \leq ch_L.$$

Choosing $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V \leq ch_L$$

where c does not depend on m . For the other terms in the right hand side of (2.7), we have

$$\begin{aligned} & \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_{L^2(D)}^2 \\ & \leq \sum_{m=1}^{M-1} \frac{1}{2} \left(\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_{L^2(D)}^2 \right. \end{aligned}$$

$$+ \left\| \frac{(u_0 - \tilde{u}_0)_m - (u_0 - \tilde{u}_0)_{m-1}}{\Delta t} \right\|_{L^2(D)}^2 \Bigg).$$

Choosing $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L^2 \left\| \frac{\partial u_0}{\partial t} \right\|_{H^2(D)}.$$

We estimate this using the procedure of [34]

$$\begin{aligned} & \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \\ &= \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) dt \right\|_H^2 (\Delta t)^{-2} \\ &\leq \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \\ &\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^2(D)} dt \right)^2 (\Delta t)^{-2} \\ &\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^2(D)}^2 dt \right) (\Delta t)^{-1}. \end{aligned}$$

From this, we deduce that

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_{L^2(D)}^2 \leq ch_L^4.$$

We then get the conclusion. $\square$

2.3.3 Sparse tensor FEs

The space of the full tensor FE product space $\bar{V}_1$ is $O(2^{(2d+1)L})$ which is very large where L is large. In this section, we use the sparse product FE spaces which is a subset of the full tensor product FE spaces with much smaller dimensions but produce essentially equal approximating accuracy.

To define the sparse tensor product FE spaces, we consider the following orthogonal projections. Let

$$P^l : L^2(D) \rightarrow V^l, \quad P_{\#\tau}^l : H_{\#}^1((0, 1)) \rightarrow V_{\#\tau}^l, \quad P_{\#}^l : H_{\#}^1(Y) \rightarrow V_{\#}^l$$

be the orthogonal projections. We then define the following detailed spaces

$$W^l = (P^l - P^{l-1})V^l, \quad W_{\#\tau}^l = (P_{\#\tau}^l - P_{\#\tau}^{l-1})V_{\#\tau}^l, \quad W_{\#}^l = (P_{\#}^l - P_{\#}^{l-1})V_{\#}^l,$$

with the convention that $P^{-1} = 0$, $P_{\#\tau}^{-1} = 0$, and $P_{\#}^{-1} = 0$. We note that

$$V^L = \bigoplus_{0 \leq l \leq L} W^l, \quad V_{\#\tau}^L = \bigoplus_{0 \leq l \leq L} W_{\#\tau}^l, \quad V_{\#}^L = \bigoplus_{0 \leq l \leq L} W_{\#}^l,$$

with respect to the norms of $L^2(D)$, $H_{\#}^1((0, 1))$ and $H_{\#}^1(Y)$ respectively.

The full tensor product FE spaces are

$$\bar{V}_1^L = \bigoplus_{0 \leq l_0, l_1, l_2 \leq L} W^{l_0} \otimes W_{\#\tau}^{l_1} \otimes W_{\#}^{l_2}.$$

We define the sparse tensor product FE spaces as

$$\hat{V}_1^L = \bigoplus_{0 \leq l_0 + l_1 + l_2 \leq L} W^{l_0} \otimes W_{\#\tau}^{l_1} \otimes W_{\#}^{l_2}.$$

To quantify the approximation of u_1 using the spaces $\hat{V}_1^L$, we define the regularity spaces $\hat{\mathcal{H}}$ that contains functions $w \in L^2(D, H_{\#}^1((0, 1), H_{\#}^1(Y)))$ so that

$$\frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} \frac{\partial^{\alpha_1}}{\partial \tau_1^{\alpha_1}} \frac{\partial^{|\alpha_2|}}{\partial y^{\alpha_2}} w \in L^2(D \times (0, 1) \times Y),$$

for all $\alpha_0 \in \mathbb{N}_0^d$ with $|\alpha_0| \leq 1$, $\alpha_1 \in \{0, 1, 2\}$ and $\alpha_2 \in \mathbb{N}_0^d$ with $|\alpha_2| \leq 2$. In other words, $\hat{\mathcal{H}} = H^1(D, H_{\#}^2((0, 1), H_{\#}^2(Y)))$. We then have the following approximation result. The space $\hat{\mathcal{H}}$ is a subspace of $\mathcal{H}$; it contains functions which possess regularity with respect t , τ and y at the same time.

Proposition 2.12. Assume that $w \in \hat{\mathcal{H}}$. Then

$$\inf_{w^L \in \hat{V}_1^L} \|w - w^L\|_{L^2(D, H_{\#}^1((0,1), H_{\#}^1(Y)))} \leq cLh_L \|w\|_{\hat{\mathcal{H}}}.$$

The proof of this proposition is similar to those for sparse tensor product FE approximations in [21] and [49]. We refer to these references for details. We denote the solution of the Crank-Nicholson scheme in (2.5) using the sparse tensor FE space $\hat{V}_1^L$ as $\hat{U}_{0,m}^L$ and $\hat{U}_{1,m}^L$; and denote by $Z_{0,M}^L$, $Z_{0,m+1/2}^L$ and $Z_{1,m+1/2}^L$ as $\hat{Z}_{0,M}^L$, $\hat{Z}_{0,m+1/2}^L$ and $\hat{Z}_{1,m+1/2}^L$ respectively. We then have

Theorem 2.13. Assume that $u_0 \in C^3([0, T], H) \cap C^2([0, T], V) \cap C^1([0, T], H^2(D))$, $u_1 \in C^2([0, T], V_1) \cap C([0, T], \hat{\mathcal{H}})$. If we choose the initial condition g^L such that $\|g - g^L\|_V \leq cLh_L$. Then

$$\|\hat{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\hat{Z}_{0,m+1/2}^L\|_V^2 + \|\hat{Z}_{1,m+1/2}^L\|_{V_1}^2) \leq c((\Delta t)^4 + L^2 h_L^2). \quad (2.12)$$

The proof is similar to that for theorem 2.11

Remark 2.14. The dimension of the full tensor product FE space $\bar{V}_1^L$ is $O(2^{(2d+1)L})$. The dimension of the sparse tensor product FE space $\hat{V}_1^L$ is $O(L^2 2^{dL})$ which is much less than the dimension of the full tensor product FE spaces $\bar{V}_1^L$.

Remark 2.15. Another way to construct the sparse tensor product FE spaces is to use the equivalent norms of wavelet basis functions. We assume that:

(i) For each $j \in \mathbb{N}_0^d$, there exists a set $I^j \subset \mathbb{N}_0^d$ and a set of basis functions ϕ^{jk} , $k \in I^j$, such that $V^l = \text{span}\{\phi^{jk} : |j|_{\infty} \leq l\}$. There are constants $c_2 > c_1 > 0$ such that if $\phi = \sum_{|j|_{\infty} \leq l, k \in I^j} \phi^{jk} c_{jk} \in V^l$, then :

$$c_1 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I^j}} c_{jk}^2 \leq \|\phi\|_{L^2(D)}^2 \leq c_2 \sum_{\substack{|j|_{\infty} \leq l \\ k \in I^j}} c_{jk}^2.$$

(ii) For each $j \in \mathbb{N}_0$, there exists a set $I_0^j \subset \mathbb{N}_0$ and a set of basis functions ϕ_0^{jk} , $k \in I_0^j$, such that $V_{\#\tau}^l = \text{span}\{\phi_0^{jk} : |j|_\infty \leq l\}$. There are constants $c_4 > c_3 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I_0^j} \phi_0^{jk} c_{jk} \in V_{\#\tau}^l$, then

$$c_3 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} c_{jk}^2 \leq \|\phi\|_{H_{\#}^1((0,1))}^2 \leq c_4 \sum_{\substack{|j|_\infty \leq l \\ k \in I_0^j}} c_{jk}^2.$$

(iii) For each $j \in \mathbb{N}_0^d$, there exists a set $I_1^j \subset \mathbb{N}_0^d$ and a set of basis functions $\phi_1^{jk} \in H_{\#}^1(Y)/\mathbb{R}$, $k \in I_1^j$, such that $V_{\#}^l/\mathbb{R} = \text{span}\{\phi_1^{jk} : |j| \leq l\}$. There are constants $c_6 > c_5 > 0$ such that if $\phi = \sum_{|j|_\infty \leq l, k \in I_1^j} \phi_1^{jk} c_{jk}$, then

$$c_5 \sum_{\substack{|j|_\infty \leq l \\ k \in I_1^j}} c_{jk}^2 \leq \|\phi\|_{H_{\#}^1(Y)/\mathbb{R}}^2 \leq c_6 \sum_{\substack{|j|_\infty \leq l \\ k \in I_1^j}} c_{jk}^2.$$

Using the norm equivalences, we define

$$\begin{aligned} W^l &= \text{span}\{\phi^{jk} : |j|_\infty = l\}, \\ W_{\#\tau}^l &= \text{span}\{\phi_0^{jk} : |j|_\infty = l\}, \\ W_{\#}^l &= \text{span}\{\phi_1^{jk} : |j|_\infty = l\}. \end{aligned}$$

Some examples of these basis can be found in [49]

Remark 2.16. We can construct the sparse tensor product by using the hierarchies $\{\mathcal{T}^l\}$, $\{\mathcal{T}_{\#\tau}^l\}$ and $\{\mathcal{T}_{\#}^l\}$. We denote by $\mathcal{S}^l$ the nodes belonging to the set of simplices $\mathcal{T}^l$. We let W^l be the set of basis functions in V^l which equals 1 at one of the nodes of $\mathcal{S}^l \setminus \mathcal{S}^{l-1}$ and equals 0 at other nodes. We construct the spaces $W_{\#\tau}^l$ and $W_{\#}^l$ similarly. The estimate for the sparse tensor product FE spaces still holds.

2.4 Numerical correctors

We employ the FE solutions for the multiscale homogenized problems (2.3) to construct numerical correctors in this section.

2.4.1 Homogenization error

We first establish the homogenized equation from (2.3). Letting $\phi_0 = 0$, we deduce that the solution u_1 can be written as

$$u_1(t, x, \tau, y) = \frac{\partial u_0}{\partial x_i}(t, x) N^i(t, x, \tau, y) \quad (2.13)$$

where

$$N^i(t, x, \tau, y) \in L^2((0, T) \times D \times (0, 1), V_{\#}) \bigcap L^2((0, T) \times D, H_{\#}^1((0, 1), V'_{\#}))$$

is the unique solution of the problem

$$\frac{\partial N^i}{\partial \tau} - \nabla_y \cdot (a(e^i + \nabla_y N^i)) = 0. \quad (2.14)$$

Here e^i is the i th unit vector in $\mathbb{R}^d$. The existence of a unique solution of (2.14) is proved in Lions and Magenes [55] Section 6.2. We can choose N^i such that $\int_0^1 \int_Y N^i(t, x, \tau, y) dy d\tau = 0$. The homogenized coefficient a^0 is defined as

$$a_{ij}^0(t, x) = \int_0^1 \int_Y a_{lk}(t, x, \tau, y) \left(\frac{\partial N^j}{\partial y_k} + \delta_{jk} \right) \left(\frac{\partial N^i}{\partial y_l} + \delta_{il} \right) dy d\tau. \quad (2.15)$$

The homogenized equation is

$$\begin{aligned} \frac{\partial u_0}{\partial t} - \nabla \cdot (a^0 \nabla u_0) &= f, \\ u_0(0, \cdot) &= g. \end{aligned}$$

2.4.2 Numerical correctors

We introduce the operator $\mathcal{T}^\varepsilon$ which reformulates time-space multiscale convergence in terms of weak convergence in $(0, T) \times D \times (0, 1) \times Y$. Using this operator, we will derive a corrector for the general time-space multiscale problem. The operator $\mathcal{T}^\varepsilon : L^1((0, T) \times D) \rightarrow L^1((0, T) \times D \times (0, 1) \times Y)$ is defined as

$$\mathcal{T}^\varepsilon(\Phi) = \Phi \left(\varepsilon^2 \left[\frac{t}{\varepsilon^2} \right]_1 + \varepsilon^2 \tau, \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y \right). \quad (2.16)$$

Let D^ε be the 2ε neighbourhood of D . We have

$$\int_0^T \int_D \Phi(t, x) dx dt = \int_{-2\varepsilon}^{T+2\varepsilon} \int_{D^\varepsilon} \int_0^1 \int_Y \mathcal{T}^\varepsilon(\Phi)(t, x, \tau, y) dy d\tau dx dt; \quad (2.17)$$

Φ is understood as 0 outside $(0, T) \times D$. If $\{w^\varepsilon\}_\varepsilon$ time-space multiscale converges to w_0 in $L^2((0, T) \times D \times (0, 1) \times Y)$ then

$$\mathcal{T}^\varepsilon(w^\varepsilon) \rightharpoonup w_0 \text{ in } L^2((0, T) \times D \times (0, 1) \times Y).$$

To establish the numerical correctors, we define the operator $\mathcal{U}^\varepsilon : L^1((0, T) \times D \times (0, 1) \times Y) \rightarrow L^1((0, T) \times D)$ as

$$\mathcal{U}^\varepsilon(\Phi) = \int_0^1 \int_Y \Phi \left(\varepsilon^2 \left[\frac{t}{\varepsilon^2} \right]_1 + \varepsilon^2 \theta, \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon z, \left\{ \frac{t}{\varepsilon^2} \right\}_1, \left\{ \frac{x}{\varepsilon} \right\} \right) dz d\theta \quad (2.18)$$

where $[\cdot]_1$ and $\{\cdot\}_1$ denote the integer and the fractional parts of a real number, and $[\cdot]$ and $\{\cdot\}$ denote the “integer” and the “fractional” part with respect to the unit cube Y of a vector in $\mathbb{R}^d$. The purpose of introducing this operator $\mathcal{U}^\varepsilon$ is to map the corrector term $u_1(t, x, \tau, y)$ to a function that on depends on t and x only, to derive an approximation of u^ε in $L^2((0, T), H^1(D))$. Let $D^{2\varepsilon}$ be the 2ε neighbourhood of D . We have that

$$\int_{-2\varepsilon^2}^{T+2\varepsilon^2} \int_{D^{2\varepsilon}} \mathcal{U}^\varepsilon(\Phi)(t, x) = \int_0^T \int_D \int_0^1 \int_Y \Phi(t, x, \tau, y) dy d\tau dx dt \quad (2.19)$$

for all $\Phi \in L^1((0, T) \times D \times (0, 1) \times Y)$. The proof of these facts can be found in [29]. We then have the following result.

Proposition 2.17. For the solution of equation (2.3)

$$\|\nabla u^\varepsilon - \nabla u_0 - \mathcal{U}^\varepsilon(\nabla_y u_1)\|_{L^2((0, T) \times D)} = 0.$$

Proof. Let

$$\begin{aligned} I &= \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \right\rangle_H \\ &\quad + \int_0^T \int_D \int_0^1 \int_Y \mathcal{T}^\varepsilon(A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})) (\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)) \\ &\quad \mathcal{T}^\varepsilon((\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)) dy d\tau dx dt. \end{aligned}$$

From (2.2), (2.3), and the fact that $\mathcal{T}^\varepsilon(A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})) \rightarrow A(t, x, \tau, y)$ point-wise, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I &= \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon \right\rangle_H - \left\langle \frac{\partial u_0}{\partial t}, u_0 \right\rangle_H dt \\ &\quad + \int_0^T \int_D A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) \nabla u^\varepsilon \cdot \nabla u^\varepsilon dx dt \\ &\quad - \int_0^T \int_D \int_0^1 \int_Y A(t, x, \tau, y) (\nabla u_0 + \nabla_y u_1) \\ &\quad \cdot (\nabla u_0 + \nabla_y u_1) dy d\tau dx dt \\ &= 0. \end{aligned}$$

As $u^\varepsilon(0) = u_0(0) = g$, we have

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon(T) - u_0(T)\|_H + \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)\|_{L^2((0, T) \times D \times (0, 1) \times Y)} = 0.$$

From (2.18), we have $(\mathcal{U}^\varepsilon(\Phi)(t, x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(t, x)$. From (2.19), we have

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^2((0, T) \times D)}^2 \leq \|\mathcal{U}^\varepsilon(\Phi^2)\|_{L^1((0, T) \times D)} \leq \|\Phi\|_{L^2((0, T) \times D \times (0, 1) \times Y)}^2.$$

We therefore have

$$\begin{aligned}
& \|\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1))\|_{L^2((0,T) \times D)} \\
& \leq \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)\|_{L^2((0,T) \times D \times (0,1) \times Y)} \\
& \rightarrow 0
\end{aligned}$$

when $\varepsilon \rightarrow 0$. Using $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon)) = \nabla u^\varepsilon$, we get the conclusion. $\square$

To approximate u^ε in the $L^2((0,T), H^1(D))$ norm using the FE solutions, we define the following function for all $t \in (0,T)$ from the FE solutions at the time t_m where $m = 1, \dots, M$. For the full tensor product FE approximation, we define the functions $\bar{U}_0^L : (0,T) \rightarrow V$ and $\bar{U}_1^L : (0,T) \rightarrow L^2(D \times (0,1), V_\#)$ as

$$\bar{U}_0^L(t) = \frac{1}{2}(\bar{U}_{0,m}^L + \bar{U}_{0,m+1}^L), \quad \bar{U}_1^L(t) = \frac{1}{2}(\bar{U}_{1,m}^L + \bar{U}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We then have the following approximation.

Theorem 2.18. Assume that the hypothesis of Theorem 2.11 hold. Then for the solution of the numerical scheme (2.5) using the full tensor product FEs, we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|\nabla u^\varepsilon - \nabla \bar{U}_0^L - \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)\|_{L^2((0,T) \times D)} = 0.$$

Proof. We note that

$$\begin{aligned}
\int_0^T \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_H^2 dt &= \sum_{m=0}^{M-1} \int_{m\Delta t}^{(m+1)\Delta t} \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_H^2 dt \\
&\leq \sum_{m=0}^{M-1} (\Delta t \|\nabla u_0(t_{m+1/2}) - \nabla \bar{U}_0^L(t_{m+1/2})\|_H^2 + c(\Delta t)^3),
\end{aligned}$$

where we have used the midpoint approximation for the integral. As $u_0 \in C^2([0,T], V)$, the constant c is independent of m . We note that

$$\left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla u_0(t_{m+1/2}) \right\|_H \leq c(\Delta t)^2$$

Therefore

$$\begin{aligned}
\int_0^T \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_H^2 dt &\leq \sum_{m=0}^{M-1} \left(\Delta t \|\nabla u_{0,m+1/2} - \nabla \bar{U}_0^L(t_{m+1/2})\|_H^2 \right. \\
&\quad \left. + c(\Delta t)^3 \right) \\
&= \Delta t \sum_{m=0}^{M-1} \|\nabla \bar{Z}_{0,m+1/2}^L\|_H^2 + O((\Delta t)^2) \\
&\leq c((\Delta t)^2 + h_L^2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_0^T \|\nabla_y u_1(t) - \nabla_y \bar{U}_1^L(t)\|_{L^2(D \times (0,1) \times Y)}^2 dt \\
&\leq \Delta t \sum_{m=0}^{M-1} \|\nabla_y \bar{Z}_{0,m+1/2}^L\|_{L^2(D \times (0,1) \times Y)}^2 + O((\Delta t)^2) \\
&\leq c((\Delta t)^2 + h_L^2).
\end{aligned}$$

From (2.18), we have that $(\mathcal{U}^\varepsilon(\Phi)(t, x))^2 \leq \mathcal{U}^\varepsilon(\Phi^2)(t, x)$. Therefore, from (2.19), we have

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^2((0,T) \times D)} \leq \|\mathcal{U}^\varepsilon(\Phi^2)\|_{L^1((0,T) \times D)} \leq \|\Phi\|_{L^2((0,T) \times D \times (0,1) \times Y)}.$$

From this, we have

$$\begin{aligned}
\|\mathcal{U}^\varepsilon(\nabla_y u_1 - \nabla_y \bar{U}_1^L)\|_{L^2((0,T) \times D)} &\leq \|\nabla_y u_1 - \nabla_y \bar{U}_1^L\|_{L^2((0,T) \times D \times (0,1) \times Y)} \\
&\leq c(\Delta t + h_L).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\nabla u^\varepsilon - \nabla \bar{U}_0^L - \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)\|_{L^2((0,T) \times D)} \\
&\leq \|\nabla u^\varepsilon - \nabla u_0 - \mathcal{U}^\varepsilon(\nabla_y u_1)\|_{L^2((0,T) \times D)} \\
&\quad + \|\nabla u_0 - \nabla \bar{U}_0^L\|_{L^2((0,T) \times D)} + \|\mathcal{U}^\varepsilon(\nabla_y u_1) - \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)\|_{L^2((0,T) \times D)} \\
&\rightarrow 0
\end{aligned}$$

when $L \rightarrow \infty$ and $\varepsilon \rightarrow 0$. □

Remark 2.19. In theorem 2.18 and in other similar corrector results later in the thesis, the limit holds when $L \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$, when $\varepsilon \rightarrow 0$ first and then $L \rightarrow \infty$, and when $L \rightarrow \infty$ and $\varepsilon \rightarrow 0$ at the same time.

For the sparse tensor product approximations, we define

$$\hat{U}_0^L(t) = \frac{1}{2}(\hat{U}_{0,m}^L + \hat{U}_{0,m+1}^L), \quad \hat{U}_1^L(t) = \frac{1}{2}(\hat{U}_{1,m}^L + \hat{U}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We have:

Theorem 2.20. Assume that the hypothesis of Theorem 2.13 hold. Then for the solution of the numerical scheme (2.5) using the sparse tensor product FEs, we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \left\| \nabla u^\varepsilon - \nabla \hat{U}_0^L - \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L) \right\|_{L^2((0,T) \times D)} = 0.$$

The proof of this theorem is similar to that for Theorem 2.18.

Remark 2.21. Geng and Shen [43] deduce a corrector in the $H^1(D)$ norm for the solution u^ε , with an explicit error. This corrector involves functions other than u_0 and u_1 , which cannot be found from problem (2.3). The corrector of [43] involves a parabolic smoothness operator for the solution u_0 . We are not aware of a simple corrector with an explicit homogenization error in terms of ε similar to that for elliptic problems. However, from Theorem 1.1 of [43], if $u_0 \in C([0, T], H^1(D))$, then

$$\|u^\varepsilon - u_0\|_{L^2((0,T) \times D)} \leq c\varepsilon.$$

Using this we will have

$$\|u^\varepsilon - \bar{U}_0^L\|_{L^2((0,T) \times D)} \leq c(\varepsilon + h_L + \Delta t)$$

for the solution of the full tensor product FE approximation, and

$$\|u^\varepsilon - \hat{U}_0^L\|_{L^2((0,T) \times D)} \leq c(\varepsilon + Lh_L + \Delta t)$$

for the solution of the sparse tensor product FE approximation.

2.5 Regularity

We show that the regularity required for obtaining the full and sparse tensor product FE errors and for obtaining the corrector hold under regularity conditions for the coefficients and the initial condition. We have the following results.

Proposition 2.22. Assume that $a \in C^3([0, T], C^3(\bar{D}, C([0, T] \times Y)))$, $f \in H^3((0, T), V')$, $f(0) \in H_0^3(D)$, $\frac{\partial f}{\partial t}(0) \in H_0^2(D)$, $\frac{\partial^2 f}{\partial t^2}(0) \in H$, and $g \in H_0^4(D)$, then $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$. Further, if $f \in H^2((0, T), H)$ and if the domain D is convex, then $u_0 \in C^1([0, T], H^2(D))$.

Proof. As $a \in C^3([0, T], C^3(\bar{D}, C([0, T] \times Y)))$, from (2.15) we deduce that $a^0 \in C^3([0, T], C^3(\bar{D}))$. From the condition, we have

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial u_0}{\partial t} - \nabla \cdot \left(a^0 \nabla \frac{\partial u_0}{\partial t} \right) &= \frac{\partial f}{\partial t} + \nabla \cdot \left(\frac{\partial a^0}{\partial t} \nabla u_0 \right) \in L^2((0, T), V'), \\ \frac{\partial u_0}{\partial t}(0) &= f(0) + \nabla \cdot (a^0 \nabla g) \in H, \end{aligned}$$

so $\frac{\partial u_0}{\partial t} \in L^2((0, T), V) \cap C([0, T], H)$. Therefore

$$-\nabla \cdot (a^0 \nabla u_0) = f - \frac{\partial u_0}{\partial t} \in C([0, T], H)$$

so $u_0 \in C([0, T], H^2(D))$. We then have

$$\frac{\partial}{\partial t} \frac{\partial^2 u_0}{\partial t^2} - \nabla \cdot \left(a^0 \nabla \frac{\partial^2 u_0}{\partial t^2} \right) = \frac{\partial^2 f}{\partial t^2} + \nabla \cdot \left(\frac{\partial^2 a^0}{\partial t^2} \nabla u_0 \right) + 2 \nabla \cdot \left(\frac{\partial a^0}{\partial t} \nabla \frac{\partial u_0}{\partial t} \right)$$

$$\in L^2((0, T), V'),$$

$$\frac{\partial^2 u_0}{\partial t^2}(0) = \frac{\partial f}{\partial t}(0) + \nabla \cdot \left(\frac{\partial a^0}{\partial t}(0) \nabla g \right) + \nabla \cdot (a^0 \nabla (f(0) + \nabla \cdot (a^0 \nabla g))) \in H.$$

Arguing as for $\frac{\partial u_0}{\partial t}$, we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^2((0, T), V) \cap C([0, T], H)$ and $\frac{\partial u_0}{\partial t} \in C([0, T], H^2(D))$. Continuing this process, we have

$$\begin{aligned} & \frac{\partial}{\partial t} \frac{\partial^3 u_0}{\partial t^3} - \nabla \cdot (a^0 \nabla \frac{\partial^3 u_0}{\partial t^3}) \\ &= \frac{\partial^3 f}{\partial t^3} + \nabla \cdot \left(\frac{\partial^3 a^0}{\partial t^3} \nabla u_0 \right) + 3 \nabla \cdot \left(\frac{\partial^2 a^0}{\partial t^2} \nabla \frac{\partial u_0}{\partial t} \right) + 3 \nabla \cdot \left(\frac{\partial a^0}{\partial t} \nabla \frac{\partial^2 u_0}{\partial t^2} \right) \\ & \in L^2((0, T), V'), \\ & \frac{\partial^3 u_0}{\partial t^3}(0) = \frac{\partial^2 f}{\partial t^2}(0) + \nabla \cdot \left(a^0 \nabla \cdot \left(\frac{\partial f}{\partial t}(0) + \nabla \cdot \left(\frac{\partial a^0}{\partial t}(0) \nabla g \right) \right. \right. \\ & \quad \left. \left. + \nabla \cdot (a^0 \nabla (f(0) + \nabla \cdot (a^0 \nabla g))) \right) \right) \\ & \quad + \nabla \cdot \left(\frac{\partial^2 a^0}{\partial t^2}(0) \nabla g \right) + 2 \nabla \cdot \left(\frac{\partial a^0}{\partial t} \nabla (f(0) + \nabla \cdot (a^0 \nabla g)) \right) \in H. \end{aligned}$$

Therefore $\frac{\partial^3 u_0}{\partial t^3} \in C([0, T], H) \cap L^2([0, T], V)$. As $u_0 \in H^3((0, T), V)$, we deduce that $u_0 \in C^2([0, T], V)$. We note that

$$\begin{aligned} -\nabla \cdot (a^0(t) \nabla \frac{\partial^2 u_0}{\partial t^2}(t)) &= \frac{\partial^2 f}{\partial t^2} - \frac{\partial^3 u_0}{\partial t^3} + \nabla \cdot \left(\frac{\partial^2 a^0}{\partial t^2} \nabla u_0 \right) + 2 \nabla \cdot \left(\frac{\partial a^0}{\partial t} \nabla \frac{\partial u_0}{\partial t} \right) \\ &\in C([0, T], H). \end{aligned}$$

Therefore for all $t \in [0, T]$, $\frac{\partial^2 u_0}{\partial t^2}(t) \in H^2(D)$ with

$$\left\| \frac{\partial^2 u_0}{\partial t^2}(t) \right\|_{H^2(D)} \leq c(\left\| \frac{\partial^2 f}{\partial t^2}(t) \right\|_H + \left\| \frac{\partial^3 u_0}{\partial t^3}(t) \right\|_H + \left\| \frac{\partial^2 a^0}{\partial t^2}(t) \right\|_H + \|u_0(t)\|_H)$$

where the constant c depends only on the domain D and the Lipschitz norm of $a^0(t)$ which is uniform for all t (see [45] Theorems 3.1.3.1 and 3.2.1.2).

Therefore $u_0 \in H^2((0, T), H^2(D)) \subset C^1([0, T], H^2(D))$. $\square$

For the regularity of N^i , we have the following result.

Proposition 2.23. Assume that $a \in C^1([0, 1] \times \bar{D}, C^3([0, 1], C(\bar{Y})))$, then $N^i \in C^1([0, 1] \times \bar{D}, H^2((0, 1), H^3(Y)/\mathbb{R}))$.

Proof. We extend N^i for all $\tau \in (0, \infty)$ periodically. It then belongs to $L^2_{\text{loc}}((0, \infty), H^1_{\#}(Y)/\mathbb{R})$. Fixing t and x , we consider problem

$$\frac{\partial N^i}{\partial \tau} - \nabla_y \cdot (a \nabla_y N^i) = \nabla_y \cdot (a e^i), \quad (2.20)$$

with the initial condition at τ_1 . When $N^i(\tau_1) \in L^2(Y)/\mathbb{R}$, we have $N^i \in H^1((\tau_1, \Theta), (H^1_{\#}(Y)/\mathbb{R})')$ for all $\Theta > \tau_1$. If $N^i(\tau_1) \in H^2_{\#}/\mathbb{R}$, we have

$$\begin{aligned} \frac{\partial}{\partial \tau} \frac{\partial N^i}{\partial \tau} - \nabla_y \cdot (a \nabla_y \frac{\partial N^i}{\partial \tau}) &= \nabla_y \cdot (\frac{\partial a}{\partial \tau} e^i) + \nabla_y \cdot (\frac{\partial a}{\partial \tau} \nabla_y N^i) \\ &\in L^2((\tau_1, \Theta), (H^1_{\#}(Y)/\mathbb{R})') \end{aligned} \quad (2.21)$$

with the initial condition $\frac{\partial}{\partial \tau} N^i(\tau_1) = \nabla_y \cdot (a(\tau_1) e^i) + \nabla_y \cdot (a(\tau_1) \nabla_y N^i(\tau_1)) \in L^2(Y)/\mathbb{R}$. Therefore $\frac{\partial}{\partial \tau} N^i \in L^2((\tau_1, \Theta), H^1_{\#}(Y)/\mathbb{R})$ so $N^i \in H^1((\tau_1, \Theta), H^1_{\#}(Y)/\mathbb{R})$. From interpolation, we deduce that if $N^i(\tau_1) \in H^1_{\#}(Y)/\mathbb{R}$, then $N^i \in H^1((\tau_1, \Theta), L^2(Y)/\mathbb{R})$. As $N^i \in L^2_{\text{loc}}((0, \infty), H^1_{\#}(Y)/\mathbb{R})$ we can choose a value τ , without loss of generality we let it be τ_1 , so that $N^i(\tau_1) \in H^1_{\#}(Y)/\mathbb{R}$ which implies $N^i \in H^1((\tau_1, \Theta), L^2(Y)/\mathbb{R})$. Therefore

$$-\nabla_y \cdot (a \nabla_y N^i) = \nabla_y \cdot (a e^i) - \frac{\partial N^i}{\partial \tau} \in L^2((\tau_1, \Theta), L^2(Y)/\mathbb{R}). \quad (2.22)$$

Thus from elliptic regularity, we deduce that $N^i \in L^2((\tau_1, \Theta), H^2_{\#}(Y)/\mathbb{R})$. As $N^i \in C_{\text{loc}}([0, \infty), H)$ is uniquely determined, without loss of generality, we assume that $N^i(\tau_1) \in H^2_{\#}(Y)/\mathbb{R}$, so equation (2.21) with the compatibility initial condition at τ_1 implies that $\frac{\partial N^i}{\partial \tau} \in L^2((\tau_1, \Theta), H^1_{\#}(Y)/\mathbb{R}) \cap C([\tau_1, \Theta], L^2(Y)/\mathbb{R})$ as shown above. Thus the right hand side of (2.22) belongs to $L^2((\tau_1, \Theta), H^1_{\#}(Y)/\mathbb{R})$ so $N^i \in L^2((\tau, \Theta), H^3_{\#}(Y)/\mathbb{R})$ due to elliptic regularity. Without loss of generality, we assume that $\frac{\partial N^i}{\partial \tau}(\tau_1) \in H^1_{\#}(Y)/\mathbb{R}$.

By interpolation as above, using the equation

$$\begin{aligned}
& \frac{\partial}{\partial \tau} \frac{\partial^2 N^i}{\partial \tau^2} - \nabla_y \cdot (a \nabla_y \frac{\partial^2 N^i}{\partial \tau^2}) \\
&= \nabla_y \cdot (\frac{\partial^2 a}{\partial \tau^2} e^i) + \nabla_y \cdot (\frac{\partial^2 a}{\partial \tau^2} \nabla_y N^i) + 2 \nabla_y \cdot (\frac{\partial a}{\partial \tau} \nabla_y \frac{\partial N^i}{\partial \tau}) \\
&\in L^2((\tau_1, \Theta), (H_{\#}^1(Y)/\mathbb{R})'), \tag{2.23}
\end{aligned}$$

we deduce that $\frac{\partial N^i}{\partial \tau} \in H^1((\tau_1, \Theta), L^2(Y)/\mathbb{R})$, so $\frac{\partial^2 N^i}{\partial \tau^2} \in L^2((\tau_1, \Theta), L^2(Y)/\mathbb{R})$. We then deduce from (2.21) that $\frac{\partial N^i}{\partial \tau} \in L^2((\tau_1, \Theta), H_{\#}^2(Y)/\mathbb{R})$. We note that $\frac{\partial N^i}{\partial \tau} \in C([0, T], L^2(Y)/\mathbb{R})$ is uniquely determined. Without loss of generality, we assume that $\frac{\partial N^i}{\partial \tau}(\tau_1) \in H_{\#}^2(Y)/\mathbb{R}$. From (2.23) with the compatible initial condition at τ_1 deduced from (2.21), we have that $\frac{\partial^2 N^i}{\partial \tau^2} \in L^2((\tau_1, \Theta), H_{\#}^1(Y)/\mathbb{R}) \cap C([0, T], L^2(Y)/\mathbb{R})$. From (2.21), using elliptic regularity, we have that $\frac{\partial N^i}{\partial \tau} \in L^2((\tau_1, \Theta), H_{\#}^3(Y)/\mathbb{R})$. Again, without loss of generality, we suppose that $\frac{\partial^2 N^i}{\partial \tau^2}(\tau_1) \in H_{\#}^1(Y)$ we deduce from equation

$$\begin{aligned}
& \frac{\partial}{\partial \tau} \frac{\partial^3 N^i}{\partial \tau^3} - \nabla_y \cdot (a \nabla_y \frac{\partial^3 N^i}{\partial \tau^3}) \\
&= \nabla_y \cdot (\frac{\partial^3 a}{\partial \tau^3} e^i) + \nabla_y \cdot (\frac{\partial^3 a}{\partial \tau^3} \nabla_y N^i) + 3 \nabla_y \cdot (\frac{\partial^2 a}{\partial \tau^2} \nabla_y \frac{\partial N^i}{\partial \tau}) \\
&\quad + 3 \nabla_y \cdot (\frac{\partial a}{\partial \tau} \nabla_y \frac{\partial^2 N^i}{\partial \tau^2}) \\
&\in L^2((\tau_1, \Theta), (H_{\#}^1(Y)/\mathbb{R})'), \tag{2.24}
\end{aligned}$$

by interpolation that $\frac{\partial^2 N^i}{\partial \tau^2} \in H^1((\tau_1, \Theta), L^2(Y)/\mathbb{R})$. Thus from (2.23), we deduce that $\frac{\partial^2 N^i}{\partial \tau^2} \in L^2((\tau_1, \Theta), H_{\#}^2(Y)/\mathbb{R})$. Without loss of generality, we assume that $\frac{\partial^2 N^i}{\partial \tau^2}(\tau_1) \in H_{\#}^2(Y)/\mathbb{R}$. Therefore, equation (2.24) with the compatible condition at τ_1 implies that $\frac{\partial^3 N^i}{\partial \tau^3} \in L^2((\tau_1, \Theta), H_{\#}^1(Y)/\mathbb{R})$. From this together with $N^i \in L^2((\tau_1, \Theta), H_{\#}^3(Y)/\mathbb{R})$, $\frac{\partial N^i}{\partial \tau} \in L^2((\tau_1, \Theta), H_{\#}^3(Y)/\mathbb{R})$ and elliptic regularity, we deduce from (2.23) that $\frac{\partial^2 N^i}{\partial \tau^2} \in L^2((\tau_1, \Theta), H_{\#}^3(Y)/\mathbb{R})$,

i.e $N^i \in H^2((\tau_1, \Theta), H_{\#}^3(Y)/\mathbb{R})$. As a is continuously differentiable with respect to x and t , we have that $N^i \in C^1([0, T] \times \bar{D}, H^2((0, 1), H_{\#}^3(Y)/\mathbb{R}))$.
 $\square$

2.6 Numerical examples

We show some numerical examples in this section to illustrate the theoretical results on the convergence of the scheme (2.5).

For a one dimensional example, we consider the domain $D = (0, 1)$. We consider the coefficient $a(t, x, \tau, y) = 3 + \cos(2\pi y) + \cos^2(2\pi \tau)$. The initial condition $u^\varepsilon(0) = 0$. Equation (2.14) cannot be solved exactly. We solve it numerically using fine mesh to compute the homogenized coefficient a^0 in (2.15). The reference solution u_1 is computed numerically. The exact solution is chosen as $u_0 = t^2(x - x^2)$. With the homogenized coefficient $a^0(t, x)$ approximated numerically as $a^0 = 3.352429824667637$, the function $f = 2t(x - x^2) + 2a^0 t^2$. For the sparse tensor product FE approximation $\hat{u}_0^L$ and $\hat{u}_1^L$, we plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1((D))}$ and $\|u_1 - \hat{u}_1^L\|_{L^2(D \times (0, 1), H_{\#}^1(Y))}$ in Figures 2.1 and 2.2 respectively where $\Delta t = \frac{1}{2} \lceil h_L^{1/2} \rceil$ at $t = 1$. The numerical results show that the errors are $O((\Delta t)^2) + O(h_L)$. When these errors hold for all t_m , we get the errors estimate (2.12). This result supports the theoretical finding. The factor L is not visible in these figures.

Figure 2.1: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$.

Figure 2.2: The error $\|u_1 - \hat{u}_1\|_{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$.

For a two dimensional example, we consider the case where the domain $D = (0, 1) \times (0, 1)$. We choose $a(t, x, \tau, y) = (3 + \sin(2\pi y') + \sin^2(2\pi \tau))(3 + \sin(2\pi y'') + \sin^2(2\pi \tau))$ for $y = (y', y'') \in Y = (0, 1)^2$. The initial condition $u^\varepsilon(0) = 0$. Cell problem (2.14) is solved numerically with fine mesh from which the homogenized coefficient $a^0 = 11.863904995808440$ is computed.

We choose $u_0(t, x) = t^2 x' x'' (1 - x') (1 - x'')$ for $x = (x', x'')$. The function $f = 2t(x' - x'^2)(x'' - x''^2) + 2a^0 t^2 (x' - x'^2 + x'' - x''^2)$. The reference solution u_1 is computed from the numerical solution for N^i and the solution u_0 . For $t = 1$, we plot the error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^2(D \times (0,1), H_{\#}^1(Y)/\mathbb{R})}$ for the sparse tensor product FE solutions in Figures 2.3 and 2.4 respectively. The numerical results agree with the error estimate (2.12).

Figure 2.3: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 2 dimensional problem at $t = 1$.

Figure 2.4: The error $\|u_1 - \hat{u}_1\|_{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 2 dimensional problem at $t = 1$.

Although we only develop the theory for the case of one microscopic spatial scale, our method is capable of treating the case of multiple spatial scales. For illustration, we solve some limiting time-space multiscale homogenized equation established in [50]. Holmbom et al. [50] consider the case of two microscopic spatial scales with the coefficient

$$a^\varepsilon = a\left(t, x, \frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)$$

where $a = a(t, x, \tau, y_1, y_2)$ is Y -periodic with respect to y_1 and y_2 , and $(0, 1)$ periodic with respect to τ . The multiscale convergence limits of u^ε are

$$\nabla u^\varepsilon \xrightarrow{\text{ts-ms}} \nabla_x u_0 + \nabla_{y_1} u_1 + \nabla_{y_2} u_2,$$

where $u_1 \in L^2((0, T) \times D, H_{\#}^1(Y)/\mathbb{R})$ and $u_2 \in L^2((0, T) \times D \times Y, H_{\#}^1(Y)/\mathbb{R})$. Holmbom et al. [50] establish the multiscale homogenized equation for $k > 0$ but the most interesting critical cases where both u_1 and u_2 depend on τ , and the derivatives of these with respect to τ appear in the equation

occur $k = 2$ and $k = 3$. When $k = 2$, we have

$$\begin{aligned}
& \left\langle \frac{\partial u_0}{\partial t}(t), \phi_0 \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial u_1}{\partial \tau}(t, x, \tau, \cdot), \phi_1 \right\rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_{Y_1} \int_{Y_2} a(t, x, \tau, y_1, y_2) \\
& \quad (\nabla_x u_0(t, x) + \nabla_{y_1} u_1(t, x, \tau, y_1) + \nabla_{y_2} u_2(t, x, \tau, y_1, y_2)) \\
& \quad \cdot (\nabla_x \phi_0(x) + \nabla_{y_1} \phi_1(x, \tau, y_1) \\
& \quad + \nabla_{y_2} \phi_2(x, \tau, y_1, y_2)) dy_2 dy_1 d\tau dx \\
& = \int_D f(t, x) \phi_0(x) dx
\end{aligned}$$

$$\forall \phi_0 \in H_0^1(D),$$

$$\forall \phi_1 \in L^2(D \times (0, 1), H_\#^1(Y)/\mathbb{R}),$$

$$\forall \phi_2 \in L^2(D \times (0, 1) \times Y, H_\#^1(Y)/\mathbb{R}),$$

with the initial condition $u_0(0) = g$.

We choose the coefficient

$$a(t, x, \tau, y_1, y_2) = (3 + \sin(2\pi y_1) + \sin(2\pi \tau))(3 + \sin(2\pi y_2) + \sin(2\pi \tau)). \quad (2.25)$$

We need to solve two separate cell problems with respect to y_1 and y_2 . In this case the cell problem for y_1 is identical to (2.14) and is solved numerically, where the cell problem with respect to y_2 is the elliptic problem, i.e. without the derivative with respect to τ , and can be solved exactly. The homogenized coefficient is computed numerically as $a^0 = 8.500245683736688$. We choose $u_0(t, x) = t^2 x(1-x)$ so that $f(t, x) = t(x-x^2) + 2a^0 t^2$. In Figures 2.5, 2.6 and 2.7, we plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$, $\|u_1 - \hat{u}_1^L\|_{L^2(D, H_\#^1(Y)/\mathbb{R})}$, and $\|u_2 - \hat{u}_2^L\|_{L^2(D \times Y_1, H_\#^1(Y_2)/\mathbb{R})}$ for the sparse tensor FE approximation respectively. The error agrees with the estimate $O((\Delta t)^2 + h_L)$ that we establish in this thesis.

Figure 2.5: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$.

Figure 2.6: The error $\|u_1 - \hat{u}_1^L\|_{L^2(D \times (0,1), H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$.

Figure 2.7: The error $\|u_2 - \hat{u}_2^L\|_{L^2(D \times (0,1) \times Y, H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 2$.

For $k = 3$, the time-space multiscale homogenized equation becomes:

$$\begin{aligned}
& \left\langle \frac{\partial u_0}{\partial t}(t), \phi_0 \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial u_1}{\partial \tau}(t, x, \tau, \cdot), \phi_1 \right\rangle_{H_{\#}} d\tau dx \\
& + \int_D \int_0^1 \int_Y \left\langle \frac{\partial u_2}{\partial \tau}(t, x, \tau, y_1, \cdot), \phi_2 \right\rangle_{H_{\#}} dy_1 d\tau dx \\
& + \int_D \int_0^1 \int_{Y_1} \int_{Y_2} a(t, x, \tau, y_1, y_2) \\
& \quad (\nabla_x u_0(t, x) + \nabla_{y_1} u_1(t, x, \tau, y_1) + \nabla_{y_2} u_2(t, x, \tau, y_1, y_2)) \\
& \quad \cdot (\nabla_x \phi_0(x) + \nabla_{y_1} \phi_1(x, \tau, y_1) \\
& \quad \quad + \nabla_{y_2} \phi_2(x, \tau, y_1, y_2)) dy_2 dy_1 d\tau dx \\
& = \int_D f(t, x) \phi_0(x) dx, \\
& \forall \phi_0 \in H_0^1(D), \phi_1 \in L^2(D \times (0, 1), H_{\#}^1(Y)/\mathbb{R}), \phi_2 \\
& \in L^2(D \times (0, 1) \times Y, H_{\#}^1(Y)/\mathbb{R}), \text{ with the initial condition } u_0(0) = g.
\end{aligned}$$

We choose the coefficient a as in (2.25). We need to solve two cell problems in the form (2.14) with respect to y_1 and y_2 respectively. They are solved numerically. The numerical value of the homogenized coefficient is $a^0 = 7.929947333234398$. We then choose $u_0 = t^2 x(1-x)$ so that $f(t, x) =$

$2t(x - x^2) + 2a^0t^2$. In Figures 2.8, 2.9 and 2.10, we plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$, $\|u_1 - \hat{u}_1^L\|_{L^2(D, H_{\#}^1(Y)/\mathbb{R})}$, and $\|u_2 - \hat{u}_2^L\|_{L^2(D \times Y_1, H_{\#}^1(Y_2)/\mathbb{R})}$ respectively. The errors again agree with the estimate $O(\Delta t)^2 + h_L$ that we establish in this thesis.

Figure 2.8: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$.

Figure 2.9: The error $\|u_1 - \hat{u}_1^L\|_{L^2(D \times (0,1), H_{\#}^1(Y)/\mathbb{R})}$ versus the mesh size h for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$.

Figure 2.10: The error $\|u_2 - \hat{u}_2^L\|_{L^2(D \times (0,1) \times Y, H_{\#}^1(Y)/\mathbb{R})}$ for 1 dimensional 3 spatial scales problem at $t = 1$ for $k = 3$

Chapter 3

Multiscale Monotone Parabolic Equations depending on Multiple Microscales in Space

Chapter 3 considers multiscale monotone parabolic equations that depend on n separated microscales. The problem is setup in Section 3.1. In this section, we also recap on the multiscale homogenization theory which is applied to the problem to derive the multiscale homogenized equation. Backward Euler and Crank-Nicholson method are developed in Section 3.2 for general FE spaces, full tensor and sparse tensor product FE spaces. A new homogenized numerical corrector in terms of the microscopic scale for two scale problems is proved in Subsection 3.3.1. Using this, a numerical corrector is derived with an explicit error in terms of the homogenization error and the numerical error. In Subsection 3.3.2, we derive a numerical corrector for the general multiscale problems, without the explicit error. Some one dimensional and two dimensional numerical results are presented in Section 3.4.

3.1 Problem setting and homogenization

3.1.1 Problem setting

Let $D \in \mathbb{R}^d$ be a bounded domain; let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. Let $T > 0$. Let n be a positive integer. Let $Y_1, \dots, Y_n$ be n copies of the unit cube Y . For conciseness, we denote by $\mathbf{y}_i = (y_1, \dots, y_i)$ a vector in $\mathbf{Y}_i = Y_1 \times \dots \times Y_i$. We denote by $\mathbf{y} = \mathbf{y}_n$ and $\mathbf{Y} = \mathbf{Y}_n$. Let $A(t, x, y_1, \dots, y_n, \xi) : (0, 1) \times D \times Y_1 \times \dots \times Y_n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be continuously differentiable that is Y_i periodic with respect to y_i . We assume that A is monotone and locally Lipschitz. In particular, we assume that there are constants $p \geq 2$, $\alpha > 0$ and $\beta > 0$ so that for all $t \in (0, T)$, $x \in D$, $y_i \in Y_i$ ($i = 1, \dots, n$), and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$(A(t, x, y_1, \dots, y_n, \xi_1) - A(t, x, y_1, \dots, y_n, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad (3.1)$$

and

$$|A(t, x, y_1, \dots, y_n, \xi_1) - A(t, x, y_1, \dots, y_n, \xi_2)| \leq \beta(|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2| \quad (3.2)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^d$. Let $\varepsilon_1, \dots, \varepsilon_n$ be n functions of a small and positive quantity ε that represent n microscopic scales on which the problem depends. We assume scale separation (see Bensoussan et al. [16]), i.e. for $i = 1, \dots, n-1$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{i+1}}{\varepsilon_i} = 0.$$

Without loss of generality, we assume that $\varepsilon_1 = \varepsilon$. The multiscale monotone function is defined as

$$A^\varepsilon(t, x, \xi) = A(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \xi).$$

Let $V = W_0^{1,p}(D)$. Let $H = L^2(D)$. We have that $V \subset H \subset V'$. By $\langle \cdot, \cdot \rangle_H$, we denote the inner product in H extended to the duality pairing between V' and V . Let $f \in L^q((0, T), V')$ where $1/p + 1/q = 1$. Let $g \in H$. Let $T > 0$. We consider the following multiscale monotone parabolic problem:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot A^\varepsilon(t, x, \nabla u^\varepsilon) &= f, \quad \text{in } D \times (0, T) \\ u^\varepsilon(0) &= g \end{aligned} \quad (3.3)$$

with the Dirichlet boundary condition on ∂D . Problem (3.3) has a unique solution that satisfies

$$\|u^\varepsilon\|_{L^p((0, T), V)} + \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^q((0, T), V')} \leq c(\|f\|_{L^q((0, T), V')} + \|g\|_H)$$

where the constant c only depends on α and β in (3.1) and (3.2), and on $\sup_{t \in [0, T], x \in D, \mathbf{y} \in \mathbf{Y}} |A(t, x, \mathbf{y}, 0)|$ (see [56] and [77]). We will study homogenization of (3.3) by multiscale convergence. We thus recall the concept of multiscale convergence in the L^p setting.

3.1.2 Multiscale homogenization

Multiscale convergence was initiated by Nguetseng in [65] and developed further by Allaire [5]. The definition is extended to functions that depend on time as follows (see [50]).

Definition 3.1. A sequence $\{w^\varepsilon\}_\varepsilon \in L^p((0, T) \times D)$ $(n+1)$ -scale converges to a function $w_0(t, x, y_1, \dots, y_n) \in L^p((0, T) \times D \times Y_1 \times \dots \times Y_n)$ if

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon(t, x) \phi(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) dx \\ &= \int_0^T \int_D \int_{Y_1} \dots \int_{Y_n} w_0(t, x, y_1, \dots, y_n) \phi(t, x, y_1, \dots, y_n) dy_n \dots dy_1 dx dt, \end{aligned}$$

for all functions $\phi \in C((0, T) \times \bar{D} \times \bar{Y}_1 \times \dots \times \bar{Y}_n)$ which are Y_i -periodic with respect to y_i , $i = 1, \dots, n$.

Definition 3.1 makes sense due to the following proposition.

Proposition 3.2. From each bounded sequence in $L^p((0, T) \times D)$, we can extract a subsequence that $(n + 1)$ -scale converges.

The proposition for the case of time independent functions is proved in [5] and [6]. For time dependent functions, the proof is similar, see, e.g. [50]. We denote by

$$V_i = L^p(D \times Y_1 \times \dots \times Y_{i-1}, W_{\#}^{1,p}(Y_i)/\mathbb{R}), \quad (i = 1, \dots, n).$$

For a bounded sequence $\{w^\varepsilon\}_\varepsilon \subset L^p((0, T), V)$ such that $\frac{\partial w^\varepsilon}{\partial t}$ is bounded in $L^q((0, T), V')$, we have the following results.

Proposition 3.3. From a bounded sequence $\{w^\varepsilon\} \subset L^p((0, T), V)$ with $\frac{\partial w^\varepsilon}{\partial t}$ being bounded in $L^q((0, T), V')$, we can extract a subsequence (not renumbered) such that ∇w^ε $(n + 1)$ -scale converges to

$$\nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i$$

where $w_0 \in L^p((0, T), V)$ and $w_i \in L^p((0, T), V_i)$ ($i = 1, \dots, n$).

The proof of this result is standard and is similar to the proof of Proposition 2.3 in Chapter 2. It follows the standard result of [5] and [6], see also [50] or [76]. We define the following space

$$\mathbf{V} = \{(\phi_0, \phi_1, \dots, \phi_n) : \phi_0 \in V_0, \phi_i \in V_i\}$$

which is equipped with the norm

$$|||(\phi_0, \{\phi_i\})||| = \|\nabla \phi_0\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} \phi_i\|_{L^p(D \times Y_1 \times \dots \times Y_i)}. \quad (3.4)$$

We have the norm equivalence:

Lemma 3.4. There are positive constants c_1 and c_2 such that for all $(\phi_0, \{\phi_i\}) \in \mathbf{V}$,

$$\begin{aligned} & c_1 |||(\phi_0, \{\phi_i\})||| \\ & \leq \left(\int_D \int_{Y_1} \dots \int_{Y_n} |\nabla_x \phi_0 + \nabla_{y_1} \phi_1 + \dots + \nabla_{y_n} \phi_n|^p dx dy_1 \dots dy_n \right)^{1/p} \\ & \leq c_2 |||(\phi_0, \{\phi_i\})|||. \end{aligned}$$

A concise proof can be found in [48]. We have the following result:

Proposition 3.5. The solution u^ε of problem (3.3) converges weakly in $L^p((0, T), V)$ to a function u_0 , and ∇u^ε $(n+1)$ -scale converges to $\nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n$ where $u_0, u_1, \dots, u_n$ satisfy problem

$$\begin{aligned} & \left\langle \frac{\partial u_0}{\partial t}(t), \phi_0 \right\rangle_H + \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0(t) + \nabla_{y_1} u_1(t) + \dots + \nabla_{y_n} u_n(t)) \\ & \quad \cdot (\nabla \phi_0 + \nabla_{y_1} \phi_1 + \dots + \nabla_{y_n} \phi_n) d\mathbf{y} dx \\ & = \int_D f(t) \phi_0 dx \end{aligned} \tag{3.5}$$

for all $(\phi_0, \phi_1, \dots, \phi_n) \in \mathbf{V}$.

The proof of this result is quite standard, see, e.g., Allaire [5] or Woukeng [76]. We present it here for completeness.

Proof. We consider a subsequence of u^ε (not renumbered) such that ∇u^ε $(n+1)$ -scale converges to $\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i$. Let $\phi_0 \in C^\infty((0, T), \mathcal{D}(D))$, $\phi_i \in C^\infty((0, T), \mathcal{D}(D, C_0^\infty(Y_1, \dots, C_\#^\infty(Y_n) \dots)))$. Let

$$\Phi^\varepsilon(t, x) = \phi_0(t, x) + \sum_{i=1}^n \varepsilon_i \phi_i(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}).$$

We have that

$$\int_0^T \left\langle f(t) - \frac{\partial u^\varepsilon}{\partial t}(t), u^\varepsilon(t) - \Phi^\varepsilon(t) \right\rangle_H dt$$

$$- \int_0^T \int_D A^\varepsilon(t, x, \nabla u^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla \Phi^\varepsilon) dx dt = 0.$$

From (3.1), we have

$$\begin{aligned} \int_0^T \int_D A^\varepsilon(t, x, \nabla u^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla \Phi^\varepsilon) dx dt \\ \geq \int_0^T \int_D A^\varepsilon(t, x, \nabla \Phi^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla \Phi^\varepsilon) dx dt. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^T \langle f(t) - \frac{\partial u^\varepsilon}{\partial t}(t), u^\varepsilon(t) - \Phi^\varepsilon(t) \rangle_H dt \\ - \int_0^T \int_D A^\varepsilon(t, x, \nabla \Phi^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla \Phi^\varepsilon) dx dt \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^T \langle f(t), u^\varepsilon(t) - \Phi^\varepsilon(t) \rangle_H dt + \int_0^T \langle \frac{\partial u^\varepsilon}{\partial t}, \Phi^\varepsilon \rangle_H \\ - \int_0^T \int_D A^\varepsilon(t, x, \nabla \Phi^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla \Phi^\varepsilon) dx dt \\ \geq \frac{1}{2} (\|u^\varepsilon(T)\|_H^2 - \|u^\varepsilon(0)\|_H^2). \end{aligned} \quad (3.6)$$

As $\frac{\partial u^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_0}{\partial t}$ in $L^2((0, T), V')$ and $u^\varepsilon \rightharpoonup u_0$ in $L^2((0, T), V)$, we have that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \|u^\varepsilon(T) - u_0(T)\|_H^2 \\ = \liminf_{\varepsilon \rightarrow 0} \int_0^T \langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \rangle_H dt \\ = \liminf_{\varepsilon \rightarrow 0} \int_0^T \left(\langle \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon \rangle_H - \langle \frac{\partial u_0}{\partial t}, u_0 \rangle_H \right) dt \\ = \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} (\|u^\varepsilon(T)\|_H^2 - \|u_0(T)\|_H^2) \end{aligned}$$

(note that $u^\varepsilon(0) = u_0(0) = g$; the proof for this is identical to that in the proof of Proposition 2.5 in Chapter 2). Thus $\liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(T)\|_H^2 \geq \|u_0(T)\|_H^2$. Passing to the limit in (3.6), we have

$$\int_0^T \langle f(t), u_0(t) - \phi_0(t) \rangle_H dt + \int_0^T \langle \frac{\partial u_0}{\partial t}, \phi_0 \rangle_H$$

$$\begin{aligned}
& - \int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) \\
& \quad \cdot (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i - \nabla \phi_0 - \sum_{i=1}^n \nabla_{y_i} \phi_i) d\mathbf{y} dx dt \\
& \geq \frac{1}{2} (\|u_0(T)\|_H^2 - \|g\|_H^2).
\end{aligned}$$

Using (3.2), by using density, this holds for all $\phi_0 \in L^p((0, T), V)$ and $\phi_i \in L^p((0, T) \times Y_1 \times \dots \times Y_{i-1}, W_{\#}^{1,p}(Y_i))$. For $v_0 \in L^p((0, T), V)$ and $v_i \in L^p((0, T) \times Y_1 \times \dots \times Y_{i-1}, W_{\#}^{1,p}(Y_i))$, let $\phi_0 = u_0 - r v_0$, $\phi_i = u_i - r v_i$ where r is a real number. We have

$$\begin{aligned}
& r \int_0^T \langle f(t), v_0(t) \rangle_H dt + \int_0^T \langle \frac{\partial u_0}{\partial t}, u_0 \rangle_H dt - r \int_0^T \langle \frac{\partial u_0}{\partial t}, v_0 \rangle_H dt \\
& - r \int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i - r \nabla v_0 - r \sum_{i=1}^n \nabla_{y_i} v_i) \\
& \quad \cdot (\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\mathbf{y} dx dt \\
& \geq \frac{1}{2} (\|u_0(T)\|_H^2 - \|g\|_H^2).
\end{aligned}$$

Thus if $r > 0$

$$\begin{aligned}
& \int_0^T \langle f(t), v_0(t) \rangle_H dt - \int_0^T \langle \frac{\partial u_0}{\partial t}, v_0 \rangle_H dt \\
& - \int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i - r \nabla v_0 - r \sum_{i=1}^n \nabla_{y_i} v_i) \\
& \quad \cdot (\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\mathbf{y} dx dt \\
& \geq 0.
\end{aligned}$$

Passing to the limit when $r \rightarrow 0$, using (3.2), we get

$$\int_0^T \langle f(t), v_0(t) \rangle_H dt - \int_0^T \langle \frac{\partial u_0}{\partial t}, v_0 \rangle_H dt$$

$$-\int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \cdot (\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\mathbf{y} dx dt \geq 0.$$

Similarly, for $r < 0$, we have the opposite inequality. We then get problem (3.5). Using (3.1), problem (3.5) has a unique solution. The whole sequence $\{u^\varepsilon\}$ $(n+1)$ -scale converges to the solution of (3.5). $\square$

3.2 FE discretization

We approximate problem (3.5) by FEs in this section. We consider both the backward Euler method and the Crank-Nicholson method.

3.2.1 Backward Euler method

We first consider the backward Euler method for general FE spaces. We then use the method for the full and sparse tensor product FE spaces.

3.2.1.1 Backward Euler method for general FE spaces

Let $V_0^L \subset V$ and $V_i^L \subset V_i$ ($i = 1, \dots, n$) be finite dimensional spaces where the superscript L indicates the level of resolution. Let M be an integer. Let $\Delta t = T/M$. We consider the time sequence $0 = t_0 < t_1 < \dots < t_M$ where $t_m = m\Delta t$ for $m = 0, 1, \dots, M$. Let $g^L \in V^L$ be an approximation of g . We consider the problem: Find $u_{0,m}^L \in V_0^L$ and $u_{i,m}^L \in V_i^L$ for $i = 1, \dots, n$ such that

$$\begin{aligned} & \left\langle \frac{u_{0,m+1}^L - u_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \int_D \int_{\mathbf{Y}} A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L) \\ & \quad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) d\mathbf{y} dx \\ & = \int_D f(t_{m+1}, x) \phi_0(x) dx \end{aligned} \tag{3.7}$$

for all $\phi_0 \in V_0^L$ and $\phi_i \in V_i^L$ ($i = 1, \dots, n$). We first show that (3.7) has a unique solution.

Proposition 3.6. Problem (3.7) has a unique solution.

Proof. Let $\mathbf{c}_m = (\mathbf{c}_{0,m}, \{\mathbf{c}_{i,m}\})$ and $\mathbf{d} = (\mathbf{d}_0, \{\mathbf{d}_i\})$ ($i = 1, \dots, n$) in $\mathbb{R}^{\dim V_0^L} \times \mathbb{R}^{\dim V_1^L} \times \dots \times \mathbb{R}^{\dim V_n^L}$ be the coordinate vectors of $(u_{0,m}^L, \{u_{i,m}^L\})$ and $(\phi_0, \{\phi_i\})$ respectively in the expansion with respect to the basis functions of $V_0^L \times V_1^L \times \dots \times V_n^L$. Let $\mathcal{A}(\mathbf{c}_{m+1})$ be the vector describing the interaction of $A(t_{m+1}, x, \mathbf{y}, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L)$ with the basis functions of $V_0^L \times V_1^L \times \dots \times V_n^L$ in the second term on the left hand side of (3.7). Let $\mathcal{B}$ be the Gram matrix describing the interaction of basis functions of V_0^L with themselves in the inner product of H . Let F_{m+1} be the interaction of $f(t_{m+1})$ with the basis functions of V_0^L with respect to the inner product of H . We can write (3.7) as

$$\frac{1}{\Delta t} \mathcal{B} \mathbf{c}_{0,m+1} \cdot \mathbf{d}_0 + \mathcal{A}(\mathbf{c}_{m+1}) \cdot \mathbf{d} = F_{m+1} \cdot \mathbf{d}_0 + \frac{1}{\Delta t} \mathcal{B} \mathbf{c}_{0,m} \cdot \mathbf{d}_0. \quad (3.8)$$

The left hand side of (3.8) represents a monotone function. Indeed, for any $(v_0, \{v_i\})$ and $(w_0, \{w_i\})$ in $V_0^L \times V_1^L \times \dots \times V_n^L$,

$$\begin{aligned} & \frac{1}{\Delta t} \langle v_0 - w_0, v_0 - w_0 \rangle_H \\ & + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \mathbf{y}, \nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) \\ & \quad - A(t_{m+1}, x, \mathbf{y}, \nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i)) \\ & \quad \cdot (\nabla(v_0 - w_0) + \sum_{i=1}^n \nabla_{y_i} (v_i - w_i)) d\mathbf{y} dx \\ & \geq \frac{1}{\Delta t} \|v_0 - w_0\|_H^2 \\ & \quad + \alpha \int_D \int_{\mathbf{Y}} |(\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) - (\nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i)|^p d\mathbf{y} dx \end{aligned}$$

i.e. for any two vectors $\mathbf{p} = (\mathbf{p}_0, \{\mathbf{p}_i\})$ and $\mathbf{q} = (\mathbf{q}_0, \{\mathbf{q}_i\})$ in $\mathbb{R}^{\dim V_0^L} \times \mathbb{R}^{\dim V_1^L} \times \dots \times \mathbb{R}^{\dim V_n^L}$,

$$\frac{1}{\Delta t} \mathcal{B}(\mathbf{p}_0 - \mathbf{q}_0) \cdot (\mathbf{p}_0 - \mathbf{q}_0) + (\mathcal{A}(\mathbf{p}) - \mathcal{A}(\mathbf{q})) \cdot (\mathbf{p} - \mathbf{q}) \geq c(\Delta t) |\mathbf{p} - \mathbf{q}|^p$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{\dim V_0^L} \times \mathbb{R}^{\dim V_1^L} \times \dots \times \mathbb{R}^{\dim V_n^L}$.

Thus problem (3.8) has a unique solution $\mathbf{c}_{m+1}$. $\square$

We denote by $u_0(t_m) = u_{0,m}$, $u_i(t_m) = u_{i,m}$, $z_{0,m}^L = u_{0,m} - u_{0,m}^L$, $z_{i,m}^L = u_{i,m} - u_{i,m}^L$ ($i = 1, \dots, n$). When u_0 is sufficiently regular with respect to t , using the approach in [34], we can show the convergence of the numerical scheme. We then have the following result.

Theorem 3.7. Assume that $u_0 \in C^2([0, T], H)$, then

$$\begin{aligned} & \|z_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|z_{0,m}^L\|_V^p + \sum_{i=1}^n \|z_{i,m}^L\|_{V_i}^p) \\ & \leq c\Delta t \left(\sum_{m=1}^M \left(\|u_{0,m} - \tilde{u}_{0,m}\|_V^{p/(p-1)} + \sum_{i=1}^n \|u_{i,m} - \tilde{u}_{i,m}\|_{V_i}^{p/(p-1)} \right) \right. \\ & \quad \left. + \sum_{m=1}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2 \right) \\ & \quad + \max_{m=1, \dots, M} \|u_{0,m} - \tilde{u}_{0,m}\|_{L^2(D)}^2 + \|g - g^L\|_{L^2(D)}^2 + c(\Delta t)^{p/p-1}. \end{aligned} \quad (3.9)$$

for all sequences $\{\tilde{u}_{0,m}, m = 1, \dots, M\} \subset V_0^L$ and $\{\tilde{u}_{i,m}, m = 1, \dots, M\} \subset V_i^L$ for $i = 1, \dots, n$.

Proof. We denote by $\rho_m = \frac{\partial u_0}{\partial t}(t_{m+1}) - (u_0(t_{m+1}) - u_0(t_m))/\Delta t$. As $u_0 \in C^2([0, T], H)$ we have that $\|\rho_m\|_H \leq c\Delta t$ for all $m = 1, \dots, M$, where c is independent of m . We then have from (3.5) and (3.7) that

$$\begin{aligned} & \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \langle \rho_m, \phi_0 \rangle_H \\ & + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}) \end{aligned}$$

$$\begin{aligned}
& -A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L)) \\
& \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) d\mathbf{y} dx = 0
\end{aligned} \tag{3.10}$$

for all $\phi_0 \in V_0^L$ and $\phi_i \in V_i^L$, $i = 1, \dots, n$. Therefore, for all $\{\tilde{u}_{0,m}\} \subset V_0^L$ and $\{\tilde{u}_{i,m}\} \subset V_i^L$, we have

$$\begin{aligned}
& \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, z_{0,m+1}^L \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L) \\
& - A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L)) \\
& \cdot (\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L) d\mathbf{y} dx \\
& = \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L) \\
& - A(t_{m+1}, x, \mathbf{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L)) \\
& \cdot ((\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}) + \sum_{i=1}^n (\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1})) d\mathbf{y} dx \\
& + \langle \rho_m, (u_{0,m+1}^L - \tilde{u}_{0,m+1}) \rangle_H.
\end{aligned}$$

Using $u_{0,m+1}^L - \tilde{u}_{0,m+1} = u_{0,m+1}^L - u_{0,m+1} + u_{0,m+1} - \tilde{u}_{0,m+1}$, from (3.2) we have

$$\begin{aligned}
& \frac{1}{2\Delta t} (\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + c \|\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}^p \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H
\end{aligned}$$

$$\begin{aligned}
& +c(\|\nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}\|_{L^p(D \times \mathbf{Y})}^{p-2} \\
& \quad + \|\nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}^{p-2}) \\
& \quad \cdot \|\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})} \\
& \quad \cdot (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} \\
& \quad \quad + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D \times \mathbf{Y})}) \\
& +c\Delta t(\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z_{0,m+1}^L\|_H).
\end{aligned}$$

From (3.7), $\|\nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}$ is uniformly bounded for all L . Thus

$$\begin{aligned}
& \frac{1}{2\Delta t}(\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + c\|\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}^p \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& \quad + c\|\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})} \\
& \quad \cdot (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} \\
& \quad \quad + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D \times \mathbf{Y})}) \\
& \quad + c\Delta t(\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z_{0,m+1}^L\|_H) \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& \quad + \delta\|\nabla z_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}^p \\
& \quad + c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} \\
& \quad \quad + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D \times \mathbf{Y})})^q \\
& \quad + c(\Delta t)^q + \delta\|z_{0,m+1}^L\|_H^p + c\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H^q + c(\Delta t)^p
\end{aligned}$$

for a constant $\delta > 0$ where we have used the Young inequality; where $q = p/(1 - p)$. From Lemma 3.4, we have

$$\|z_{0,m+1}^L\|_V^p + \sum_{i=1}^n \|z_{i,m+1}^L\|_{V_i}^p \leq c \|\nabla z_{0,m+1}^L\| + \sum_{i=1}^n \|\nabla_{y_i} z_{i,m+1}^L\|_{L^p(D \times \mathbf{Y})}^p.$$

Thus for δ sufficiently small,

$$\begin{aligned} & \frac{1}{2\Delta t} (\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + \|z_{0,m+1}^L\|_V^p + \sum_{i=1}^n \|z_{i,m+1}^L\|_{V_i}^p \\ & \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\ & \quad + c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)}) \\ & \quad + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D \times \mathbf{Y})}^q + c(\Delta t)^q. \end{aligned}$$

Therefore for any $P = 1, \dots, M$, we deduce

$$\begin{aligned} & \|z_{0,P}^L\|_H^2 + c\Delta t \sum_{m=1}^P (\|z_{0,m}^L\|_V^p + \sum_{i=1}^n \|z_{i,m}^L\|_{V_i}^p) \\ & \leq c\Delta t \sum_{m=1}^P \left(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\ & \quad \left. + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|_{L^p(D \times \mathbf{Y})}^q \right) \\ & \quad + c(\Delta t)^q + c\Delta t \sum_{m=1}^P \left\langle \frac{z_{0,m}^L - z_{0,m-1}^L}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H + \|g - g^L\|_H^2. \end{aligned}$$

We have

$$\begin{aligned} & \Delta t \sum_{m=1}^P \left\langle \frac{z_{0,m}^L - z_{0,m-1}^L}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H \\ & = -\langle z_{0,0}^L, u_{0,1} - \tilde{u}_{0,1} \rangle_H + \langle z_{0,P}^L, u_{0,P} - \tilde{u}_{0,P} \rangle \\ & \quad + \sum_{m=1}^{P-1} \langle z_{0,m}^L, (u_{0,m} - \tilde{u}_{0,m}) - (u_{0,m+1} - \tilde{u}_{0,m+1}) \rangle \\ & \leq c\|z_{0,0}^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + \delta\|z_{0,P}^L\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \end{aligned}$$

$$+\delta\Delta t \sum_{m=1}^{P-1} \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2.$$

We have $\Delta t \sum_{m=1}^{P-1} \|z_{0,m}^L\|_H^2 \leq \Delta t M \max_m \|z_{0,m}^L\|_H \leq T \max_m \|z_{0,m}^L\|_H$. Choosing δ sufficiently small, we deduce that

$$\begin{aligned} & \|z_{0,P}^L\|_H^2 \\ \leq & c\Delta t \sum_{m=1}^P \left(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\ & \quad \left. + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|_{L^p(D \times \mathbf{Y})}^q \right) \\ & + c(\Delta t)^q + c\|g - g^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \\ & + \delta T \max_m \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2. \end{aligned}$$

Thus

$$\begin{aligned} & \max_m \|z_{0,m}^L\|_H^2 \\ \leq & c\Delta t \sum_{m=1}^M \left(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\ & \quad \left. + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|_{L^p(D \times \mathbf{Y})}^q \right) \\ & + c(\Delta t)^q + c\|g - g^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \\ & + \delta T \max_m \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2 \end{aligned}$$

so

$$\begin{aligned} & \max_m \|z_{0,m}^L\|_H^2 \\ \leq & c\Delta t \sum_{m=1}^M \left(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\ & \quad \left. + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|_{L^p(D \times \mathbf{Y})}^q \right) \end{aligned}$$

$$\begin{aligned}
& +c(\Delta t)^q + c\|g - g^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \\
& +c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2
\end{aligned}$$

The conclusion follows. $\square$

3.2.1.2 Backward Euler method for full tensor product FE spaces

We construct a hierarchy of FE spaces. We assume that the domain D is a polygon. Let $\{\mathcal{T}^l\}$ for $l \geq 0$ be the hierarchy of regular triangular simplices of mesh size $h_l = O(2^{-l})$ in D . The set of simplices $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles in the two dimension case, and is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 8 tetrahedra in the three dimension case. In the periodic cube Y , in a similar manner, we construct the hierarchy $\{\mathcal{T}_\#^l\}$ of regular triangular simplices of mesh size $h_l = O(2^{-l})$ which are periodically distributed in Y . Let $P^1(T)$ be the set of linear polynomials in $T \in \mathcal{T}_\#^l$. We define the FE spaces.

$$\begin{aligned}
V_0^l &= \{\phi \in W_0^{1,p}(D) : \phi \in P^1(T), \forall T \in \mathcal{T}^l\}; \\
V^l &= \{\phi \in W^{1,p}(D) : \phi \in P^1(T), \forall T \in \mathcal{T}^l\}; \\
V_\#^l &= \{\phi \in W_\#^{1,p}(D) : \phi \in P^1(T), \forall T \in \mathcal{T}_\#^l\}.
\end{aligned}$$

The following approximations hold. (see [28], [20] and [18])

$$\begin{aligned}
\inf_{w^l \in V_0^l} \|w - w^l\|_{W_0^1(D)} &\leq ch_l \|w\|_{W^{2,p}(D)}, \quad \forall w \in W_0^{1,p}(D) \cap W^{2,p}(D); \\
\inf_{w^l \in V^l} \|w - w^l\|_{L^p(D)} &\leq ch_l \|w\|_{W^{1,p}(D)}, \quad \forall w \in W^{1,p}(D); \\
\inf_{w^l \in V_\#^l} \|w - w^l\|_{W_\#^{1,p}(Y)} &\leq ch_l \|w\|_{W_\#^{2,p}(Y)}, \quad \forall w \in W_\#^{2,p}(Y); \\
\inf_{w^l \in V_\#^l} \|w - w^l\|_{L^p(Y)} &\leq ch_l \|w\|_{W_\#^{1,p}(Y)}, \quad \forall w \in W_\#^{1,p}(Y).
\end{aligned}$$

As $V_i = L^p(D \times Y_1 \times \dots \times Y_{i-1}, W_{\#}^{1,p}(Y_i)) \cong L^p(D) \otimes L^p(Y_1) \otimes \dots \otimes L^p(Y_{i-1}) \otimes W_{\#}^{1,p}(Y_i)$, we employ the space

$$\bar{V}_i^L = V^L \otimes \underbrace{V_{\#}^L \otimes \dots \otimes V_{\#}^L}_{i \text{ times}} \quad (3.11)$$

to approximate u_i .

To quantify the approximations of functions in V_i by functions in V_i^L , we define the following regularity spaces. Let $\mathcal{W}_i$ be the space of functions $w \in L^p(D \times Y_1 \times \dots \times Y_{i-1}, W^{2,p}(Y_i))$ that belong to $L^p(Y_1 \times \dots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))$ and $L^p(D \times \prod_{1 \leq j \leq i-1, j \neq k} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))$ for all $k = 1, \dots, i-1$. The space $\mathcal{W}_i$ is equipped with the norm

$$\begin{aligned} \|w\|_{\mathcal{W}_i} &= \|w\|_{L^p(D \times Y_1 \times \dots \times Y_{i-1}, W^{2,p}(Y_i))} + \|w\|_{L^p(Y_1 \times \dots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))} \\ &\quad + \sum_{k=1}^{i-1} \|w\|_{L^p(D \times \prod_{1 \leq j \leq i-1, j \neq k} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))}. \end{aligned}$$

The space $\mathcal{W}_i$ contains functions which possess regularity with respect to $y_1, \dots, y_i$ but the regularity needs not occur at the same time. We then have the following approximation.

Lemma 3.8. For $w \in \mathcal{W}_i$,

$$\inf_{w^L \in \bar{V}_i^L} \|w - w^L\|_{V_i} \leq ch_L \|w\|_{\mathcal{W}_i}.$$

The proof can be found in [48].

We employ the FE spaces V_0^L and $\bar{V}_i^L$ in the places of V_0^L and V_i^L in the backward Euler approximation (3.10). We denote the solution $u_{0,m}^L$ and $u_{i,m}^L$ by $\bar{u}_{0,m}^L$ and $\bar{u}_{i,m}^L$ respectively, and $z_{0,m}^L$ and $z_{i,m}^L$ by $\bar{z}_{0,m}^L$ and $\bar{z}_{i,m}^L$ respectively. We have the following result.

Theorem 3.9. Assume that $u_0 \in C^2([0, T], H) \cap H^1((0, T), W^{2,p}(D))$, $u_i \in C([0, T], \mathcal{W}_i)$ for $i = 1, \dots, n$, and $\|g - g^L\|_H \leq ch_L^{p/(2(p-1))}$, then

$$\|\bar{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|\bar{z}_{0,m}^L\|_V^p + \sum_{i=1}^n \|\bar{z}_{i,m}^L\|_{V_i}^p) \leq c(h_L^{p/(2(p-1))} + (\Delta t)^{p/(p-1)}).$$

Proof. We bound the right hand side of (3.9). As $u_i \in C([0, T], \mathcal{W}_i)$, we can choose $\tilde{u}_{i,m} \in V_i^L$ for $m = 1, \dots, M$ and a constant c independent of m such that

$$\|(u_i - \tilde{u}_i)_m\|_{V_i} \leq ch_L \|u_i(t_m)\|_{\mathcal{W}_i} \leq ch_L.$$

Let $I^L u_0(t) \in V^L$ be the interpolation operator whose value at each node equals the value of $u_0(t)$ (notes $u_0 \in C([0, T], W^{2,p}(D)) \subset C([0, T], C(\bar{D}))$) (see Section 3.2.1.3). We have

$$\|u_0(t) - I^L u_0(t)\|_V \leq ch_L \|u_0(t)\|_{W^{2,p}(D)} \leq ch_L.$$

Let $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\|(u_0 - \tilde{u}_0)_m\|_V \leq ch_L$$

where c is independent of m . With $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L^2 \left\| \frac{\partial u_0}{\partial t} \right\|_{W^{2,p}(D)}.$$

Using the procedure of [34]

$$\begin{aligned} & \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \\ &= \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) dt \right\|_H^2 (\Delta t)^{-2} \\ &\leq \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \end{aligned}$$

$$\begin{aligned}
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{2,p}(D)} dt \right)^2 (\Delta t)^{-2} \\
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{2,p}(D)}^2 dt \right) (\Delta t)^{-1}.
\end{aligned}$$

Therefore

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_{L^2(D)}^2 \leq ch_L^4.$$

We then get the conclusion. $\square$

3.2.1.3 Backward Euler method for sparse tensor product FE spaces

The dimension of the full tensor FE spaces is very large when L is large. In this section, we employ the sparse tensor product FE spaces with much smaller dimensions. However, for this approach to work, we need more regularity for the solution. For $p = 2$, as both $L^2(D)$ and $H_{\#}^1(Y)$ are both Hilbert spaces, we define the sparse tensor product FE spaces as in [49] using orthogonal projections. We define

$$P^{l0} : L^2(D) \rightarrow V^l, \quad P_{\#}^{l0} : L^2(Y) \rightarrow V_{\#}^l, \quad P_{\#}^{l1} : H_{\#}^1(Y) \rightarrow V_{\#}^l.$$

in the norm of $L^2(D)$, $L^2(Y)$ and $H_{\#}^1(Y)$ respectively. The increment spaces are defined as

$$W^l = (P^{l0} - P^{(l-1)0})V^l, \quad W_{\#}^{l0} = (P_{\#}^{l0} - P_{\#}^{(l-1)0})V_{\#}^l, \quad W_{\#}^{l1} = (P_{\#}^{l1} - P_{\#}^{(l-1)1})V_{\#}^l.$$

We then have

$$V^l = \bigoplus_{0 \leq i \leq l} W^i, \quad V_{\#}^l = \bigoplus_{0 \leq i \leq l} W_{\#}^{i0}, \quad V_{\#}^l = \bigoplus_{0 \leq i \leq l} W_{\#}^{i1}.$$

The full tensor product space $\bar{V}_i^L$ can be written as

$$\bar{V}_i^L = \bigoplus_{\substack{0 \leq j_k \leq L \\ k=0,1,\dots,i}} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{i-1} 0} \otimes W_{\#}^{j_i 1}.$$

. We define the sparse tensor product space $\hat{V}_i^L$ as

$$\hat{V}_i^L = \bigoplus_{0 \leq j_0 + j_1 + \dots + j_i \leq L} W^{j_0} \otimes W_{\#}^{j_1 0} \otimes \dots \otimes W_{\#}^{j_{i-1} 0} \otimes W_{\#}^{j_i 1}.$$

For $p > 2$, as $L^p(D)$ and $W_{\#}^{1,p}(Y)$ are no longer Hilbert spaces, the sparse tensor product FE spaces need to be defined differently. For $p > d$, then $W^{1,p}(D) \subset C(D)$ and $W_{\#}^{1,p}(Y_i) \subset C_{\#}(Y_i)$. Let $\mathcal{S}^l$ be the set of nodes of the triangulation $\mathcal{T}^l$. We have that $\mathcal{S}^l \subset \mathcal{S}^{l+1}$. Following [48], we consider the basis of V^l that consists of functions ϕ_x^l for $x \in \mathcal{S}$ such that $\phi_x^l(x) = 1$ and $\phi_x^l(x') = 0$ where $x' \in \mathcal{S}, x' \neq x$. For continuous functions w in D , we defined the interpolation operator $I^l : C(D) \rightarrow V^l$ as

$$I^l w = \sum_{x \in \mathcal{S}^l} w(x) \phi_x^l. \quad (3.12)$$

Similarly, we define the set $\mathcal{S}_{\#}^l$ of triangulation nodes of $V_{\#}^l$, and for $y \in \mathcal{S}_{\#}^l$, the basis function $\phi_{\#y}^l$ which equals 1 at $y \in \mathcal{S}_{\#}^l$ and 0 at other nodes. We define the interpolation operator $I_{\#}^l : C_{\#}(Y) \rightarrow V_{\#}^l$ as

$$I_{\#}^l w = \sum_{y \in \mathcal{S}_{\#}^l} w(y) \phi_{\#y}^l. \quad (3.13)$$

We define the subspaces $W^l \subset V^l$ and $W_{\#}^l \subset V_{\#}^l$ as

$$W^l = (I^l - I^{l-1})C(\bar{D}) \quad \text{and} \quad W_{\#}^l = (I_{\#}^l - I_{\#}^{l-1})C_{\#}(\bar{Y})$$

with $W^0 = V^0$ and $W_{\#}^0 = V_{\#}^0$. The space W^l contains the linear combinations of basis function ϕ_x^l of V^l where $x \in \mathcal{S}^l \setminus \mathcal{S}^{l-1}$; and the space $W_{\#}^l$

contains the linear combinations of basis functions $\phi_{\#y}^l$ for $y \in \mathcal{S}_{\#}^l \setminus \mathcal{S}_{\#}^{l-1}$.

We then have that

$$W^l = \bigoplus_{0 \leq l' \leq l} W^{l'}, \quad W_{\#}^l = \bigoplus_{0 \leq l' \leq l} W_{\#}^{l'}.$$

The full tensor product FE space $\bar{V}_i^L$ is of the form

$$\bar{V}_i^L = \bigoplus_{\substack{0 \leq l_j \leq L \\ j=0, \dots, i}} W^{l_0} \otimes W_{\#}^{l_1} \otimes \dots \otimes W_{\#}^{l_i}.$$

The sparse tensor product FE space $\hat{V}_i^L$ is defined as

$$\hat{V}_i^L = \bigoplus_{0 \leq \sum_{j=0}^L l_j \leq L} W^{l_0} \otimes W_{\#}^{l_1} \otimes \dots \otimes W_{\#}^{l_i}. \quad (3.14)$$

To quantify the approximating properties of the spaces $\hat{V}_i^L$, we define the regularity spaces $\hat{\mathcal{W}}_i$ of functions $w(x, y_1, \dots, y_i)$ which are Y_j periodic with respect to y_j for $j = 0, \dots, i$ such that for all $\alpha_j \in \mathbb{R}^d$ ($j = 0, \dots, i-1$) with $|\alpha_j| \leq 1$ and $\alpha_i \in \mathbb{R}^d$ with $|\alpha_i| \leq 2$, we have $\partial^{\sum_{j=0}^i |\alpha_j|} w / (\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i) \in L^p(D \times Y_1 \times \dots \times Y_i)$. In other words,

$$\begin{aligned} \hat{\mathcal{W}}_i &= W^{1,p}(D, W_{\#}^{1,p}(Y_1, \dots, W_{\#}^{1,p}(Y_{i-1}, W_{\#}^{2,p}(Y_i)))) \\ &\cong W^{1,p}(D) \otimes W_{\#}^{1,p}(Y_1) \otimes \dots \otimes W_{\#}^{1,p}(Y_{i-1}) \otimes W_{\#}^{2,p}(Y_i). \end{aligned}$$

This space is equipped with the norm

$$\|w\|_{\hat{\mathcal{W}}_i} = \sum_{\substack{0 \leq |\alpha_i| \leq 2 \\ 0 \leq |\alpha_0|, \dots, |\alpha_{i-1}| \leq 1}} \left\| \frac{\partial^{\sum_{j=0}^i |\alpha_j|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i} \right\|_{L^p(D \times Y_1 \times \dots \times Y_i)}.$$

The space $\hat{\mathcal{W}}_i$ contains functions that possess regularity with respect $x, y_1, \dots, y_i$ and the regularity is required to hold at the same time. We have the following approximation properties

Lemma 3.10. For $w \in \hat{\mathcal{W}}_i$, when $p = 2$

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^p(D \times Y_1 \times \dots \times Y_i)} \leq cL^{i/2} h_L \|w\|_{\hat{\mathcal{W}}_i};$$

and when $p > d$

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^p(D \times Y_1 \times \dots \times Y_i)} \leq cL^i h_L \|w\|_{\hat{\mathcal{W}}_i};$$

The proofs for these are presented [49] and [48]. We employ the sparse tensor product FEs for the backward Euler approximating problem (3.7), i.e. we let $V_i^L = \hat{V}_i^L$ for $i = 1, \dots, n$. We denote the solution as $\hat{u}_{0,m}^L, \hat{u}_{i,m}^L$. We denote $z_{0,m}^L$ by $\hat{z}_{0,m}^L$ and $z_{1,m}^L$ by $\hat{z}_{1,m}^L$. We then have:

Theorem 3.11. Assume that $u_0 \in C^2([0, T], H) \cap H^1((0, 1), W^{2,p}(D))$, $u_i \in C([0, T], \hat{\mathcal{W}}_i)$ for $i = 1, \dots, n$ and $\|g - g^L\|_H \leq cL^{np/(2(p-1))} h_L^{p/(2(p-1))}$. Then

$$\|\hat{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}_{0,m}^L\|_V^2 + \sum_{i=1}^n \|\hat{z}_{i,m}^L\|_{V_i}^2) \leq c(L^n h_L^2 + (\Delta t)^2)$$

when $p = 2$; and

$$\|\hat{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}_{0,m}^L\|_V^p + \sum_{i=1}^n \|\hat{z}_{i,m}^L\|_{V_i}^p) \leq c(L^{np/(p-1)} h_L^{p/(p-1)} + (\Delta t)^{p/(p-1)})$$

when $p > d$.

The proof of this theorem is similar to that for Theorem 3.9.

Remark 3.12. The requirement of $p > d$ is to ensure that $W^{1,p}(D) \subset C(D)$ and $W_{\#}^{1,p}(Y) \subset C_{\#}(Y)$. If u_i is smoother than $W^{1,p}(D)$ with respect to x and $W_{\#}^{1,p}(Y_i)$ with respect to y_i so that they are continuous with respect to x and y_i , then we can remove this requirement.

3.2.2 Crank-Nicholson method

We use the Crank-Nicholson discretizing scheme to solve problem (3.5) in this section. We first consider the scheme for general FE spaces, and prove the convergence of the scheme. We then use the full tensor product FEs and sparse tensor product FEs for the Crank-Nicholson scheme, and deduce the error of convergence.

3.2.2.1 Crank-Nicholson method for general FE spaces

We employ the partition $0 \leq t_1 \leq \dots \leq t_M = T$ as in the previous section. We consider general FE spaces as in Section 3.2.1.1. The discretized problem is: For $m = 1, \dots, M$, find $U_{0,m}^L \in V_0^L$ and $U_{i,m}^L \in V_i^L$ for $i = 1, \dots, n$ such that

$$\begin{aligned}
& \left\langle \frac{U_{0,m+1}^L - U_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} A \left(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2} ((\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\
& \quad \left. + \sum_{i=1}^n (\nabla_{y_i} U_{i,m}^L + \nabla_{y_i} U_{i,m+1}^L)) \right) \\
& \quad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) d\mathbf{y} dx \\
& = \int_D f(t_{m+1/2}) \phi_0 dx
\end{aligned} \tag{3.15}$$

for all $\phi_i \in V_0^L$ and $\phi_i \in V_i^L$. We then have:

Proposition 3.13. Problem (3.15) has a unique solution.

The proof of this proposition is similar to that of Proposition 3.6.

We have the following approximation result. Let

$$Z_{0,m}^L = u_0(t_m) - U_{0,m}^L,$$

$$\begin{aligned}
Z_{i,m}^L &= u_i(t_m) - U_{i,m}^L, \\
Z_{0,m+1/2}^L &= \frac{1}{2}(Z_{0,m}^L + Z_{0,m+1}^L), \\
Z_{i,m+1/2}^L &= \frac{1}{2}(Z_{i,m}^L + Z_{i,m+1}^L).
\end{aligned}$$

When u_0 and u_i are sufficiently regular with respect to t , using the approach in [34], we show the convergence of the numerical scheme.

Theorem 3.14. Assume that $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_i \in C^2([0, T], V_i)$. Then

$$\begin{aligned}
& \|Z_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|Z_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p) \\
& \leq c\Delta t \left(\sum_{m=0}^{M-1} (\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^{p/(p-1)} + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^{p/(p-1)}) \right. \\
& \quad \left. + \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \right) \\
& \quad + \max_{m=1, \dots, M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_H^2 + \|g - g^L\|_H^2 + c(\Delta t)^{2p/(p-1)}. \quad (3.16)
\end{aligned}$$

for all $\{\tilde{u}_{0m}, m = 0, \dots, M\} \subset V^L$ and $\{\tilde{u}_{i,m}, m = 1, \dots, M\} \subset V_i^L$ for $i = 1, \dots, n$.

Proof. Let $\rho_{0,m} = \frac{1}{\Delta t}(u_0(t_{m+1}) - u_0(t_m)) - \frac{\partial u_0}{\partial t}(t_{m+1/2})$, $\zeta_{0,m} = \frac{1}{2}(u_0(t_{m+1}) + u_0(t_m)) - u_0(t_{m+1/2})$, $\zeta_{i,m} = \frac{1}{2}(u_i(t_{m+1}) + u_i(t_m)) - u_i(t_{m+1/2})$. Since $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_i \in C^2([0, T], V_i)$, we deduce that

$$\|\rho_{0,m}\|_H \leq c(\Delta t)^2, \quad \|\zeta_{0,m}\|_V \leq c(\Delta t)^2, \quad \|\zeta_{i,m}\|_{V_i} \leq c(\Delta t)^2$$

where the constant c does not depend on m . From (3.5) and (3.15) considered at $t = t_{m+1/2}$ we deduce that

$$\left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H - \langle \rho_{0,m}, \phi_0 \rangle_H$$

$$\begin{aligned}
& + \int_D \int_{\mathbf{Y}} \left(A \left(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2} (\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^n \frac{1}{2} (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1}) - \nabla \zeta_{0,m} - \sum_{i=1}^n \nabla_{y_i} \zeta_{i,m} \right) \right. \\
& \quad \left. - A \left(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2} (\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^n \frac{1}{2} (\nabla_{y_i} U_{i,m}^L + \nabla_{y_i} U_{i,m+1}^L) \right) \right) \\
& \quad \cdot (\nabla_x \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) d\mathbf{y} dx \\
& = 0, \quad \forall \phi_0 \in V_0^L \text{ and } \phi_i \in V_i^L, \ i = 1, \dots, n
\end{aligned} \tag{3.17}$$

Consider

$$\begin{aligned}
I & = \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} \left(A \left(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2} (\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1}) \right) \right. \\
& \quad \left. - A \left(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2} (\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} U_{i,m}^L + \nabla_{y_i} U_{i,m+1}^L) \right) \right) \\
& \quad \cdot \left(\nabla_x \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \sum_{i=1}^n \nabla_{y_i} \frac{Z_{i,m}^L + Z_{i,m+1}^L}{2} \right) d\mathbf{y} dx.
\end{aligned} \tag{3.18}$$

For $\{\tilde{u}_{0,m}, m = 0, \dots, M\} \subset V_0^L$ and $\{\tilde{u}_{i,m}, m = 1, \dots, M\} \subset V_i^L$, we have

$$\begin{aligned}
I & = \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& + \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (\tilde{u}_0 - U_0^L)_{m+1/2} \right\rangle_H
\end{aligned}$$

$$\begin{aligned}
& + \int_D \int_{\mathbf{Y}} \left(A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\
& \quad \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1})) \right. \\
& \quad \left. - A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\
& \quad \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} U_{i,m}^L + \nabla_{y_i} U_{i,m+1}^L)) \right) \\
& \quad \cdot \left((\nabla(u_0 - \tilde{u}_0)_{m+1/2} + \sum_{i=1}^n \nabla_{y_i} (u_i - \tilde{u}_i)_{m+1/2}) \right. \\
& \quad \left. + (\nabla(\tilde{u}_0 - U_0^L)_{m+1/2} \right. \\
& \quad \left. + \sum_{i=1}^n \nabla_{y_i} (\tilde{u}_i - U_i^L)_{m+1/2}) \right) d\mathbf{y} dx.
\end{aligned}$$

From (3.15) we have

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&+ \int_D \int_{\mathbf{Y}} \left(A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\
& \quad \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1})) \right. \\
& \quad \left. - A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\
& \quad \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} U_{i,m}^L + \nabla_{y_i} U_{i,m+1}^L)) \right) \\
& \quad \cdot \left((\nabla_x(u_0 - \tilde{u}_0)_{m+1/2} + \sum_{i=1}^n \nabla_{y_i} (u_i - \tilde{u}_i)_{m+1/2}) \right) d\mathbf{y} dx \\
&+ \langle \rho_{0,m}, (\tilde{u}_0 - U_0^L)_{m+1/2} \rangle_H \\
&- \int_D \int_{\mathbf{Y}} \left(A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\
& \quad \left. + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1}) - \nabla \zeta_{0,m} - \sum_{i=1}^n \nabla_{y_i} \zeta_{i,m} \right.
\end{aligned}$$

$$\begin{aligned}
& -A(t_{m+1/2}, x, \mathbf{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \\
& + \frac{1}{2} \sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1})) \\
& \cdot (\nabla(\tilde{u}_0 - U_0^L)_{m+1/2} + \sum_{i=1}^n \nabla_{y_i}(\tilde{u}_i - U_i^L)_{m+1/2}) d\mathbf{y} dx.
\end{aligned}$$

We note that $(\tilde{u}_0 - U_0)_{m+1/2} = (\tilde{u}_0 - u_0)_{m+1/2} + Z_{0,m+1/2}^L$ and $(\tilde{u}_i - U_i)_{m+1/2} = (\tilde{u}_i - u_i)_{m+1/2} + Z_{i,m+1/2}^L$. For a positive constant $\delta > 0$, using the Young inequality, we have

$$\begin{aligned}
I & \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q \\
& + c \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q + c \|\rho_{0,m}\|_H^q + \delta \|Z_{0,m+1/2}^L\|_H^p \\
& + c \|\rho_{0,m}\|_H^p + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_H^q + c \|\zeta_{0,m}\|_V^q \\
& + c \sum_{i=1}^n \|\zeta_{i,m}\|_{V_i}^q + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p + c \|\zeta_{0,m}\|_V^p \\
& + c \sum_{i=1}^n \|\zeta_{i,m}\|_{V_i}^p + c \|(\tilde{u}_0 - u_0)_{m+1/2}\|_V^q + c \sum_{i=1}^n \|(\tilde{u}_i - u_i)_{m+1/2}\|_{V_i}^q \\
& \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + c \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q + c(\Delta t)^{2q} \\
& + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p
\end{aligned}$$

where we have used the fact that $\|w\|_H \leq \|w\|_V$ for all $w \in V$. From (3.2)

and Lemma 3.4, we have

$$I \geq \frac{1}{2\Delta t} (\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) + c(\|Z_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p).$$

Choosing δ sufficiently small, we have

$$\begin{aligned}
& \frac{1}{2\Delta t} (\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) + c(\|Z_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p) \\
& \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& \quad + c\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + c \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q + c(\Delta t)^{2q}.
\end{aligned}$$

Fixing an integer $P \leq M$, taking the sum for $m = 0, \dots, P-1$, we have

$$\begin{aligned}
& \|Z_{0,P}^L\|_H^2 - \|Z_{0,0}^L\|_H^2 + c\Delta t \sum_{m=0}^{P-1} (\|Z_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|Z_{i,m+1/2}^L\|_{V_i}^p) \\
& \leq c\Delta t \sum_{m=0}^{P-1} \left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q \right) \\
& \quad + cP(\Delta t)^{2q+1} + 2\Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H. \quad (3.19)
\end{aligned}$$

We note that

$$\begin{aligned}
& \Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& = \langle Z_{0,P}^L, (u_0 - \tilde{u}_0)_{P-1/2} \rangle_H - \langle Z_{0,0}^L, (u_0 - \tilde{u}_0)_{1/2} \rangle_H \\
& \quad + \Delta t \sum_{m=1}^{P-1} \left\langle \frac{Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m-1/2} - (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& \leq \delta \|Z_{0,P}^L\|_H^2 + c\|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + c\|Z_{0,0}^L\|_H^2 + c\|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
& \quad + \delta \Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2
\end{aligned}$$

where we have employed the Cauchy-Schwartz inequality; δ is a constant which we choose to be small. From the fact that $\Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 \leq T \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2$ and (3.19), we have

$$\|Z_{0,P}^L\|_H^2$$

$$\begin{aligned}
&\leq c\Delta t \sum_{m=0}^{P-1} \left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q \right) \\
&\quad + c(\Delta t)^{2q} + c\|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + c\|Z_{0,0}^L\|_H^2 + \|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
&\quad + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \\
&\quad + \delta T \max_{m=0,\dots,M} \|Z_{0,M}^L\|_H^2
\end{aligned}$$

for all $P = 1, \dots, M$. Choosing δ sufficiently small, we have

$$\begin{aligned}
&\max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2 \\
&\leq c\Delta t \sum_{m=0}^{M-1} \left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^q \right) \\
&\quad + c(\Delta t)^{2q} + c \max_{m=1,\dots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_H^2 + c\|Z_{0,0}^L\|_H^2 \\
&\quad + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2.
\end{aligned}$$

From this, we get the conclusion.

3.2.2.2 Crank-Nicholson method for full tensor product FE spaces

We deduce the error for the Crank-Nicholson method where we use the full tensor product FE space $\bar{V}_i^L$ defined in (3.11) for V_i in Subsection 3.2.2.1.

We denote the solution as $\bar{U}_{0,m}^L, \bar{U}_{i,m}^L$. We denote $Z_{0,m+1/2}^L$ and $Z_{i,m+1/2}^L$ by $\bar{Z}_{0,m+1/2}^L$ and $\bar{Z}_{i,m+1/2}^L$ respectively. We then have the following result.

Theorem 3.15. Assume that

$$\begin{aligned}
u_0 &\in C^3([0, T], H) \cap C^2([0, T], V) \cap H^1([0, T], W^{2,p}(D)), \\
u_i &\in C^2([0, T], V_i) \cap C([0, T], \mathcal{W}_i).
\end{aligned}$$

Then

$$\begin{aligned} \|\bar{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\bar{Z}_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|\bar{Z}_{i,m+1/2}^L\|_{V_i}^p) \\ \leq c(h_L^{p/(p-1)} + (\Delta t)^{2p/(p-1)}). \end{aligned}$$

Proof. Since $u_i \in C([0, T], \mathcal{W}_i)$, we choose $\tilde{u}_{i,m} \in \bar{V}_i^L$ for $m = i, \dots, M$ such that

$$\|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i} \leq ch_L(\|u_i(t_m)\|_{\mathcal{W}_i} + \|u_i(t_{m+1})\|_{\mathcal{W}_i}) \leq ch_L,$$

where c is independent of t . Define the interpolation $I^L u_0(t) \in V^L$ such that the value of $I^L u_0(t)$ at each node equals the value of $u_0(t)$. We have

$$\|u_0(t) - I^L u_0(t)\|_V \leq ch_L \|u_0(t)\|_{W^{2,p}(D)} \leq ch_L.$$

Choosing $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V \leq ch_L$$

where c does not depend on m . We then have

$$\begin{aligned} & \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \\ & \leq c \sum_{m=1}^{M-1} \left(\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \right. \\ & \quad \left. + \left\| \frac{(u_0 - \tilde{u}_0)_m - (u_0 - \tilde{u}_0)_{m-1}}{\Delta t} \right\|_H^2 \right). \end{aligned}$$

With $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L^2 \left\| \frac{\partial u_0}{\partial t} \right\|_{W^{2,p}(D)}.$$

Thus

$$\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2$$

$$\begin{aligned}
&= \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) dt \right\|_H^2 (\Delta t)^{-2} \\
&\leq \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \\
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{2,p}(D)} dt \right)^2 (\Delta t)^{-2} \\
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{2,p}(D)}^2 dt \right) (\Delta t)^{-1}.
\end{aligned}$$

Therefore

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \leq ch_L^4.$$

We then get the conclusion. $\square$

3.2.2.3 Crank-Nicholson method for sparse tensor product FE spaces

Using the sparse tensor product FE, i.e. we use the sparse tensor product FE space $\hat{V}_i^L$ defined in (3.14) for V_i in Subsection 3.2.2.1, for $i = 1, \dots, n$, we denote the solution as $\hat{U}_{0,m}^L$ and $\hat{U}_{i,m}^L$, and $Z_{0,m+1/2}^L$ and $Z_{i,m+1/2}^L$ by $\hat{Z}_{0,m+1/2}^L$ and $\hat{Z}_{i,m+1/2}^L$ respectively. We then have the following result.

Theorem 3.16. Assume that

$$\begin{aligned}
u_0 &\in C^3([0, T], H) \cap C^2([0, T], V) \cap H^1([0, T], W^{2,p}(D)), \\
u_i &\in C^2([0, T], V_i) \cap C([0, T], \hat{\mathcal{W}}_i)
\end{aligned}$$

for $i = 1, \dots, n$. Then

$$\|\hat{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\hat{Z}_{0,m+1/2}^L\|_V^2 + \sum_{i=1}^n \|\hat{Z}_{i,m+1/2}^L\|_{V_i}^2)$$

$$\leq c(L^n h_L^2 + (\Delta t)^4) \quad (3.20)$$

when $p = 2$; and when $p > d$

$$\begin{aligned} \|\hat{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\hat{Z}_{0,m+1/2}^L\|_V^p + \sum_{i=1}^n \|\hat{Z}_{i,m+1/2}^L\|_{V_i}^p) \\ \leq c((L^n h_L)^{p/(p-1)} + (\Delta t)^{2p/(p-1)}). \end{aligned} \quad (3.21)$$

The proof of this theorem is similar to that of Theorem 3.15.

3.3 Numerical correctors

We derive the numerical correctors in this section. First we derive the homogenized equation from (3.5).

Let $\phi_0 = 0$ and $\phi_i = 0$ for $i = 1, \dots, n-1$ in (3.5), we have

$$\int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n) \cdot \nabla_{y_n} \phi_n \, dy_n \dots dy_1 dx = 0.$$

For each vector $\xi \in \mathbb{R}^d$, we denote by $N^n(t, x, \mathbf{y}_{n-1}, y_n, \xi) \in W_{\#}^{1,p}(Y_n)/\mathbb{R}$, as a function of y_n , the solution of the problem

$$\nabla_{y_n} \cdot A(t, x, \mathbf{y}_{n-1}, y_n, \xi + \nabla_{y_n} N^n(t, x, \mathbf{y}, \xi)) = 0.$$

We then have $u_n = N^n(t, x, \mathbf{y}, \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_{n-1}} u_{n-1})$. The $(n-1)$ th homogenized operator is determined as

$$A^{n-1}(t, x, \mathbf{y}_{n-1}, \xi) = \int_{Y_n} A(t, x, \mathbf{y}_{n-1}, y_n, \xi + \nabla_{y_n} N^n(t, x, \mathbf{y}_{n-1}, y_n, \xi)) dy_n.$$

It can be shown that A^{n-1} satisfies the monotone and local Lipschitz conditions similar to those of (3.1) and (3.2) (see, e.g., [24], [32]) and [19]). Inductively, let $A^n(t, x, \mathbf{y}, \xi) = A(t, x, \mathbf{y}, \xi)$. Let $N^i(t, x, \mathbf{y}_{i-1}, y_i, \xi) \in W_{\#}^{1,p}(Y_i)/\mathbb{R}$ as a function of y_i be the solution of the problem

$$\nabla_{y_i} \cdot A^i(t, x, \mathbf{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(t, x, \mathbf{y}_{i-1}, y_i, \xi)) = 0.$$

The $(i - 1)$ th homogenized operator is defined as

$$A^{i-1}(t, x, \mathbf{y}_{i-1}, \xi) = \int_{Y_i} A^i(t, x, \mathbf{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(t, x, \mathbf{y}_{i-1}, y_i, \xi)) dy_i$$

which satisfies the monotone and local Lipschitz conditions similar to (3.1) and (3.2). The homogenized equation is:

$$\left\langle \frac{\partial u_0}{\partial t}, \phi_0 \right\rangle_H + \int_D A^0(t, x, \nabla u_0) \cdot \nabla \phi_0 \, dx = \int_D f \phi \, dx, \quad \forall \phi \in V. \quad (3.22)$$

3.3.1 Two scale problems

For problems of two scales, we can deduce an explicit homogenization error in terms of the microscopic scale. We then derive a numerical corrector with an error that is the sum of the FE error and the homogenization error. For $\xi \in \mathbb{R}^d$, let $N(t, x, y, \xi)$ belong to $W_{\#}^{1,p}(Y)/\mathbb{R}$ as a function of y and satisfy the cell problem

$$\nabla_y \cdot A(t, x, y, \xi + \nabla_y N(t, x, y, \xi)) = 0. \quad (3.23)$$

The homogenized operator is determined by

$$A^0(t, x, \xi) = \int_Y A(t, x, y, \xi + \nabla_y N(x, y, \xi)) dy. \quad (3.24)$$

We have the following homogenization result:

Proposition 3.17. Assume that

$$\begin{aligned} u_0 &\in C([0, T], C^2(\bar{D})), \\ u_1 &\in C([0, T], C^1(\bar{D}, C^1(\bar{Y}))), \end{aligned}$$

we then have

$$\|\nabla u^\varepsilon - [\nabla u_0 + \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon})]\|_{L^p((0,T) \times D)} \leq c\varepsilon^{1/(p(p-1))}.$$

Proof. Let

$$u_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon u_1(t, x, \frac{x}{\varepsilon}).$$

We have from the fact that $A \in C^1([0, T] \times \bar{D} \times \bar{Y} \times \mathbb{R}^d)$ and (3.2) that

$$\left| A\left(t, x, y, \nabla u_0(t, x) + \nabla_y u_1(t, x, \frac{x}{\varepsilon})\right) - A(t, x, y, \nabla u_1^\varepsilon(t, x)) \right| \leq c\varepsilon. \quad (3.25)$$

For $i = 1, \dots, d$, we define the function

$$g_i(t, x, y) = A_i(t, x, y, \nabla u_0(x) + \nabla_y u_1(t, x, y)) - A_i^0(t, x, \nabla u_0(x)).$$

From (3.23) we have

$$\frac{\partial}{\partial y_i} g_i(t, x, y) = 0;$$

and from (3.24),

$$\int_Y g_i(t, x, y) dy = 0.$$

From Chapter 1 in [53], there are functions $\alpha_{ij}(t, x, y)$ for $i, j = 1, \dots, d$ such that $\alpha_{ij} = -\alpha_{ji}$ and

$$g_i(t, x, y) = \frac{\partial}{\partial y_j} \alpha_{ij}(t, x, y).$$

As $g_i \in C^1([0, T] \times \bar{D}, C(\bar{Y}))$, we have $\alpha_{ij} \in C^1([0, T] \times \bar{D}, C^1(\bar{Y}))$. We then have

$$\begin{aligned} & A_i\left(t, x, \frac{x}{\varepsilon}, \nabla_x u_0(t, x) + \nabla_y u_1\left(t, x, \frac{x}{\varepsilon}\right)\right) - A_i^0(t, x, \nabla u_0(x)) \\ &= \varepsilon \frac{d}{dx_j} \alpha_{ij}\left(t, x, \frac{x}{\varepsilon}\right) - \varepsilon \frac{\partial}{\partial x_j} \alpha_{ij}\left(t, x, \frac{x}{\varepsilon}\right), \end{aligned}$$

where $\frac{d}{dx_j}$ denotes the total partial derivative of $\alpha_{ij}(t, x, \frac{x}{\varepsilon})$ as a function of x only. Thus for any functions $\phi \in W_0^{1,p}(D)$ we have

$$\int_D \left(A_i\left(t, x, \frac{x}{\varepsilon}, \nabla_x u_0(t, x) + \nabla_y u_1\left(t, x, \frac{x}{\varepsilon}\right)\right) \right)$$

$$\begin{aligned}
& -A_i^0(t, x, \nabla u_0(x)) \frac{\partial \phi}{\partial x_i}(x) dx \\
&= -\varepsilon \int_D \alpha_{ij} \left(t, x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx - \varepsilon \int_D \frac{\partial}{\partial x_j} \alpha_{ij} \left(t, x, \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_i} dx \\
&= -\varepsilon \int_D \frac{\partial}{\partial x_j} \alpha_{ij} \left(t, x, \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_i} dx
\end{aligned}$$

due to $\alpha_{ij}(t, x, y) = -\alpha_{ji}(t, x, y)$. From this and (3.25) we have

$$\left\| \left(\frac{\partial u_1^\varepsilon}{\partial t}(t) - \nabla \cdot (A^\varepsilon(t, \cdot, \nabla u_1^\varepsilon)) \right) - \left(\frac{\partial u_0}{\partial t}(t) - A^0(t, \cdot, \nabla u_0) \right) \right\|_{W^{-1,q}(D)} \leq c\varepsilon. \quad (3.26)$$

Let $\eta^\varepsilon \in \mathcal{D}(D)$ be such that $\eta^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\varepsilon |\nabla_x \eta^\varepsilon(x)| \leq c$ where c is independent of ε . We consider the function

$$w_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon \eta^\varepsilon(x) u_1(t, x, x/\varepsilon).$$

The motivation of introducing the boundary layer function η^ε is to make w_1^ε satisfy the zero boundary condition. We note that

$$\begin{aligned}
& \nabla_x(u_1^\varepsilon(t, x) - w_1^\varepsilon(t, x)) \\
&= -\varepsilon \nabla_x \eta^\varepsilon(x) u_1 \left(t, x, \frac{x}{\varepsilon} \right) + \varepsilon (1 - \eta^\varepsilon(x)) \nabla_x u_1 \left(t, x, \frac{x}{\varepsilon} \right) \\
& \quad + (1 - \eta^\varepsilon(x)) \nabla_y u_1 \left(t, x, \frac{x}{\varepsilon} \right).
\end{aligned}$$

As the support of $\nabla_x(u_1^\varepsilon - w_1^\varepsilon)$ is in an ε neighbourhood of ∂D , we deduce that

$$\|u_1^\varepsilon - w_1^\varepsilon\|_{W^{1,p}(D)} \leq c\varepsilon^{1/p}. \quad (3.27)$$

From (3.2), we have

$$\|A^\varepsilon(\cdot, \nabla u_1^\varepsilon(\cdot)) - A^\varepsilon(\cdot, \nabla w_1^\varepsilon(\cdot))\|_{L^q(D)} \leq c\varepsilon^{1/p}.$$

Therefore

$$\left\| \left(\frac{\partial u_1^\varepsilon}{\partial t}(t) - \nabla \cdot (A^\varepsilon(\cdot, \nabla u_1^\varepsilon)) \right) - \left(\frac{\partial w_1^\varepsilon}{\partial t}(t) - A^\varepsilon(\cdot, \nabla w_1^\varepsilon) \right) \right\|_{W^{-1,q}(D)} \leq c\varepsilon^{1/p}.$$

From this and (3.26), we have

$$\left\| \left(\frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (A^\varepsilon(t, \cdot, \nabla u^\varepsilon)) \right) - \left(\frac{\partial w_1^\varepsilon}{\partial t} - A^\varepsilon(t, \cdot, \nabla w_1^\varepsilon) \right) \right\|_{W^{-1,q}(D)} \leq c\varepsilon^{1/p}. \quad (3.28)$$

Therefore

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial}{\partial t} (u^\varepsilon - w_1^\varepsilon), (u^\varepsilon - w_1^\varepsilon) \right\rangle_H dt \\ & + \int_0^T \int_D (A^\varepsilon(t, \cdot, \nabla u^\varepsilon) - A^\varepsilon(t, \cdot, \nabla w_1^\varepsilon)) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon) dx dt \\ & \leq c\varepsilon^{1/p} \int_0^T \|u^\varepsilon - w_1^\varepsilon\|_V dt \\ & \leq c\varepsilon^{1/p} \left(\int_0^T \|u^\varepsilon - w_1^\varepsilon\|_V^p dt \right)^{1/p} \end{aligned}$$

Using (3.1), we have

$$\begin{aligned} & \|u^\varepsilon(T) - w_1^\varepsilon(T)\|_H^2 + \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p((0,T) \times D)}^p \\ & \leq c\varepsilon^{1/p} \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p((0,T) \times D)} + \|u^\varepsilon(0) - w_1^\varepsilon(0)\|_H^2. \end{aligned}$$

As $u_1 \in C([0, T] \times \bar{D} \times \bar{Y})$ and $u^\varepsilon(0) = u_0(0) = g$, we have $\|u^\varepsilon(0) - w_1^\varepsilon(0)\|_H \leq c\varepsilon$. Therefore

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p((0,T) \times D)} \leq c\varepsilon^{1/(p(p-1))}.$$

From (3.27), we have

$$\|\nabla u^\varepsilon - \nabla u_1^\varepsilon\|_{L^p((0,T) \times D)} \leq c\varepsilon^{1/(p(p-1))}.$$

From this we get the conclusion. $\square$

To construct a numerical corrector, we employ the following operator.

For $\Phi \in L^1(D \times Y)$ we define

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi \left(\varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\} \right) dz, \quad (3.29)$$

where $[x/\varepsilon]$ denotes the “integer” part of x/ε with respect to the unit cube Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$. The operator $\mathcal{U}^\varepsilon$ satisfies:

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x) dx = \int_{D \times Y} \Phi(x, y) dy dx, \quad (3.30)$$

for all $\Phi \in L^1(D \times Y)$, where D^ε is the 2ε neighbourhood of D ; Φ is regarded as 0 when x is outside D . The proof of this proposition is quite straightforward; we refer to [29] for details.

For the solution of the backward Euler approximation using full tensor product FEs, let $\bar{u}_0^L : (0, T) \rightarrow V$ and $\bar{u}_1^L : (0, T) \rightarrow V_1$ be defined as

$$\bar{u}_0^L(t) = \frac{1}{2}(\bar{u}_{0,m}^L + \bar{u}_{0,m+1}^L), \quad \bar{u}_1^L(t) = \frac{1}{2}(\bar{u}_{1,m}^L + \bar{u}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We then have the following corrector result.

Theorem 3.18. Assume that $u_0 \in C^2([0, T], V)$ and $u_1 \in C^2([0, T], V_1) \cap C([0, T], C^1(\bar{D} \times \bar{Y}))$ and the hypothesis of Theorem 3.9 holds. For the solution of the full tensor product FE backward Euler approximation in (3.9), we have

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)(\cdot)]\|_{L^p((0,T) \times D)} \\ & \leq c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)} + (\Delta t)^{1/(p-1)}). \end{aligned}$$

Proof. Using the midpoint rule, we have

$$\begin{aligned} \int_0^T \|\nabla u_0(t) - \nabla \bar{u}_0^L(t)\|_{L^p(D)}^p dt &= \sum_{m=0}^{M-1} \int_{m\Delta t}^{(m+1)\Delta t} \|\nabla u_0(t) - \nabla \bar{u}_0^L(t)\|_{L^p(D)}^p dt \\ &\leq \sum_{m=0}^{M-1} (\Delta t \|\nabla u_0(t_{m+1/2}) - \nabla \bar{u}_0^L(t_{m+1/2})\|_{L^p(D)}^p + c(\Delta t)^3) \end{aligned}$$

where the constant c is independent of Δt and $t_{m+1/2} = t_m + \frac{1}{2}\Delta t$. We have

$$\left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla u_0(t_{m+1/2}) \right\|_{L^p(D)} \leq c(\Delta t)^2.$$

From this we deduce

$$\begin{aligned}
& \int_0^T \|\nabla u_0(t) - \nabla \bar{u}_0^L(t)\|_{L^p(D)}^p dt \\
& \leq \sum_{m=0}^{M-1} \left(\Delta t \left\| \frac{1}{2} (\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \frac{1}{2} (\nabla \bar{u}_{0,m}^L + \nabla \bar{u}_{0,m+1}^L) \right\|_{L^p(D)}^p \right. \\
& \quad \left. + c(\Delta t)^3 \right) \\
& \leq \Delta t \sum_{m=1}^{M-1} (\|\nabla \bar{z}_{0,m}^L\|_{L^p(D)}^p + \|\nabla \bar{z}_{0,m+1}^L\|_{L^p(D)}^p) + c(\Delta t)^2 \\
& \leq c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).
\end{aligned}$$

By the same argument, we have

$$\int_0^T \|\nabla_y u_1(t) - \nabla_y \bar{u}_1^L(t)\|_{L^p(D \times Y)}^p dt \leq c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).$$

From (3.30) we deduce

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1 - \nabla_y \bar{u}_1^L)\|_{L^p((0,T) \times D)} \leq \|\nabla_y u_1 - \nabla_y \bar{u}_1^L\|_{L^p((0,T) \times D \times Y)}.$$

Further, as $u_1 \in C([0, T], C^1(\bar{D} \times \bar{Y}))$,

$$|\nabla_y u_1(t, x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u_1)(t, x)| \leq c\varepsilon.$$

Thus

$$\begin{aligned}
& \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)]\|_{L^p((0,T) \times D)} \\
& \leq \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}^\varepsilon(\nabla_y u_1)]\|_{L^p((0,T) \times D)} \\
& \quad + \|\nabla u_0 - \nabla \bar{u}_0^L\|_{L^p((0,T) \times D)} + \|\mathcal{U}^\varepsilon(\nabla_y u_1) - \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)\|_{L^p((0,T) \times D)} \\
& \leq c(\varepsilon^{1/(p-1)} + (\Delta t)^{1/(p-1)} + h_L^{1/(p-1)}).
\end{aligned}$$

□

Similarly, for the solution of backward Euler scheme using the sparse tensor product FEs, we define $\hat{u}_0^L : [0, T] \rightarrow V$ and $\hat{u}_1^L : [0, T] \rightarrow V_1$ as

$$\hat{u}_0^L(t) = \frac{1}{2}(\hat{u}_{0,m}^L + \hat{u}_{0,m+1}^L), \quad \hat{u}_1^L(t) = \frac{1}{2}(\hat{u}_{1,m}^L + \hat{u}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We then have the following corrector result.

Theorem 3.19. Assume that $u_0 \in C^2([0, T], V)$ and $u_1 \in C^2([0, T], V_1) \cap C([0, T], C^1(\bar{D} \times \bar{Y}))$ and the hypothesis of Theorem 3.11 holds. For the solution of the sparse tensor product FE backward Euler approximation in (3.7), we have

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^2((0,T) \times D)} \\ & \leq c(\varepsilon^{1/2} + L^{1/2}h_L + \Delta t) \end{aligned}$$

when $p = 2$; and when $p > d$

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^p((0,T) \times D)} \\ & \leq c(\varepsilon^{1/(p(p-1))} + (Lh_L)^{1/(p-1)} + (\Delta t)^{1/(p-1)}). \end{aligned}$$

The results for the Crank-Nicholson scheme are similar. For the full tensor product FEs, we define $\bar{U}_0^L : [0, T] \rightarrow V$ and $\bar{U}_1^L : [0, T] \rightarrow V_1$ as

$$\bar{U}_0^L(t) = \frac{1}{2}(\bar{U}_{0,m}^L + \bar{U}_{0,m+1}^L), \quad \bar{U}_1^L(t) = \frac{1}{2}(\bar{U}_{1,m}^L + \bar{U}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

Theorem 3.20. Assume that $u_0 \in C^2([0, T], V)$ and $u_1 \in C^2([0, T], V_1) \cap C([0, T], C^1(\bar{D} \times \bar{Y}))$ and the hypothesis of Theorem 3.15 holds. For the solution of the full tensor product FE Crank-Nicholson approximation in (3.9), we have

$$\|\nabla u^\varepsilon - [\nabla \bar{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)(\cdot)]\|_{L^p((0,T) \times D)} \leq c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)} + (\Delta t)^{2/p}).$$

Similarly, for the solution of Crank-Nicholson scheme using the sparse tensor product FEs, we define $\hat{U}_0^L : [0, T] \rightarrow V$ and $\hat{U}_1^L : [0, T] \rightarrow V_1$ as

$$\hat{U}_0^L(t) = \frac{1}{2}(\hat{U}_{0,m}^L + \hat{U}_{0,m+1}^L), \quad \hat{U}_1^L(t) = \frac{1}{2}(\hat{U}_{1,m}^L + \hat{U}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We then have the following corrector result.

Theorem 3.21. Assume that $u_0 \in C^2([0, T], V)$ and $u_1 \in C^2([0, T], V_1) \cap C([0, T], C^1(\bar{D} \times \bar{Y}))$ and the hypothesis of Theorem 3.16 holds. For the solution of the sparse tensor product FE Crank-Nicholson approximation in (3.7), we have

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \hat{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L(\cdot))]\|_{L^2((0,T) \times D)} \\ & \leq c(\varepsilon^{1/2} + L^{1/2}h_L + \Delta t) \end{aligned}$$

when $p = 2$; and when $p > d$

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \hat{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L(\cdot))]\|_{L^p((0,T) \times D)} \\ & \leq c(\varepsilon^{1/(p(p-1))} + (Lh_L)^{1/(p-1)} + (\Delta t)^{2/p}). \end{aligned}$$

3.3.2 Multiscale problems

For general problems with more than two scales, a homogenization error in terms of the microscopic scales is not available. However, we can establish a numerical corrector from the FE solutions. We assume that $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \dots, n$. We first define the operator $\mathcal{T}_n^\varepsilon : L^1(D) \rightarrow L^1(D \times \mathbf{Y})$ as:

$$\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y}) = \phi\left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left\lfloor \frac{y_1}{\varepsilon_2/\varepsilon_1} \right\rfloor + \dots + \varepsilon_n \left\lfloor \frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}} \right\rfloor + \varepsilon_n y_n\right) \quad (3.31)$$

where the function ϕ is understood as 0 outside D , and $\lfloor \cdot \rfloor$ denotes the “integer” part with respect to Y . Let D^ε be the 2ε neighbourhood of D .

For all functions $\phi \in L^1(D)$ which are understood as 0 outside D , we have

$$\int_D \phi dx = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\mathbf{y} dx. \quad (3.32)$$

We can also show that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n \text{ in } L^p(D \times \mathbf{Y}). \quad (3.33)$$

We define the operator $\mathcal{U}_n^\varepsilon : L^1(D \times \mathbf{Y}) \rightarrow L^1(D)$ as

$$\begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \dots \int_{Y_n} \Phi \left(\varepsilon_1 \left[\frac{x}{\varepsilon_1} \right] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1} \left[\frac{\varepsilon_1}{\varepsilon_2} \left\{ \frac{x}{\varepsilon_1} \right\} \right] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \dots, \right. \\ \left. \frac{\varepsilon_n}{\varepsilon_{n-1}} \left[\frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{ \frac{x}{\varepsilon_{n-1}} \right\} \right] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \left\{ \frac{x}{\varepsilon_n} \right\} \right) dt_n \dots dt_1 \end{aligned} \quad (3.34)$$

where $\{\cdot\} = \cdot - [\cdot]$; the function Φ is understood as 0 outside D . We have $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\Phi)) = \Phi \forall \Phi \in L^1(D)$. We can show that

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) dx = \int_D \int_{\mathbf{Y}} \Phi d\mathbf{y} dx \quad (3.35)$$

where Φ is understood as 0 outside D . The proofs of these facts can be found in [29].

We have the following corrector results.

Proposition 3.22. We have

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i} u_i)]\|_{L^p((0,T) \times D)} = 0.$$

Proof. As u^ε is uniformly bounded in V , from (3.2), $A(t, x, \frac{x}{\varepsilon}, \dots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)$ is bounded in $L^q((0, T) \times D)$. Let $\chi \in L^q((0, T) \times D \times \mathbf{Y})$ be the $(n+1)$ -scale convergence limit of a subsequence of $A(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)$. From (3.5), we have that

$$\int_0^T \int_D \int_{\mathbf{Y}} \chi(\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\mathbf{y} dx dt$$

$$\begin{aligned}
&= \int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \\
&\quad \cdot (\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\mathbf{y} dx dt. \tag{3.36}
\end{aligned}$$

for all $v_0 \in V$ and $v_i \in V_i$. We consider:

$$\begin{aligned}
I &= \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \right\rangle_H \\
&\quad + \int_0^T \int_D \int_{\mathbf{Y}} \left(\mathcal{T}^\varepsilon(A(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)) - A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \right) \\
&\quad \cdot (\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)) d\mathbf{y} dx dt.
\end{aligned}$$

From (3.3), (3.5), (3.36) we have

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} I \\
&= \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon \right\rangle_H - \left\langle \frac{\partial u_0}{\partial t}, u_0 \right\rangle dt \\
&\quad + \int_0^T \int_D A(t, x, \frac{x}{\varepsilon}, \dots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon) \cdot \nabla u^\varepsilon dx dt \\
&\quad - \int_0^T \int_D \int_{\mathbf{Y}} A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \\
&\quad \cdot (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) d\mathbf{y} dx dt \\
&= 0.
\end{aligned}$$

Using the smoothness of A , we have

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} I \\
&= \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \right\rangle_H \\
&\quad + \int_0^T \int_D \int_{\mathbf{Y}} \left(A(t, x, \mathbf{y}, \mathcal{T}^\varepsilon(\nabla u^\varepsilon)) - A(t, x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \right)
\end{aligned}$$

$$\cdot (\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)) d\mathbf{y} dx dt.$$

From (3.1) and $u^\varepsilon(0) = u_0(0) = g$, we have

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon(T) - u_0(T)\|_H + \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)\|_{L^p((0,T) \times D \times \mathbf{Y})} = 0.$$

From (3.34), we have $(\mathcal{U}^\varepsilon(\Phi)(t, x))^p \leq \mathcal{U}^\varepsilon(\Phi^p)(t, x)$. From (3.35), we have

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^p((0,T) \times D)}^p \leq \|\mathcal{U}^\varepsilon(\Phi^p)\|_{L^1((0,T) \times D)} \leq \|\Phi\|_{L^p((0,T) \times D \times \mathbf{Y})}^p.$$

We therefore have

$$\begin{aligned} & \|\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i))\|_{L^p((0,T) \times D)} \\ & \leq \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)\|_{L^p((0,T) \times D \times \mathbf{Y})} \rightarrow 0 \end{aligned}$$

when $\varepsilon \rightarrow 0$. Using $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon)) = \nabla u^\varepsilon$, we get the conclusion. $\square$

We construct the numerical correctors from the FE solutions of problem (3.5). We present the results for the backward Euler approximations. The results for the Crank-Nicholson approximations are similar. For the full tensor product FE approximations, let $\bar{u}_0 : (0, T) \rightarrow V$ and $\bar{u}_i : (0, T) \rightarrow V_i$ be defined as

$$\bar{u}_0^L(t) = \frac{1}{2}(\bar{u}_{0,m}^L + \bar{u}_{0,m+1}^L), \quad \bar{u}_i^L(t) = \frac{1}{2}(\bar{u}_{i,m}^L + \bar{u}_{i,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We have:

Theorem 3.23. Assume that the hypothesis of Theorem 3.9 hold. For the solution of the approximating problems (3.7) using the full tensor product FE approximation for u_i , we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i} \bar{u}_i^L)]\|_{L^p((0,T) \times D)} = 0.$$

Proof. Using the midpoint rule, we have

$$\begin{aligned}
& \int_0^T \|\nabla u_0(t) - \nabla \bar{u}_0^L(t)\|_{L^p(D)}^p dt \\
&= \sum_{m=0}^{M-1} \int_{m\Delta t}^{(m+1)\Delta t} \|\nabla u_0(t) - \nabla \bar{u}_0^L(t)\|_{L^p(D)}^p dt \\
&\leq \sum_{m=0}^{M-1} (\Delta t \|\nabla u_0(t_{m+1/2}) - \nabla \bar{u}_0^L(t_{m+1/2})\|_{L^p(D)}^p + c(\Delta t)^3)
\end{aligned}$$

where the constant c is independent of Δt . We have

$$\left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla u_0(t_{m+1/2}) \right\|_{L^p(D)} \leq c(\Delta t)^2.$$

From this we deduce

$$\begin{aligned}
& \int_0^T \|\nabla u_0(t) - \nabla \bar{u}_0(t)\|_{L^p(D)}^p dt \\
&\leq \sum_{m=0}^{M-1} (\Delta t \left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \frac{1}{2}(\nabla \bar{u}_{0,m}^L + \nabla \bar{u}_{0,m+1}^L) \right\|_{L^p(D)}^p \\
&\quad + c(\Delta t)^3) \\
&= c\Delta t \sum_{m=1}^{M-1} (\|\bar{z}_{0,m}^L\|_{L^p(D)}^p + \|\bar{z}_{0,m+1}^L\|_{L^p(D)}^p) + c(\Delta t)^2 \\
&\leq c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).
\end{aligned}$$

By the same argument, for all $i = 1, \dots, n$, we have

$$\int_0^T \|\nabla_{y_i} u_i(t) - \nabla_{y_i} \bar{u}_i(t)\|_{L^p(D \times (0,1) \times \mathbf{Y})}^p dt \leq c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).$$

From (3.35) we deduce

$$\|\mathcal{U}^\varepsilon(\nabla_{y_i} u_i - \nabla_{y_i} \bar{u}_i^L)\|_{L^p((0,T) \times D)} \leq \|\nabla_{y_i} u_i - \nabla_{y_i} \bar{u}_i^L\|_{L^p((0,T) \times D \times \mathbf{Y})}.$$

Thus

$$\|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i} \bar{u}_i^L)]\|_{L^p((0,T) \times D)}$$

$$\begin{aligned}
&\leq \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i} u_i)]\|_{L^p((0,T) \times D)} \\
&\quad + \|\nabla u_0 - \nabla \bar{u}_0^L\|_{L^p((0,T) \times D)} + \|\mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla u_i) - \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla \bar{u}_i^L)\|_{L^p((0,T) \times D)}
\end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$ and $L \rightarrow \infty$. $\square$

Similarly, for the sparse tensor product FE approximation, we have:

Theorem 3.24. Assume that the hypothesis of Theorem 3.11 hold. For the solution of the approximating problem (3.7) using the sparse tensor product FE approximation for u_i for $i = 1, \dots, n$, we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L + \mathcal{U}^\varepsilon(\sum_{i=1}^n \nabla_{y_i} \hat{u}_i^L)]\|_{L^p((0,T) \times D)} = 0.$$

3.4 Numerical examples

We present some one dimensional and two dimensional numerical results in this section to illustrate the sparse tensor product FE method.

We first consider a problem on the domain $D = (0, 1)$ where the monotone nonlinear function

$$\begin{aligned}
A(t, x, y, \xi) = & \xi + \frac{1}{2} \sin(x \sin(2\pi y) \xi) - t^2 x \cos(2\pi y) \\
& - \frac{1}{2} \sin(t^2 x \sin(2\pi y) (1 - 2x + x \cos(2\pi y))) \quad (3.37)
\end{aligned}$$

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. In this case $p = 2$. The problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y) = \frac{t^2}{2\pi} x \sin(2\pi y).$$

We apply the backward Euler and Crank-Nicholson method with sparse tensor product FE spaces to the multiscale homogenized problem. We use the Broyden's method ([22]) and the Polak-Ribière method ([27]) at each time step to solve the simultaneous nonlinear problems.

For the backward Euler method with sparse tensor product FE spaces, we use a timestep of $\Delta t = 1/2^L$. We plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^2(D, H_{\#}^1(Y))}$ in Figures 3.1 and 3.2 respectively at $t = 1$. The numerical results show that the errors are $O(\Delta t) + O(h_L)$. When these errors hold for all t_m , we get the errors estimate as in Theorem 3.11. This result supports the theoretical finding.

In Figures 3.3 and 3.4, we plot the numerical errors for the Crank-Nicholson method on sparse tensor product FE spaces for the same problem. For the Crank-Nicholson method, we choose a timestep of $\Delta t = 1/(\lceil 2^{(L/2)} \rceil)$. The numerical results show that the errors are $O((\Delta t)^2) + O(h_L)$. This result supports the theoretical finding in Theorem 3.16.

Figure 3.1: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.

Figure 3.2: The error $\|u_1 - \hat{u}_1\|_{L^2(D, H^1_\#(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.

Figure 3.3: The error $\|u_0 - \hat{u}_0^L\|_{H^1_0(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.

Figure 3.4: The error $\|u_1 - \hat{u}_1\|_{L^2(D, H^1_\#(Y))}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for Crank-Nicholson method.

We then consider the multiscale monotone problem for $p = 4$ on $D = (0, 1)$ with the monotone function

$$\begin{aligned} A(t, x, y, \xi) = & \xi + \frac{1}{10}(2 + \sin(2\pi y))\xi^3 - t^2 x \cos(2\pi y) \\ & - \frac{t^6}{10}(2 + \sin(2\pi y))(1 - 2x + x \cos(2\pi y))^3 \end{aligned} \quad (3.38)$$

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. This problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y) = \frac{t^2}{2\pi} x \sin(2\pi y).$$

We plot the numerical errors $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}((D))}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D, W_\#^{1,4}(Y))}$ for $t = 1$ in Figures 3.5 and 3.6 respectively for the backward Euler method, using sparse tensor product FE spaces. The numerical results show that the error behaves like $O(\Delta t) + O(h_L)$.

The numerical results for the Crank-Nicholson scheme are plotted in Figures 3.7 and 3.8. The numerical results show that the error behaves like $O((\Delta t)^2) + O(h_L)$.

Figure 3.5: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for backward Euler method.

Figure 3.6: The error $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for backward Euler method.

Figure 3.7: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.

Figure 3.8: The error $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for Crank-Nicholson method.

For the two dimensional domain $D = (0, 1) \times (0, 1)$, we consider the problem for $p = 4$ with coefficient

$$\begin{aligned}
 & A(t, x, y, \xi) \\
 = & \xi' + \xi'' + \frac{1}{10}(2 + \sin(2\pi y_2))\xi'^3 + \frac{1}{10}(2 + \sin(2\pi y_1))\xi''^3
 \end{aligned}$$

$$\begin{aligned}
& -t^2(x' + x'')(\cos(2\pi y') \sin(2\pi y'') + \sin(2\pi y') \cos(2\pi y'')) \\
& -\frac{t^6}{10}(2 + \sin(2\pi y_2))((1 - 2x')(x'' - x''^2) \\
& \quad + (x' + x'') \cos(2\pi y') \sin(2\pi y''))^3 \\
& -\frac{t^6}{10}(2 + \sin(2\pi y_1))((x' - x'^2)(1 - 2x'') \\
& \quad + (x' + x'') \sin(2\pi y') \cos(2\pi y''))^3,
\end{aligned}$$

and

$$f(t, x) = 2t^2(x' + x'' - x'^2 - x''^2) + 2t(x' - x'^2)(x'' - x''^2)$$

with $x = (x', x'') \in D$, $y = (y', y'') \in Y$ and $\xi = (\xi', \xi'') \in \mathbb{R}^2$. This problem has the exact solution

$$u_0(t, x) = t^2(x' - x'^2)(x'' - x''^2)$$

and

$$u_1(t, x, y) = \frac{t^2}{2\pi}(x' + x'') \sin(2\pi y') \sin(2\pi y'')$$

For $t = 1$, we plot the error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D, W_{\#}^{1,4}(Y)/\mathbb{R})}$ for the backward Euler method with sparse tensor product FE spaces in Figures 3.9 and 3.10 respectively. The error for the Crank-Nicholson method for sparse tensor product FE spaces are plotted in Figures 3.11 and 3.12. Once more the results show that the error behaves like $O(\Delta t) + O(h_L)$ and $O((\Delta t)^2) + O(h_L)$ for the backward Euler and Crank-Nicholson method respectively.

Figure 3.9: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem at $t = 1$ for backward Euler method.

Figure 3.10: The error $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for two dimensional problem at $t = 1$ for backward Euler method.

Figure 3.11: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem at $t = 1$ for Crank-Nicholson method.

Figure 3.12: The error $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y))}$ versus the mesh size h for two dimensional problem at $t = 1$ for Crank-Nicholson method.

The results for the one dimensional and two dimensional cases for $p = 4$ show that the observed numerical errors are of the optimal orders of Δt and h_L which are far better than what we can show theoretically. This is in agreement with the well known facts for numerical solutions of monotone problems (see [48]).

We now study the stability of the backward Euler and Crank-Nicolson methods. We solve equations (3.37) and (3.38) for $L = 3$ for $t = 1$ up to $t = 20$. Tables 3.1 and 3.3 show the relative errors for u_0 and u_1 for the backward Euler method when $p = 2$ and $p = 4$ respectively. Tables 3.2 and 3.4 show the relative errors for u_0 and u_1 for the Crank Nicolson method when $p = 2$ and $p = 4$ respectively. The results indicate that the methods are stable in the examples that we test.

t	Relative Error for u_0	Relative Error for u_1
1	0.125854735644064	0.224617913458754
2	0.125078447593275	0.225141266722862
4	0.125009432238565	0.224720796094935
6	0.125001870638104	0.224477483978783
8	0.125001169194599	0.224451696958569
10	0.125000586251790	0.224426812679338
12	0.125000351751539	0.224413816320202
14	0.125000237900600	0.224408791284751
16	0.125000173975774	0.224408934346191
18	0.125000128277011	0.224408338046937
20	0.125000100207190	0.224408633128799

Table 3.1: Backward Euler method for $p = 2$ and $L = 3$

t	Relative Error for u_0	Relative Error for u_1
1	0.125016171960009	0.224702762406534
2	0.125008017696875	0.225138751812743
4	0.125002327418120	0.224713712201364
6	0.125000485195110	0.224479745189368
8	0.125000481755306	0.224449687602161
10	0.125000288360970	0.224426681687281
12	0.125000184174021	0.224413974904643
14	0.125000148381969	0.224408827157406
16	0.125000115555598	0.224408963910168
18	0.125000087345132	0.224408372384859
20	0.125000067573447	0.224408744771673

Table 3.2: Crank Nicolson method for $p = 2$ and $L = 3$

t	Relative Error for u_0	Relative Error for u_1
1	0.126666035775885	0.260055621153219
2	0.741333700387852	0.741260075908698
4	0.811465732497316	0.932120676540197
6	0.815041503410363	0.946201241262360
8	0.815648883631344	0.948812179364110
10	0.815819840981795	0.949534685427530
12	0.815879556981405	0.949809542120457
14	0.815904597633041	0.949926391538875
16	0.815916463338271	0.949982568828778
18	0.815922638461955	0.950012222289596
20	0.815926105813247	0.950029034323637

Table 3.3: Backward Euler method for $p = 4$ and $L = 3$

t	Relative Error for u_0	Relative Error for u_1
1	0.125243236516428	0.260248573309431
2	0.511849388020212	0.594792319365976
4	0.685901961457734	0.827390450179001
6	0.692784892883401	0.843413959251363
8	0.694829225010516	0.849012313634601
10	0.695134632413850	0.849801026674036
12	0.695281928978166	0.850300709938052
14	0.695342609037959	0.850437378671294
16	0.695374900005737	0.850739020541571
18	0.695416569737956	0.851006995588748
20	0.695393122871507	0.851133258966387

Table 3.4: Crank Nicolson method for $p = 4$ and $L = 3$

Chapter 4

Multiscale Monotone Parabolic Equations depending on Microscales in Space and Time

Chapter 4 considers multiscale monotone parabolic equations that depend on a microscopic time scale and a microscopic spatial scale. Similar to Chapter 2, we consider the critical case when the derivative to the fast time variable plays a role in the cell problems. The multiscale monotone parabolic problem together with the multiscale homogenization theory required to derive the multiscale homogenized equation are introduced in Section 4.1. In Section 4.2, backward Euler and Crank-Nicholson methods are developed for general FE spaces, full tensor and sparse tensor product FE spaces. A numerical corrector is derived from the FE solutions in Section 4.3. Some numerical results are presented in Section 4.4 to verify the theory.

4.1 Problem setting and homogenization

4.1.1 Problem setting

Let $D \in \mathbb{R}^d$ be a bounded domain. Let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. Let $A(t, x, \tau, y, \xi) : [0, T] \times \bar{D} \times [0, 1] \times \bar{Y} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuously differentiable function that is Y periodic with respect to y and $[0, 1]$ periodic with respect to τ . The function A is monotone and locally Lipschitz with respect to ξ , i.e. there are constants $p \geq 2$, $\alpha > 0$ and $\beta > 0$ so that for all $t \in [0, T]$, $x \in \bar{D}$, $y \in \bar{Y}$, and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$(A(t, x, \tau, y, \xi_1) - A(t, x, \tau, y, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad (4.1)$$

and

$$|A(t, x, \tau, y, \xi_1) - A(t, x, \tau, y, \xi_2)| \leq \beta(|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2| \quad (4.2)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^d$. Let ε be a positive value which represents the small scale that the problem depends on. We define the multiscale monotone function as

$$A^\varepsilon(t, x, \xi) = A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \xi).$$

We denote by $V = W_0^{1,p}(D)$ and $H = L^2(D)$. We have $V \subset H \subset V'$. Let $f \in L^q((0, T), V')$ where $1/p + 1/q = 1$. Let $g \in H$. We consider the following multiscale monotone parabolic problem:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (A^\varepsilon(t, x, \nabla u^\varepsilon)) &= f, \quad \text{in } D \times (0, T) \\ u^\varepsilon(0) &= g \end{aligned} \quad (4.3)$$

with the Dirichlet boundary condition on ∂D . Problem (4.3) has a unique solution that satisfies

$$\|u^\varepsilon\|_{L^p((0,T),V)} + \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^q((0,T),V')} \leq c(\|f\|_{L^q((0,T),V')} + \|g\|_H)$$

where the constant c only depends on α and β in (4.1) and (4.2) and on $\sup_{t \in [0,T], x \in D, \tau \in [0,1], y \in \bar{Y}} |A(t, x, \tau, y, 0)|$ (see Lions [56] and [77]). We will study homogenization of (4.3) by multiscale convergence. We thus recall the concept of multiscale convergence in the L^p setting for functions depending on microscopic scales in both time and space.

4.1.2 Multiscale homogenization

The concept of multiscale convergence of Nguetseng [65] is extended to functions depending on microscopic scales in both time and space in [50] in the L^2 setting. The definition in the L^p setting below is a simplified version of the one in [76] where more than one microscopic time scales are considered.

Definition 4.1. A sequence $\{w^\varepsilon\}$ time-space multiscale converges to a function $w_0 \in L^p((0,T) \times D)$ if for all functions $\phi \in C([0,T] \times \bar{D} \times [0,1] \times \bar{Y})$ which are $(0,1) \times Y$ periodic with respect to τ and y

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D w^\varepsilon(t, x) \phi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\ = \int_0^T \int_D \int_0^1 \int_Y w_0(t, x, \tau, y) \phi(t, x, \tau, y) dy d\tau dx dt. \end{aligned}$$

The definition makes sense due to the following result

Proposition 4.2. From a bounded sequence in $L^p((0,T) \times D)$ we can extract a time-space multiscale convergent subsequence.

The proof of this result is similar to those in [5] and [50]. For a bounded sequence in $L^p((0, T), W_0^{1,p}(D)) \cap L^q((0, T), W^{-1,q}(D))$, we have the following result.

Proposition 4.3. For a bounded sequence $\{w^\varepsilon\}$ in $L^p((0, T), W_0^{1,p}(D)) \cap L^q((0, T), W^{-1,q}(D))$, we can extract a subsequence (not renumbered) such that

$$\nabla w^\varepsilon \xrightarrow{\text{ts-ms}} \nabla w_0 + \nabla_y w_1$$

for $w_0 \in L^p((0, T) \times D)$ and $w_1 \in L^p((0, T) \times D \times (0, 1), W_{\#}^{1,p}(Y)/\mathbb{R})$.

The proof of this result is standard. It is similar to the proof in Chapter 2. To establish the multiscale homogenized equation of (4.3), we need the following result.

Proposition 4.4. Let $\{w^\varepsilon\}$ be a bounded sequence in $L^p((0, T), W^{1,p}(D))$ such that

$$\nabla w^\varepsilon \xrightarrow{\text{ts-ms}} \nabla w_0 + \nabla_y w_1.$$

Then for all smooth functions $\psi(t, x, \tau, y)$ which are $(0, 1) \times Y$ periodic with respect to τ and y and

$$\int_Y \psi(t, x, \tau, y) dy = 0$$

for all t, x and τ , then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T \int_D \frac{1}{\varepsilon} w^\varepsilon(t, x) \psi(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) dx dt \\ = \int_0^T \int_D \int_0^1 \int_Y w_1(t, x, \tau, y) \psi(t, x, \tau, y) dy d\tau dx dt. \end{aligned}$$

The proof of this proposition is similar to that of Proposition 2.3 in Chapter 2. We have the following norm equivalence.

Lemma 4.5. There are positive constants c_1 and c_2 such that for all $w_0 \in W_0^{1,p}(D)$ and $w_1 \in L^p(D \times (0, 1), W_{\#}^{1,p}(Y))$,

$$\begin{aligned} & c_1(\|\nabla w_0\|_{L^p(D)}^p + \|\nabla_y w_1\|_{L^p(D \times (0,1) \times Y)}^p) \\ & \leq \int_D \int_0^1 \int_Y |\nabla w_0 + \nabla_y w_1|^p d\mathbf{y} d\tau dx \\ & \leq c_2(\|\nabla w_0\|_{L^p(D)}^p + \|\nabla_y w_1\|_{L^p(D \times (0,1) \times Y)}^p) \end{aligned}$$

A proof can be found in [48].

From these results, we establish the multiscale homogenized problem of (4.3).

4.1.3 Multiscale homogenization of (4.3)

We denote by $V_{\#} = W_{\#}^{1,p}(Y)/\mathbb{R}$ and by $H_{\#} = L^2(Y)/\mathbb{R}$. In the limiting equation (4.4) below, as $\frac{\partial u_1}{\partial \tau}(t, x, \tau, \cdot)$ belongs to $V'_{\#}$ we use the evolution triple $V_{\#} \subset H_{\#} \subset V'_{\#}$. We denote by $\langle \cdot, \cdot \rangle_{H_{\#}}$ the inner product in $H_{\#}$ extended by density to the duality parity between $V'_{\#}$ and $V_{\#}$. We then have the following result.

Proposition 4.6. We can extract a subsequence from $\{u^{\varepsilon}\}$ (not renumbered) such that $u^{\varepsilon} \rightharpoonup u_0$ in $W_0^{1,p}(D)$ and

$$\nabla u^{\varepsilon} \xrightarrow{\text{ts-ms}} \nabla u_0 + \nabla_y u_1$$

with $u_1 \in L^p((0, T) \times D \times (0, 1), V_{\#}) \cap L^q(D, W_{\#}^{1,q}((0, 1), V'_{\#}))$. The functions u_0 and u_1 satisfy the problem

$$\begin{aligned} & \left\langle \frac{\partial u_0}{\partial t}(t), \phi_0 \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial u_1}{\partial \tau}(t, x, \tau, \cdot), \phi_1(x, \tau, \cdot) \right\rangle_{H_{\#}} d\tau dx \\ & + \int_D \int_0^1 \int_Y A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1) \cdot (\nabla \phi_0 + \nabla_y \phi_1) dy d\tau dx \end{aligned}$$

$$= \int_D f(t, x) \phi_0(x) dx \quad (4.4)$$

for all $\phi_0 \in V$ and $\phi_1 \in L^p(D \times (0, 1), V_\#)$. The initial condition is $u_0(0) = g$.

The proof of the proposition is similar to that of Proposition 3.5 in Chapter 3. We then have

Proposition 4.7. Problem (4.4) has a unique solution. The whole sequence $\{u^\varepsilon\}$ time-space multiscale converges to the solution of this equation.

Proof. Assume that (4.4) has two solutions (u_0, u_1) and (u_0^*, u_1^*) . We have

$$\begin{aligned} & \left\langle \frac{\partial(u_0 - u_0^*)}{\partial t}, \phi_0(\cdot) \right\rangle_H \\ & + \int_D \int_Y \left\langle \frac{\partial(u_1 - u_1^*)}{\partial \tau}(t, x, \tau, \cdot), \phi_1(t, x, \tau, \cdot) \right\rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y (A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1) - A(t, x, \tau, y, \nabla u_0^* + \nabla_y u_1^*)) \\ & \quad \cdot (\nabla \phi_0 + \nabla_y \phi_1) dy d\tau dx \\ & = 0. \end{aligned}$$

Let $\phi_0 = u_0 - u_0^*$ and $\phi_1 = u_1 - u_1^*$. As $u_0(0) - u_0^*(0) = 0$, using (4.1), we have

$$\begin{aligned} & \|u_0(T) - u_0^*(T)\|_H^2 + \int_D \int_0^1 \int_Y |(\nabla u_0 + \nabla_y u_1) \\ & \quad - (\nabla u_0^* + \nabla_y u_1^*)|^p dy d\tau dx = 0. \end{aligned}$$

Thus $\nabla u_0 = \nabla u_0^*$ and $\nabla_y u_1 = \nabla_y u_1^*$. We get the conclusion. $\square$

4.2 FE discretization

We consider the FE approximation of problem (4.4) in this section. We consider both the backward Euler method and the Crank-Nicholson method. We denote by V_1 the space $L^p(D \times (0, 1), V_\#)$.

4.2.1 Backward Euler method

As with Chapter 3, we first consider the backward Euler method for general FE spaces. We then apply the method for the full and sparse tensor product FE spaces. The main difficulty is due to the fact that the second term in the left hand side of the equation (4.4) is not symmetric. However, using the approach in [34] assuming regularity with respect to t , we can show the convergence of the numerical scheme.

4.2.1.1 Backward Euler method for general FE spaces

Let $V_0^L \subset V$ and $V_1^L \subset V_1 \cap L^q(D, W_{\#}^{1,q}((0,1), V'_{\#}))$ be finite dimensional spaces where the superscript L indicates the level of resolution. Let M be an integer. Let $\Delta t = T/M$. We consider the time sequence $0 = t_0 < t_1 < \dots < t_M$ where $t_m = m\Delta t$ for $m = 0, 1, \dots, M$. Let $g^L \in V_0^L$ be an approximation of g . We consider the problem: Find $u_{0,m}^L \in V_0^L$ and $u_{1,m}^L \in V_1^L$ such that

$$\begin{aligned} & \left\langle \frac{u_{0,m+1}^L - u_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial u_{1,m+1}^L}{\partial \tau}, \phi_1 \right\rangle_{H_{\#}} d\tau dx \\ & + \int_D \int_0^1 \int_Y A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L) \\ & \quad \cdot (\nabla \phi_0 + \nabla_{y_1} \phi_1) dy d\tau dx \\ & = \int_D f(t_{m+1}, x) \phi_0(x) dx, \end{aligned} \tag{4.5}$$

for all $\phi_0 \in V_0^L$ and $\phi_1 \in V_1^L$. We first show that (4.5) has a unique solution.

Proposition 4.8. Problem (4.5) has a unique solution.

Proof. Let $\mathbf{c}_m = (\mathbf{c}_{0,m}, \mathbf{c}_{1,m})$ and $\mathbf{d} = (\mathbf{d}_0, \mathbf{d}_1)$ in $\mathbb{R}^{\dim V_0^L} + \mathbb{R}^{\dim V_1^L}$ be the coordinate vectors of $(u_{0,m}^L, u_{1,m}^L)$ and (ϕ_0, ϕ_1) respectively in the expansion with respect to the basis functions of $V_0^L \times V_1^L$. Let $\mathcal{A}(\mathbf{c}_{m+1})$ be the vector

describing the interaction of $A(t_{m+1}, x, \tau, y, \nabla u_{0,m}^L + \nabla_y u_{1,m}^L)$ with the basis functions of $V_0^L \times V_1^L$ in the third term on the left hand side of (4.5). Let $\mathcal{B}$ be the Gram matrix describing the interaction of basis functions of V_0^L with themselves in the inner product of H . Let $\mathcal{M}$ be the matrix describing the interaction of the basis functions of V_1^L and themselves in the second term on the left hand side of (4.5). Let F_{m+1} be the interaction of $f(t_{m+1})$ with the basis functions of V_0 with respect to the inner product of H . We can write (4.5) as

$$\frac{1}{\Delta t} \mathcal{B} \mathbf{c}_{0,m+1} \cdot \mathbf{d}_0 + \mathcal{M} \mathbf{c}_{1,m+1} \cdot \mathbf{d}_1 + \mathcal{A}(\mathbf{c}_{m+1}) \cdot \mathbf{d} = F_{m+1} \cdot \mathbf{d}_0 + \frac{1}{\Delta t} \mathcal{B} \mathbf{c}_{0,m} \cdot \mathbf{d}_0. \quad (4.6)$$

The left hand side of (4.6) represents a monotone function. Indeed, for any (v_0, v_1) and (w_0, w_1) in $V_0^L \times V_1^L$,

$$\begin{aligned} & \frac{1}{\Delta t} \langle v_0 - w_0, v_0 - w_0 \rangle_H + \int_D \int_0^1 \left\langle \frac{\partial(v_1 - w_1)}{\partial \tau}, v_1 - w_1 \right\rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y (A(t_{m+1}, x, \tau, y, \nabla v_0 + \nabla_y v_1) - A(t_{m+1}, x, \tau, y, \nabla w_0 + \nabla_y w_1)) \\ & \quad \cdot (\nabla(v_0 - w_0) + \nabla_{y_1}(v_1 - w_1)) dy d\tau dx \\ & \geq \frac{1}{\Delta t} \|v_0 - w_0\|_H^2 + \alpha \int_D \int_0^1 \int_Y |(\nabla v_0 + \nabla_y v_1) - (\nabla w_0 + \nabla_y w_1)|^p dy d\tau dx \end{aligned}$$

i.e. for any two vectors $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1)$ and $\mathbf{q} = (\mathbf{q}_0, \mathbf{q}_1)$ in $\mathbb{R}^{\dim V_0^L} \times \mathbb{R}^{\dim V_1^L}$,

$$\begin{aligned} & \frac{1}{\Delta t} \mathcal{B}(\mathbf{p}_0 - \mathbf{q}_0) \cdot (\mathbf{p}_0 - \mathbf{q}_0) + \mathcal{M}(\mathbf{p}_1 - \mathbf{q}_1) \cdot (\mathbf{p}_1 - \mathbf{q}_1) + (\mathcal{A}(\mathbf{p}) - \mathcal{A}(\mathbf{q})) \cdot (\mathbf{p} - \mathbf{q}) \\ & \geq c(\Delta t) |\mathbf{p} - \mathbf{q}|^2 \quad (4.7) \end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{\dim V_0^L} + \mathbb{R}^{\dim V_1^L}$. Thus problem (4.6) has a unique solution $\mathbf{c}_{m+1}$. $\square$

We denote by $u_0(t_m) = u_{0,m}$, $u_1(t_m) = u_{1,m}$, $z_{0,m}^L = u_{0,m} - u_{0,m}^L$, $z_{1,m}^L = u_{1,m} - u_{1,m}^L$. We then have the following result.

Theorem 4.9. Assume that $u_0 \in C^2([0, T], H)$, then

$$\begin{aligned}
& \|z_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|z_{0,m}^L\|_V^p + \|z_{1,m}^L\|_{V_1}^p) \\
& \leq c\Delta t \left(\sum_{m=1}^M \|u_{0,m} - \tilde{u}_{0,m}\|_V^{p/(p-1)} \right. \\
& \quad + \|(u_{1,m} - \tilde{u}_{1,m})\|_{V_1}^{p/(p-1)} + \left\| \frac{\partial}{\partial \tau}(u_{1,m} - \tilde{u}_{1,m}) \right\|_{V_1'}^{p/(p-1)} \\
& \quad + \sum_{m=0}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2 \Bigg) \\
& \quad + \max_{m=1,\dots,M} \|u_{0,m} - \tilde{u}_{0,m}\|_{L^2(D)}^2 + \|g - g^L\|_{L^2(D)}^2 + c(\Delta t)^{p/(p-1)}. \quad (4.8)
\end{aligned}$$

for all sequences $\{\tilde{u}_{0,m}, m = 1, \dots, M\} \subset V_0^L$ and $\{\tilde{u}_{1,m}, m = 1, \dots, M\} \subset V_1^L$

Proof. We denote by $\rho_m = \frac{\partial u_0}{\partial t}(t_{m+1}) - (u_0(t_{m+1}) - u_0(t_m))/\Delta t$. As $u_0 \in C^2([0, T], H)$ we have that $\|\rho_m\|_H \leq c\Delta t$ for all $m = 1, \dots, M$. We then have from (4.4) and (4.5) that

$$\begin{aligned}
& \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \langle \rho_m, \phi_0 \rangle_H + \int_D \int_0^1 \left\langle \frac{\partial z_{1,m+1}^L}{\partial \tau}, \phi_1 \right\rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_Y (A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1} + \nabla_y u_{1,m+1}) \\
& \quad - A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L)) \\
& \quad \cdot (\nabla \phi_0 + \nabla_y \phi_1) dy d\tau dx \\
& = 0 \quad (4.9)
\end{aligned}$$

for all $\phi_0 \in V_0^L$ and $\phi_1 \in V_1^L$. Therefore, for all $\{\tilde{u}_{0,m}\} \subset V_0^L$ and $\{\tilde{u}_{1,m}\} \subset V_1^L$, we have

$$\left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, z_{0,m+1}^L \right\rangle_H + \int_D \int_0^1 \left\langle \frac{\partial z_{1,m+1}^L}{\partial \tau}, z_{1,m+1}^L \right\rangle_{H_\#} d\tau dx$$

$$\begin{aligned}
& + \int_D \int_0^1 \int_Y (A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1} + \nabla_y u_{1,m+1}) \\
& \quad - A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L)) \\
& \quad \cdot (\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L) dy d\tau dx \\
& = \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& \quad + \int_D \int_0^1 \left\langle \frac{\partial z_{1,m+1}^L}{\partial \tau}, u_{1,m+1} - \tilde{u}_{1,m+1} \right\rangle_{H\#} d\tau dx \\
& \quad + \int_D \int_0^1 \int_Y (A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1} + \nabla_y u_{1,m+1}) \\
& \quad - A(t_{m+1}, x, \tau, y, \nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L)) \\
& \quad \cdot ((\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}) + (\nabla_y u_{1,m+1} - \nabla_y \tilde{u}_{1,m+1})) dy d\tau dx \\
& \quad + \langle \rho_m, (u_{0,m+1}^L - \tilde{u}_{0,m+1}) \rangle_H.
\end{aligned}$$

Using $u_{0,m+1}^L - \tilde{u}_{0,m+1} = u_{0,m+1}^L - u_{0,m+1} + u_{0,m+1} - \tilde{u}_{0,m+1}$, from (4.2) we have

$$\begin{aligned}
& \frac{1}{2\Delta t} (\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + c \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}^p \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& \quad + \|z_{1,m+1}^L\|_{V_1} \left\| \frac{\partial}{\partial \tau} (u_{1,m+1} - \tilde{u}_{1,m+1}) \right\|_{V_1'} \\
& \quad + c (\|\nabla u_{0,m+1} + \nabla_y u_{1,m+1}\|_{L^p(D \times (0,1) \times Y)}^{p-2} \\
& \quad \quad + \|\nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}^{p-2}) \\
& \quad \cdot \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)} \\
& \quad \cdot (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \|\nabla_y u_{1,m+1} - \nabla_y \tilde{u}_{1,m+1}\|_{L^p(D \times (0,1) \times Y)}) \\
& \quad + c \Delta t (\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z_{0,m+1}^L\|_H).
\end{aligned}$$

From (4.5), $\|\nabla u_{0,m+1}^L + \nabla_y u_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}$ is uniformly bounded for all

L . Thus

$$\begin{aligned}
& \frac{1}{2\Delta t} (\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + c \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}^p \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& \quad + \|z_{1,m+1}^L\|_{V_1} \left\| \frac{\partial}{\partial \tau} (u_{1,m+1} - \tilde{u}_{1,m+1}) \right\|_{V_1'} \\
& \quad + c \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)} \\
& \quad \cdot (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \|\nabla_y u_{1,m+1} - \nabla_y \tilde{u}_{1,m+1}\|_{L^p(D \times (0,1) \times Y)}) \\
& \quad + c\Delta t (\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z_{0,m+1}^L\|_H) \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H + \delta \|z_{1,m+1}^L\|_{V_1}^p \\
& \quad + c \left\| \frac{\partial}{\partial \tau} (u_{1,m+1} - \tilde{u}_{1,m+1}) \right\|_{V_1'}^q \\
& \quad + \delta \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}^p \\
& \quad + c (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \|\nabla_y u_{1,m+1} - \nabla_y \tilde{u}_{1,m+1}\|_{L^p(D \times (0,1) \times Y)})^q \\
& \quad + c(\Delta t)^q + \delta \|z_{0,m+1}^L\|_H^p + c(\Delta t)^p + c \|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H^q
\end{aligned}$$

for a constant $\delta > 0$ where we have used the Young inequality. From Lemma 4.5, we have

$$\|\nabla z_{0,m+1}^L\|_V^p + \|\nabla_y z_{1,m+1}^L\|_{V_1}^p \leq c \|\nabla z_{0,m+1}^L + \nabla_y z_{1,m+1}^L\|_{L^p(D \times (0,1) \times Y)}^p.$$

Thus

$$\begin{aligned}
& \frac{1}{2\Delta t} (\|z_{0,m+1}^L\|_H^2 - \|z_{0,m}^L\|_H^2) + \|z_{0,m+1}^L\|_V^p + \|z_{1,m+1}^L\|_{V_1}^p \\
& \leq \left\langle \frac{z_{0,m+1}^L - z_{0,m}^L}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H + c \left\| \frac{\partial}{\partial \tau} (u_{1,m+1} - \tilde{u}_{1,m+1}) \right\|_{V_1'}^q \\
& \quad + c (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \|\nabla_y u_{1,m+1} - \nabla_y \tilde{u}_{1,m+1}\|_{L^p(D \times (0,1) \times Y)})^q \\
& \quad + c(\Delta t)^q.
\end{aligned}$$

Therefore for any $P = 1, \dots, M$, we deduce

$$\begin{aligned}
& \|z_{0,P}^L\|_H^2 + c\Delta t \sum_{m=1}^P (\|z_{0,m}^L\|_V^p + \|z_{1,m}\|_{V_1}^p) \\
& \leq c\Delta t \sum_{m=1}^P \left(\left\| \frac{\partial}{\partial \tau}(u_{1,m} - \tilde{u}_{1,m}) \right\|_{V_1'}^q + \|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\
& \quad \left. + \|\nabla_y u_{1,m} - \nabla_y \tilde{u}_{1,m}\|_{L^p(D \times (0,1) \times Y)}^q \right) \\
& \quad + (\Delta t)^q + c\Delta t \sum_{m=1}^P \left\langle \frac{z_{0,m}^L - z_{0,m-1}^L}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H + \|g - g^L\|_H^2.
\end{aligned}$$

We have

$$\begin{aligned}
& \Delta t \sum_{m=1}^P \left\langle \frac{z_{0,m}^L - z_{0,m-1}^L}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H \\
& = -\langle z_{0,0}^L, u_{0,1} - \tilde{u}_{0,1} \rangle_H + \langle z_{0,P}^L, u_{0,P} - \tilde{u}_{0,P} \rangle \\
& \quad + \sum_{m=1}^{P-1} \langle z_{0,m}^L, (u_{0,m} - \tilde{u}_{0,m}) - (u_{0,m+1} - \tilde{u}_{0,m+1}) \rangle \\
& \leq c\|z_{0,0}^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + \delta\|z_{0,P}^L\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \\
& \quad + \delta\Delta t \sum_{m=1}^{P-1} \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2.
\end{aligned}$$

We have $\Delta t \sum_{m=1}^{P-1} \|z_{0,m}^L\|_H^2 \leq \Delta t M \max_m \|z_{0,m}^L\|_H \leq T \max_m \|z_{0,m}^L\|_H$. Choosing δ sufficiently small, we deduce that

$$\begin{aligned}
& \|z_{0,P}^L\|_H^2 \\
& \leq c\Delta t \sum_{m=1}^P \left(\left\| \frac{\partial}{\partial \tau}(u_{1,m} - \tilde{u}_{1,m}) \right\|_{V_1'}^q + \|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\
& \quad \left. + \|\nabla_y u_{1,m} - \nabla_y \tilde{u}_{1,m}\|_{L^p(D \times (0,1) \times Y)}^q \right) \\
& \quad + c(\Delta t)^q + c\|g - g^L\|_H^2 + c\|u_{0,1} - \tilde{u}_{0,1}\|_H^2 + c\|u_{0,P} - \tilde{u}_{0,P}\|_H^2 \\
& \quad + \delta T \max_m \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& \max_m \|z_{0,m}^L\|_H^2 \\
& \leq c\Delta t \sum_{m=1}^M \left(\left\| \frac{\partial}{\partial \tau} (u_{1,m} - \tilde{u}_{1,m}) \right\|_{V_1'}^q + \|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|_{L^p(D)}^q \right. \\
& \quad \left. + \|\nabla_y u_{1,m} - \nabla_y \tilde{u}_{1,m}\|_{L^p(D \times (0,1) \times Y)}^q \right) \\
& \quad + c(\Delta t)^q + c\|g - g^L\|_H^2 + c \max_{m=1, \dots, M} \|u_{0,m} - \tilde{u}_{0,m}\|_H^2 \\
& \quad + \delta T \max_m \|z_{0,m}^L\|_H^2 + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|_H^2.
\end{aligned}$$

The conclusion follows. $\square$

4.2.1.2 Backward Euler for full tensor product FE spaces

We construct the backward Euler method for full tensor product FE spaces similar to the previous chapters. We first construct a hierarchy of FE spaces. We assume that the domain D is a polygon. Let $\{\mathcal{T}^l\}$ for $l \geq 0$ be the hierarchy of regular triangular simplices of mesh size $h_l = O(2^{-l})$ in D . The set of simplices $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles in the two dimension case, and is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 8 tetrahedral in the three dimension case. In the periodic cube Y , in a similar manner, we construct the hierarchy $\{\mathcal{T}_\#^l\}$ of regular triangular simplices of mesh size $h_l = O(2^{-l})$ which are periodically distributed in Y . For $l = 0, 1, 2, \dots$, we divide the interval $(0, 1)$ into the set $\mathcal{T}_{\#\tau}^l$ of intervals of length 2^{-l} . Let $P^1(K)$ be the set of linear polynomials in a domain K . We define the spaces:

$$\begin{aligned}
V_0^l &= \{\phi \in W_0^{1,p}(D) : \phi \in P^1(T) \forall T \in \mathcal{T}^l\}; \\
V^l &= \{\phi \in W^{1,p}(D) : \phi \in P^1(T) \forall T \in \mathcal{T}^l\};
\end{aligned}$$

$$V_{\#}^l = \{\phi \in W_{\#}^{1,p}(Y) : \phi \in P^1(T) \forall T \in \mathcal{T}_{\#}^l\};$$

$$V_{\#\tau}^l = \{\phi \in W_{\#}^{1,p}(0,1) : \phi \in P^1(T) \forall T \in \mathcal{T}_{\#\tau}^l\}.$$

The following approximations hold.

$$\inf_{w^l \in V_0^l} \|w - w^l\|_{W_0^{1,p}(D)} \leq ch_l \|w\|_{W^{2,p}(D)}, \quad \forall w \in W_0^{1,p}(D) \cap W^{2,p}(D);$$

$$\inf_{w^l \in V^l} \|w - w^l\|_{L^p(D)} \leq ch_l \|w\|_{W^{1,p}(D)}, \quad \forall w \in W^{1,p}(D);$$

$$\inf_{w^l \in V_{\#}^l} \|w - w^l\|_{W_{\#}^{1,p}(Y)} \leq ch_l \|w\|_{W_{\#}^{2,p}(Y)}, \quad \forall w \in W_{\#}^{2,p}(Y);$$

$$\inf_{w^l \in V_{\#}^l} \|w - w^l\|_{L^p(Y)} \leq ch_l \|w\|_{W_{\#}^{1,p}(Y)}, \quad \forall w \in W_{\#}^{1,p}(Y)$$

$$\inf_{w^l \in V_{\#\tau}^l} \|w - w^l\|_{W_{\#}^{1,p}((0,1))} \leq ch_l \|w\|_{W_{\#}^{2,p}((0,1))}, \quad \forall w \in W_{\#}^{2,p}(0,1)$$

$$\inf_{w^l \in V_{\#}^l} \|w - w^l\|_{L^p((0,1))} \leq ch_l \|w\|_{W_{\#}^{1,p}(Y)}, \quad \forall w \in W_{\#}^{1,p}((0,1)).$$

As $u_1(t) \in V_1 = L^p(D \times (0,1), W_{\#}^{1,p}(Y)/\mathbb{R}) \cong L^p(D) \otimes L^p((0,1)) \otimes W_{\#}^{1,p}(Y)/\mathbb{R}$, to approximate it, we use the following full tensor product FE space

$$\bar{V}_1^L = V^L \otimes V_{\#\tau}^L \otimes V_{\#}^L. \quad (4.10)$$

Let $\mathcal{W}$ be the regularity space

$$W^{1,p}(D, W_{\#}^{1,p}((0,1), W_{\#}^{1,p}(Y))) \cap L^p(D, W_{\#}^{2,p}((0,1), W_{\#}^{1,p}(Y)))$$

$$\cap L^p(D, W_{\#}^{1,p}((0,1), W_{\#}^{2,p}(Y)))$$

with the norm

$$\|w\|_{\mathcal{W}} = \|w\|_{W^{1,p}(D, W_{\#}^{1,p}((0,1), W_{\#}^{1,p}(Y)))} + \|w\|_{L^p(D, W_{\#}^{2,p}(L^p(0,1), W_{\#}^{1,p}(Y)))}$$

$$+ \|w\|_{L^p(D, W_{\#}^{1,p}((0,1), W_{\#}^{2,p}(Y)))}.$$

We then have the following approximation properties.

Proposition 4.10. For $w \in \mathcal{W}$

$$\inf_{w^L \in V_1^L} \|w - w^L\|_{L^p(D, W^{1,p}((0,1), V_{\#}))} \leq ch_L \|w\|_{\mathcal{W}}.$$

The proof for this estimate is similar to that for tensor product approximations of L^p spaces in [48]. We employ the FE spaces V_0^L and $\bar{V}_1^L$ in the places of V_0^L and V_1^L in the backward Euler approximation (4.9). We have the following result.

Theorem 4.11. Assume that $u_0 \in C^2([0, T], H) \cap H^1((0, T), W^{2,p}(D))$, $u_1 \in C([0, T], \mathcal{W})$ and $\|g - g^L\|_H \leq ch_L^{p/(2(p-1))}$, then

$$\|\bar{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M \|\bar{z}_{0,m}^L\|_V^p + \|\bar{z}_{1,m}^L\|_{V_1}^p \leq c(h_L^{p/(p-1)} + (\Delta t)^{p/(p-1)}).$$

Proof. We bound the right hand side of (4.8). As $u_1 \in C([0, T], \mathcal{W})$, we can choose $\tilde{u}_{1,m} \in V_1^L$ for $m = 1, \dots, M$ and a constant c independent of m such that

$$\|(u_1 - \tilde{u}_1)_m\|_{L^p(D, W^{1,p}((0,1), V_{\#}))} \leq ch_L \|u_1(t_m)\|_{\mathcal{W}} \leq ch_L.$$

Therefore

$$\|(u_1 - \tilde{u}_1)_m\|_{V_1} + \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_m \right\|_{V_1'} \leq ch_L.$$

Let $I^L u_0(t) \in V^L$ be the interpolation operator whose value at each node equals the value of $u_0(t)$ (notes $u_0 \in C([0, T], W^{2,p}(D)) \subset C([0, T], C(\bar{D}))$).

We have

$$\|u_0(t) - I^L u_0(t)\|_V \leq ch_L \|u_0(t)\|_{W^{2,p}(D)} \leq ch_L.$$

Let $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\|(u_0 - \tilde{u}_0)_m\|_V \leq ch_L$$

where c is independent of m . With $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L^2 \left\| \frac{\partial u_0}{\partial t} \right\|_{W^{2,p}(D)}.$$

Using the procedure of [34]

$$\begin{aligned}
& \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \\
&= \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) dt \right\|_H^2 (\Delta t)^{-2} \\
&\leq \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \\
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{1,p}(D)} dt \right)^2 (\Delta t)^{-2} \\
&\leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{W^{1,p}(D)}^2 dt \right) (\Delta t)^{-1}.
\end{aligned}$$

Therefore

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_{H(D)}^2 \leq ch_L^4.$$

We then get the conclusion. $\square$

4.2.1.3 Backward Euler method for sparse tensor product FE spaces

We first construct the sparse tensor product FE spaces. When $p = 2$, we employ the sparse tensor product FE space $\hat{V}_1^L$ in Subsection 2.3.3. When $p > d$, we construct the sparse tensor product FE spaces following the procedure in Chapter 3. Let $\mathcal{S}^l$ as the set of nodes of the triangulation $\mathcal{T}^l$. The basis of V^l consists of functions ϕ_x^l for $x \in \mathcal{S}$ such that $\phi_x^l(x) = 1$ and $\phi_x^l(x') = 0$ where $x' \in \mathcal{S}, x' \neq x$. We consider the interpolation operator $I^l : C(D) \rightarrow V^l$ as

$$I^l w = \sum_{x \in \mathcal{S}^l} w(x) \phi_x^l. \quad (4.11)$$

In the cube Y , let $\mathcal{S}_\#^l$ be the set of triangulation nodes of $\mathcal{T}_\#^l$. The basis functions of $V_\#^l$ are $\phi_{\#y}^l$ for $y \in \mathcal{S}_\#^l$ which equals 1 at $y \in \mathcal{S}_\#^l$ and 0 at other nodes. The interpolation operator $I_\#^l : C_\#(Y) \rightarrow V_\#^l$ satisfies

$$I_\#^l w = \sum_{y \in \mathcal{S}_\#^l} w(y) \phi_{\#y}^l. \quad (4.12)$$

Similarly, let $\mathcal{S}_{\#\tau}^l$ be the set of triangulation nodes of $\mathcal{T}_{\#\tau}^l$. The basis functions of $V_{\#\tau}^l$ are $\phi_{\#\tau}^l$ for $\tau \in \mathcal{S}_{\#\tau}^l$ which equals 1 at τ and equals 0 at other nodes in $\mathcal{S}_{\#\tau}^l$. We define the interpolation operator $I_{\#\tau}^l : C_\#((0, 1)) \rightarrow V_{\#\tau}^l$ satisfies

$$I_{\#\tau}^l w = \sum_{\tau \in \mathcal{S}_{\#\tau}^l} w(y) \phi_{\#\tau}^l. \quad (4.13)$$

To construct the sparse tensor product FE spaces for $l \geq 1$, we define the subspaces $W^l \subset V^l$ and $W_\#^l \subset V_\#^l$ and $W_{\#\tau}^l \subset V_{\#\tau}^l$ as

$$\begin{aligned} W^l &= (I^l - I^{l-1})C(\bar{D}), \\ W_\#^l &= (I_\#^l - I_\#^{l-1})C_\#(\bar{Y}), \\ W_{\#\tau}^l &= (I_{\#\tau}^l - I_{\#\tau}^{l-1})C_\#([0, 1]). \end{aligned}$$

with $W^0 = V^0$, $W_\#^0 = V_\#^0$, and $W_{\#\tau}^0 = V_{\#\tau}^0$. The space W^l contains the linear combinations of basis function ϕ_x^l of V^l where $x \in \mathcal{S}^l \setminus \mathcal{S}^{l-1}$; the space $W_\#^l$ contains the linear combinations of basis functions $\phi_{\#y}^l$ for $y \in \mathcal{S}_\#^l \setminus \mathcal{S}_\#^{l-1}$; and the space $W_{\#\tau}^l$ consists of the linear combinations of basis function $\phi_{\#\tau}^l$ for $\tau \in \mathcal{S}_{\#\tau}^l \setminus \mathcal{S}_{\#\tau}^{l-1}$. We then have that

$$W^l = \bigoplus_{0 \leq l' \leq l} W^{l'}, \quad W_\#^l = \bigoplus_{0 \leq l' \leq l} W_\#^{l'}, \quad W_{\#\tau}^l = \bigoplus_{0 \leq l' \leq l} W_{\#\tau}^{l'}.$$

The full tensor product FE space $\bar{V}_1^L$ can be written as

$$\bar{V}_1^L = \bigoplus_{0 \leq l_0, l_1, l_2 \leq L} W^{l_0} \otimes W_{\#\tau}^{l_1} \otimes W_\#^{l_2}.$$

We define the sparse tensor product FE space $\hat{V}_i^L$ as

$$\hat{V}_1^L = \bigoplus_{0 \leq l_0 + l_1 + l_2 \leq L} W^{l_0} \otimes W_{\#\tau}^{l_1} \otimes W_{\#}^{l_2}. \quad (4.14)$$

To quantify the error for the approximation of functions in V_1 by the spaces $\hat{V}_1^L$, we define the regularity spaces $\hat{\mathcal{W}}$ which contains functions $w \in L^p(D, W_{\#}^{1,p}((0,1), W_{\#}^{1,p}(Y)))$ so that

$$\frac{\partial^{|\alpha_0|}}{\partial x^{\alpha_0}} \frac{\partial^{\alpha_1}}{\partial \tau_1^{\alpha_1}} \frac{\partial^{|\alpha_2|}}{\partial y^{\alpha_2}} w \in L^p(D \times (0,1) \times Y),$$

for all $\alpha_0 \in \mathbb{N}_0^d$ with $|\alpha_0| \leq 1$, $\alpha_1 \in \{0,1,2\}$ and $\alpha_2 \in \mathbb{N}_0^d$ with $|\alpha_2| \leq 2$. We can write $\hat{\mathcal{W}}$ as $\hat{\mathcal{W}} = W^{1,p}(D, W_{\#\tau}^{2,p}((0,1), W_{\#}^{2,p}(Y)))$. The following approximation holds.

Proposition 4.12. Assume that $w \in \hat{\mathcal{W}}$. Then for $p = 2$

$$\inf_{w^L \in \hat{V}_1^L} \|w - w^L\|_{L^2(D, W_{\#}^{1,2}((0,1), W_{\#}^{1,2}(Y)))} \leq cLh_L \|w\|_{\hat{\mathcal{W}}};$$

and when $p > d$

$$\inf_{w^L \in \hat{V}_1^L} \|w - w^L\|_{L^p(D, W_{\#}^{1,p}((0,1), W_{\#}^{1,p}(Y)))} \leq cL^2h_L \|w\|_{\hat{\mathcal{W}}}.$$

The proof for this proposition is similar to those in [49] and [48]. We employ the space $\hat{V}_1^L$ for the backward Euler approximating problem (4.5), i.e. we let $V_1^L = \hat{V}_1^L$. We denote the solution as $\hat{u}_{0,m}$, $\hat{u}_{1,m}$. We denote $z_{0,m}^L$ by $\hat{z}_{0,m}^L$ and $z_{1,m}^L$ by $\hat{z}_{1,m}^L$. We then have:

Theorem 4.13. Assume that $u_0 \in C^2([0, T], H) \cap H^1((0, T), W^{2,p}(D))$, $u_1 \in C([0, T], \hat{\mathcal{W}})$ and $\|g - g^L\|_H \leq cL^{p/(p-1)}h_L^{p/(2(p-1))}$. Then

$$\|\hat{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}_{0,m}^L\|_V^2 + \|\hat{z}_{1,m}^L\|_{V_1}^2) \leq c(L^2h_L^2 + (\Delta t)^2)$$

when $p = 2$; and when $p > d$

$$\|\hat{z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}_{0,m}^L\|_V^p + \|\hat{z}_{1,m}^L\|_{V_1}^p) \leq c(L^{2p/(p-1)}h_L^{p/(p-1)} + (\Delta t)^{p/(p-1)}).$$

The proof of this theorem is similar to that for Theorem 4.11.

Remark 4.14. Similar to Chapter 3, the requirement $p > d$ is to ensure that $W^{1,p}(D) \subset C(D)$ so that the interpolation operator can be defined. If u_1 is smoother than $W^{1,p}(D)$ with respect to x so that it is continuous with respect to x , then we can remove this requirement.

4.2.2 Crank-Nicholson method

We use the Crank-Nicholson discretizing scheme to solve problem (4.4) in this section. We first consider the scheme for general FE spaces, and prove the convergence of the scheme. We then use the full tensor product FEs and sparse tensor product FEs for the Crank-Nicholson scheme, and deduce the error of convergence.

4.2.2.1 Crank-Nicholson method for general FE spaces

We consider the Crank-Nicholson method in this section. We also consider general FE spaces as in Section 4.2.1. The discretized problem is: For $m = 1, \dots, M$, find $U_{0,m}^L \in V_0^L$ and $U_{1,m}^L \in V_1^L$ such that

$$\begin{aligned}
& \left\langle \frac{U_{0,m+1}^L - U_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H + \frac{1}{2} \int_D \int_0^1 \left\langle \frac{\partial}{\partial \tau} (U_{1,m+1}^L + U_{1,m}^L), \phi_1 \right\rangle_{H_\#} d\tau dx \\
& + \int_D \int_0^1 \int_Y A \left(t_{m+1/2}, x, \tau, y, \frac{1}{2} ((\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\
& \quad \left. + (\nabla_y U_{1,m}^L + \nabla_y U_{1,m+1}^L)) \right) \\
& \quad \cdot (\nabla \phi_0 + \nabla_{y_1} \phi_1) dy d\tau dx \\
& = \int_D f(t_{m+1/2}) \phi_0 dx, \quad \forall \phi_0 \in V_0^L \text{ and } \phi_1 \in V_1^L.
\end{aligned} \tag{4.15}$$

We then have:

Proposition 4.15. Problem (4.15) has a unique solution.

The proof of this proposition is similar to that of Proposition 4.8.

We have the following approximation result. Let

$$\begin{aligned} Z_{0,m}^L &= u_0(t_m) - U_{0,m}^L, \\ Z_{1,m}^L &= u_1(t_m) - U_{1,m}^L, \\ Z_{0,m+1/2}^L &= \frac{1}{2}(Z_{0,m}^L + Z_{0,m+1}^L), \\ Z_{1,m+1/2}^L &= \frac{1}{2}(Z_{1,m}^L + Z_{1,m+1}^L). \end{aligned}$$

Theorem 4.16. Assume that $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_1 \in C^2([0, T], V_1)$ and $\frac{\partial u_1}{\partial \tau} \in C^2([0, T], V_1')$. Then

$$\begin{aligned} & \|Z_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|Z_{0,m+1/2}^L\|_V^p + \|Z_{1,m+1/2}^L\|_{V_1}^p) \\ & \leq c\Delta t \left(\sum_{m=0}^{M-1} \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^{p/(p-1)} + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^{p/(p-1)} \right. \\ & \quad + \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^{p/(p-1)} \\ & \quad + \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \Bigg) \\ & \quad + \max_{m=1, \dots, M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_H^2 + \|g - g^L\|_H^2 + c(\Delta t)^{2p/(p-1)}. \end{aligned} \quad (4.16)$$

for all $\{\tilde{u}_{0,m}, m = 0, \dots, M\} \subset V^L$ and $\{\tilde{u}_{1,m}, m = 1, \dots, M\} \subset V_1^L$.

Proof. Let $\rho_{0,m} = \frac{1}{\Delta t}(u_0(t_{m+1}) - u_0(t_m)) - \frac{\partial u_0}{\partial t}(t_{m+1/2})$, $\zeta_{0,m} = \frac{1}{2}(u_0(t_{m+1}) + u_0(t_m)) - u_0(t_{m+1/2})$, $\zeta_{1,m} = \frac{1}{2}(u_1(t_{m+1}) + u_1(t_m)) - u_1(t_{m+1/2})$ and $\xi_{1,m} = \frac{1}{2}(\frac{\partial u_1}{\partial \tau}(t_{m+1}) + \frac{\partial u_1}{\partial \tau}(t_m)) - \frac{\partial u_1}{\partial \tau}(t_{m+1/2})$. Since $u_0 \in C^3([0, T], H) \cap C^2([0, T], V)$, $u_1 \in C^2([0, T], V_1)$ and $\frac{\partial u_1}{\partial \tau} \in C^2([0, T], V_1')$, we deduce that

$$\|\rho_{0,m}\|_H \leq c(\Delta t)^2,$$

$$\|\zeta_{0,m}\|_V \leq c(\Delta t)^2,$$

$$\|\zeta_{1,m}\|_{V_1} \leq c(\Delta t)^2,$$

$$\|\xi_{1,m}\|_{V'_1} \leq c(\Delta t)^2$$

where the constant c does not depend on m . From (4.4) and (4.15) considered at $t = t_{m+1/2}$ we deduce that

$$\begin{aligned} & \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H - \langle \rho_{0,m}, \phi_0 \rangle_H \\ & + \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), \phi_1 \right\rangle_{H_\#} d\tau dx \\ & - \int_D \int_0^1 \langle \xi_{1,m}, \phi_1 \rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y \left(A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\ & \quad \left. + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1}) - \nabla \zeta_{0,m} - \nabla_y \zeta_{1,m}) \right. \\ & \quad \left. - A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\ & \quad \left. + \frac{1}{2}(\nabla_y U_{1,m}^L + \nabla_y U_{1,m+1}^L)) \right) \cdot (\nabla_x \phi_0 + \nabla_y \phi_y) dy d\tau dx \\ & = 0, \quad \forall \phi_0 \in V_0^L \text{ and } \phi_1 \in V_1^L. \end{aligned} \tag{4.18}$$

Consider

$$\begin{aligned} I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} \right\rangle_H \\ & + \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right\rangle_{H_\#} d\tau dx \\ & + \int_D \int_0^1 \int_Y \left(A \left(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \right. \\ & \quad \left. \left. + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1}) \right) \right. \\ & \quad \left. - A \left(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right) \right) \end{aligned} \tag{4.19}$$

$$\begin{aligned}
& + \frac{1}{2}(\nabla_y U_{1,m}^L + \nabla_y U_{1,m+1}^L) \Big) \Big) \\
& \cdot \left(\nabla \frac{Z_{0,m+1}^L + Z_{0,m}^L}{2} + \nabla_y \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2} \right) dy d\tau dx.
\end{aligned}$$

For $\{\tilde{u}_{0,m}, m = 0, \dots, M\} \subset V^L$ and $\{\tilde{u}_{1,m}, m = 1, \dots, M\} \subset V_1^L$, we have

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&+ \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (\tilde{u}_0 - U_0^L)_{m+1/2} \right\rangle_H \\
&+ \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), (u_1 - \tilde{u}_1)_{m+1/2} \right\rangle_{H_\#} d\tau dx \\
&+ \int_D \int_0^1 \left\langle \frac{1}{2} \left(\frac{\partial Z_{1,m+1}^L}{\partial \tau} + \frac{\partial Z_{1,m}^L}{\partial \tau} \right), (\tilde{u}_1 - U_1^L)_{m+1/2} \right\rangle_{H_\#} d\tau dx \\
&+ \int_D \int_0^1 \int_Y \left(A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\
&\quad \left. + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1})) \right. \\
&\quad \left. - A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \right. \\
&\quad \left. + \frac{1}{2}(\nabla_y U_{1,m}^L + \nabla_y U_{1,m+1}^L)) \right) \\
&\quad \cdot ((\nabla(u_0 - \tilde{u}_0)_{m+1/2} + \nabla_y(u_1 - \tilde{u}_1)_{m+1/2}) \\
&\quad + (\nabla(\tilde{u}_0 - U_0^L)_{m+1/2} + \nabla_y(\tilde{u}_1 - U_1^L)_{m+1/2})) dy d\tau dx.
\end{aligned}$$

From (4.15) we have

$$\begin{aligned}
I &= \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&- \int_D \int_0^1 \left\langle \frac{Z_{1,m+1}^L + Z_{1,m}^L}{2}, \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\rangle_{H_\#} \\
&+ \int_D \int_0^1 \int_Y \left(A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1})) \\
& - A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla U_{0,m}^L + \nabla U_{0,m+1}^L) \\
& + \frac{1}{2}(\nabla_y U_{1,m}^L + \nabla_y U_{1,m+1}^L))) \\
& \cdot ((\nabla_x(u_0 - \tilde{u}_0)_{m+1/2} + \nabla_y(u_1 - \tilde{u}_1)_{m+1/2})) dy d\tau dx \\
& + \langle \rho_{0,m}, (\tilde{u}_0 - U_0^L)_{m+1/2} \rangle_H + \int_D \int_0^1 \langle \xi_{1,m}, (\tilde{u}_1 - U_1^L)_{m+1/2} \rangle_{H_\#} d\tau dx \\
& - \int_D \int_0^1 \int_Y \left(A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \right. \\
& \quad + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1}) - \nabla \zeta_{0,m} - \nabla_y \zeta_{1,m}) \\
& \quad - A(t_{m+1/2}, x, \tau, y, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) \\
& \quad \left. + \frac{1}{2}(\nabla_y u_{1,m} + \nabla_y u_{1,m+1})) \right) \\
& \cdot (\nabla(\tilde{u}_0 - U_0^L)_{m+1/2} + \nabla_y(\tilde{u}_1 - U_1^L)_{m+1/2}) dy d\tau dx.
\end{aligned}$$

We note that $(\tilde{u}_0 - U_0^L)_{m+1/2} = (\tilde{u}_0 - u_0)_{m+1/2} + Z_{0,m+1/2}^L$ and $(\tilde{u}_1 - U_1^L)_{m+1/2} = (\tilde{u}_1 - u_1)_{m+1/2} + Z_{1,m+1/2}^L$. For a positive constant $\delta > 0$, using the Young inequality, we have

$$\begin{aligned}
I & \leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& + \delta \|Z_{1,m+1/2}^L\|_{V_1}^p + c \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q \\
& + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \|Z_{1,m+1/2}^L\|_{V_1}^p \\
& + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + c \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q \\
& + c \|\rho_{0,m}\|_H^q + \delta \|Z_{0,m+1/2}^L\|_H^p + c \|\rho_{0,m}\|_H^p + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_H^q \\
& + c \|\xi_{1,m}\|_{V_1'}^q + \delta \|Z_{1,m+1/2}^L\|_{V_1}^p + c \|\xi_{1,m}\|_{V_1'}^p + c \|(\tilde{u}_1 - u_1)_{m+1/2}\|_{V_1}^q \\
& + c \|\zeta_{0,m}\|_V^q + c \|\zeta_{1,m}\|_{V_1}^q + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \|Z_{1,m+1/2}^L\|_{V_1}^p \\
& + c \|\zeta_{0,m}\|_V^p + c \|\zeta_{1,m}\|_{V_1}^p + c \|(\tilde{u}_0 - u_0)_{m+1/2}\|_V^q + c \|(\tilde{u}_1 - u_1)_{m+1/2}\|_{V_1}^q
\end{aligned}$$

$$\begin{aligned}
&\leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H + c \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q \\
&\quad + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + c \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q + c(\Delta t)^{2q} \\
&\quad + \delta \|Z_{0,m+1/2}^L\|_V^p + \delta \|Z_{1,m+1/2}^L\|_{V_1}^p
\end{aligned}$$

where we have used the fact that $\|w\|_H \leq \|w\|_V$ for all $w \in V$. From (4.2), we have

$$I \geq \frac{1}{2\Delta t} (\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) + c \|Z_{0,m+1/2}^L\|_V^p + c \|Z_{1,m+1/2}^L\|_{V_1}^p.$$

Choosing δ sufficiently small, we have

$$\begin{aligned}
&\frac{1}{2\Delta t} (\|Z_{0,m+1}^L\|_H^2 - \|Z_{0,m}^L\|_H^2) + c (\|Z_{0,m+1/2}^L\|_V^p + \|Z_{1,m+1/2}^L\|_{V_1}^p) \\
&\leq \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H + c \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q \\
&\quad + c \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q + c \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q + c(\Delta t)^{2q}.
\end{aligned}$$

Fixing an integer $P \leq M$, taking the sum for $m = 0, \dots, P-1$, we have

$$\begin{aligned}
&\|Z_{0,P}^L\|_H^2 - \|Z_{0,0}^L\|_H^2 + c\Delta t \sum_{m=0}^{P-1} (\|Z_{0,m+1/2}^L\|_V^p + \|Z_{1,m+1/2}^L\|_{V_1}^p) \\
&\leq c\Delta t \sum_{m=0}^{P-1} \left(\left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q \right. \\
&\quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q \right) + cP(\Delta t)^{2q+1} \\
&\quad + 2\Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H. \tag{4.20}
\end{aligned}$$

We note that

$$\begin{aligned}
&\Delta t \sum_{m=0}^{P-1} \left\langle \frac{Z_{0,m+1}^L - Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
&= \langle Z_{0,P}^L, (u_0 - \tilde{u}_0)_{P-1/2} \rangle_H - \langle Z_{0,0}^L, (u_0 - \tilde{u}_0)_{1/2} \rangle_H
\end{aligned}$$

$$\begin{aligned}
& +\Delta t \sum_{m=1}^{P-1} \left\langle \frac{Z_{0,m}^L}{\Delta t}, (u_0 - \tilde{u}_0)_{m-1/2} - (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H \\
& \leq \delta \|Z_{0,P}^L\|_H^2 + c \|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + \|Z_{0,0}^L\|_H^2 + \|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
& \quad + \delta \Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 + c \Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2
\end{aligned}$$

where we have employed the Cauchy-Schwartz inequality; δ is a constant which we choose to be small. From the fact that $\Delta t \sum_{m=1}^{P-1} \|Z_{0,m}^L\|_H^2 \leq T \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2$ and (4.20), we have

$$\begin{aligned}
& \|Z_{0,P}^L\|_H^2 \\
& \leq c \Delta t \sum_{m=0}^{P-1} \left(\left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q \right. \\
& \quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q \right) \\
& \quad + c(\Delta t)^{2q} + c \|(u_0 - \tilde{u}_0)_{P-1/2}\|_H^2 + 2\|Z_{0,0}^L\|_H^2 + \|(u_0 - \tilde{u}_0)_{1/2}\|_H^2 \\
& \quad + c \Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 + \delta T \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2
\end{aligned}$$

for all $P = 1, \dots, M$. Choosing δ sufficiently small, we have

$$\begin{aligned}
& \max_{m=0,\dots,M} \|Z_{0,m}^L\|_H^2 \\
& \leq c \Delta t \sum_{m=0}^{M-1} \left(\left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'}^q + \|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^q \right. \\
& \quad \left. + \|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1}^q \right) \\
& \quad + c(\Delta t)^{2q} + c \max_{m=1,\dots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|_H^2 + c \|Z_{0,0}^L\|_H^2 \\
& \quad + c \Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2.
\end{aligned}$$

From this, we get the conclusion.

4.2.2.2 Crank-Nicholson method for full tensor product FE spaces

We deduce the error for the Crank-Nicholson method where we use the full tensor product FE space $\bar{V}_1^L$ defined in (4.10) for V_1 in Subsection 4.2.2.1. We denote the solution as $\bar{U}_{0,m}^L, \bar{U}_{1,m}^L$. We denote $Z_{0,m+1/2}^L$ and $Z_{1,m+1/2}^L$ by $\bar{Z}_{0,m+1/2}^L$ and $\bar{Z}_{1,m+1/2}^L$ respectively. We then have the following result.

Theorem 4.17. Assume that

$$\begin{aligned} u_0 &\in C^3([0, T], H) \cap C^2([0, T], V) \cap H^1((0, T), W^{2,p}(D)), \\ u_1 &\in C^2([0, T], V_1) \cap C([0, T], \mathcal{W}). \end{aligned}$$

Then

$$\|\bar{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\bar{Z}_{0,m+1/2}^L\|_V^p + \|\bar{Z}_{1,m+1/2}^L\|_{V_1}^p) \leq c(h_L^{p/(p-1)} + (\Delta t)^{2p/(p-1)}).$$

Proof. Since $u_1 \in C([0, T], \mathcal{W})$, we choose $\tilde{u}_{1,m} \in \bar{V}_1^L$ for $m = 1, \dots, M$ such that

$$\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{L^p(D, W^{1,p}((0,1), V_{\#}))} \leq ch_L (\|u_1(t_m)\|_{\mathcal{W}} + \|u_1(t_{m+1})\|_{\mathcal{W}}) \leq ch_L,$$

where c depends only on $\sup_{t \in [0, T]} \|u_1(t)\|_{\mathcal{W}}$. Therefore

$$\|(u_1 - \tilde{u}_1)_{m+1/2}\|_{V_1} + \left\| \frac{\partial}{\partial \tau} (u_1 - \tilde{u}_1)_{m+1/2} \right\|_{V_1'} \leq ch_L.$$

Define the interpolation $I^L u_0(t) \in V^L$ such that the value of $I^L u_0(t)$ at each node equals the value of $u_0(t)$. We have

$$\|u_0(t) - I^L u_0(t)\|_V \leq ch_L \|u_0(t)\|_{W^{2,p}(D)} \leq ch_L.$$

Choosing $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V \leq ch_L$$

where c does not depend on m . We then have

$$\begin{aligned} & \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_H^2 \\ & \leq c \sum_{m=1}^{M-1} \left(\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \right. \\ & \quad \left. + \left\| \frac{(u_0 - \tilde{u}_0)_m - (u_0 - \tilde{u}_0)_{m-1}}{\Delta t} \right\|_H^2 \right). \end{aligned}$$

With $\tilde{u}_0(t) = I^L u_0(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L \left\| \frac{\partial u_0}{\partial t} \right\|_{W^{2,p}(D)}.$$

Thus

$$\begin{aligned} & \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \\ & = \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) dt \right\|_H^2 (\Delta t)^{-2} \\ & \leq \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial(u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \\ & \leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^1(D)} dt \right)^2 (\Delta t)^{-2} \\ & \leq ch_L^4 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^1(D)}^2 dt \right) (\Delta t)^{-1}. \end{aligned}$$

Therefore

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|_{L^2(D)}^2 \leq ch_L^4.$$

We then get the conclusion. $\square$

4.2.2.3 Crank-Nicholson method for sparse tensor product FE spaces

Using the sparse tensor product FEs, i.e. we use the sparse tensor product FE space $\bar{V}_1^L$ defined in (4.14) for V_1 in Subsection 4.2.2.1, we denote the solution as $\hat{U}_{0,m}^L$ and $\hat{U}_{1,m}^L$. We denote $Z_{0,m+1/2}^L$ and $Z_{1,m+1/2}^L$ by $\hat{Z}_{0,m+1/2}^L$ and $\hat{Z}_{1,m+1/2}^L$ respectively. We then have the following result.

Theorem 4.18. Assume that

$$\begin{aligned} u_0 &\in C^3([0, T], H) \cap C^2([0, T], V) \cap H^1((0, T), W^{2,p}(D)), \\ u_1 &\in C^2([0, T], V_1) \cap C([0, T], \hat{\mathcal{W}}). \end{aligned}$$

Then, for $p = 2$

$$\|\hat{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\hat{Z}_{0,m+1/2}^L\|_V^2 + \|\hat{Z}_{1,m+1/2}^L\|_{V_1}^2) \leq c((Lh_L)^2 + (\Delta t)^4);$$

and when $p > d$

$$\|\hat{Z}_{0,M}^L\|_H^2 + \Delta t \sum_{m=0}^{M-1} (\|\hat{Z}_{0,m+1/2}^L\|_V^p + \|\hat{Z}_{1,m+1/2}^L\|_{V_1}^p) \leq c((L^2 h_L)^{p/(p-1)} + (\Delta t)^{2p/(p-1)}).$$

The proof of this theorem is similar to that of Theorem 4.17.

4.3 Numerical corrector

We construct the numerical correctors using the FE solutions in this section. First we establish the homogenized equation. For each vector $\xi \in \mathbb{R}^d$, let $N(t, x, \tau, y, \xi)$ be the $(0, 1) \times Y$ periodic solution with respect to τ and y solution of the problem

$$\frac{\partial N}{\partial \tau} - \nabla \cdot A(t, x, \tau, y, \xi + \nabla_y N) = 0. \quad (4.21)$$

From (4.4), we deduce that the homogenized equations is

$$\frac{\partial u_0}{\partial t} - \nabla \cdot A^0(t, x, \nabla u_0) = f, \quad u_0(0) = g,$$

where the homogenized monotone operator is

$$A^0(t, x, \xi) = \int_0^1 \int_Y A(t, x, \tau, y, \xi) dy d\tau.$$

We have that

$$u_1(t, x, \tau, y) = N(t, x, \tau, y, \nabla u_0(t, x)).$$

We define the operator $\mathcal{T}^\varepsilon : L^1((0, T) \times D) \rightarrow L^1((0, T) \times D \times (0, 1) \times Y)$ as

$$\mathcal{T}^\varepsilon(\Phi) = \Phi \left(\varepsilon^2 \left[\frac{t}{\varepsilon^2} \right]_1 + \varepsilon^2 \tau, \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y \right). \quad (4.22)$$

Let D^ε be the 2ε neighbourhood of D . We have

$$\int_0^T \int_D \Phi(t, x) dx dt = \int_{-2\varepsilon}^{T+2\varepsilon} \int_{D^\varepsilon} \int_0^1 \int_Y \mathcal{T}^\varepsilon(\Phi)(t, x, \tau, y) dy d\tau dx dt \quad (4.23)$$

where the function Φ is understood as 0 outside $(0, T) \times D$. We also have that if $\{w^\varepsilon\}_\varepsilon$ time-space multiscale converges to w_0 in $L^p((0, T) \times D \times (0, 1) \times Y)$ then

$$\mathcal{T}^\varepsilon(w^\varepsilon) \rightharpoonup w_0 \text{ in } L^p((0, T) \times D \times (0, 1) \times Y).$$

We define the operator $\mathcal{U}^\varepsilon : L^1((0, T) \times D \times (0, 1) \times Y) \rightarrow L^1((0, T) \times D)$ as

$$\mathcal{U}^\varepsilon(\Phi) = \int_0^1 \int_Y \Phi \left(\varepsilon^2 \left[\frac{t}{\varepsilon^2} \right]_1 + \varepsilon^2 \theta, \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon z, \left\{ \frac{t}{\varepsilon^2} \right\}_1, \left\{ \frac{x}{\varepsilon} \right\} \right) dz d\theta \quad (4.24)$$

where $[\cdot]_1$ and $\{\cdot\}_1$ denote the integer and the fractional parts of a real number, and $[\cdot]$ and $\{\cdot\}$ denote the “interger” and the “fractional” parts of a vector in $\mathbb{R}^d$ with respect to Y . We then have

$$\int_{-2\varepsilon^2}^{T+2\varepsilon^2} \int_{D^{2\varepsilon}} \mathcal{U}^\varepsilon(\Phi)(t, x) = \int_0^T \int_D \int_0^1 \int_Y \Phi(t, x, \tau, y) dy d\tau dx dt. \quad (4.25)$$

The proofs of (4.23) and (4.25) are similar to the corresponding results in [29]. We then have the following corrector result.

Proposition 4.19. For the solution of equation (4.4)

$$\|\nabla u^\varepsilon - \nabla u_0 - \mathcal{U}^\varepsilon(\nabla_y u_1)\|_{L^p((0,T) \times D)} = 0.$$

Proof. We note that $A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon)$ is bounded in $L^q((0, T) \times D)$. Let $\chi \in L^q((0, T) \times D \times (0, 1) \times Y)$ be the time-space multiscale convergence limit of a subsequence of $A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon)$. From (4.4), we have that

$$\begin{aligned} & \int_0^T \int_D \int_0^1 \int_Y \chi(\nabla v_0 + \nabla_y v_1) dy d\tau dx dt \\ &= \int_0^T \int_D \int_0^1 \int_Y A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1) \\ & \quad \cdot (\nabla v_0 + \nabla_y v_1) dy d\tau dx dt. \end{aligned} \quad (4.26)$$

for all $v_0 \in V$ and $v_1 \in V_1$. We consider

$$\mathcal{T}^\varepsilon \left(A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon) \right) = A \left(\varepsilon^2 \left[\frac{t}{\varepsilon^2} \right]_1 + \varepsilon^2 \tau, \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y, \tau, y, \mathcal{T}^\varepsilon(\nabla u^\varepsilon) \right).$$

As $\mathcal{T}^\varepsilon (A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon))$ converges weakly to χ in $L^q((0, T) \times D \times (0, 1) \times Y)$, using the smoothness of A , we deduce that $A(t, x, \tau, y, \mathcal{T}^\varepsilon(\nabla u^\varepsilon))$ converges weakly to χ in $L^q((0, T) \times D \times (0, 1) \times Y)$. We consider:

$$\begin{aligned} I &= \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \right\rangle_H \\ &+ \int_0^T \int_D \int_0^1 \int_Y \left(\mathcal{T}^\varepsilon(A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon)) - A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1) \right) \\ & \quad \cdot (\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)) dy d\tau dx dt. \end{aligned}$$

From (4.3), (4.4), (4.26) we have

$$\lim_{\varepsilon \rightarrow 0} I$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon \right\rangle_H - \left\langle \frac{\partial u_0}{\partial t}, u_0 \right\rangle_H dt \\
&\quad + \int_0^T \int_D A(t, x, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \nabla u^\varepsilon) \cdot \nabla u^\varepsilon dx dt \\
&\quad - \int_0^T \int_D \int_0^1 \int_Y A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1) \cdot (\nabla u_0 + \nabla_y u_1) dy d\tau dx dt \\
&= 0.
\end{aligned}$$

Using the smoothness of A , we have

$$\begin{aligned}
&\lim_{\varepsilon \rightarrow 0} I \\
&= \int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0 \right\rangle_H \\
&\quad + \int_0^T \int_D \int_0^1 \int_Y (A(t, x, \tau, y, \mathcal{T}^\varepsilon(\nabla u^\varepsilon)) - A(t, x, \tau, y, \nabla u_0 + \nabla_y u_1)) \\
&\quad \cdot (\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)) dy d\tau dx dt.
\end{aligned}$$

From (4.1) and $u^\varepsilon(0) = u_0(0) = g$, we have

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon(T) - u_0(T)\|_H + \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)\|_{L^p((0,T) \times D \times (0,1) \times Y)} = 0.$$

From (4.24), we have $(\mathcal{U}^\varepsilon(\Phi)(t, x))^p \leq \mathcal{U}^\varepsilon(\Phi^p)(t, x)$. From (4.25), we have

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^p((0,T) \times D)}^p \leq \|\mathcal{U}^\varepsilon(\Phi^p)\|_{L^1((0,T) \times D)} \leq \|\Phi\|_{L^p((0,T) \times D \times (0,1) \times Y)}^p.$$

We therefore have

$$\begin{aligned}
&\|\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1))\|_{L^p((0,T) \times D)} \\
&\leq \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \nabla_y u_1)\|_{L^p((0,T) \times D \times (0,1) \times Y)} \\
&\rightarrow 0
\end{aligned}$$

when $\varepsilon \rightarrow 0$. Using $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon)) = \nabla u^\varepsilon$, we get the conclusion. $\square$

We construct the numerical correctors from the FE solutions of problem (4.4). We present the results for the Crank-Nicholson approximations. The

results for the backward Euler approximations are similar. For the full tensor product FE approximations, let $\bar{U}_0 : (0, T) \rightarrow V$ and $\bar{U}_1 : (0, T) \rightarrow V_1$ be defined as

$$\bar{U}_0^L(t) = \frac{1}{2}(\bar{U}_{0,m}^L + \bar{U}_{0,m+1}^L), \quad \bar{U}_1^L(t) = \frac{1}{2}(\bar{U}_{1,m}^L + \bar{U}_{1,m+1}^L) \quad \text{for } t \in [t_m, t_{m+1}).$$

We have:

Theorem 4.20. Assume that the hypothesis of Theorem 4.17 hold. For the solution of the approximating problems (4.15) using the full tensor product FE approximation for u_1 , we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|\nabla u^\varepsilon - [\nabla \bar{U}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)]\|_{L^p((0,T) \times D)} = 0.$$

Proof. Using the midpoint rule, we have

$$\begin{aligned} \int_0^T \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_{L^p(D)}^p dt &= \sum_{m=0}^{M-1} \int_{m\Delta t}^{(m+1)\Delta t} \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_{L^p(D)}^p dt \\ &\leq \sum_{m=0}^{M-1} (\Delta t \|\nabla u_0(t_{m+1/2}) - \nabla \bar{U}_0^L(t_{m+1/2})\|_{L^p(D)}^p + c(\Delta t)^3) \end{aligned}$$

where the constant c is independent of Δt . We have

$$\left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla u_0(t_{m+1/2}) \right\|_{L^p(D)} \leq c(\Delta t)^2.$$

From this we deduce

$$\begin{aligned} &\int_0^T \|\nabla u_0(t) - \nabla \bar{U}_0^L(t)\|_{L^p(D)}^p dt \\ &\leq \sum_{m=0}^{M-1} (\Delta t \left\| \frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla \bar{U}_0^L(t_{m+1/2}) \right\|_{L^p(D)}^p + c(\Delta t)^3) \\ &= \Delta t \sum_{m=1}^{M-1} \|\nabla \bar{Z}_{0,m+1/2}\|_{L^p(D)}^p + c(\Delta t)^2 \\ &\leq c((\Delta t)^{2p/(p-1)} + h_L^{p/(p-1)}). \end{aligned}$$

By the same argument, we have

$$\int_0^T \|\nabla_y u_1(t) - \nabla_y \bar{U}_1^L(t)\|_{L^p(D \times (0,1) \times Y)}^p dt \leq c((\Delta t)^{2p/(p-1)} + h_L^{p/(p-1)}).$$

From (4.25) we deduce

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1 - \nabla_y \bar{U}_1^L)\|_{L^p((0,T) \times D)} \leq \|\nabla_y u_1 - \nabla_y \bar{U}_1^L\|_{L^p((0,T) \times D \times (0,1) \times Y)}.$$

Thus

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla \bar{U}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)]\|_{L^p((0,T) \times D)} \\ & \leq \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}^\varepsilon(\nabla_y u_1)]\|_{L^p((0,T) \times D)} + \|\nabla u_0 - \nabla \bar{U}_0^L\|_{L^p((0,T) \times D)} \\ & \quad + \|\mathcal{U}^\varepsilon(\nabla u_1) - \mathcal{U}^\varepsilon(\nabla \bar{U}_1^L)\|_{L^p((0,T) \times D)} \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$ and $L \rightarrow \infty$. $\square$

Similarly, for the sparse tensor product FE approximation, we have:

Theorem 4.21. Assume that the hypothesis of Theorem 4.18 hold. For the solution of the approximating problem (4.15) using the sparse tensor product FE approximation for u_1 , we have

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \|\nabla u^\varepsilon - [\nabla \hat{U}_0^L + \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L)]\|_{L^p((0,T) \times D)} = 0.$$

4.4 Numerical examples

We conclude this chapter with some one dimensional and two dimensional numerical results to illustrate the sparse tensor product FE method.

We first consider a one dimensional problem on the domain $D = (0, 1)$ where the monotone nonlinear function

$$A(t, x, y, \tau, \xi) = 2\xi + \sin(x \sin(2\pi y)\xi) - 2t^2 x \cos(2\pi y) \cos(2\pi \tau)$$

$$\begin{aligned}
& -\sin(t^2 x \sin(2\pi y)(1 - 2x + x \cos(2\pi y) \cos(2\pi \tau))) \\
& -\frac{t^2}{2\pi} x \cos(2\pi y) \sin(2\pi \tau)
\end{aligned}$$

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. In this case $p = 2$. The problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y, \tau) = \frac{t^2}{2\pi} x \sin(2\pi y) \cos(2\pi \tau).$$

Similarly as with Chapter 3, we use the Broyden's method and the Polak-Ribière method at each time step for the backward Euler and Crank-Nicholson method. For the backward Euler and Crank-Nicholson method, we use a timestep of $\Delta t = 1/2^L$ and $\Delta t = 1/(\lceil 2^{(L/2)} \rceil)$ respectively.

For the backward Euler method with sparse tensor product FE spaces, we plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^2(D \times (0,1), H_{\#}^1(Y))}$ in Figures 4.1 and 4.2 respectively at $t = 1$. The figures indicate that the error is $O(\Delta t) + O(h_L)$.

Figures 4.3 and 4.4 shows the numerical error for the Crank-Nicholson scheme on sparse tensor product FE spaces at $t = 1$. The figures indicate that the error is $O((\Delta t)^2) + O(h_L)$.

Figure 4.1: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method.

Figure 4.2: The error $\|u_1 - \hat{u}_1\|_{L^2(D \times (0,1), H_\#^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method.

Figure 4.3: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method.

Figure 4.4: The error $\|u_1 - \hat{u}_1\|_{L^2(D \times (0,1), H_{\#}^1(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method.

The numerical results support the theoretical findings.

We then consider the one dimensional multiscale monotone problem for $p = 4$ on $D = (0, 1)$ with the monotone function

$$A(t, x, y, \tau, \xi) = 2\xi + \frac{1}{5}(2 + \sin(2\pi y))\xi^3 - 2t^2 x \cos(2\pi y) \cos(2\pi \tau)$$

$$-\frac{t^6}{5}(2 + \sin(2\pi y))(1 - 2x + x \cos(2\pi y) \cos(2\pi\tau))^3 \\ - \frac{t^2}{2\pi} x \cos(2\pi y) \sin(2\pi\tau)$$

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. In this case $p = 4$. This problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y, \tau) = \frac{t^2}{2\pi} x \sin(2\pi y) \cos(2\pi\tau).$$

We plot the numerical results for the error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}((D))}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ for $t = 1$ in Figures 4.5 and 4.6 for the backward Euler method respectively. The numerical results for the Crank-Nicholson method are shown in Figures 4.7 and 4.8.

Figure 4.5: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for the backward Euler method.

Figure 4.6: The error $\|u_1 - \hat{u}_1\|_{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the backward Euler method.

Figure 4.7: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for 1 dimensional problem at $t = 1$ for the Crank-Nicholson method.

Figure 4.8: The error $\|u_1 - \hat{u}_1\|_{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ for 1 dimensional problem at $t = 1$ versus the mesh size h for the Crank-Nicholson method.

For the two dimensional domain $D = (0, 1) \times (0, 1)$, we consider the problem when $p = 4$ with coefficient

$$\begin{aligned}
& A(t, x, y, \tau, \xi) \\
&= 2(\xi' + \xi'') + \frac{1}{5}(2 + \sin(2\pi y_2))\xi'^3 + \frac{1}{5}(2 + \sin(2\pi y_1))\xi''^3 \\
&\quad - 2t^2(x' + x'')(\cos(2\pi y') \sin(2\pi y'') + \sin(2\pi y') \cos(2\pi y'') \cos(2\pi\tau)) \\
&\quad - \frac{t^6}{5}(2 + \sin(2\pi y_2))((1 - 2x')(x'' - x''^2) \\
&\quad \quad \quad + (x' + x'') \cos(2\pi y') \sin(2\pi y'') \cos(2\pi\tau))^3 \\
&\quad - \frac{t^6}{5}(2 + \sin(2\pi y_1))((x' - x'^2)(1 - 2x'') \\
&\quad \quad \quad + (x' + x'') \sin(2\pi y') \cos(2\pi y'') \cos(2\pi\tau))^3 \\
&\quad - \frac{t^2}{2\pi}(x' + x'') \cos(2\pi y') \sin(2\pi y'') \sin(2\pi\tau) \\
&\quad - \frac{t^2}{2\pi}(x' + x'') \sin(2\pi y') \cos(2\pi y'') \sin(2\pi\tau),
\end{aligned}$$

and

$$f(t, x) = 2t^2(x' + x'' - x'^2 - x''^2) + 2t(x' - x'^2)(x'' - x''^2)$$

with $x = (x', x'') \in D$, $y = (y', y'') \in Y$ and $\xi = (\xi', \xi'') \in \mathbb{R}^2$. In this case

$p = 4$. This problem has the exact solution

$$u_0(t, x) = t^2(x' - x'^2)(x'' - x''^2)$$

and

$$u_1(t, x, y, \tau) = \frac{t^2}{2\pi}(x' + x'') \sin(2\pi y') \sin(2\pi y'') \cos(2\pi \tau)$$

For $t = 0.5$, we plot the error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D \times (0,1), W_{\#}^{1,4}(Y)/\mathbb{R})}$ for the backward Euler sparse tensor product FE solutions in Figures 4.9 and 4.10 respectively. Here we use a timestep of $\Delta t = 1/(2^{L+1})$.

Figure 4.9: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for 2 dimensional problem at $t = 0.5$ for the backward Euler method.

Figure 4.10: The error $\|u_1 - \hat{u}_1\|_{L^4(D \times (0,1), W_{\#}^{1,4}(Y))}$ versus the mesh size h for 2 dimensional problem at $t = 0.5$ for the backward Euler method.

As with the numerical examples in Chapter 3, the numerical results for the one dimensional and two dimensional problems for $p = 4$ show the optimal convergence rate which is better than the theoretical results.

Chapter 5

Multiscale Monotone Variational Inequalities

Chapter 5 considers multiscale variational inequalities of monotone type. In Section 5.1, we formulate the multiscale monotone variational inequality and apply the multiscale convergence to derive the monotone multiscale homogenized problem. In Section 5.2, we consider the full tensor and sparse tensor product FE methods. A new homogenization error for two scale monotone variational inequality is derived in Section 5.3 and a numerical corrector is constructed with an explicit convergence rate. For general multiscale problems, we derive a numerical corrector without an error. We then show some numerical examples in Section 5.4.

5.1 Multiscale variational inequalities

We first recall the notations of Chapter 3. Let $D \in \mathbb{R}^d$ be a bounded domain; and let $Y = (0, 1)^d$ be the unit cube in $\mathbb{R}^d$. Let n be a positive integer. Let $Y_1, \dots, Y_n$ be n copies of the unit cube Y . For conciseness, we

denote by $\mathbf{y}_i = (y_1, \dots, y_i)$ a vector in $Y_1 \times \dots \times Y_i$. We denote by $\mathbf{y} = \mathbf{y}_n$ and $\mathbf{Y} = Y_1 \times \dots \times Y_n$. Let $A(x, y_1, \dots, y_n, \xi) : D \times Y_1 \times \dots \times Y_n \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Y_i periodic with respect to y_i and continuously differentiable. We assume that A is monotone and locally Lipschitz. In particular, we assume that there are constants $p \geq 2$, $\alpha > 0$ and $\beta > 0$ so that for all $x \in D$, $y_i \in Y_i$ ($i = 1, \dots, n$), and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$(A(x, y_1, \dots, y_n, \xi_1) - A(x, y_1, \dots, y_n, \xi_2), \xi_1 - \xi_2) \geq \alpha |\xi_1 - \xi_2|^p, \quad (5.1)$$

and

$$|A(x, y_1, \dots, y_n, \xi_1) - A(x, y_1, \dots, y_n, \xi_2)| \leq \beta(|\xi_1| + |\xi_2|)^{p-2} |\xi_1 - \xi_2| \quad (5.2)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^d$. Let $\varepsilon_1, \dots, \varepsilon_n$ be n functions of $\varepsilon > 0$ that represent n microscopic scale on which the problem depends. As in the previous chapters, we assume scale separation

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon_{i+1}}{\varepsilon_i} = 0,$$

for $i = 1, \dots, n-1$. The multiscale monotone function is defined as

$$A^\varepsilon(x, \xi) = A(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \xi).$$

Let

$$K = \{\phi \in W_0^{1,p}(D), \phi(x) \geq 0 \text{ for a.a } x \in D\}.$$

Let $f \in L^q(D)$ where $1/p + 1/q = 1$. We consider the following variational inequality: Find $u^\varepsilon \in K$ such that

$$\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot \nabla(\phi^\varepsilon - u^\varepsilon) dx \geq \int_D f(\phi^\varepsilon - u^\varepsilon), \quad \forall \phi^\varepsilon \in K. \quad (5.3)$$

Problem (5.3) has a unique solution that is uniformly bounded in the $W^{1,p}(D)$ norm for all $\varepsilon > 0$ (see, e.g., [54]). We use multiscale homogenization to study homogenization of (5.3).

5.1.1 Homogenization problem

To study homogenization of problem (5.3), we employ $n + 1$ -scale convergence. The following definitions in the L^p setting is introduced in [6] (see also [65] and [5]).

Definition 5.1. A sequence $\{u^\varepsilon\}_\varepsilon \in L^p(D)$ $n + 1$ -scale converges to a function $u_0(x, y_1, \dots, y_n) \in L^p(D \times Y_1 \times \dots \times Y_n)$ if

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_D u^\varepsilon(x) \phi(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) dx \\ &= \int_D \int_{Y_1} \dots \int_{Y_n} u_0(x, y_1, \dots, y_n) \phi(x, y_1, \dots, y_n) dy_n \dots dy_1 dx, \end{aligned}$$

for all functions $\phi \in C(\bar{D} \times \bar{Y}_1 \times \dots \times \bar{Y}_n)$ which are Y_i -periodic with respect to y_i .

Definition 5.1 makes sense due to the following proposition, which is shown in [6].

Proposition 5.2. From each bounded sequence in $L^p(D)$, we can extract a subsequence that $n + 1$ -scale converges.

We denote by

$$V_0 = W_0^{1,p}(D), \quad V_i = L^p(D \times Y_1 \times \dots \times Y_{i-1}, W_{\#}^{1,p}(Y_i)/\mathbb{R}), \quad (i = 1, \dots, n).$$

For a bounded sequence in $W_0^{1,p}(D)$ we have the following results.

Proposition 5.3. From a bounded sequence $\{w^\varepsilon\} \subset W_0^{1,p}(D)$, we can extract a subsequence (not renumbered) such that ∇w^ε $(n + 1)$ -scale converges to

$$\nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i.$$

for $w_0 \in V_0$ and $w_i \in V_i$ ($i = 1, \dots, n$).

We define the following space

$$\mathbf{V} = \{(\phi_0, \phi_1, \dots, \phi_n) : \phi_0 \in V_0, \phi_i \in V_i\}$$

which is equipped with the norm

$$|||(\phi_0, \phi_1, \dots, \phi_n)||| = \|\nabla \phi_0\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} \phi_i\|_{L^p(D \times Y_1 \times \dots \times Y_{i-1})}. \quad (5.4)$$

We have the norm equivalence (see [48]):

Lemma 5.4. There are positive constants c_1 and c_2 such that for all $(\phi_0, \{\phi_i\}) \in \mathbf{V}$,

$$\begin{aligned} & c_1 |||(\phi_0, \{\phi_i\})||| \\ & \leq \left(\int_D \int_{Y_1} \dots \int_{Y_n} |\nabla_x \phi_0 + \nabla_{y_1} \phi_1 + \dots + \nabla_{y_n} \phi_n|^p dx dy_1 \dots dy_n \right)^{1/p} \\ & \leq c_2 |||(\phi_0, \{\phi_i\})|||. \end{aligned}$$

For the limiting homogenized variational inequality, we consider the following convex subset of $\mathbf{V}$:

$$\mathcal{K} = \{(\phi_0, \phi_1, \dots, \phi_n) \in \mathbf{V} : \phi_0 \in K\}.$$

For $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbf{V}$ and $\mathbf{w} = (w_0, w_1, \dots, w_n) \in \mathbf{V}$, we define bilinear form $B : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ as

$$\begin{aligned} B(\mathbf{v}; \mathbf{w}) = & \int_D \int_{Y_1} \dots \int_{Y_n} A(x, y_1, \dots, y_n, \nabla v_0(x) + \sum_{i=1}^n \nabla_{y_i} v_i(x, y_1, \dots, y_i)) \cdot \\ & (\nabla w_0(x) + \sum_{i=1}^n \nabla_{y_i} w_i(x, y_1, \dots, y_i)) dy_n \dots dy_1 dx. \end{aligned}$$

We have the following result:

Proposition 5.5. The solution u^ε of problem (5.3) converges weakly in $W_0^{1,p}(D)$ to a function u_0 , and ∇u^ε $n+1$ scale converges to $\nabla u_0 + \nabla_{y_1} u_1 +$

$\dots + \nabla_{y_n} u_n$ where $\mathbf{u} = (u_0, u_1, \dots, u_n) \in \mathcal{K}$ is the unique solution of the variational inequality:

$$B(\mathbf{u}; \phi - \mathbf{u}) \geq \int_D f(\phi_0(x) - u_0(x)) dx, \quad \forall \phi = (\phi_0, \phi_1, \dots, \phi_n) \in \mathcal{K}. \quad (5.5)$$

This result is established in Sandrakov [73] for two scale variational inequalities, using two scale convergence. Sandrakov only considers the case of a two scale monotone operator $A : Y \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ posed in $H^1(\Omega)$ which does not depend on the slow variable x . The proof for more than two scales is similar; we present it here.

Proof of proposition 5.5 First we show that u^ε is uniformly bounded in $W_0^{1,p}(D)$ for all ε . From (5.3) and (5.1), we have that

$$\begin{aligned} \int_D |\nabla u^\varepsilon|^p dx &\leq \left| \int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot \nabla \phi^\varepsilon dx \right| + \left| \int_D A^\varepsilon(x, 0) \cdot \nabla u^\varepsilon dx \right| \\ &\quad + \left| \int_D f(u^\varepsilon - \phi^\varepsilon) dx \right|. \end{aligned}$$

From (5.2), we have

$$|A^\varepsilon(x, \nabla u^\varepsilon)| \leq c |\nabla u^\varepsilon|^{p-1} + |A^\varepsilon(x, 0)|$$

so

$$\begin{aligned} \|\nabla u^\varepsilon\|_{L^p(D)}^p &\leq c \int_D |\nabla u^\varepsilon|^{p-1} |\nabla \phi^\varepsilon| dx + c \int_D |\nabla u^\varepsilon| dx \\ &\quad + \left| \int_D f \phi^\varepsilon dx \right| + \int_D f u^\varepsilon dx \\ &\leq \|\nabla u^\varepsilon\|_{L^p(D)}^{p-1} \|\nabla \phi^\varepsilon\|_{L^p(D)} + c \|\nabla u^\varepsilon\|_{L^p(D)} \\ &\quad + \|f\|_{L^q(D)} \|u^\varepsilon\|_{L^p(D)} + \int_D f \phi^\varepsilon dx. \end{aligned}$$

Let $\phi^\varepsilon = 0$, we deduce that u^ε is uniformly bounded in $W^{1,p}(D)$. Therefore we can extract a subsequence (not renumbered) so that ∇u^ε ($n+1$)-scale converges to $\nabla_x u_0 + \sum_{i=1}^n \nabla_{y_i} u_i$ for $u_0 \in V_0$ and $u_i \in V_i$ for $i = 1, \dots, n$.

From Minti theorem (see [54]), problem (5.3) is equivalent to

$$\begin{aligned} & \int_D A^\varepsilon(x, \nabla \phi^\varepsilon(x)) \cdot \nabla(\phi^\varepsilon(x) - u^\varepsilon(x)) dx \\ & \geq \int_D f(x)(\phi^\varepsilon(x) - u^\varepsilon(x)) dx, \quad \forall \phi^\varepsilon \in K. \end{aligned} \quad (5.6)$$

The main advantage of using Minti theorem is that we can pass to the limit of the nonlinear term $A^\varepsilon(x, \nabla \phi^\varepsilon(x))$. For each $i = 1, \dots, n$, let $\phi_i \in C_0^\infty(D, C^\infty(\bar{Y}_1 \times \dots \times \bar{Y}_i))$ which are Y_j -periodic in y_j for $j = 1, \dots, i$. Let $\chi(x)$ be a non-negative function in $C_0^\infty(D)$ which equals 1 in the supports (with respect to x) of all ϕ_i . Let δ be a positive number. The motivation of introducing $\chi(x)$ and δ is to make the test function $\phi^\varepsilon(x)$ below non-negative for a.a. x . Let $\phi_0 \in C_0^\infty(\Omega)$ be in K . Then when ε is sufficiently small,

$$\phi^\varepsilon(x) = \phi_0(x) + \sum_{i=1}^n \varepsilon_i \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) + \delta \chi(x)$$

is in K . We note that

$$\begin{aligned} \nabla \phi^\varepsilon(x) &= \nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) \\ &+ \sum_{i=1}^n \left(\varepsilon_i \nabla_x \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) + \sum_{j=1}^{i-1} \frac{\varepsilon_i}{\varepsilon_j} \phi_{y_j} \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) \right) \\ &+ \delta \nabla \chi. \end{aligned}$$

From condition (5.2), we have that

$$\begin{aligned} & |A^\varepsilon(x, \nabla \phi^\varepsilon) - A^\varepsilon(x, \nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) + \delta \nabla \chi(x))| \\ & \leq c \left(\max_{i=1, \dots, n} \varepsilon_i + \max_{j < i} \frac{\varepsilon_i}{\varepsilon_j} \right). \end{aligned}$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \int_D A^\varepsilon(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \nabla \phi^\varepsilon(x)) \cdot (\nabla \phi^\varepsilon - \nabla u^\varepsilon) dx$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0} \int_D A^\varepsilon\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \nabla \phi_0(x)\right) \\
&\quad + \sum_{i=1}^n \nabla_{y_i} \phi_i\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}\right) + \delta \nabla \chi(x) \cdot (\nabla \phi^\varepsilon - \nabla u^\varepsilon) dx.
\end{aligned}$$

Passing to the $(n+1)$ -scale limit, we have

$$\begin{aligned}
&\int_D \int_{\mathbf{Y}} A\left(x, y_1, \dots, y_n, \nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i + \delta \nabla \chi\right) \cdot \\
&\quad \left(\left(\nabla \phi_0(x) + \sum_{i=1}^n \nabla_{y_i} \phi_i + \delta \nabla \chi\right) - \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right)\right) d\mathbf{y} dx \\
&\geq \int_D f(\phi_0 - u_0) dx.
\end{aligned}$$

Passing to the limit when $\delta \rightarrow 0$, using (5.2) again, we get

$$\begin{aligned}
&\int_D \int_{\mathbf{Y}} A\left(x, y_1, \dots, y_n, \nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i\right) \\
&\quad \cdot \left(\left(\nabla \phi_0(x) + \sum_{i=1}^n \nabla_{y_i} \phi_i\right) - \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i\right)\right) d\mathbf{y} dx \\
&\geq \int_D f(\phi_0 - u_0) dx.
\end{aligned}$$

Using density and (5.2), this holds for all $(\phi_0, \phi_1, \dots, \phi_n) \in \mathcal{K}$. We deduce (5.5) from Minti's theorem. $\square$

5.2 FE discretization

We develop the tensor product FE method to solve the multiscale homogenized problem (5.5) in this section. The FE full and sparse tensor product FE spaces are defined as in Chapter 3. To make the chapter self contained, we recall the definitions of these spaces here. As with Chapter 3, we follow the setting of [48].

5.2.1 Hierarchical FE spaces

Assuming that D is a polygonal domain. We consider a hierarchy $\{\mathcal{T}^l\}$ for $l \geq 0$ of regular triangular simplices of mesh size $h_l = O(2^{-l})$ in D . When $d = 2$ the simplices in $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles; and when $d = 3$, $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 8 tetrahedral. Let V^l be the space of continuous functions in D which are linear in each simplex of $\mathcal{T}^l$. We have that $V^0 \subset V^1 \subset \dots \subset V^l \subset V^{l+1} \dots$. We define by $V_0^l = V^l \cap W_0^{1,p}(D)$. Similarly, we consider a hierarchy $\{\mathcal{T}_\#^l\}$ of regular triangular simplices of mesh size $h_l = O(2^{-l})$ which are periodically distributed in Y ; and a hierarchy of spaces of continuous periodic piecewise linear functions in Y $V_\#^0 \subset V_\#^1 \subset \dots$

We then use these spaces to define the full and sparse tensor product FE spaces in the next sections.

5.2.2 Full tensor product FE spaces

We note that $L^p(D \times Y_1 \times \dots \times Y_{i-1}, W_\#^{1,p}(Y_i)) \cong L^p(D) \otimes L^p(Y_1) \otimes \dots \otimes L^p(Y_{i-1}) \otimes W_\#^{1,p}(Y_i)$. We therefore employ the spaces

$$V_i^L = V^L \otimes \underbrace{V_\#^L \otimes \dots \otimes V_\#^L}_{i \text{ times}}$$

to approximate u_i . To approximate $u_0 \in W_0^{1,p}(D)$, we employ the space $V_0^L = V^L \cap W_0^{1,p}(D)$, and the set

$$K^L = \{\phi^L \in V_0^L, \phi^L(x) \geq 0, \text{ a.a. } x \in D\}.$$

We define the FE space

$$\mathbf{V}^L = \{(u_0^L, \{u_i^L\}) : u_0^L \in V_0^L, u_i^L \in V_i^L, i = 1, \dots, n\}$$

We recall the regularity space $\mathcal{W}_i$ defined in Chapter 3. Let $\mathcal{W}_i$ be the space of functions $w \in L^p(D \times Y_1 \times \dots \times Y_{i-1}, W^{2,p}(Y_i))$ that belong to $L^p(Y_1 \times \dots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))$ and $L^p(D \times \prod_{1 \leq j \leq i-1, j \neq k} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))$ for all $k = 1, \dots, i-1$. The space $\mathcal{W}_i$ is equipped with the norm

$$\begin{aligned} \|w\|_{\mathcal{W}_i} &= \|w\|_{L^p(D \times Y_1 \times \dots \times Y_{i-1}, W^{2,p}(Y_i))} + \|w\|_{L^p(Y_1 \times \dots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))} \\ &\quad + \sum_{k=1}^{i-1} \|w\|_{L^p(D \times \prod_{1 \leq j \leq i-1, j \neq k} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))}. \end{aligned}$$

We then have the following approximation.

Lemma 5.6. For $w \in \mathcal{W}_i$,

$$\inf_{w^L \in V_i^L} \|w - w^L\|_{V_i} \leq ch_L \|w\|_{\mathcal{W}_i}.$$

The proof can be found in [48]. We denote by

$$\mathcal{K}^L = \{(\phi_0^L, \phi_1^L, \dots, \phi_n^L) : \phi_0^L \in K^L, \phi_i^L \in V_i^L, (i = 1, \dots, n)\}.$$

The FE approximating problem of (5.5) is: find $(u_0^L, u_1^L, \dots, u_m^L) \in \mathcal{K}^L$ such that

$$\begin{aligned} &B(u_0^L, u_1^L, \dots, u_n^L; \phi_0^L - u_0^L, \phi_1^L - u_1^L, \dots, \phi_n^L - u_n^L) \\ &\geq \int_D f(\phi_0^L - u_0^L) dx, \quad \forall (\phi_0^L, \phi_1^L, \dots, \phi_n^L) \in \mathcal{K}^L. \end{aligned} \quad (5.7)$$

To prove the FE error estimate, we assume the following condition for the function A , in addition to (5.1) and (5.2). We assume further that

$$|\nabla_x A(x, y_1, \dots, y_n, \xi)| \leq c(1 + |\xi|^{p-1}) \quad (5.8)$$

and

$$|\nabla_\xi A(x, y_1, \dots, y_n, \xi)| \leq c(1 + |\xi|^{p-2}). \quad (5.9)$$

We have the following error estimate.

Theorem 5.7. When $u_0 \in W^{2,p}(D)$ and $u_i \in \mathcal{W}_i$ for all $i = 1, \dots, n$, for the full tensor product FE approximating solution of problem (5.7), we have the error estimate:

$$|||(u_0 - u_0^L, u_1 - u_1^L, \dots, u_n - u_n^L)||| \leq ch_L^{1/(p-1)}. \quad (5.10)$$

Proof. We denote by $\mathbf{u}^L = (u_0^L, u_1^L, \dots, u_n^L) \in \mathcal{K}^L$. We consider

$$\begin{aligned} & B(\mathbf{u}; \mathbf{u} - \mathbf{u}^L) - B(\mathbf{u}^L; \mathbf{u} - \mathbf{u}^L) \\ &= B(\mathbf{u}; \phi - \mathbf{u}^L) + B(\mathbf{u}^L; \phi^L - \mathbf{u}) + B(\mathbf{u}; \mathbf{u} - \phi) + B(\mathbf{u}^L; \mathbf{u}^L - \phi^L) \end{aligned}$$

for all $\phi = (\phi_0, \phi_1, \dots, \phi_n) \in \mathcal{K}$ and $\phi^L = (\phi_0^L, \phi_1^L, \dots, \phi_n^L) \in \mathcal{K}^L$. From (5.5) and (5.7), we have that

$$B(\mathbf{u}; \mathbf{u} - \phi) \leq \int_D f(u_0 - \phi_0) dx, \quad B(\mathbf{u}^L; \mathbf{u}^L - \phi^L) \leq \int_D f(u_0^L - \phi_0^L) dx.$$

Therefore

$$\begin{aligned} & B(\mathbf{u}; \mathbf{u} - \mathbf{u}^L) - B(\mathbf{u}^L; \mathbf{u} - \mathbf{u}^L) \\ &\leq B(\mathbf{u}; \phi - \mathbf{u}^L) + B(\mathbf{u}^L; \phi^L - \mathbf{u}) \\ &\quad + \int_D f(u_0 - \phi_0) dx + \int_D f(u_0^L - \phi_0^L) dx. \end{aligned}$$

Letting $\phi_0 = u_0$ in (5.5), we have

$$\int_D \int_{\mathbf{Y}} A(x, y_1, \dots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \cdot \left(\sum_{i=1}^n \nabla_{y_i} (\phi_i - u_i) \right) d\mathbf{y} dx \geq 0,$$

$\forall \phi_i \in V_i, i = 1, \dots, n$. Thus for all $v_i \in V_i, i = 1, \dots, n$, we have

$$\int_D \int_{\mathbf{Y}} A(x, y_1, \dots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \cdot \left(\sum_{i=1}^n \nabla_{y_i} v_i \right) d\mathbf{y} dx = 0. \quad (5.11)$$

We now choose $\phi_0 = u_0^L$. Then $B(\mathbf{u}; \phi - \mathbf{u}^L) = 0$. We have from (5.1)

$$\alpha \|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}^p \leq B(\mathbf{u}; \phi^L - \mathbf{u}) + B(\mathbf{u}^L; \phi^L - \mathbf{u})$$

$$-B(\mathbf{u}; \phi^L - \mathbf{u}) + \int_D f(u_0 - \phi_0^L) dx. \quad (5.12)$$

From (5.2), we have

$$\begin{aligned} & B(\mathbf{u}^L; \phi^L - \mathbf{u}) - B(\mathbf{u}; \phi^L - \mathbf{u}) \\ & \leq \beta \int_D \int_{\mathbf{Y}} \left(\left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| + \left| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right| \right)^{p-2} \\ & \quad \left| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \right| \\ & \quad \left| \left(\nabla_x u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla_x \phi_0^L + \sum_{i=1}^n \nabla_{y_i} \phi_i^L \right) \right| d\mathbf{y} dx \\ & \leq c \left(\left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|_{L^p(D \times \mathbf{Y})}^{p-2} + \left\| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right\|_{L^p(D \times \mathbf{Y})}^{p-2} \right) \\ & \quad \cdot \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \right\|_{L^p(D \times \mathbf{Y})} \\ & \quad \cdot \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla \phi_0^L + \sum_{i=1}^n \nabla_{y_i} \phi_i^L \right) \right\|_{L^p(D \times \mathbf{Y})}. \end{aligned}$$

From (5.1) we have

$$\begin{aligned} & \left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|_{L^p(D \times \mathbf{Y})}^p \\ & \leq \int_D \int_{\mathbf{Y}} \left(A \left(x, y_1, \dots, y_n, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) - A(x, y_1, \dots, y_n, 0) \right) \cdot \\ & \quad \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) d\mathbf{y} dx \\ & \leq \int_D \int_{\mathbf{Y}} A \left(x, y_1, \dots, y_n, \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \cdot \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) + \\ & \quad c \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| d\mathbf{y} dx. \end{aligned}$$

Therefore from (5.7), choosing $\phi_0^L = 0$, $\phi_i^L = 0$ for $i = 1, \dots, n$, we deduce

$$\begin{aligned}
& \left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|_{L^p(D \times \mathbf{Y})}^p \\
& \leq B(\mathbf{u}^L; \mathbf{u}^L) + c \int_D \int_{\mathbf{Y}} \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| d\mathbf{y} dx \\
& \leq c \int_D \int_{\mathbf{Y}} \left| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right| d\mathbf{y} dx + \int_D f u_0^L dx \\
& \leq c \left\| \nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right\|_{L^p(D \times \mathbf{Y})} + c \|u_0^L\|_{W_0^{1,p}}.
\end{aligned}$$

Thus $\|\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L\|_{L^p(D \times \mathbf{Y})}$ is uniformly bounded with respect to L .

We then have

$$\begin{aligned}
& B(\mathbf{u}^L; \phi^L - \mathbf{u}) - B(\mathbf{u}; \phi^L - \mathbf{u}) \\
& \leq c \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla u_0^L + \sum_{i=1}^n \nabla_{y_i} u_i^L \right) \right\|_{L^p(D \times \mathbf{Y})} \\
& \quad \cdot \left\| \left(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right) - \left(\nabla \phi_0^L + \sum_{i=1}^n \nabla_{y_i} \phi_i^L \right) \right\|_{L^p(D \times \mathbf{Y})} \quad (5.13)
\end{aligned}$$

From (5.11)

$$\begin{aligned}
& B(\mathbf{u}; \phi^L - \mathbf{u}) \\
& = \int_D \int_{\mathbf{Y}} A(x, y_1, \dots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \cdot \nabla_x (\phi_0^L - u_0) d\mathbf{y} dx \\
& = - \int_D \int_{\mathbf{Y}} \left[\nabla_x \cdot A(x, y_1, \dots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \right. \\
& \quad \left. + \sum_{j=1}^n \frac{\partial}{\partial \xi_j} A_k(x, y_1, \dots, y_n, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \right. \\
& \quad \left. \frac{\partial}{\partial x_k} \left(\frac{\partial u_0}{\partial x_j} + \sum_{i=1}^n \frac{\partial u_i}{(\partial y_i)_j} \right) \right] (\phi_0^L - u_0) d\mathbf{y} dx.
\end{aligned}$$

From (5.8) and (5.9), we have

$$B(\mathbf{u}; \phi^L - \mathbf{u})$$

$$\begin{aligned}
&\leq c \left(1 + \left\| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right\|_{L^p(D \times \mathbf{Y})}^{p-1} \right) \|\phi_0^L - u_0\|_{L^p(D)} \\
&\quad + \sum_{k=1}^n \sum_{j=1}^n \left(1 + \left\| \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i \right\|_{L^p(D \times \mathbf{Y})}^{p-2} \right) \\
&\quad \quad \left\| \frac{\partial}{\partial x_k} \left(\frac{\partial u_0}{\partial x_j} + \sum_{i=1}^n \frac{\partial u_i}{\partial (y_i)_j} \right) \right\|_{L^p(D \times \mathbf{Y})} \|\phi_0^L - u_0\|_{L^p(D)} \\
&\leq c \|u_0 - \phi_0^L\|_{L^p(D)} \tag{5.14}
\end{aligned}$$

due to $u_0 \in W^{2,p}(D)$ and $u_i \in \mathcal{W}_i$ for $i = 1, \dots, n$. From (5.12), (5.13) and (5.14) we have

$$\alpha \|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}}^p \leq c \|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}} \|\mathbf{u} - \phi^L\|_{\mathbf{V}} + c \|u_0 - \phi_0^L\|_{L^p(D)}.$$

As $u_0 \in W^{2,p}(D) \subset C(D)$ for $p \geq 2$ ($d = 2, 3$), we choose $\phi_0^L = I^L u_0$ so $\phi_0^L \in K$. Then $\|u_0 - \phi_0^L\|_{L^p(D)} \leq ch_L^2$ and $\|u_0 - \phi_0^L\|_{W^{1,p}(D)} \leq ch_L$. Further, we choose ϕ_i^L such that $\|u_i - \phi_i^L\|_{V_i} \leq ch_L$. Then

$$\alpha \|u - u^L\|_{\mathbf{V}}^p \leq c \|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}} h_L + ch_L^2.$$

Therefore

$$\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}} \leq c(h_L^{1/(p-1)} + h_L^{2/p}) \leq ch_L^{1/(p-1)}.$$

□

Remark 5.8. The error estimate (5.10) is equivalent to that for the error of the FE approximation of a monotone partial differential equation, see, e.g. [28] and [25]. If conditions (5.8) and (5.9) do not hold, then we only have that

$$B(\mathbf{u}; \phi - \mathbf{u}^L) \leq c \|u_0 - \phi_0^L\|_{W^{1,p}(D)} \leq ch_L.$$

We then get a weaker estimate

$$\|\mathbf{u} - \mathbf{u}^L\|_{\mathbf{V}} \leq ch_L^{1/p}.$$

5.2.3 Sparse tensor product FE spaces

We employ the sparse tensor product FE spaces $\hat{V}_i^L$ defined in Section 3.2.1.3. We recall the regularity spaces $\hat{\mathcal{W}}_i$ defined in Chapter 3. It consists of functions $w(x, y_1, \dots, y_i)$ which are Y_j periodic with respect to y_j for $j = 0, \dots, i$ such that for all $\alpha_j \in \mathbb{R}^d$ ($j = 0, \dots, i-1$) with $|\alpha_j| \leq 1$ and $\alpha_i \in \mathbb{R}^d$ with $|\alpha_i| \leq 2$, we have $\partial^{\sum_{j=0}^i |\alpha_j|} w / (\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i) \in L^p(D \times Y_1 \times \dots \times Y_i)$. In other words, $\mathcal{W}_i = W^{1,p}(D, W_{\#}^{1,p}(Y_1, \dots, W_{\#}^{1,p}(Y_{i-1}, W_{\#}^{2,p}(Y_i)))) \cong W^{1,p}(D) \otimes W_{\#}^{1,p}(Y_1) \otimes \dots \otimes W_{\#}^{1,p}(Y_{i-1}) \otimes W_{\#}^{2,p}(Y_i)$. This space is equipped with the norm

$$\|w\|_{\mathcal{W}_i} = \sum_{\substack{0 \leq |\alpha_i| \leq 2 \\ 0 \leq |\alpha_0|, \dots, |\alpha_{i-1}| \leq 1}} \left\| \frac{\partial^{\sum_{j=0}^i |\alpha_j|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \dots \partial^{\alpha_i} y_i} \right\|_{L^p(D \times Y_1 \times \dots \times Y_i)}.$$

We have the following approximation properties

Lemma 5.9. For $w \in \hat{\mathcal{W}}_i$, for $p = 2$

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^2(D \times Y_1 \times \dots \times Y_i)} \leq cL^{i/2} h_L \|w\|_{\hat{\mathcal{W}}_i};$$

and for $p > d$

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^p(D \times Y_1 \times \dots \times Y_i)} \leq cL^i h_L \|w\|_{\hat{\mathcal{W}}_i}.$$

We refer to [49] and [48] for a proof. We define the convex set

$$\hat{\mathcal{K}}^L = \{(\hat{\phi}_0^L, \hat{\phi}_1^L, \dots, \hat{\phi}_n^L) : \hat{\phi}_0^L \in K^L, \hat{\phi}_i^L \in \hat{V}_i^L, (i = 1, \dots, n)\}.$$

The sparse tensor product FE approximating problem is: Find

$$\hat{\mathbf{u}}^L = (\hat{u}_0^L, \hat{u}_1^L, \dots, \hat{u}_n^L) \in \hat{\mathcal{K}}^L$$

such that for all $\hat{\phi}^L = (\hat{\phi}_0^L, \hat{\phi}_1^L, \dots, \hat{\phi}_n^L) \in \hat{\mathcal{K}}^L$:

$$B(\hat{u}_0^L, \hat{u}_1^L, \dots, \hat{u}_n^L; \hat{\phi}_0^L - \hat{u}_0^L, \hat{\phi}_1^L - \hat{u}_1^L, \dots, \hat{\phi}_n^L - \hat{u}_n^L) \geq \int_D f(\hat{\phi}_0^L - \hat{u}_0^L). \quad (5.15)$$

We have the following error estimate

Theorem 5.10. Assume conditions (5.1), (5.2), (5.8) and (5.9). If $u_0 \in W^{2,p}(D)$ and $u_i \in \hat{\mathcal{W}}_i$ for $i = 1, \dots, n$, then for the sparse tensor product FE problem (5.15), we have

$$|||(u_0 - \hat{u}_0^L, u_1 - \hat{u}_1^L, \dots, u_n - \hat{u}_n^L)||| \leq cL^{n/2}h_L$$

when $p = 2$; and when $p > d$

$$|||(u_0 - \hat{u}_0^L, u_1 - \hat{u}_1^L, \dots, u_n - \hat{u}_n^L)||| \leq cL^{n/(p-1)}h_L^{1/(p-1)}.$$

The proof is identical to that for Theorem 5.7.

Remark 5.11. We require $p > d$ so that $W^{1,p}(D) \subset C(D)$ and $W^{1,p}(Y) \subset C(Y)$ and the interpolation $I^L w$ and $I_{\#}^L w$ can be defined. As noted previously, when $p \leq d$, if u_i is smoother than $W^{1,p}(D)$ with respect to x and $W^{1,p}(Y_i)$ with respect to y_i so that it is continuous with respect to x and y_i , the interpolation can be defined and we have the same results for sparse tensor product FE approximations.

5.3 Numerical correctors

In the two scale case, Sandrakov [73] established the homogenized variational inequality from the multiscale homogenized equation (5.5). We can write the corrector terms u_i in terms of the solution of the cell problems. For each vector $\xi \in \mathbb{R}^d$, we denote by $N(x, \mathbf{y}_{n-1}, y_n, \xi) \in W_{\#}^{1,p}(Y_n)/\mathbb{R}$, as a function of y_n , the solution of the problem

$$\nabla_{y_n} \cdot A(x, \mathbf{y}_{n-1}, y_n, \xi + \nabla_{y_n} N(x, \mathbf{y}, \xi)) = 0.$$

The $(n-1)$ th homogenized operator is determined as

$$A^{n-1}(x, \mathbf{y}_{n-1}, \xi) = \int_{Y_n} A(x, \mathbf{y}_{n-1}, y_n, \xi + \nabla_{y_n} N(x, \mathbf{y}_{n-1}, y_n, \xi)) dy_{n-1}.$$

It can be shown that A^{n-1} satisfies the monotone and local Lipschitz conditions similar to those of (5.1) and (5.2) (see, e.g., [24], [32]) and [19]). As in Chapter 3, inductively, let $A^n(x, \mathbf{y}, \xi) = A(x, \mathbf{y}, \xi)$. For each $\xi \in \mathbb{R}^d$, let $N^i(x, \mathbf{y}_{i-1}, y_i, \xi) \in W_{\#}^{1,p}(Y_i)/\mathbb{R}$ as a function of y_i be the solution of the problem

$$\nabla_{y_i} \cdot A^i(x, \mathbf{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(x, \mathbf{y}_{i-1}, y_i, \xi)) = 0.$$

The $(i-1)$ th homogenized operator is defined as

$$A^{i-1}(x, \mathbf{y}_{i-1}, \xi) = \int_{Y_i} A^i(x, \mathbf{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(x, \mathbf{y}_{i-1}, y_i, \xi)) dy_i$$

which satisfies the monotone and local Lipschitz conditions similar to (5.1) and (5.2). The homogenized variational inequality is: Find $u_0 \in K$ such that for all $\phi \in K$,

$$\int_D A^0(x, \nabla u_0) \cdot (\nabla \phi - \nabla u_0) dx \geq \int_D f(\phi - u_0) dx, \quad \forall \phi \in K. \quad (5.16)$$

5.3.1 Two-scale problems

In the case of two scales, as for multiscale partial differential equations, we are able to derive a homogenization error in terms of the microscopic scale. In this case, the function $N(x, y, \xi)$ satisfies the cell problem

$$\nabla_y \cdot A(x, y, \xi + \nabla_y N(x, y, \xi)) = 0, \quad (5.17)$$

and $N(x, \cdot, \xi) \in W_{\#}^{1,p}(Y)/\mathbb{R}$ as a function of y . The homogenized operator is

$$A^0(x, \xi) = \int_Y A(x, y, \xi + \nabla_y N(x, y, \xi)) dy. \quad (5.18)$$

We then have the following result on homogenization error.

Proposition 5.12. Assume that $u_0 \in C^2(\bar{D})$, $u_1 \in C^1(\bar{D}, C^1(\bar{Y}))$. There is a constant c independent of ε such that

$$\left\| \nabla u^\varepsilon - \nabla u_0 - \nabla_y u_1 \left(\cdot, \frac{\cdot}{\varepsilon} \right) \right\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

Proof. Let

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right).$$

We have from the fact that $A \in C^1(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$ and (5.2) that

$$|A(x, y, \nabla u_0(x) + \nabla_y u_1(x, \frac{x}{\varepsilon}) - A(x, y, \nabla u_1^\varepsilon(x)))| \leq c\varepsilon. \quad (5.19)$$

For $i = 1, \dots, d$, we define the function

$$g_i(x, y) = A_i(x, y, \nabla u_0(x) + \nabla_y u_1(x, y)) - A_i^0(x, \nabla u_0(x)).$$

From (5.17) we have

$$\frac{\partial}{\partial y_i} g_i(x, y) = 0,$$

and from (5.18),

$$\int_Y g_i(x, y) dy = 0.$$

From [53], there are functions $\alpha_{ij}(x, y)$ for $i, j = 1, \dots, d$ such that $\alpha_{ij} = -\alpha_{ji}$ and

$$g_i(x, y) = \frac{\partial}{\partial y_j} \alpha_{ij}(x, y).$$

As $g_i \in C^1(\bar{D}, C(\bar{Y}))$, we have $\alpha_{ij} \in C^1(\bar{D}, C^1(\bar{Y}))$. We then have

$$\begin{aligned} & A_i \left(x, \frac{x}{\varepsilon}, \nabla_x u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) \right) - A_i^0(x, \nabla u_0(x)) \\ &= \varepsilon \frac{d}{dx_j} \alpha_{ij} \left(x, \frac{x}{\varepsilon} \right) - \varepsilon \frac{\partial}{\partial x_j} \alpha_{ij} \left(x, \frac{x}{\varepsilon} \right), \end{aligned}$$

where $\frac{d}{dx_j}$ denotes the total partial derivative of $\alpha_{ij}(x, \frac{x}{\varepsilon})$ as a function of x only. Thus for any functions $\phi \in W_0^{1,p}(D)$ we have

$$\int_D \left(A_i \left(x, \frac{x}{\varepsilon}, \nabla_x u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) \right) - A_i^0(x, \nabla u_0(x)) \right) \frac{\partial \phi}{\partial x_i}(x) dx$$

$$\begin{aligned}
&= -\varepsilon \int_D \alpha_{ij} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx - \varepsilon \int_D \frac{\partial}{\partial x_j} \alpha_{ij} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_i} dx \\
&= -\varepsilon \int_D \frac{\partial}{\partial x_j} \alpha_{ij} \left(x, \frac{x}{\varepsilon} \right) \frac{\partial \phi}{\partial x_i} dx
\end{aligned}$$

due to $\alpha_{ij}(x, y) = -\alpha_{ji}(x, y)$. From this and (5.19) we have

$$\|\nabla \cdot (A^\varepsilon(\cdot, \nabla u_1^\varepsilon) - A^0(\cdot, \nabla u_0))\|_{W^{-1,q}(D)} \leq c\varepsilon \quad (5.20)$$

where $q = p/(p-1)$ with $1/p + 1/q = 1$. Let $\eta^\varepsilon \in \mathcal{D}(D)$ be such that $\eta^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\varepsilon |\nabla_x \eta^\varepsilon(x)| \leq c$ where c is independent of ε . Let $\delta > 0$ be sufficiently large. We consider the function

$$w_1^\varepsilon(x) = u_0(x) + \varepsilon \eta^\varepsilon(x) (u_1(x, x/\varepsilon) + \delta).$$

The motivation of introducing the δ is to ensure that w_1^ε is non-negative for a.a. x . Due to the factor η^ε , w_1^ε satisfies the zero boundary condition. As $u_1(x, y) \in C(\bar{D} \times \bar{Y})$, it is bounded for all $x \in D$ and $y \in Y$ so when δ is sufficiently large, $w_1^\varepsilon \in W_0^{1,p}(D) \cap K$. We note that

$$\begin{aligned}
&\nabla_x(u_1^\varepsilon(x) - w_1^\varepsilon(x)) \\
&= -\varepsilon \nabla_x \eta^\varepsilon(x) u_1 \left(x, \frac{x}{\varepsilon} \right) + \varepsilon (1 - \eta^\varepsilon(x)) \nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) \\
&\quad + (1 - \eta^\varepsilon(x)) \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) - \delta \nabla_x (\varepsilon \eta^\varepsilon(x)).
\end{aligned}$$

As the support of $\nabla_x(u_1^\varepsilon - w_1^\varepsilon)$ is in an ε neighbourhood of ∂D , we deduce that

$$\|u_1^\varepsilon - w_1^\varepsilon\|_{W^{1,p}(D)} \leq c\varepsilon^{1/p}. \quad (5.21)$$

From (5.2), we have

$$\|A^\varepsilon(\cdot, \nabla u_1^\varepsilon(\cdot)) - A^\varepsilon(\cdot, \nabla w_1^\varepsilon(\cdot))\|_{L^q(D)} \leq c\varepsilon^{1/p}.$$

Therefore

$$\|\nabla \cdot (A^\varepsilon(\cdot, \nabla u_1^\varepsilon) - A^\varepsilon(\cdot, \nabla w_1^\varepsilon))\|_{W^{-1,q}(D)} \leq c\varepsilon^{1/p}.$$

From this and (5.20), we have

$$\|\nabla \cdot (A^\varepsilon(\cdot, \nabla w_1^\varepsilon) - A^0(\cdot, \nabla u_0))\|_{W^{-1,q}(D)} \leq c\varepsilon^{1/p}. \quad (5.22)$$

From (5.16) we have that for all $\phi \in K$

$$\begin{aligned} & \int_D A^0(x, \nabla u_0) \cdot (\nabla \phi - \nabla w_1^\varepsilon) dx \\ &= \int_D A^0(x, \nabla u_0) \cdot (\nabla \phi - \nabla u_0) - \int_D A^0(x, \nabla u_0) \cdot (\nabla w_1^\varepsilon - \nabla u_0) \\ &\geq \int_D f(\phi - u_0) dx + \int_D \nabla \cdot A^0(x, \nabla u_0) (w_1^\varepsilon - u_0) dx. \end{aligned}$$

As $A^0(x, \xi) \in C^1(\bar{D} \times \mathbb{R}^d)$ and $u_0 \in C^2(\bar{D})$, we have that $|\nabla \cdot A^0(x)(x, \nabla u_0(x))| \leq c$ for all $x \in D$. Using $|w_1^\varepsilon(x) - u_0(x)| \leq c\varepsilon$ we deduce that

$$\int_D A^0(x, \nabla u_0) \cdot (\nabla \phi - \nabla w_1^\varepsilon) dx \geq \int_D f(\phi - w_1^\varepsilon) dx - c\varepsilon.$$

for all $\phi \in K$. From (5.22) we deduce

$$\int_D A^\varepsilon(x, \nabla w_1^\varepsilon) \cdot (\nabla \phi - \nabla w_1^\varepsilon) dx \geq \int_D f(\phi - w_1^\varepsilon) dx - c\varepsilon - c\varepsilon^{1/p} \|\nabla \phi - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

Let $\phi = u^\varepsilon$, we have

$$\begin{aligned} & \int_D A^\varepsilon(x, \nabla w_1^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon) dx \\ &\geq \int_D f(u^\varepsilon - w_1^\varepsilon) dx - c\varepsilon - c\varepsilon^{1/p} \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}. \end{aligned} \quad (5.23)$$

As $w_1^\varepsilon \in K$, from (5.3), we have

$$\int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot (\nabla w_1^\varepsilon - \nabla u^\varepsilon) dx \geq \int_D f(w_1^\varepsilon - u^\varepsilon) dx. \quad (5.24)$$

From (5.23) and (5.24),

$$\int_D (A^\varepsilon(x, \nabla w_1^\varepsilon) - A^\varepsilon(x, \nabla u^\varepsilon)) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon) dx \geq -c\varepsilon - c\varepsilon^{1/p} \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}$$

so

$$\int_D (A^\varepsilon(x, \nabla u^\varepsilon) - A^\varepsilon(x, \nabla w_1^\varepsilon)) \cdot (\nabla u^\varepsilon - \nabla w_1^\varepsilon) dx \leq c\varepsilon + c\varepsilon^{1/p} \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

From the monotone condition (5.1), we have

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}^p \leq c\varepsilon + c\varepsilon^{1/p} \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)}.$$

Therefore

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

From (5.21), we have that

$$\|\nabla w_1^\varepsilon - \nabla u_0 - \nabla_y u_1(\cdot, \frac{\cdot}{\varepsilon})\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

We then get the conclusion. $\square$

To construct a numerical corrector, we employ the following operator.

For $\Phi \in L^1(\Omega \times Y)$ we define

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}\right) dz, \quad (5.25)$$

where $[x/\varepsilon]$ denotes the “integer” part of x/ε with respect to the unit cube Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$. The operator $\mathcal{U}^\varepsilon$ satisfies:

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x) dx = \int_{D \times Y} \Phi(x, y) dy dx, \quad (5.26)$$

for all $\Phi \in L^1(D \times Y)$, where D^ε is the 2ε neighbourhood of D ; Φ is regarded as 0 when x is outside D . The proof of this proposition is quite straightforward; we refer to [29] for details. We then have the following corrector result.

Theorem 5.13. Assume that $u_0 \in C^2(\bar{D})$ and $u_1 \in C^1(\bar{D}, C^1(\bar{Y}))$. For the solution of the full tensor product FE approximation in (5.7), we have

$$\|\nabla u^\varepsilon(\cdot) - [\nabla u_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y u_1^L(\cdot))]\|_{L^p(D)} \leq c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)}).$$

Proof. As $u_1 \in C^1(\bar{D}, C^1(\bar{Y}))$, we have from (5.25)

$$\left| \nabla_y u_1(x, \frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u_1)(x) \right| \leq c\varepsilon.$$

for all $x \in D$. Therefore

$$\|\nabla u^\varepsilon(x) - \nabla u_0(x) - \mathcal{U}^\varepsilon(\nabla_y u_1)(x)\|_{L^p(D)} \leq c\varepsilon^{1/(p(p-1))}.$$

For all functions $\Phi \in L^p(D \times Y)$, from the Cauchy-Schwartz inequality, we have

$$(\mathcal{U}^\varepsilon(\Phi)(x))^p \leq \mathcal{U}^\varepsilon(\Phi^p)(x).$$

Therefore, from (5.26)

$$\|\mathcal{U}^\varepsilon(\Phi)\|_{L^p(D)} \leq \|\Phi\|_{L^p(D \times Y)}.$$

Thus

$$\begin{aligned} & \|\mathcal{U}^\varepsilon(\nabla_y u_1)(x) - \mathcal{U}^\varepsilon(\nabla_y u_1^L)(x)\|_{L^p(D)} \\ & \leq \|\nabla_y u_1(x, y) - \nabla_y u_1^L(x, y)\|_{L^p(D \times Y)} \leq ch_L^{1/(p-1)}. \end{aligned}$$

We then get the result. $\square$

Similarly, for the solution of the sparse tensor product FE approximation problem (5.15), we have

Theorem 5.14. Assume that $u_0 \in C^2(\bar{D})$ and $u_1 \in C^1(\bar{D}, C^1(\bar{Y}))$. For the solution of the sparse tensor product FE approximation in (5.15), we have

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^2(D)} \leq c(\varepsilon^{1/2} + L^{1/2}h_L)$$

when $p = 2$; and when $p > d$

$$\|\nabla u^\varepsilon(\cdot) - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^p(D)} \leq c(\varepsilon^{1/(p(p-1))} + L^{1/(p-1)}h_L^{1/(p-1)}).$$

The proof of this theorem is indential to that of the previous one.

5.3.2 Numerical corrector for the multiscale problem

For general problems with more than two scales, we deduce a numerical corrector without an explicit error. We assume that $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \dots, n$. As in Chapter 3, we define the operator $\mathcal{T}_n^\varepsilon : L^1(D) \rightarrow L^1(D \times \mathbf{Y})$ as:

$$\mathcal{T}_n^\varepsilon(\phi)(x, \mathbf{y}) = \phi\left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_2 \left\lfloor \frac{y_1}{\varepsilon_2/\varepsilon_1} \right\rfloor + \dots + \varepsilon_n \left\lfloor \frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}} \right\rfloor + \varepsilon_n y_n\right)$$

where the function ϕ is understood as 0 outside D , and $\lfloor \cdot \rfloor$ denotes the “integer” part with respect to Y . For all functions $\phi \in L^1(D)$ which are understood as 0 outside D , we have

$$\int_D \phi dx = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\mathbf{y} dx. \quad (5.27)$$

where D^ε is the 2ε neighbourhood of D . We can also show that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n \text{ in } L^p(D \times \mathbf{Y}). \quad (5.28)$$

We recall the operator $\mathcal{U}_n^\varepsilon : L^1(D \times \mathbf{Y}) \rightarrow L^1(D)$ defined in Chapter 3

$$\begin{aligned} \mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \dots \int_{Y_n} \Phi\left(\varepsilon_1 \left\lfloor \frac{x}{\varepsilon_1} \right\rfloor + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1} \left\lfloor \frac{\varepsilon_1}{\varepsilon_2} \left\{ \frac{x}{\varepsilon_1} \right\} \right\rfloor + \frac{\varepsilon_2}{\varepsilon_1} t_2, \dots, \right. \\ \left. \frac{\varepsilon_n}{\varepsilon_{n-1}} \left\lfloor \frac{\varepsilon_{n-1}}{\varepsilon_n} \left\{ \frac{x}{\varepsilon_{n-1}} \right\} \right\rfloor + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \left\{ \frac{x}{\varepsilon_n} \right\} \right) dt_n \dots dt_1 \end{aligned} \quad (5.29)$$

where $\{\cdot\} = \cdot - \lfloor \cdot \rfloor$; the function Φ is understood as 0 outside D . We have $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\Phi)) = \Phi \forall \Phi \in L^1(D)$. Further,

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) dx = \int_D \int_{\mathbf{Y}} \Phi d\mathbf{y} dx \quad (5.30)$$

where Φ is understood as 0 outside D . The proofs of these facts can be found in [29].

We then have

Proposition 5.15. Under conditions (5.1) and (5.2), we have

$$\lim_{\varepsilon \rightarrow 0} \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n)]\|_{L^p(D)} = 0.$$

Proof. As $\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \dots + \nabla_{y_n} u_n$ in $L^p(D)$,

$$\lim_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) \cdot (\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)) d\mathbf{y} dx = 0. \quad (5.31)$$

Let $\phi_0 \in C_0^\infty(D)$, $\phi_i \in C_0^\infty(D, C_\#^\infty(Y_1, \dots, C_\#^\infty(Y_i) \dots))$ for $i = 1, \dots, n$. Let $\chi \in C_0^\infty(D)$ which is nonnegative and equal to 1 in the support of ϕ_i with respect to x (which does not depend on ε). For $\delta > 0$, we have

$$\phi^\varepsilon(x) = \phi_0(x) + \sum_{i=1}^n \varepsilon_i \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) + \delta \chi(x) \in K$$

when ε is sufficiently small. Therefore from (5.3)

$$\begin{aligned} & \int_D A^\varepsilon(x, \nabla u^\varepsilon) \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i})) \\ & \quad + \sum_{i=1}^n \sum_{j=1}^{i-1} \frac{\varepsilon_i}{\varepsilon_j} \nabla_{y_j} \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) + \delta \nabla \chi - \nabla u^\varepsilon dx \\ & \geq \int_D f(\phi_0 + \sum_{i=1}^n \varepsilon_i \phi_i(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_i}) + \delta \chi - u^\varepsilon) ds. \end{aligned}$$

Using (5.27) and the fact that

$$\mathcal{T}_n^\varepsilon(\nabla \phi_0) \rightarrow \nabla \phi_0 \text{ in } L^p(D \times \mathbf{Y}),$$

and

$$\mathcal{T}_n^\varepsilon(\nabla_{y_i} \phi_i(\cdot, \frac{\cdot}{\varepsilon_1}, \dots, \frac{\cdot}{\varepsilon_i})) \rightarrow \nabla_{y_i} \phi_i(x, y_1, \dots, y_i),$$

which can be shown by the smoothness of $\phi_0, \phi_1, \dots, \phi_n$, we have

$$\liminf_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon(x, \nabla u^\varepsilon))$$

$$\begin{aligned}
& \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, y_1, \dots, y_n) \\
& + \delta \nabla \chi(x) - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) d\mathbf{y} dx \\
& \geq \int_D f(\phi_0 + \delta \chi - u_0) dx.
\end{aligned}$$

As this holds for all $\delta > 0$, we have

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon(x, \nabla u^\varepsilon)) \\
& \quad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, y_1, \dots, y_n) - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) d\mathbf{y} dx \\
& \geq \int_D f(\phi_0 - u_0) dx.
\end{aligned}$$

Using density, we can replace ϕ_0 by u_0 and ϕ_i by u_i for $i = 1, \dots, n$. Further, from the smoothness of A and condition (5.2), we have

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} (\mathcal{T}_n^\varepsilon(A^\varepsilon(x, \nabla u^\varepsilon)) - A(x, \mathbf{y}, \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon))) \\
& \quad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i(x, y_1, \dots, y_n) - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) d\mathbf{y} dx = 0
\end{aligned}$$

Therefore,

$$\liminf_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} A(x, \mathbf{y}, \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) d\mathbf{y} dx \geq 0$$

This together with (5.31) implies

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \int_D \int_{\mathbf{Y}} (A(x, \mathbf{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i) - A(x, \mathbf{y}, \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon))) \\
& \quad \cdot ((\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i - \mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) d\mathbf{y} dx \leq 0.
\end{aligned}$$

From (5.1) we get

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) - [\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i]\|_{L^p(D \times \mathbf{Y})} = 0.$$

As $\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) = \nabla u^\varepsilon$, from (5.30), we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \|\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon)) - [\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i]\|_{L^p(D)} \\ &= \lim_{\varepsilon \rightarrow 0} \|\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) - \mathcal{U}_n^\varepsilon(\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)\|_{L^p(D \times \mathbf{Y})} \\ &= 0. \end{aligned}$$

As $\lim_{\varepsilon \rightarrow 0} \|\mathcal{U}_n^\varepsilon(\nabla u_0) - \nabla u_0\|_{L^p(D)} = 0$, we get the conclusion. $\square$

As an error of convergence for the corrector is not available, we do not distinguish the two cases of full and sparse tensor product FE approximations. We denote the FE solutions as u_0^L and u_i^L generally. We then have:

Theorem 5.16. Under conditions (5.1) and (5.2), we have:

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ L \rightarrow \infty}} \|\nabla u^\varepsilon - [\nabla u_0^L + \sum_{i=1}^n \mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i^L)]\|_{L^p(D)} = 0.$$

Proof. From the definition (5.29), we have that $\mathcal{U}_n^\varepsilon(\Phi)^p \leq \mathcal{U}_n^\varepsilon(\Phi^p)$ so

$$|(\mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i - \nabla_{y_i} u_i^L)(x))|^p \leq \mathcal{U}_n^\varepsilon(|\nabla_{y_i} u_i - \nabla_{y_i} u_i^L|^p)(x).$$

Thus from (5.30) we get

$$\begin{aligned} & \|\mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i - \nabla_{y_i} u_i^L)\|_{L^p(D)} \leq \|\mathcal{U}_n^\varepsilon(|\nabla_{y_i} u_i - \nabla_{y_i} u_i^L|^p)\|_{L^1(D)} \\ & \leq \|\nabla_{y_i} u_i - \nabla_{y_i} u_i^L\|_{L^p(D \times \mathbf{Y})} \leq ch_L^{1/(p-1)}. \end{aligned}$$

Therefore

$$\begin{aligned} & \|\nabla u^\varepsilon - [\nabla u_0^L + \sum_{i=1}^n \mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i^L)]\|_{L^p(D)} \\ & \leq \|\nabla u^\varepsilon - [\nabla u_0 + \sum_{i=1}^n \mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i)]\|_{L^p(D)} + \|\nabla u_0 - \nabla u_0^L\|_{L^p(D)} \\ & \quad + \|\sum_{i=1}^n \mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i) - \sum_{i=1}^n \mathcal{U}_n^\varepsilon(\nabla_{y_i} u_i^L)\|_{L^p(D)} \end{aligned}$$

which converges to 0 when $\varepsilon \rightarrow 0$ and $L \rightarrow \infty$.

The conclusion then follows. $\square$

5.4 Numerical examples

We solve some two scale variational inequalities in one and two dimensional domains to illustrate the theoretical results.

We first consider a problem on the one dimensional domain $D = (0, 1)$ where the monotone nonlinear function

$$\begin{aligned} A(x, y, \xi) = & \xi + \frac{1}{2} \sin(x \sin(2\pi y) \xi) - x \cos(2\pi y) \\ & - \frac{1}{2} \sin(x \sin(2\pi y) (1 - 2x + x \cos(2\pi y))) \end{aligned}$$

and $f(x) = 2$. In this case $p = 2$. The problem has the exact solution

$$u_0(x) = x - x^2$$

and

$$u_1(x, y) = \frac{1}{2\pi} x \sin(2\pi y).$$

This variational problem with constrain corresponds to an optimization problem with constraint conditions in the convex set $\mathcal{K}$ ([27]). We therefore use the relaxation method ([27] Section 8.6) to solve the variational inequality problem (5.15). In Figures 5.1 and 5.2 we plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ and $\|u_1 - \hat{u}_1\|_{L^2(D, H_{\#}^1(Y)/\mathbb{R})}$ respectively. The numerical results agree with the theoretical results.

Figure 5.1: The $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ error versus the mesh size h .

Figure 5.2: The $\|u_1 - \hat{u}_1\|_{L^2(D, H_{\#}^1(Y)/\mathbb{R})}$ error versus the mesh size h .

We then consider the problem on $D = (0, 1)$ with the monotone function

$$A(x, y, \xi) = \xi + \frac{1}{10}(2 + \sin(2\pi y))\xi^3 - x \cos(2\pi y) - \frac{1}{10}(2 + \sin(2\pi y))(1 - 2x + x \cos(2\pi y))^3$$

and $f(x) = 2$. In this case $p = 4$. This problem has the exact solution

$$u_0(x) = x - x^2$$

and

$$u_1(x, y) = \frac{1}{2\pi}x \sin(2\pi y).$$

We again use the relaxation method to find the minimizer of the corresponding optimization problem with constraints. In Figures 5.3 and 5.4, we plot the error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ and $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y)/\mathbb{R})}$ respectively. The numerical results show that these errors behave like $O(h_L)$, i.e. the theoretical results are pessimistic. This is in agreement with the well known fact in solving (single macroscopic scale) monotone problems that the observed numerical error is the optimal $O(h_L)$ rate which is far better than what we can show theoretically, see, e.g., [48].

Figure 5.3: The $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ error versus the mesh size h .

Figure 5.4: The $\|u_1 - \hat{u}_1\|_{L^4(D, W_{\#}^{1,4}(Y)/\mathbb{R})}$ error versus the mesh size h .

For the two dimensional domain $D = (0, 1) \times (0, 1)$, we consider monotone function

$$\begin{aligned}
 & A(x, y, \xi) \\
 = & \xi' + \xi'' + \frac{1}{20}x'x'' \sin(2\pi y'') \sin(\xi') + \frac{1}{20}x'x'' \sin(2\pi y') \sin(\xi'') \\
 & - (x' + x'')(\cos(2\pi y') \sin(2\pi y'') + \sin(2\pi y') \cos(2\pi y'')) \\
 & - \frac{1}{20}x'x'' \sin(2\pi y'') \sin((1 - 2x')(x'' - x''^2)) \\
 & - \frac{1}{20}x'x'' \sin(2\pi y') \sin((x' - x'^2)(1 - 2x''))
 \end{aligned}
 ,$$

and

$$f(x) = 2(x' + x'' - x'^2 - x''^2)$$

with $x = (x', x'') \in D$, $y = (y', y'') \in Y$ and $\xi = (\xi', \xi'') \in \mathbb{R}^2$. In this case $p = 2$. This problem has the exact solution

$$u_0(x) = (x' - x'^2)(x'' - x''^2)$$

and

$$u_1(x, y) = \frac{1}{2\pi}(x' + x'') \sin(2\pi y') \sin(2\pi y'')$$

In Figures 5.5 and 5.6, we plot the error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ and $\|u_1 - \hat{u}_1\|_{L^2(D, H_{\#}^1(Y))}$ respectively. The numerical results show that these errors behave like $O(h_L)$.

Figure 5.5: The $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ error versus the mesh size h .

Figure 5.6: The $\|u_1 - \hat{u}_1\|_{L^2(D, H_{\#}^1(Y))}$ error versus the mesh size h .

References

- [1] A. Abdulle, W. E, B. Engquist, and E. Vanden-Eijnden. The heterogeneous multiscale method. *Acta Numerica*, 21:1–87, 5 2012.
- [2] A. Abdulle and M. E. Huber. Error estimates for finite element approximations of nonlinear monotone elliptic problems with application to numerical homogenization. *Numer. Methods Partial Differential Equations*, 32(3):955–969, 2016.
- [3] A. Abdulle, M. E. Huber, and G. Vilmart. Linearized numerical homogenization method for nonlinear monotone parabolic multiscale problems. *Multiscale Model. Simul.*, 13(3):916–952, 2015.
- [4] D. S. Ahluwalia, E. L. Reiss, and S. E. Stone. A uniform asymptotic analysis of dispersive wave motion. *Arch. Rational Mech. Anal.*, 54:340–355, 1974.
- [5] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.
- [6] G. Allaire and M. Briane. Multiscale convergence and reiterated ho-

- mogenisation. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(2):297–342, 1996.
- [7] T. Arbogast. Computational aspects of dual-porosity models. In *Homogenization and porous media*, volume 6 of *Interdiscip. Appl. Math.*, pages 203–223, 259–275. Springer, New York, 1997.
- [8] T. Arbogast, J. Douglas, Jr., and U. Hornung. Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.*, 21(4):823–836, 1990.
- [9] H. Attouch and C. Picard. Variational inequalities with varying obstacles: the general form of the limit problem. *J. Funct. Anal.*, 50(3):329–386, 1983.
- [10] I. Babuvska. Solution of interface problems by homogenization. III. *SIAM J. Math. Anal.*, 8(6):923–937, 1977.
- [11] N. Bakhvalov and G. Panasenko. *Homogenisation: Averaging Processes in Periodic Media: Mathematical Problems in the Mechanics of Composite Materials*. Mathematics and its Applications. Springer Netherlands, 2012.
- [12] J. W. Barrett and W. B. Liu. Finite element approximation of the parabolic p -Laplacian. *SIAM J. Numer. Anal.*, 31(2):413–428, 1994.
- [13] G. Bayada, B. Cid, and C. Vázquez. Two-scale homogenization study of a Reynolds-rod elastohydrodynamic model. *Math. Models Methods Appl. Sci.*, 13(2):259–293, 2003.
- [14] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. Sur quelques

- phénomènes asymptotiques d'évolution. *C. R. Acad. Sci. Paris Sér. A-B*, 281(10):Ai, A317–A322, 1975.
- [15] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. Homogénéisation, correcteurs et problèmes non-linéaires. *C. R. Acad. Sci. Paris Sér. A-B*, 282(22):Aii, A1277–A1282, 1976.
- [16] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.
- [17] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou. Boundary layers and homogenization of transport processes. *Publ. Res. Inst. Math. Sci.*, 15(1):53–157, 1979.
- [18] C. Bernardi and V. Girault. A local regularization operator for triangular and quadrilateral finite elements. *SIAM J. Numer. Anal.*, 35(5):1893–1916, 1998.
- [19] A. Braides and A. Defranceschi. *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998.
- [20] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics. Springer, 2008.
- [21] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.
- [22] R. L. Burden, J. D. Faires, and A. M. Burden. *Numerical analysis*. Boston, MA : Cengage Learning, [2016].

-
- [23] Z. Chen and T. Y. Hou. A mixed multiscale finite element method for elliptic problems with oscillating coefficients. *Math. Comp.*, 72(242):541–576 (electronic), 2003.
- [24] V. Chiadò Piat and A. Defranceschi. Homogenization of monotone operators. *Nonlinear Anal.*, 14(9):717–732, 1990.
- [25] S.-S. Chow. Finite element error estimates for nonlinear elliptic equations of monotone type. *Numer. Math.*, 54(4):373–393, 1989.
- [26] E. T. Chung, Y. Efendiev, R. L. Gibson, and M. Vasilyeva. A generalized multiscale finite element method for elastic wave propagation in fractured media. *GEM - International Journal on Geomathematics*, 7(2):163–182, 2016.
- [27] P. G. Ciarlet. *Introduction to numerical linear algebra and optimisation*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1989. With the assistance of Bernadette Miara and Jean-Marie Thomas, Translated from the French by A. Buttigieg.
- [28] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].
- [29] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis*, 40(4):1585–1620, 2008.
- [30] G. Dal Maso. Integral representation on $BV(\Omega)$ of Γ -limits of variational integrals. *Manuscripta Math.*, 30(4):387–416, 1979/80.

-
- [31] G. Dal Maso. *An Introduction to Γ -Convergence*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, 1993.
- [32] G. Dal Maso and A. Defranceschi. Correctors for the homogenization of monotone operators. *Differential Integral Equations*, 3(6):1151–1166, 1990.
- [33] G. Dal Maso and P. Trebeschi. Γ -limit of periodic obstacles. *Acta Appl. Math.*, 65(1-3):207–215, 2001. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday.
- [34] J. Douglas, Jr. and T. Dupont. Galerkin methods for parabolic equations. *SIAM J. Numer. Anal.*, 7:575–626, 1970.
- [35] W. E and B. Engquist. The heterogeneous multiscale methods. *Commun. Math. Sci.*, 1(1):87–132, 2003.
- [36] W. E and B. Engquist. Multiscale modeling and computation. *Notices Amer. Math. Soc.*, 50(9):1062–1070, 2003.
- [37] Y. Efendiev, J. Galvis, and T. Y. Hou. Generalized multiscale finite element methods (GMsFEM). *Journal of Computational Physics*, 251:116 – 135, 2013.
- [38] Y. Efendiev and T. Y. Hou. *Multiscale Finite Element Methods: Theory and Applications*. Surveys and Tutorials in the Applied Mathematical Sciences. Springer, 2009.
- [39] Y. Efendiev and A. Pankov. Numerical homogenization of monotone elliptic operators. *Multiscale Model. Simul.*, 2(1):62–79, 2003.

- [40] Y. Efendiev and A. Pankov. Numerical homogenization and correctors for nonlinear elliptic equations. *SIAM J. Appl. Math.*, 65(1):43–68, 2004.
- [41] Y. Efendiev and A. Pankov. Numerical homogenization of nonlinear random parabolic operators. *Multiscale Model. Simul.*, 2(2):237–268, 2004.
- [42] K. Gao, S. Fu, R. L. Gibson, E. T. Chung, and Y. Efendiev. Generalized multiscale finite-element method (GMsFEM) for elastic wave propagation in heterogeneous, anisotropic media. *Journal of Computational Physics*, 295:161 – 188, 2015.
- [43] J. Geng and Z. Shen. Convergence rates in parabolic homogenization with time-dependent periodic coefficients. *J. Funct. Anal.*, 272(5):2092–2113, 2017.
- [44] A. Gloria. An analytical framework for the numerical homogenization of monotone elliptic operators and quasiconvex energies. *Multiscale Model. Simul.*, 5(3):996–1043, 2006.
- [45] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.
- [46] F. Hellman and A. Malqvist. Numerical homogenization of time-dependent diffusion. *Computing Research Repository*, (1703.08857).
- [47] P. Henning and M. Ohlberger. Error control and adaptivity for heterogeneous multiscale approximations of nonlinear monotone problems. *Discrete Contin. Dyn. Syst. Ser. S*, 8(1):119–150, 2015.

- [48] V. H. Hoang. Sparse finite element method for periodic multiscale nonlinear monotone problems. *Multiscale Modeling & Simulation*, 7(3):1042–1072, 2008.
- [49] V. H. Hoang and C. Schwab. High-dimensional finite elements for elliptic problems with multiple scales. *Multiscale Model. Simul.*, 3(1):168–194, 2004/05.
- [50] A. Holmbom, N. Svanstedt, and N. Wellander. Multiscale convergence and reiterated homogenization of parabolic problems. *Appl. Math.*, 50(2):131–151, 2005.
- [51] T. Y. Hou and X.-H. Wu. A multiscale finite element method for elliptic problems in composite materials and porous media. *Journal of Computational Physics*, 134(1):169 – 189, 1997.
- [52] T. Y. Hou, X.-H. Wu, and Z. Cai. Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients. *Math. Comp.*, 68(227):913–943, 1999.
- [53] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. [electronic resource]. Berlin, Heidelberg : Springer Berlin Heidelberg, 1994.
- [54] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.
- [55] J. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*. Springer Berlin Heidelberg, 1972.

-
- [56] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.
- [57] J. L. Lions, D. Lukkassen, L. E. Persson, and P. Wall. Reiterated homogenization of nonlinear monotone operators. *Chinese Annals of Mathematics*, 22(1):1, 2001.
- [58] A. Malqvist and A. Persson. Multiscale techniques for parabolic equations. *Numerische Mathematik*, Jul 2017.
- [59] A. Malqvist and D. Peterseim. Localization of elliptic multiscale problems. *Math. Comp.*, 83(290):2583–2603, 2014.
- [60] J. C. Maxwell. *A Treatise on Electricity and Magnetism*. A Treatise on Electricity and Magnetism. Volume 2. Clarendon Press, 1873.
- [61] G. W. Milton. *The theory of composites*, volume 6 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.
- [62] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.
- [63] F. Murat. *H-Convergence*. Séminaire d’Analyse Fonctionnelle et Numérique de l’Université d’Alger, Algiers, Algeria, 1978.
- [64] F. Murat and L. Tartar. *H-convergence*. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.

-
- [65] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.
- [66] G. Nguetseng and J. L. Woukeng. Σ -convergence of nonlinear parabolic operators. *Nonlinear Anal.*, 66(4):968–1004, 2007.
- [67] M. Ohlberger. A posteriori error estimates for the heterogeneous multiscale finite element method for elliptic homogenization problems. *Multiscale Modeling & Simulation*, 4(1):88–114, 2005.
- [68] H. Owhadi and L. Zhang. Homogenization of parabolic equations with a continuum of space and time scales. *SIAM Journal on Numerical Analysis*, 46(1):1–36, 2008.
- [69] G. C. Papanicolaou and S. R. S. Varadhan. Diffusions with random coefficients. In *Statistics and probability: essays in honor of C. R. Rao*, pages 547–552. North-Holland, Amsterdam, 1982.
- [70] D. Peterseim and S. Sauter. Finite elements for elliptic problems with highly varying, nonperiodic diffusion matrix. *Multiscale Model. Simul.*, 10(3):665–695, 2012.
- [71] J. Rayleigh. On the influence of obstacles arranged in rectangular order upon the properties of a medium. *Philosophical Magazine Series 5*, 34(211):481–502, 1892.
- [72] E. Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin-New York, 1980.

-
- [73] G. V. Sandrakov. Homogenization of variational inequalities for problems with an obstacle. *Mat. Sb.*, 196(4):79–98, 2005.
- [74] N. Svanstedt, N. Wellander, and J. Wyller. A numerical algorithm for nonlinear parabolic equations with highly oscillating coefficients. *Numer. Methods Partial Differential Equations*, 12(4):423–440, 1996.
- [75] J. Wloka. *Partial Differential Equations*. Cambridge University Press, 1987.
- [76] J. L. Woukeng. Periodic homogenization of nonlinear non-monotone parabolic operators with three time scales. *Ann. Mat. Pura Appl. (4)*, 189(3):357–379, 2010.
- [77] E. Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.