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2015

Yang, Z., Tan, E. L., & Heh, D. Y. Second-order temporal-accurate scheme for 3-D LOD-FDTD method with three split matrices. IEEE Antennas and Wireless Propagation Letters, 14, 1105-1108.

<https://hdl.handle.net/10356/79345>

<https://doi.org/10.1109/LAWP.2015.2394773>

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Second-Order Temporal-Accurate Scheme for 3-D LOD-FDTD Method with Three Split Matrices

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Abstract—This letter presents second-order temporal-accurate scheme for three-dimensional (3-D) LOD-FDTD method with three split matrices. The main iterations of the proposed scheme comprise only one more procedure in addition to the existing three procedures. Using proper (initial only) input and (often infrequent, field-point only) output processings, the second-order temporal accuracy is achieved and verified analytically. To further enhance the efficiency, the proposed scheme is implemented based on the principle of fundamental schemes for unconditionally-stable FDTD methods, which feature a matrix-operator-free right-hand side (RHS). Stability analysis is provided to ascertain the unconditional stability of the proposed scheme and numerical results are demonstrated to justify its higher accuracy.

Index Terms—LOD-FDTD, second-order temporal accuracy, unconditional stability.

I. INTRODUCTION

Alternative to the alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method [1], [2], the locally one-dimensional (LOD-FDTD) method [3] - [9] is another unconditionally stable method that is readily amenable to higher number of split steps or procedures [10]. The conventional scheme of LOD-FDTD has two split matrices denoted as **A** and **B** of LOD1 in [11]. Its temporal order accuracy is of first order and can be improved to second order when the noncommutative terms **AB** and **BA** are omitted efficiently as in [6] and [7]. An alternative scheme of LOD-FDTD has been developed in [12] and involves three split matrices **A**, **B** and **C**, whereby each split matrix contains only spatial operators along one direction. Here, **A** and **B** in [12] are different from (more sparse than) those in [11]. While the temporal accuracy of LOD-FDTD method with three split matrices is also of first order, it tends to be much more costly to realize second-order temporal accuracy in the traditional (Strang) way due to many more noncommutative terms **AC**, **CA**, **BC** and **CB** [13]. The associated noncommutative error may also be more significant compared to those of LOD-FDTD with two split matrices **A** and **B** [14], [15].

In this letter, we propose the second-order temporal-accurate scheme for three-dimensional (3-D) LOD-FDTD method with three split matrices. The main iterations of the proposed scheme comprise only one more procedure in addition to the existing three procedures in [12]. Using proper (initial only) input and (often infrequent, field-point only) output processings, the second-order temporal accuracy is achieved which is to be verified analytically. To further enhance the

efficiency, the proposed scheme is implemented based on the principle of fundamental schemes in [11], which features a matrix-operator-free right-hand side (RHS). Stability analysis is provided to ascertain the unconditional stability of the proposed scheme and numerical results are demonstrated to justify its higher accuracy.

II. 3-D LOD-FDTD METHOD WITH THREE SPLIT MATRICES

A. First-order temporal-accurate scheme

The conventional LOD-FDTD with three split matrices can be expressed as [12]

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{3}} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^n \quad (1a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{u}^{n+\frac{2}{3}} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{u}^{n+\frac{1}{3}} \quad (1b)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{C}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{C}\right)\mathbf{u}^{n+\frac{2}{3}} \quad (1c)$$

where

$$\mathbf{u} = [E_x, E_y, E_z, H_x, H_y, H_z]^T. \quad (2)$$

and **A**, **B** and **C** are the three split matrices defined therein [12, (1)]. From (1), three procedures make up one main iteration and each procedure contains only spatial operators along *x*, *y*, or *z* directions. The above scheme is only of first-order temporal accuracy.

B. Second-order temporal-accurate scheme

To realize second-order temporal accuracy, (1b) is split into two procedures and executed before and after (1c), resulting in the following four update equations:

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^{n,1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\mathbf{u}^{n,0} \quad (3a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n,2} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n,1} \quad (3b)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{C}\right)\mathbf{u}^{n,3} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{C}\right)\mathbf{u}^{n,2} \quad (3c)$$

$$\left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n+1,0} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{B}\right)\mathbf{u}^{n,3} \quad (3d)$$

with the following input and output processings:

$$\text{Input: } \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{1,0} = \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^0 \quad (4a)$$

$$\text{Output: } \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{u}^{n+1,0} \quad (4b)$$

Note that the main iterations of the proposed scheme comprise only one more procedure in addition to the three procedures in (1). Furthermore, the input processing is only executed for once when the initial fields are (seldom) nonzero. The output processing also needs to be executed only when the result of a particular component at certain time step (often infrequent) and certain location (field-point only) is needed. The overall isotropy of numerical dispersion will not be much degraded if the input and output processings are invoked properly.

C. Verification of second-order temporal accuracy

To verify the second-order temporal accuracy, we first write down the amplification matrix after n main iterations (denoted as G^n):

$$G^n = \{(\mathbf{I} - (\Delta t/4)\mathbf{B})^{-1}(\mathbf{I} + (\Delta t/4)\mathbf{B})(\mathbf{I} - (\Delta t/2)\mathbf{C})^{-1}(\mathbf{I} + (\Delta t/2)\mathbf{C})(\mathbf{I} - (\Delta t/4)\mathbf{A})^{-1}(\mathbf{I} + (\Delta t/4)\mathbf{A})\}^n \quad (5)$$

Taylor series expansion up to second-order is used as follows:

$$(\mathbf{I} - (\Delta t/4)\mathbf{A})^{-1} = \mathbf{I} + (\Delta t/4)\mathbf{A} + (\Delta t/4)^2\mathbf{A}^2 + O(\Delta t^3) \quad (6a)$$

$$(\mathbf{I} - (\Delta t/4)\mathbf{B})^{-1} = \mathbf{I} + (\Delta t/4)\mathbf{B} + (\Delta t/4)^2\mathbf{B}^2 + O(\Delta t^3) \quad (6b)$$

$$(\mathbf{I} - (\Delta t/2)\mathbf{C})^{-1} = \mathbf{I} + (\Delta t/2)\mathbf{C} + (\Delta t/2)^2\mathbf{C}^2 + O(\Delta t^3) \quad (6c)$$

Next, the above Taylor series expansions are applied to (5), we obtain

$$G^n = \mathbf{I} + n\Delta t(\mathbf{A} + \mathbf{B} + \mathbf{C}) + (1/2)n^2\Delta t^2[(\mathbf{A} + \mathbf{B} + \mathbf{C})^2 + (\mathbf{BA} + \mathbf{CA} - \mathbf{AB} - \mathbf{AC})] + O(\Delta t^3). \quad (7)$$

After input and output processings are performed, the final amplification matrix is derived as

$$\begin{aligned} & \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right) G^n \left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{A}\right)^{-1} \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right) \\ &= \mathbf{I} + n\Delta t(\mathbf{A} + \mathbf{B} + \mathbf{C}) + \frac{n^2\Delta t^2}{2}(\mathbf{A} + \mathbf{B} + \mathbf{C})^2 + O(\Delta t^3). \end{aligned} \quad (8)$$

It can be seen that the final amplification matrix (8) has the same temporal accuracy as the analytical solution up to second-order. Thus, the limitation related to first order temporal accuracy in [16] would be eliminated. Note that in (8), more noncommutative terms $\mathbf{AC}$, $\mathbf{CA}$, $\mathbf{BC}$, $\mathbf{CB}$ are involved compared to only $\mathbf{AB}$, $\mathbf{BA}$ in [4, (26)]. The iterative method in [15], [17] could be attempted to make the proposed second-order accurate method more accurate by approaching the Crank-Nicolson scheme, but the procedures could be more complicated due to more noncommutative terms.

D. Efficient fundamental scheme

To further enhance the efficiency, the proposed scheme is implemented based on the principle of fundamental schemes in [11], which features a matrix-operator-free RHS. We first introduce an auxiliary vector $\mathbf{v}$ as

$$\mathbf{v} = [e_x, e_y, e_z, h_x, h_y, h_z]^T. \quad (9)$$

Following [11], the fundamental scheme of (3) can be denoted as FLOD-FDTD and given by

$$\left(\frac{1}{2}\mathbf{I} - \frac{\Delta t}{4}\mathbf{A}\right)\mathbf{v}^{n,1} = \mathbf{u}^{n,0} \quad (10a)$$

$$\mathbf{u}^{n,1} = \mathbf{v}^{n,1} - \mathbf{u}^{n,0} \quad (10b)$$

$$\left(\frac{1}{2}\mathbf{I} - \frac{\Delta t}{8}\mathbf{B}\right)\mathbf{v}^{n,2} = \mathbf{u}^{n,1} \quad (10c)$$

$$\mathbf{u}^{n,2} = \mathbf{v}^{n,2} - \mathbf{u}^{n,1} \quad (10d)$$

$$\left(\frac{1}{2}\mathbf{I} - \frac{\Delta t}{4}\mathbf{C}\right)\mathbf{v}^{n,3} = \mathbf{u}^{n,2} \quad (10e)$$

$$\mathbf{u}^{n,3} = \mathbf{v}^{n,3} - \mathbf{u}^{n,2} \quad (10f)$$

$$\left(\frac{1}{2}\mathbf{I} - \frac{\Delta t}{8}\mathbf{B}\right)\mathbf{v}^{n+1,0} = \mathbf{u}^{n,3} \quad (10g)$$

$$\mathbf{u}^{n+1,0} = \mathbf{v}^{n+1,0} - \mathbf{u}^{n,3}. \quad (10h)$$

where the RHS of (10) is now matrix-operator-free. Note that $\mathbf{u}$ and $\mathbf{v}$ are used to store the fields whereas for conventional method in Section II-A, two $\mathbf{u}$'s are needed. Hence, the fields memory arrays are the same.

The implementations of (10a) and (10b) read:

1) Implicit updating for $e_y^{n,1}$ and $e_z^{n,1}$

$$\frac{1}{2}e_y^{n,1} - \frac{\Delta t^2}{8\varepsilon\mu}\partial_x^2 e_y^{n,1} = E_y^{n,0} - \frac{\Delta t}{2\varepsilon}\partial_x H_z^{n,0} \quad (11a)$$

$$\frac{1}{2}e_z^{n,1} - \frac{\Delta t^2}{8\varepsilon\mu}\partial_x^2 e_z^{n,1} = E_z^{n,0} + \frac{\Delta t}{2\varepsilon}\partial_x H_y^{n,0} \quad (11b)$$

2) Explicit updating for $E_y^{n,1}$, $E_z^{n,1}$, $H_y^{n,1}$ and $H_z^{n,1}$

$$E_y^{n,1} = e_y^{n,1} - E_y^{n,0} \quad (12a)$$

$$E_z^{n,1} = e_z^{n,1} - E_z^{n,0} \quad (12b)$$

$$H_y^{n,1} = H_y^{n,0} + \frac{\Delta t}{2\mu}\partial_x e_z^{n,1} \quad (12c)$$

$$H_z^{n,1} = H_z^{n,0} - \frac{\Delta t}{2\mu}\partial_x e_y^{n,1} \quad (12d)$$

Note that it is unnecessary to update E_x and H_x in (10a) as they are unchanged using the fundamental scheme for this procedure. Other update equations can be similarly obtained by expanding and manipulating (10c)-(10h).

E. Stability Analysis

The von Neumann method is used to verify the unconditional stability of the proposed scheme. The eigenvalues of amplification matrix in the main iteration can be solved as:

$$\lambda_1 = \lambda_2 = 1, \quad (13a)$$

$$\lambda_3 = \lambda_4 = \frac{X + jY}{Z} \quad (13b)$$

$$\lambda_5 = \lambda_6 = \lambda_3^* = \lambda_4^* \quad (13c)$$

where

$$\begin{aligned} X = & 4096\varepsilon^4\mu^4 - 512\Delta t^2\varepsilon^3\mu^3(2K_x^2 + 3K_y^2 + 2K_z^2) \\ & + 16\Delta t^4\varepsilon^2\mu^2(K_y^4 - 8K_x^2K_y^2 - 16K_x^2K_z^2 - 8K_y^2K_z^2) \\ & - 4\varepsilon\mu\Delta t^6(K_x^2K_y^4 + 8K_x^2K_y^2K_z^2 + K_y^4K_z^2) \\ & - K_x^2K_y^4K_z^2\Delta t^8 \end{aligned} \quad (14)$$

$$\begin{aligned} Y = & 8\{\Delta t^2\varepsilon^2\mu^2(16\varepsilon\mu - K_y^2\Delta t^2)^2[K_x^2K_y^4K_z^2\Delta t^6 \\ & + 32K_x^2K_y^2K_z^2\Delta t^4\varepsilon\mu + 4\Delta t^4\varepsilon\mu K_y^4(K_x^2 + K_z^2) \\ & + 256K_x^2K_z^2\Delta t^2\varepsilon^2\mu^2 + 1024\varepsilon^3\mu^3(K_x^2 + K_y^2 + K_z^2) \\ & + 128K_y^2\Delta t^2\varepsilon^2\mu^2(K_z^2 + K_x^2)]\}^{1/2} \end{aligned}$$

$$K_\xi = 2\sin(k_\xi\Delta\xi/2)/\Delta\xi, \quad \xi = x, y, z \quad (15)$$

and k_ξ is the wavenumber along ξ direction. Since X, Y, Z are real, it can be shown that

$$\frac{X^2 + Y^2}{Z^2} = 1. \quad (16)$$

Hence, all eigenvalues have magnitude of one and the scheme is unconditionally stable.

III. NUMERICAL RESULTS

An air-filled $100 \times 60 \times 60$ mm³ cavity meshed with uniform cells of 2 mm is simulated to demonstrate the accuracy of the proposed fundamental LOD-FDTD method. A differentiated Gaussian pulse source along the z -direction: $(t - t_0)\exp[-(t - t_0)^2/\tau^2]$ ($\tau = 350$ ps, $t_0 = 3\tau$) is used to excite the EM field at the center of the cavity. Table I illustrates the normalized norm error of LOD-FDTD in [12] and proposed second-order temporal-accurate FLOD-FDTD scheme for various CFLN = $\Delta t/\Delta t_{CFL}$. The normalized norm error is taken with respect to explicit FDTD results in the whole computational domain at $t = 2440\Delta t_{CFL}$. Note that the output processing for FLOD-FDTD is only required once at $t = 2440\Delta t_{CFL}$. It is observed that the proposed FLOD-FDTD scheme have lower normalized norm error than LOD-FDTD in [12] due to higher-order temporal accuracy. The flops count (RHS) of multiplication/division (M/D) and addition/subtraction (A/S) for the two methods are given in Table II. It is shown that while featuring higher order temporal accuracy, the proposed FLOD-FDTD scheme still achieve comparable (slightly higher) efficiency with respect to LOD-FDTD method in [12]. We further implement both schemes in an air-filled cavity with $120 \times 120 \times 120$ computational domain and the relative CPU time and memory requirement are also shown in Table II. The programs are run using Matlab on Windows 7 platform with 3.1 GHz Intel i5 processor and 4 GB RAM. Although having one additional procedure with extra cost of solving tridiagonal systems, the CPU time incurred by the second-order FLOD-FDTD scheme is still around 0.93 that of first-order LOD-FDTD scheme in [12] due to RHS flops count reduction. Compared with the method in [13], the proposed fundamental scheme is almost two times faster while maintaining second-order temporal accuracy due to one substep fewer and matrix-operator-free RHS. The memory requirements of all the three schemes are also the same.

TABLE I
NORMALIZED NORM ERROR OF LOD-FDTD IN [12] AND PROPOSED SECOND-ORDER TEMPORAL-ACCURATE FLOD-FDTD SCHEME

CFLN	1	2	4	8
LOD-FDTD in [12]	0.0103	0.0165	0.0400	0.1874
FLOD-FDTD Scheme	0.0066	0.0099	0.0294	0.0547

TABLE II
FLOPS COUNT (RHS), RELATIVE CPU TIME AND MEMORY FOR LOD-FDTD IN [12] [13] AND PROPOSED SECOND-ORDER TEMPORAL-ACCURATE FLOD-FDTD SCHEME

Arithmetic Operations	LOD-FDTD		Proposed Scheme
	[12]	[13]	
Implicit	M/D	12	20
	A/S	30	50
Explicit	M/D	6	10
	A/S	24	40
Total (RHS)	72	120	56
Temporal Accuracy	First-order	Second-Order	Second-Order
Relative CPU Time	1	1.71	0.93
Memory	158MBytes	158MBytes	158MBytes

Next, we simulate a reverberation chamber with two stirrers. The geometry of the chamber with two stirrers is shown in Fig. 1, along with the source and observation point. The two stirrers have the same dimension. One is placed on the top of the reverberation chamber and the other is one the left. Both two stirrers are modeled as PEC and staircasing is used to approximate the slanted geometries. Each stirrer has four segments which are perpendicular to each other. Each segment has length of 10 cells, width of 10 cells and thickness of 1 cell. The computational domain is $100 \times 60 \times 60$ and cell size is 2 mm. A differentiated Gaussian pulse source is excited along the z -direction at the center of the chamber: $(t - t_0)\exp[-(t - t_0)^2/\tau^2]$, $\tau = 350$ ps, $t_0 = 3\tau$. Fig. 2 shows the electric field E_z at the observation point (60,30,15) computed using explicit FDTD method, the proposed FLOD-FDTD scheme and the first-order LOD-FDTD scheme in [12] for various CFLNs when the angles of the stirrers are 0 and 90 degrees respectively. To display Fig. 2, the output processing is executed for each time step at the field observation point only. The spatial grids employed for the explicit FDTD and first-order LOD-FDTD methods are the same as the proposed FLOD-FDTD method. At CFLN=1, the normalized norm error with respect to time at the observation point for LOD-FDTD [12] is 0.0124 compared with explicit FDTD method, while the error of the proposed FLOD-FDTD method is 0.0081. For CFLN = 2, 4 and 8, the normalized norm errors for LOD-FDTD [12] are 0.0233, 0.0795 and 0.3012 respectively and 0.0128, 0.0390, 0.1423 for the proposed LOD-FDTD method. The computation time of the proposed method (including the output processing only for the observation point) when CFLN=8 is 3.7 times faster than the explicit FDTD method. The results of FLOD-FDTD scheme agree well with that of explicit FDTD method, which further validates our proposed scheme.

Fig. 1. Geometry of reverberation chamber with two stirrers.

IV. CONCLUSION

In this letter, we have proposed second-order temporal-accurate scheme for 3-D LOD-FDTD method with three split matrices. The main iterations of the proposed scheme comprise only one more procedure in addition to the existing three procedures. Using proper (initial only) input and (often infrequent, field-point only) output processings, the second-order temporal accuracy is achieved which has been verified analytically. To further enhance the efficiency, the proposed scheme has been implemented based on the principle of fundamental schemes, which features a matrix-operator-free RHS. Stability analysis has been provided to ascertain the unconditional stability of the proposed scheme and numerical results have been demonstrated to justify its higher accuracy. It is anticipated that the scheme might be extended to other types of partial differential equations (e.g. thermal diffusion equation).

ACKNOWLEDGMENT

The authors would like to acknowledge the research project DSOCL12016 and MINDEF/TL-9013101679-01.

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(a)

(b)

Fig. 2. Electric field E_z at field point computed using explicit FDTD and FLOD-FDTD method for various CFLNs when the angles of the stirrers are (a) 0° and (b) 90° .

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