Nonlinear dynamic analysis and feedback control of thermoacoustic instability

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

SINGAPORE

**NONLINEAR DYNAMIC ANALYSIS
AND FEEDBACK CONTROL OF
THERMOACOUSTIC INSTABILITY**

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SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

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NONLINEAR DYNAMIC ANALYSIS AND FEEDBACK CONTROL OF THERMOACOUSTIC INSTABILITY

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Abstract

This dissertation mainly studies the nonlinear dynamic behaviors of thermoacoustic instability and feedback control of thermoacoustic oscillations numerically and experimentally. Rijke-type thermoacoustic systems are modeled for numerical investigation, and experimental investigations are conducted on a Rijke tube. Coupling the heat input with Galerkin series expansion of flow disturbances provides a platform to gain insights on the time evolution of system's response.

The effects of the heat source on the process of transition to thermoacoustic instability are studied. Two different premixed flame models are proposed to characterize the relation between the unsteady heat release and acoustic pressure: a 3rd-order polynomial and a 5th-order polynomial. Experiment is conducted to examine the transition process on a Rijke tube, within which both subcritical and supercritical bifurcations are observed. By comparison between the systems with different order heat source model, it is found that the nonlinearity of heat source determines the transition process of thermoacoustic system, i.e., via subcritical or supercritical bifurcation.

The effects of the noisy fluctuation on the stability behaviors of thermoacoustic systems are studied. The noisy fluctuation is characterized by a Gaussian white noise, and the stochastic stability is measured by stationary probability function. With the

increase of noise intensity, it is found that the thermoacoustic system near the hysteresis area shows a stochastic P -bifurcation. In addition, the theoretical results show that increasing the noise intensity would decrease the hysteresis region and expand the unstable area. Then the noise-induced motions near the bifurcation point are experimentally and numerically studied. Resonance-like behaviors of signal to noise ratio (SNR) are observed, denoting the occurrence of coherence resonance. When the system approaches the critical bifurcation point, SNR is found to become larger, and the optimal noise intensity is decreased to a smaller value. Qualitative agreements are obtained in the theoretical and experimental results in terms of SNR , the peak height of power spectrum, and the width of the frequency. Coherence resonance is also observed in supercritical thermoacoustic systems. By comparison, it is found that the characteristics of coherence resonance in thermoacoustic systems depend on whether the system loses stability via subcritical bifurcation or supercritical one.

To suppress large-amplitude thermoacoustic oscillations in Rijke tube, two different state-feedback controllers are proposed. One is a Lyapunov-function-based controller (LFC), and the other is a sliding mode controller (SMC). The non-uniformity of the temperature field within the combustor is considered, and it is approximated according to the experimental measurement. First, it is found that implementing the developed LFC can make the system achieve strict dissipativity, and reduce approximately 25 dB sound pressure level (SPL). Then, the performances of SMC are evaluated in two thermoacoustic systems with either 1 eigenmode or 4 eigenmodes. Both thermoacoustic systems are stabilized by reducing around 40 dB SPL, and the proposed controller shows a good performance in the system with multiple unstable modes.

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Nomenclature

Throughout this thesis, bold letters denote vectors and matrices; lower case letters denote variables and parameters.

Symbols

$ \cdot $	Absolute value of quantity
$\ \cdot\ $	Norm of matrix
$\langle\cdot\rangle$	Expectation operator
$\dot{(\cdot)}$	First order time derivative
$\Delta(\cdot)$	Difference operator
$\hat{(\cdot)}$	Fourier transform of quantity
$\bar{(\cdot)}$	Time averaged quantity
$(\cdot)'$	Fluctuating quantity
$(\cdot)^T$	Transpose operator
$(\cdot)_l$	Variable for the upstream region of heat source
$(\cdot)_r$	Variable for the downstream region of heat source
A	Amplitude of oscillation
c	Speed of sound

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D	Intensity of noise
E_s	Total acoustic energy per unit cross-sectional area
H	Maximum of power spectrum density
i	Unit imaginary number $\sqrt{-1}$
j	Integer variable
K	Non-dimensional heating power
L	Total length of tube
M	Mach number
m, n	Integer variables
n_w	white Gaussian noise
N	Number of eigenmodes
N_1	Number of actuators
p	Pressure
P	Probability transition function
P_s	Stationary density function
Q	Rate of heat release
S	Cross-sectional area of tube
S_{aj}	Cross-sectional area of the j -th actuator
$\mathbf{S}_w$	Switch function
R_j, S_j	Actuator parameters
t	Time
T	Temperature
Tr	Transfer function
u	Velocity

U	Control input
V	Volume of combustor
x	Axial position
η	Time-dependent variable
ψ	Basis function
δ	Dirac function
ρ	Flow density
γ	Specific heat ratio
C_v	Specific heat at constant volume
ξ	Acoustic damping coefficient
ω	Frequency of acoustic waves
λ	Spectrum
φ	Phase lag
τ	Time delay
ϕ_f	Fuel flow rate
α_i	non-physical variable
α_{ai}	Ratio of S_{ai} to S
$\mathcal{L}$	Lyapunov function

Acronyms

LFC	Lyapunov-Function-Based Control
LQR	Linear Quadratic Regulator
PSD	Power Spectrum Density
SNR	Signal-to-Noise Ratio

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SMC	Sliding Mode Control
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

Chapter 1

Introduction

1.1 Thermoacoustic Instability

Thermoacoustic instability often occurs in many thermoacoustic engines [4–6] and combustion systems. It is desirable in thermoacoustic engines, because the generated pressure oscillations can be utilized to produce electrical power via piezoelectric generators [7–9]. However, it is unwanted in propulsion systems (such as rocket motors [10], ramjets, and afterburners), land-based gas turbines [11–13], and furnaces. In different types of combustion systems, thermoacoustic instability manifests itself as different characteristic pressure oscillations, such as surging (very low frequencies), chugging (on the order of 100 Hz), screaming (high frequencies of thousand Hz), buzzing, and squealing. The present studies focus on the negative influences of thermoacoustic instability in combustion systems.

Generally, thermoacoustic instability is caused by a coupling between the unsteady heat release and acoustic waves. First, the unsteady heat release works as an efficient energy source to produce the acoustic waves. Then, the acoustic waves propagate

within the combustor, part of which will be reflected from surface boundaries. Finally, the reflected acoustic waves reach the combustion zone and in turn perturb the heat release rate. If the unsteady heat release rate from the heat source is in phase with the acoustic pressure, the pressure oscillation is intensified. Otherwise, the pressure oscillation is dampened.

Thermoacoustic oscillations also occur on a Rijke tube at laboratory scale. A Rijke tube is a simple open-ended tube with a compact heat source inside. The nature of the heat source varies depending on specific experiments, such as electrically-heated wires or a premixed flame. At certain conditions, a Rijke tube can generate a loud sound which exhibits a similar process of thermoacoustic instability with large-scale practical devices. Considering that the practical combustion is associated with several couplings between the flow, the pressure, and the entropy fields, directly analyzing the real operation conditions is too complex. Therefore, considerable experimental, analytical and numerical researches have been done on a Rijke tube to elucidate the processes accounting for these instabilities. Our study is restricted only to Rijke tube.

1.2 Control of Thermoacoustic Instability

1.2.1 Passive Control

Many attempts have been made to prevent the onset of thermoacoustic instabilities or dampen self-sustained oscillations [14]. These efforts usually include two categories: 1) passive control and 2) active control. Passive control involves either absorbing acoustic energy by utilizing acoustic dampers such as half or quarter wave tubes [15,16],

acoustic liners [17], Helmholtz resonators [18–20] and perforated plates [21–23], or decreasing the sensitivity of the combustion process to acoustic excitation by structure modifications, such as the structure asymmetry [24], modifying the fuel injection system [11, 25], changing the position of heat sources and the length of the resonator. In some practical combustion systems, passive control has been widely used with a good performance. Lack of movement makes them high durability and low maintenance. In addition, high robustness makes it impossible to enhance an instability or destabilize a stable combustor. However, extensive testing and modifications in a trial-and-error procedure are required, so implementing this technique is costly and time consuming. Furthermore, the structure modification might meet the space and weight limitations. Most importantly, passive control is only effective over a narrow high frequency range for some specific combustors.

1.2.2 Feedback Control

As an alternative to passive control, active control of thermoacoustic instability [26, 27] shows large potential in the regime where passive control is weak, such as handling low-frequency oscillations. It can also enhance the efficiency of the combustion. Active control is comprised of open-loop control and closed-loop control (feedback control). Technically, open-loop control modulates the system without guidance, while feedback control is configured in a way that the combustion system can be actively modified in response to a measured signal by the addition of external forcing perturbations [28].

There are three main components in implementing feedback control to suppress thermoacoustic oscillations: sensors, actuators, and controllers. Sensors are used to

provide some information of a dynamic quantity concerning combustion oscillations, such as pressure transducers and photomultipliers. The first sensors are used to measure acoustic pressures with large bandwidth and reasonable robustness. Considering that acoustic waves propagate throughout the whole combustor, pressure transducers can be placed away from the combustion zone, but pressure nodes should be avoided. Photomultipliers show a large freedom in choosing its location, because the information they measured is independent of mode shapes. They can detect the radical light emission of flame, which is interpreted as being proportional to the heat release in the combustor. However, the photomultiplier is sensitive to the variation in flame location, and it requires an optic access for data collection. The state of the art in sensors is reviewed in [28]. Actuators are used to produce perturbations to modulate the combustion condition. The typical control actuators involve flow injectors [29,30] and acoustic drivers. The first one is mainly used to perturb the heat release rate, and it only needs a change in the fuel or air feed system. This actuation is generally used in full-scale combustion systems. However, it suffers from limited bandwidth. The second actuators are used to change the flow field with the sacrifice of modification to the combustor structure, such as loudspeakers [31,32]. Easy implementation and large bandwidth make loudspeakers popular at laboratory scale, but limited power capacity makes loudspeakers unpopular in industrial combustion systems [33,34]. Last, a controller determines the relation between the measured signal and the signal for actuation. Detailed information are reviewed in Sec. 2.3.

1.3 Main Motivation

The strongest drive of studying the mechanism of thermoacoustic instability comes from the incurred damaging influences on combustion systems.

In the development of high-performance propulsion systems, thermoacoustic instability was first observed in the testing of large-thrust rocket motors, and the amplitude of the emergent pressure oscillations was up to 100% mean pressure [10], which leads to a severe damaging to the combustor structure, just as shown in Fig. 1.1. It was also found that such oscillations can increase the heat transfer to the combustor wall, and damage the material of the combustor.

Figure 1.1: Damaged components of NASA rocket motors by thermoacoustic oscillations [1]

Thermoacoustic oscillations present in gas turbines are not so intense like in rocket motors [2]. Generally, the incurred acoustic oscillations in gas turbines account for ten percentage of mean pressure. This oscillation can cause high-cycle mechanical

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vibrations and deteriorate the components inside the combustors by fatigue fracture, as shown in Fig. 1.2.

Figure 1.2: Damaged components of gas turbines by thermoacoustic oscillations [2]

While irrelevant to structural damages, some combustion systems, such as industrial gas, fuel or coal fired furnaces, which operate close to the atmospheric pressure, may involve thermoacoustic oscillations that lead to unacceptable noise. These noises are undesirable and may adversely affect the health of human beings.

Currently, with increasingly stringent requirements of low NO_x emission and higher combustion efficiency, the problem of thermoacoustic instability becomes more severe in new generation of gas turbines. It may be due to two reasons. First, considering that the amount of NO_x emission is exponentially proportional to the mean temperature inside the combustor, one will aim to decrease the mean temperature for low pollutant emissions, at the expense of higher probability of flame blowout or flashback. Second, lean premixed pre-vaporized techniques are widely used to meet the requirements of low emission. However, burning at lean operating condition is especially prone to thermoacoustic instability, which can be seen in Fig. 1.3.

Figure 1.3: Effect of equivalence ratio on NOx production [3]

To sum up, the deeper understanding of the mechanism of thermoacoustic instability is of central importance, in particular for the avoidance of the thermoacoustic instability at the early stage of engine design and efficient mitigation of the emergent thermoacoustic oscillations.

1.4 Objectives

A good understanding of the complicated behaviors of thermoacoustic instability can not only make the prediction of the system's response increasingly reliable, but also provides more useful information for controller design. For this, the following works are conducted:

1. The processes of transition to instability of thermoacoustic system are numerically and experimentally studied on a Rijke tube with a premixed flame, as the nonlinearity of flame and system's main parameters are changed.

2. The effects of the noisy fluctuation on the stability boundaries and coherent motions are numerically studied. Experiments are conducted to validate the numerical findings about the characteristics of coherent motions.

Due to the destructive consequences of thermoacoustic instability to combustion system, exploring proper approaches to dampen thermoacoustic oscillations is important. For this, the following works are conducted:

1. Lyapunov-function-based controller is proposed to eliminate the destructive large-amplitude thermoacoustic oscillations, with an emphasis on constraining the ‘triggering’ of initial low-amplitude perturbations.
2. Sliding mode controller is proposed to stabilize the thermoacoustic instability in combustion systems, due to its successful applications in dealing with the influence of modeling nonlinearity, uncertainty, and disturbances.

1.5 Structure and Research Programme

The structure of this dissertation is organized as follows. In Chapter 1, the background of thermoacoustic instability and control of thermoacoustic oscillations are introduced. Chapter 2 reviews the previous studies about thermoacoustic instabilities and feedback control of thermoacoustic oscillations. In Chapter 3, the nonlinear dynamic behaviors of Rijke-type thermoacoustic systems are studied. Considering the destructive consequences of thermoacoustic oscillations, two different feedback controllers are developed, and their performances of suppressing thermoacoustic oscillations are evaluated in Chapter 4. In Chapter 5, a summary of the present findings and some potential future works are provided.

Chapter 2

Literature Review

In the present thesis, our studies focus on the nonlinear dynamic analysis of the thermoacoustic instability and feedback control of thermoacoustic oscillations [35]. The papers of most interest and relevance to our topics are briefly reviewed.

2.1 Physics of Thermoacoustic Instability

Before the discovery of Rijke-type oscillations [36, 37], two similar thermoacoustic oscillations had been observed. The first one is ‘singing flame’ which was observed by Higgins (1802), when a jet of flame was inserted into a vertical glass tube with upper end closed [38]. The frequency of sound is found to be close to the first natural frequency of the tube. The other is ‘Sondhauss oscillations’ (1850) that occurred in a tube by heating the closed bulb end [39]. In 1859, Rijke [40] heard a strong booming, when a heated wire screen was anchored in an open-ended vertical pipe. In addition, the maximum oscillations were obtained, when the heater was placed at one-fourth of the tube’s length. The original configuration of a Rijke tube is shown in Fig. 2.1,

Figure 2.1: Original configuration of the Rijke tube

and its geometry has a large length-to-diameter ratio. A criterion for the necessary condition of such oscillations [41] was advanced by Lord Rayleigh in 1878 [41]:

“If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged”.

Pflaum attributed the Rijke oscillation to friction between the convective upright flow and the heated wires [42]. If his theory had been correct, a Rijke tube with no heating source would have song by friction from forced convection. However, this phenomenon has not been observed in the experiments. Blackshear [43] verified the Rayleigh’s criterion by an analytical derivation for low Mach number and constant temperature distribution. Putnam and Dennis [44] experimentally confirmed the Rayleigh’s criterion in three different hydrogen-air configurations: a Rijke tube with

Figure 2.2: A control volume of perfect gas within a combustor. The arrow denotes the direction of the wave traveling in the tube. The red color is for the incident waves which are generated by the unsteady heat release, while the green color is for the reflected waves from tube end.

a premixed flame, a Schmidt tube (closed-open tube) with a premixed flame, and a Schmidt tube with a diffusion flame. Chu [45] generalized Rayleigh's criterion by incorporating the effects of boundary conditions, and provided a mathematical insight into thermoacoustic instability, as shown in Eq. (2.1). In his work, a perfect gaseous reacting medium is considered in a volume V and bounded by a surface S , as shown in Fig. 2.2. For simplicity, the effects of mean flow and mean heat release are assumed to be neglected. The instantaneous variables, including pressure p , velocity u , density ρ , and speed of sound c , are a combination of a mean value denoted by an overbar and a fluctuating value by a prime. The acoustic energy equation can be presented as

$$\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \bar{\rho} u^2 + \frac{1}{2} \frac{p'^2}{\bar{\rho} \bar{c}^2} \right) dV = \int_V \frac{(\gamma - 1) p' q}{\bar{\rho} \bar{c}^2} dV - \int_S p' \mathbf{u} \cdot d\mathbf{S} \quad (2.1)$$

The term on the left-hand side (LHS) represents the rate of change of total acoustic energy in the combustor volume V . The first term on the right-hand side accounts for the exchange of energy between combustion and acoustic pressure. If the phase between the heat release rate and acoustic pressure lies between -90° and 90° , the gain of acoustic energy is increased. The second term on the right-hand side represents the

energy losses across the bounding surface for the dissipation of fluid. If the acoustic energy gain from combustion is larger than acoustic energy losses across the surface over one oscillation cycle, the total acoustic energy increases and thermoacoustic instability occurs. Therefore, the condition for thermoacoustic instability [46] can be transformed as

$$\int_V \frac{(\gamma - 1)\overline{p'q}}{\overline{\rho c^2}} dV > \int_S \overline{\rho' \mathbf{u}} \cdot d\mathbf{S} \quad (2.2)$$

where the overbar here denotes an average over one acoustic oscillation cycle. If the inequality is satisfied, the amplitude of acoustic pressure would grow exponentially and finally saturate to limit cycle due to the effects of nonlinearity. More early studies about Rijke-type oscillations are reviewed in [36, 37]

2.2 Nonlinear Dynamic Analysis of Thermoacoustic Instability

Over the last several decades, many numerical, experimental, and theoretical studies have been conducted to know the complex behaviors of thermoacoustic systems. Han [47] used a network modelling to perform a limit cycle analysis in a combustor driven by a partially premixed flame. The nonlinear flame response is characterized by large eddy simulation, and it is combined with the wave-based linear model. The predicted frequency and forcing amplitude of the limit cycle oscillations agree well with the experimental measurement. Sterling et al. [48] used the phase space reconstruction to characterize the pressure oscillation in a lab combustor, and a series of bifurcations is reported, including a Hopf bifurcation to period motions, period-doubling bifurcations, and transition to chaos. Turan et al. [49] employed the Poincare map to

analyze the stability of thermoacoustic oscillations in a simplified rocket motor. Kabiraj et al. [50] conducted experimental bifurcation analysis of pressure oscillations and reported several complicated nonlinear responses, such as periodic, quasi-periodic, frequency-locked, and chaotic oscillations in a simple duct with open-closed ends. By comparison, using Galerkin's expansion to theoretically analyze the nonlinear behaviors of combustion would save much time and efforts. It was firstly introduced by Powell and Zinn [51] in liquid-fueled rocket motors. Later Culick [52] developed this approach to derive an analytical condition for the occurrence of nonlinear limit cycle oscillations. The unsteady flow is decoupled in the spatial and time domain, and the governing equations for acoustic waves are represented by a set of ordinary differential equations (ODEs). Based on the obtained ODEs, Culick proposed to use the method of averaging to study the nonlinear growth and finite amplitude of acoustic waves in solid propellant rockets with minimum efforts and expense. In the present work, the method of multiple scales is proposed to extract the essential features of thermoacoustic oscillations in Sec. 3.2. On the other hand, the nonlinearity of the unsteady heat release rate may due to the vortex shedding, equivalence ratio fluctuation, vortex-flame interaction, or oscillatory flame area variations. Since our interest is about the nonlinear effects, the heat input is assumed to be concentrated over a short axial location.

The accurate prediction of stability of a given combustion system is important for the design of stable combustors and the prevention of the occurrence of the damaging thermoacoustic oscillations. Two main approaches are generally used: linear stability analysis [53] and nonlinear analysis [49, 54]. The linear stability analysis often relies on the calculation of the eigenmodes and eigenfrequencies of the linearized

2.2. NONLINEAR DYNAMIC ANALYSIS OF THERMOACOUSTIC INSTABILITY

thermoacoustic systems. With the real parts of all eigenvalues being negative, the thermoacoustic system is linearly stable. This approach can provide useful information about the growth rates and frequencies of oscillations for acoustic damper design. In practice, the stability boundaries of the rocket motors are obtained by conducting pulsed test [55]. An interesting dynamic phenomenon has been observed, for some fixed operating conditions, where the thermoacoustic system is stable for small-amplitude disturbances, but becomes unstable for large-amplitude ones. This phenomenon is called ‘pulsed instability’ or ‘triggering’ [55–57], and it is a persistent problem in rocket motors and aircraft engines. Apparently, the classical linear analysis fails to explain triggering. To understand the mechanism of triggering in thermoacoustic systems, nonlinear bifurcation analysis [58–60] and transient growth analysis [57, 61, 62] are needed.

A bifurcation [63] is defined to describe a dynamic behavior that a small smooth change in one system parameter can cause a sudden qualitative change in the behavior of system. When a Hopf bifurcation occurs, a fixed steady state would switch into a limit cycle oscillation. There are mainly two types of Hopf bifurcations: subcritical and supercritical Hopf bifurcations [64]. With the critical parameter smoothly going through the bifurcation point, if the amplitude of induced limit cycle oscillation is increased/decreased continuously, a supercritical Hopf bifurcation occurs. While for a subcritical bifurcation, a little change to the critical parameter may lead to an abrupt large-amplitude oscillation. If the critical parameter is smoothly varied on the inverse direction, the limit cycle oscillation decrease initially, and suddenly jumps to a steady state at a second bifurcation point. Generally, the two critical bifurcation points do not overlap, and the region between them is called a hysteresis region, where a stable

fixed state and a stable limit cycle coexist. If the system is configured in the hysteresis region and forced by large external disturbances, the stable system would jump from one steady state to the basin of the periodic attractor, and stay there. This is the prerequisite for the occurrence of triggering.

Both supercritical and subcritical bifurcation have been experimentally observed in unstable gas turbine combustors [64] and Rijke tubes [65]. Typically, the state resulting from Hopf bifurcation in thermoacoustic systems is a limit cycle at a frequency close to one of the acoustic frequencies. Kabiraj [50] conducted an experimental bifurcation analysis of pressure oscillations in a Rijke type system. Subramanian [66] used DDE-BIFTOOL to study the subcritical bifurcation of a Rijke tube [67]. For the same reduced-order time-delayed system, Yang [60] applied the method of Lines to decouple the governing PDEs into a set of ODEs, and the predicted subcritical bifurcation agrees well with Subramanian's work [66]. Laera [59] used a Helmholtz solver to track the bifurcation diagram of thermoacoustic instabilities in longitudinal and annular combustors.

Transient growth is found to be able to trigger thermoacoustic oscillations in the hysteresis region [61] through the non-normality of thermoacoustic system [68–70]. Mathematically, a system is non-normal if the linearized operator of the system does not commute with its adjoint. This non-normality would lead the eigenmodes of the linearized system to be non-orthogonal [69]. As is known in a normal system with orthogonal eigenmodes, if the real parts of eigenvalues are negative, the external disturbances would decay monotonically at each eigenmode direction in time. However, in a non-normal system with non-orthogonal eigenmodes, the transient growth may occur for a suitable initial condition, hence the perturbations can grow transiently

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before eventually decay. Therefore, the external large-amplitude disturbances, which have potential to trigger thermoacoustic instability, can be obtained by amplifying the initial small perturbations through the transient growth.

In a thermoacoustic system, the non-normality can be caused by several factors, including unsteady heat release [54], general complex impedance boundary conditions [71], and the mean temperature ratio across the heat source [69, 72]. The non-normal phenomenon was first studied by Balasubramanian and Sujith in a Burke-Schumann tube with a diffusion flame confined [54], and later in a Rijke tube with heated wires [68]. Due to the presence of a heat source, the thermoacoustic system is non-normal. In addition, non-orthogonality leads to an exchange of energy between different acoustic modes, i.e., the energy can be transferred from one mode to another. Considering that the classical linear eigenvalue analysis can only predict the stability of non-normal system for a long-term period, pseudospectra analysis was proposed [68] to study the short-term dynamic behaviors of system, i.e., the transient growth. The maximum transient growth rate in the Rijke type systems was found to be around 1 to 10. In addition, the ‘dangerous disturbance’, which denotes the optimum initial condition for the maximum transient growth, is obtained by singular value decomposition. Different from the linear optimal condition for the maximum transient growth, Juniper [61] figured out the most dangerous condition with minimum energy for triggering. Mariappan et al. [66] experimentally confirmed the non-normal nature of thermoacoustic interaction in the Rijke tube. The measured maximum transient growth rate is on the same order of previous theoretical analysis [68]. Wieczoreck et al. [73] conducted a transient growth analysis in the tube with a choked isentropic nozzle outlet. Compared with using the classic acoustic energy as

the measure of transient growth, it is found that the total disturbance energy, which includes the contribution of the entropy waves, would give more accurate evaluation of transient energy growth. In addition, with the increase of the mean flow velocity and the strength of the thermoacoustic coupling, the degree of non-normality is increased. Different from the previous works with a compact heat source, Li et al. [74] conducted a transient energy growth analysis in the systems with distributed mean heat input. With the outlet end choked, the maximum transient growth of acoustic energy is found to be at least one order of magnitude larger than that with open-ended outlet.

Besides the transient growth, additive stochastic perturbations can also trigger thermoacoustic instability [61, 75]. In particular, pink noise is found to be more effective at triggering thermoacoustic oscillations than white noise and blue noise [76]. Besides the noise-induced triggering, numerous measured experimental data [77] suggest that thermoacoustic oscillations have random features. For example, the amplitude and phase of limit cycle oscillation vary from cycle to cycle [3], and the parameters defining stability boundaries of the combustor change from test to test. Since noise is inevitably present in thermoacoustic systems, the thermoacoustic system with stochastic driving will be more appropriate to represent its response. Generally, flow fluctuations [78] originating from aerodynamic noise [79], turbulence, and random modulations in the operating parameters are typically associated with heat release process [80]. Interest in combustion noise was initially raised in solid propellant rockets and ramjet engines [81]. Taking the effects of turbulent flow into account, Culick [79] performed an excellent stochastic modelling study with noise and combustion instability coupled. Lieuwen [82] studied the statistical characteristics of pressure

oscillations in a premixed combustor. A good prediction in probability density function of oscillation amplitude is obtained. Lieuwen [83] also studied the effects of the background noise on the combustion stability from a linear aspect, and explained many random features of combustion by external noise perturbation, such as cyclic variability. The experimental investigations [84] show that large-intensified noise would suppress the hysteresis region. Noise-induced coherence motions and stochastic bifurcations were experimentally observed in prototypical combustors [85, 86].

2.3 Feedback Control of Thermoacoustic Instability

Implementing active control to extend the stability domain needs three main components: sensors, actuators, and controllers. Researches in this topic usually fall into two categories. One is related to sensors and actuators, addressing how they can be properly chosen to effectively measure and excite the combustion system. The other is in regard to controller design, addressing how an effective and robust controller can be designed for complex combustion systems. Although our focus is on the second subject, a few examples of the first group is given first to know the efforts in practical application for completeness.

The analysis of controlling low-frequency oscillations was first proposed by Tsien in 1952 in a liquid mono-propellant rocket motor [87]. At the early stage, experimental investigations of active control of thermoacoustic oscillations were rare in real engines with critical operating conditions, such as high pressure and power density, large Reynolds numbers, multiple inlet systems, and complex fuels. Since the 1990s,

the improvement of technology in fast-response sensors and actuators made active control more practical. Seume et al. [29] implemented active control in a full-scale heavy duty gas turbine to suppress ‘humming’. Multiple water-cooled pressure transducers were placed around the combustor circumference to measure the azimuthal pressure oscillations. The fuel flow rate of pilot flame was actuated by a direct drive valve with a separate control system for each burner. With a simple gain/phase-shift controller implemented, the dominant mode oscillation was dampened by 17 dB. Later, a high-speed solenoid valve actuator was installed in a full-scale liquid-fueled gas turbine to modulate the fuel flow rate by Cohen [88] and Hibshman [89]. Respectively, around 16 dB and 6.5 dB reduction of the dominant oscillations were achieved. Moeck et al. [90] proposed to use a combination of a secondary fuel injection and acoustic perturbation in an atmospheric swirl-stabilized combustor. The secondary fuel was introduced by an on-off valve to dampen the main instability, and the upstream air mass flow was perturbed by loudspeakers to maintain stability. It was found that using two actuators simultaneously can give a better suppression than using only one of the two actuators, and a total reduction of 24 dB oscillations was obtained. Bennewitz et al. [14] proposed to use acoustic modulation in a simulated liquid rocket combustor to dampen longitudinal instability. A JBL 2446J compression driver is housed at the base of the oxidizer supply line of a single injector, and band-limited white noise and single frequency waves were chosen as actuation signal. Both acoustic modulations can achieve 34 dB reduction of oscillations. At laboratory scale, Dines [91] performed the first tests of active control of thermoacoustic oscillations on a Rijke tube. The light emission from CH radicals in the flames were measured by a photomultiplier to obtain the heat release rate, and a loudspeaker is used to modify the boundary

condition. A similar implementation was also conducted by Heckl, and instead of photomultiplier, a microphone was used to measure acoustic pressure. Since these primary demonstrations, loudspeakers, microphones, and photomultiplier are widely used in the laboratory investigations.

As the heart of feedback control for mitigation of thermoacoustic instabilities, satisfactory design of controller is not easy to achieve in practical combustors. First, the controller must be robust to high background noise levels from turbulent flow. In addition, more than one unstable modes may occur, thus requiring the controller to eliminate several unstable modes simultaneously.

A common practice to active control is a straightforward gain/phase shift controller. The measured signal is filtered, delayed, amplified, and fed into the actuation. Generally, the optimum control parameters are obtained by an offline testing, such as actuation frequency sweep tests and phase sweep tests. Besides the enormous testing work, this controller can only work well in the combustor with one single dominant unstable oscillation, and its effectiveness drops dramatically, if multiple-frequency oscillations occur.

Another approach to feedback control is adaptive controllers. It is designed in a way that the parameters of the controller are continuously updated in response to operating conditions change. ‘Self-calibrating’ can relax *a priori* knowledge of the characteristics of unstable combustors, and the real-time self-adjusting makes adaptive control effective over a wide range of operating conditions. There are mainly two types of adaptive controllers. One treats the combustion system as a ‘black box’. Control parameters are determined by online system identification [92], such as Least

Mean Squares (LMS) controller [93–96] with a form of Infinite Impulse Response (IIR) filters, or a real-time observer [97, 98]. For the first controller, the parameters of IIR filters are regulated by the LMS algorithm. A real-time observer is generally used to obtain the frequencies, amplitudes, and phases of thermoacoustic oscillations for controller design. The phase-lag for the controller can be determined by optimum schemes, such as Rayleigh’s criterion [99] and extremum-seeking algorithm [100]. The other efficient adaptive controller is a self-tuning regulator [101–103]. Some knowledge of the combustion system is given by a model. The control parameters were developed to follow changes in operating condition by some stability schemes, such as a Lyapunov’s direct method [104], a Lyapunov-Krasvoskii adaptation criterion [105], and a particular adaptation scheme in Ref. [106]. The main limitation for adaptive controller is the lack of guarantee of convergence.

Different from the preceding controllers, implementing a state-feedback controller lays more emphasis on the theoretical analysis of active instability control of thermoacoustic oscillations. It has large potential in application, if accurate system modelling and robust control strategies are obtained. State-feedback control was first proposed by Yang [107]. The external acoustic perturbations produced by several point actuators were incorporated to a generalized wave equation. By using the modal expansion and spatial averaging, the controlled combustion system was represented by a series of ordinary differential equations. Based on pole placement argument, a simple linear controller was designed, in which the system states were estimated by a Luenberger observer from the sensor measurement. Several important factors are related to controller design, including the locations of sensors and actuators, controllability, and observability, were investigated. Instead of employing acoustic perturbations, Fung

et al. [108] employed a secondary fuel to actuate the systems, and these actuators were distributed within the combustor. For simplicity, a proportional-plus-integral (PI) controller was designed. It was found that only a small fraction of energy related with oscillatory flow was needed to control thermoacoustic instabilities. Other examples of state-feedback controller for suppressing thermoacoustic instabilities include the linear quadratic Gaussian-loop transfer recovery (LQG-LTR) and H_∞ controller by Annaswamy [109] and Hong [110], linear parameter varying (LPV)- L_2 gain robust controller [111] by Hong, and active compensation based controller by Wei [112].

In the present work, two different nonlinear state-feedback controllers are proposed: Lyapunov-function-based controller (LFC) and sliding mode controller (SMC). The transient growth has been seen to occur not only in open-loop thermoacoustic systems [61, 68], but also in controlled thermoacoustic systems [70, 113]. In order to achieve the suppression of thermoacoustic oscillations and strict dissipativity simultaneously, the transient growth controller LFC is proposed in Sec. 4.2. As for SMC, it has been successfully used to guide the unmanned underwater vehicles and perform several combined tasks [114]. SMC pertains to one of the variable structure control approaches, and offers an effective and robust means of controlling nonlinear plants. Considering that the combustion system involves many complicated processes, including flow perturbations and chemical reactions, the modeling of the thermoacoustic system is certainly challenging, thus raising a high requirement for the robustness of the controller to modelling uncertainty and disturbances in the operation condition. Therefore, the potential of applying SMC on stabilizing the unstable thermoacoustic oscillations with multiple unstable modes is evaluated in Sec. 4.3.

Chapter 3

Nonlinear Dynamic Analysis of Rijke-Type Thermoacoustic Systems

In this chapter, the nonlinear dynamic behaviors of Rijke type thermoacoustic systems are theoretically and experimentally studied. In Sec. 3.1, a theoretical model of a 1D Rijke type thermoacoustic system with a premixed flame is developed. Based on the obtained governing equation for acoustic waves, Sec. 3.2 applies the method of multiple scales and continuation method to study the process of transition to instability in the absence of noisy perturbation. In Sec. 3.3, with the noisy influence considered, the stochastic transitions of the thermoacoustic system are studied near the bifurcation point. In addition, the noise-involved coherence responses of thermoacoustic systems are also investigated.

3.1 Modelling of Rijke-Type Thermoacoustic Systems

Figure 3.1: (a) Schematic of the numerically-modeled thermoacoustic combustor. $\tilde{x}_f$ denotes the location of the premixed flame, and L represents the total length of the tube; (b) the Rijke tube with the length of 750 mm and inner diameter of 49 mm.

The configuration of the Rijke-type thermoacoustic system with a premixed flame is shown in Fig. 3.1. The 1D momentum and energy equations for acoustic perturbations are given as [46]

$$\bar{\rho} \frac{\partial \tilde{u}'}{\partial \tilde{t}} + \frac{\partial \tilde{p}'}{\partial \tilde{x}} = 0 \quad (3.1a)$$

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} + \gamma \bar{p} \frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1) \tilde{Q}'(\tilde{x}, \tilde{t}) \delta(\tilde{x} - \tilde{x}_f) \quad (3.1b)$$

where overbar denotes a mean value, and the prime describes a perturbation. $\tilde{x}$ denotes the axial distance from the upstream end. ρ , p , u , and γ denote the density,

pressure, velocity, and the specific heat ratio respectively. $\delta(\cdot)$ is the Dirac delta function, and $\tilde{Q}'$ denotes unsteady heat release rate from the heat source to incoming air flow. Due to the low Mach number we considered ($\bar{M} < 0.01$), the effects of the mean flow in the tube are negligible. In order to generalize our analysis, Eqs. (3.1a) and (3.1b) are rescaled to a nondimensional form by introducing the reference scales as follows:

$$\tilde{u}' = \bar{u}u', \quad \tilde{p}' = \bar{p}p', \quad \tilde{Q}' = \bar{Q}Q', \quad \tilde{x} = Lx, \quad \tilde{t} = (L/\bar{c})t, \quad \delta(\tilde{x} - \tilde{x}_f) = \delta(x - x_f)/L, \quad (3.2)$$

where $\bar{c}$ denotes speed of sound. After nondimensionalizing Eqs. (3.1a) and (3.1b), then differentiating Eq. (3.1a) in space x and Eq. (3.1b) in time t , and finally eliminating their cross term lead to

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = (\gamma - 1)K \frac{\partial Q'}{\partial t} \delta(x - x_f), \quad (3.3)$$

where $K \equiv \bar{Q}/\bar{p}\bar{c}$, and it is treated as non-dimensional heat power. Galerkin series is applied to expand the acoustic disturbances here, so that the behaviors of the thermoacoustic system can be studied by using the dynamic theory. For convenience, it is assumed that the unsteady pressure oscillation is dominated by a single mode, and this assumption is supported by several experimental [3, 69] and theoretical investigations [56, 82, 115]. Accordingly, the acoustic pressure can be represented as $p'(x, t) = \psi(kx)\eta(t)$, where $\eta(t)$ denotes the time-dependent variable, $\psi(kx)$ is an acoustic mode function, and its specific form is determined by the boundary condition: $k = \omega/\bar{c}$ and the acoustic frequency ω . Substituting the Galerkin expansion of

pressure perturbation into Eq. (3.3) gives

$$(\ddot{\eta} + k^2\eta)\psi(kx) = K(\gamma - 1)\frac{\partial Q'}{\partial t}\delta(x - x_f), \quad (3.4)$$

where overdot denotes time derivative. The basis function $\psi(kx)$ satisfies the orthogonality condition, i.e.,

$$\int_0^1 \psi(kx)\psi(mx)dx = \begin{cases} 0, & k \neq m, \\ 1/2, & k = m. \end{cases}$$

Multiplying Eq. (3.4) with $\psi(mx)$ and integrating the term from $x = 0$ to $x = 1$ thus gives

$$\ddot{\eta} + k^2\eta = 2K(\gamma - 1)\dot{Q}'(x_f, t)\psi(kx_f). \quad (3.5)$$

Due to heat conduction and viscosity in the acoustic boundary layer and sound radiation at both ends of the tube, the acoustic loss/damping effect may become significant, and Eq. (3.5) has to be revised to [3]

$$\ddot{\eta} + 2\xi k\dot{\eta} + k^2\eta = 2K(\gamma - 1)\dot{Q}'(x_f, t)\psi(kx_f) \quad (3.6)$$

where ξ denotes the overall damping coefficient. Since only a small amount of heat release is used to generate acoustic oscillations, and the damping effect is limited, a small parameter ϵ is introduced into the two forcing terms, leading to

$$\ddot{\eta} + 2\epsilon\xi k\dot{\eta} + k^2\eta = 2\epsilon K(\gamma - 1)\dot{Q}'(x_f, t)\psi(kx_f) \quad (3.7)$$

After the preceding derivations, the governing equations for acoustic waves are simplified to a second-order differential equation, and it provides a basis for the further investigation in the next two sections.

3.2 Deterministic Bifurcation Study of Rijke-Type Thermoacoustic Systems

3.2.1 Overview

In this section, two different models are proposed to characterize the response of premixed flame to acoustic waves: model 1 is formulated with 3rd order polynomial equation, while model 2 is 5th order one. The method of multiple scales is proposed to extract the essential dynamic behaviors from the obtained ordinary differential equation. Transition to instability of the thermoacoustic system is evaluated by continuation methods, as one of the main system parameters is varied, including the heating power, the axial location of heat source, and the acoustic damping/loss. The roles of the heat source on the stability behaviors of thermoacoustic systems are studied.

3.2.2 Modelling of Unsteady Heat Release and Bifurcation Analysis

Due to the fact that the unsteady heat release is the main driving source to produce acoustic pressure [46,53], the relation between the unsteady heat release and acoustic pressure is significant for characterizing the dynamic behaviors of thermoacoustic

3.2. DETERMINISTIC BIFURCATION STUDY OF RIJKE-TYPE THERMOACOUSTIC SYSTEMS

system. Based on Merk's work [116], the flame-acoustic coupling can be described in frequency domain, as shown as

$$\hat{Q}' = Tr_{q,u'}\hat{u}' + Tr_{q,p}\hat{p}' \quad (3.8)$$

where Tr denotes the transfer function. $\hat{p}'$ and $\hat{u}'$ represent the Fourier transform of acoustic pressure and velocity. Replacing $\hat{u}'$ by $\hat{p}'/Z_b(\omega)$, where $Z_b(\omega)$ is the burner impedance, one has

$$\hat{Q}' = \left(\frac{Tr_{q,u'}}{Z_b(\omega)} + Tr_{q,p} \right) \hat{p}' = G(\omega)\hat{p}' \quad (3.9)$$

Therefore, the unsteady heat release can be a function of acoustic pressure [115, 117]. In Refs. [77] and [115], the unsteady heat release rate of a premixed flame is characterized by a third-order polynomial, as given as

$$\text{Model 1 :} \quad |\tilde{Q}'/\bar{Q}| = a_1\mathcal{A} + a_2\mathcal{A}^3 \quad (3.10)$$

where $\mathcal{A} \equiv |\tilde{p}'/\bar{p}|$. The coefficients a_1 and a_2 are two non-zero constants. Considering the fact that large-amplitude acoustic forcing would enhance the nonlinearity of the flame response, it is reasonable to increase the order of the unsteady heat release model, as shown as

$$\text{Model 2 :} \quad |\tilde{Q}'/\bar{Q}| = b_1\mathcal{A} + b_2\mathcal{A}^3 + b_3\mathcal{A}^5 \quad (3.11)$$

where the coefficients b_1 , b_2 , and b_3 are non-zero constants. The second, the forth and the sixth order terms are omitted, due to the symmetry condition [118]. The models proposed here characterize these experimental results [119–122] in a qualitative way,

i.e., the unsteady heat release rate increases linearly at small-amplitude forcing, and then gradually ‘saturates’ at high-amplitude forcing [86, 123, 124], as shown in Fig. 3.2. For completeness and comparison, both models are studied in this work.

Figure 3.2: Saturation of the normalized heat release rate fluctuation $|\tilde{Q}'/\bar{Q}|$. (a) Model 1: $a_1=2$, $a_2=-2.2$; (b) Model 2: $b_1=1.2$, $b_2=1$, $b_3=-3$

Following Sec. 3.1, the governing equation for acoustic waves has been simplified as follows:

$$\ddot{\eta} + 2\epsilon\xi k\dot{\eta} + k^2\eta = 2\epsilon K(\gamma - 1)\dot{Q}'(x_f, t)\psi(kx_f) \quad (3.7)$$

Substituting the unsteady heat release rate model Eq. (3.11) into Eq. (3.7) leads to

$$\ddot{\eta} + 2\epsilon\dot{\eta}(\alpha_1 + \alpha_2\eta^2 + \alpha_3\eta^4) + k^2\eta = 0, \quad (3.12)$$

where $\alpha_1 = \xi k - b_1 K(\gamma - 1)\psi^2(kx_f)$, $\alpha_2 = -3b_2 K(\gamma - 1)\psi^4(kx_f)$, $\alpha_3 = -5b_3 K(\gamma - 1)\psi^6(kx_f)$. To extract main features of the dynamic responses in Eq. (3.12), the

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method of multiple scales is applied to expand η in the form of

$$\eta(t, \epsilon) = \eta_0(t_0, t_1, t_2) + \epsilon \eta_1(t_0, t_1, t_2) + \epsilon^2 \eta_2(t_0, t_1, t_2) + \dots$$

where $t_0 = t$, $t_1 = \epsilon t_0$, and $t_2 = \epsilon^2 t_0$. t_0 is a fast scale time, while t_1 and t_2 are slower scales associated with modulations in the amplitude and phase of the oscillations. It can be shown that the time derivative $d\{\cdot\}/dt$ in terms of t_j can be written as

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots \quad (3.13a)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 (2D_0 D_2 + D_1^2) + \dots \quad (3.13b)$$

where $D_j = \partial\{\cdot\}/\partial t_j$. Substituting Eqs. (3.13a) and (3.13b) into Eq. (3.12) leads to

$$D_0^2 \eta_0 + k^2 \eta_0 = 0; \quad (3.14a)$$

$$D_0^2 \eta_1 + k^2 \eta_1 = -2(\alpha_1 + \alpha_2 \eta_0^2 + \alpha_3 \eta_0^4) D_0 \eta_0 - 2 D_0 D_1 \eta_0 \quad (3.14b)$$

$$\begin{aligned} D_0^2 \eta_2 + k^2 \eta_2 = & -2(\alpha_1 + \alpha_2 \eta_0^2 + \alpha_3 \eta_0^4)(D_0 \eta_1 + D_1 \eta_0) - 2 D_0 D_2 \eta_0 - D_1^2 \eta_0 \\ & - 4(\alpha_2 \eta_0 \eta_1 + 2\alpha_3 \eta_0^3 \eta_1) D_0 \eta_0 - 2 D_0 D_1 \eta_1 \end{aligned} \quad (3.14c)$$

The solution to Eq. (3.14a) can be written as

$$\eta_0 = C(t_1, t_2) e^{ik t_0} + \bar{C}(t_1, t_2) e^{-ik t_0} \quad (3.15)$$

where $\bar{C}(t_1, t_2)$ is complex conjugate of $C(t_1, t_2)$. Substituting Eq. (3.15) into Eq. (3.14b) and eliminating the secular terms as well as the zeros divisors enable η_1 to be determined. By repeating the same procedure to Eq. (3.14c) and introducing a polar

coordinate transformation as

$$C = \frac{1}{2}A(t) e^{i\varphi(t)} \quad (3.16)$$

Eq. (3.12) can be rewritten as

$$\frac{dA}{dt} = -\epsilon \left(\alpha_1 A + \frac{\alpha_2}{4} A^3 + \frac{\alpha_3 A^5}{8} \right) + \epsilon^2 f_1(A, \alpha_1, \alpha_2, \alpha_3) \quad (3.17a)$$

$$\frac{d\varphi}{dt} = \epsilon^2 f_2(A, \alpha_1, \alpha_2, \alpha_3) \quad (3.17b)$$

where f_1 and f_2 are two functions with small limited upper bounds. Here only the terms with accuracy of $O(\epsilon)$ are considered. If higher-order accuracy is needed, the terms with accuracy of $O(\epsilon^2)$ should be added. The steady states of thermoacoustic system as described by using Eqs. (3.17a) and (3.17b) can be obtained by setting $\dot{A} = 0$, $\dot{\varphi} = 0$. Because Eq. (3.17a) is independent of φ , it can be solved alone. To track the Hopf bifurcation diagrams, continuation method [125,126] is applied in this work.

3.2.3 Results and Discussion

The nonlinear dynamic behaviors of the Rijke-type thermoacoustic system are discussed. First, the model of the unsteady heat release rate with the 5th order polynomial is used. Fig. 3.3(a) shows the calculated bifurcation diagram in term of the normalized heat power K by integrating Eq. (3.17a) with time. It should be noted that, in the bifurcation diagram, a point with zero amplitude denotes a steady state, while a point with nonzero amplitude denotes a periodic oscillation. From Fig. 3.3(a), it can be seen that when the normalized heat release K is small, there is a stable fixed point with zero amplitude. As K is increased to a critical value (Hopf

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Figure 3.3: (a) The bifurcation diagram as a function of the normalized heat power K , as $x_f = 1/4$, $\bar{c}=400$ m/s, $\xi = 0.24$, $\epsilon = 0.1$, Model 2: $b_1 = 1.2$, $b_2=1$, $b_3=-3$; (b) The experiment measurement [3].

bifurcation point), the steady state jumps into large-amplitude oscillations abruptly, not smoothly. Further increasing heat power K leads to a monotonic growth of the oscillation amplitude. However, as K is decreased, the amplitude of system's response A decreases continuously first and then jumps back to the steady state at the fold bifurcation point. In summary, both Hopf and fold bifurcations are observed, and the hysteresis zone is unambiguously distinguished. The similar behavior is experimentally measured as shown in Fig. 3.3(b), when the unsteady heat release is provided by electric heated wires. Comparing the predicted bifurcation diagram with the measured one reveals good qualitative agreement in the main structure of system's response, when the thermoacoustic system is driven by different types of heat source.

In order to verify the analytical solution as shown in Fig. 3.3(a), the governing Eq. (3.12) is integrated directly, as the normalized heating power K is set to 3 different

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values. In Fig. 3.3(a), it shows that the amplitude of system's response equals zero

Figure 3.4: (a) Phase plane portraits of the steady-state solution by Eq. (3.12); (b)-(d) The thermoacoustic system's transient response at different heat power. $x_f = 1/4$, $\bar{c}=400$ m/s, $\xi = 0.24$, $\epsilon = 0.1$, Model 2: $b_1 = 1.2$, $b_2=1$, $b_3=-3$.

when $K = 2.0$, i.e., $A = 0$. Under the same operating condition, a fixed point is observed in the phase diagram of the thermoacoustic system's steady response (Fig. 3.4(a)), and it indicates a zero amplitude. In addition, the thermoacoustic system's transient response is given in Fig. 3.4(b), and it shows that acoustic disturbances present in the system decays quickly to zero with a given initial condition. Therefore, in the globally stable region, the analytical result can describe the thermoacoustic system's behavior well.

As K is increased to 2.59, the thermoacoustic system is configured in the hysteresis region in Fig. 3.3(a), where the thermoacoustic system is linearly stable but

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nonlinearly unstable. In Fig. 3.4(a), a red circle (for unstable oscillations) and a blue circle (for stable oscillations) with different amplitudes are observed in phase plane, when $K = 2.59$. The corresponding time evolution of the two circles are shown in Fig. 3.4(c), where two periodic oscillations are observed. Furthermore, the amplitudes of these oscillations agree well with the predicted values in Fig. 3.3(a). Therefore, in the hysteresis region, the predicted results can accurately describe the thermoacoustic system's behavior. It is noteworthy that the red circular phase diagram represents an unstable periodic oscillation. The unstable oscillation generally serves as a boundary between the basins of the two stable solutions (one is stable steady state, and the other is a stable periodic one denoted by the blue circle). In addition, this type of oscillations cannot be observed in the practical experiment, because a little bit small perturbation can drive the system away from the unstable periodic trajectory, and finally into either the stable periodic oscillation or stable steady state.

Further increasing the heat power K to 2.75 leads to only one stable solution with $A = 0.95$, as shown in Fig. 3.3(a). The green circular shaped phase diagram as shown in Fig. 3.4(a) indicates the periodic solution. And the corresponding time-domain signal is illustrated in Fig. 3.4(d). Again, the amplitude of acoustic oscillations predicted by the method of multiple scales (denoted by the dash line) agrees well with the green numerical results.

After the verification of the analytical prediction in term of the normalized heat power K , the effects of the axial location of heat source x_f , and the acoustic damping/loss ξ on the nonlinear dynamic behaviors of the thermoacoustic system are also examined. The numerical results are summarized in Fig. 3.5. It can be seen from Fig. 3.5(a) that the nonlinear thermoacoustic oscillation is difficult to trigger, when

Figure 3.5: (a) The bifurcation diagram as a function of (a) the flame location x_f , as $\xi=0.24$, (b) the acoustic damping ξ , as $x_f = 1/4$, respectively. $\epsilon = 0.1$, $K=2.75$, Model 2: $b_1 = 1.2$, $b_2=1$, $b_3=-3$.

the heat source is placed close to the upstream open end, i.e. $x_f \rightarrow 0$. However, as $x_f = 1/4$, the loudest sound occurs. These predicted behaviors agree well with the experimental measurement [127]. Fig. 3.5(b) examines the effect of the acoustic damping ξ , as $x_f = 1/4$. Generally, when the heat addition to the acoustic waves is greater than the acoustic loss ξ , the acoustic energy is increased. Therefore, if the heat addition is fixed, a decrease in the acoustic loss can lead to an increase in the acoustic energy. In addition, it has been known that large damping can stabilize the unstable system, so no acoustic oscillations will be present. It can be observed that Fig. 3.5(b) captures the two characteristic behaviors very well. Furthermore, the hysteresis region can be identified both in Fig. 3.5(a) and in 3.5(b). Therefore, the variation of the heater location x_f and acoustic damping ξ can lead to the thermoacoustic system being unstable via subcritical bifurcation.

Foregoing analysis is based on the unsteady heat model with higher nonlinearity,

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Figure 3.6: (a) The bifurcation diagram as a function of the normalized heating power K , as $x_f = 1/4$, $\bar{c}=400$ m/s, $\epsilon = 0.1$, $\xi = 0.24$, Model 1: $a_1=2$, $a_2=-2.2$; (b) The bifurcation diagram with respect to heating power in the experiment measurement [3]

Figure 3.7: (a) The bifurcation diagram as a function of heater location x_f , as $\epsilon = 0.1$, $\xi = 0.24$, $\bar{c}=400$ m/s, Model 1: $a_1=2$, $a_2=-2.2$; (b) The bifurcation diagram as a function of acoustic damping ξ , as $x_f = 1/4$, $\epsilon = 0.1$, $K=2.75$, $\bar{c}=400$ m/s, Model 1: $a_1=2$, $a_2=-2.2$.

and subcritical bifurcation is clearly observed, as did in the experiment. When the lower-order model 1 is used, a supercritical bifurcation in the thermoacoustic system is observed in Fig. 3.6(a), as the heating power K is varied. When K is greater than a certain value ($K_c = 2.0$), a tunable sound is generated. The transition to instability is ‘soft’, and the hysteresis zone is indiscernible. The similar supercritical behaviors are observed in the experimental measurement as shown in Fig. 3.6(b).

In addition to the heating power K , the effects of the axial location of the heat source x_f and the overall acoustic damping ξ on the dynamic behaviors of the thermoacoustic system are studied and illustrated in Fig. 3.7. Supercritical bifurcations are also clearly observed with the variation of heater location or the acoustic damping.

3.2.4 Experimental Investigation

In order to validate our theoretical findings, a simple experimental investigation on a lab-scale Rijke tube with premixed flame confined is conducted in China. The experimental setup is shown in Fig. 3.1(b). The Rijke tube is made of quartz glass. Its both ends are acoustically open (i.e. pressure nodes). The overall length of the tube is 75 cm with a measurement uncertainty of ± 0.5 mm. The inner diameter of the tube is 4.9 cm, and it is measured by vernier caliper. A Bunsen burner is placed at the bottom half of the tube to anchor an acoustically compact flame, as shown in Fig. 3.8(a). Methane is mixed with air before combustion. The temperature contour is measured by using an infrared thermal imaging camera as shown in Fig. 3.8(b). The ‘hottest’ zone is corresponding to the heat source. In this experiment, the pressure transducer is used to measure the acoustic pressure inside the tube. The hot environment requires the pressure transducers to be capable of working properly during long experimental runs. To alleviate the hot temperature flow, the pressure transducer is connected to the side branch along the combustor. To reduce the background noise effects, the pressure transducer is placed at $x_s/L=0.5$, closer to the first mode pressure antinode. The flow rate of Methane is controlled and measured by a digital flow meter, i.e., MC Standard Series Mass Flow Controller with a measurement uncertainty of $\pm 0.8\%$ of reading. In order to provide accurate measurements of the acoustic pressures, a fast sampling system is required. The data acquisition system NI USB-6343 is used with a sampling rate of 5 kHz.

There are two basic parameters that can be mechanically controlled: the heater location and fuel flow rate. Due to the grid mesh on the exit of the barrel for holding

Figure 3.8: (a) A premixed flame of Methane; (b) Measured temperature contour by using an infrared thermal image camera. $L=75$ cm,

the flame, the thickness of the premixed flame and the variation of the flame thickness is rather smaller than the dominant wavelength of sound. Therefore, the specific shape of the flame is not considered. Here the location of the premixed flame refers to the flame holder location. Before commencing an experimental run, the tube should undergo a warm-up procedure. It should be noted that the data can only be collected after the system reaches the steady state, especially for the variation in the fuel flow rate, where the temperature field in the tube responds slowly. The measured time series of the self-excited oscillation in the Rijke tube are given in Fig. 3.9(a) and (c), as the flame is placed at 2 different locations. Clearly, the pressure perturbations are significantly different. The stable response in Fig. 3.9(a) demonstrates a noisy character, and large-amplitude oscillations with little variations from cycle to cycle are observed in Fig. 3.9(c). In order to understand the nature of the thermoacoustic oscillations, the 3D phase diagrams of the measured pressure signal are plotted as shown in Fig. 3.9 (b) and (d). 5000 cycles of experimental pressure data are reconstructed to characterize the attractors, when the thermoacoustic system reaches the

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Figure 3.9: (a) and (c) Measured thermoacoustic oscillations from the Rijke tube; (b) and (d) Phase portrait reconstructed from pressure time series. For (a) and (b), $x_f=0.06$; For (c) and (d) $x_f=0.25$. The time delay chosen for reconstructing the phase space is $\tau=0.353$ period, and 254 cycles of the steady times series are present.

steady state. The minimal embedding dimension for the phase space is estimated by using the false nearest neighbour method, and the optimal time delay τ is calculated by the method of average mutual information [128]. By comparing Fig. 3.9 (b) and (d), an isolated closed orbit can be observed, when limit-cycle oscillations occur in the thermoacoustic system.

The dependence of the system response on the fuel flow rate and the flame location x_f are experimentally evaluated. Fig. 3.10 shows the bifurcation diagram of the Rijke-type thermoacoustic system in terms of the amplitude of pressure oscillations A

and ϕ_f . It can be seen that the tunable sound is suddenly generated at the fuel flow rate 2.8 l/min. Further increase of the fuel flow rate ϕ_f leads to a small increase of acoustic amplitude. As ϕ_f is decreased from 3 l/min to 2.2 l/min, the intensity of the tone decreases gradually first and then drops abruptly to the silence. The hysteresis is clearly observed, as ϕ_f is within 2.4 l/min and 2.8 l/min. Two representative fuel flow rates are selected to evaluate the effects of flame location x_f : one is in the hysteresis region ($\phi_f = 2.5$ l/min), and the other is out of the hysteresis region ($\phi_f = 3.0$ l/min). In Fig. 3.11(a), the presence of a gap between the two critical locations may

Figure 3.10: The bifurcation diagram as a function of fuel flow rate as $x_f=1/4$.

imply the hysteresis. Fig. 3.11(b) shows that the intensity of acoustic disturbances is increased, and then decreased. No hysteresis is discernible. Therefore, a supercritical Hopf bifurcation occurs. Comparing Fig. 3.7 with Fig. 3.11(b) reveals that there is a difference in the critical bifurcation point between the theoretical prediction and the experimental measurement. The most likely reason for this discrepancy is a noisy environment in the vicinity of the heater inside the tube. Such perturbations may

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initiate early transitions between stable and excited states. Meanwhile, the higher temperature gradients in the experimental system also account from this discrepancy. It is worth noticing that the analytical prediction from model 1 can only be used to study the nonlinear behaviors of supercritical thermoacoustic systems. Using the low-order model to predict the subcritical thermoacoustic systems may lead to severe error.

Figure 3.11: Measured bifurcation diagram as a function of the axial location x_f of the flame, as (a) $\phi_f = 2.5$ l/min, and (b) $\phi_f = 3$ l/min.

3.2.5 Summary

In this work, the effects of the nonlinearity of the flame's response on the process of transition to instability is investigated via bifurcation analysis in Rijke-type thermoacoustic systems. The unsteady heat release rate from an acoustically compact

premixed flame is modeled by 2 different formulations. One is a 3rd order polynomial equation. The other is a 5th order one. It is found that the nonlinearity of the flame response determines the process of thermoacoustic system's transition, as any one of these three main parameters (the heating power, the heater axial location and the overall acoustic damping/loss) is varied. The flame with low-order nonlinearity generates large-amplitude thermoacoustic oscillations through a supercritical Hopf bifurcation, while subcritical bifurcation occurs in the thermoacoustic system with high-order nonlinear premixed flame. Experimental investigations on a Rijke tube with premixed flame are also conducted. Both subcritical and supercritical bifurcation are observed.

3.3 Stochastic Bifurcation and Coherence Resonance Studies of a Rijke-Type Thermoacoustic System

3.3.1 Overview

In this section, the effects of external noisy flow fluctuations on the stability boundaries of a Rijke type thermoacoustic system are studied first, and then noise-induced behaviors are investigated near subcritical and supercritical bifurcation points. As for the stochastic dynamic system, there are mainly two different ways [129] to characterize the instability: one is denoted by the sign change of the maximum Lyapunov exponent, while the other is by a qualitative change in the probability density function. In this work, the random system response is characterized by the probability density function.

3.3.2 Modelling of a Stochastic Thermoacoustic System

In this section, the 5th order polynomial equation is used to characterize the response of the premixed flame to acoustic pressure, as shown as

$$Q' = b_1 p' + b_2 p'^3 + b_3 p'^5, \quad (3.18)$$

The governing equation for acoustic pressure in the Rijke-type thermoacoustic system has been simplified in Sec. 3.1, as given as

$$\ddot{\eta} + 2\epsilon\xi k\dot{\eta} + k^2\eta = 2\epsilon K(\gamma - 1)\dot{Q}'(x_f, t)\psi(kx_f) \quad (3.7)$$

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Substituting Eq. (3.18) into the preceding governing equation leads to

$$\ddot{\eta} + 2\epsilon\dot{\eta}(\alpha_1 + \alpha_2\eta^2 + \alpha_3\eta^4) + k^2\eta = 0, \quad (3.19)$$

where $\alpha_1 = \xi k - b_1 K(\gamma - 1)\psi^2(kx_f)$, $\alpha_2 = -3b_2 K(\gamma - 1)\psi^4(kx_f)$, $\alpha_3 = -5b_3 K(\gamma - 1)\psi^6(kx_f)$. Based on the work in Ref. [130], the effect of the flow perturbation $n_w(t)$ can be incorporated into the governing equation as follows:

$$\ddot{\eta} + 2\epsilon\dot{\eta}(\alpha_1 + \alpha_2\eta^2 + \alpha_3\eta^4) + k^2\eta = n_w(t), \quad (3.20)$$

where $n_w(t)$ is a white Gaussian noise: $\langle n_w(t) \rangle = 0$, $\langle n_w(t)n_w(t + \tau) \rangle = D\delta(\tau)$. $\langle \cdot \rangle$ is the expectation operator, and D is the noise intensity. In order to approximate the nonlinear stochastic response to be a Markov process by stochastic averaging, a pair of slowly varying process is needed. For this, the following transformations are implemented:

$$\eta(t) = A(t) \cos \theta, \quad \dot{\eta}(t) = -kA(t) \sin \theta, \quad \theta = kt + \varphi(t), \quad (3.21)$$

where $A(t)$ denotes the acoustic wave amplitude, and $\varphi(t)$ denotes the phase. Eq. (3.20) can be cast into the following pair of exact equations in term of A and φ :

$$\dot{A} = -2\epsilon A \sin^2 \theta (\alpha_1 + \alpha_2 A^2 \cos^2 \theta + \alpha_3 A^4 \cos^4 \theta) - \frac{\sin \theta}{k} n_w(t), \quad (3.22a)$$

$$\dot{\varphi} = -2\epsilon \sin \theta \cos \theta (\alpha_1 + \alpha_2 A^2 \cos^2 \theta + \alpha_3 A^4 \cos^4 \theta) - \frac{\cos \theta}{Ak} n_w(t). \quad (3.22b)$$

Due to the fact that ϵ is small, (A, φ) can be approximated as a joint Markov process [131]. By using stochastic averaging [132], the corresponding Itô equations [133] can

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be obtained as

$$dA = - \left(\epsilon\alpha_1 A + \frac{\epsilon\alpha_2 A^3}{4} + \frac{\epsilon\alpha_3 A^5}{8} \right) dt + \sqrt{\frac{D}{2k^2}} dW_1, \quad (3.23a)$$

$$d\varphi = \sqrt{\frac{D}{2k^2 A^2}} dW_2, \quad (3.23b)$$

where dW_1 and dW_2 denote two independent Wiener process (Brownian motion) [134]. It should be noted that the amplitude $A(t)$ does not depend on φ in Eq. (3.23a), so it will be convenient to develop a 1D probability transition function for A , rather than a joint function for A and φ , when $D \neq 0$. The probability transition function $P(A, t)$ of the 1D Markov process A is governed by the following Fokker-Planck-Kolmogorov (FPK) equation [133]:

$$\frac{\partial P(A, t)}{\partial t} = \frac{\partial}{\partial A} \left\{ \left(\epsilon\alpha_1 A + \frac{\epsilon\alpha_2 A^3}{4} + \frac{\epsilon\alpha_3 A^5}{8} - \frac{D}{4k^2 A} \right) P(A, t) \right\} + \frac{D}{4k^2} \frac{\partial^2 P(A, t)}{\partial A^2}. \quad (3.24)$$

The stationary density function $P_s(A)$ can be obtained by setting $\partial P(A, t)/\partial t = 0$, as shown as

$$P_s(A) = \mathcal{N} \exp \left\{ -\epsilon k^2 A^2 (\alpha_3 A^4 + 3\alpha_2 A^2 + 24\alpha_1) / 12D \right\} \quad (3.25)$$

where $\mathcal{N}$ is a normalization constant, so that $\int_{-\infty}^{\infty} P_s(A) dA = 1$. In addition, the peaks of $P_s(A)$ is determined by $dP_s(A)/dA = 0$, leading to

$$f(A_e) = \alpha_3 A_e^6 + 2\alpha_2 A_e^4 + 8\alpha_1 A_e^2 - \frac{2D}{\epsilon k^2} = 0, \quad (3.26)$$

where A_e is the amplitude corresponding to the extremum of the amplitude density distribution.

3.3.3 Description of Experiments

Figure 3.12: Structure of the experimental apparatus.

In order to validate the theoretical findings, experimental tests on a Rijke tube with loudspeakers actuated are performed in Nanyang Technological University. The structure of the experimental apparatus is shown in Fig. 3.12. The Rijke tube is made of silica glass. Both ends are acoustically open. The overall length of the tube is 75 cm and the inner diameter of the tube is 4.9 cm. This structure ensures that a longitudinal standing wave can be generated by a heat source at certain condition. A Bunsen burner with 3 cm diameter and 25 cm height is used to provide a premixed propane-fueled flame, which is enclosed in the bottom half of the tube. The burner is anchored on a workbench. The volumetric fuel flow rate of gas ϕ_f is measured by Dwyer

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Figure 3.13: (a) Brü & Kjær 4957 Microphone; (b) NI PXIe-1062Q chassis.

Instrument RMA series flow meter with uncertainty of $\pm 4\%$. The location of the flame from the bottom end is a bifurcation parameter, and it is varied with a resolution of 1 mm. The loudspeakers (Vintage Altec Lansing 414-16E, 12 inch) are connected to a power amplifier (Crown D-75A), and the white Gaussian noise is generated by using a Digital function generator (Sony Tektronix AFG320). The pressure transducer Brüel & Kjær 4957 is used to measure the acoustic pressure with the sensitivity of 11.2 mV/Pa, as shown in Fig. 3.13(a). It has specified and stable phase-matching over wide range of temperature and humidity. The frequency range is from 50 to 10000 Hz, which can cover the minimum frequency of thermoacoustic oscillations. To avoid the damage from the hot temperature flow, the pressure transducer is attached to the side branch of the tube at 35 cm from the inlet end. The measured signal is acquired with a sampling rate of 10 kHz, and it is logged by National Instruments (NI-PXIe1062Q chassis, PXI4498 sound module), as shown in Fig. 3.13(b).

3.3.4 Results and Discussion

Before embarking on the study of stochastic thermoacoustic systems, a deterministic bifurcation diagram is given in Fig. 3.14, as the heating power K is varied. It can

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be seen that in the absence of stochasticity, i.e., the noise intensity $D = 0$, there are a fold bifurcation and an Andronov-Hopf bifurcation [135]. Furthermore, the bi-stable behavior is clearly observed in the shaded region, where a stable steady state and a stable limit cycle coexist.

Figure 3.14: Calculated deterministic bifurcation diagram in the phase plane of (K, A) for the simplified 1D Rijke-type thermoacoustic system, as $b_1=1$, $b_2=1.2$, $b_3=-3$, $k = \pi$, $x_f = 1/4$, $\xi=0.24$, $\bar{c}=400$ m/s, $D=0$. s-s: steady state, l-c: limit cycle.

In order to describe the stochastic transition process more clearly, the variance of system response $\sigma^2(A)$, which is often used to measure the intensity of perturbation, is introduced in the stationary probability density distribution $P_s(A)$ by considering their product [136] as shown in Fig. 3.15(a). It can be seen that $P_s(A)\sigma^2(A)$ has only one local maximum in the vicinity of zero for small intensified noise (e.g., $D = 5 \times 10^{-3}$). In this case, the nonlinear effect can be neglected. When the noise intensity is increased (e.g., $D = 1.2 \times 10^{-2}$), bimodal distribution with two peaks can be clearly identified. A transition from unimodal to bimodal distributions occurs.

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Figure 3.15: (a) Stationary probability distribution $P_s(A)\sigma^2(A)$ varied with the amplitude A and the noise intensity D ; (b) Stationary probability distribution $P_s(A)$ with respect to three different noise intensity D , as $K=2.475$, $b_1=1$, $b_2=1.2$, $b_3=-3$, $k = \pi$, $x_f = 1/4$, $\xi=0.24$, $\bar{c}=400$ m/s.

This indicates that the thermoacoustic system turns to oscillate between two local regions. In this bimodal regime, the nonlinearity of the thermoacoustic system becomes increasingly important. Further increasing the noise intensity (e.g., $D = 0.03$) leads to the thermoacoustic system being ‘locked’ in a large-amplitude limit cycle, and at the same time the peak at the local small amplitude disappears. The amplitude distribution becomes unimodal again. It can be found that the stochastic P -bifurcation occurs twice, as the noise intensity is increased.

It should be noted that the averaged model (Eq. (3.23a)) cannot capture the stochastic properties of the original thermoacoustic system (Eq. (3.20)), if the intensity of the noisy perturbation is too large. For verifying the results from the averaged model, Eq. (3.20) is numerically integrated by modified Euler scheme [134], and the simulation results are illustrated by the black gray dot in Fig. 3.15(b). The time

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step of numerical simulation is 2^{-6} , and the number of iteration path is 300. It is observed from Fig. 3.15(a) that the dotted amplitude distributions track the profile of the surface very well. The more detailed overlap in Fig. 3.15(b) confirms that the averaged model is applicable to study stochastic properties of the thermoacoustic system in the interested noise range.

Figure 3.16: Stochastic bifurcation diagram in the phase plane of (K, D) for the simplified 1D Rijke-type thermoacoustic system, as $b_1=1$, $b_2=1.2$, $b_3=-3$, $k = \pi$, $x_f = 1/4$, $\bar{c}=400$ m/s.

Based on the preceding findings that the variation of the noise intensity leads to a significant change in the probability density distribution, the effects of the noise intensity D on the stability boundaries of the thermoacoustic system is investigated next. The stochastic bifurcation diagram of the noisy thermoacoustic system is shown in Fig. 3.16. It is obtained by evaluating the dependence of the root number of Eq. (3.26) on the heating power K and the noise intensity D . Curve l_1 denotes the critical boundary about the presence of the peak of $P_s(A)$ at a larger amplitude A . Below l_1 , the peak at a large amplitude A disappears. Curve l_2 corresponds to

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Figure 3.17: The simplified 1D Rijke-type thermoacoustic system with $b_1=1$, $b_2=1.2$, $b_3=-3$, $k = \pi$, $x_f = 1/4$, $\bar{c}=400$ m/s: (a) Power spectrum density of the normalized acoustic pressure with respect to $\bar{p}$ at 3 different noise intensities D , as $K = 2.475$; (b) Power spectrum density, as $K = 2.475$, $D = 0.01$; (c) A bell-shaped SNR , as $K = 2.475$; (d) The role of precursor by SNR , as the heating power K is set to 3 different values.

the critical curve for the other peak at the smaller amplitude A . Above l_2 , the peak at small amplitude A disappears. The shaded region in red represents the bimodal probability distribution, and the outside region denotes an uni-modal distribution. The main trends in Fig. 3.16 show a large similarity with the experimental measurement [86]. Apparently the transitions either from outside region to the shaded region or in the opposite direction indicate that stochastic P -bifurcation occurs. The vertical dashed lines stand for the bi-stable boundary for the deterministic system. It can be seen that the bimodal region for $D \neq 0$ and bi-stable region for $D=0$ do

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not overlap. Furthermore, the bimodal region shifts to a decreased heating power K , and becomes narrower with increased noise intensity D . Further increment of D (e.g., $D > 0.02$) leads to the stochastic P -bifurcation being unobserved for any given heating power K . Thus, the bimodal region does not appear anymore. This reveals that adding ‘moderate’ intensified noise would expand the unstable regime and lead to the thermoacoustic system being more prone to instability.

With previous physical findings obtained, the coherence motions induced by noise in the thermoacoustic system are studied. The power spectrum density of acoustic pressure (PSD) is applied to study the stochastic responses in frequency domain. It has been known [137, 138] that noise excitation of a self-sustained oscillator would lead to the well-known effect of washing the spectral line out [139], if the system is far from the critical bifurcation point. In the present work, the thermoacoustic system is set near the subcritical bifurcation point. Fig. 3.17(a) shows the power spectrum density of acoustic pressure at 3 different intensified noise, as $K=2.475$. A small-amplitude peak appears as $D=0$. If the noise intensity is increased further (i.e., $D=0.012$), the maximum power spectrum is increased, i.e., the amplitude of the peak is increased and the peak becomes more pronounced. This indicates that strong coherent motion is stimulated by the added white Gaussian noise. Further increasing noise to $D=1$ gives rise to the peak being so flattened (i.e., lower gradient) that it is difficult to distinguish the peak from the background noise. In these conditions, the thermoacoustic system is dominated by the noise only.

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Figure 3.18: (a)-(d) Numerical prediction for the simplified 1D Rijke tube, as $b_1 = 1$, $b_2 = 1.2$, $b_3 = -3$, $\bar{c} = 400$ m/s, $k = \pi$, $K = 2.75$: (a) the bifurcation diagram in term of the heat location x_f , as $D = 0$, s-s: steady state, l-c: limit cycle; (b)-(d) SNR , the frequency width $\Delta\omega$, and the peak height of power spectrum H with respect to D , as $x_f = 0.23$; (e)-(h) Experimental measurement for the lab-scale Rijke tube, as the fuel flow rate $\phi_f = 250$ ml/min: (e) the bifurcation diagram in term of x_f , as $D = 0$; (f)-(h) SNR , $\Delta\omega$, and H with respect to D , as $x_f = 0.06$. The fuel flow rate ϕ_f is fixed at 250 ml/min

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In order to quantify the regularity of the noise-induced system response, the signal-to-noise ratio (SNR) is introduced. It is defined as [140]

$$SNR = H\omega_p/\Delta\omega, \quad (3.27)$$

where H denotes the peak value of the power spectrum density, as illustrated in Fig. 3.17(b). ω_p is the corresponding peak frequency. $\Delta\omega$ is the width of the peak at the height $h_1 = e^{-1/2}H$. Thus, $\omega_p/\Delta\omega$ corresponds to the relative width of the peak, which is often used as a quality factor. For a pure sinusoidal signal, $SNR \rightarrow \infty$, and for a white noise signal, $SNR \rightarrow 0$. Fig. 3.17(c) shows SNR of the Rijke-type thermoacoustic system with respect to the noise intensity D . It can be seen that SNR is maximized, when the noise involves an ‘optimum’ intensity D . A resonance-like behavior can be clearly observed. This reveals that coherence resonance occurs. Fig. 3.17(d) shows the variation of SNR with respect to noise intensity, as the heating power is set to 3 different values. Comparing SNR at different heating powers reveals that the optimal noise intensity is decreased, and the maximum SNR is increased, as the heating power is approaching the critical bifurcation point $K = 2.495$. This trend shows a qualitative similarity with the experiment results [85] in a prototypical combustor.

In order to know more about the noise-induced motions of the Rijke-type system, the frequency width $\Delta\omega$ and the peak H are also investigated in Fig. 3.18. Meanwhile, experimental tests are conducted on a lab-scale Rijke tube to validate the numerical predictions. Because the numerical investigations are conducted near the subcritical bifurcation point, the lab-scale Rijke tube should be configured properly to ensure that the system loses its stability via subcritical bifurcation. The measured

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bifurcation diagram varied with x_f is shown in Fig. 3.18(e). It can be seen that a distinguished hysteresis zone is present. The hysteresis is also predicted by the model as shown in Fig. 3.18(a). It is worth noting that the experimental critical point for transition to instability do not overlap the modeled prediction. The inconsistency may be due to combustion noise around the heat source and higher temperature gradients in the experimental system. As the noise intensity is increased, a bell-shaped SNR is predicted in the modeled thermoacoustic system in Fig. 3.18(b), and such behavior is also observed in the experimental measurement in Fig. 3.18(f). Fig. 3.18(c) shows that the width of the power spectrum $\Delta\omega$ is decreased first and then increased, as the noise intensity is increased. And there is a minimum $\Delta\omega$. However, the height of the power spectrum H exhibits a distinguished peak with increased D as shown in Fig. 3.18(d). Similar characteristics are identified from our experiments as shown in Figs. 3.18(g) and (h). The qualitative agreements validate the numerical predictions for the noise-induced motions of the simplified Rijke-type thermoacoustic model.

With the subcritical transition of the modeled thermoacoustic system validated, the stochastic properties of a Rijke-type thermoacoustic system with supercritical bifurcation are then numerically studied. The bifurcation diagram of the modeled system with respect to the axial location of the heater source x_f is shown in Fig. 3.19(a). Fig. 3.19(b) shows that coherence resonance can also occur, when the transition to instability is supercritical. Different from the case with subcritical system, the width $\Delta\omega$ and the peak height H are found to increase monotonically with the increased noise intensity D , as illustrated in Fig. 3.19(c) and Fig. 3.19(d). In addition, the height H is found to ‘saturate’ at a higher intensified noise. By comparing Fig.

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3.18(a)-(d) with Fig. 3.19, it reveals that the noise-induced coherent motions in thermoacoustic systems may involve obviously different behaviors, depending on whether the system loses its stability via subcritical bifurcation or supercritical bifurcation.

Figure 3.19: Numerical predictions for the simplified 1D Rijke tube, as $b_1 = 2$, $b_2 = -2.2$, $b_3 = 0$, $\bar{c} = 400$ m/s, $k = \pi$, $K = 2.75$: (a) the deterministic bifurcation diagram, as the heater location x_f is varied. $D = 0$. s-s: steady state, l-c: limit cycle; (b)-(d) SNR , $\Delta\omega$, and H with respect to D , as $x_f = 0.15$.

3.3.5 Summary

In the present section, the effects of noisy fluctuations on the stability boundaries and nonlinear behaviors of 1D Rijke-type thermoacoustic systems are systematically investigated. It is found that the transition to instability of the thermoacoustic system near subcritical bifurcation point can be described well by stochastic P -bifurcation, as the external noise intensity is varied. The transitions from unimodal mode to the

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bimodal mode are clearly presented. If the noise intensity is chosen as a bifurcation parameter, the increased noise intensity is found to shift the hysteresis region and reduce the hysteresis area. Meanwhile, the noise-induced coherence motions are also examined in the system with subcritical bifurcation. A bell-shaped signal to noise ratio (SNR) is clearly observed, which indicates the presence of coherence resonance in the thermoacoustic system. In addition, SNR is found to become larger and the optimal noise intensity is decreased to a smaller value, when the system approaches the critical bifurcation point. This property can be used as a precursor to the Hopf bifurcation in thermoacoustic systems. To validate the theoretical findings, experimental tests are conducted on a Rijke tube with 3 loudspeakers implemented. Transition to instability is found to be subcritical. Comparing experimental results with theoretical predictions reveals qualitative agreements in terms of SNR , the peak height of power spectrum, and the width of the frequency. Furthermore, the characteristics of noise-induced motions are also examined in the modeled Rijke-type systems with a supercritical bifurcation. Coherence resonance is also observed, but its stochastic properties exhibit different behaviors when the system is subcritical.

Chapter 4

Feedback Control of Rijke-Type Thermoacoustic Instability

In this chapter, two different feedback controllers are designed, and their performances of stabilizing thermoacoustic instability are evaluated. As is known in Sec. 3.2, the Rijke-type thermoacoustic system loses its stability via subcritical or supercritical bifurcation. Compared with supercritical bifurcation, subcritical bifurcation poses more severe threat to steady operation of combustion system for abrupt jump in system's response. Therefore, a Rijke-type thermoacoustic system with subcritical bifurcation is considered here. Heated wires are chosen as the heat source, which can not only destabilize the thermoacoustic system via subcritical bifurcation [66], but also achieve convenient implementation for the future experimental work. In order to improve the performance of state-feedback controllers, a more accurate modelling scheme is developed.

4.1 Modelling of Actuated Rijke-Type Thermoacoustic Systems

Figure 4.1: Schematic of closed-loop thermoacoustic system with N_1 distributed actuators. L denotes the length of the tube. S and S_{an} denote the cross-sectional area of the duct and the n th actuator, respectively. $\bar{T}$, $\bar{\rho}$, and $\bar{c}$ denote the mean temperature, mean air density, and mean speed of sound, respectively. The subscripts l and r denote the flow variables in the upstream and downstream regions of the heat wires.

Following the previous works [61, 69], a simplified mathematical model of a 1D thermoacoustic system with non-uniform mean temperature and multiple monopole-like actuators is considered, as shown in Fig. 4.1. Electric heated wires are axially placed at the plane $x = x_f$, and work as an acoustically compact heat source.

Generally, the presence of the heat source causes the mean temperature $\bar{T}$, the mean flow $\bar{u}$, and gas density $\bar{\rho}$ undergoing a variation from the region $0 \leq x < x_f$ to $x_f < x \leq L$. This variation has been experimentally measured by Matveev [65, 141], as denoted by blue cross dots in Fig. 4.2. However, this variation is often neglected in some works relating to the physics studies of thermoacoustic instability, such as in Heckl's work [67], where an uniform temperature configuration is assumed. Considering that the efficiencies of state-feedback controllers relies significantly on the

accuracy of the thermoacoustic system model, the non-uniformity of mean temperature is considered in the present work, and its distribution is approximated by a sudden jump from upstream to downstream, as denoted by green dash lines.

Figure 4.2: Measured mean temperature varied with axial position at atmospheric pressure.

Apparently, the approximation fits the experimental data (by Matveev) better than the uniform temperature configuration. By comparison of the normalized mode shapes and eigenfrequencies of the first and second acoustic pressure and velocity in Fig. 4.3, the predictions from the temperature jump configuration are much closer to the experimental measurement than that from uniform temperature assumption.

The instantaneous flow variables in the thermoacoustic system consist of a mean and a fluctuating part as

$$u(x, t) = \bar{u}(x) + u'(x, t), \quad \rho(x, t) = \bar{\rho}(x) + \rho'(x, t), \quad p(x, t) = \bar{p} + p'(x, t) \quad (4.1)$$

The overbar denotes a mean value and the prime describes a perturbation. Since

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Figure 4.3: Comparison of normalized mode shapes of acoustic pressure $p'(x_f, t)$ and velocity $u'(x_f, t)$, as $L=1.0$ m, and $x_f=0.25$ m.

the acoustic pressure is small in amplitude with respect to the ambient pressure, it is assumed to behave linearly. The thermoacoustic system with N_1 distributed actuators are governed by the following momentum and energy conservation equations [69, 142]

$$\frac{\partial u'}{\partial t} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0 \quad (4.2a)$$

$$\frac{1}{\bar{c}^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = \frac{\gamma - 1}{\bar{c}^2} \frac{\partial}{\partial t} (Q' \delta(x - x_f)) + \underbrace{\frac{\gamma \bar{p}}{\bar{c}^2} \frac{\partial}{\partial t} \left(\sum_{n=1}^{N_1} \alpha_{an} v_{an} \delta(x - x_{an}) \right)}_{Q_a} \quad (4.2b)$$

where γ and $\bar{c}$ denote the ratio of specific heat and sound speed. $\delta(x)$ is the Dirac delta function which is used to define the location of the heat source and the actuators. Q_a denotes the actuation perturbation. The unsteady heat release rate $Q'(x, t)$ is

characterized as [54, 67]

$$Q'(x, t) = K \left[\sqrt{\left| \frac{1}{3} + \frac{u'(x, t - \tau)}{\bar{u}_l} \right|} - \sqrt{\frac{1}{3}} \right] \quad (4.3)$$

where $K = 2L_w(T_w - \bar{T})\sqrt{\pi\lambda C_\nu \bar{\rho}_1 \bar{u}_l d_w}/\sqrt{6}S$. Meanwhile T_w , d_w , and L_w denote the temperature, diameter, and length of the heated wires, respectively. S is cross-sectional area of the duct. C_v is the specific heat at constant volume, and $\bar{u}_l$ is the incoming mean flow velocity. The subscripts l and r denote the flow variables in the upstream and downstream regions of the heat wires. τ is the time lag between the moment when the incoming velocity perturbation acts and that when the corresponding heated release is felt. It is mainly caused by the process of surface heat transfer and subsequent thermal diffusion between the heated wires and the ambient fluid. For Reynolds number greater than 10 and frequencies smaller than $20\bar{u}/d_w$, τ can be estimated [143] by $\tau = 0.2d_w/\bar{u}$. In Eq. (4.2b), α_{an} denotes the ratio of the cross-sectional area S_{an} of the n_{th} actuator [142] to the cross-sectional area of the duct S , i.e., $\alpha_{an} = S_{an}/S$. v_{an} denotes the diaphragm velocity of the n_{th} actuator [70, 142]. By modulating the diaphragm velocity v_{an} , the acoustic field can be modified to lead the Rayleigh integral [144] to be negative. This is how an unstable thermoacoustic system may be stabilized via feedback control approach.

In order to solve Eqs. (4.2a) and (4.2b), Galerkin series [46] is applied to expand the acoustic pressure as a superposition of natural modes of the duct as

$$p'(x, t) = \sum_{j=1}^N \frac{-1}{z_j \omega_j} \dot{\eta}_j(t) \psi_j(x) \quad (4.4)$$

where $z_j^2 = \int_0^L \psi_j^2(x) dx$, and ω_j denotes the eigenfrequency of acoustic waves. N

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is the number of eigenmodes. The basis function $\psi_j(x)$ is the eigen-solution of the classical acoustic wave equation with no damping and heat source, as given as

$$\frac{1}{\bar{c}^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0 \quad (4.5)$$

Since the presence of the heat source leads to the mean temperature undergoing a jump from the upstream region and downstream of the heated wires, substituting Eq.(4.4) into Eq.(4.5) yields

$$\frac{d^2 \psi_j(x)}{dx^2} + \frac{\omega_j^2}{\bar{c}_l^2} \psi_j(x) = 0, \quad x \in [0, x_f]; \quad \frac{d^2 \psi_j(x)}{dx^2} + \frac{\omega_j^2}{\bar{c}_r^2} \psi_j(x) = 0, \quad x \in (x_f, L]. \quad (4.6)$$

The open-ended boundary conditions and match conditions across the heat source at $x = x_f$ are given as [46]

$$\psi_j(0) = \psi_j(L) = 0, \quad \psi_j(x) \Big|_{x=x_f^-}^{x=x_f^+} = 0, \quad \frac{1}{\bar{\rho}(x)} \frac{d\psi_j(x)}{dx} \Big|_{x=x_f^-}^{x=x_f^+} = 0 \quad (4.7)$$

With these conditions applied, the basis function $\psi_j(x)$ can be presented as

$$\psi_j(x) = \begin{cases} \sin\left(\frac{\omega_j x}{\bar{c}_l}\right), & 0 \leq x < x_f \\ -\sin\left(\frac{\omega_j(x-L)}{\bar{c}_r}\right) \frac{\sin \beta_{1j}}{\sin \beta_{2j}}, & x_f < x \leq L \end{cases} \quad (4.8)$$

where $\beta_{1j} = \omega_j x_f / \bar{c}_l$, and $\beta_{2j} = \omega_j (L - x_f) / \bar{c}_r$. The eigenfrequency ω_j is determined by the characteristic equation, i.e., $\bar{c}_r \sin \beta_{1j} \cos \beta_{2j} = -\bar{c}_l \sin \beta_{2j} \cos \beta_{1j}$. It can be further shown that the basis function $\psi_j(x)$ satisfies the condition of orthogonality,

i.e.,

$$\int_0^L \psi_j(x) \psi_m(x) dx = \begin{cases} 0, & j \neq m \\ z_j^2, & j = m \end{cases}$$

By replacing the acoustic pressure $p'(x, t)$ in the moment of Eq. (4.2a), the acoustic velocity $u'(x, t)$ can be obtained as

$$u'(x, t) = \begin{cases} \sum_{j=1}^N \frac{\eta_j(t)}{\bar{\rho}_l \bar{c}_l z_j} \cos\left(\frac{\omega_j x}{\bar{c}_l}\right) & 0 \leq x < x_f \\ -\sum_{j=1}^N \frac{\eta_j(t)}{\bar{\rho}_r \bar{c}_r z_j} \cos\left(\frac{\omega_j (L-x)}{\bar{c}_r}\right) \frac{\sin \beta_{1j}}{\sin \beta_{2j}} & x_f < x \leq L \end{cases} \quad (4.9)$$

Substituting Eq.(4.4) into Eq.(4.2b) leads to

$$\begin{aligned} \sum_{j=1}^N \frac{-1}{z_j \omega_j} \left(\frac{d^2 \dot{\eta}_j}{dt^2} + \omega_j^2 \dot{\eta}_j \right) \psi_j(x) &= (\gamma - 1) \frac{\partial}{\partial t} (Q'(x, t) \delta(x - x_f)) \\ &+ \gamma \bar{p} \frac{\partial}{\partial t} \left(\sum_{n=1}^{N_1} \alpha_{an} v_{an} \delta(x - x_{an}) \right) \end{aligned} \quad (4.10)$$

Multiplying Eq.(4.10) with $\psi_m(x)$ and then integrating the term over the whole spatial domain, and applying the property of orthogonality, give rise to

$$\frac{z_j}{\omega_j} \frac{d^2 \eta_j}{dt^2} + z_j \omega_j \eta_j = (1 - \gamma) Q'(x_f, t) \sin\left(\frac{\omega_j x_f}{\bar{c}_l}\right) - \gamma \bar{p} \sum_{n=1}^{N_1} \alpha_{an} v_{an}(x_{an}, t) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) \quad (4.11)$$

Due to heat conduction and viscosity in the acoustic boundary layer and sound radiation at both ends of the tube, the acoustic loss/damping effect is taken into account,

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and Eq. (4.11) can be further modified as [3]

$$\begin{aligned} \frac{z_j}{\omega_j} \frac{d^2 \eta_j}{dt^2} + 2\xi_j z_j \frac{d\eta_j}{dt} + z_j \omega_j \eta_j = & (1 - \gamma) Q'(x_f, t) \sin\left(\frac{\omega_j x_f}{\bar{c}_l}\right) \\ & - \gamma \bar{p} \sum_{n=1}^{N_1} \alpha_{an} v_{an}(x_{an}, t) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) \end{aligned} \quad (4.12)$$

where the damping coefficient ξ_j is frequency-dependent, and it is given as

$$\xi_j = \left(g_1 \omega_j / \omega_1 + g_2 \sqrt{\omega_1 / \omega_j} \right) / 2\pi, \text{ where } g_1 \text{ and } g_2 \text{ are constants.}$$

Based on Eq. (4.12), two nonlinear feedback controllers are developed. The parameters of the heat source are set as follows: $L_w = 1$ m, $T_w = 1680$ K, $S = 1.56 \times 10^{-3}$ m², $\lambda = 0.0328$ W/m·K, $C_v = 719$ J/Kg·K, and $\gamma = 1.4$. The thermoacoustic system is configured at atmospheric condition, i.e., $\bar{p} = 1.01325 \times 10^5$ Pa. The inlet temperature is $T_l = 300$ K, and the outlet temperature $T_r = 400$ K.

4.2 Lyapunov-Function-Based Control of Thermoacoustic Instability

4.2.1 Overview

Transient energy growth of acoustic disturbances may trigger thermoacoustic instability in an open-ended Rijke-type system. The transient energy may also appear in the controlled thermoacoustic system with traditional linear controller implemented. In this section, it is first shown that implementing a Linear Quadratic Regulator (LQR) works well in suppressing thermoacoustic instability, but fails in eliminating the transient growth. This finding is consistent with pseudospectra analysis of the present closed-loop system with two eigenmodes. In order to achieve strict dissipativity (i.e., unity maximum transient growth), a Lyapunov-function-based unity maximum transient growth controller is developed.

4.2.2 Definition of Transient Acoustical Energy Growth

To characterize the transient growth of the flow disturbances in a 1D thermoacoustic system, the total acoustic energy $E_s(x_f, t)$ per unit cross-sectional area is chosen as a measure, which comprises a kinetic component and a potential component. The definition of the total acoustic energy is given as follows:

$$\begin{aligned} E_s(x_f, t) &= \int_0^{x_f} \left(\frac{p_l'^2(x, t)}{2\gamma\bar{p}} + \frac{\bar{\rho}_l u_l'^2(x, t)}{2} \right) dx + \int_{x_f}^L \left(\frac{p_r'^2(x, t)}{2\gamma\bar{p}} + \frac{\bar{\rho}_r u_r'^2(x, t)}{2} \right) dx \\ &= \frac{1}{2\gamma\bar{p}} \sum_{j=1}^N \left(\eta_j^2 + \frac{\dot{\eta}_j^2}{\omega_j^2} \right) = \sum_{j=1}^N E_j(t) \end{aligned} \quad (4.13)$$

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The acoustical transient energy growth is defined as the maximum growth ratio of acoustical energy at instant moment to the initial status, i.e.,

$$G(x_f, t) = \max_{t \geq 0} \frac{E_s(x_f, t)}{E_s(x_f, 0)} \quad (4.14)$$

And the maximum transient growth is given as

$$G_{\max} = \max_{t \rightarrow \infty} \{G(x_f, t)\} \quad (4.15)$$

If $G_{\max} = 1$, there is no transient growth of the acoustical energy, which is also known as strict dissipativity. The energy of acoustic disturbances is decaying, if

$$\frac{dE_s}{dt} < 0 \quad (4.16)$$

It is worth noting that the designed Lyapunov function is based on Eq. (4.16) to achieve unity maximum transient growth.

4.2.3 Design of Lyapunov-Function-Based Transient Growth Controller

Based on the simplified governing equation of the 1D Rijke-type thermoacoustic system in Sec. 4.1, Eq. (4.12) can be rewritten as

$$\frac{d^2 \eta_j}{dt^2} + 2\xi_j \omega_j \frac{d\eta_j}{dt} + \omega_j^2 \eta_j = (1 - \gamma) \frac{\omega_j}{z_j} Q'(x_f, t) \sin\left(\frac{\omega_j x_f}{\bar{c}_l}\right) + U_j \quad (4.17)$$

U_j is defined as the control input and it is written as

$$U_j = -\gamma \bar{p} \frac{\omega_j}{z_j} \sum_{n=1}^{N_1} \alpha_{an} v_{an}(x_{an}, t) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) \quad (4.18)$$

The diaphragm velocity v_{an} is given as [70, 142]

$$v_{an} = R_n \frac{u'(x_{an}, t)}{\bar{u}_l} + S_n \frac{p'(x_{an}, t)}{\gamma \bar{M}_l \bar{p}}, \quad (4.19)$$

where R_n , S_n are control parameters which need to be determined, and $\bar{M}_l = \bar{u}_l / \bar{c}_l$. By replacing the acoustic pressure and velocity via Galerkin series Eq. (4.4) and Eq. (4.9), the diaphragm velocity v_{an} can be shown as

$$v_{an} = \frac{R_n}{\bar{u}_l} \sum_{j=1}^N \frac{\eta_j(t)}{\bar{\rho}_l \bar{c}_l z_j} \cos\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) - \frac{S_n}{\gamma \bar{M}_l \bar{p}} \sum_{j=1}^N \frac{\dot{\eta}_j(t)}{z_j \omega_j} \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) \quad (4.20)$$

If $\tilde{R}_n$, $\tilde{S}_n$, A_{mjn} , and B_{mjn} are defined as

$$\begin{aligned} \tilde{R}_n &= -\frac{\alpha_{an} R_n}{\bar{M}_l}, & A_{mjn} &= \cos\left(\frac{\omega_m x_{an}}{\bar{c}_l}\right) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) / z_m, \\ \tilde{S}_n &= \frac{\alpha_{an} S_n}{\bar{M}_l}, & B_{mjn} &= \sin\left(\frac{\omega_m x_{an}}{\bar{c}_l}\right) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) / z_m \omega_m, \end{aligned} \quad (4.21)$$

then the control input as involved in Eq. (4.18) can be rewritten as

$$U_j = \sum_{n=1}^{N_1} \left(\tilde{R}_n \frac{\omega_j}{z_j} \sum_{m=1}^N A_{mjn} \eta_m + \tilde{S}_n \frac{\omega_j}{z_j} \sum_{m=1}^N B_{mjn} \dot{\eta}_m \right) \quad (4.22)$$

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According to Eq. (4.17), the control input is proposed as

$$U_j = (2\xi_j\omega_j - b_{j2})\dot{\eta}_j + (\omega_j^2 - b_{j1})\eta_j + (\gamma - 1)\frac{\omega_j}{z_j}Q'(x_f, t) \sin\left(\frac{\omega_j x_f}{\bar{c}_l}\right) \quad (4.23)$$

The actuated thermoacoustic system Eq. (4.29) can be further simplified as

$$\ddot{\eta}_j + b_{j2}\dot{\eta}_j + b_{j1}\eta_j = 0, \quad j = 1, 2, \dots, N \quad (4.24)$$

The analytical solution to Eq.(4.24) can be shown as

$$\eta_j = \vartheta_{j1} \exp\left(-\frac{b_{j2} + \sqrt{b_{j2}^2 - 4b_{j1}}}{2}t\right) + \vartheta_{j2} \exp\left(-\frac{b_{j2} - \sqrt{b_{j2}^2 - 4b_{j1}}}{2}t\right) \quad (4.25)$$

$\vartheta_{j1}, \vartheta_{j2}$ are coefficients depending on the initial conditions. The control parameters b_{j1} and b_{j2} can be modified to ensure the monotonic decrease of the acoustic transient energy growth. This is achievable by constructing a proper Lyapunov function $\mathcal{L}$ as given as

$$\mathcal{L} = \frac{1}{2\gamma\bar{p}} \sum_{j=1}^N \left(\eta_j^2 + \frac{\dot{\eta}_j^2}{b_{j1}} \right) \quad (4.26)$$

Differentiating the proposed Lyapunov function with respect to time leads to

$$\frac{d\mathcal{L}}{dt} = -\frac{1}{\gamma\bar{p}} \sum_{j=1}^N \frac{b_{j2}}{b_{j1}} \dot{\eta}_j^2 \leq 0 \quad (4.27)$$

The non-positive sign of Eq. (4.27) guarantees that the system is asymptotically stable by LaSalle's invariance principle [145]. If we set $b_{j1} = \omega_j^2$, the proposed Lyapunov function $\mathcal{L}$ can be converted into the acoustical energy, as defined in Eq. (4.13). Therefore, the nonpositive gradient of $\mathcal{L}$ ensures the strict dissipativity (unity maximum transient growth) to be achieved.

In order to obtain $\hat{R}_n$ and $\hat{S}_n$, it is assumed that U_j can be written in the following form,

$$U_j = \sum_{s=1}^N q_{js} \mathcal{I}_s \quad \text{for} \quad j = 1, 2, \dots, N \quad (4.28)$$

where $\mathcal{I}_s$ are unknown variables. The governing equations for the actuated system can then be expressed in a matrix form as

$$\begin{bmatrix} \ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 \\ \ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 \\ \vdots \\ \ddot{\eta}_N + 2\xi_N\omega_N\dot{\eta}_N + \omega_N^2\eta_N \end{bmatrix} = \begin{bmatrix} (1-\gamma)\frac{\omega_1}{z_1}Q'(x_f, t) \sin\left(\frac{\omega_1 x_f}{\bar{c}_l}\right) \\ (1-\gamma)\frac{\omega_2}{z_2}Q'(x_f, t) \sin\left(\frac{\omega_2 x_f}{\bar{c}_l}\right) \\ \vdots \\ (1-\gamma)\frac{\omega_N}{z_N}Q'(x_f, t) \sin\left(\frac{\omega_N x_f}{\bar{c}_l}\right) \end{bmatrix} + \underbrace{\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & \cdots & q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \cdots & q_{NN} \end{bmatrix}}_{\mathbf{Q}} \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \\ \mathcal{I}_N \end{bmatrix} \quad (4.29)$$

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Matrix $\mathbf{Q}$ is proposed to be constructed as

$$q_{js} = \frac{\omega_j A_{mjs}}{z_j A_{m1s}} \quad \text{for } j = 1 \dots N \quad (4.30)$$

which can be written as

$$\mathbf{Q}_{N \times N} = \begin{bmatrix} \frac{\omega_1}{z_1} & \frac{\omega_1}{z_1} & \dots & \frac{\omega_1}{z_1} \\ \frac{\omega_2 \sin(\omega_2 x_{a1}/\bar{c}_l)}{z_2 \sin(\omega_1 x_{a1}/\bar{c}_l)} & \frac{\omega_2 \sin(\omega_2 x_{a2}/\bar{c}_l)}{z_2 \sin(\omega_1 x_{a2}/\bar{c}_l)} & \dots & \frac{\omega_2 \sin(\omega_2 x_{aN}/\bar{c}_l)}{z_2 \sin(\omega_1 x_{aN}/\bar{c}_l)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_N \sin(\omega_N x_{a1}/\bar{c}_l)}{z_N \sin(\omega_1 x_{a1}/\bar{c}_l)} & \frac{\omega_N \sin(\omega_N x_{a2}/\bar{c}_l)}{z_N \sin(\omega_1 x_{a2}/\bar{c}_l)} & \dots & \frac{\omega_N \sin(\omega_N x_{aN}/\bar{c}_l)}{z_N \sin(\omega_1 x_{aN}/\bar{c}_l)} \end{bmatrix} \quad (4.31)$$

Note that $\mathbf{Q}$ is invertible, if the locations of actuators x_{an} are properly chosen. Substituting Eq. (4.23) into Eq. (4.18) leads to

$$\sum_{n=1}^{N_1} \left(\tilde{R}_n \frac{\omega_j}{z_j} \sum_{m=1}^N A_{mjn} \eta_m + \tilde{S}_n \frac{\omega_j}{z_j} \sum_{m=1}^N B_{mjn} \dot{\eta}_m \right) = \sum_{s=1}^N q_{js} \mathcal{I}_s \quad (4.32)$$

By replacing q_{js} with Eq. (4.30), Eq. (4.32) can be simplified to

$$\tilde{R}_n \sum_{m=1}^N A_{m1n} \eta_m + \tilde{S}_n \sum_{m=1}^N B_{m1n} \dot{\eta}_m = \mathcal{I}_n, \quad n = 1, 2, \dots, N. \quad (4.33)$$

Figure 4.4: Phase diagram of acoustic pressure $p'(x_f, t)$ and velocity $u'(x_f, t)$, as the actuators are not actuated (i.e., open-loop), as $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_l=0.5$ m/s, $g_1=0.3$, and $g_2=0.03$.

where $\tilde{R}_n$ and $\tilde{S}_n$ can be obtained as :

$$\tilde{R}_n = \frac{\sum_{m=1}^N A_{m1n} \eta_m}{\left(\sum_{m=1}^N A_{m1n} \eta_m \right)^2 + \left(\sum_{m=1}^N B_{m1n} \dot{\eta}_m \right)^2} \mathcal{I}_n; \quad (4.34a)$$

$$\tilde{S}_n = \frac{\sum_{m=1}^N B_{m1n} \dot{\eta}_m}{\left(\sum_{m=1}^N A_{m1n} \eta_m \right)^2 + \left(\sum_{m=1}^N B_{m1n} \dot{\eta}_m \right)^2} \mathcal{I}_n; \quad (4.34b)$$

Therefore, to stabilize N unstable modes, only $N_1 = N$ actuators are needed. The number of actuators is at least 50% less than that required in the previous works of Kulkarni et al. [70] and Zhao & Reyhanoglu [113]. This is another key contribution of the present work in comparison with the previous ones [70, 113].

4.2.4 Results and Discussion

To examine the performance of the Lyapunov-function-based controller, the thermoacoustic system with two eigenmodes ($N=2$) is considered. Before the designed

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Figure 4.5: Pseudospectra of the thermoacoustic system in open-loop and closed-loop configurations. The contour values are calculated by using $\log \varsigma = \log_{10} \|\lambda \mathbf{I} - \mathbf{A}^{-1}\|$, and the maximum value corresponds to eigenvalues, as $N=2$, $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_l=0.5$ m/s, $g_1=0.3$, $g_2=0.03$, $b_{12} = 100$, $b_{22} = 200$. (a) open-loop, (b) closed-loop with LQR controller implemented. (c) closed-loop with Lyapunov-based controller applied.

transient growth controller is actuated, the open-loop thermoacoustic system needs to be studied first. Fig. 4.4 shows the phase portraits of acoustic pressure and velocity, when the actuators are not actuated. It can be seen that initial small-amplitude pressure perturbations grow into large-amplitude limit cycle oscillations, as denoted by the outer circle. This means that the open-loop system is unstable. From the pseudospectra analysis [146] in Fig. 4.5(a), it is apparent that there is one pair of complex eigenvalues with positive real parts. The dash line describes the numerical

Figure 4.6: The transient growth of the total energy in the open-loop thermoacoustic system, as $L=1.0$ m, $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_1=0.5$ m/s, $g_1=0.3$, $g_2=0.03$.

abscissa [147]. Part of it protrudes into the right half plane. This indicates that the system is non-normal, and transient growth [69] occurs. Fig. 4.6 shows the transient growth behavior of the system. It can be seen that the maximum transient growth $G_{\max} \rightarrow \infty$, as $t \rightarrow \infty$. This finding from the transient growth analysis is consistent with the pseudospectra analysis.

Now the designed Lyapunov-function-based controller is actuated. For comparison, a conventional linear controller, i.e., LQR is also implemented. The performances of these controllers are evaluated and compared in terms of minimizing transient growth and sound level reduction. Fig. 4.7 shows phase plots of acoustic pressure and velocity, when LQR and Lyapunov-based controllers are implemented respectively. It can be seen that the limit cycle oscillations are quickly dampened by the two controllers. Furthermore, LQR controller works more rapidly and effectively in minimizing thermoacoustic oscillations.

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Figure 4.7: The phase diagram of pressure $p'(t)$ and velocity oscillation $u'(t)$ in an actuated thermoacoustic system, as $N=2$, $L=1.0$ m, $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_l=0.5$ m/s, $g_1=0.3$, $g_2=0.03$, $b_{12} = 100$, and $b_{22} = 200$.

Figure 4.8: Comparison of the transient growth of acoustical energy in the actuated thermoacoustic system with LQR and Lyapunov-based controller implemented, as $N=2$, $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_l=0.5$ m/s, $g_1=0.3$, $g_2=0.03$, $b_{12} = 100$, $b_{22} = 200$.

However, implementing the LQR controller is associated with large transient growth of acoustic energy, as denoted by the red dash line in Fig. 4.8. The Lyapunov-based controller is found to be able to achieve unity maximum transient growth, as

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denoted by the blue cross line in Fig. 4.8. Pseudospectra analysis of the actuated system, as shown in Fig. 4.5(b) and (c) reveals that both controllers lead to all eigenmode being stable (negative real parts). However, part of numerical abscissa line for the application of LQR controller is protruding in the right half plane. By comparison, implementing Lyapunov-based controller is associated with no part of numerical abscissa in the right half plane. This explains how strict dissipativity is achieved.

The performance of these controllers are further evaluated by assessing the sound pressure level before and after control. Fig. 4.9 shows that the sound pressure level is reduced by approximately 25 dB, as the Lyapunov-based controller is applied. By comparison, LQR is more effective in reducing the sound pressure level than Lyapunov-based controller.

Figure 4.9: Variation of sound pressure level before and after the LQR and Lyapunov-based controllers are implemented.

In order to gain insight on the eigenmodes interaction, acoustical energy analysis is conducted. Fig. 4.10 illustrates time evolution of acoustical energy at ω_1 and ω_2 .

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Figure 4.10: Phase diagram of acoustical energy, as $N=2$, $x_f=0.65$ m, $\tau=4$ ms, $\bar{u}_l=0.5$ m/s, $g_1=0.3$, $g_2=0.03$, $b_{12} = 100$, $b_{22} = 200$. E_1 and E_2 denote the acoustical energy at ω_1 and ω_2 respectively. Note that $E_s = E_1 + E_2$ denotes the total acoustical energy. (a)-(c) for open-loop system. (d)-(f) for closed-loop system with Lyapunov-based controller implemented.

It can be seen from Figs. 4.10(a)-(b) that the acoustical energy of both modes are ‘saturated’, when there are no controllers implemented. In addition, the magnitude of $\dot{E}_s(t)$ in Fig. 4.10(c) is comparable with that of $\dot{E}_1(t)$, while it is much greater than that of $\dot{E}_2$. This indicates that the generated limit cycle oscillations are dominated by the lower frequency mode. When the actuators are turned on, the acoustical energy at both eigenfrequencies are simultaneously minimized, as shown in Fig. 4.10(d)-(f).

Figure 4.11: The time evolution of actuation signals v_{a1} and v_{a2} , as $x_{a1}=0.25$ m, $x_{a2}=0.35$ m. $\alpha_{a1}=\alpha_{a2}=0.01$

For the controlled system responses in Fig. 4.7, two actuation signals are generated by the proposed Lyapunov-based controller to suppress the system with two eigenmodes. Fig. 4.11 shows the time evolution of the actuation signals. It can be seen that initial large-amplitude actuation signals decay quickly, once the unstable thermoacoustic oscillations are dampened. This is one key difference from ‘anti-sound’ mechanism that needs the large-amplitude actuation to work during the whole control period.

4.2.5 Summary

In this section, transient growth control of acoustical energy is achieved via properly designing a Lyapunov-function-based controller. The number of actuators is

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greatly reduced in comparison with that suggested in the previous works [70, 113]. In addition, the dominant two modes are simultaneously stabilized. When the number of unstable oscillation modes is increased, the number of actuators increases linearly. It would be challenging to ensure all actuators being placed away from acoustic pressure nodes. Thus there is a compromise between the modes to be controlled [148, 149] and actuators to be implemented, especially to achieve unity maximum transient growth, i.e., strict dissipativity.

4.3 Sliding Mode Control of Thermoacoustic Instability

4.3.1 Overview

Considering that the combustion system involves many complicated processes, including the flow perturbation and the combustion itself, it is almost impossible to accurately model the thermoacoustic system, which raises a high requirement for the robustness of the controller to modeling uncertainty and disturbances. In the present work, sliding mode controller is proposed to stabilize the thermoacoustic instability in combustion systems, due to its successful applications in dealing with the influence of modeling nonlinearity, uncertainty, and disturbances. The main idea of a sliding mode controller is to control an unstable system in a way that the system can always move towards a predefined manifold in finite time. Once the manifold is reached, the controlled system is forced to ‘slide’ on it and finally converge to the predicted status, which is fairly robust to disturbances. The sign of the control is switched due to the relative location of the system to the manifold. The motion of the system along the manifold is called ‘sliding mode’. The performance of sliding mode controller for eliminating thermoacoustic oscillations is explored here.

4.3.2 Design of Sliding Mode Controller

Based on the series of derivations for the actuated Rijke-type thermoacoustic system in Sec. 4.1, the governing equation for acoustic waves can be presented as

$$\begin{aligned} \frac{z_j}{\omega_j} \frac{d^2 \eta_j}{dt^2} + 2\xi_j z_j \frac{d\eta_j}{dt} + z_j \omega_j \eta_j = & (1 - \gamma) Q'(x_f, t) \sin\left(\frac{\omega_j x_f}{\bar{c}_l}\right) \\ & - \gamma \bar{p} \sum_{n=1}^{N_1} \alpha_{an} v_{an}(x_{an}, t) \sin\left(\frac{\omega_j x_{an}}{\bar{c}_l}\right) \end{aligned} \quad (4.12)$$

And the unsteady heat release rate is modeled as

$$Q'(x_f, t) = K \left[\sqrt{\left| \frac{1}{3} + \frac{u'(x_f, t - \tau)}{\bar{u}_l} \right|} - \sqrt{\frac{1}{3}} \right] \quad (4.3)$$

In order to implement the feedback controller, the developed governing equations are formulated into a state-space form. For this, the nonlinear heat source term needs to be linearized. If it is assumed that $|u'(x_f, t - \tau)/\bar{u}_l| < 1/3$, the nonlinear term in the heat release rate can be approximated as

$$\left(\sqrt{\left| \frac{1}{3} + \frac{u'(x_f, t - \tau)}{\bar{u}_l} \right|} - \sqrt{\frac{1}{3}} \right) \approx \frac{\sqrt{3}}{2} \frac{u'(x_f, t - \tau)}{\bar{u}_l} \quad (4.35)$$

Assuming that the time delay is relatively small compared with the period of the fundamental mode $2\pi/\omega_1$, Taylor series can be used to expand the velocity to first order in τ as

$$u'(x_f, t - \tau) \approx u'(x_f, t) - \tau \frac{\partial u'(x_f, t)}{\partial t} = \mathbf{P}_f^T \mathbf{x} - \tau \mathbf{P}_f^T \dot{\mathbf{x}} \quad (4.36)$$

where $(\cdot)^T$ denotes the operator of matrix transpose, $\mathbf{x}^T = [\eta_1, \eta_2, \dots, \eta_N]$, and

$\mathbf{P}_f^T = [\cos(\omega_1 x_f / \bar{c}_l) / \bar{\rho}_l \bar{c}_l z_1, \cos(\omega_2 x_f / \bar{c}_l) / \bar{\rho}_l \bar{c}_l z_2, \dots, \cos(\omega_N x_f / \bar{c}_l) / \bar{\rho}_l \bar{c}_l z_N]$. Substituting Eq. (4.36) into Eq. (4.3) gives

$$Q'(x_f, t) \approx \frac{\sqrt{3}K}{2\bar{u}_l} (\mathbf{P}_f^T \mathbf{x} - \tau \mathbf{P}_f^T \dot{\mathbf{x}}) \quad (4.37)$$

The system Eq. (4.12) can be rewritten as

$$\mathbf{\Omega}^{-1} \ddot{\mathbf{x}} + \boldsymbol{\zeta} \dot{\mathbf{x}} + \mathbf{\Omega} \mathbf{x} = \frac{\sqrt{3}K(\gamma - 1)}{2\bar{u}_l} (\mathbf{P}_f^T \mathbf{x} - \tau \mathbf{P}_f^T \dot{\mathbf{x}}) \mathbf{\Omega} \mathbf{Q}_f + \mathbf{u} + \mathbf{h}(\mathbf{x}, \dot{\mathbf{x}}) \quad (4.38)$$

where $\mathbf{\Omega}$, $\boldsymbol{\zeta}$, and $\mathbf{Q}_f^T$ are given in Appendix. $\mathbf{h}(\mathbf{x}, \dot{\mathbf{x}})$ is the vector of nonlinear terms, which is assumed to be negligible. $\mathbf{u}$ is the control input. Define $\mathbf{y}_1 = \mathbf{x}$ and $\mathbf{y}_2 = \mathbf{\Omega}^{-1} \dot{\mathbf{x}}$. The actuated thermoacoustic system Eq. (4.38) can be rewritten as

$$\begin{aligned} \dot{\mathbf{y}}_1 &= \mathbf{\Omega} \mathbf{y}_2 \\ \dot{\mathbf{y}}_2 &= \left(\frac{\sqrt{3}K(\gamma - 1)}{2\bar{u}_l} \mathbf{\Omega} \mathbf{Q}_f \mathbf{P}_f^T - \mathbf{\Omega} \right) \mathbf{y}_1 - \left(\frac{\sqrt{3}K(\gamma - 1)\tau}{2\bar{u}_l} \mathbf{\Omega} \mathbf{Q}_f \mathbf{P}_f^T \mathbf{\Omega} + \boldsymbol{\zeta} \mathbf{\Omega} \right) \mathbf{y}_2 + \mathbf{u} \end{aligned} \quad (4.39)$$

Eq. (4.39) can be rephrased in a state-space form as

$$\frac{d\mathbf{x}_e}{dt} = \mathbf{A} \mathbf{x}_e + \mathbf{B} \mathbf{u} \quad (4.40)$$

where

$$\mathbf{x}_e = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{\Omega} \\ \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{N \times N} \end{bmatrix}, \mathbf{A}_1 = \frac{\sqrt{3}K(\gamma - 1)}{2\bar{u}_l} \mathbf{\Omega} \mathbf{Q}_f \mathbf{P}_f^T - \mathbf{\Omega},$$

$$\mathbf{A}_2 = -\frac{\sqrt{3}K(\gamma - 1)\tau}{2\bar{u}_l} \mathbf{\Omega} \mathbf{Q}_f \mathbf{P}_f^T \mathbf{\Omega} - \boldsymbol{\zeta} \mathbf{\Omega}.$$

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The control input $\mathbf{u}$ can be shown to depend on $\mathbf{y}_1$ and $\mathbf{y}_2$ as

$$\mathbf{u} = \frac{\gamma \bar{p}}{\bar{u}_l} \sum_{n=1}^{N_1} \alpha_{an} R_n \boldsymbol{\Omega} \mathbf{Q}_{an} \mathbf{P}_{an}^T \mathbf{y}_1 + \frac{1}{\bar{M}_l} \sum_{n=1}^{N_1} \alpha_{an} S_n \boldsymbol{\Omega} \mathbf{Q}_{an} \mathbf{Q}_{an}^T \boldsymbol{\Omega} \mathbf{y}_2 \quad (4.41)$$

where $\mathbf{Q}_{an}^T$ and $\mathbf{P}_{an}^T$ are given in Appendix.

Before implementing the sliding mode control, the controllability of the thermoacoustic system is examined and confirmed

$$\text{rank} [\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \dots, \mathbf{A}^{2N-1}\mathbf{B}] = 2N \quad (4.42)$$

If a switching surface is chosen as

$$\mathbf{s}_w = [s_1, s_2, \dots, s_N]^T = \mathbf{y}_2 + \boldsymbol{\Lambda} \mathbf{y}_1 \quad (4.43)$$

Then the sliding mode controller [150] can be designed as

$$\mathbf{u} = -\mathbf{M}(t) \text{sign}(\mathbf{s}_w) \quad (4.44)$$

where $\mathbf{M}(t) = \text{diag}(m_1(t), m_2(t), \dots, m_N(t))$, and $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. $m_j(t) > 0$ and $\lambda_j > 0$, for $j = 1, 2, \dots, N$.

The existence of the sliding manifold can be confirmed by examining the stability of Lyapunov function $\mathcal{L} = \mathbf{s}_w^T \mathbf{s}_w / 2$. Differentiating Lyapunov function $\mathcal{L}$ with respect to time t leads to $\dot{\mathcal{L}} = \mathbf{s}_w^T (\mathbf{A}_1 \mathbf{y}_1 + (\mathbf{A}_2 + \boldsymbol{\Lambda} \boldsymbol{\Omega}) \mathbf{y}_2 - \mathbf{M}(t) \text{sign}(\mathbf{s}_w))$.

If $\dot{\mathcal{L}}$ satisfies

$$\dot{\mathcal{L}} < 0 \quad \text{i.e.,} \quad \|\mathbf{M}(t)\| > \|\mathbf{A}_1 \mathbf{y}_1 + (\mathbf{A}_2 + \boldsymbol{\Lambda} \boldsymbol{\Omega}) \mathbf{y}_2\| \quad (4.45)$$

then the control input can force the system trajectory to reach the manifold $\mathbf{s}_w = \mathbf{0}$ within finite time. According to Eq. (4.45), it can be defined that $m_j(t) = \alpha|\mathbf{A}_1\mathbf{y}_1 + (\mathbf{A}_2 + \mathbf{\Lambda}\mathbf{\Omega})\mathbf{y}_2|_j$, $\alpha > 1$. Once the thermoacoustic system arrives at the manifold, it would ‘slide’ along the trajectory $\dot{\mathbf{y}}_2 = -\mathbf{\Lambda}\mathbf{y}_1$, and becomes asymptotically stable, i.e., $\mathbf{y}_2 \rightarrow \mathbf{0}$, as $t \rightarrow \infty$.

In order to actuate the practical actuators such as loudspeakers, the sign function as defined in the control input $\mathbf{u}$ is approximated as

$$\text{sign}(\mathbf{s}_w) \approx \left[\frac{s_1}{|s_1| + \epsilon_1}, \frac{s_2}{|s_2| + \epsilon_2}, \dots, \frac{s_N}{|s_N| + \epsilon_N} \right]^T \quad (4.46)$$

where ϵ_i are small parameters. By substituting Eq. (4.46) into Eq. (4.44), the control input $\mathbf{u}$ can be expressed as

$$\mathbf{u} = -\mathbf{K}_1\mathbf{y}_1 - \mathbf{K}_2\mathbf{y}_2 \quad (4.47)$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are given in Appendix. Comparing Eq. (4.47) with Eq. (4.41) yields

$$\mathbf{K}_1 = -\frac{\gamma\bar{p}}{\bar{u}_l} \sum_{n=1}^{N_1} \alpha_{an} R_n \mathbf{\Omega} \mathbf{Q}_{an} \mathbf{P}_{an}^T, \quad \mathbf{K}_2 = -\frac{1}{\bar{M}_l} \sum_{n=1}^{N_1} \alpha_{an} S_n \mathbf{\Omega} \mathbf{Q}_{an} \mathbf{Q}_{an}^T \mathbf{\Omega} \quad (4.48)$$

According to Eq. (4.48), there are $2N^2$ linear equations in term of control parameters R_n and S_n . It means that N^2 actuators are needed to stabilize N -modes unstable thermoacoustic system. Therefore, the relationship between the numbers of the actuators N_1 and eigenmodes N is $N_1 = N^2$.

4.3.3 Results and Discussion

Figure 4.12: Phase diagram of acoustic pressure $p'(x_f, t)/\bar{p}$ and velocity $u'(x_f, t)/\bar{u}_l$ in the uncontrolled thermoacoustic system, as $x_f=0.45$ m, $\tau=0.0014$ s, $g_1 = 0.135$, $g_2 = 0.015$: (a) and (b) $N=1$; (c) and (d) $N=4$.

The performance of the sliding mode controller is evaluated by implementing it in two thermoacoustic systems. One is associated with $N = 1$ mode. The other is with $N = 4$ modes. When the monopole-like actuators are not actuated, the thermoacoustic system is corresponding to an open-loop one. Initial small-amplitude disturbances are observed to grow into nonlinear limit cycles, as indicated by the circle-shaped phase diagram in Fig. 4.12. By making a comparison of Figs. 4.12(a) and (b) to Figs. 4.12(c) and (d), the nonlinearity of the thermoacoustic system is found to be increased, as the mode number N is increased from $N = 1$ to $N = 4$. This might be due to the interaction between low-frequency and high-frequency modes.

When the sliding mode controller is actuated, the thermoacoustic systems are

corresponding to closed-loop ones. The performance of implementing the sliding mode controller on stabilizing the unstable system is shown in Fig. 4.13. It can be seen that initial flow disturbances do not grow into limit cycle oscillations. This indicates that the controller can prevent the onset of self-sustained thermoacoustic oscillations. By comparison with the popular linear quadratic regulator (LQR), the sliding mode controller is associated with faster responding and more effectively dampening the flow disturbances.

Figure 4.13: Time evolution of acoustic pressure $p'(x_f, t)/\bar{p}$ and velocity $w'(x_f, t)/\bar{u}_1$ in the closed-loop thermoacoustic system, as $\alpha=1.5$, $\mathbf{\Lambda} = \mathbf{I}$, $x_f=0.45$ m, $\tau=0.0014$ s, $\bar{u}_l=0.3$ m/s, $g_1 = 0.135$, $g_2 = 0.015$. (a) and (b) $N=1$, (c) and (d) $N=4$.

The actuation signals from LQR and SMC controllers needed to prevent the onset of thermoacoustic instability in the system with 4 modes are shown in Fig. 4.14. It can be seen that the sliding mode controller attempts to control the 4 modes more effectively at the same time than that of LQR.

In order to gain insight on the mode selection, energy exchange analysis between

Figure 4.14: Time evolution of control input for the first four mode oscillations, as $\alpha=1.5$, $\mathbf{\Lambda} = \mathbf{I}$, $x_f=0.45$ m, $\tau=0.0014$ s, $\bar{u}_l=0.3$ m/s, $g_1 = 0.135$, $g_2 = 0.015$, $N=4$.

the neighboring modes is conducted on the thermoacoustic system with 4 eigenmodes, i.e., $N = 4$. For this, the total acoustical energy E_s is chosen as the measure, and it is given as

$$\begin{aligned}
 E_s(x_f, t) &= \int_0^{x_f} \left(\frac{p_l'^2(x, t)}{2\gamma\bar{p}} + \frac{\bar{\rho}_l u_l'^2(x, t)}{2} \right) dx + \int_{x_f}^L \left(\frac{p_r'^2(x, t)}{2\gamma\bar{p}} + \frac{\bar{\rho}_r u_r'^2(x, t)}{2} \right) dx \\
 &= \sum_{j=1}^N \frac{1}{2\gamma\bar{p}} \left(\eta_j^2 + \frac{\dot{\eta}_j^2}{\omega_j^2} \right) = \sum_{j=1}^N E_j(t)
 \end{aligned} \tag{4.14}$$

The time evolutions of acoustical energy at each eigenmode $E_i(t)$ are determined and shown in Fig. 4.15. It can be seen from Figs. 4.15(a)-(d) that the energy at the four different modes grows quickly and finally saturates, when the actuators are not actuated. It indicates that the limit cycle oscillations are produced. By comparison in the oscillation amplitude, the dominant mode is at ω_1 . As the flow conditions are fixed and the controller is actuated, the energy at ω_1 is quickly reduced, and no other

Figure 4.15: The variation of acoustic energy with time, (a)-(d) for the uncontrolled system, (e)-(h) for the controlled system $t > 0$, as $N=4$, $\alpha=1.5$, $\mathbf{\Lambda} = \mathbf{I}$, $x_f=0.45$ m, $\tau=0.0014$ s, $\bar{u}_l=0.3$ m/s, $g_1 = 0.135$, $g_2 = 0.015$

mode limit cycle oscillations are destabilized, as seen in Fig. 4.15(e)-(h). Therefore, the sliding mode control can prevent the onset of thermoacoustic oscillations.

Another main feature of the sliding mode controller is to dampen nonlinear limit cycle oscillations. This is clearly shown in Fig. 4.16. Before the controller is actuated, i.e., $t < 0.65$ s, large-amplitude oscillations are produced. As the sliding mode controller is actuated at $t = 0.65$ s, the limit cycle oscillations are quickly dampened as $t \geq 0.65$ s in both thermoacoustic systems with different number of eigenmodes.

Figure 4.16: Time evolution of acoustic pressure $p'(x_f, t)/\bar{p}$ and velocity $u'(x_f, t)/\bar{u}_l$, as the controller is actuated at $t = 0.65$ s, $\alpha = 1.5$, $\Lambda = \mathbf{I}$, $x_f = 0.45$ m, $\tau = 0.0014$ s, $\bar{u}_l = 0.3$ m/s, $g_1 = 0.135$, $g_2 = 0.015$: (a) and (b) $N=1$; (c) and (d) $N=4$.

The damping performance of implementing the sliding mode controller is further evaluated and compared with that of LQR controller in Fig. 4.17. By comparison with using LQR controller to achieve 19 dB sound pressure level, using sliding mode controller can reduce sound pressure level by approximate 40 dB in both thermoacoustic systems. This confirms that the sliding mode controller has great potential to be applied to stabilize an unstable thermoacoustic system, even with multiple eigenmodes.

4.3.4 Summary

In this section, a sliding mode controller is designed and implemented to stabilize unstable thermoacoustic systems. For this, a 1D thermoacoustic model is developed. An acoustically compact heat source is confined in an axial open-ended pipe. It is modeled by using a modified form of King's Law. Two actuated thermoacoustic

Figure 4.17: Variation of sound pressure level before and after LQR and SMC controllers implemented, as $x_f=0.45$ m, $\tau=0.0014$ s, $\bar{u}_l=0.3$ m/s, $g_1 = 0.135$, $g_2 = 0.015$: (a) $N=1$ and (b) $N=4$.

systems with monopole-like actuators implemented are considered. One is associated with one mode. The other is with four modes. The linearized thermoacoustic systems are shown to be controllable. Implementing the sliding mode controller results in both systems being stabilized. Sound pressure level is reduced by approximately 40 dB in both thermoacoustic systems. Comparison is made between the performance of the sliding mode controller and the popular LQR one. This confirms that the sliding mode controller has great potential to be applied to stabilize unstable combustion systems. In addition, the acoustical energy exchange between neighboring eigenmodes is studied to gain insight on the thermoacoustic mode selection. The successful demonstration indicates that the sliding mode controller can be applied to stabilize unstable thermoacoustic systems, even with multiple eigenmodes.

Chapter 5

Conclusions and Future Work

5.1 Conclusions

In this work, both nonlinear dynamic analysis and feedback control of thermoacoustic instability are conducted on Rijke-type combustors. The effects of heat source and additive noise on the process of the system's transition to instability are first investigated. The main findings obtained are summarized as follows:

1. The nonlinearity of the flame response determines the process of thermoacoustic system transition. The flame model with a 3rd-order nonlinearity generates large-amplitude thermoacoustic oscillations through a supercritical Hopf bifurcation, while subcritical bifurcation occurs in the thermoacoustic system driven by high-order flame.
2. For the stochastic thermoacoustic system near subcritical bifurcation region, it is found that the stochastic transition process can be better described by stochastic P -bifurcation, as the external noise intensity is varied. Furthermore,

large intensified noise is found to smoothly shift the hysteresis region and reduce the stability region. Meanwhile, increasing the noise intensity can initially reduce the hysteresis area and finally eliminate the region. Noise-induced coherence resonance is observed by the presence of bell-shape signal-to-noise ratio (SNR). SNR related characteristics, including the peak height of power spectrum and the width of the frequency, are found to depend on whether the system loses stability via subcritical bifurcation or supercritical one.

In order to prevent the onset of thermoacoustic instability, nonlinear feedback control strategies are developed and implemented on modeled Rijke-type combustors. Mean temperature jump across the heat source is considered to improve the accuracy of the modelling. Nonlinear controllers, including Lyapunov-based transient growth controller and sliding mode controller, are designed and evaluated in two actuated thermoacoustic systems with two different number of eigenmodes. One system is associated with two modes and four monopole-like actuators. The other system is with one mode and one actuator. The main findings obtained from nonlinear feedback control studies are summarized as follows:

1. The linearized open-loop thermoacoustic systems are shown to be controllable.
2. By implementing the Lyapunov-function-based nonlinear controller, thermoacoustic oscillations are dampened, and around 25 dB reduction of sound pressure level is achieved. Meanwhile, the unity maximum transient growth is attained for the controlled thermoacoustic system. Last, it is found that this controller is applicable, if a large time delay is present in the system.

3. Implementing the sliding mode controller results in both systems being stabilized. Sound pressure level is reduced by more than 40 dB in both thermoacoustic systems with different number of unstable modes. This confirms that the sliding mode controller has great potential to be applied to stabilize unstable combustion systems.
4. Compared with sliding mode controller, Lyapunov-function-based controller needs less actuators. For sliding mode controller, $N_1 = N^2$ actuators are needed. while for the Lyapunov-based controller, only $N_1 = N$ actuators are needed.

In summary, the present works shed lights on the nonlinear dynamic behaviors of thermoacoustic combustors. In addition, feedback control strategies are demonstrated to be able to stabilize successfully Rijke-type systems.

5.2 Future Work

Several suggestions for future work on nonlinear dynamic studies and feedback control of thermoacoustic instability are:

1. Triggering is a persistent problem in dynamic combustion systems. Most previous works have been found that either transient growth, or flow/acoustic disturbances can trigger thermoacoustic oscillations in the linear stable but nonlinear stable region. It is known that transient growth is caused by the non-normality of thermoacoustic system. Meanwhile, random flow fluctuation is always present in the process of unsteady combustion. It is reasonable to consider that transient growth and flow/acoustic disturbances are present simultaneously. Therefore, it is interesting and meaningful to study the effects of the noise along with non-normality on the amplification of small disturbances.
2. Random fluctuations are present in combustion systems. Most previous analytical investigations focus mainly on the effects of white Gaussian noise on the thermoacoustic system's behaviors, as demonstrated in Sec. 3.3. Actually, compared with white noise, the colored noise would be more realistic and accurate to characterize the turbulent-induced stochastic forcing [151, 152]. The colored stochastic forcing is determined by both the noisy intensity and correlation time. It has been found that the correlation time of the colored noise had a significant influence on the statistical properties in some dynamic systems [153]. Therefore, further works need to be done to study the effects of the turbulence-induced colored noise on the coherence motion of the premixed combustor in the vicinity of Hopf bifurcation point.

3. In order to implement designed feedback controllers on Rijke tube, experimental works should be done. Sensors and actuators should be chosen properly and implemented carefully on the tested combustor. In addition, the system parameters of each unstable mode, including the frequency and acoustic losses [154], should be obtained from the experiment and used in the modeled combustor. As for the frequency of acoustic waves, it is determined by the geometry of combustors and the inner temperature distribution. Considering that the mean temperature changes significantly for different fuel constitutions and flow condition, for a specific heat source and operating condition, the mean temperature distribution should be measured and modelled.
4. The flow field in the combustors involves different types of unsteady motions: pressure, entropy, and vortical mode. In the Rijke tube system, only acoustic waves are considered. Entropy and vorticity are neglected due to low mean flow velocity through the combustor. However, practical combustors are associated with non-negligible mean flow and choked boundary conditions. The effects of entropy and vorticity should be considered [73, 155]. Thus, evaluating the performance of active control in suppressing acoustic and entropy waves are more practical and interesting.

Appendix A

The matrix for the linearized system

Ω , Φ , ζ , $\mathbf{P}_{an}$, $\mathbf{Q}_f^T$ and $\mathbf{Q}_{an}^T$ are given as

$$\begin{aligned}\Omega &= \text{diag}(\omega_1, \omega_2, \dots, \omega_N), \quad \Phi = \text{diag}(z_1, z_2, \dots, z_N), \quad \zeta = \text{diag}(2\xi_1, 2\xi_2, \dots, 2\xi_N), \\ \mathbf{P}_{an}^T &= \left[\frac{\cos(\omega_1 x_{an}/\bar{c}_l)}{\bar{\rho}_l \bar{c}_l z_1}, \frac{\cos(\omega_2 x_{an}/\bar{c}_l)}{\bar{\rho}_l \bar{c}_l z_2}, \dots, \frac{\cos(\omega_N x_{an}/\bar{c}_l)}{\bar{\rho}_l \bar{c}_l z_N} \right], \\ \mathbf{Q}_f^T &= \left[-\frac{\sin(\omega_1 x_f/\bar{c}_l)}{z_1 \omega_1}, -\frac{\sin(\omega_2 x_f/\bar{c}_l)}{z_2 \omega_2}, \dots, -\frac{\sin(\omega_N x_f/\bar{c}_l)}{z_N \omega_N} \right], \\ \mathbf{Q}_{an}^T &= \left[-\frac{\sin(\omega_1 x_{an}/\bar{c}_l)}{z_1 \omega_1}, -\frac{\sin(\omega_2 x_{an}/\bar{c}_l)}{z_2 \omega_2}, \dots, -\frac{\sin(\omega_N x_{an}/\bar{c}_l)}{z_N \omega_N} \right].\end{aligned}$$

$\mathbf{K}_1$ and $\mathbf{K}_2$ are given as

$$\begin{aligned}\mathbf{K}_1 &= \text{diag} \left(\frac{m_1(t)\lambda_1}{|s_1(\mathbf{X})| + \epsilon_1}, \frac{m_2(t)\lambda_2}{|s_2(\mathbf{X})| + \epsilon_2}, \dots, \frac{m_N(t)\lambda_N}{|s_N(\mathbf{X})| + \epsilon_N} \right), \\ \mathbf{K}_2 &= \text{diag} \left(\frac{m_1(t)}{|s_1(\mathbf{X})| + \epsilon_1}, \frac{m_2(t)}{|s_2(\mathbf{X})| + \epsilon_2}, \dots, \frac{m_N(t)}{|s_N(\mathbf{X})| + \epsilon_N} \right).\end{aligned}$$

Appendix B

Derivation of the stochastic thermoacoustic system

The 1D momentum and energy equations for the Rijke type acoustic perturbations are given as [46]

$$\bar{\rho} \frac{\partial \tilde{u}'}{\partial \tilde{t}} + \frac{\partial \tilde{p}'}{\partial \tilde{x}} = 0 \quad (3.1a)$$

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} + \gamma \bar{p} \frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1) \tilde{Q}' \delta(\tilde{x} - \tilde{x}_f) \quad (3.1b)$$

It is assumed that the Mach number is less than 0.01, then, the convective effect on the acoustics appears much smaller than the influence from the unsteady combustion forcing. Therefore, the convective effect is implicitly expressed in the combustion terms [3]. Taking the external flow fluctuation into account [130], Eq. (3.1b) can be modified as

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} + \gamma \bar{p} \frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1)(\tilde{Q}' + \tilde{Q}'_e) \delta(\tilde{x} - \tilde{x}_f) \quad (B.1)$$

where $\tilde{Q}'$ denotes the unsteady heat release resulted from acoustic pressure, and $\tilde{Q}'_e(\tilde{x}, \tilde{t})$ denotes the non-coherence heat release induced by external fluctuation. By introducing the reference scales in Eq. (3.2), the governing equation for acoustic pressure can be non-dimensionalized as

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = (\gamma - 1)K \frac{\partial Q'}{\partial t} \delta(x - x_f) + (\gamma - 1)K \frac{\partial Q'_e}{\partial t} \delta(x - x_f), \quad (\text{B.2})$$

It is assumed that the unsteady pressure oscillation is dominated by a single mode [3, 56, 82, 115], then the Galerkin expansion of the acoustic pressure can be written as $p'(x, t) = \psi(kx)\eta(t)$. The response of the unsteady heat release to the acoustic pressure is modeled as

$$Q' = b_1 p' + b_2 p'^3 + b_3 p'^5, \quad (\text{3.18})$$

Substituting Eq. (3.18) and the Galerkin expansion into Eq. (B.2) leads to

$$\ddot{\eta} + 2\epsilon\dot{\eta}(\alpha_1 + \alpha_2\eta^2 + \alpha_3\eta^4) + k^2\eta = 2K(\gamma - 1)\dot{Q}'_e\psi(kx_f), \quad (\text{B.3})$$

For the correlation time of the external fluctuation is much smaller than the time scale of thermoacoustic oscillations, the external fluctuation is approximated as a random, white noise source. Taking the term on the right hand of Eq. (B.3) to be $n_w(t)$, Eq. (3.20) can be obtained.

List of Publications

Journal papers:

- [1] **Xinyan LI**, Dan ZHAO, “Experimental and Theoretical Bifurcation Study of a Nonlinear Standing-wave Thermoacoustic System”, *International Journal of Energy*, 135:553-562, 2017.
- [2] **Xinyan LI**, Dan ZHAO, “Unity maximum transient energy growth of heat-driven acoustic oscillations”, *Energy Conversion and Management*, 116:1-10, 2016.
- [3] **Xinyan LI**, Dan ZHAO, “Feedback control of self-sustained nonlinear combustion oscillations”, *Journal of Engineering for Gas Turbines and Power*, 138.6:061505, 2016.
- [4] **Xinyan LI**, Dan ZHAO, “Stochastic Bifurcation and Coherence Resonance Studies of Standing-wave Thermoacoustic system”, under 2nd revision in the *Journal of Sound and Vibration*.
- [5] **Xinyan LI**, Dan ZHAO, “Effects of Background Noises on Nonlinear Dynamics of a Modelled Thermoacoustic Combustor”, *Journal of the Acoustical Society of America*, 143:60, 2018.

Conference paper:

- [1] **Xinyan LI**, Dan ZHAO, “Feedback Control of Self-Sustained Nonlinear Combustion Oscillations”. In *ASME Turbo Expo 2015: Turbine Technical Conference and Exposition, American Society of Mechanical Engineers*, Montreal, June, 2015.

- [2] **Xinyan LI**, Dan ZHAO, “Coherence Resonance in a Premixed Combustor Driven by a Turbulence-Induced Colored Noise”. In *the 24th International Congress on Sound and Vibration*, London, July, 2017.

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