Robust model predictive control of nonlinear and time-delay systems

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NANYANG
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**ROBUST MODEL PREDICTIVE CONTROL OF
NONLINEAR AND TIME-DELAY SYSTEMS**

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ENERGY RESEARCH INSTITUTE @ NTU (ERI@N)**

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Robust Model Predictive Control of Nonlinear and Time-Delay Systems

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University in partial fulfilment of the requirement for
the degree of
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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Student Name

Abstract

Industrial systems usually suffer from severe nonlinearities and large time delays, which may cause degradation of the control performance or even instability of the whole system. Control of such systems is not an easy problem especially if the two characteristics exist simultaneously. The thesis is to develop advanced control methodologies for such systems.

This thesis is divided into three parts: (1) robust model predictive control of nonlinear systems modeled as T-S fuzzy systems with nonlinear local models; (2) robust fuzzy model predictive control of nonlinear systems with time delays; (3) model reference tracking control of networked systems subjected to time delays and package dropouts, and its practical application to a linear brushless dc motor.

In the first part of this thesis, robust model predictive control of a recently developed T-S fuzzy system that relies on nonlinear local models is investigated. Due to the advantage of the T-S fuzzy system with nonlinear local models, the computational burden of online optimization for model predictive control is decreased when compared to an existing fuzzy model predictive control approach of T-S systems with linear local models.

In the second part of this thesis, we derive robust model predictive control of nonlinear time-delay systems, with a special focus on the Lyapunov Razumikhin function rather than the Lyapunov Krasovskii functional. Both online and efficient off-line model predictive control algorithms are considered for systems with multiple delays and time-varying delays, respectively. We show that the proposed methods have better performances as well as computational advantages over several existing model predictive control approaches for time-delay systems.

In the third part, taking the advantages of the Lyapunov Razumikhin approach in dealing with systems with time-varying delays which is also mentioned in the second part, model reference tracking control of a networked linear brushless dc

motor is investigated, with both networked induced communication delays and data package dropouts involved. Simulation and experimental results are provided to verify the effectiveness of the proposed method.

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List of Symbols

The following list only contains symbols which are used throughout the thesis. Symbols which are defined locally are not listed below.

Acronyms

MPC	model predictive control
RMPC	robust model predictive control
FMPC	fuzzy model predictive control
RFMPC	robust fuzzy model predictive control
NMPC	nonlinear model predictive control
CLF	control Lyapunov function
LMI	linear matrix inequality
BMI	bilinear matrix inequality
CSTR	continuous stirred tank reactor
DDE	delay difference equation
DDI	delay difference inclusion
RPI	robust positive invariance
ISS	input-to-state stability
LKF	Lyapunov-Krasovskii functional
LRF	Lyapunov-Razumikhin function
NCS	networked control system
LBDCM	linear brushless dc motor

Sets

$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non-negative reals
$\mathbb{Z}$	set of integers

$\mathbb{Z}_+$	set of non-negative integers
$\mathbb{Z}_{[m,n]}$	set of integers in the interval $[m,n]$
$\mathbb{N}$	set of natural numbers
$\mathbb{R}^n$	n -dimensional Euclidean space
$\mathbf{x}_{[c1,c2]}$	with $c1, c2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c1,c2]}$, i.e., $\mathbf{x}_{[c1,c2]} := \{x(l)\}_{l \in \mathbb{Z}_{[c1,c2]}}$
$x \in \Omega_{a1} \setminus \Omega_{a2}$	$x \in \Omega_{a1}$ and $x \notin \Omega_{a2}$
$\mathcal{K}$	A function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$
$\mathcal{K}_\infty$	A function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$
$\mathcal{KL}$	The function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is strictly decreasing and $\lim_{s \rightarrow \infty} \beta(r, s) = 0$

Vectors, Matrices, and Norms

I	identity matrix
0	matrix of zeroes
A^T	transpose of matrix A
$\begin{bmatrix} A & * \\ B & C \end{bmatrix}$	$*$ denotes the symmetric part of a matrix, i.e.,
$\begin{bmatrix} A & * \\ B & C \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$	
$\ x\ $	Euclidean norm of a vector $x \in \mathbb{R}^n$
$A > 0, (A \geq 0)$	matrix A is positive (semi-)definite

$A < 0, (A \leq 0)$ matrix A is negative (semi-)definite

$\rho_{\max}(A)$ largest real part of the eigenvalues of matrix A

$\rho_{\min}(A)$ smallest real part of the eigenvalues of matrix A

Chapter 1

Introduction

In this chapter, the literature review on model predictive control, time-delays systems, and networked control systems is provided. Then, an overview of this thesis as well as the summary of contributions is outlined.

1.1 Model Predictive Control

Model predictive control (MPC) has attracted many research interests from both the academic community and the industry for decades. Usually, at each sampling instant, an optimal control sequence are computed via optimization of a finite horizon cost function under some constraints, and stability and recursive feasibility are guaranteed [1]. Then the first control law is merely implemented, and the optimization is repeated at the next instant with the newly obtained system information. For the context of robust model predictive control (RMPC), extensive works have been developed focusing on several aspects in the past two decades, e.g., that with disturbances [2]-[4], with parametric uncertainties [5]-[7], and with both [8]-[10]. On the other hand, the T-S fuzzy model is firstly proposed in [11] and has been widely investigated for the control of nonlinear systems [12]-[19].

The industrial processes are mostly nonlinear systems. Designing predictive controllers for such systems has attracted many research interests that several different MPC strategies have been developed. One method is to directly deal with nonlinear systems. However, when the nonlinearities are very complex, nonlinear model predictive control (NMPC) may not be able to execute effectively [20]. In addition, the combination of MPC and T-S fuzzy systems has become a very good alternative. Several T-S fuzzy control techniques have been investigated for fuzzy model predictive control (FMPC), e.g., common and piecewise Lyapunov functions [21], and parallel distributed compensation (PDC) and non-PDC techniques [22], [23]. Especially, structural uncertainties and persistent disturbances are considered simultaneously in [9], with both online and off-line approaches are provided. In [24], FMPC of nonlinear systems modeled as stochastic T-S fuzzy models subjected to data package loss in a network environment is investigated. Output feedback predictive control of T-S systems has been considered in [25], [26], in case that the state information cannot be measured. In [27], the cooperative fuzzy model predictive control is

investigated for input-coupled multi-agent systems. In addition, MPC with a hybrid T-S fuzzy model is investigated in [28]-[29], and the results show its advantages over the approach which adopts a hybrid linear model. In summary, the motivations of FMPC are that, compared with traditional fuzzy control algorithms, the implementation of MPC to T-S fuzzy systems provides a systematic method to handle constraints and calculates optimal control inputs at each sampling instant by optimizing the cost function.

Many contributions have been achieved to reduce the conservativeness of traditional T-S fuzzy controllers. As it is mentioned above that, which has already been utilized in FMPC, the piecewise Lyapunov functions are developed in replace of common Lyapunov functions to obtain less conservative results [30]. Compared with the PDC technique for T-S fuzzy systems [31], the control performance can be improved by adopting non-PDC control laws and nonquadratic Lyapunov functions [32]. T-S fuzzy system with imperfect premise matching is investigated in [33], the computational cost induced by the complexity of fuzzy controller under the PDC technique is alleviated. Moreover, in contrast to the conventional fuzzy control methods which adopt T-S fuzzy systems with linear local models (LLMs), a new T-S fuzzy modeling and control approach using nonlinear local models (NLMs) and nonlinear local controllers is developed in [34] for continuous-time nonlinear systems and extended to the discrete-time counterpart in [35]. Compared with the LLM-based T-S fuzzy approach, a less conservative result can be obtained via the NLM-based fuzzy approach. At the same time, the computational cost can be significantly reduced. In addition, nonlinear systems are modeled as polynomial fuzzy systems other than the T-S method [36], and the sum-of-squares (SOS) techniques are utilized, which shows that stability analysis and control design in terms of SOS are more general and relaxed than those of T-S approaches using matrix inequalities.

Therefore, extensive results have been achieved to improve control performance and reduce computational cost of the classical T-S fuzzy control

methods. However, for the context of FMPC, although some techniques have been discussed to reduce conservatism, such as the utilization of piecewise Lyapunov functions and non-PDC techniques for FMPC in [21] and [22], there still are some drawbacks for existing FMPC approaches that restrict their implementations in practical applications. It is noted that the aforementioned FMPC methods only adopt T-S fuzzy systems with LLMs. Based on which, MPC is designed for each LLM and the fuzzy blendings of each control law are also taken into account. Under such circumstances, one problem is the conservatism due to the LLM-based T-S fuzzy models especially for systems with severe nonlinearities. Moreover, it may result in a heavy computational burden due to the structural complexity of the fuzzy controller by solving the matrix inequalities problem. In particular, as demonstrated in [9], if disturbances and uncertainties are involved in RMPC, the whole computational burden of the optimization problem may become even worse. As a result, it may not be appropriate for real-time application considering the sampling restrictions. As such, it is worthwhile to investigate more efficient FMPC algorithms.

1.2 Time-delay Systems

Time delay can be frequently seen in many industrial systems, see [37] and references therein. Since time delay can cause degradation of the control performance or even instability of the whole systems, it has attracted extensive research interests. Utilization of Lyapunov theory for time-delay systems can be generally categorized as two different approaches, the Lyapunov-Krasovskii functional (LKF) and the Lyapunov-Razumikhin function (LRF) [38]. An LKF investigates the stability of a high dimensional augmented system where the state vectors are with all delayed states [39], [40]. Compared with LKF, although LRF is conservative, it reflects the Lyapunov function for original non-augmented systems, thus the set theory of positive invariance for the

original systems can be directly employed [41]-[43]. As such, LRF-based stability analysis and stabilization synthesis approaches attract some research interests in recent years [42]-[45].

As a kind of optimization control methods, model predictive control (MPC) has been widely investigated and utilized in industrial process systems. However, only a few works can be seen on MPC for time-delay systems. Early results on related problems can be seen in [46]-[48] where only LKF is adopted. In recent years, some progresses have been achieved for continuous- and discrete-time linear and nonlinear systems, respectively. LRF-based MPC method is proposed for linear delay difference inclusions (DDIs) with polytopic uncertainties in [44]. For the nonlinear case, investigations of continuous time-delay systems can be seen in [49]-[53]. Particularly, in [51], a combination of LRF and LKF conditions is adopted to decide the terminal constraint set and terminal cost function, respectively. The economic MPC of nonlinear systems with state and input delay is investigated in [53].

However, in the aforementioned works on MPC of time-delay systems, the disturbance has not been considered, despite that it exists extensively in practical applications. When persistent and bounded disturbances are involved, the traditional asymptotical stability to the origin cannot be achieved, instead, a robust constraint set is adopted so that the system state is converged to this set.

1.3 Networked Control Systems

Networked control systems (NCSs) are a class of distributed systems that the system plant, sensors, controllers and actuators can be located in different places. And communication networks, either wired or wireless, are used to transmit system information and control action among them. Taking the advantages of the prosperous development of information and communication technologies, NCSs have attracted more and more research interests in the past decade. Compared with traditional control systems where the aforementioned

components are put together, it has been recognized that NCSs have a lot of advantages in terms of cost reduction, simple installation and maintenance, higher reliability and greater flexibility [54]. Various applications can be seen on the utilization of NCSs, such as robot manipulator [55], intelligent transportation [56], process control [57], networked surgery [58], and power electronics [59] and so on.

Despite the advantages of NCSs, communication through networks also brings several new and challenging control problems, such as time delays [60], packet dropouts (losses) [61], quantization [62], scheduling [63], and security [64]. Among which, time delays and package dropouts are two main issues that can not be ignored, otherwise, the control performance may be deteriorated or even stability of the whole system is not guaranteed. Considering practical applications of NCSs to motor control, various contributions have been achieved to deal with time delays and package dropouts [65]-[71]. In particular, output tracking control of a networked dc motor subjected to communication delays and package dropouts is investigated in [65] that, by optimizing a cost function with respect to over shoot and settling time, the control performance is improved. In [67] and [70], H_2 and H_∞ tracking control methods for NCSs subjected to random stochastic time delays modeled by Markov chains are investigated that, control problems are represented by solving a set of linear matrix inequalities (LMIs). In addition, networked predictive control (NPC) is investigated to compensate the effect of delays and packet dropouts [66], [68], [69]. To reduce traffics in communication networks, event-triggered control scheme is considered for NCSs with one-step package dropout [71], and a switched system comprised of systems with normal transmission and dropouts is adopted. However, to deal with time delay, most of the aforementioned works rely on an augmented system with all possibly delayed states, which may lead to serious complexity in controller synthesis [70], or controller computation [65], [67], [69], [71], especially for systems with large or unknown delays. In addition, the upper bound of delay is required to be known as well.

1.4 Thesis Overview

This thesis addresses control approaches for systems with severe nonlinearities, time delays, and disturbances, with a special focus on developing model predictive control algorithms for such systems. The rest part of this thesis is outlined as follows:

Chapter 2 investigates a computation-efficient fuzzy model predictive control algorithm for nonlinear systems. A combination of T-S fuzzy systems with nonlinear local models and an existing robust fuzzy model predictive control method is investigated that, the computational burden of online optimization is reduced significantly when compared to a typical robust fuzzy model predictive control approach which is based on T-S fuzzy systems with linear local models. And a better control performance can be achieved as well.

Chapter 3 discusses online robust model predictive control algorithms for nonlinear systems with multiple state delays and disturbances. The Lyapunov Razumikhin function, rather than the classical Lyapunov Krasovskii functional, is adopted that the controller synthesis and computation can be greatly simplified especially for systems subjected to multiple delays and disturbances simultaneously. It can be seen that the robust positive invariance and input-to-state stability for such systems can be easily achieved.

As a counterpart of Chapter 3, *Chapter 4* investigates efficient off-line model predictive control approaches for nonlinear systems with multiple state delays and disturbances. By taking the advantage of the Lyapunov Razumikhin function, a robust terminal constraint set and a sequence of robust constraint sets, as well as corresponding control laws, are built in the system's original state space. It is shown that the system state can be steered to the terminal set in finite steps when given an appropriate initial value. However, the specific finite control performance and its rationale is very different from the case without delays. In addition, the robust positive invariance and input-to-state stability for such systems are achieved as well.

Chapter 5 expands the results of Chapter 3 and Chapter 4 to systems with time-varying delays. Both online and efficient off-line model predictive control algorithms are provided for nonlinear systems with time-varying delays and disturbances. And the time-varying delay can be either known or unknown.

Chapter 6 investigates a practical application of the Lyapunov Razumikhin approach to model reference tracking control of networked systems subjected to communication delays and package dropouts. Owing to the Lyapunov Razumikhin approach, the proposed method is delay-independent such that the upperbound of delay is not needed to be known. Communication delays and package dropouts in both forward and backward channels are considered.

Chapter 7 provides conclusions and outlines the directions of future work for this thesis.

1.5 Publications

List of journal papers:

1. L. Teng, Y. Wang, W. Cai, and H. Li, "Fuzzy model predictive control of discrete systems with time-varying delay and disturbances," IEEE Transactions on Fuzzy Systems, 2017, in publication. (DOI: 10.1109/TFUZZ.2017.2717798)
2. L. Teng, Y. Wang, W. Cai, and H. Li, "Robust model predictive control of discrete nonlinear systems with time delays and disturbances via T-S fuzzy approach," Journal of Process Control, vol. 53, pp. 70-79, 2017.
3. L. Teng, Y. Wang, W. Cai, and H. Li, "Efficient robust fuzzy model predictive control of discrete nonlinear time-delay systems via Razumikhin approach," submitted to IEEE Transactions on Fuzzy Systems, 2017. Accepted.
4. L. Teng, Y. Wang, W. Cai, and H. Li, "Robust fuzzy model predictive control of discrete-time Takagi-Sugeno systems with nonlinear local models," submitted to IEEE Transactions on Fuzzy Systems, 2017. Accepted.
5. L. Teng, H. Yang, Y. Wang, W. Cai, and H. Li, "Model reference tracking control of switched networked systems: application to a linear brushless dc motor," submitted to a journal, 2017.

List of conference papers:

1. L. Teng, Y. Wang, C. Chen, W. Cai, and H. Li, "Application of T-S fuzzy controllers on an HVAC system," In 7th International Conference on Information and Automation for Sustainability, Colombo, Sri Lanka, 2014.
2. L. Teng, Y. Wang, W. Cai, and H. Li, "Robust model predictive control for discrete T-S fuzzy systems with nonlinear local models," In 12th IEEE International Conference on Control and Automation, Kathmandu, Nepal, 2016.
3. L. Teng, Y. Wang, W. Cai, and H. Li, "Model predictive control of discrete T-S fuzzy systems with time-varying delay," In 14th International Conference on Control, Automation, Robotics and Vision, Phuket, Thailand, 2016.
4. L. Teng, H. Yang, and Y. Wang, "Model reference tracking control of linear motor with dead-zone via switched systems subjected to time-varying delay," In 42nd Annual Conference of the IEEE Industrial Electronics Society, Florence, Italy, 2016.
5. L. Teng, Y. Wang, W. Cai, and H. Li, "Decentralized robust fuzzy controller with nonlinear local models for large-scale interconnected systems," In 13th IEEE International Conference on Control and Automation, Ohrid, Macedonia, 2017.

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Chapter 2

Robust Model Predictive Control for Discrete-time T-S Fuzzy Systems with Nonlinear Local Models

Robust fuzzy model predictive control of discrete nonlinear systems is investigated in this chapter. A recently developed Takagi-Sugeno (T-S) fuzzy approach which uses nonlinear local models is adopted to approximate nonlinear systems. A critical issue that restricts the practical application of classical model predictive control is the online computational cost. For model predictive control of T-S fuzzy systems, the online computational burden is even worse. Especially for complex systems with severe nonlinearities, parametric uncertainties, and disturbances, existing model predictive control of T-S fuzzy systems usually leads to a very conservative solution or even no solution in some occasions. However, more relaxed results can be achieved by the proposed fuzzy model predictive control approach which adopts T-S systems with nonlinear local models. Another advantage is that, online computational cost of the optimization problem through solving matrix inequalities can be significantly reduced at the same time. Simulations on a numerical example and a two-tank system are presented to verify the effectiveness and advantages of the proposed method. Comparisons among several T-S fuzzy approaches are illustrated and show that the best settling time is achieved via the proposed method.

2.1 Introduction

From the review of MPC for nonlinear systems in Chapter 1, it can be seen that the NLM-based fuzzy approach provides a relatively easier way to approximate and stabilize nonlinear systems, while a good control performance can be achieved and the computational cost can be drastically reduced when compared to the classical LLM-based method. In this chapter, based on the existing results, RMPC of nonlinear systems which adopts NLM-based T-S fuzzy systems is investigated. And it shows that the proposed method outperforms several typical fuzzy control and FMPC algorithms in stabilization. In particular, the computational cost, especially for RMPC where parametric uncertainties and persistent disturbances are considered, can be significantly reduced. Thus it will benefit the practical implementation of FMPC.

It is noted that this work is based on the NLM-based fuzzy approach in [1] and [2], however, only H_∞ and H_2 control problems are investigated in these works that require merely off-line computation to obtain the control law. Thus the lower computational cost advantage of NLM-based T-S approaches is not fully utilized. Moreover, the combination of MPC and NLM-based fuzzy approach achieves an optimal control performance. In general, it can be regarded as an approach with both control performance and computational cost considered simultaneously.

2.2 Preliminaries and Problem Statement

2.2.1 Useful Definitions and Lemmas

Definition 2.1 (Input-to-state Stability (ISS)): Consider $x(k+1) \in F(x(k), d(k))$, it is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \leq \beta(x(0), k) + \delta(\|d\|) \quad (2.1)$$

where $x(0)$ is the initial state, $\mathbf{d} = \{d(0), d(1), \dots, d(k-1)\}$ is the disturbance sequence.

Definition 2.2 (ISS Lyapunov Function): Suppose that there exists a positive definite function $V(x(k))$ such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|) \quad (2.2)$$

for all $(x(k), d(k), x(1)) \in \mathbb{R}^n \times \mathbb{R}^c \times F(x(0), d(0))$;

(ii) For any initial condition, there exists a function $\alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfying

$$V(x(k+1)) - V(x(k)) \leq -\alpha_3(\|x(k)\|) + \sigma(\|d(k)\|) \quad (2.3)$$

then $V(x(k))$ is called an ISS Lyapunov Function for $x(k+1) \in F(x(k), d(k))$.

Lemma 2.1 ([2]): Let $x(k) = [x_1(k) \ x_2(k) \ \dots \ x_n(k)]$, $1 \leq p \leq n$. For

$\Lambda^{-1} = \text{diag}[\rho_1 \ \rho_2 \ \dots \ \rho_q]^{-1}$, where ρ_m , $1 \leq m \leq q$, are arbitrary positive

scalars, if there exist $W = [W_1^T \ W_2^T \ \dots \ W_q^T]^T$, where

$W_m = [0 \ \dots \ 0 \ w_{mp} \ 0 \ \dots \ 0]$, w_{mp} are positive scalars, and

$\phi(k) = [\phi_1(k) \ \phi_2(k) \ \dots \ \phi_q(k)]^T$, where $\phi_m(k) = F_m(x_p(k))$ with $F_m(0) = 0$,

such that $\phi_m(k) \in \text{co}\{0, W_m x(k)\}$, then

$$\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi \geq 0 \quad (2.4)$$

Lemma 2.2: Let N , F , S be real matrices of appropriate dimensions with

$F(k)$ being a matrix function. For any $\sigma > 0$ and $F^T(k)F(k) \leq I$,

$$NF(k)S + S^T F^T(k)N^T \leq \frac{1}{\sigma} NN^T + \sigma S^T S \quad (2.5)$$

2.2.2 NLM-based T-S Fuzzy System

Firstly, to simplify the problem description, the parametric uncertainties are not

involved. Thus we consider the following discrete nonlinear system with disturbances:

$$x(k+1) = f(x(k)) + g(x(k))u(k) + w(x(k))d(k) \quad (2.6)$$

where $x(k+1)$, $x(k) \in \mathbb{R}^n$ are the states, $u(k) \in \mathbb{R}^m$ and $d(k) \in \mathbb{R}^c$ are the control input and external disturbance, respectively. $f(\cdot)$, $g(\cdot)$, and $w(\cdot)$ are functions of $x(k)$. $f(\cdot)$ often contains a lot of nonlinear terms especially for systems with severe nonlinearities. Then the aforementioned system can be described by the following NLM-based T-S fuzzy system:

Plant Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$x(k+1) = A_l x(k) + B_l u(k) + G_l \phi(k) + E_l d(k) \quad (2.7)$$

where, $z := [z_1, \dots, z_v]$ are premise variables. $F_1^l, \dots, F_v^l$ are fuzzy sets. $l \in \mathbb{Z}_{[1, L]}$, and L is the number of fuzzy rules. $\phi(k) \in co\{0, Wx(k)\}$ is a nonlinear term which is separated from $f(x(k))$,

$$f(x(k)) = \sum_{l=1}^L \mu_l(z) [A_l x(k) + G_l \phi(k)] \quad (2.8)$$

where $\mu_l(z)$ is normalized membership function. The following notations are introduced,

$$A_\mu := \sum_{l=1}^L \mu_l(z) A_l$$

$$G_\mu := \sum_{l=1}^L \mu_l(z) G_l$$

$$B_\mu := \sum_{l=1}^L \mu_l(z) B_l$$

$$E_\mu := \sum_{l=1}^L \mu_l(z) E_l$$

Then the system can be rewritten as follows,

$$x^+ = A_\mu x + B_\mu u + G_\mu \phi + E_\mu d \quad (2.9)$$

where x^+ denotes the system state in the next instant for simplicity purpose.

The nonlinear local controller corresponding to each NLM is given as follows [2]:

Control Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN: $u_l(k) = K_{al}x(k) + K_{bl}\phi(k)$

By using the fuzzy inference method with a singleton fuzzifier, product inference and center average defuzzifiers, the final control output can be obtained as follows:

$$u(k) = K_{a\mu}x(k) + K_{b\mu}\phi(k) \quad (2.10)$$

where

$$K_{a\mu} = \sum_{l=1}^L K_{al}$$

$$K_{b\mu} = \sum_{l=1}^L K_{bl}$$

2.2.3 Robust Fuzzy Model Predictive Control

The prediction model for the above T-S fuzzy system is

$$x(k+i+1|k) = A_\mu x(k+i|k) + B_\mu u(k+i|k) + G_\mu \phi(k+i|k) + E_\mu d(k+i|k) \quad (2.11)$$

MPC optimizes a cost function under some constraints. Usually, a finite horizon cost function is adopted,

$$J(k) = \sum_{i=0}^{N-1} \ell(k+i|k) + V_t(x(k+N|k)) \quad (2.12)$$

where $\ell(k+i|k)$ and $V_t(x(k+N|k))$ are called the stage cost and the terminal cost, respectively. The stage cost is chosen as

$$\ell(k+i|k) = x^T(k+i|k)Qx(k+i|k) + u^T(k+i|k)Ru(k+i|k) - \tau d^T(k+i|k)d(k+i|k)$$

(2.13)

where Q , R are positive matrices, τ is a positive scalar. One can see that the disturbance is involved in the stage cost. It is noted that the choice of cost function is inspired by the H_∞ control, see Section 4.7 in [3] for the so called “ H_∞ model predictive control”. And similar settings of cost functions can be found in [3]-[5]. However, the cost function cannot be optimized directly as it contains disturbance. Instead, a min-max approach is adopted that the worst-case cost function is minimized [6]. In addition, in the presence of persistent disturbance, asymptotic stability to the origin cannot be reached, instead, asymptotic convergence to an invariant set can be achieved [7], [8]. As such, a robust positively invariant set is employed as the terminal constraint set that, the system state is guaranteed to enter the terminal set at the end of prediction. The online optimization problem is shown as follows,

$$\begin{aligned} \min_{u(k+i|k)} \max_{d(k+i|k)} J(k), \\ u(k+i|k) \in U, \\ d(k+i|k) \in D, \\ x(k+N|k) \in \Omega_t, \end{aligned}$$

where Ω_t is the terminal constraint set. The disturbance and control input satisfy that

$$\begin{aligned} d \in D &:= \{d \mid d^T d \leq \gamma\} \\ u \in U &:= \{u \mid |u_s| \leq u_{s,\max}\} \end{aligned}$$

where γ is a positive scalar, u_s is the s -th element of the inputs, $s \in \mathbb{Z}_{[1,m]}$.

2.3 Main Results

In this section, the RPI property and terminal constraint set are introduced firstly, based on which the whole algorithm of FMPC is summarized. Then Analysis of the recursive feasibility and the ISS are provided.

2.3.1 RPI Set

Denote Ω as an RPI set and $h(x)$ as the corresponding control law. The RPI property can be described as, $\forall x \in \Omega$, the control effort will guarantee $x^+ \in \Omega$ for all allowable uncertainties and disturbances. Define Ω as follows:

$$\Omega := \{x \mid x^T P_\mu x \leq \xi\} \quad (2.14)$$

where $P_\mu = \sum_{l=1}^L \mu_l(x) P_l$. The controller for T-S fuzzy systems with NLMs is selected as

$$h(x) := K_{a\mu}x + K_{b\mu}\phi \quad (2.15)$$

Lemma 2.3 [4]: The set Ω is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\frac{1}{\xi} x^{+T} P_\mu^+ x^+ - \frac{1-\lambda}{\xi} x^T P_\mu x - \frac{\lambda}{\gamma} d^T d \leq 0 \quad (2.16)$$

with $P_\mu^+ = \sum_{l=1}^L \mu_l(x^+) P_l$, for all $x^+ \in A_\mu x + B_\mu u + G_\mu \phi + E_\mu d$, $u \in U$, and $d \in D$.

Proof: A simple proof is also given here. Move the last two items in (2.16) to the right side. As $x^T P_\mu x \leq \xi$ and $d^T d \leq \gamma$, then $\frac{1}{\xi} x^{+T} P_\mu^+ x^+ \leq (1-\lambda) + \lambda = 1$, thus $x^{+T} P_\mu^+ x^+ \leq \xi$, which implies $x^+ \in \Omega$.

Theorem 2.1: Consider the T-S fuzzy system (2.9), if there exist positive definite matrices X_i (or X_j , X_l), matrices L_j , Y_{aj} , Y_{bj} , Z , $\Lambda^{-1} = \text{diag}[\rho_1 \ \rho_2 \ \cdots \ \rho_q]^{-1}$ with $\rho_1, \rho_2, \dots, \rho_q > 0$, and positive scalar $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ WL_j & -2\Lambda & * & * \\ 0 & 0 & -\lambda/\gamma & * \\ A_i L_j + B_i Y_{aj} & G_i \Lambda + B_i Y_{bj} & E_i & -X_l \end{bmatrix} \leq 0 \quad (2.17)$$

$$\begin{bmatrix} Z & * & * \\ Y_{aj}^T & \Phi_{22} & * \\ Y_{bj}^T & -WL_j & 2\Lambda \end{bmatrix} \geq 0, Z_{ss} \leq u_{s,\max}^2, s \in \mathbb{Z}_{[1,m]} \quad (2.18)$$

where $\Phi_{11} = (\lambda - 1)(L_j + L_j^T - X_i)$, $\Phi_{22} = L_j + L_j^T - X_j$. Z_{ss} is the s -th diagonal element of matrix Z , and $i, j, l \in \mathbb{Z}_{[1,L]}$, then the set $\Omega = \{x \mid x^T P_\mu x \leq \xi\}$, with $P_\mu = \sum_{i=1}^L \mu_i(x) P_i$ and $P_i = \xi X_i^{-1}$, is an RPI set for the fuzzy system (2.9) corresponding to the feedback control law $h(x) = K_{a\mu}x + K_{b\mu}\phi$, with $K_{ai} = Y_{ai}L_i^{-1}$, $K_{bi} = Y_{bi}\Lambda^{-1}$.

Proof: Resorting to the dilation lemma [9],

$$-L_j^T X_i^{-1} L_j \leq X_i - L_j^T - L_j \quad (2.19)$$

then multiplying $\text{diag}\{L_j^{-T}, \Lambda^{-1}, I, I\}$ and its transpose from both sides of (2.17), respectively, substituting $Y_{ai}L_i^{-1}$ with K_{ai} , and $Y_{bi}\Lambda^{-1}$ with K_{bi} , yields that

$$\begin{bmatrix} (\lambda - 1)X_i^{-1} & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * \\ 0 & 0 & -\lambda / \gamma & * \\ A_i + B_i K_{aj} & G_i + B_i K_{bj} & E_i & -X_l \end{bmatrix} \leq 0 \quad (2.20)$$

Substituting X_i^{-1} with P_i / ξ , then rewrite (2.20) as follows,

$$\begin{bmatrix} (\lambda - 1)P_\mu / \xi & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * \\ 0 & 0 & -\lambda / \gamma & * \\ A_\mu + B_\mu K_{a\mu} & G_\mu + B_\mu K_{b\mu} & E_\mu & -X_l \end{bmatrix} \leq 0 \quad (2.21)$$

Applying schur complement to (2.21), then

$$\begin{bmatrix} (\lambda - 1)P_\mu / \xi & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * \\ 0 & 0 & -\lambda / \gamma \end{bmatrix} + \Psi^T X_l^{-1} \Psi \leq 0 \quad (2.22)$$

where $\Psi = [A_\mu + B_\mu K_{a\mu} \quad G_\mu + B_\mu K_{b\mu} \quad E_\mu]$. Multiplying $[x^T \quad \phi^T \quad d^T]$ and its

transpose from both sides of (2.22), respectively, it holds that

$$x^{+T} X_l^{-1} x^+ - \frac{1}{\xi} x^T P_\mu x - \lambda \left(\frac{1}{\gamma} d^T d - \frac{1}{\xi} x^T P_\mu x \right) + 2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \leq 0 \quad (2.23)$$

where $x^+ = (A_\mu + B_\mu K_{a\mu})x + (G_\mu + B_\mu K_{b\mu})\phi + E_\mu d$. Substituting X_l^{-1} with P_l / ξ into the above inequality, with $2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \geq 0$ from Lemma 2.1, then

$$\frac{1}{\xi} (x^+)^T P_\mu^+ x^+ - \frac{1}{\xi} x^T P_\mu x - \lambda \left(\frac{1}{\gamma} d^T d - \frac{1}{\xi} x^T P_\mu x \right) \leq 0 \quad (2.24)$$

Thus the RPI property of the set Ω can be established if (2.17) holds. Moreover, the input constraint can be satisfied by (2.18), and proof is shown below.

From (2.19), the following inequality can be got if (2.18) holds,

$$\begin{bmatrix} Z & * & * \\ Y_{aj}^T & L_j^T X_j^{-1} L_j & * \\ Y_{bj}^T & -W L_j & 2\Lambda \end{bmatrix} \geq 0 \quad (2.25)$$

Multiplying $\text{diag}\{I, L_j^{-1}, \Lambda^{-1}\}$ and its transpose from both sides of (2.25), respectively, then substituting $Y_{ai} L_i^{-1}$ with K_{ai} , and $Y_{bi} \Lambda^{-1}$ with K_{bi} , yields that

$$\begin{bmatrix} Z & * & * \\ K_{aj}^T & X_j^{-1} & * \\ K_{bj}^T & -\Lambda^{-1} W & 2\Lambda^{-1} \end{bmatrix} \geq 0 \quad (2.26)$$

(2.26) can be written as

$$\begin{bmatrix} Z & * & * \\ K_{a\mu}^T & X_\mu^{-1} & * \\ K_{b\mu}^T & -\Lambda^{-1} W & 2\Lambda^{-1} \end{bmatrix} \geq 0 \quad (2.27)$$

Applying schur complement to (2.27), then

$$\begin{bmatrix} X_\mu^{-1} & * \\ -\Lambda^{-1} W & 2\Lambda^{-1} \end{bmatrix} - \frac{1}{Z} \begin{bmatrix} K_{a\mu}^T \\ K_{b\mu}^T \end{bmatrix} \begin{bmatrix} K_{a\mu} & K_{b\mu} \end{bmatrix} \geq 0 \quad (2.28)$$

Multiplying $\begin{bmatrix} x^T & \phi^T \end{bmatrix}$ and its transpose from both sides of (2.28), respectively, and substituting X_μ^{-1} with P_μ / ξ , yields that

$$\frac{1}{Z} (K_{a\mu}x + K_{b\mu}\phi)^T (K_{a\mu}x + K_{b\mu}\phi) - \frac{1}{\xi} x^T P_\mu x + 2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \leq 0 \quad (2.29)$$

Thus it holds that

$$\frac{1}{Z} (K_{a\mu}x + K_{b\mu}\phi)^T (K_{a\mu}x + K_{b\mu}\phi) - \frac{1}{\xi} x^T P_\mu x \leq 0 \quad (2.30)$$

As $u = K_{a\mu}x + K_{b\mu}\phi$, thus the following inequality is got,

$$\frac{1}{Z} u^T u \leq \frac{1}{\xi} x^T P_\mu x \quad (2.31)$$

Due to $\frac{1}{\xi} x^T P_\mu x \leq 1$, thus $u^T u \leq Z$. Proof is completed.

2.3.2 Terminal Constraint Set

The terminal constraint set Ω_t should satisfy two requirements. Firstly, it should be an RPI set. Secondly, there exists a positive definite function (terminal cost function) $V(x)$ that $\forall x \in \Omega_t$,

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|) \quad (2.32)$$

$$V(x^+) - V(x) < -(x^T Q x + h^T(x) R h(x) - \tau d^T d) \quad (2.33)$$

where α_3 and α_4 are $\mathcal{K}_\infty$ functions, $V(x)$ is given as

$$V(x) := \sum_{l=1}^L \mu_l(x) x^T P_l x \quad (2.34)$$

Theorem 2.2: Consider the T-S fuzzy system (9), if (17), (18) and the following matrix inequality is feasible

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\tau\xi & * & * & * \\ \Xi_{41} & \Xi_{42} & \xi E_i & -X_l & * & * \\ QL_j & 0 & 0 & 0 & -\xi Q & * \\ RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (2.35)$$

where $\Xi_{11} = X_i - L_j - L_j^T$, $\Xi_{41} = A_i L_j + B_i Y_{aj}$, $\Xi_{42} = G_i \Lambda + B_i Y_{bj}$. X_i (or X_j , X_l), L_j , Y_{aj} , Y_{bj} are the same as mentioned in Theorem 2.1, then Ω_t is a terminal constraint set corresponding to the terminal cost function $V(x)$.

Proof: Since (2.32) can be easily got by resorting to the eigenvalues. The main concentration is focused on the condition for (2.33).

Considering (2.19), substituting Y_{ai} with $K_{ai} L_i$, and Y_{bi} with $K_{bi} \Lambda$, then multiplying $\text{diag}\{L_j^{-1}, \Lambda^{-1}, I, I, I, I\}$ and its transpose to both sides of (2.35), respectively, yields that

$$\begin{bmatrix} -X_i^{-1} & * & * & * & * & * \\ \Lambda^{-1} W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau\xi & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & \xi E_i & -X_l & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_{aj} & RK_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (2.36)$$

where $\Upsilon_{41} = A_i + B_i K_{aj}$, $\Upsilon_{42} = G_i + B_i K_{bj}$. Substituting X_i^{-1} with $\xi^{-1} P_i$ and X_l^{-1} with $\xi^{-1} P_l$, (2.36) is equivalent to

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ \Lambda^{-1} W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & E_i & -P_l^{-1} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_{aj} & RK_{bj} & 0 & 0 & 0 & -R \end{bmatrix} < 0 \quad (2.37)$$

(2.37) can be written as

$$\begin{bmatrix} -P_\mu & * & * & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau & * & * & * \\ \Pi_{41} & \Pi_{42} & E_\mu & -P_l^{-1} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_{a\mu} & RK_{b\mu} & 0 & 0 & 0 & -R \end{bmatrix} < 0 \quad (2.38)$$

where $\Pi_{41} = A_\mu + B_\mu K_{a\mu}$, $\Pi_{42} = G_\mu + B_\mu K_{b\mu}$. Denoting P_l as P_μ^+ , and then applying schur complement, (2.38) is equivalent to

$$\begin{bmatrix} Q - P_\mu & * & * \\ \xi \Lambda^{-1}W & -2\xi \Lambda^{-1} & * \\ 0 & 0 & -\tau \end{bmatrix} + \Theta^T P_\mu^+ \Theta + \begin{bmatrix} K_{a\mu}^T \\ K_{b\mu}^T \\ 0 \end{bmatrix} R \begin{bmatrix} K_{a\mu} & K_{b\mu} & 0 \end{bmatrix} < 0 \quad (2.39)$$

where $\Theta = [\Pi_{41} \quad \Pi_{42} \quad E_\mu]$. Multiplying $[x^T \quad \phi^T \quad d^T]$ and its transpose to both sides of (2.39), respectively, yields that

$$\begin{aligned} & x^{+T} P_\mu^+ x^+ - x^T P_\mu x + x^T Q x + (K_{a\mu} x + K_{b\mu} \phi)^T R (K_{a\mu} x + K_{b\mu} \phi) \\ & - \tau d^T d + 2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) < 0 \end{aligned} \quad (2.40)$$

where $x^+ = (A_\mu + B_\mu K_{a\mu})x + (G_\mu + B_\mu K_{b\mu})\phi + E_\mu d$.

As $2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \leq 0$ and $h(x) = K_{a\mu} x + K_{b\mu} \phi$, then (2.33) is got. Proof is thus completed.

2.3.3 Control Algorithm

Based on the above results, online control algorithm is discussed in this subsection to complete the FMPC design.

From the aforementioned computation of terminal constraint set, the terminal cost function $V(x(k))$, see (2.34), should satisfy the following condition,

$$V(x(k)) = \sum_{l=1}^L \mu_l(x(k)) x^T(k) P_l x(k) \leq \xi \quad (2.41)$$

Then the following problem is considered to minimize $V(x(k))$,

$$\min \xi, \text{ subjected to } V(x(k)) \leq \xi \quad (2.42)$$

Moreover, a sufficient condition for $V(x(k)) \leq \xi$ is

$$\begin{bmatrix} 1 & x^T(k) \\ x(k) & X_l \end{bmatrix} \geq 0 \quad (2.43)$$

In addition, consider the condition for terminal constraint set in (2.35), τ also can be optimized. Taking the same idea as stated in [4], a new variable ε is defined as: $\varepsilon = \tau\xi$, thus ε is minimized instead. Then (2.35) is rewritten as

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\varepsilon & * & * & * \\ \Xi_{41} & \Xi_{42} & \xi E_i & -X_l & * & * \\ QL_j & 0 & 0 & 0 & -\xi Q & * \\ RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (2.44)$$

In summary, the MPC algorithm is given as follows. It can be seen that bilinear matrix inequalities (BMIs) are introduced in (2.17) due to the variable λ . To reduce the computational burden, firstly, we try different values of λ and find a feasible one, then it is utilized all the time. Although it may be conservative to set λ as a constant value, the optimization problem is reduced to solving linear matrix inequalities (LMIs).

Algorithm:

Step 1: Obtain the system state $x(k)$;

Step 2: Solve the optimization problem

$$\min_{\xi, \varepsilon, X_i, L_j, \Lambda, Y_{aj}, Y_{bj}, Z} \varepsilon, \quad (2.45)$$

subject to (2.17), (2.18), (2.43), (2.44). Go to Step 3;

Step 3: Implement the control input $u(k) = K_{a\mu}x(k) + K_{b\mu}\phi(k)$, where

$K_{a\mu} = \sum_{l=1}^L \mu_l(x(k)) Y_{al} L_l^{-1}$, and $K_{b\mu} = \sum_{l=1}^L \mu_l(x(k)) Y_{bl} \Lambda^{-1}$. Move the time instant from k to $k+1$ and go to Step 1.

Theorem 2.3: For system (2.9), if the optimization problem has a solution at time 0, it will always be solvable. Furthermore, recursive feasibility can be achieved.

Proof: Assume that (2.45) is satisfied at time k , then $x(k) \in \Omega(k)$, where $\Omega(k)$ represents the terminal set Ω obtained at time k . As $\Omega(k)$ is an RPI set, thus $x(k+1) \in \Omega(k)$ is got. Then it can be concluded that (2.45) is solvable at time $k+1$. Moreover, the solution obtained at time k is feasible at time $k+1$. Applying the same process, it is observed that the optimization problem is feasible at all time.

Theorem 2.4: If the optimization problem is feasible at the initial time 0, system (2.9) is ISS with respect to disturbance d .

Proof: Consider the Lyapunov function $V(k, x) = x^T(k)P_\mu^*(k)x(k)$, where $P_\mu^*(k) = \sum_{l=1}^L \mu_l(z)P_l^*(k)$, $P_l^*(k)$ is an optimal value of $P_l(k)$ at time k , then it can be easily got

$$\rho_{\min}^* \|x(k)\|^2 \leq V(k, x) \leq \rho_{\max}^* \|x(k)\|^2 \quad (2.46)$$

where

$$\rho_{\min}^* := \min \{\rho_{\min}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1, L]}, k \in \mathbb{N}\}$$

$$\rho_{\max}^* := \max \{\rho_{\max}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1, L]}, k \in \mathbb{N}\}$$

where $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ represent the maximal and minimal eigenvalues, respectively.

Moreover, (2.33) implies that

$$V_k(x(k+1)) - V(k, x) < -(x^T(k)Qx(k) + u^T(k)Ru(k) - \tau d^T d) \quad (2.47)$$

where $V_k(x(k+1)) = x^T(k+1)P_\mu^*(k)x(k+1)$. Thus one has

$$V_k(x(k+1)) - V(k, x) < -x^T(k)Qx(k) + \tau d^T d \quad (2.48)$$

Owing to the optimality of (2.45) at time $k+1$, it holds

$$V_{k+1}(x(k+1)) \leq V_k(x(k+1)) \quad (2.49)$$

Then it holds that

$$V_{k+1}(x(k+1)) - V(k, x) < -x^T(k)Qx(k) + \tau d^T(k)d(k) \quad (2.50)$$

From Definition 2.2, (2.46) and (2.50) together yield that $V(x(k))$ is an ISS

Lyapunov function. Hence the closed-loop system is ISS with respect to disturbances. The proof is complete.

2.4 Extension to Systems with Parametric Uncertainties

In the aforementioned algorithm for FMPC with T-S fuzzy systems with NLMs, the disturbances and input constraints have been considered. To simplify the synthesis, the parametric uncertainties have not been investigated. However, it is noted that parametric uncertainties can be easily dealt with. And it is shown in this section.

Considering T-S fuzzy systems with structured uncertainties, then the T-S fuzzy system (2.7) is rewritten as

$$x(k+1) = (A_l + \Delta A_l)x(k) + (B_l + \Delta B_l)u(k) + (G_l + \Delta G_l)\phi(k) + E_l d(k) \quad (2.51)$$

where the structural uncertainties are defined as follows:

$$\begin{bmatrix} \Delta A_l & \Delta B_l & \Delta G_l \end{bmatrix} = MH \begin{bmatrix} N_{Al} & N_{Bl} & N_{Gl} \end{bmatrix}$$

where H is an unknown matrix function satisfying $H^T H \leq I$, M , N_{Al} , N_{Bl} , and N_{Gl} are known real constant matrices that structure the uncertainty.

Under such circumstances, (2.17) and (2.44) are replaced by (2.52) and (2.53), respectively, where η_{ijl} and ν_{ijl} are positive scalars. Proof can be easily got via a similar procedure for (2.17) and (2.44). And Lemma 2.2 is utilized to deal with the structural uncertainties. Thus it is omitted here.

$$\begin{bmatrix} (\lambda-1)(L_j + L_j^T - X_l) & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\lambda/\gamma & * & * & * \\ A_l L_j + B_l Y_{aj} & G_l \Lambda + B_l Y_{bj} & E_l & -X_l & * & * \\ N_{Al} L_j + N_{Bl} Y_{aj} & N_{Gl} \Lambda + N_{Bl} Y_{bj} & 0 & 0 & -\eta_{ijl} I & * \\ 0 & 0 & 0 & \eta_{ijl} M^T & 0 & -\eta_{ijl} I \end{bmatrix} \leq 0 \quad (2.52)$$

$$\begin{bmatrix}
X_i - L_j - L_j^T & * & * & * & * & * & * & * \\
WL_j & -2\Lambda & * & * & * & * & * & * \\
0 & 0 & -\varepsilon & * & * & * & * & * \\
A_i L_j + B_i Y_{aj} & G_i \Lambda + B_i Y_{bj} & \xi E_i & -X_i & * & * & * & * \\
QL_j & 0 & 0 & 0 & -\xi Q & * & * & * \\
RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R & * & * \\
N_{Ai} L_j + N_{Bi} Y_{aj} & N_{Gi} \Lambda + N_{Bi} Y_{bj} & 0 & 0 & 0 & 0 & -v_{ijl} I & * \\
0 & 0 & 0 & v_{ijl} M^T & 0 & 0 & 0 & -v_{ijl} I
\end{bmatrix} < 0
\quad (2.53)$$

2.5 Simulation Examples

In this section, two examples are introduced to show the effectiveness of the proposed method. The simulations are implemented with MATLAB and all LMIs are solved with MATLAB LMI toolbox. The simulations are executed on HP desktop (3.40 GHz Intel Core i7, 8GB RAM).

Two typical fuzzy control and FMPC methods, and the proposed method will be executed:

1. NFC: The NLM-based fuzzy control (NFC) method proposed in [2] considering disturbances and input constraints.
2. LFMPC: The LLM-based fuzzy model predictive control (LFMPC) method proposed in [4];
3. NFMPC: The proposed NLM-based fuzzy model predictive control (NFMPC) method;

2.5.1 Example 2.1

The model is given by

$$\begin{aligned}
x_1(k+1) &= (1 - 0.2 \sin^2(x_1(k)))x_1(k) - 0.3x_2(k) + u_1(k) \\
x_2(k+1) &= 0.4x_1(k) + (1 - 1.2 \sin^2(x_1(k)))\sin(x_2(k)) + 0.6x_2(k) + u_2(k) + d(k)
\end{aligned}$$

The input constraints are $|u_1(k)| \leq 1.5$, and $|u_2(k)| \leq 1$. And the disturbance is a white noise satisfying $|d(k)| \leq 0.01$. $x_1(k), x_2(k) \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.

It can be seen that nonlinearities are caused by $z_1(k) = \sin^2(x_1(k))$, $z_2(k) = \sin(x_2(k))$. To utilize the proposed FMPC method, the model is rewritten as follows

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} f_1(x_1(k)) & -0.3 \\ 0.4 & f_2(x_1(k)) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ f_3(x_1(k)) \end{bmatrix} \phi(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(k)$$

where $f_1(x_1(k)) = 1 - 0.2 \sin^2(x_1(k))$, $f_2(x_1(k)) = 0.6 + \frac{2}{\pi}(1 - 1.2 \sin^2(x_1(k)))$,

$f_3(x_1(k)) = 1 - 1.2 \sin^2(x_1(k))$, $\phi(k) = \sin(x_2(k)) - \frac{2}{\pi} x_2(k)$. From Figure 2.1, one

can get that $\phi_0(k) = \sin(x_2(k)) \in \text{co}\{\frac{2}{\pi} x_2(k), x_2(k)\}$, thus

$\phi(k) \in \text{co}\{0, (1 - \frac{2}{\pi})x_2(k)\}$.

Figure 2.1 The bound of ϕ_0 .

Then the following T-S fuzzy model is constructed:

Rule 1:

IF $z_1(k)$ is 0,

THEN $x(k+1) = A_1x(k) + B_1u(k) + G_1\phi(k) + E_1d(k)$

Rule 2:

IF $z_1(k)$ is 1,

THEN $x(k+1) = A_2x(k) + B_2u(k) + G_2\phi(k) + E_2d(k)$

$$\text{where } A_1 = \begin{bmatrix} 1 & -0.3 \\ 0.4 & 0.6 + \frac{2}{\pi} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & -0.3 \\ 0.4 & 0.6 - \frac{0.4}{\pi} \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The membership functions are defined as follows,

$$F_1 = \cos^2(x_2(k))$$

$$F_2 = 1 - F_1$$

Let $Q = \text{diag}\{1, 1\}$, $R = \text{diag}\{1, 1\}$, $\lambda = 0.4688$. Simulation results of the aforementioned three methods are shown in Figure 2.2. One can see that, among all approaches, the best control performance is achieved via the NFMPC. Figure 2.2 also shows that the input constraints are guaranteed with all methods. On the other hand, Table 2.1 shows the computational cost of different approaches. The control law for NLM-based fuzzy approach in [2] is off-line computed, while the other two approaches are based on online optimization. In addition, compared with the LFMPC in [4], the cputime in simulation is reduced significantly by adopting the proposed NFMPC. Moreover, the numerical complexity, which can reflect the computational cost, represented by the number of decision variables and the number of lines in LMI, are also shown.

Figure 2.2 Trajectories of states and inputs.

Table 2.1 Computational Complexities of Different Approaches.

Methods	NFC in [2]	LFMPC in [4]	Proposed NFMPC
The number of fuzzy rules	2	4	2
The number of decision variables (D)	29	44	29
Lines of the LMI (L)	142	870	142
Numerical complexity ($C = D^3 L$)	3463238	74110080	3463238
Total cputime	off-line	20.3694s	4.3366s
Average cputime	-	1.8518s	0.3942s

2.5.2 Example 2.2

Figure 2.3 The two-tank system.

Considering the two-tank system shown in Figure 2.3, where h_1 and h_2 denote the water level of the two tanks, respectively. The control objective is to keep the water level of the two tanks in the desired operation point.

The dynamic equations of the system are given as follows [10]:

$$\begin{aligned}\dot{h}_1(t) &= \frac{1}{A} \left(ku(t) - a_1 \sqrt{2g(h_1(t) - h_2(t))} \right) \\ \dot{h}_2(t) &= \frac{1}{A} \left(a_1 \sqrt{2g(h_1(t) - h_2(t))} - a_2 \sqrt{2gh_2(t)} \right)\end{aligned}$$

where the horizontal section is $A = 100\text{cm}^2$, the section of the valve connecting the two tanks is $a_1 = 1\text{cm}^2$, and the section of the outlet valve is $a_2 = 0.7\text{cm}^2$.

Let $g = 981\text{cm/s}^2$. From the aforementioned dynamic equations, a desired setpoint is chosen as $[1.49; 1]$, with the reference input being 0.3101.

An incremental model of the dynamic system can be acquired as follows:

$$\begin{aligned}\dot{x}_1 &= -\frac{a_1}{A} \frac{2g(x_1 - x_2)}{\sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}} + \frac{k}{A} \Delta u \\ \dot{x}_2 &= \frac{a_1}{A} \frac{2g(x_1 - x_2)}{\sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}} - \frac{a_2}{A} \frac{2gx_2}{\sqrt{2g(x_{2s} + x_2)} + \sqrt{2gx_{2s}}}\end{aligned}$$

where $x_s = [x_{1s}; x_{2s}]$ is the setpoint, Δu is the relative input corresponding to

the reference input. The model is rewritten as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{a_1}{A} \frac{\sqrt{2g}}{f(x_1, x_2)} & \frac{a_1}{A} \frac{\sqrt{2g}}{f(x_1, x_2)} \\ \frac{a_1}{A} \frac{\sqrt{2g}}{f(x_1, x_2)} & -\frac{a_1}{A} \frac{\sqrt{2g}}{f(x_1, x_2)} - \frac{a_2}{A} \frac{\sqrt{2g}}{t(x_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{A} \\ 0 \end{bmatrix} \Delta u$$

where $f(x_1, x_2) = \sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}$,

$t(x_2) = \sqrt{2g(x_{2s} + x_2)} + \sqrt{2gx_{2s}}$.

It can be seen that the nonlinearities are introduced by $z_1 = x_1 - x_2$, $z_2 = x_2$.

Suppose $z_1 \in [-0.3, 0.3]$, $z_2 \in [-0.5, 0.5]$. To build the NLM-based fuzzy model,

denote $\phi(t) = \left(\frac{1}{t(x_2)} - \frac{1}{\sqrt{x_{2s} + 0.5} + \sqrt{x_{2s}}} \right) x_2$. Figure 2.4 shows the bound of $\phi(t)$, which indicates $\phi(t) \in co\{0, 0.1363x_2\}$.

Then the continuous-time T-S fuzzy system can be built. The discrete-time system model can be got by setting the sampling time as $T_s = 5s$, and the

following discretization method is adopted: $A_l = e^{A_{lc}T_s}$, $B_l = \left(\int_0^{T_s} e^{A_{lc}\tau} d\tau \right) B_{lc}$,

$G_l = \left(\int_0^{T_s} e^{A_{lc}\tau} d\tau \right) G_{lc}$, $E_l = \left(\int_0^{T_s} e^{A_{lc}\tau} d\tau \right) E_{lc}$. The discrete T-S fuzzy system is

shown as follows, where disturbance and structural uncertainties are introduced.

Figure 2.4 The bound of ϕ .

Rule 1:

IF z_1 is -0.3,

THEN

$$x(k+1) = (A_1 + MHN_{A1})x(k) + (B_1 + MHN_{B1})\Delta u(k) + (G_1 + MHN_{G1})\phi(k) + E_1 d(k)$$

Rule 2:

IF z_1 is 0.3,

THEN

$$x(k+1) = (A_2 + MHN_{A2})x(k) + (B_2 + MHN_{B2})\Delta u(k) + (G_2 + MHN_{G2})\phi(k) + E_2 d(k)$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.4337 & 0.3515 \\ 0.3515 & 0.3081 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.4735 & 0.3372 \\ 0.3372 & 0.3049 \end{bmatrix}, & B_1 &= \begin{bmatrix} 2.9936 \\ 1.5414 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 3.2474 \\ 1.3587 \end{bmatrix}, & G_1 &= \begin{bmatrix} -0.4779 \\ -0.7574 \end{bmatrix}, & G_2 &= \begin{bmatrix} -0.4213 \\ -0.7963 \end{bmatrix}, & E_1 = E_2 &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \\ M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & N_{A1} = N_{A2} &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, & N_{B1} = N_{B2} &= \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \\ N_{G1} = N_{G2} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The membership functions are defined as follows,

$$\begin{aligned} F_1 &= \frac{0.3 - (x_1 - x_2)}{0.6} \\ F_2 &= 1 - F_1 \end{aligned}$$

Let $Q = \text{diag}\{0.1, 0.1\}$, $R = 0.1$, $\lambda = 0.3135$, $|\Delta u(k)| \leq 0.05$, $|d(k)| \leq 0.1$.

Figure 2.5 Trajectories of states and inputs.

Table 2.2 Computational Complexities of Different Approaches.

Methods	NFC in [2]	LFMPC in [4]	Proposed NFMPC
The number of fuzzy rules	2	4	2
The number of decision variables (D)	38	163	38
Lines of the LMI (L)	222	1508	222
Numerical complexity ($C = D^3 L$)	12181584	6530766476	12181584
Total cputime	off-line	64.8501s	14.0058s
Average cputime	-	4.3233s	0.9337s

Simulation results of aforementioned three methods are shown in Figure 2.5. It shows that the proposed NFMPC achieves best control performance among all methods. As persistent disturbances are involved, the system states are converged to a small region near the origin. The disturbance is eliminated after 60s, then it can be seen that asymptotical convergence to the origin can be

achieved. It can be seen that the input constraint is satisfied with all methods.

And computational cost for all methods are shown in Table 2.2. Both the numerical complexity and the cputime cost for simulation are demonstrated. It shows that the computational cost can be significantly reduced by adopting the NFMPC when compared to the LFMPC. In addition, the average cputime of LFMPC for each instant is 4.3233s, which is much closed to the sampling time. As a result, the LFMPC may not be used effectively for the stabilization control of the two-tank system. Moreover, in Example 2.2, one can see that the cputime cost increases a lot compared with that of Example 2.1 due to the structural uncertainties.

In addition, the region of attraction is illustrated in Figure 2.6 to further verify the relaxation provided by the NFMPC when compared to the LFMPC. One can see that a larger region of attraction is achieved via the NFMPC. It is noted that uncertainties are not involved to simplify the plant model for comparison.

Moreover, as λ is determined off-line, to find a best value, we run simulations with different values of λ . The results are shown in Figure 2.7. It can be seen that the best settling time is achieved when $\lambda = 0.1$.

Figure 2.6 Region of attraction.

Figure 2.7 Trajectories of states with different λ .

2.6 Conclusions

The T-S fuzzy systems with NLMs are adopted to approximate nonlinear systems. Compared with the LLM-based FMPC, especially for systems with complex nonlinearities, parametric uncertainties and disturbances which make the optimization problem become very complicated, less conservative results and better control performance can be achieved through the proposed NLM-based FMPC. In particular, the computational cost of the online optimization problem by solving LMIs can be significantly reduced at the same time. Therefore, both control performance and computational cost are considered in the proposed method.

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Chapter 3

Robust Model Predictive Control of Discrete Nonlinear Systems with Time Delays and Disturbances via T-S Fuzzy Approach

In this chapter, robust fuzzy model predictive control of a class of nonlinear discrete systems subjected to time delays and persistent disturbances is investigated. Based on the modeling method of delay difference inclusions, nonlinear discrete time-delay systems can be represented by T-S fuzzy systems comprised of piecewise linear delay difference equations. Moreover, Lyapunov-Razumikhin function (LRF), instead of Lyapunov-Krasovskii functional (LKF), is employed for time-delay systems due to its ability to reflect system original state space and its advantages in controller synthesis and computation. The robust positive invariance and input-to-state stability with respect to disturbance under such circumstances are investigated. A robust constraint set is adopted that the system state is converged to this set round the desired point. In addition, the controller synthesis conditions are derived by solving a set of matrix inequalities. Simulation results show that the proposed approach can be successfully applied to the well-known continuous stirred tank reactor (CSTR) systems subjected to time delay.

3.1 Introduction

In this work, the T-S fuzzy systems are adopted for modeling of nonlinear time-delay systems. Then LRF is adopted that conditions of robust positive invariance and input-to-state stability (ISS) with respect to disturbances are provided. It is worth to note that ISS with LKF for MPC of continuous systems is discussed in [1]. The contributions of this work can be summarized as follows: (1) MPC of discrete systems with multiple delays and bounded disturbances are investigated simultaneously. Although time delay and disturbances have been solely considered in some existing MPC methods as shown above. However, for MPC of time-delay systems, disturbances have been rarely investigated. The reason is that only LKF has been widely investigated for time-delay systems. However, controller synthesis and computation via LKF are usually very complex as it relies on a functional that uses trajectories segments rather than state trajectory. The complexity exists even for systems with single time delay, and will become greater if multiple delays and disturbances are considered simultaneously. In this work, it is shown that multiple delays and disturbances can be easily dealt with through the proposed LRF-based MPC method. (2) FMPC for time-delay systems is investigated and applied to the process systems. Although FMPC has been widely investigated in non-delayed systems, few result can be seen on FMPC of time-delay systems. Furthermore, the proposed method is successfully applied to the famous continuous stirred tank reactor (CSTR) system subjected to time delay, and robust tracking of variant reference signals is achieved. There are extensive results on MPC of CSTR system [2], [3]-[8], however, the time delay effect has not been investigated in most of aforementioned works. Although time delay is considered in [4], only LKF is employed that the synthesis is very complex and disturbance is absent. (3) New conditions of robust positive invariance and ISS under such circumstances are provided based on several existing results.

It should be noted that this work extends the results in [2] that persistent disturbances are investigated for time-delay systems. However, it is shown that the proposed method is more general than the one in [2] where time delay is not considered. And we will show that the proposed method can perform better even for non-delayed systems as investigated in [2].

3.2 System Description and Problem Statement

In this section, basic knowledge about DDEs, T-S fuzzy systems, and MPC control scheme for time-delay systems are introduced sequentially.

3.2.1 DDEs

Consider the following

$$x(k+1) \in F(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}, d(k)), \quad k \in \mathbb{Z}_+ \quad (3.1)$$

where $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$, $\mathbf{u}_{[k-h,k]} \in \mathbb{U}^{h+1}$ and $d(k) \in \mathbb{D}$, with $\mathbb{X} \in \mathbb{R}^n$, $\mathbb{U} \in \mathbb{R}^m$ and $\mathbb{D} \in \mathbb{R}^c$, represent delayed and current states, inputs, and disturbances, respectively. $h \in \mathbb{Z}_+$ stands for the maximal value of time delay. To stabilize DDE (3.1), a control law π will be adopted. The closed-loop system with the control law yields

$$x(k+1) \in F_\pi(\mathbf{x}_{[k-h,k]}, d(k)), \quad k \in \mathbb{Z}_+, \quad (3.2)$$

where

$$F_\pi(\mathbf{x}_{[-h,0]}, d) := \{F(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}, d) \mid u(0) \in \pi(\mathbf{x}_{[-h,0]})\}$$

and $\mathbf{x}_{[-h,0]}$ and $\mathbf{u}_{[-h,-1]}$ is assumed to be known. Furthermore, suppose that $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ are available at each sampling instant. The dependence of F_π on $\mathbf{u}_{[k-h,k-1]}$ is omitted in (3.2) because it is assumed that the input sequence $\mathbf{u}_{[k-h,k-1]}$ is dependent on the state sequence $\mathbf{x}_{[k-h,k]}$ only. It is noted that more details about the DDEs can be seen in [9].

Definition 3.1 (LRF-type Robust Positively Invariant (RPI) Set, D-invariance): Consider DDE (3.1), a set $\Omega \in \mathbb{X}$ is called an RPI set for the closed-loop system corresponding to the control law π , for $i \in \mathbb{Z}_{[-h,0]}$, if $\forall x(k+i) \in \Omega$ and $\forall d(k) \in \mathbb{D}$, it holds $x(k+1) \in \Omega$.

Definition 3.2 (Input-to-State Stability (ISS) [10]): The DDE (3.2) is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \leq \beta\left(\|\mathbf{x}_{[-h,0]}\|, k\right) + \delta\left(\|\mathbf{d}_{[0,k-1]}\|\right) \quad (3.3)$$

where $\mathbf{d}_{[0,k-1]} \in \mathbb{D}^k$ is the disturbance sequence.

Definition 3.3 (LRF-type ISS Lyapunov Function [10]): For LRF, suppose that there exists a positive definite function such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying for all $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (3.4)$$

(ii) For all initial conditions, there exists a function $\delta \in \mathcal{K}$ satisfying

$$V(x(k+1)) \leq \bar{V}(x(k)) + \delta\|d(k)\| \quad (3.5)$$

where $\bar{V}(x(k)) = \max_{i \in \mathbb{Z}_{[-h,0]}} \{V(x(k+i))\}$, then $V(x(k))$ is called an ISS Lyapunov function for DDE (3.2).

It is noted that detailed descriptions of LRF-type ISS and ISS Lyapunov function for DDEs are introduced in [10].

In what follows, to simplify the whole problem, we only consider the following system with state delays and disturbances,

$$x(k+1) \in F(\mathbf{x}_{[k-h,k]}, u(k), d(k)), \quad k \in \mathbb{Z}_+ \quad (3.6)$$

3.2.2 T-S Fuzzy Time-delay Systems

Based on the aforementioned DDEs with disturbances, nonlinear time-delay system (6) can be represented by T-S fuzzy time-delay systems comprised of

fuzzy blendings of the piecewise linear DDEs:

Plant Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$x(k+1) = \sum_{i=-h}^0 A_{i,l} x(k+i) + B_l u(k) + E_l d(k) \quad (3.7)$$

where, $z := [z_1, \dots, z_v]$ are premise variables. $F_1^l, \dots, F_v^l$ are fuzzy sets.

$l \in \mathbb{Z}_{[1,L]}$, and L is the number of fuzzy rules. The T-S fuzzy system (3.7) can

be written as,

$$x(k+1) = \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k) + E_\mu d(k) \quad (3.8)$$

where $A_{i,\mu} := \sum_{l=1}^L \mu_l(z) A_{i,l}$, $B_{i,\mu} := \sum_{l=1}^L \mu_l(z) B_{i,l}$, $E_\mu := \sum_{l=1}^L \mu_l(z) E_l$, $\mu_l(z)$ is normalized membership function.

Control method for above T-S fuzzy system is given as follows:

Control Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$u(k) = K_l x(k)$$

Thus the final control output is given as,

$$u(k) = K_\mu x(k) \quad (3.9)$$

where $K_\mu = \sum_{l=1}^L K_l$.

3.2.3 MPC scheme

As a model-based control method, MPC utilizes the system model to predict the future system states and generate a series of control inputs, usually the first computed input is adopted. A cost function over the prediction horizon is

minimized to get optimized control actions. And the optimization algorithm is repeated in the next instant.

In this subsection, control scheme about MPC for the aforementioned T-S fuzzy DDEs with disturbances is introduced. The prediction model is introduced as

$$x(k+s+1|k) = \sum_{i=-h}^0 A_{i,\mu} x(k-i+s|k) + B_{\mu} u(k+s|k) + E_{\mu} d(k+s|k) \quad (3.10)$$

Usually, MPC optimizes a finite horizon cost function under some constraints,

$$J(k) = \sum_{s=0}^{N-1} \ell(k+s|k) + V_T(x(k+N|k)) \quad (3.11)$$

where $\ell(k+s|k)$ and $V_T(x(k+N|k))$ are called the stage cost and the terminal cost, respectively. The stage cost is given by

$$\ell(k+s|k) = x^T(k+s|k) Q x(k+s|k) + u^T(k+s|k) R u(k+s|k) - \tau d^T(k+s|k) d(k+s|k) \quad (3.12)$$

where Q and R are weighting matrices, τ is a positive scalar. It is noted that the disturbance is involved in the stage cost. This kind of cost function is inspired by the H_{∞} control, see Section 4.7 in [11] for the so-called “ H_{∞} model predictive control”. And similar settings of cost functions can be found in [2], [12], [13]. However, the cost function cannot be optimized directly as it contains disturbance. Instead, a min-max approach is adopted that the worst-case cost function is minimized [14]. In addition, in the presence of persistent disturbances, asymptotic stability to the origin cannot be reached, instead, asymptotic convergence to an invariant set can be achieved [15], [16]. As such, a robust positively invariant set is employed as the terminal constraint set that, the system state is guaranteed to enter the terminal set at the end of prediction. The online optimization problem can be summarized as

$$\begin{aligned} \min_{u(k+s|k)} \max_{d(k+s|k)} J(k), \\ u(k+s|k) \in \mathbb{U}, \\ d(k+s|k) \in \mathbb{D}, \\ x(k+N|k) \in \Omega_t, \end{aligned} \quad (3.13)$$

where $s \in \mathbb{Z}_{[1,N]}$, Ω_t is the terminal constraint set. The disturbance and the control input satisfy that

$$d \in \mathbb{D} := \{d \mid d^T d \leq \eta^2\}$$

$$u \in \mathbb{U} := \{u \mid |u_t| \leq u_{t,\max}\}$$

where η is a known constant. u_t is the t -th element of the input vector, and $t \in \mathbb{Z}_{[1,m]}$.

3.3 Main Results

In this section, the conditions of RPI property for MPC of T-S fuzzy DDEs with disturbances are firstly discussed. Based on which the computation of terminal constraint set is given. In the end, the whole control algorithm is introduced, and stability and feasibility are analyzed.

3.3.1 RPI Set

From Definition 3.1, denote $\Omega_k \in \mathbb{X}$ as an RPI set Ω at time instant k , and π as the corresponding control law. Let quadratic LRF $V(x) := x^T P_\mu x$. And define Ω_k as the following set

$$\Omega_k := \bigcup_{j \in \mathbb{Z}_{[-h,0]}} \Omega_k^j \quad (3.14)$$

where $\Omega_k^j = \{x \mid V(x(k-j)) \leq \xi\}$ is the intersection of several ellipsoidal sets due to the utilization of piecewise Lyapunov function with respect to $P_\mu = \sum_{l=1}^L \mu_l P_l$ for the fuzzy systems in (3.8), and with the corresponding control law

$$\pi(\mathbf{x}_{[k-h,k]}) := \{K_\mu x(k)\} \quad (3.15)$$

It can be seen that Ω_k is a sublevel set of $\bar{V}(x(k))$ described in (3.5).

Lemma 3.1: Consider system (3.8), the set Ω_k is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{[0,1]}$ such that

$$\frac{1}{\xi} V(x(k+1)) - \frac{1-\lambda}{\xi} \bar{V}(x(k)) - \frac{\lambda}{\eta^2} d^T(k) d(k) \leq 0 \quad (3.16)$$

and $u(k) \in \mathbb{U}$, $d(k) \in \mathbb{D}$.

Proof: Move the last two items in (3.16) to the right side. As $\bar{V}(x(k)) \leq \xi$ and $d^T(k) d(k) \leq \eta^2$, then $\frac{1}{\xi} V(x(k+1)) \leq (1-\lambda) + \lambda = 1$, thus $V(x(k+1)) \leq \xi$, proof is completed. It is noted that a similar RPI approach involving disturbances for non-delayed systems can be seen in [2].

Conditions of RPI property and input constraint will be given as matrix inequalities in the following theorem.

Theorem 3.1: Consider system (3.8), if there exist positive definite matrix X_a (or X_b , X_l), matrices Y_b , G_b , Z , and positive scalar $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Xi_0 & * & * & * & * & * \\ 0 & \Xi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \dots & \Xi_h & * & * \\ 0 & 0 & \dots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Theta_0 & \Theta_1 & \dots & \Theta_h & E_a & -X_l \end{bmatrix} \leq 0 \quad (3.17)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_b \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (3.18)$$

where $\Xi_{-i} = \gamma_i(\lambda-1)(G_b + G_b^T - X_a)$, $i \in \mathbb{Z}_{[-h,0]}$, $\Theta_0 = A_{0,a}G_b + B_{0,a}Y_b$,

$\Theta_{-j} = A_{j,a}G_b$, $j \in \mathbb{Z}_{[-h,-1]}$, $a, b, l \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of

matrix Z , γ_i are pre-set valued positive numbers that satisfy $\sum_{i=-h}^0 \gamma_i = 1$, then

the set Ω_k , with $P_\mu = \sum_{l=1}^L \mu_l P_l$ and $P_l = \xi X_l^{-1}$, is an RPI set for system (3.8)

corresponding to the feedback control law $\pi = K_\mu x(k)$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$

and $K_l = Y_l G_l^{-1}$.

Proof: Resorting to the dilation lemma [17],

$$-G_b^T X_a^{-1} G_b \leq X_a - G_b^T - G_b \quad (3.19)$$

The following inequality is got from (3.17),

$$\begin{bmatrix} \Psi_0 & * & * & * & * & * \\ 0 & \Psi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Psi_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & E_a & -X_l \end{bmatrix} \leq 0 \quad (3.20)$$

where $\Psi_{-i} = \gamma_i(\lambda-1)G_b^T X_a^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, \dots, G_b^{-T}, I, I\}$

and its transpose from both sides of (3.20), respectively, yields that

$$\begin{bmatrix} \Pi_0 & * & * & * & * & * \\ 0 & \Pi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Pi_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -X_l \end{bmatrix} \leq 0 \quad (3.21)$$

where $\Pi_{-i} = \gamma_i(\lambda-1)X_a^{-1}$, $\tilde{A}_0 = A_{0,a} + B_{0,a}K_b$, $\tilde{A}_j = A_{j,a}$, $j \in \mathbb{Z}_{[-h,-1]}$.

(3.21) can be rewritten as

$$\begin{bmatrix} \Lambda_0 & * & * & * & * & * \\ 0 & \Lambda_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Lambda_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \tilde{A}_{0,\mu} & \tilde{A}_{-1,\mu} & \cdots & \tilde{A}_{-h,\mu} & E_\mu & -X_\mu \end{bmatrix} \leq 0 \quad (3.22)$$

where $\Lambda_{-i} = \gamma_i(\lambda - 1)X_\mu^{-1}$, $\tilde{A}_{0,\mu} = A_{0,\mu} + B_{0,\mu}K_\mu$, $\tilde{A}_{j,\mu} = A_{j,\mu}$, $j \in \mathbb{Z}_{[-h,-1]}$.

Multiplying $[x^T(k), x^T(k-1), \dots, x^T(k-h), d^T(k)]$ and its transpose from both sides of (3.22), respectively, yields that

$$\varpi \begin{bmatrix} \Lambda_0 & & & & \\ & \Lambda_1 & & & \\ & & \ddots & & \\ & & & \Lambda_h & \\ & & & & -\frac{\lambda}{\eta^2}I \end{bmatrix} \varpi^T + \varpi \Delta X_\mu^{-1} \Delta^T \varpi^T \leq 0 \quad (3.23)$$

where $\varpi = [x^T(k), x^T(k-1), \dots, x^T(k-h), d^T(k)]$, $\Delta = [\tilde{A}_{0,\mu}, \tilde{A}_{-1,\mu}, \dots, \tilde{A}_{-h,\mu}, E]^T$.

Substituting X_μ^{-1} with P_μ / ξ , (3.23) is equivalent to

$$\frac{1}{\xi} V(x(k+1)) - \frac{\lambda}{\eta^2} d^T(k) d(k) \leq \frac{1-\lambda}{\xi} \sum_{i=-h}^0 \gamma_i V(x(k+i)) \quad (3.24)$$

As $\sum_{i=-h}^0 \gamma_i V(x(k+i)) \leq \bar{V}(x(k))$ due to $\sum_{i=-h}^0 \gamma_i = 1$, thus (3.16) is got.

(3.18) is very commonly used to guarantee the input constraint, proof is thus completed.

3.3.2 Terminal Constraint Set

Definition 3.4: The terminal constraint set $\Omega_{t,k}$ should satisfy two requirements. Firstly, it is an RPI set, i.e., Ω_k stated in the last subsection. Secondly, there exist $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$, and a positive definite function $V(x(k))$ that $\forall x \in \Omega_{t,k}$,

$$\alpha_3(\|x(k)\|) \leq V(x(k)) \leq \alpha_4(\|x(k)\|) \quad (3.25)$$

$$V(x(k+1)) - \bar{V}(x(k)) \leq -x^T(k) Q x(k) - u^T(k) R u(k) + \tau d^T(k) d(k) \quad (3.26)$$

In the following theorem, the condition which guarantees Definition 3.4 is provided.

Theorem 3.2: Consider system (3.8), if (3.17), (3.18) and the following matrix inequality are feasible

$$\begin{bmatrix} \Gamma_0 & * & * & * & * & * & * & * \\ 0 & \Gamma_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \cdots & \Gamma_h & * & * & * & * \\ 0 & 0 & \cdots & 0 & -\tau\xi I & * & * & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & \xi E_a & -X_l & * & * \\ QG_b & 0 & \cdots & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & \cdots & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (3.27)$$

where $\Gamma_{-i} = \gamma_i(X_a - G_b - G_b^T)$, Θ_i , $i \in \mathbb{Z}_{[-h,0]}$, X_a (or X_b , X_l), Y_b , G_b , are the same as defined in Theorem 3.1, then $\Omega_{t,k}$ is a terminal constraint set.

Proof: (3.25) can be easily got by resorting to the eigenvalues of the positive definite matrices. Then the main effort is focused on the condition for (3.26).

Based on (3.17), multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, \dots, G_b^{-T}, I, I, I, I\}$ and its transpose from both sides of (3.27), respectively, yields that

$$\begin{bmatrix} \Upsilon_0 & * & * & * & * & * & * & * \\ 0 & \Upsilon_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \cdots & \Upsilon_h & * & * & * & * \\ 0 & 0 & \cdots & 0 & -\tau\xi I & * & * & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & \xi E_a & -X_l & * & * \\ Q & 0 & \cdots & 0 & 0 & 0 & -\xi Q & * \\ RK_b & 0 & \cdots & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (3.28)$$

where $\Upsilon_{-i} = -\gamma_i X_a^{-1}$.

(3.28) is equivalent to

$$\begin{bmatrix} H_0 & * & * & * & * & * & * & * \\ 0 & H_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \cdots & H_h & * & * & * & * \\ 0 & 0 & \cdots & 0 & -\tau I & * & * & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -\frac{X_l}{\xi} & * & * \\ Q & 0 & \cdots & 0 & 0 & 0 & -Q & * \\ RK_b & 0 & \cdots & 0 & 0 & 0 & 0 & -R \end{bmatrix} \leq 0 \quad (3.29)$$

where $H_{-i} = -\gamma_i \xi X_a^{-1}$.

Substituting $X^{-1} = \xi^{-1} P$ to (3.29), and rewriting (3.29) as

$$\begin{bmatrix} \Phi & * & * & * & * & * \\ 0 & -\gamma_{-1} P_\mu & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & -\gamma_{-h} P_\mu & * & * \\ 0 & 0 & \cdots & 0 & -\tau I & * \\ \tilde{A}_{0,\mu} & \tilde{A}_{-1,\mu} & \cdots & \tilde{A}_{-h,\mu} & E_\mu & -P_\mu^{-1} \end{bmatrix} \leq 0 \quad (3.30)$$

where $\Phi = Q + K_\mu^T RK_\mu - \gamma_0 P_\mu$. Applying schur complement to (3.30), then

multiplying $[x^T(k), x^T(k-1), \dots, x^T(k-h), d^T(k)]$ and its transpose, respectively, yields that

$$\begin{aligned} & \left(\sum_{i=-h}^0 \tilde{A}_{i,\mu} x(k+i) + E_\mu d(k) \right)^T P_\mu \left(\sum_{i=-h}^0 \tilde{A}_{i,\mu} x(k+i) + E_\mu d(k) \right) + x^T(k) Q x(k) + x^T(k) K_\mu^T RK_\mu x(k) \\ & - \tau d^T(k) d(k) - \sum_{i=-h}^0 \gamma_i x^T(k+i) P_\mu x(k+i) \leq 0 \end{aligned} \quad (3.31)$$

As $\sum_{i=-h}^0 \gamma_i V(x(k+i)) \leq \bar{V}(x(k))$ due to $\sum_{i=-h}^0 \gamma_i = 1$, then it follows that,

$$x^T(k+1) P_\mu x(k+1) - \bar{V}(x(k)) \leq -x^T(k) Q x(k) - u^T(k) R u(k) + \tau d^T(k) d(k) \quad (3.32)$$

The proof is thus completed.

3.3.3 Control Algorithm

Based on the above results, the whole control algorithm is summarized in this subsection, then feasibility and stability are analyzed.

From the aforementioned RPI property, the terminal constraint set $\Omega_{t,k}$ should satisfy the following condition,

$$\bar{V}(x(k)) \leq \xi \quad (3.33)$$

thus $x(k+i) \in \Omega_{t,k}$, $i \in \mathbb{Z}_{[-h,0]}$. The following optimization problem is considered,

$$\min \xi, \text{ subjected to } \bar{V}(x(k)) \leq \xi \quad (3.34)$$

A sufficient condition for $\bar{V}(x(k)) \leq \xi$ is

$$\begin{bmatrix} 1 & x^T(k+i) \\ x(k+i) & X_l \end{bmatrix} \geq 0, i \in \mathbb{Z}_{[-h,0]}, l \in \mathbb{Z}_{[1,L]} \quad (3.35)$$

Consider the condition for terminal constraint set in (3.27), τ can also be optimized. Taking the same idea as stated in [2], a new variable ε is defined as: $\varepsilon = \tau\xi$, thus ε is minimized instead. Then (3.27) is rewritten as

$$\begin{bmatrix} \Gamma_0 & * & * & * & * & * & * & * \\ 0 & \Gamma_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \dots & \Gamma_h & * & * & * & * \\ 0 & 0 & \dots & 0 & -\varepsilon I & * & * & * \\ \Theta_0 & \Theta_1 & \dots & \Theta_h & \xi E_a & -X_l & * & * \\ QG_b & 0 & \dots & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & \dots & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (3.36)$$

As bilinear matrix inequalities (BMIs) are introduced in (3.17), see Ξ_{-i} . To reduce the computational cost, a similar procedure stated in [2] is adopted that the feasible value of λ in the initial sampling instant is saved and utilized all the time. Thus the optimization problem is reduced to solving linear matrix inequalities (LMIs). In summary, the overall MPC algorithm is given as follows,

MPC Algorithm:

Step 1. Obtain the system all (delayed) states $x(k+i)$, $i \in \mathbb{Z}_{[-h,0]}$;

Step 2. If $k=0$, solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z, \lambda} \mathcal{E}, \quad (3.37)$$

subject to (3.17), (3.18), (3.35), (3.36), and $\lambda \in \mathbb{R}_{(0,1)}$. Obtain the optimal λ

and denotes it as λ^* . Go to Step 3;

Else if $k \geq 1$, replace λ in (3.17) with λ^* , and solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z} \mathcal{E}, \quad (3.38)$$

subject to (3.17), (3.18), (3.35), (3.36). Go to Step 3;

Step 3. Obtain the control input $u(k) = K_\mu x(k)$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$. Then implement the control input to system (3.8). Move the time instant from k to $k+1$ and go to Step 1.

Remark 1: It is noted that the results in [2] for non-delayed systems can be regarded as a special case of proposed method for time-delay systems. In another word, the proposed method can be directly applied to systems without delay that, in which case one only needs to set the state matrices for delayed states as $A_{j,l} = 0$, $j \in \mathbb{Z}_{[-h,-1]}$, $l \in \mathbb{Z}_{[1,L]}$.

Theorem 3.3: For system (3.8), if the aforementioned constrained optimization problem has a solution at the initial time, it will always have a solution. And recursive feasibility is guaranteed.

Proof: Assume that an optimal solution is available at time k , then $x(k+i) \in \Omega_{t,k}$, $\forall i \in \mathbb{Z}_{[-h,0]}$. Since $\Omega_{t,k}$ is an RPI set due to satisfaction of (3.16), $x(k+1) \in \Omega_{t,k}$ is got. With $x(k+1+j) \in \Omega_{t,k}$, $\forall j \in \mathbb{Z}_{[-h,-1]}$, thus $x(k+1+i) \in \Omega_{t,k}$, $\forall i \in \mathbb{Z}_{[-h,0]}$. Then it can be concluded that the control problem has a solution at time $k+1$. Moreover, the solution obtained at time k is feasible at time $k+1$. Applying the same process, it is observed that the optimization problem is feasible at all time.

Theorem 3.4: If the constrained optimization problem is feasible at the initial time 0, system (3.8) is ISS with respect to disturbance d .

Proof: Consider the LRF $V(k, x)$ which represents optimal $V(x(k))$ at time k , suppose that it gets its maximum value at the delayed states $x(k + \theta)$, $\theta \in \mathbb{Z}_{[-h, 0]}$, that is $\bar{V}(k, x) = x^T(k + \theta)P_\mu^*(k)x(k + \theta)$, $P_\mu^*(k) = \sum_{l=1}^L \mu_l P_l^*(k)$, and $P_l^*(k)$ is an optimal value of $P_l(k)$ at time k , then it can be easily got

$$\rho_{\min}^* \|x(k)\|^2 \leq V(k, x) \leq \rho_{\max}^* \|x(k)\|^2 \quad (3.39)$$

where

$$\begin{aligned} \rho_{\min}^* &:= \min \left\{ \rho_{\min}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1, L]}, k \in \mathbb{N} \right\} \\ \rho_{\max}^* &:= \max \left\{ \rho_{\max}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1, L]}, k \in \mathbb{N} \right\} \end{aligned}$$

where $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ represent the maximal and minimal eigenvalues, respectively.

Moreover, (3.26) implies that

$$V(k, x(k+1)) - \bar{V}(k, x) \leq -(x^T(k)Qx(k) + u^T(k)Ru(k) - \tau d^T(k)d(k)) \quad (3.40)$$

where $V(k, x(k+1)) = x^T(k+1)P_\mu^*(k)x(k+1)$. Thus one has

$$V(k, x(k+1)) - \bar{V}(k, x) \leq -x^T(k)Qx(k) + \tau d^T(k)d(k) \quad (3.41)$$

Owing to the optimality of the control problem at time $k+1$, it holds

$$V(k+1, x(k+1)) \leq V(k, x(k+1)) \quad (3.42)$$

Then it holds that

$$V(k+1, x(k+1)) - \bar{V}(k, x) \leq -x^T(k)Qx(k) + \tau d^T(k)d(k) \quad (3.43)$$

Thus

$$V(k+1, x(k+1)) \leq \bar{V}(k, x) + \tau d^T(k)d(k) \quad (3.44)$$

Letting $\delta \|d(k)\| = \tau d^T(k)d(k)$, thus $\delta \|\cdot\| \in \mathcal{K}$ for $\tau > 0$, which implies that (3.44) is equivalent to (3.5). According to Definition 3.3, (3.39) and (3.44) together yield that $V(k, x)$ is an ISS Lyapunov function. Hence the closed-loop system is ISS with respect to disturbances. The proof is completed.

3.4 Simulation Examples

In this section, simulations on a numerical example and a well-known time-delay continuous stirred-tank reactor (CSTR) system are given to show the effectiveness of the proposed method. The simulations are implemented with MATLAB and all matrix inequalities are solved by YALMIP toolbox [18] with PENBMI solver [19].

3.4.1 Example 3.1

Consider the following T-S fuzzy time-delay systems with maximal delay $h=2$,

Rule 1: If $x_1(k)$ is F_1 , then

$$x(k+1) = \sum_{j=-2}^0 A_{j,1} x(k+j) + B_1 u(k) + E_1 d(k)$$

Rule 2: If $x_1(k)$ is F_2 , then

$$x(k+1) = \sum_{j=-2}^0 A_{j,2} x(k+j) + B_2 u(k) + E_2 d(k)$$

$$\text{where } A_{0,1} = \begin{bmatrix} 1.2 & 0.8 \\ 0 & 1.2 \end{bmatrix}, \quad A_{0,2} = \begin{bmatrix} 0.8 & 0.5 \\ 0 & 0.8 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$A_{-1,1} = A_{-2,1} = 0.1A_{0,1}, \quad A_{-1,2} = A_{-2,2} = 0.1A_{0,2}.$$

The membership functions are given as follows:

$$F_1 = 0.5 + \sin(x_1) \cos(x_1)$$

$$F_2 = 1 - F_1$$

Two RFMPC strategies are investigated:

Strategy 1: The proposed RFMPC approach for systems with time delay and disturbance. In this case, the weighting matrices Q and R in the cost function are selected as $\text{diag}\{1,1\}$ and 1, respectively. As $\sum_{j=-2}^0 \gamma_j = 1$ should be satisfied, let $\gamma_0 = 0.7$, $\gamma_{-1} = 0.15$, and $\gamma_{-2} = 0.15$.

Strategy 2: The RFMPC for the augmented system without delay. Inspired by [20], the time-delay system is modeled as an augmented system without delay, and the RFMPC approach in [2] is applied to the augmented system. In this case, Q and R for the augmented system are selected as $\text{diag}\{1,1,1,1,1,1\}$ and 1, respectively.

The input constraint and the bounded disturbance are given as: $\|u\| \leq 3$, $\|d\| \leq 1$.

The initial system (delayed) states are chosen as $x(i) = [-2.5; 1.7]$ for all $i \in \mathbb{Z}_{[-2,0]}$. The simulation results are shown in Figure 3.1. One can see that the proposed method (Strategy 1) achieves a better control performance than Strategy 2 where the augmented system without delay is adopted. It can also be seen that the system states vary in a very small range near the origin when persistent disturbance is added. The disturbance is eliminated after $k=16$, then asymptotic stability to the origin is achieved. In addition, Figure 3.1 shows that the input constraint is guaranteed. The computational complexity of both methods is shown in Table 3.1. It can be seen that the numerical complexity of Strategy 2 is much greater than that of proposed method.

Figure 3.1 Trajectories of states and inputs.

Table 3.1 Computational Complexities of Different Methods.

Methods	Strategy 1	Strategy 2
The number of decision variables (D)	19	99
Lines of the LMI (L)	199	307
Numerical complexity ($C = D^3L$)	1364941	297881793

3.4.2 Example 3.2

Consider the well-known CSTR system in [21], [22] where time delay is introduced. Suppose the time delay is $t_d = 0.2$ minutes. Then the sampling time is set as: $T_s = 0.2$ minutes, the discrete T-S fuzzy time-delay systems can be described as follows, where parameters are obtained from [22] after discretization.

Rule 1: If $x_2(k)$ is F_1 , then

$$x(k+1) = \sum_{j=-1}^0 A_{j,1} x(k+j) + B_1 u(k) + E_1 d(k)$$

Rule 2: If $x_2(k)$ is F_2 , then

$$x(k+1) = \sum_{j=-1}^0 A_{j,2} x(k+j) + B_2 u(k) + E_2 d(k)$$

Rule 3: If $x_2(k)$ is F_3 , then

$$x(k+1) = \sum_{j=-1}^0 A_{j,3} x(k+j) + B_3 u(k) + E_3 d(k)$$

where

$$A_{0,1} = \begin{bmatrix} 0.7500 & 0.0119 \\ -0.2238 & 0.8262 \end{bmatrix}, A_{0,2} = \begin{bmatrix} 0.6203 & 0.0762 \\ -1.2337 & 1.3265 \end{bmatrix},$$

$$A_{0,3} = \begin{bmatrix} 0.3068 & 0.0442 \\ -3.6621 & 1.0765 \end{bmatrix}, A_{-1,1} = \begin{bmatrix} 0.0435 & 0.0003 \\ -0.0061 & 0.0455 \end{bmatrix},$$

$$A_{-1,2} = \begin{bmatrix} 0.0403 & 0.0019 \\ -0.0312 & 0.0581 \end{bmatrix}, A_{-1,3} = \begin{bmatrix} 0.0310 & 0.0013 \\ -0.1037 & 0.0528 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0004 \\ 0.0546 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0023 \\ 0.0698 \end{bmatrix}, B_3 = \begin{bmatrix} 0.0015 \\ 0.0634 \end{bmatrix}, E_1 = E_2 = E_3 = \begin{bmatrix} 0 \\ 0.00001 \end{bmatrix}.$$

Figure 3.2 Membership functions.

The membership functions shown in Figure 3.2 are given as follows:

$$F_1 = \begin{cases} 1 & \text{if } x_2 \leq 0.8882 \\ \frac{2.7520 - x_2}{2.7520 - 0.8882} & \text{if } 0.8882 < x_2 \leq 2.7520 \\ 0 & \text{if } x_2 > 2.7520 \end{cases}$$

$$F_2 = \begin{cases} 1 - F_1 & \text{if } 0.8882 < x_2 \leq 2.7520 \\ 1 - F_3 & \text{if } 2.7520 < x_2 \leq 4.7052 \end{cases}$$

$$F_3 = \begin{cases} 0 & \text{if } x_2 \leq 2.7520 \\ \frac{x_2 - 2.7520}{4.7052 - 2.7520} & \text{if } 2.7520 < x_2 \leq 4.7052 \\ 1 & \text{if } x_2 > 4.7052 \end{cases}$$

The input constraint and bounded disturbance are set as: $\|u\| \leq 10$, $\|d\| \leq 1$.

There are three nominal states when $u=0$: $x_{s1}=[0.1440; 0.8862]$, $x_{s2}=[0.4472; 0.7520]$, and $x_{s3}=[0.7646; 4.7052]$. The reference signal is

changed from x_{s3} , to x_{s2} , x_{s1} , and x_{s2} . The initial system (delayed) states are chosen as $x(i) = [0.6972; 3.652]$ for $i \in \mathbb{Z}_{[-1,0]}$.

Case 1: The CSTR system with time delay is investigated.

The following RFMPC strategies are applied:

Strategy 1 for Case 1: The proposed RFMPC approach for systems with time delay and disturbance. In this case, the weighting matrices Q and R in the cost function are selected as $\text{diag}\{1e-6, 1e-9\}$ and 0.001, respectively. As

$\sum_{j=-1}^0 \gamma_j = 1$ should be satisfied, let $\gamma_0 = 0.8$, $\gamma_{-1} = 0.2$.

Strategy 2 for Case 1: The RFMPC for the augmented system without delay. Inspired by [20], the time-delay system is modeled as an augmented system without delay, and the RFMPC approach in [2] is applied to the augmented system. In this case, Q and R for the augmented system are selected as $\text{diag}\{1e-6, 1e-8, 1e-6, 1e-8\}$ and 0.001, respectively.

Strategy 3 for Case 1: The RFMPC in [2] without considering the time delay effect. In this case, Q and R are selected the same as in Strategy 1 for Case 1.

The simulation results for Case 1 by adopting the aforementioned approaches are shown in Figure 3.3. It shows that the proposed method (Strategy 1 for Case 1) achieves best control performance among all approaches, while the RFMPC without considering delay (Strategy 3 for Case 1) is worst. It also shows that input constraint is guaranteed with all approaches. The computational complexities of all approaches are shown in Table 3.2. One can see that RFMPC for the augmented system without delay (Strategy 2 for Case 1) is worst in terms of computational complexity. The computational complexity via proposed method is greater than that of RFMPC without considering delay. However, compared with Strategy 2, the difference of computational complexity between Strategy 1 and Strategy 3 can be ignored. As such, it can be concluded that a good control performance can be achieved via the proposed approach, and the computational complexity is mild.

Figure 3.3 Trajectories of states and input.

Table 3.2 Computational Complexities of Different Methods.

Methods	Strategy 1	Strategy 2	Strategy 3
The number of decision variables (D)	27	75	27
Lines of the LMI (L)	495	666	378
Numerical complexity ($C = D^3 L$)	9743085	280968750	7440174

Case 2: The CSTR system without time delay is investigated.

The following RFMPC strategies are applied:

Strategy 1 for Case 2: The proposed RFMPC approach for systems without time delay. In this case, the system without time delay is modeled as a time-delay system where the state matrices for delayed states are set as $A_{-1,1} = A_{-1,2} = A_{-1,3} = 0$. Q and R are selected as $\text{diag}\{1e-6, 1e-9\}$ and 0.001, respectively. And let $\gamma_0 = 0.8$, $\gamma_{-1} = 0.2$.

Strategy 2 for Case 2: The proposed RFMPC approach for systems without time delay. The parameters are selected the same as in Strategy 1 for Case 2, except that $\gamma_0 = 1$, $\gamma_{-1} = 0$.

Strategy 3 for Case 2: The RFMPC in [2]. In this case, Q and R are selected the same as in Strategy 1 for Case 2.

The simulation results for Case 2 are shown in Figure 3.4. It can be seen that, for systems without delay, the proposed method with $\gamma_0 = 0.8$ and $\gamma_{-1} = 0.2$ performs better than the other two approaches. In addition, if $\gamma_0 = 1$ and $\gamma_{-1} = 0$ are adopted instead, then the proposed method has the same control performance as [2], which implies that the proposed approach is equivalent to the method in [2] under such circumstances. Therefore, the proposed method regarding time delay is more general than the case that time delay is not considered.

3.5 Conclusions

FMPC for a class of discrete nonlinear systems with time delays, persistent disturbances, and input constraints is investigated in this chapter. The nonlinear time-delay systems are represented by T-S fuzzy systems comprised of linear DDEs. Based on the Lyapunov-Razumikhin approach, new conditions for input-to-state stability and robust positive invariance property under such circumstances are provided. Simulation results show the effectiveness and advantages of proposed method when applied to the CSTR system.

Figure 3.4 Trajectories of states and input.

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Chapter 4

Efficient Robust Fuzzy Model Predictive Control of Discrete Nonlinear Time-Delay Systems via Razumikhin Approach

In this chapter, two efficient robust fuzzy model predictive control algorithms are investigated for discrete nonlinear time-delay systems with bounded disturbances. Lyapunov-Razumikhin function is employed for time-delay systems because it involves invariant sets in the original state space of system dynamics. A sequence of explicit control laws corresponding to a sequence of constraint sets are computed off-line that the online computational burden associated with classical model predictive control algorithms is significantly reduced. In particular, the set invariance theory behind Razumikhin approach, which is more complicated than the one for non-delayed systems, can be directly observed. And it is proved that system all (delayed) states can enter the terminal set in finite time. Moreover, robust positive invariance and input-to-state stability for time-delay systems concerning disturbances are realized. In addition, an online optimization algorithm is also provided based on the off-line computed ellipsoidal sets. Thus the conservatism induced by Razumikhin approach is relaxed, while the computational cost is not significantly increased.

4.1 Introduction

One of the main drawbacks of classical MPC strategies is that it requires solving online an optimization problem, which prevents its applications to some contexts due to the high computational cost. Thus several MPC schemes have been developed to reduce the online computational burden, i.e., the explicit MPC and efficient MPC algorithms, see [1] and references therein. For the efficient implementation of MPC, a sequence of explicit control laws corresponding to a sequence of constraint sets are computed off-line. As a consequence, the online computation is reduced to search the smallest constraint set where the state belongs and apply the pre-determined controller. It is noted that reasonable works can be seen on efficient MPC for non-delayed systems, see, e.g., [1]-[4]. However, exploiting efficient solutions for time-delay systems, there are only a few works for linear time-delay systems [5]-[7], where only the Krasovskii approach is investigated that these stabilization approaches are very complex.

In contrast to Chapter 3 which requires online optimization, two efficient robust MPC approaches for time-delay systems are provided in this work. Compared with LKF, LRF is not a simple extension of traditional Lyapunov function to time-delay systems, thus new conditions for robust positive invariance property and finite-time control performance should be established for efficient MPC of time-delay systems. Firstly, an off-line MPC method is proposed for discrete nonlinear systems where time delays and disturbances are involved simultaneously. The LRF is adopted that a sequence of robust constraint sets associated with a sequence of explicit control laws are constructed in the original state space of system dynamics, with which system all (delayed) states are guaranteed to enter the terminal constraint set in finite time and ISS is achieved. Furthermore, based on the obtained robust constraint sets, a simple online optimization algorithm is also introduced to reduce the conservatism caused by Razumikhin approach. Compared with several existing FMPC

approaches, a less conservative result is achieved via the proposed algorithms. Particularly, the special control theory and positive invariance behind Razumikhin approach is revealed. The relevant constraint sets, where system current state and all (delayed) states belong, can be observed directly by the proposed method. Thus one can see the non-monotonic characteristic of LRF which makes it more flexible and easier for systems with severe complexities and constraints where it may be difficult to find a monotonic Lyapunov function, when compared to LKF.

4.2 System Description

In this work, the off-line MPC approach of the T-S fuzzy system with time delays and disturbances in Chapter 3 is investigated. The T-S fuzzy system is given as:

Plant Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$x(k+1) = \sum_{i=-h}^0 A_{i,l} x(k+i) + B_l u(k) + E_l d(k) \quad (4.1)$$

where, $z := [z_1, \dots, z_v]$ are premise variables. $F_1^l, \dots, F_v^l$ are fuzzy sets. $l \in \mathbb{Z}_{[1,L]}$, and L is the number of fuzzy rules. The T-S fuzzy system (3.7) can be written as,

$$x(k+1) = \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k) + E_\mu d(k) \quad (4.2)$$

where $A_{i,\mu} := \sum_{l=1}^L \mu_l(z) A_{i,l}$, $B_{i,\mu} := \sum_{l=1}^L \mu_l(z) B_{i,l}$, $E_\mu := \sum_{l=1}^L \mu_l(z) E_l$, $\mu_l(z)$ is normalized membership function.

Control method for above T-S fuzzy system is given as follows:

Control Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$u(k) = K_l x(k)$$

Thus the final control output is given as,

$$u(k) = K_\mu x(k) \quad (4.3)$$

where $K_\mu = \sum_{l=1}^L K_l$.

4.3 LRF-based Off-line MPC Approach

In off-line MPC schemes, the approximation constraint set can be either ellipsoidal or polytopic [8]. In this work, the former one is utilized for the construction of terminal constraint set and robust constraint sets concerning its simple representation. However, it should be noted that the latter one is more flexible than the former one.

For non-delayed systems, selection of an appropriate control law only requires to look up all acquired constraint sets and find the smallest one where system current state belongs, see [2]. In contrast, in this work where Razumikhin approach is adopted for time-delay systems, the appropriate control law is determined not only by the system current state but also by all delayed states. Firstly, the smallest constraint set where all (delayed) states belong should be found out, then the corresponding control law is applied.

Moreover, for non-delayed systems it requires only one step to steer the system state from one constraint set to a neighboring one via the corresponding off-line control law, see details of “robust one-step set” for non-delayed systems in [9]. As a consequence, the system state can enter the terminal constraint set in finite time steps. The finite-time control performance is still effective for the proposed LRF-based off-line MPC approach, however, the principle behind is more complicated and it will be proved and verified in this work.

Remark 4.1: As the Krasovskii approach for time-delay systems can be regarded as an extension of the classical Lyapunov function for non-delayed

systems, thus both of the two share the same principles in Lyapunov function and set invariance theory. To better clarify the special properties of the proposed efficient MPC algorithm based on the Razumikhin approach, above and in what follows, the review of non-delayed systems which have been well recognized is provided for comparison purpose.

4.3.1 Terminal Constraint Set

The terminal constraint set should satisfy the following two requirements:

Firstly, the terminal constraint set should be an RPI set. Let quadratic LRF $V(x) := x^T P_t x$, denote $\Omega_t \in \mathbb{R}^n$ as the following ellipsoidal set

$$\Omega_t := \{x \mid \bar{V}(x(k)) \leq \xi\} \quad (4.4)$$

where $\bar{V}(x(k))$ is defined in (3.5), with $i \in \mathbb{Z}_{[-h,0]}$. From the definition of the quadratic LRF $V(x)$ and $\bar{V}(x)$, It can be seen that Ω_t is a convex set.

Lemma 4.1: The set Ω_t is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\frac{1}{\xi} V(x^+) - \frac{1-\lambda}{\xi} \bar{V}(x(k)) - \frac{\lambda}{\eta^2} w^T(k)w(k) \leq 0 \quad (4.5)$$

Proof: According to Definition 3.1, move the last two items in (4.5) to the right side. As $\bar{V}(x(k)) \leq \xi$ and $w^T(k)w(k) \leq \eta^2$, then $\frac{1}{\xi} V(x^+) \leq (1-\lambda) + \lambda = 1$, thus

$V(x^+) \leq \xi$, proof is completed. It can be seen that disturbance is involved through this way, and this result is extended from the one for non-delayed systems in [9].

Secondly, there exist $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$, and a positive definite function $V(x)$ that $\forall x \in \Omega_t$,

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|) \quad (4.6)$$

$$V(x^+) - \bar{V}(x(k)) \leq -x^T(k)Qx(k) - u^T(k)Ru(k) \quad (4.7)$$

In the following, the conditions which satisfy the aforementioned requirements are provided.

Theorem 4.1: Consider T-S fuzzy system (4.2), if there exist a positive definite matrix X_t , general matrices Y_a (or Y_b , Y_l presented below), G_b , Z , and positive scalars τ , $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Xi_0 & * & * & * & * & * \\ 0 & \Xi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \dots & \Xi_h & * & * \\ 0 & 0 & \dots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \Theta_0 & \Theta_1 & \dots & \Theta_h & E_a & -X_t \end{bmatrix} \leq 0 \quad (4.8)$$

$$\begin{bmatrix} \Gamma_0 & * & * & * & * & * & * & * \\ 0 & \Gamma_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \dots & \Gamma_h & * & * & * & * \\ 0 & 0 & \dots & 0 & -\tau\xi I & * & * & * \\ \Theta_0 & \Theta_1 & \dots & \Theta_h & \xi E_a & -X_t & * & * \\ QG_b & 0 & \dots & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & \dots & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (4.9)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_t \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (4.10)$$

where $\Xi_{-i} = \gamma_i(\lambda - 1)(G_b + G_b^T - X_t)$, $\Gamma_{-i} = \gamma_i(X_t - G_b - G_b^T)$, $i \in \mathbb{Z}_{[-h,0]}$,

$\Theta_0 = A_{0,a}G_b + B_aY_b$, $\Theta_{-j} = A_{j,a}G_b$, $j \in \mathbb{Z}_{[-h,-1]}$, $a, b, l \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th

diagonal element of matrix Z , γ_i are pre-set valued positive numbers that

$\sum_{i=-h}^0 \gamma_i = 1$, then a terminal constraint set $\Omega_t = \{x | x^T P_t x \leq \xi\}$, with

$P_t = \xi X_t^{-1}$, is obtained for system (4.2), and the corresponding feedback

control law is given as $\pi_t = K_\mu x$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$.

Proof: Firstly, we show that (4.8) implies (4.5). Resorting to the dilation lemma [10],

$$-G_b^T X_t^{-1} G_b \leq X_t - G_b^T - G_b \quad (4.11)$$

The following inequality is obtained from (4.8),

$$\begin{bmatrix} M_0 & * & * & * & * & * \\ 0 & M_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & M_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & E_a & -X_t \end{bmatrix} \leq 0 \quad (4.12)$$

where $M_{-i} = \gamma_i(\lambda-1)G_b^T X_t^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, \dots, G_b^{-T}, I, I\}$

and its transpose from both sides of (4.12), respectively, yields that

$$\begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -X_t \end{bmatrix} \leq 0 \quad (4.13)$$

where $\Delta_{-i} = \gamma_i(\lambda-1)X_t^{-1}$, $\tilde{A}_0 = A_{0,a} + B_a K_b$, $\tilde{A}_j = A_{j,a}$, $j \in \mathbb{Z}_{[-h,-1]}$. Based on

which one has

$$\sum_{a=1}^L \sum_{b=1}^L \mu_a \mu_b \begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -X_t \end{bmatrix} \leq 0 \quad (4.14)$$

where $\sum_{a=1}^L \mu_a = 1$, $\sum_{b=1}^L \mu_b = 1$.

The above inequality can be written as

$$\begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \tilde{A}_{0,\mu} & \tilde{A}_{-1,\mu} & \cdots & \tilde{A}_{-h,\mu} & E_\mu & -X_t \end{bmatrix} \leq 0 \quad (4.15)$$

where $\tilde{A}_{0,\mu} = A_{0,\mu} + B_\mu K_\mu$, $\tilde{A}_{j,\mu} = A_{j,\mu}$, $j \in \mathbb{Z}_{[-h,-1]}$. Applying schur complement to (4.15), and multiplying $[x^T(k), x^T(k-1), \dots, x^T(k-h), w^T(k)]$ and its transpose from both sides of the resulting matrix inequality, respectively, yields that

$$\varpi \begin{bmatrix} \Delta_0 & & & & & \\ & \Delta_1 & & & & \\ & & \ddots & & & \\ & & & \Delta_h & & \\ & & & & -\frac{\lambda}{\eta^2}I & \end{bmatrix} \varpi^T + \varpi \Lambda X_t^{-1} \Lambda^T \varpi^T \leq 0 \quad (4.16)$$

where $\varpi = [x^T(k), x^T(k-1), \dots, x^T(k-h), w^T(k)]$,

$$\Lambda = [\tilde{A}_{0,\mu}, \tilde{A}_{-1,\mu}, \dots, \tilde{A}_{-h,\mu}, E]^T.$$

By substituting X_t^{-1} with P_t / ξ , (4.16) is equivalent to

$$\begin{aligned} & (\lambda - 1) \sum_{i=-h}^0 \gamma_i x^T(k+i) X_t^{-1} x(k+i) - \frac{\lambda}{\eta^2} w^T(k) w(k) \\ & + \left(\sum_{i=-h}^0 x^T(k+i) \tilde{A}_{i,\mu}^T \right) X_t^{-1} \left(\sum_{i=-h}^0 x^T(k+i) \tilde{A}_{i,\mu}^T \right)^T \\ & = \frac{\lambda - 1}{\xi} \sum_{i=-h}^0 \gamma_i x^T(k+i) P_t x(k+i) - \frac{\lambda}{\eta^2} w^T(k) w(k) + \frac{1}{\xi} x^{+T} P_t x^+ \\ & = \frac{\lambda - 1}{\xi} \sum_{i=-h}^0 \gamma_i V(x(k+i)) - \frac{\lambda}{\eta^2} w^T(k) w(k) + \frac{1}{\xi} V(x^+) \leq 0 \end{aligned} \quad (4.17)$$

As $\sum_{i=-h}^0 \gamma_i V(x(k+i)) \leq \bar{V}(x(k))$ due to $\sum_{i=-h}^0 \gamma_i = 1$, thus (4.5) can be got, which verifies that (4.5) is guaranteed by (4.8).

It is noted that (4.6) can be easily got by resorting to the eigenvalues of P_t . (4.9) which guarantees (4.7) can be verified by following a similar proof for (4.8). The input constraint is satisfied by (4.10) and the derivation can be seen in [11]. The proof is thus complete.

Theorem 4.2: With the computed terminal constraint set Ω_t and the corresponding state feedback control law $u(k) = K_{\mu}x(k)$, the closed-loop system (4.2) is ISS with respect to disturbance w .

Proof: It is noted that (4.8) guarantees the RPI property of Ω_t described by (4.5), and (4.9) guarantees the satisfaction of the optimization problem (4.7). From (4.8)-(4.10), one can see that the computation of Ω_t and the corresponding state feedback control law is independent of a specific state, which indicates that the corresponding state feedback control law is feasible for any state in Ω_t . Meanwhile, the RPI property is guaranteed that system states are trapped in the terminal set for all time. And the feedback control law guarantees the satisfaction of (4.7) at each time instant. It is noted that (4.7) leads to

$$V(x^+) < \bar{V}(x(k)) + w^T(k)w(k) \quad (4.18)$$

which implies (3.5) is guaranteed. In addition, (3.4) can be easily satisfied by resorting to the eigenvalues of P_t . Therefore, from Definition 3.3 it can be got that $V(x)$ is an ISS Lyapunov function for the closed-loop system (4.2). It is noted that Definition 3.3 leads to Definition 3.2, see details in [12], thus ISS is achieved.

4.3.2 A Sequence of Robust Constraint Sets

Denote the obtained terminal set Ω_t and corresponding control law π_t , respectively, as Ω_0 and π_0 . A robust constraint set Ω_1 ($\Omega_1 \supseteq \Omega_0$) is employed

such that system all (delayed) states $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_1$. According to Definition 3.1, a control law π_1 is given such that the system state in the next instant $x^+ \in \Omega_0$, and it is easy to see that the positive invariance is guaranteed due to $\Omega_0 \subseteq \Omega_1$. When Ω_1 and corresponding control law π_1 are obtained, in a similar way, then a sequence of robust constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_e, \dots, \Omega_{\tilde{N}}\}$ and corresponding control laws $\{\pi_0, \pi_0, \dots, \pi_e, \dots, \pi_{\tilde{N}}\}$ can be got, $\Omega_e \in \mathbb{R}^n$ with $e \in \mathbb{Z}_{[1, \tilde{N}]}$, $\tilde{N}$ ($\tilde{N} \geq 1$) represents the number of constraint sets.

Firstly, without loss of generality, at instant k , assume system all (delayed) states $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$, the design principle requires that $x^+ \in \Omega_{e-1}$. It can be seen that the RPI property is guaranteed as $\Omega_{e-1} \subseteq \Omega_e$. Therefore, based on which one has

$$x^{+T} X_{e-1}^{-1} x^+ - (1 - \lambda) \max_{i \in \mathbb{Z}_{[-h,0]}} \{x^T(k+i) X_e^{-1} x(k+i)\} - \frac{\lambda}{\eta^2} w^T(k) w(k) \leq 0 \quad (4.19)$$

where X_e and X_{e-1} are associated with two neighboring robust constraint sets $\Omega_e = \{x \mid x^T X_e^{-1} x \leq 1\}$ and $\Omega_{e-1} = \{x \mid x^T X_{e-1}^{-1} x \leq 1\}$, respectively. Similar to (4.5), (4.19) guarantees $x^{+T} X_{e-1}^{-1} x^+ \leq 1$, provided that $\max_{i \in \mathbb{Z}_{[-h,0]}} \{x^T(k+i) X_e^{-1} x(k+i)\} \leq 1$.

Secondly, the optimization control is formulated as follows,

$$x^{+T} P_e x^+ - \max_{i \in \mathbb{N}_{[-h,0]}} \{x^T(k+i) P_e x(k+i)\} < -x^T(k) Q x(k) - u^T(k) R u(k) \quad (4.20)$$

Theorem 4.3: Assume Ω_{e-1} is already known, if there exist positive definite matrix X_e , general matrices Y_a (or Y_b , Y_l presented below), Z , and positive scalars ξ_e , $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Psi_0 & * & * & * & * & * \\ 0 & \Psi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \dots & \Psi_h & * & * \\ 0 & 0 & \dots & 0 & -\frac{\lambda}{\eta^2} I & * \\ \Pi_0 & \Pi_1 & \dots & \Pi_h & E_a & -X_e \end{bmatrix} \leq 0 \quad (4.21)$$

$$\begin{bmatrix} \Phi_0 & * & * & * & * & * & * \\ 0 & \Phi_1 & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * \\ 0 & 0 & \dots & \Phi_h & * & * & * \\ \Pi_0 & \Pi_1 & \dots & \Pi_h & -X_e & * & * \\ QX_e & 0 & \dots & 0 & 0 & -\xi_e Q & * \\ RY_b & 0 & \dots & 0 & 0 & 0 & -\xi_e R \end{bmatrix} \leq 0 \quad (4.22)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & X_e \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (4.23)$$

where $\Psi_{-i} = \gamma_i(\lambda - 1)X_e$, $\Phi_{-i} = -\gamma_i X_e$, $i \in \mathbb{Z}_{[-h,0]}$, $\Pi_0 = A_{0,a}X_e + B_a Y_b$, $\Pi_{-j} = A_{j,a}X_e$, $j \in \mathbb{Z}_{[-h,-1]}$, $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ_i are pre-set valued positive numbers that $\sum_{i=-h}^0 \gamma_i = 1$, then the set $\Omega_e := \{x \mid x^T P_e x \leq \xi_e\}$, with $P_e = \xi_e X_e^{-1}$, is a robust constraint set for T-S fuzzy system (4.2), and the corresponding feedback control law π_e will steer system state to Ω_{e-1} at the next instant. π_e is given as $\pi_e = K_\mu x$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l X_e^{-1}$.

Proof: It is noted that (4.21) and (4.22) satisfy (4.19) and (4.20), respectively. (4.23) fulfills the input constraint. The proof is very similar to that of Theorem 4.1. Thus it is omitted here.

4.3.3 Control Algorithm

Based on the above results, the whole control algorithm is summarized to complete the off-line FMPC method, and then the finite-time control performance is analyzed.

Algorithm 1:

1. Off-line computation

Step 1. Compute a terminal constraint set Ω_0 and the corresponding control law π_0 by adopting Theorem 4.1;

Step 2. Based on the obtained terminal constraint set, a sequence of constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_e, \dots, \Omega_{\tilde{N}}\}$ together with corresponding control laws can be got by Theorem 4.3;

2. Online computation

Step 1. Consider an initial condition $x(i)_{i \in \mathbb{Z}_{[-h,0]}}$ which satisfies $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$.

Let $k=0$ and go to Step 2;

Step 2. At time instant k , obtain system all (delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}}$, implement the search to decide $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e \setminus \Omega_{e-1}$, if $\Omega_e = \Omega_0$, then the corresponding control law for Ω_0 is applied all the time. Otherwise, the corresponding control law for Ω_e is applied, let $k=k+1$ and repeat Step 2.

Remark 4.2: In Step 2 of off-line computation of robust constraint set, we use $-\log \det(X_e)$ as the objective function and minimize it to get the constraint set with maximal size.

Theorem 4.4: If a sequence of robust constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_e, \dots, \Omega_{\tilde{N}}\}$ can be computed by Theorem 4.1 and Theorem 4.3, with the initial conditions satisfying $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$, then it requires no more than $\tilde{N} \cdot (h+1)$ steps to steer system all (delayed) states to the terminal constraint set.

Proof: Without loss of generality, in time instant k , suppose the system all (delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$. Firstly, the control law π_e is applied to drive $x(k+1)$ to Ω_{e-1} . As a result, at time instant $k+1$, it can be seen that $x(k+1) \in \Omega_{e-1}$, while $x(k+1+i)_{i \in \mathbb{Z}_{[-h,-1]}} \in \Omega_e$. Therefore, the control law π_e , rather than π_{e-1} , will still be adopted at instant $k+1$ to drive $x(k+2)$ to Ω_{e-1} . By repeating the process, it can be concluded that it takes $h+1$ steps to drive system all (delayed) states from Ω_e to Ω_{e-1} , which is different from the context for non-delayed systems where only one step is required. Therefore, in the case of $\tilde{N}$ robust constraint sets, it takes no more than $\tilde{N} \cdot (h+1)$ steps to drive system all (delayed) states to the terminal constraint set. Proof is thus completed.

Remark 4.3: It can be seen that the system states can be converged to a small region near the origin regarding persistent disturbances. From Theorem 4.4, we can conclude that system all (delayed) states can be steered to the terminal constraint set, given the initial states in a specific region. After that, as mentioned in Theorem 4.2, the control law that corresponds with the terminal set will steer system all states into a small region near the origin and ISS is achieved.

Remark 4.4: It is noted that ellipsoidal MPC algorithms have been widely investigated for non-delayed systems [1]-[4], [9]. For time-delay systems, a few results can be seen for linear parameter-varying (LPV) systems [5]-[7], where LKF is utilized that the control synthesis and computation are usually complex, and disturbance is not involved in these works.

Remark 4.5: Utilization of LKF conditions for the construction of ellipsoidal sets for time-delay systems can be regarded as an extension of the classical Lyapunov theory for non-delayed systems [5], although the LKF-based solutions are more complicated as LKF contains several segments. In contrast, ellipsoidal off-line MPC algorithms based on LRF is very different from the contexts for both non-delayed systems and time-delay systems based on LKF.

As mentioned in Theorem 4.4, in the case of h delayed states, it takes $h+1$ steps to steer system all (delayed) states to the neighboring subset, thus it is more complicated than the concept of “robust one-step set” for non-delayed systems in [9]. We will show the special RPI property and finite-time control theory based on LRF in the following example.

4.3.4 Example 4.1

In this subsection, simulation on a numerical example is presented to verify the effectiveness of the proposed method. The simulation is implemented with MATLAB and all matrix inequalities are solved by YALMIP toolbox [13].

Consider the following T-S fuzzy time-delay systems with $h=2$,

Rule 1: If $x_1(k)$ is F_1 , then

$$x^+ = \sum_{i=-2}^0 A_{i,1}x(k+i) + B_1u(k) + E_1w(k)$$

Rule 2: If $x_1(k)$ is F_2 , then

$$x^+ = \sum_{i=-2}^0 A_{i,2}x(k+i) + B_2u(k) + E_2w(k)$$

where

$$\begin{aligned} A_{0,1} &= \begin{bmatrix} 1.5 & 2.5 \\ 0 & 1.5 \end{bmatrix}, \quad A_{0,2} = \begin{bmatrix} 0.8 & 1.5 \\ 0 & 0.8 \end{bmatrix}, \\ A_{-1,1} &= A_{-2,1} = 0.05A_{0,1}, \quad A_{-1,2} = A_{-2,2} = 0.05A_{0,2}, \\ B_1 &= B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \end{aligned}$$

The membership functions are given as follows:

$$\begin{aligned} F_1 &= 0.5 + \sin(x_1) \cos(x_1) \\ F_2 &= 1 - F_1 \end{aligned}$$

The matrices Q and R in the cost function are selected as $\text{diag}\{1,1\}$ and 0.1, respectively. The input constraint and bounded disturbance are given as: $\|u\| \leq 2$,

$\|w\| \leq 0.1$. As the constraint $\sum_{i=-2}^0 \gamma_i = 1$ should be satisfied, let $\gamma_0 = 0.7$, $\gamma_{-1} = 0.15$, $\gamma_{-2} = 0.15$.

The simulation results are shown in Figure 4.1, where the initial conditions are given as $x(i)_{i \in \mathbb{Z}_{[-2,0]}} = [3; -1.5]$, $x(i)_{i \in \mathbb{Z}_{[-2,0]}} = [-3; 1.5]$, $x(i)_{i \in \mathbb{Z}_{[-2,0]}} = [0.12; -0.68]$, $x(i)_{i \in \mathbb{Z}_{[-2,0]}} = [-0.12; 0.68]$, respectively. The number of robust constraint sets including the terminal constraint set is given as 25. It can be seen that, despite different initial conditions, the system all (delayed) states enter the terminal constraint set in the end.

Figure 4.1 Trajectories in state space for four different initial conditions.

Figure 4.2 provides a clearer view of the relevant constraint sets where all (delayed) states and the current state belong, respectively, at each time instant. Set 1 represents the smallest constraint set (terminal constraint set) in Figure 4.1, while set 25 represents the largest constraint set. It can be seen that:

- (1) In the existence of delayed states, at each instant, the constraint set for current state is always a (non-strict) subset of the constraint set for all (delayed) states.

- (2) Consider the constraint set for all (delayed) states at each instant, except the terminal set, the corresponding control effort for this set will steer system current state to a strict subset at the next instant.
- (3) It takes no more than 3 (i.e., $h+1$, here $h=2$) steps to drive system all (delayed) states from one constraint set to a strict subset. Thus the finite-time control performance in Theorem 4.4 is verified.
- (4) The second histogram also shows that system current state at instant 2 enters a superset at the next instant. Thus it verifies that the Lyapunov function can be non-monotone in the Razumikhin approach.

Figure 4.2 Constraint sets for all (delayed) states and the current state at each time instant for two different initial conditions.

Therefore, from above example, one can observe the special control theory and positive invariance behind the Razumikhin approach that all these characteristics are different from those for non-delayed systems. And therefore,

as discussed in Remarks 4.1 and 4.5, they are also different from the contexts of the Krasovskii approach for time-delay systems.

However, in the aforementioned LRF-based ellipsoidal offline FMPC approach, the obtained result is conservative due to several reasons: One reason is the intrinsic conservatism induced by the Razumikhin approach; Another one is that, the control law is off-line computed thus the system information (the delayed states $\mathbf{x}_{[k-h,k]}$) is not completely used for controller design at each instant. In the next section, a combination of the aforementioned off-line computed robust constraint sets and an online optimization algorithm is investigated to reduce the conservatism.

4.4 Online Optimization Approach

4.4.1 Online Optimization Algorithm

Consider the following disturbance-free prediction model,

$$x(k+s+1|k) = \sum_{i=-h}^0 (A_{i,\mu} x(k+i+s|k) + B_{\mu} u(k+s|k)) \quad (4.24)$$

A finite horizon cost function is optimized at each time instant, that is

$$J_N(k) = \sum_{s=0}^{N-1} \ell(k+s|k) + V_T(x(k+N|k)) \quad (4.25)$$

Denote

$$J_1(k) := x^T(k) Q x(k) + u^T(k) R u(k) + x^T(k+1|k) P_{e-1} x(k+1|k) \quad (4.26)$$

and minimization of $J_1(k)$ is carried out online. Without loss of generality, assume all (delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$ at time k . The optimization problem can be formulated as

$$\min_{u(k)} \psi, \quad (4.27)$$

subject to

$$J(k) \leq \psi, \quad (4.28)$$

$$x(k+1|k) \in \Omega_{e-1}, \quad (4.29)$$

$$u(k) \in U. \quad (4.30)$$

where $\Omega_{e-1} := \{x \mid x^T P_{e-1} x \leq \xi_{e-1}\}$. In the following, we will show that the constraints of the optimization problem (4.27) can be represented by some linear matrix inequalities (LMIs).

Substituting $x^+ = \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k)$ into (4.28), then it is not difficult to get that (4.28) is equivalent to the following inequality,

$$\begin{bmatrix} -\psi + x^T(k) Q x(k) & * & * \\ \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k) & -P_{e-1}^{-1} & * \\ u(k) & 0 & -R^{-1} \end{bmatrix} \leq 0 \quad (4.31)$$

Considering (4.29), which is equivalent to

$$x^{+T} P_{e-1} x^+ \leq \xi_{e-1} \quad (4.32)$$

Taking into account the disturbance, then (4.32) holds if

$$x^{+T} P_{e-1} x^+ - \xi_{e-1} - \lambda(w^T(k)w(k) - \eta^2) \leq 0 \quad (4.33)$$

As $x^+ = \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k) + E_\mu w(k)$, the above equation holds if

$$\begin{bmatrix} -\xi_{e-1} + \lambda\eta^2 & * & * \\ 0 & -\lambda & * \\ \sum_{i=-h}^0 A_{i,\mu} x(k+i) + B_\mu u(k) & E_\mu & -P_{e-1}^{-1} \end{bmatrix} \leq 0 \quad (4.34)$$

The input constraint is satisfied by

$$\begin{bmatrix} Z & u(k) \\ u^T(k) & I \end{bmatrix} \geq 0 \quad (4.35)$$

In summary, the constrained online optimization problem (4.27) is given as

$$\min_{u(k), \psi, \lambda, Z} \psi, \quad (4.36)$$

subject to (4.31), (4.34), and (4.35), with the delayed states $\mathbf{x}_{[k-h,k]}$ known.

Therefore, the whole algorithm can be summarized as follows:

Algorithm 2:**1. Off-line computation**

The off-line computation is focused on generating a sequence of robust constraint sets, the steps are the same as the ones in Algorithm 1.

2. Online computation

Step 1. Similar to Algorithm 1, we consider an initial condition which satisfies $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\bar{N}}$. Let $k=0$ and go to Step 2;

Step 2. At time instant k , obtain the system all (delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}}$, execute the search to decide $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e \setminus \Omega_{e-1}$, if $\Omega_e = \Omega_0$, then the corresponding control law for Ω_0 is applied all the time. Otherwise, go to Step 3.

Step 3. Solve the optimization problem (4.36), obtain the control input and then feed the plant with it, let $k=k+1$ and return to Step 2.

Remark 4.6: Although only a finite horizon cost function is minimized at each time instant, the finite-time control performance derived in Theorem 4.4 is still satisfied because an additional constraint (4.29) is also added to the online optimization problem. Thus the system state is guaranteed to enter the neighboring subset at the next instant.

4.4.2 Example 4.2

Consider the well-known CSTR system in [14] and [15], which is also investigated in last chapter.

The following FMPC strategies are adopted in the simulation:

CS1: The online robust FMPC algorithm for time-delay systems in Chapter 3. The prediction horizon is chosen as 1, and the weighting matrices Q and R for the cost function are selected as $\text{diag}\{1e-6, 1e-9\}$ and 0.001, respectively.

Since the constraint $\sum_{i=-h}^0 \gamma_i = 1$ should be satisfied, let $\gamma_0 = 0.8$, $\gamma_{-1} = 0.2$.

CS2: The robust FMPC for the augmented system without delay. Inspired by [16], the time-delay system is modeled as an augmented system without delay, and then the online robust FMPC approach in [9] is applied to the augmented system. In which case, Q and R for the cost function of the augmented system are selected as $\text{diag}\{1e-6, 1e-8, 1e-6, 1e-8\}$ and 0.001, respectively.

CS3: The proposed off-line FMPC algorithm (Algorithm 1). The simulation parameters are the same as the ones in CS1.

CS4: The proposed off-line FMPC algorithm with online optimization (Algorithm 2). The prediction horizon is chosen as 1, and the simulation parameters are the same as those in CS1.

Two different initial conditions are considered that the simulation results are shown in Figures 4.3 and 4.4. Table 4.1 shows the computational cost of different FMPC approaches. Comparisons of control performance via different approaches are shown in Figure 4.3, where the initial conditions are given as: (a) $x(i)_{i \in \mathbb{Z}_{[-1,0]}} = [0.6972; 3.652]$, (b) $x(i)_{i \in \mathbb{Z}_{[-1,0]}} = [0.2; 3.0533]$, respectively. It can be seen that, among all approaches, the best control performance is achieved by adopting CS4, while CS2 which makes use of the augmented system has longest settling time. In addition, from Table 4.1 one can see that, compared with CS3 and CS4, the computational cost of CS1 and CS2 are much heavier, especially for CS2 where augmented system dynamics are involved. Compared with CS3, an additional online optimization algorithm is involved in CS4, however, the online optimization is not very complex. Therefore, CS4 can be regarded as an MPC method in which both control performance and computational burden are considered. Moreover, with the online optimization added to the off-line MPC approach in CS4, the online optimization is reduced to solving LMI problem, while both CS1 and CS2 have to solve bilinear matrix inequality (BMI) problems. Figure 4.4 shows the trajectories in the state space by adopting the proposed two algorithms, where the number of robust constraints sets are set as 25. From Figure 4.4 one can see that system all

(delayed) states are guaranteed to enter the terminal set by adopting CS3 and CS4.

Figure 4.3 State responses for two different initial conditions.

Figure 4.4 Trajectories in state space.

Table 4.1 Computational Complexities of Different MPC Strategies.

Methods	Types	Dimension of the matrix inequality
CS1	online (BMI)	495×495
CS2	online (BMI)	666×666
CS3	off-line (BMI)	--
CS4	off-line (BMI) + online (LMI)	13×13 (for online part)

4.5 Conclusions

Based on the Lyapunov-Razumikhin approach, an efficient FMPC approach is developed that a sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line. The system all (delayed) states are guaranteed to enter the terminal constraint set in finite time, given the initial states in a specific region. In addition, an online optimization algorithm is investigated to reduce the conservatism of the off-line approach. It can be seen that the robust positive invariance and input-to-state stability are realized. Simulation results show that the proposed two methods are effective and have advantages in control performance and computational cost over several MPC strategies. Particularly, the special control theory and positive invariance behind Razumikhin approach is revealed by the proposed method.

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Chapter 5

Fuzzy Model Predictive Control of Discrete-Time Systems With Time-Varying Delay and Disturbances

In this chapter, model predictive control of discrete T-S fuzzy systems subjected to bounded time-varying delay and persistent disturbances is investigated. The Razumikhin approach is adopted for time-delay systems because it involves a Lyapunov function associated with the original non-augmented state space of system dynamics when compared to the Krasovskii approach. Based on which, both online and off-line model predictive control approaches for systems with time-varying delay are provided. In addition, persistent disturbances are considered that robust positive invariance and input-to-state stability under such circumstances are realized. In the off-line approach, a sequence of explicit control laws that correspond to a sequence of robust constraints sets are computed off-line. And it is proved that system states including all possibly delayed states can be steered to the terminal constraint set in finite time. Moreover, it allows the exact time delay to be unknown in the proposed two approaches. In particular, for systems with time-varying delay the special positively invariant set theory and finite-time control theory based on the Razumikhin approach are directly revealed via the proposed off-line approach.

5.1 Introduction

Compared with time-delay systems with constant delay, a more natural aspect is to consider systems with time-varying delay. However, LKF is utilized in most existing works for this kind of systems [1]-[3]. Only a few results are based on LRF and they are only applied to continuous systems [4], [5].

In this chapter, fuzzy model predictive control (FMPC) is investigated for discrete nonlinear systems where time-varying delay and persistent disturbances are involved simultaneously. Nonlinear systems with time-varying delay are modeled as T-S fuzzy systems. Then LRF is adopted that conditions of robust positive invariance and input-to-state stability (ISS) with respect to disturbances are derived.

Furthermore, an efficient off-line FMPC approach for discrete nonlinear systems with timevarying delay and persistent disturbances is also provided. A sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line that guarantee the system all (possibly delayed) states will enter the terminal constraint set in finite time.

5.2 System Description

5.2.1 System Model

Consider the following discrete-time nonlinear system with time-varying state delay

$$x(k+1) \in F(x(k), x_d(k), u(k), w(k)), \quad k \in \mathbb{Z}_+ \quad (5.1)$$

where $x(k), x_d(k) \in \mathbb{X}$, $u(k) \in \mathbb{U}$ and $w(k) \in \mathbb{W}$, with $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{W} \subseteq \mathbb{R}^c$, represent system current and delayed states, inputs, and disturbances, respectively. $x_d(k) \in \mathbb{X}$ is defined as follows:

$$x_d(k) := x(k + d(k)), \quad d(k) \in \mathbb{Z}_{[-h, -1]} \quad (5.2)$$

where $d(k)$ denotes the number of delay, h denotes the upper bound of delay, and the minimal delay is set as 1.

Next, for time-delay systems some definitions of robust positive invariance and input-to-state stability are introduced.

Definition 5.1 (Robust Positively Invariant (RPI) Set, Dinvariance): For LRF conditions, considering system (5.1), a set Ω is called an RPI set for the closed-loop system corresponding to the control law π , and for $d(k) \in \mathbb{Z}_{[-h,-1]}$, if $\forall x \in \Omega$, $\forall x_d \in \Omega$, and $\forall w \in \mathbb{W}$, it holds $x^+ \in \Omega$.

Above and in what follows, x , x_d , x^+ , and w are used to denote, at arbitrary time instant, the current state, delayed state, state in the next instant, and disturbance, respectively.

Definition 5.2 (Input-to-state Stability (ISS)): Consider $x^+ \in F(x, x_d, w)$, it is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k) + \delta(\|\mathbf{w}_{[0,k-1]}\|) \quad (5.3)$$

where $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ is the initial (delayed) states, $\mathbf{w}_{[0,k-1]} \in \mathbb{W}^k$ is the disturbance sequence.

Definition 5.3 (ISS Lyapunov-Razumikhin Function (ISS-LRF)): For LRF conditions, suppose that there exists a positive definite function $V(x)$ such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (5.4)$$

for all $(x, x_d, w, x_1) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times F(x_0, x_d, w)$;

(ii) For any initial condition, there exists a function $\sigma \in \mathcal{K}$ satisfying

$$V(x^+) \leq \max\{\bar{V}(x), \sigma\|w\|\} \quad (5.5)$$

where $\bar{V}(x) = \max\{V(x), V(x_d)\}$, then $V(x)$ is called an ISS-LRF for

$$x^+ \in F(x, x_d, w).$$

It is worth to mention that detailed descriptions of ISS and ISS-LRF for discrete time-delay systems are introduced in [6].

5.2.2 T-S Fuzzy Time-delay Systems

Based on the aforementioned system model, nonlinear systems with time-varying delay can be represented by T-S fuzzy time-delay systems:

Plant Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$x^+ = A_l x + A_{dl} x_d + B_l u + E_l w \quad (5.6)$$

where, $z := [z_1, \dots, z_v]$ are premise variables. $F_1^l, \dots, F_v^l$ are fuzzy sets. $l \in \mathbb{Z}_{[1, L]}$, and L is the number of fuzzy rules. The T-S fuzzy system (5.6) can be written as,

$$x^+ = A_\mu x + A_{d\mu} x_d + B_\mu u + E_\mu w \quad (5.7)$$

where $A_\mu := \sum_{l=1}^L \mu_l(z) A_l$, $A_{d\mu} := \sum_{l=1}^L \mu_l(z) A_{dl}$, $B_\mu := \sum_{l=1}^L \mu_l(z) B_l$, $E_\mu := \sum_{l=1}^L \mu_l(z) E_l$, $\mu_l(z)$ is normalized membership function.

Control method for above T-S fuzzy system is given as follows:

Control Rule l :

IF: z_1 is F_1^l and $\dots$ z_v is F_v^l

THEN:

$$u = K_l x$$

Thus the final control output is given as,

$$u = K_\mu x \quad (5.8)$$

where $K_\mu = \sum_{l=1}^L K_l$.

5.3 Online MPC Approach

Online MPC methods require solving an optimization problem over a finite horizon at each sampling instant. Then in the next instant, the optimization problem is repeated based on the updated state information over a shifted horizon. Thus it is also called the receding horizon control (RHC).

In this section, the online MPC approach of systems with time-varying delay is investigated. The online control scheme about MPC for the aforementioned T-S fuzzy time-delay systems with disturbances is firstly introduced. Then the conditions of RPI set are discussed, based on which the computation of terminal constraint set is provided. In the end, the whole control algorithm is constructed, and then stability and feasibility properties are analyzed.

5.3.1 MPC Scheme

For T-S fuzzy system (5.7), the prediction model for online approach is given by

$$x(k+s+1|k) = A_{\mu}x(k+s|k) + A_{d\mu}x_d(k+s|k) + B_{\mu}u(k+s|k) + E_{\mu}w(k+s|k) \quad (5.9)$$

MPC optimizes a cost function under some constraints. Usually, a finite horizon cost function is adopted,

$$J(k) = \sum_{s=0}^{N-1} \ell(k+s|k) + V_T(x(k+N|k)) \quad (5.10)$$

where N is the prediction length, $\ell(k+s|k)$ is called the stage cost at the predicted time instant, and $V_T(x(k+N|k))$ is called the terminal cost, with $V_T(\cdot)$ being a positive definite function. The stage cost is chosen as

$$\ell(k+s|k) = x^T(k+s|k)Qx(k+s|k) + u^T(k+s|k)Ru(k+s|k) - \tau w^T(k+s|k)w(k+s|k) \quad (5.11)$$

where Q and R are positive matrices, τ is a positive scalar. One can see that

the disturbance is involved in the stage cost. It is noted that the choice of cost function is inspired by the H_∞ control, see Section 4.7 in [7] for the so-called “ H_∞ model predictive control”. And similar settings of cost functions can be found in [8], [9], [10]. However, the cost function cannot be optimized directly as it contains disturbance. Instead, a min-max approach is adopted that the worst-case cost function is minimized [11]. In addition, in the presence of persistent disturbance, asymptotic stability to the origin cannot be reached, instead, asymptotic convergence to an invariant set can be achieved [12], [13]. As such, a robust positively invariant set is employed as the terminal constraint set that, the system state is guaranteed to enter the terminal set at the end of prediction. The online optimization problem can be summarized as

$$\begin{aligned} \min_{u(k+s|k)} \max_{w(k+s|k)} J(k), \\ u(k+s|k) \in \mathbb{U}, \\ w(k+s|k) \in \mathbb{W}, \\ x(k+N|k) \in \Omega_T, \end{aligned}$$

where $s \in \mathbb{Z}_{[1, N-1]}$, $\Omega_T \subseteq \mathbb{X}$ represents the terminal constraint set. The disturbance and control input satisfy that

$$\begin{aligned} w \in \mathbb{W} &:= \{w \mid w^T w \leq \eta^2\} \\ u \in \mathbb{U} &:= \{u \mid |u_t| \leq u_{t, \max}\} \end{aligned}$$

where η is a known constant. u_t is the t -th element of the input vector, and $t \in \mathbb{Z}_{[1, m]}$.

5.3.2 RPI Set for Online Approach

Denote $\Omega \subseteq \mathbb{X}$ as an RPI set which is described in Definition 5.1, thus $x, x_d \in \Omega$, and π as the corresponding control law. Define Ω as follows

$$\Omega := \{ \{x, x_d\} \mid \max \{x^T P_\mu x, x_d^T P_\mu x_d\} \leq \xi \} \quad (5.12)$$

and the corresponding control law

$$\pi(x, x_d) := \{K_\mu x\} \quad (5.13)$$

where $P_\mu = \sum_{l=1}^L \mu_l(x) P_l$, $K_\mu = \sum_{l=1}^L \mu_l(x) K_l$.

Lemma 5.1: Consider system (5.7), the set Ω is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{[0,1]}$ such that

$$\frac{1}{\xi} x^{+T} P_\mu^+ x^+ - \frac{1-\lambda}{\xi} \max\{x^T P_\mu x, x_d^T P_\mu x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (5.14)$$

where $P_\mu^+ = \sum_{l=1}^L \mu_l(x^+) P_l$.

Proof: For (5.14), move the last two items from the left side to the right side.

According to the definition of Ω in (5.12) and $w^T w \leq \eta^2$, then

$\frac{1}{\xi} x^{+T} P_\mu^+ x^+ \leq (1-\lambda) + \lambda = 1$, thus $x^{+T} P_\mu^+ x^+ \leq \xi$, which indicates that $x^+ \in \Omega$.

Proof is thus completed. It is noted that the RPI approach regarding disturbance is extended from the case of non-delayed systems in [8].

Conditions of RPI set and input constraint are given as solving bilinear matrix inequalities (BMIs) and linear matrix inequalities (LMIs) as follows.

Theorem 5.1: Consider T-S fuzzy system (5.7), if there exist positive definite matrix X_a (or X_b , X_l presented below), matrices Y_b , G_b , Z , and positive scalar $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma\Phi & * & * & * \\ 0 & \gamma_d\Phi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a G_b + B_a Y_b & A_{da} G_b & E_a & -X_l \end{bmatrix} \leq 0 \quad (5.15)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_b \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (5.16)$$

where $\Phi = (1-\lambda)(X_a - G_b - G_b^T)$, $a, b, l \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are preset valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the set $\Omega = \{x, x_d \mid \max\{x^T P_\mu x, x_d^T P_\mu x_d\} \leq \xi\}$, with

$P_\mu = \sum_{l=1}^L \mu_l(x) P_l$ and $P_l = \xi X_l^{-1}$, is an RPI set for system (5.7) corresponding to the feedback control law $\pi = K_\mu x$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$.

Proof: Resorting to the dilation lemma [14],

$$-G_b^T X_a^{-1} G_b \leq X_a - G_b^T - G_b \quad (5.17)$$

The following inequality is got from (5.15),

$$\begin{bmatrix} \gamma \Delta & * & * & * \\ 0 & \gamma_d \Delta & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_a G_b + B_a Y_b & A_{da} G_b & E_a & -X_l \end{bmatrix} \leq 0 \quad (5.18)$$

where $\Delta = (\lambda - 1) G_b^T X_a^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I\}$ and its transpose from both sides of (5.18), respectively, yields that

$$\begin{bmatrix} \gamma(\lambda - 1) X_a^{-1} & * & * & * \\ 0 & \gamma_d(\lambda - 1) X_a^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_a + B_a K_b & A_{da} & E_a & -X_l \end{bmatrix} \leq 0 \quad (5.19)$$

Based on which one has

$$\sum_{a=1}^L \sum_{b=1}^L \mu_a \mu_b \begin{bmatrix} \gamma \Theta & * & * & * \\ 0 & \gamma_d \Theta & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_a + B_a K_b & A_{da} & E_a & -X_l \end{bmatrix} \leq 0 \quad (5.20)$$

where $\Theta = (\lambda - 1) X_a^{-1}$.

Substituting X_a^{-1} with P_a / ξ , it holds that

$$\begin{bmatrix} \gamma(\lambda - 1) P_\mu / \xi & * & * & * \\ 0 & \gamma_d(\lambda - 1) P_\mu / \xi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2} I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -X_l \end{bmatrix} \leq 0 \quad (5.21)$$

Applying schur complement to (5.21), and then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides, respectively, yields that

$$\varpi \begin{bmatrix} \gamma(\lambda-1)P_\mu / \xi & & \\ & \gamma_d(\lambda-1)P_\mu / \xi & \\ & & -\frac{\lambda}{\eta^2}I \end{bmatrix} \varpi^T + \varpi \Lambda X_l^{-1} \Lambda^T \varpi^T \leq 0 \quad (5.22)$$

where $\varpi = [x^T, x_d^T, w^T]$, $\Lambda = [A_\mu + B_\mu K_\mu, A_{d\mu}, E_\mu]^T$.

By substituting X_l^{-1} with P_l / ξ , (5.22) is equal to

$$\frac{1}{\xi} x^{+T} P_\mu^+ x^+ - \frac{\lambda}{\eta^2} w^T w \leq \frac{1-\lambda}{\xi} (\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d) \quad (5.23)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d \leq \max\{x^T P_\mu x, x_d^T P_\mu x_d\}$, put it to (5.23) and then (5.14) is got.

(5.16) guarantees the input constraint and the proof can be seen in [15]. The proof is thus completed.

From the notation of Φ in (5.15), it can be seen that BMIs are introduced. Thus the solver PENBMI [16] is utilized to solve BMIs problems.

5.3.3 Terminal Constraint Set for Online Approach

Definition 5.4: The terminal constraint set Ω_T should satisfy two conditions.

Firstly, it should be an RPI set. Secondly, there exist $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$, and a positive definite function $V(x) := x^T P_\mu x$, with $P_\mu = \sum_{l=1}^L \mu_l P_l$, that $\forall x \in \Omega_T$,

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|) \quad (5.24)$$

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (5.25)$$

where $\bar{V}(x) = \max\{x^T P_\mu x, x_d^T P_\mu x_d\}$. Definition 5.4 is concluded based on the results in [6]. In the following, the condition which assures the requirement of Definition 5.4 is provided.

Theorem 5.2: Consider T-S fuzzy system (5.7), if (5.15), (5.16) and the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma\Xi & * & * & * & * & * \\ 0 & \gamma_d\Xi & * & * & * & * \\ 0 & 0 & -\tau\xi I & * & * & * \\ \Psi & A_{da}G_b & \xi E_a & -X_l & * & * \\ QG_b & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (5.26)$$

where $\Xi = X_a - G_b - G_b^T$, $\Psi = A_a G_b + B_a Y_b$. matrix X_a (or X_b , X_l presented below), matrices Y_b , G_b , are the same as those defined in Theorem 5.1, then Ω_r is a terminal constraint set regarding the terminal cost function $\bar{V}(x)$.

Proof: Firstly, (5.24) can be easily got by resorting to the eigenvalues of the positive definite matrices P_l , $l \in \mathbb{Z}_{[1,L]}$. Let $\rho_{\min} := \min\{\rho_{\min}(P_l) | l \in \mathbb{Z}_{[1,L]}\}$, and $\rho_{\max} := \max\{\rho_{\max}(P_l) | l \in \mathbb{Z}_{[1,L]}\}$, thus $\rho_{\min} \|x\|^2 \leq V(k, x) \leq \rho_{\max} \|x\|^2$. As such, the requirement of (5.24) is satisfied.

Then the main effort is focused on the satisfaction of optimization problem (5.25). Considering (5.17), then multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I, I, I\}$ and its transpose from both sides of (5.26), respectively, yields that

$$\begin{bmatrix} -\gamma X_a^{-1} & * & * & * & * & * \\ 0 & -\gamma_d X_a^{-1} & * & * & * & * \\ 0 & 0 & -\tau\xi I & * & * & * \\ Y & A_{da} & \xi E_a & -X_l & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (5.27)$$

where $Y = A_a + B_a K_b$. The above equation is equivalent to

$$\begin{bmatrix} -\gamma\xi X_a^{-1} & * & * & * & * & * \\ 0 & -\gamma_d\xi X_a^{-1} & * & * & * & * \\ 0 & 0 & -\tau I & * & * & * \\ Y & A_{da} & E_a & -\frac{X_l}{\xi} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_b & 0 & 0 & 0 & 0 & -R \end{bmatrix} \leq 0 \quad (5.28)$$

Substituting X_a^{-1} with P_a / ξ to (5.28), and rewriting (5.28) as

$$\begin{bmatrix} Q + K_\mu^T R K_\mu - \gamma P_\mu & * & * & * \\ 0 & -\gamma_d P_\mu & * & * \\ 0 & 0 & -\tau I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -P_l^{-1} \end{bmatrix} \leq 0 \quad (5.29)$$

Applying schur complement to (5.29), then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of the resulting matrix inequality, respectively, noticing that $P_\mu^+ = \sum_{l=1}^L \mu_l(x^+) P_l$, then it holds that

$$x^{+T} P_\mu^+ x^+ + x^T Q x + x^T K_\mu^T R K_\mu x - \tau w^T w - (\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d) \leq 0 \quad (5.30)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_\mu x + \gamma_d x_d^T P_\mu x_d \leq \bar{V}(x)$, thus

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (5.31)$$

The proof is completed.

5.3.4 Online MPC Algorithm

Based on the above results, the whole algorithm is discussed in this subsection to complete the online MPC controller design, and then feasibility and stability are analyzed.

The terminal constraint set Ω_T should satisfy the requirement of RPI set according to (5.12), which indicates,

$$\bar{V}(x) \leq \xi \quad (5.32)$$

thus $x(k), x_d(k) \in \Omega_T$. The following problem is considered,

$$\min \quad \xi, \text{ subjected to } \bar{V}(x) \leq \xi \quad (5.33)$$

A sufficient condition for $\bar{V}(x) \leq \xi$ is given as

$$\begin{bmatrix} 1 & x^T(k+j) \\ x(k+j) & X_l \end{bmatrix} \geq 0, j \in \mathbb{Z}_{[-h,0]}, l \in \mathbb{Z}_{[1,L]} \quad (5.34)$$

Remark 5.1: For clarity of exposition, at time instant k , it can be seen that the

terminal constraint set only involves $x(k)$ and $x_d(k)$ according to (5.5), i.e., $x(k), x_d(k) \in \Omega_k$. However, all possibly delayed states are employed in (5.34), which indicates $x(k+j) \in \tilde{\Omega}_k$, $j \in \mathbb{Z}_{[-h,0]}$. Thus $\tilde{\Omega}_k$, rather than Ω_k , represents the terminal constraint set Ω_T computed at instant k , and it is easy to know that $\Omega_k \subseteq \tilde{\Omega}_k$. Although this choice may be less optimal because it may lead to a larger terminal constraint set, it is essential to guarantee the positive invariance of terminal constraint set as the delay is time-varying. And it will be utilized further for the feasibility analysis subsequently.

Remark 5.2: (5.34) indicates that all possibly delayed states which are available at each time instant are involved. Moreover, the upper bound of delay is assumed to be known. As such, it does not need to know the exact delay in the controller design.

Consider the condition for terminal constraint set in (5.26), τ can also be optimized. Taking the same idea as stated in [8], a new variable ε is defined as: $\varepsilon = \tau\xi$, thus ε is minimized instead. Then (5.26) is rewritten as

$$\begin{bmatrix} \gamma\Xi & * & * & * & * & * \\ 0 & \gamma_d\Xi & * & * & * & * \\ 0 & 0 & -\varepsilon I & * & * & * \\ \Psi & A_{da}G_b & \xi E_a & -X_l & * & * \\ QG_b & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (5.35)$$

In summary, the MPC algorithm is given as follows. As BMIs are introduced in (5.15), to reduce the computational burden, the feasible value of λ got in the initial time is saved and utilized all the time. By taking this measure and the one stated in (5.35), the whole control problem is reduced to solving LMIs.

Algorithm 1:

Step 1. Obtain the system all (delayed) states $x(k+j)$, $j \in \mathbb{Z}_{[-h,0]}$;

Step 2. If $k=0$, solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z, \lambda} \mathcal{E}, \quad (5.36)$$

subject to (5.15), (5.16), (5.34), (5.35), and $\lambda \in \mathbb{R}_{(0,1)}$. Obtain the optimal λ and

denotes it as λ^* . Go to Step 3;

Else if $k \geq 1$, replace λ in (5.15) with λ^* , and solve

$$\min_{\xi, \varepsilon, X_a, G_b, Y_b, Z} \mathcal{E}, \quad (5.37)$$

subject to (5.15), (5.16), (5.34), and (5.35). Go to Step 3;

Step 3. Obtain the control input $u(k) = K_\mu x(k)$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$. Then implement the control input to system (5.7). Move the time instant from k to $k+1$ and go to Step 1.

Remark 5.3: Consider the optimization problem (5.37), where (5.34) implies that the number of LMIs will increase with the number of delay. However, the other conditions, i.e., (5.15), (5.16) and (5.35), will remain unchanged with the delay. Therefore, the computational burden of the online MPC algorithm will slightly increase with the number of delay.

Theorem 5.3: For system (5.7), if the constrained optimization problem has a solution at time 0, it will always be solvable. Furthermore, recursive feasibility can be achieved.

Proof: Without loss of generality, assume that (5.37) is satisfied at time k , according to the description in Remark 5.1, thus $x(k+j) \in \tilde{\Omega}_k$, $j \in \mathbb{Z}_{[-h,0]}$, with $\tilde{\Omega}_k$ being the terminal constraint set at time k . Since $x(k) \in \Omega_k$ and $x_d(k) \in \Omega_k$ as stated in Remark 5.1, thus $x(k+1) \in \Omega_k$. Considering $\Omega_k \subseteq \tilde{\Omega}_k$, thus $x(k+1+j) \in \tilde{\Omega}_k$, $j \in \mathbb{Z}_{[-h,0]}$. Then it can be concluded that (5.37) is solvable at time $k+1$. Moreover, the solution obtained at time k is feasible at time $k+1$. Applying the same process, it is observed that the optimization problem is feasible at all time.

Theorem 5.4: If the constrained optimization problem is feasible at the initial time 0, system (5.7) is ISS with respect to disturbance w .

Proof: Consider the LRF $V(x)$, at time k , suppose that it gets its maximum value at the delayed states $x(k + \theta)$, $\theta = \{d(k), 0\}$ with $d(k)$ defined in (5.2), i.e., $\bar{V}^*(x(k)) = x^T(k + \theta)P_\mu^*(k)x(k + \theta)$, $P_\mu^*(k) = \sum_{l=1}^L \mu_l P_l^*(k)$, $P_l^*(k)$ is an optimal value of $P_l(k)$ at time k . In addition, as $V^*(x(k)) = x^T(k)P_\mu^*(k)x(k)$, then it can be easily got

$$\rho_{\min}^* \|x(k)\|^2 \leq V^*(x(k)) \leq \rho_{\max}^* \|x(k)\|^2 \quad (5.38)$$

where

$$\rho_{\min}^* := \min \left\{ \rho_{\min}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{N} \right\}$$

$$\rho_{\max}^* := \max \left\{ \rho_{\max}(P_l^*(k)) \mid l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{N} \right\}$$

where $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ represent the maximal and minimal eigenvalues, respectively.

Moreover, (5.25) implies

$$V_k^*(x(k+1)) - \bar{V}^*(x(k)) \leq -(x^T(k)Qx(k) + u^T(k)Ru(k) - \tau w^T(k)w(k)) \quad (5.39)$$

where $V_k^*(x(k+1)) = x^T(k+1)P_\mu^*(k)x(k+1)$. Thus one has

$$V_k^*(x(k+1)) - \bar{V}^*(x(k)) \leq -x^T(k)Qx(k) + \tau w^T(k)w(k) \quad (5.40)$$

Owing to the optimality of (5.37) at time $k+1$, it holds

$$V_{k+1}^*(x(k+1)) \leq V_k^*(x(k+1)) \quad (5.41)$$

where $V_{k+1}^*(x(k+1))$ denotes the optimal $V(x)$ at time $k+1$, then it holds that

$$V_{k+1}^*(x(k+1)) - \bar{V}^*(x(k)) \leq -x^T(k)Qx(k) + \tau w^T(k)w(k) \quad (5.42)$$

According to Definition 5.3, from (5.38) and (5.42) it follows that $V(x)$ is an ISS-Lyapunov function. Hence the closed-loop system is ISS with respect to disturbance. The proof is completed.

5.4 Off-line MPC Approach

In this section, computation of a terminal constraint set for off-line approach is firstly introduced, based on which a sequence of robust constraint sets corresponding to a sequence of explicit control laws are constructed. Then the whole control algorithm is summarized, and the finite-time control performance is analyzed.

5.4.1 Terminal Constraint Set for Off-line Approach

As it is discussed in the last section, the terminal constraint set should be an RPI set firstly. Define the terminal constraint set as $\Omega_t := \{x, x_d \mid \max\{x^T P_t x, x_d^T P_t x_d\} \leq \xi\}$, with P_t a positive definite matrix, then Ω_t is an RPI set if there exists $\lambda \in \mathbb{R}_{[0,1]}$ such that

$$\frac{1}{\xi} x^{+T} P_t x^+ - \frac{1-\lambda}{\xi} \max\{x^T P_t x, x_d^T P_t x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (5.43)$$

Similar to (5.14), (5.43) guarantees $x^+ \in \Omega_t$, provided that $x, x_d \in \Omega_t$.

Secondly, there exist $\alpha_5, \alpha_6 \in \mathcal{K}_\infty$, and a positive definite function $V(x) := x^T P_t x$ that $\forall x \in \Omega_t$,

$$\alpha_5(\|x\|) \leq V(x) \leq \alpha_6(\|x\|) \quad (5.44)$$

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (5.45)$$

where $\bar{V}(x) = \max\{x^T P_t x, x_d^T P_t x_d\}$. In the following, the conditions which guarantee the aforementioned requirements are provided.

Theorem 5.5: Consider system (5.7), if there exist positive definite matrix X_t , matrices Y_a (or Y_b presented below), G_b , Z , positive scalars τ , $\lambda \in \mathbb{R}_{[0,1]}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma\Pi & * & * & * \\ 0 & \gamma_d\Pi & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ \Gamma & A_{da}G_b & E_a & -X_t \end{bmatrix} \leq 0 \quad (5.46)$$

$$\begin{bmatrix} \gamma H & * & * & * & * & * \\ 0 & \gamma_d H & * & * & * & * \\ 0 & 0 & -\tau\xi I & * & * & * \\ \Gamma & A_{da}G_b & \xi E_a & -X_t & * & * \\ QG_b & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (5.47)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & G_b + G_b^T - X_t \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (5.48)$$

where $\Pi = (1-\lambda)(X_t - G_b - G_b^T)$, $H = X_t - G_b - G_b^T$, $\Gamma = A_d G_b + B_a Y_b$, $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are preset valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the terminal constraint set Ω_t is got for system (5.7), with $P_t = \xi X_t^{-1}$, and the corresponding feedback control law is given as $\pi_t = K_\mu x$, with $K_\mu = \sum_{l=1}^L \mu_l K_l$ and $K_l = Y_l G_l^{-1}$.

Proof: It is first shown that (5.46) implies (5.43). Resorting to the dilation lemma [14],

$$-G_b^T X_t^{-1} G_b \leq X_t - G_b^T - G_b \quad (5.49)$$

The following inequality is got from (5.46),

$$\begin{bmatrix} \gamma M & * & * & * \\ 0 & \gamma_d M & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ \Gamma & A_{da}G_b & E_a & -X_t \end{bmatrix} \leq 0 \quad (5.50)$$

where $M = (\lambda-1)G_b^T X_t^{-1} G_b$. By multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I\}$ and its transpose from both sides of (5.50), respectively, yields that

$$\begin{bmatrix} \gamma(\lambda-1)X_t^{-1} & * & * & * \\ 0 & \gamma_d(\lambda-1)X_t^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a + B_a K_b & A_{da} & E_a & -X_t \end{bmatrix} \leq 0 \quad (5.51)$$

(5.51) can be rewritten as

$$\begin{bmatrix} \gamma(\lambda-1)X_t^{-1} & * & * & * \\ 0 & \gamma_d(\lambda-1)X_t^{-1} & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -X_t \end{bmatrix} \leq 0 \quad (5.52)$$

Applying schur complement to (5.52) and then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of resulting inequality, respectively, yields that

$$\varpi \begin{bmatrix} \gamma(\lambda-1)X_t^{-1} & & \\ & \gamma_d(\lambda-1)X_t^{-1} & \\ & & -\frac{\lambda}{\eta^2}I \end{bmatrix} \varpi^T + \varpi \Lambda X_t^{-1} \Lambda^T \varpi^T \leq 0 \quad (5.53)$$

By substituting X_t^{-1} with P_t / ξ , (5.53) is equivalent to

$$\frac{1}{\xi} x^{+T} P_t x^+ - \frac{\lambda}{\eta^2} w^T w \leq \frac{1-\lambda}{\xi} (\gamma x^T P_t x + \gamma_d x_d^T P_t x_d) \quad (5.54)$$

With $\gamma x^T P_t x + \gamma_d x_d^T P_t x_d \leq \max\{x^T P_t x, x_d^T P_t x_d\}$ due to $\gamma + \gamma_d = 1$, thus (5.43) is got.

Since (5.44) can be easily got by resorting to the eigenvalues of P_t , next we show that (5.45) is guaranteed by (5.47). Considering (5.49), multiplying $\text{diag}\{G_b^{-T}, G_b^{-T}, I, I, I, I\}$ and its transpose from both sides of (5.47), respectively, yields that

$$\begin{bmatrix} -\gamma X_t^{-1} & * & * & * & * & * \\ 0 & -\gamma_d X_t^{-1} & * & * & * & * \\ 0 & 0 & -\tau \xi I & * & * & * \\ \bar{O} & A_{da} & \xi E_a & -X_t & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_b & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} \leq 0 \quad (5.55)$$

where $\bar{O} = A_a + B_a K_b$. (5.55) is equivalent to

$$\begin{bmatrix} -\gamma \xi X_t^{-1} & * & * & * & * & * \\ 0 & -\gamma_d \xi X_t^{-1} & * & * & * & * \\ 0 & 0 & -\tau I & * & * & * \\ \bar{O} & A_{da} & E_a & -\frac{X_t}{\xi} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_b & 0 & 0 & 0 & 0 & -R \end{bmatrix} \leq 0 \quad (5.56)$$

Substituting X_t^{-1} with P_t / ξ and then applying schur complement to (5.56), thus one has

$$\begin{bmatrix} \bar{W} & * & * & * \\ 0 & -\gamma_d P_t & * & * \\ 0 & 0 & -\tau I & * \\ A_\mu + B_\mu K_\mu & A_{d\mu} & E_\mu & -P_t^{-1} \end{bmatrix} \leq 0 \quad (5.57)$$

where $\bar{W} = Q + K_\mu^T RK_\mu - \gamma P_t$. Applying schur complement to (5.57), then multiplying $[x^T, x_d^T, w^T]$ and its transpose from both sides of resulting inequality, respectively, yields that

$$x^{+T} P_t x^+ + x^T Q x + x^T K_\mu^T RK_\mu x - \tau w^T w - (\gamma x^T P_t x + \gamma_d x_d^T P_t x_d) \leq 0 \quad (5.58)$$

With $\gamma + \gamma_d = 1$, it is easy to get $\gamma x^T P_t x + \gamma_d x_d^T P_t x_d \leq \bar{V}(x)$, thus

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u + \tau w^T w \quad (5.59)$$

(5.48) guarantees the input constraint and the proof can be seen in [15]. The proof is thus completed.

5.4.2 A Sequence of Robust Constraint Sets

Based on the obtained terminal constraint set Ω_t in the last subsection, a robust constraint set Ω_e ($\Omega_e \supseteq \Omega_t$) is employed that the system current state and delayed state $\{x, x_d\} \in \Omega_e$. A control law π_e is given that assures the system state at next instant $x(k+1) \in \Omega_t$, and it is easy to see that the positive invariance is guaranteed since $\Omega_t \subseteq \Omega_e$. When Ω_e and corresponding control law π_e are computed, then in a similar way, a sequence of constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_e, \dots, \Omega_{\tilde{N}}\}$ and corresponding control laws $\{\pi_0, \pi_0, \dots, \pi_e, \dots, \pi_{\tilde{N}}\}$ can be got, where Ω_t and π_t denote the terminal set Ω_t and corresponding control law π_t , respectively, $\Omega_e \subseteq \mathbb{X}$, with $e \in \mathbb{Z}_{[2, \tilde{N}]}$, $\tilde{N} \geq 2$ represents the number of robust constraint sets, where the terminal set is included.

Firstly, without loss of generality, consider two neighbouring robust constraint sets Ω_e and Ω_{e-1} , with $\Omega_{e-1} \subseteq \Omega_e$. When provided that $x, x_d \in \Omega_e$, then $x^+ \in \Omega_{e-1}$ can be satisfied by the following inequality,

$$x^{+T} X_{e-1}^{-1} x^+ - (1 - \lambda) \max \{x^T X_e^{-1} x, x_d^T X_e^{-1} x_d\} - \frac{\lambda}{\eta^2} w^T w \leq 0 \quad (5.60)$$

where X_e and X_{e-1} are positive definite matrices associated with the two neighbouring robust constraint sets, $\Omega_e = \{x \mid x^T X_e^{-1} x \leq 1\}$ and $\Omega_{e-1} = \{x \mid x^T X_{e-1}^{-1} x \leq 1\}$, respectively. $x, x_d \in \Omega_e$ implies that $\max \{x^T X_e^{-1} x, x_d^T X_e^{-1} x_d\} \leq 1$. Similar to (5.43), then $x^{+T} X_{e-1}^{-1} x^+ \leq 1$ can be got from (5.60), which indicates that $x^+ \in \Omega_{e-1}$.

Secondly, denote $V(x) := x^T P_e x$, thus the optimization control strategy is formulated as follows,

$$V(x^+) - \bar{V}(x) \leq -x^T Q x - u^T R u \quad (5.61)$$

where $V(x^+) = x^{+T} P_e x^+$, $\bar{V}(x) = \max \{x^T P_e x, x_d^T P_e x_d\}$.

Theorem 5.6: Assume the set $\Omega_{e-1} = \{x | x^T P_{e-1} x \leq \xi_{e-1}\}$ is known that $X_{e-1} = \xi_{e-1} P_{e-1}^{-1}$, if there exist positive definite matrix X_e , general matrices Y_a (or Y_b presented below), Z , positive scalars ξ_e , $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \gamma(\lambda-1)X_e & * & * & * \\ 0 & \gamma_d(\lambda-1)X_e & * & * \\ 0 & 0 & -\frac{\lambda}{\eta^2}I & * \\ A_a X_e + B_a Y_b & A_{da} X_e & E_a & -X_{e-1} \end{bmatrix} \leq 0 \quad (5.62)$$

$$\begin{bmatrix} -\gamma X_e & * & * & * & * \\ 0 & -\gamma_d X_e & * & * & * \\ A_a X_e + B_a Y_b & A_{da} X_e & -X_e & * & * \\ QX_e & 0 & 0 & -\xi_e Q & * \\ RY_b & 0 & 0 & 0 & -\xi_e R \end{bmatrix} \leq 0 \quad (5.63)$$

$$\begin{bmatrix} Z & * \\ Y_b^T & X_e \end{bmatrix} \geq 0, Z_{tt} \leq u_{t,\max}^2, t \in \mathbb{Z}_{[0,m]} \quad (5.64)$$

where $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t -th diagonal element of matrix Z , γ and γ_d are preset valued positive numbers that satisfy $\gamma + \gamma_d = 1$, then the set $\Omega_e = \{x | x^T P_e x \leq \xi_e\}$, with $P_e = \xi_e X_e^{-1}$, is a robust constraint set for T-S fuzzy system (5.7), and the corresponding feedback control law π_e will steer system state to Ω_{e-1} at the next instant. Moreover, π_e is given as $\pi_e = K_\mu x$, with $K_\mu = \sum_{a=1}^L \mu_a K_a$ and $K_a = Y_a X_e^{-1}$.

Proof: It is noted that (5.62) and (5.63) guarantee (5.60) and (5.61), respectively. (5.64) satisfies the input constraint. The proof is very similar to that of Theorem 5.5 for terminal constraint set. Thus is omitted for simplicity.

5.4.3 Off-line MPC Algorithm

Based on the above results, the whole control algorithm is formulated in this

subsection to complete the off-line MPC design, and then the finite-time control performance is analyzed.

Algorithm 2:

1. Off-line computation

Step 1. Compute a terminal constraint set Ω_t and the corresponding control law π_t using Theorem 5.5;

Step 2. Base on the obtained terminal constraint set, a sequence of constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_{\tilde{N}}\}$ together with corresponding control laws $\{\pi_0, \pi_0, \dots, \pi_{\tilde{N}}\}$ are computed by Theorem 5.6;

1. Online computation

Step 1. Assume the initial conditions $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$. Let $k=0$ and go to Step 2;

Step 2. At time instant k , if $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_1$ which indicates that all (possibly delayed) states are in the terminal constraint set Ω_1 , then the terminal control law π_1 is utilized all the time. Otherwise, go to Step 3;

Step 3. Implement the search to decide $\{x(k), x_d(k)\} \in \Omega_e \setminus \Omega_{e-1}$, then the corresponding control law π_e is applied, let $k=k+1$ and return to Step 2.

Remark 5.5: In Step 2 of off-line computation of robust constraint set, we use $-\log \det(X_e)$ as the objective function and minimize it to get the constraint set with maximal size.

Remark 5.6: As it is shown in Remark 2 that, it does not need to know the exact delay for the online MPC approach. In contrast, from Step 3 of online computation of Algorithm 2, it can be seen that the delayed state $x_d(k)$ should be known to decide the constraint set Ω_e . An alternative method to avoid the dependence of the exact delay for the off-line MPC algorithm is that, we can resort to another constraint set Ω_g , which is used to replace Ω_e , decided by $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_g \setminus \Omega_{g-1}$. Thus it does not need to know exactly the delayed

state $x_d(k)$. However, this choice of constraint set also has some disadvantages. One is that it will lead to a less optimal control performance since a larger constraint set Ω_g (it is easy to know that $\Omega_g \supseteq \Omega_e$) is adopted. In addition, the computational cost of online search of constraint set Ω_g will increase because all (possibly delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}}$ are involved in the search.

Remark 5.7: Consider the computational cost of the off-line MPC approach, the off-line computation of a sequence of robust constraint sets in Algorithm 2 is not related to the size of delay. For the online computation part of Algorithm 2, if the time-varying delay is known that Ω_e in Algorithm 2 is used as the constraint set, then the online computational cost will not increase with the number of delay. However, if the time-varying delay is unknown that the constraint set Ω_g in Remark 5.6 is adopted instead, then the online computational cost will increase with the number of delay.

Theorem 5.7: It is assumed that the initial conditions $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$, for the aforementioned systems with time-varying delay if a sequence of robust constraint sets $\{\Omega_0, \Omega_0, \dots, \Omega_{\tilde{N}}\}$, where Ω_1 is the terminal constraint set, can be computed by Theorem 5.5 and Theorem 5.6, then the system all (possibly delayed) states will be steered to the terminal constraint set in no more than $(h+1) \cdot (\tilde{N}-1)$ steps.

Proof: Without loss of generality, suppose that system all (possibly delayed) states $x(k+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$ at time instant k . Firstly, the control law π_e is applied to drive $x(k+1)$ to Ω_{e-1} . As a result, at time instant $k+1$, one can see that $x(k+1) \in \Omega_{e-1}$, while $x_d(k+1) \in \Omega_e$. Thus the control law π_e , rather than π_{e-1} , will still be adopted at instant $k+1$ to steer $x(k+2)$ to Ω_{e-1} . Therefore, considering the delay is time-varying and the upper bound of delay is h , in the worst case, it can be concluded that it requires no more than $h+1$ steps to drive

system all (possibly delayed) states from Ω_e to Ω_{e-1} , i.e., $x(k+h+1+i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{e-1}$. As a consequence, in the case of $\tilde{N}$ robust constraint sets and the initial conditions satisfying $x(i)_{i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$, it takes no more than $(h+1) \cdot (\tilde{N}-1)$ steps to drive system all (possibly delayed) states to the terminal constraint set. Proof is completed.

Remark 5.8: If the constraint set Ω_g in Remark 6 is adopted, the finite-time control theory revealed in Theorem 5.7 will still be effective. Actually, the result can be easily got by referring to the proof of Theorem 5.7.

5.5 Simulation Examples

In this section, simulations on a numerical example and the time-delay CSTR system are provided to verify the effectiveness of the proposed two algorithms. The simulations are implemented with MATLAB and all matrix inequalities are solved via YALMIP toolbox [17].

5.5.1 Example 5.1

Consider the following T-S fuzzy time-delay systems revised from [18], with the upper bound of delay $h=3$,

Rule 1: If $x_1(k)$ is F_1 , then

$$x(k+1) = A_1 x(k) + A_{d1} x_d(k) + B_1 u(k) + E_1 w(k)$$

Rule 2: If $x_1(k)$ is F_2 , then

$$x(k+1) = A_2 x(k) + A_{d2} x_d(k) + B_2 u(k) + E_2 w(k)$$

where

$$A_1 = \begin{bmatrix} 1.5 & 2.5 \\ 0 & 1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & 1.5 \\ 0 & 0.8 \end{bmatrix},$$

$$A_{d1} = 0.05 A_1, \quad A_{d2} = 0.05 A_2,$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The membership functions are given as follows:

$$\begin{aligned} F_1 &= 0.5 + \sin(x_1) \cos(x_1) \\ F_2 &= 1 - F_1 \end{aligned}$$

The matrices Q and R in the cost function are selected as $\text{diag}\{1, 1\}$ and 0.1, respectively. The input constraint and bounded disturbance are given as: $\|u\| \leq 3$, $\|w\| \leq 0.1$. As the constraint $\gamma + \gamma_d = 1$ should be satisfied, set $\gamma = 0.8$, $\gamma_d = 0.2$. The simulation result of Algorithm 1 (online MPC approach) is shown in Figure 5.1. The initial condition is given as $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [1; -1.5]$. It can be seen that with the control effort, the system states come into a very small region near the origin in the presence of persistent disturbance. And the input constraint is guaranteed. Figure 5.1 also shows the number of time-varying delay at each time instant.

The simulation result of Algorithm 2 (off-line MPC approach) is shown in Figure 5.2, where the initial conditions are given as: $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-1.64; -0.479]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [2.6; 0.12]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-0.23; 1.08]$, respectively. The number of robust constraint sets including the terminal constraint set is given as 11. It can be seen that, despite the different initial conditions, system all (delayed) states enter the terminal constraint set in the end.

Figure 5.1 Control performance of Algorithm 1.

Figure 5.2 Control performance using Algorithm 2.

Figure 5.3 Constraint sets for all (delayed) states and the current state at each instant for the three different initial conditions.

Figure 5.4 Time-varying delay for the three initial conditions.

Figure 5.3 gives a clearer view of the relevant constraint sets where all (delayed) states and the current state belong, respectively, at each time instant. The upper, middle, and lower histograms correspond to $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-1.64; -0.479]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [2.6; 0.12]$, $x(i)_{i \in \mathbb{Z}_{[-3,0]}} = [-0.23; 1.08]$, respectively. Set 1 represents the smallest constraint set (terminal constraint set) in Fig. 2, while Set 11 represents the largest constraint set. It can be seen that:

- (1) In the existence of delayed states, at each instant, the constraint set for current state is always a (non-strict) subset of the constraint set for all (delayed) states, which implies that the fundamental idea of Razumikhin approach is satisfied.
- (2) Consider the constraint set for all (delayed) states at each instant, except the terminal set, the control effort will steer the system current state to a strict subset at the next instant.
- (3) Moreover, the constraint set for all (delayed) states is not monotone decreasing at all time, due to time-varying delay.
- (4) The middle histogram in Figure 5.3 also shows that the system current state at instant 2 enters a superset at the next instant. And the same phenomenon can

also be seen in the lower histogram. Thus it verifies that the Lyapunov function is non-monotone in the Razumihkin approach.

(5) It takes no more than 4 (i.e., $h+1$, with $h=3$ in this example) steps to drive system all (possibly delayed) states from one constraint set to a strict subset. Based on which, it can be easily got that system all (possibly delayed) states can be steered to the terminal set in finite time. Thus the finitetime control performance introduced in Theorem 5.7 is verified by simulation.

Therefore, one can observe the special set invariance theory and finite-time control theory behind the Razumihkin approach that all the mentioned characteristics are different from those for non-delayed systems. As such, as shown in Remark 5.4, they are also different from the contexts of Krasovskii approach for time-delay systems. In addition, the number of time-varying delay for the three initial conditions are shown in Figure 5.4.

5.5.2 Example 5.2

Consider the well-known CSTR system modeled as a nonlinear time-delays system in [19]. In [20], the CSTR system is represented by discrete T-S fuzzy systems and the sampling time is set as $T_s = 0.2$ minutes. However, the time delay is set as a constant number in [20]. In contrast, a more natural condition is given in this work that the time delay is allowed to be time-varying, and the upper bound of delay is set as $h = 10$.

There are three nominal states when $u=0$: $x_{s1}=[0.1440;0.8862]$, $x_{s2}=[0.4472;0.7520]$, and $x_{s3}=[0.7646;4.7052]$. Then the discrete T-S fuzzy time-delay systems can be constructed as follows:

Rule 1: If $x_2(k)$ is F_1 , then

$$x(k+1) = A_1x(k) + A_{d1}x_d(k) + B_1u(k) + E_1w(k)$$

Rule 2: If $x_2(k)$ is F_2 , then

$$x(k+1) = A_2x(k) + A_{d2}x_d(k) + B_2u(k) + E_2w(k)$$

Rule 3: If $x_2(k)$ is F_3 , then

$$x(k+1) = A_3x(k) + A_{d3}x_d(k) + B_3u(k) + E_3w(k)$$

where

$$A_1 = \begin{bmatrix} 0.7500 & 0.0119 \\ -0.2238 & 0.8262 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6203 & 0.0762 \\ -1.2337 & 1.3265 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.3068 & 0.0442 \\ -3.6621 & 1.0765 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0.0435 & 0.0003 \\ -0.0061 & 0.0455 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.0403 & 0.0019 \\ -0.0312 & 0.0581 \end{bmatrix},$$

$$A_{d3} = \begin{bmatrix} 0.0310 & 0.0013 \\ -0.1037 & 0.0528 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0004 \\ 0.0546 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0023 \\ 0.0698 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.0015 \\ 0.0634 \end{bmatrix},$$

$$E_1 = E_2 = E_3 = \begin{bmatrix} 0 \\ 0.00001 \end{bmatrix}.$$

Figure 5.5 Membership functions.

The membership functions shown in Figure 5.5 are given as follows:

$$\begin{aligned}
F_1 &= \begin{cases} 1 & \text{if } x_2 \leq 0.8882 \\ \frac{2.7520 - x_2}{2.7520 - 0.8882} & \text{if } 0.8882 < x_2 \leq 2.7520 \\ 0 & \text{if } x_2 > 2.7520 \end{cases} \\
F_2 &= \begin{cases} 1 - F_1 & \text{if } 0.8882 < x_2 \leq 2.7520 \\ 1 - F_3 & \text{if } 2.7520 < x_2 \leq 4.7052 \end{cases} \\
F_3 &= \begin{cases} 0 & \text{if } x_2 \leq 2.7520 \\ \frac{2.7520 - x_2}{4.7052 - 2.7520} & \text{if } 2.7520 < x_2 \leq 4.7052 \\ 1 & \text{if } x_2 > 4.7052 \end{cases}
\end{aligned}$$

The weighting matrices Q and R are selected as $\text{diag}\{1e-6, 1e-9\}$ and 0.001, respectively. The input constraint and bound of disturbance are set as $\|u\| \leq 10$, $\|w\| \leq 1$. As the constraint $\gamma + \gamma_d = 1$ should be satisfied, set $\gamma = 0.8$, $\gamma_d = 0.2$.

The simulation result for Algorithm 1 (online MPC approach) is shown in Figure 5.6 where the initial (delayed) states are set as $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.6972; 3.652]$. The piecewise reference signal changes from $[0.4472; 2.7520]$, to $[0.1440; 0.8862]$, to $[0.4472; 2.7520]$, and then to $[0.7646; 4.7052]$. Figure 5.6 demonstrates that the closed-loop state response can track the piecewise reference well via the proposed Algorithm 1. It also shows that the input constraint is satisfied. The number of time-varying delay is also displayed in Figure 5.6. It can be seen that large delays are involved that the upper bound of delay is 10. As such, if the Krasovskii approach is utilized instead, it can be easily got that the controller synthesis and computation will become very complex.

The simulation result of Algorithm 2 (off-line MPC approach) is shown in Figure 5.7, where the initial conditions are given as: $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.8573; 4]$ and $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.126; 3.0533]$, respectively. The number of robust constraint sets including the terminal constraint set is given as 25. From Figure 5.7, it can be seen that, despite different initial conditions, the system all (possibly

delayed) states are steered to the terminal set in the end. The number of time-varying delay for the two initial conditions are shown in Figure 5.8.

Figure 5.6 Simulation results of Algorithm 1.

Figure 5.7 Control performance of Algorithm 2.

Figure 5.8 The number of time-varying delays.

Figure 5.9 Constraint sets for all (delayed) states and the current state at each time instant for the two different initial conditions.

Figure 5.9 gives a clearer view of the differences via histograms, in constraint sets where all (delayed) states and the current state belong to, respectively. The upper and the lower histograms correspond to $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.8573; 4]$ and $x(i)_{i \in \mathbb{Z}_{[-10,0]}} = [0.126; 3.0533]$, respectively. Set 1 represents the smallest constraint set (terminal constraint set) in Fig. 10, while Set 25 represents the

largest constraint set. One can see that system all (possibly delayed) states are steered to the terminal set in finite time. And the implied positively invariant set theory and finite-time control theory are the same as those demonstrated in Example 5.1.

5.6 Conclusions

MPC of discrete-time nonlinear systems with time-varying delay is investigated in this chapter. Both online and off-line MPC approaches are provided. Based on the Lyapunov-Razumikhin approach, new conditions of robust positive invariance and input-to-state stability with respect to disturbances are realized. In the off-line approach, a sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line. Therefore, the online computational burden of classical MPC can be significantly reduced. Moreover, the exact time delay can be allowed unknown in the proposed two approaches. However, if the time delay is known, a less conservative result can be provided in the off-line MPC approach.

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Chapter 6

Model Reference Tracking Control of Networked Systems: Application to a Linear Brushless DC Motor

This chapter investigates control of networked systems subjected to communication delays and packet dropouts, with delays and dropouts in both sensor-to-controller and controller-to-actuator channels taken into account. Model reference tracking control of a linear motor motion platform, which is modeled as a networked control system subjected to time-varying delay, is the control objective. Based on the Lyapunov Razumikhin function, a novel delay-independent model reference tracking control approach is proposed to deal with systems subjected to unknown time-varying delay that, the upper bound of delay is allowed to be unknown. Therefore, it is more flexible than the case that the upper bound of delay is required to be known. In addition, it has a simpler representation in controller synthesis when compared to traditional approaches based on the Lyapunov Krasovskii functional. The control problem can be formulated as solving matrix inequalities. Simulation and experimental results on the linear motor motion platform are provided to illustrate the effectiveness of the proposed method.

6.1 Introduction

Apart from the Lyapunov-Krasovskii functional (LKF) adopted in [1], another approach of Lyapunov theory for time-delay systems is the Lyapunov-Razumikhin function (LRF) [2]–[6]. The Razumikhin approach involves a Lyapunov function for the original non-augmented system. Therefore, it has a good potential to get rid of the complexity associated with the Krasovskii approach. It is noted that Razumikhin approach has been applied to networked control of dc motor subjected to time-varying input delay [7], [8], however, only the regulation control problem is considered in these works, and the maximal delay is less than one sampling instant.

In this chapter, the Razumikhin approach is adopted for control of a linear brushless dc motor (LBDCM) motion platform subjected to network-induced communication delays and package dropouts, with the model reference tracking control problem [9]–[11] being considered. In particular, to deal with system suffered from time-varying or unknown delay, a simple controller design is provided by taking the advantage of LRF which has an easier representation when compared to LKF. And an H_∞ -like control problem is employed that the control law is computed by solving matrix inequalities.

It should be noted that although only an H_∞ -like control problem is investigated since we have not found a way to guarantee the typical H_∞ performance with the proposed LRF-based approach. However, the H_∞ tracking performance is only optimal with respect to a prescribed cost function that it does not necessarily lead to a best controller for tracking. Moreover, a perfect tracking control performance cannot be achieved even if the H_∞ tracking performance is satisfied. Whether the transient control performance satisfies practical requirements is more important in motor control when compared to satisfaction of the H_∞ performance which only denotes the disturbance attenuation level over a large range of time interval.

The proposed method is delay-independent such that the upper bound of delay

is not required to be known, which has advantages especially for systems with unknown or large delays. Moreover, it should be noted that the tracking performance will be deteriorated with the number of delays. Thus it cannot be applied to systems with infinite delays. There should be an upper bound of delays but it is not required to be known for controller design.

6.2 Preliminaries

6.2.1 System Model

The frequency response of an LBDCM driven by a motor drive working in torque control mode is defined by

$$\frac{Y(s)}{U(s)} = \frac{b}{s(s+a)} \quad (6.1)$$

where $Y(s)$ is the rotor position. $U(s)$ is the input direct voltage. From (6.1), the following continuous-time model in state space is obtained:

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c w(t) \quad (6.2)$$

where $w(t)$ represents the additional external disturbances and parameter uncertainties, and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ b \end{bmatrix}, E_c = \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

where x_1 and x_2 represent the position and the speed, respectively. Discretize the above model with sampling time T_s using the following method

$$A = e^{A_c T_s}, B = \int_0^{T_s} e^{A_c \tau} d\tau B_c, E = \int_0^{T_s} e^{A_c \tau} d\tau E_c \quad (6.3)$$

which leads to the following discrete-time model

$$x(k+1) = Ax(k) + Bu(k) + Ew(k) \quad (6.4)$$

In addition, we consider model reference tracking control of NCSs in this work. As such, the linear desired reference model is given as follows:

$$x_r(k+1) = A_r x_r(k) + B_r r(k) \quad (6.5)$$

where $x_r(k)$ and $r(k)$ represent the reference state and reference input signal, respectively. A_r denotes a specific asymptotically stable matrix.

Furthermore, for zero initial condition, the H_∞ control problem can be described as:

$$\sum_{k=0}^{\infty} \{(x(k) - x_r(k))^T Q (x(k) - x_r(k))\} \leq \rho^2 \sum_{k=0}^{\infty} \hat{w}(k)^T \hat{w}(k) \quad (6.6)$$

where $x(k) - x_r(k)$ represents the tracking error. Q is a positive definite weighting matrix. ρ describes the disturbance attenuation level, and $\hat{w}(k) = [w^T(k), r^T(k)]^T$ represents a vector that contains disturbance and reference input.

6.2.2 NCSs

The networked system model is given by

$$x(k+1) = Ax(k) + Bu_d(k) + Ew(k) \quad (6.7)$$

where $u_d(k)$ denotes the delayed input.

Based on which, the structure of model reference tracking control of the networked LBDCM is shown in Figure 6.1. It can be seen that the whole NCS is divided by three parts: 1) the physical level, which contains the actuator, the LBDCM motion platform, and the sensor; 2) the controller level, which receives sensor information, generates control law and sends it to the actuator; 3) the supervisory level, which provides the whole system with the reference input signal and the reference model. It can be seen that communication between the physical level and the controller level is achieved via network, while the networked communication is not considered between the supervisory level and the controller level in this work. It is another problem that deserves investigation, for example, one may find it in the control problem of remote supervisory and control [12] and tele-operations [13].

In addition, there are three buffers in this control scheme, see Figure 6.1. The first one is buffer 1 with the controller, which restores a sequence of past reference states (or reference outputs in output feedback). The second one is buffer 2 in the controller, which always restores the newly arrived state with its time stamp sent from sensor. The third one is buffer 3 in the actuator, and it always restores the latest control action sent from the controller. At each sampling instant, the control law is generated by utilizing the state information from buffer 1 and buffer 2. After that, the control law together with the time stamp of the used feedback state, are sent to actuator via network. As such, it is assumed that all the states are available in the networked system.

Figure 6.1 Networked tracking control scheme of LBDCM.

Moreover, one can see that there are two channels of communication networks: one is the backward network from sensor to controller, the other one is the forward network from controller to actuator. In Figure 6.1, τ_{sc} and τ_{ca} are used

to represent sensor-to-controller and controller-to-actuator delays, respectively. There are two switches in the networks, S1 and S2, which are used to model the data package dropouts in two channels.

6.2.3 Lyapunov-Razumikhin Approach

Compared with the Lyapunov-Krasovskii approach which adopts a Lyapunov function for an augmented system with all possibly delayed states [1], the Lyapunov-Razumikhin approach offers a different stability condition that is based on the original state space of system dynamics. In this subsection, some fundamental knowledge about delay difference equations (DDE) and its stability based on LRF is introduced.

Firstly, consider the DDE with multiple delays described in [8],

$$x(k+1) \in F(\mathbf{x}_{[k-h,k]}) \quad (6.8)$$

where $\mathbf{x}_{[k-h,k]} \in (\mathbb{R}^n)^{h+1}$ is a sequence of (delayed) states, h denotes the maximal delay, $F: (\mathbb{R}^n)^{h+1} \Rightarrow \mathbb{R}^n$ is a set-valued map with the origin as equilibrium point, i.e., $F(\mathbf{0}_{[-h,0]}) := \{0\}$. Then the stability criterion based on the LRF can be described as Lemma 6.1.

Lemma 6.1 [8]: If there exist a $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$, some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a $\kappa \in \mathbb{R}_{[0,1]}$ such that

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|) \quad (6.9a)$$

$$V(x(k+1)) \leq \kappa \max_{i \in \mathbb{Z}_{[-h,0]}} V(x(k+i)) \quad (6.9b)$$

for all $x \in \mathbb{R}^n$, then the DDE (6.8) is $\mathcal{KL}$ -stable. And $V(x)$ is called an LRF for (6.8).

Based on the aforementioned DDE with multiple delays and its stability criterion, then we consider a discrete linear system with unknown time-varying delay

$$x(k+1) \in F(x(k), x_d(k)) \quad (6.10)$$

where $x(k), x_d(k) \in \mathbb{R}^n$, $x_d(k)$ denotes unknown timevarying delayed state at instant k , i.e., $x_d(k) = x(k-d)$, then the stability criterion based on the LRF can be described as Lemma 6.2.

Lemma 6.2: If there exist a $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$, some $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and a $\tau \in \mathbb{R}_{[0,1]}$ such that

$$\alpha_3(\|x(k)\|) \leq V(x(k)) \leq \alpha_4(\|x(k)\|) \quad (6.11a)$$

$$V(x(k+1)) \leq \tau \max\{V(x(k)), V(x_d(k))\} \quad (6.11b)$$

for all $x \in \mathbb{R}^n$, then the DDE (6.10) is $\mathcal{KL}$ -stable. And $V(x)$ is called an LRF for (6.10).

Proof: Given an arbitrary upper bound of the unknown time-varying delay discussed in system (6.10), it can be seen that the stability criterion (6.11b) leads to the satisfaction of (6.9b), i.e., assume the upper bound of delay is $\lambda > 0$, thus $d \in \mathbb{Z}_{[1,\lambda]}$. When (6.11b) is guaranteed, then it can be got that

$$\begin{aligned} & V(x(k+1)) \\ & \leq \tau \max\{V(x(k)), V(x_d(k))\} \\ & \leq \tau \max_{i \in \mathbb{Z}_{[-\lambda,0]}} V(x(k+i)) \end{aligned} \quad (6.12)$$

After this, the proof is very similar to that of Lemma 6.1 that one can find it in [8], thus it is omitted here.

6.3 Main Results

6.3.1 LRF-based Tracking control

In this section, the stabilization condition for the networked system (6.7) is introduced. Consider system (6.7), if the feedback control law for the subsystem with input delay is given as:

$$u_d(k) = \bar{K}(x_d(k) - x_{rd}(k)) \quad (6.13)$$

where $x_d(k)$ and $x_{rd}(k)$ are delayed state and delayed reference state,

respectively, then an augmented system can be got by combining (6.5), (6.7) and (6.13):

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{A}_d\bar{x}_d(k) + \bar{E}\bar{w}(k) \quad (6.14)$$

where

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ x_r(k) \end{bmatrix}, \quad \bar{x}_d(k) = \begin{bmatrix} x_d(k) \\ x_{rd}(k) \end{bmatrix}, \quad \bar{w}(k) = \begin{bmatrix} w(k) \\ r(k) \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} BK & -BK \\ 0 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & B_r \end{bmatrix}.$$

In this work, to reduce the complexity of controller synthesis, the LRF rather than the LKF is adopted for stabilization of the closed-loop time-delay system (6.14). As disturbance is involved, an H_∞ -like control problem is considered to attenuate the impact of the disturbance. Let Lyapunov function $\bar{V}(x(k)) = x^T(k)\bar{P}x(k)$, from Lemma 6.2, it comes out that

$$\bar{V}(\bar{x}(k+1)) - \tau \max\{\bar{V}(\bar{x}(k)), \bar{V}(\bar{x}_d(k))\} + \bar{x}^T(k)\bar{Q}\bar{x}(k) - \rho^2\bar{w}^T(k)\bar{w}(k) < 0 \quad (6.15)$$

where $\tau \in \mathbb{R}_{(0,1)}$ and

$$\bar{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} \quad (6.16)$$

Theorem 6.2: If there exist a positive definite matrix $\bar{P}$, and a positive scalar $\tau \in \mathbb{R}_{(0,1)}$, such that the following matrix inequality is feasible

$$\begin{bmatrix} \bar{A}^T\bar{P}\bar{A} - \tau\gamma\bar{P} + \bar{Q} & * & * \\ \bar{A}_d^T\bar{P}\bar{A} & \bar{A}_d^T\bar{P}\bar{A}_d - \tau\gamma_d\bar{P} & * \\ \bar{E}^T\bar{P}\bar{A} & \bar{E}^T\bar{P}\bar{A}_d & \bar{E}^T\bar{P}\bar{E} - \rho^2I \end{bmatrix} < 0 \quad (6.17)$$

where γ and γ_d are pre-set valued positive scalars satisfying $\gamma + \gamma_d = 1$, then the closed-loop augmented system (6.14) is $\mathcal{KL}$ -stable under the control law $\pi = K(x_d(k) - x_{rd}(k))$.

Proof: Multiplying $\begin{bmatrix} \bar{x}^T & \bar{x}_d^T & \bar{w}^T \end{bmatrix}$ and its transpose from both sides of (6.17), respectively, yields that

$$\begin{aligned}
& (\bar{A}\bar{x}(k) + \bar{A}_d\bar{x}_d(k) + \bar{E}\bar{w}(k))^T \bar{P} (\bar{A}\bar{x}(k) + \bar{A}_d\bar{x}_d(k) + \bar{E}\bar{w}(k)) \\
& - \tau\gamma\bar{x}^T(k)\bar{P}\bar{x}(k) - \tau\gamma_d\bar{x}_d^T(k)\bar{P}\bar{x}_d(k) + \bar{x}^T(k)\bar{Q}\bar{x}(k) - \rho^2\bar{w}^T(k)\bar{w}(k) \leq 0
\end{aligned} \quad (6.18)$$

Due to $\gamma + \gamma_d = 1$, thus

$$\tau\gamma\bar{x}^T(k)\bar{P}\bar{x}(k) + \tau\gamma_d\bar{x}_d^T(k)\bar{P}\bar{x}_d(k) \leq \tau \max\{\bar{V}(\bar{x}(k)), \bar{V}(\bar{x}_d(k))\} \quad (6.19)$$

Then (6.30) and (6.31) together yields (6.15).

In addition, in the disturbance-free case, i.e., $\bar{w}(k) = 0$, then from (6.15) one can get that

$$\bar{V}(\bar{x}(k+1)) \leq \tau \max\{\bar{V}(\bar{x}(k)), \bar{V}(\bar{x}_d(k))\} \quad (6.20)$$

From Lemma 6.2, it implies that the closed-loop system (6.14) is $\mathcal{KL}$ -stable.

The proof is thus complete.

For convenience, $\bar{P}$ can be described as

$$\bar{P} = \begin{bmatrix} \bar{P}_c & \\ & \bar{P}_r \end{bmatrix} \quad (6.21)$$

Then (6.17) can be rewritten as

$$\begin{bmatrix}
\Theta_{11} & * & * & * & * & * \\
-Q & \Theta_{22} & * & * & * & * \\
\Theta_{31} & 0 & \Theta_{33} & * & * & * \\
\Theta_{41} & 0 & \Theta_{43} & \Theta_{44} & * & * \\
E^T \bar{P}_c A & 0 & \Theta_{53} & \Theta_{54} & \Theta_{55} & * \\
0 & B_r^T \bar{P}_r A_r & 0 & 0 & 0 & \Theta_{66}
\end{bmatrix} < 0 \quad (6.22)$$

where $\Theta_{11} = A^T \bar{P}_c A - \tau\gamma \bar{P}_c + Q$, $\Theta_{22} = A_r^T \bar{P}_r A_r - \tau\gamma_d \bar{P}_r + Q$,

$\Theta_{31} = -\Theta_{41} = (B\bar{K})^T \bar{P}_c A$, $\Theta_{33} = (B\bar{K})^T \bar{P}_c B\bar{K} - \tau\gamma_d \bar{P}_c$, $\Theta_{43} = -(B\bar{K})^T \bar{P}_c B\bar{K}$,

$\Theta_{44} = (B\bar{K})^T \bar{P}_c B\bar{K} - \tau\gamma_d \bar{P}_r$, $\Theta_{53} = -\Theta_{54} = E^T \bar{P}_c B\bar{K}$, $\Theta_{55} = E^T \bar{P}_c E - \rho^2 I$,

$\Theta_{66} = B_r^T \bar{P}_r B_r - \rho^2 I$.

6.3.2 Control Algorithm

The control algorithm of the proposed method is shown as follows:

Algorithm:

- 1: Initialization: At instant k , it is recognized that the delayed reference states $\mathbf{x}_{r[k-h,k]}$ restored in buffer 1 of the controller, the state $x_c(k-1)$ restored in buffer 2 of the controller, and the control law $u_a(k-1)$ restored in buffer 3 of the actuator are available.
- 2: If: Package dropout in the backward channel happens,
- 3: The restored state $x_c(k)$ with its time stamp $\alpha(k)$ in buffer 2 of the controller is not updated, i.e., $x_c(k) = x_c(k-1)$, $\alpha(k) = \alpha(k-1)$, and $u_c(k) = \bar{K}(x_c(k) - x_r(\alpha(k)))$.
- 4: Else: The sensor information $x(k - \tau_{sc}(k))$ is sent to controller, $x_c(k) = x(k - \tau_{sc}(k))$, $\alpha(k) = k - \tau_{sc}(k)$, $u_c(k) = K_{delay}(x_c(k) - x_r(\alpha(k)))$.
- 5: End
- 6: If: Package dropout in the forward channel happens,
- 7: The control law $u_a(k)$ with the time stamp $\beta(k)$ of corresponding feedback state in buffer 3 of the actuator is not updated, i.e., $u_a(k) = u_a(k-1)$, $\beta(k) = \beta(k-1)$.
- 8: Else: The control law $u_c(k - \tau_{ca}(k))$ with the time stamp ε of corresponding feedback state is sent to the actuator,
- 9: If: $\varepsilon > \beta(k-1)$
- 10: $u_a(k) = u_c(k - \tau_{ca}(k))$, $\beta(k) = \varepsilon$.
- 11: Else: $u_a(k)$ and $\beta(k)$ are not updated, i.e., $u_a(k) = u_a(k-1)$, $\beta(k) = \beta(k-1)$.
- 12: End
- 13: End
- 14: The control law $u_a(k)$ is applied to the actuator. Let $k = k+1$ and return to Step 1.

6.4 Simulation Results

The model of the LBDCM system is first identified by using MATLAB system identification toolbox. The identified system plant model is

$$\frac{Y(s)}{U(s)} = \frac{1.995}{s(s+10.46)} \quad (6.23)$$

By adding additional external disturbance signals, then select the sampling time as $T_s = 0.001s$, the discrete-time model can be got,

$$x(k+1) = Ax(k) + Bu(k) + Ew(k) \quad (6.24)$$

where $A = [1 \ 9.948e-4; 0 \ 0.9896]$, $B = [9.94e-7; 0.00198]$, $E = [5.494e-8; 0.0001]$.

The reference system is constructed as follows:

$$x_r(k+1) = A_r x_r(k) + B_r r(k) \quad (6.25)$$

where $A_r = A - BL_f$, $B_r = pB$, with $L_f = [0.77887 \ 0.01204]$, and $p = 0.77887$.

The controller gain $\bar{K}$ for the closed-loop delayed subsystem (6.14) can be calculated by solving matrix inequality (6.22) with $\gamma = 0.8$, $\gamma_d = 0.2$, $Q = \text{diag}\{0.00001, 0.00000001\}$, and $\rho = 1$, thus $\bar{K} = [-95.9431 \ -86.4853]$.

In the simulation, the maximum value of allowable delays in both the sensor-to-controller channel and the controller-toactuator channel are set as 3. The simulation results are shown in Figure 6.2. In Figure 6.2, simulation results for tracking of square and sinusoidal source signals are provided. The dashed lines in the first and second subfigures represent the reference signals, and similar settings can be seen in the following figures. The amplitude, the frequency and the duty cycle of the square signal are set as 0.03m, 1Hz, and 0.5, respectively. From the simulation results, we can see that the actual position and speed can track well with their respective reference signals. The total delay in Figure 6.2 denotes the number of delay of the state used for feedback at each time instant. The package dropouts in the backward and forward channels are also shown in

Figure 6.2, where 1 and 0 represent regular transmission and package dropout, respectively. In addition, simulation results for tracking of a sinusoidal signal are shown. The amplitude and the frequency of the sinusoidal signal are set as 0.03m and 1Hz, respectively. The input voltage for each case is also demonstrated in Figure 6.2.

Figure 6.2 Simulation results.

Figure 6.3 The LBDCM system.

6.5 Experimental Results

Besides the simulation results provided in the last section, experimental exhibitions on an LBDCM are demonstrated in this section. The LBDCM servo system is shown in Figure 6.3. The system consists of an isolated power supply, dSpace DS1103, a computer, an LBDCM and a current amplifier which works in torque control mode. The controller is designed within MATLAB Simulink, which receives states information and generates the control law, at each instant. Then the control law is sent to dSpace DS1103, and then to the current amplifier.

In the experiment, we adopt a simulated network, which has also been widely used in a lot of works [1], [14], [15], to generate transmission delays and package dropouts between the computer controller and other components.

Figure 6.4 Experimental results.

Similar to the simulation, at each instant, the maximum value of allowable delays in both sensor-to-controller and controller-to-actuator channels are set as

3. The experimental results are shown in Figure 6.4, where tracking control of square and sinusoidal signals are achieved, respectively. In both two experiments, the amplitude of the position reference signal is set as 0.03m, and its frequency is changed from 0.25Hz, to 0.5Hz and 0.75Hz in sequence. The input voltage, the total number of time delay of the state for feedback, and the package dropouts are also demonstrated. It can be seen that the system states can track well with the reference signals in the presence of time delays and package dropouts. In addition, own to the proposed method which is delay-independent, one can see that the number of delay can be very large in our experiments that the maximum value of total delay can reach 20. However, it should be noted that the tracking performance will be deteriorated with the number of delays.

6.6 Conclusions

Model reference tracking control of a networked linear motor is investigated in this chapter. A new model reference tracking control algorithm for time-delay system is proposed based on the Lyapunov Razumikhin function. It is an alternative method that does not rely on an augmented system with all possibly delayed states when compared to existing approaches based on the Lyapunov Krasovskii functional. In addition, the proposed method is delay-independent such that the upper bound of delay is not required to be known, which has advantages especially for systems with unknown or large delays. The effectiveness of the proposed method is verified through simulation and experimental exhibition.

It is noted that only an H_∞ -like stabilization control is adopted that the typical H_∞ performance is not guaranteed. However, for the tracking control of motor, the H_∞ performance is not as important as it is used in some other applications. Because the H_∞ performance focus on the tracking performance with disturbances attenuation over a large time interval, while the transient tracking

error is more critical in motor control. Thus the main focus should be given to the actual tracking performance even though the H_∞ performance was satisfied. However, as a theoretical problem, the H_∞ problem based on the Lyapunov Razumikhin approach is still an interesting research topic that deserves further investigation. And according to our best knowledge, it has not been investigated yet. Thus it can be regarded as one of our future work.

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Chapter 7

Conclusions and Future Work

In this chapter, the main results and contributions of this thesis are summarized, as well as some possible directions of future work are provided.

7.1 Conclusions

This thesis investigates control of discrete systems subjected to nonlinearities, time delays, and disturbances, with a special focus on model predictive control of such a kind of systems, and both theoretical derivations and practical applications are provided. The main efforts and contributions can be summarized as following aspects.

Computation efficient algorithms.

The computational aspect has been taken into account throughout this thesis. In Chapter 2, a combination of model predictive control and T-S fuzzy systems with nonlinear local models are investigated. By taking the advantage of T-S fuzzy systems with nonlinear local models rather than classical T-S fuzzy systems that based on linear local models, the computational burden of online optimization for robust fuzzy model predictive control is decreased significantly. In Chapters 3-5, robust model predictive control of time-delay systems based on the Lyapunov Razumikhin function are investigated. Comparisons with different approaches are presented to show the computational advantage of proposed methods. In Chapter 6, model reference tracking control of a networked linear motor subjected to time delays and package dropouts is considered. Based on the Lyapunov Razumikhin function which is also investigated in Chapters 3-5, a delay-independent tracking control approach for networked systems is proposed that the upper bound of delays is not required to be known. Thus the controller synthesis and computation is simplified.

Lyapunov Razumikhin function.

Utilization of Lyapunov theory for time-delay systems can be categorized as two different approaches, the Lyapunov-Krasovskii functional (LKF) and Lyapunov-Razumikhin function (LRF). Compared with the Krasovskii approach, the Razumikhin approach is conservative but it involves a Lyapunov function for the original non-augmented system. Therefore, it has a good potential to get rid of the complexity induced by the Krasovskii approach

especially for systems with large delays and disturbances. It is noted that extensive works can be seen on stabilization of time-delay systems adopting LKF, while LRF-based approaches have not been widely investigated until in recent years, see [1]-[5], except for a few early works [6]-[8]. Therefore, we make use of the Lyapunov-Razumikhin function to deal with systems with time delays. In Chapters 3 and 4, online and efficient algorithms for fuzzy model predictive control of nonlinear systems with multiple delays are investigated, respectively. In Chapter 5, we extend the results in Chapters 3 and 4 to the case with time-varying delays. In Chapter 6, based on the Razumikhin approach, model reference tracking control of networked systems subjected to time delays and package dropouts are investigated.

Robust model predictive control of nonlinear time-delay systems.

Although time delays and disturbances have been investigated separately for many times, however, investigation of systems with both time delays and disturbances is still a challenging work. Due to the complexity of the classical Lyapunov-Krasovskii functional, the controller synthesis and computation are usually complicated especially for systems with large or unknown delays. It is not difficult to find that the complexity will increase if time delays and disturbance are considered simultaneously. Owing to the Lyapunov-Razumikhin function, robust fuzzy model predictive control of nonlinear systems with time delays and persistent disturbances is investigated in this thesis. And the input-to-state stability and the robust positive invariance of the constraint sets are achieved.

Efficient MPC algorithms.

One of the main drawbacks of classical MPC strategies is that it requires solving online an optimization problem, which prevents its applications to some contexts due to the high computational cost. Thus several MPC schemes have been developed to reduce the online computational burden, i.e., the explicit MPC and efficient MPC algorithms. As a counterpart of Chapter 3 where online optimization algorithms of fuzzy model predictive control of time-delay

systems is provided, Chapter 4 investigates efficient off-line MPC algorithms for systems with time delays and persistent disturbances. In Chapter 5, both online and efficient FMPC algorithms are provided for nonlinear systems with time-varying delays and disturbances.

Tracking control of networked control systems and its applications.

In this thesis, not only the theoretical results are derived, but also an application to a practical system is provided. Based on the Lyapunov-Razumikhin function, model reference tracking control of networked systems subjected to time delays and package dropouts is investigated. The proposed method is delay-independent such that the upper bound of delays is not required to be known. Simulation and experimental results on a linear motor are provided to verify the effectiveness of the proposed method.

7.2 Future Work

Some possible directions for future research are outlined below.

Distributed model predictive control of interconnected systems with local and communication delays.

Distributed control of network control systems has become a hot spot in the control area. With the fast development of communication and computing techniques, distributed control becomes more and more reliable and preferable compared with centralized control and decentralized control. Particularly, consensus of multi-agent systems has attracted a lot of research interest. Distributed MPC (DMPC) for multi-agent systems have been investigated in recent years. Both theoretical and experimental achievements have been made. DMPC of interconnected systems with local and communication delays and disturbances will be considered. It is noted that time delay can be happened in both single system and communication between systems. With the small gain theory, it shows that the communication delay will not affect the stability of the whole system if some conditions are hold. Then only local delay and

disturbances will be considered which is based on the results for MPC of time-delay systems with persistent disturbances in previous chapters. Meanwhile, involvement of constraints and uncertainties will be considered further.

Stochastic model predictive control of time-delay systems.

In recent years, control of systems with stochastic disturbances has attracted many research interests. In particular, several works can be seen on stochastic MPC [9]-[11]. However, there are still a lot of things to do with stochastic MPC for time-delay systems. In stochastic control scheme, the disturbance vector is usually modeled as a discrete-time Markov chain with a transition probability matrix. It is noted that with time delays and disturbances considered simultaneously, the controller synthesis and computation are more complicated.

Model predictive control of networked systems.

Various progresses have been achieved on networked control systems, such as communication delays [12], packet dropouts [13], quantization [14], scheduling [15], and security [16]. Among which, the time delays and dropouts will reduce the control performance and or even cause instability of the system if they were not considered appropriately. Compared with the Lyapunov-Krasovskii approach, the Lyapunov-Razumikhin approach provides with a different way to deal with time delays. The over conservatism induced by the former approach can be relieved with the latter one.

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