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Active Contour with A Tangential Component

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Abstract Conventional edge-based active contours often require the normal component of an edge indicator function on the optimal contours to approach zero, while the tangential component can still be significant. In real images, the full gradients of the edge indicator function along the object boundaries are often small. Hence, the curve evolution of edge-based active contours can terminate early before converging to the object boundaries with a careless contour initialization. We propose a novel Geodesic Snakes (GeoSnakes) active contour that requires the full gradients of the edge indicator to vanish at the optimal solution. Besides, the conventional curve evolution approach for minimizing active contour energy cannot fully solve the Euler-Lagrange equation of our GeoSnakes active contour, causing a Pseudo Stationary Phenomenon (PSP). To address the PSP problem, we propose an auxiliary curve evolution equation, named the equilibrium flow (EF) equation. Based on the EF and the conventional curve evolution, we obtain a solution to the full Euler-Lagrange equation of GeoSnakes active contour. Experimental results validate the proposed geometrical interpretation of the early termination problem, and they also show that the proposed method is able to overcome the problem.

1 Introduction

Energy minimization provides a principled framework for various fundamental computer vision problems. Active contour was proposed for object segmentation based on energy minimization. The essential idea of the active contour is to model the object boundaries by the contour curves that minimize the functional energy which measures the error of boundary detection. The active contour has been adopted in many

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Fig. 1: The full gradients of the edge indicator near the object boundary are close to zero. (a) is an image containing an object of interest. In (b), the center is the edge indicator function for (a). The intensity of the edge indicator is visualized by the darkness in figure (b). The surrounding patches in (b) show the gradient vector field of the edge indicator along the object boundary (dashed curve). (b) is better seen by zooming in.

application domains of computer vision, such as surveillance video analysis [1,2] and medical image analysis [3,4] etc.

There are both edge-based and region-based active contours [5,6,3,7–12]. It is arguable that in general neither the region-based nor the edge-based methods can outperform the other. There are some recent efforts on integrating the region-based and edge-based active contours [13,14], and improving the edge-based active contours [15,8,16]. Our work addresses a prominent problem with edge-based active contours reported in [17,18,15,19,16], which was not yet explored in depth. The problem was often stated as: *when the curve is initialized relatively arbitrarily, the active contour can stop early and some part of the converged curve can still be far from the boundaries of the objects of interest*. This problem is known to exist only in local optimization based active contour frameworks. Parallel to the local optimization based frameworks, global optimization technique has been developed for edge-based active contours [20,21,2], which might have broken new ground in active contours. It is still, however, unclear if global optimization is always advantageous to local optimization in terms of efficiency and convenience, especially in the context of interactive segmentation.

In this paper, we address a problem with the local optimization based active contours framework. We investigate the cause of the early termination of curve evolution in general edge-based active contours. We observe that the full gradients of the edge indicator function along the object boundaries are often small. However, conventional edge-based active contours, such as the Geodesic Active Contour (GAC), often only require the normal component of the edge indicator function on the optimal contours to approach zero, while the tangential component can still be significant. An example of this phenomenon is shown in Figure 1. The arrows in the vector fields represent both the magnitudes and directions of the vectors. The empty vector fields are due

to zero vectors. We can observe the empty fields along the edge boundary. This phenomenon can be observed in many natural and medical images. Based on this observation, we propose a novel active contour model: the *Geodesic Snakes* (GeoSnakes) active contour model. The derived Euler-Lagrange equation of the GeoSnakes model requires the full gradients of the edge indicator to be close to zero on the optimal contours. However, the conventional curve evolution method does not fully solve the EL equation of the GeoSnakes model, although the curve evolution can still converge stationarily. This phenomenon is named the Pseudo Stationary Phenomenon (PSP). To address the PSP problem, we propose an auxiliary curve evolution equation, named the Equilibrium Flow (EF). The full EL equation of the GeoSnakes for boundary extraction is solved by using both the EF and the conventional gradient descent curve evolution.

From our point of view, our contributions are listed as follows:

- we observe that, in addition to the normal component, the tangential component of the gradient of the edge indicator function along the boundary curve is also important for boundary locating, and this observation contradicts the conventional view that the tangential component is merely a useless reparameterization force;
- we formulate a new active contour model based on our observation;
- we propose a curve evolution method to solve the Euler-Lagrange equations containing both the normal and tangential components.

This work is an substantial extension of the preliminary version presented at BMVC 2008 [22]. The major difference between this work and the conference version [22] is that we have developed the entire theory for incorporating tangential components in edge-based active contours in this work, while the conference version was only focused on solving an Euler-Lagrange equation that contains a tangential component.

The rest of the paper is organized as follows. In Section 2, we introduce the geodesic active contour and the works relevant to the problem of early termination of curve evolution. In Section 3, we present our problem statement followed by our formulation and solution. We also propose a novel theoretical framework for analyzing general curve evolutions in Section 4. We applied this theory to analyzing and improving the convergence of the proposed method. In Section 5, experiments are conducted to evaluate our proposed method and to also valid our theory. In Section 6, we conclude the paper with suggestions on possible future work. In the Appendix, we present the lengthy proofs of the theoretical claims made in the paper.

2 Background and related works

2.1 The Geodesic Active Contour

Our discussions are mostly based on the original Geodesic Active Contour (GAC) model. The energy functional of GAC is as follows:

$$\begin{aligned} C^* &= \arg \min_C \mathcal{L}(C) \\ \mathcal{L}(C) &= \int_0^1 \left[C_p^T \begin{pmatrix} g(\nabla I) & 0 \\ 0 & g(\nabla I) \end{pmatrix} C_p \right]^{\frac{1}{2}} dp \\ &= \int_0^1 g \|C_p\| dp = \int_C g ds, \end{aligned} \quad (1)$$

where $C = C(p)$, p is an arbitrary contour parametrization, $ds = \|C_p\| dp$ is the arclength parametrization and the edge indicator function g can be defined below:

$$g(x, y) = \frac{1}{1 + \|G_\sigma * \nabla I\|^q}, \quad (2)$$

in which G_σ is a Gaussian filter of width σ , and we may assume $q = 2$. The justifications of this choice of edge indicator can be found in [5, 23, 6].

We rewrite the Euler-Lagrange equation of GAC as follows:

$$\frac{\delta \mathcal{L}}{\delta C} = \langle \nabla g, \mathbf{N} \rangle \mathbf{N} - g\kappa \mathbf{N} = 0, \quad (3)$$

where κ is the curvature of curve C , and $\mathbf{N}$ is the normal vector of C . The smoothing term $g\kappa$ is close to zero on a smooth curve. Hence, $\langle \nabla g, \mathbf{N} \rangle$ will be also close to zero if the solution is such a curve.

2.2 Previous works on the early termination problem in curve evolution

In this subsection, we review some pioneering works on the early termination problem. Their methods have been proven effective for addressing this problem to various degrees. Our approach differs from them in that we aim to not only remedy the problem but also investigate the cause of the problem and, thus, develop a theory of the problem.

The early termination of edge-based active contours was first reported by Cohen in [17]. He proposed the Balloon term to push the curve either to shrink or to expand before reaching the object boundary. Generally, the Balloons can be applied when there is little attraction sensed by the evolving curve. Xu and Prince [18] observed that the Snakes might not converge to concave boundary even when there is large attraction. We quote their statement below.

Fig. 2: The U-shape and the opposite gradient vectors

Although the external forces¹ correctly point toward the object boundary, within the boundary concavity the forces point horizontally in opposite directions. Therefore, the active contour is pulled apart toward each of the fingers of the U-shape, but not made to progress downward into the concavity

We also reproduce a U-shape as well as the associated normalized velocity field in Figure 2 to visualize the statement above. Based on this observation, Xu and Prince [18] also proposed the Gradient Vector Flow (GVF) to extend and smooth the gradient vector field for extracting objects with moderately concave boundaries.

The celebrated level set method was introduced to this subarea due to its capability of handling topology changes in a curve evolution. Paragios et al. [15] applied the GVF to GAC with the level set method for extracting multiple objects. However, they identified a problem which we quote below.

...the proposed flow does not perform propagation when the NGVF² is close to orthogonal to the inward normal ... propagation will not take place, as well as change of the topology, even if they are supported by the level set technique.

We can compare this statement above with the observations of Xu and Prince [18] to conclude that they reported the same problem. Paragios and his colleagues proposed the adaptive balloon to force the active contour to evolve when the normal projection of the gradient is close to zero. As shown in our experiments, this idea works for images of relatively simple topology, but for images of complex topology, the contour might be oscillating near the the positions where the normal projection of the gradient is close to zero.

There are a few works on solving the problem, such as the splitting method [19], or avoiding the problem, such as the Magnetostatic active contour (MAC) [16] as well as the region-based active contours. Li et al. [19] proposed to pre-segment the image before curve evolution. The segmentation is based on the geometry of the GVF. By segmenting the image, the closed curve can also be cut into smaller ones. However, the segmentation of GVF is not in the active contour framework. Xie and Mirmehdi [16] formulated a novel edge-based curve evolution equation motivated by the mathematical formulation of magnetostatic / Lorentz force, and the curve evolution model

¹ the gradient vectors

² normalized gradient vector flow

is called the MAC. However, the curve evolution does not necessarily minimize an energy functional. Hence, MAC was not formulated under the energy minimization framework. Besides, the curve evolution of MAC behaves like region based active contours as observed in the experiments presented in this paper.

2.3 Alternatives to common edge-based active contours

There exists an interesting reformulation of GAC model in literature, proposed by Kimmel et al. [7]. They proposed an active contour formulation based on a principle of edge detection identified by Haralick [24]. The Haralick edge detector requires the image gradients to be large on the edges while the second order derivative of image values in the direction of the image gradients to be zero, i.e. forming degenerate saddle points. Our work differs from [7] in that we target at solving the problem of early termination of curve evolution of general edge-based active contours whereas Kimmel et al. [7] did not. As an actual fact, the method in [7] also suffers from the problem of early termination of curve evolution because of the omission of tangential component in curve evolution, which is shown in our experiment.

In the rich literature of active contours, there exist alternatives to level set based active contour for handling topology changes, such as the T-Snakes [25]. Broadly speaking, these frameworks are equivalent for segmentation, given the same parametric/geometric active contour model, while each has its pros and cons. For example, the level set method based active contour is more theoretically sound than T-Snakes, while the T-Snakes framework allows for more flexibility in incorporation of priors on the evolving curve. This paper is focused on the active contour based on level set method.

The research on graph cut based segmentation is flourishing and influential. Graph cut is able to compute the global optimal solution to the underlying energy minimization problem. This method is commonly viewed as a more desirable alternative to the active contours. However, in fact, both of the active contour and graph cut require careful initialization. The difference between the two methods is that the initialization is involved in the formulation in graph cut, whereas the initialization is part of the solution in active contours. Boykov and Kolmogorov [26] has also proposed a purely edge-based graph cut formulation, namely the geocut. We also compare our method with geocut in the experiment.

3 Active contour with a tangential component

3.1 An interpretation of the early termination problem

The object boundaries perceived by human are located at the peaks of the magnitude of image gradients according to the theory of edge detection [27]. Based on the magnitude of image gradients, an edge indicator function can be formulated as in Eq. (2). A prototypical edge-based active contour based on edge-indicator function is the GAC. Variants of the edge indicator functions have been considered in [7]. In GAC

model, the boundary is considered as the contour corresponding to the minimal GAC energy. The GAC energy will be low, if the edge indication on the contour is strong.

The Euler-Lagrange equation for minimizing the GAC energy requires the gradients of the edge indicator in the normal direction of the optimal curve to approach zero. However, this requirement is insufficient. It has been observed that the curve evolution for solving the EL equation often converges at the non-boundary positions where the tangential component is still significant [18, 15]. In fact, the *full* gradients of the edge indicator along the boundaries are often small, i.e. both the normal and tangential components of the gradients of the edge indicator along the boundary are close to zero, such as in Figure 1. In practice, the gradient field of the edge indicator functions of images containing noise or spurious edges are topologically complex. In such gradient fields, we can easily find the non-boundary curves on which the normal component of the gradients is zero but the tangential is non-zero. Some more examples can be found in the experiment section. Note that the assumption of small gradients along object boundary is not always valid. For example, the inhomogeneous region may lead to non-zero gradients on the boundary. The intensity inhomogeneity in medical images can often be corrected by preprocessing [28].

To conclude, we are not able to obtain the object boundaries without requiring the tangential component of the gradients of the edge indicator to be close to zero, which causes the early termination problem. Figure 3 shows a possible situation of early termination of the curve evolution in GAC. Let us consider the curve segments at A and B as the same curve segment from two successive iterations of the curve evolution in a vector field. The black arrows in Figure 3 (b) denote the actual attraction velocity on the contour curves. The attraction velocity is the projection of the vector field, visualized by the red arrows, in the normal direction of the curve. As the curve evolves, the curve at A at first moves towards position B in the first iteration. Then, in the second iteration, the curve at B will move towards A . By using a sufficiently small time step, the curve will finally stop at some place where the gradients are nearly orthogonal to the inward/outward normal, but not necessarily zero. This curve evolution terminates too early since the curve converges to the place in-between the boundaries.

Fig. 3: The curve stops (or oscillates) when the gradient is nearly orthogonal to the inward normal of the curve. (a) is a 3D visualization of a typical edge indication function overlaid by the negative gradient field; (b) demonstrates the curve evolution in the negative gradient field within the gray-shaded box region in (a). See text for details.

3.2 The GeoSnakes and the Pseudo Stationary Phenomenon

In the following, we propose a new active contour model, called the geodesic snakes(GeoSnakes) active contour, that requires the full gradients of an edge indicator to approach zero at the optimal solution. The model is given as

$$C^* = \operatorname{argmin}_C \mathcal{L}(C) = \int_C \ln \circ g(C) dp + \int_C \|C_p\| dp, \quad (4)$$

where $\circ$ is the function composition operator, $\ln \circ g(C) = \ln(g(C))$ is the composition of the three functions, g is the edge indicator function defined in Eq. (2).

At the first glance, this formulation is equivalent to that of GAC. The optimal contour is required to be smooth, and the edge indicator function on the optimal contour is required to be as small as possible. These concerns are in line with those leading to the GAC formulation. Nevertheless, the Euler-Lagrange equation of proposed model is different from that of the GAC. The Euler-Lagrange equation of (4) is

$$\begin{aligned} \frac{\delta \mathcal{L}(C)}{\delta C} &= \frac{1}{g} \nabla g - \kappa \mathbf{N} = 0 \Leftrightarrow \nabla g - g\kappa \mathbf{N} = 0 \\ &\Leftrightarrow \langle \nabla g, \mathbf{T} \rangle \mathbf{T} + \langle \nabla g, \mathbf{N} \rangle \mathbf{N} - g\kappa \mathbf{N} = 0, \end{aligned} \quad (5)$$

where $\mathbf{T}$ is the tangential of the contour curve and $\mathbf{N}$ is the normal. The first equivalence holds since g is guaranteed positive. Comparing (5) with (3), we can observe that (5) contains an additional tangential component. Additionally, this tangential component is exactly the tangential component of the gradient of the edge indicator function. By solving the Euler-Lagrange equation of GeoSnakes, we can obtain a smooth contour on which the full gradients of edge indicator approach zero, and, thus, we can hopefully obtain the object boundary while avoiding early termination of curve evolution according to our observations presented previously.

There exists geometric and parametric active contours. GAC is a typical geometric active contour. A typical parametric active contour is the original Snakes active contour [29]. A discussion regarding the relationship between the geometric active contours and the parametric active contours is in [30]. The major difference between the two kinds of active contours is that the curve evolution equation of parametric active contours may contain a tangential component whereas the other one contains only the normal component.

Note that the Euler-Lagrange equation of GeoSnakes contains a tangential component. We can identify that the GeoSnakes model is a parametric active contour. Moreover, the normal component in the Euler-Lagrange equation of GeoSnakes is the same as that of the GAC. Hence, the proposed model is related to both the GAC and Snakes, which motivates us to name it the *Geodesic Snakes (GeoSnakes)* active contour model.

Now let us return to the solution to the GeoSnakes model. The gradient descent curve evolution equation for GeoSnakes active contour is

$$\partial_t C = g\kappa \mathbf{N} - \langle \nabla g, \mathbf{T} \rangle \mathbf{T} - \langle \nabla g, \mathbf{N} \rangle \mathbf{N}. \quad (6)$$

We are left to solve this curve evolution equation. Unfortunately, this curve evolution equation cannot be easily solved by using the existing numerical methods, such

as the level set method. This is because the tangential component in the curve evolution equation is omitted by curve evolution. This behavior can be explained with the help of the Lemma of curve evolution stated in [31].

Lemma 1 *Given the closed curve $C(\tilde{p}, t)$ parameterized by arbitrary $\tilde{p} \in \tilde{B}$ at an artificial time t with the normal $\mathbf{N}$, the tangent $\mathbf{T}$ of the curve, and given the geometric flow of a curve evolution by*

$$\partial_t C(\tilde{p}, t) = \alpha(\tilde{p}, t) \mathbf{T}(C(\tilde{p}, t)) + \beta(\tilde{p}, t) \mathbf{N}(C(\tilde{p}, t)) \quad (7)$$

If β does not depend on the parametrization, meaning that β is a geometric intrinsic characteristic of the curve, then the image of $C(\tilde{p}, t)$ that satisfies Equation(7) is identical to the image of the family of curves $C(p, t)$, parameterized by $p \in B$, that satisfies

$$\begin{cases} \partial_t C(p, t) = \beta(p, t) \mathbf{N}(p, t) & (8a) \\ \frac{\partial p}{\partial t} = -\frac{\alpha}{\|C_p\|} & (8b) \end{cases}$$

The above lemma states that the geometry of the curve evolution only depends on the velocity in the normal direction of the curve, although the reparametrization by $\frac{\partial p}{\partial t}$ may last forever due to the possibly non-vanishing α . A brief proof of this lemma can be found in the book written by [32]. Here we include a proof for self-completeness.

Proof Since the curve parametrization p is parameterized by t , we have the following new curve evolution equation of Eq. (7).

$$\frac{dC}{dt} = \alpha \mathbf{T} + \beta \mathbf{N} + \frac{\partial C}{\partial p} \frac{dp}{dt}$$

Note that $\frac{dp}{dt} = -\frac{\alpha}{\|C_p\|}$, and $\frac{C_p}{\|C_p\|} = \mathbf{T}$. We have that the first and last terms of the above cancel out, which completes the proof. $\square$

The omission of the tangential component in the original Euler-Lagrange (or curve evolution) equation can also be seen in the level set method for implementing general curve evolution. Recall the level set method for curve evolution [33]

$$\begin{aligned} \partial_t \Phi|_{\Phi=0} &= \langle \nabla \Phi, \partial_t C \rangle|_{\Phi=0} \\ &= \langle \nabla \Phi, \beta \mathbf{N} + \alpha \mathbf{T} \rangle|_{\Phi=0}, \text{ by Equation(7)} \\ &= \beta \|\nabla \Phi\| \Big|_{\Phi=0} \\ &\text{by setting } \mathbf{N} = \frac{\nabla \Phi}{\|\nabla \Phi\|} \Rightarrow \langle \nabla \Phi, \mathbf{T} \rangle = 0. \end{aligned} \quad (9)$$

All the above arguments conclude that the curve evolution is independent of the tangential velocity on the closed curve. It also implies that the converged solution of Eq. (8) may not necessarily be the stationary solution of Eq. (7). To be specific, there may exists a non-vanishing reparametrization velocity, i.e. the tangential velocity, on the converged curve. This can be ascertained by rephrasing Lemma 1 as follows.

Corollary 1 (Pseudo Stationary Phenomenon) *The converged curve of Eq. (8), denoted by $C(\tilde{p}, \infty)$ which satisfies*

$$\partial_t C(\tilde{p}, \infty) = \beta(C(\tilde{p}, \infty))\mathbf{N}(C(\tilde{p}, \infty)) = \mathbf{0}$$

can lead to

$$\begin{aligned} \partial_t C(p, \infty) &= \alpha(C(p, \infty))\mathbf{T}(C(p, \infty)) \\ &\quad + \beta(C(p, \infty))\mathbf{N}(C(p, \infty)) \\ &\neq \mathbf{0}, \end{aligned} \tag{10}$$

where $p = p(\tilde{p}, t)$, $\alpha = -\|C_p\| \frac{\partial p}{\partial t}$, $\frac{\partial p}{\partial t}$ is the parametrization and $\alpha(C(p, \infty))$ can be large. $C(p, \infty)$ is the converged curve.

To prove this corollary, we can define a full curve evolution equation with both tangential and normal component, e.g. the curve evolution equation defined in Eq. (10). We may substitute a converged solution of the general curve evolution in Eq. (7) into Eq. (10). We will see the term $\alpha(C(p, \infty))\mathbf{T}(C(p, \infty))$ is arbitrary immediately. Since the proof of this claim is straightforward, we omit it for conciseness.

The above property is called the Pseudo Stationary Phenomenon (PSP), since the stationarily converged curve may not be a solution of the original Euler-Lagrange equation such as (5). Interestingly, the early termination of GAC is corresponded to the PSP of GeoSnakes.

3.3 The Equilibrium Flow and the Alternating Curve Evolution

On the one hand, the curve evolutions with only the normal component is insufficient for boundary locating and the tangential component should also be considered. On the other hand, the curve evolutions can only solve for the normal component of the curve evolution equation, even when the curve evolution equation contains a tangential component. Thus, we find ourselves in a dilemma.

In the following, we propose a solution to the curve evolution equation with a tangential component, i.e., a solution to the PSP problem. The curve evolution equation concerned can be written in the most general form as follows:

$$\partial_t C = \beta(p, t)\mathbf{N}(p, t) + \langle \mathbf{F}(p, t), \mathbf{T}(p, t) \rangle \mathbf{T}(p, t), \tag{11}$$

where $\mathbf{F}$ is a vector field. The objective is to ensure that both the normal component, β , and the tangential component, $\langle \mathbf{F}(p, t), \mathbf{T}(p, t) \rangle$, vanish when the curve evolution stops. Since the curve evolution is independent of the tangential velocity, the tangential component plays no role in the whole process of curve evolution. Therefore, we propose an auxiliary flow, named the *Equilibrium Flow* (EF), to ensure the tangential velocity to be zero when the curve evolution converges. The EF equation is defined as follows:

$$\partial_t C(p, t) = \langle \mathbf{F}(p, t), \mathbf{T}(p, t) \rangle \mathbf{N}(p, t). \tag{12}$$

This equations is named the Equilibrium Flow due to the following property.

Proposition 1 *The stationary solution $C^* = C(p, t^*)$ of the Eq. (12), i.e. $\partial_t C(p, t)|_{t=t^*} = \langle \mathbf{F}(p, t^*), \mathbf{T}(p, t^*) \rangle \mathbf{N}(p, t^*) = 0$, is either on stationary points in $\mathbf{F}$ or on level sets of the potential g corresponding to the gradient field $\mathbf{F}$.*

This proposition holds true, because $\langle \mathbf{F}(C^*), \mathbf{T}(C^*) \rangle = 0$, meaning that either $\mathbf{N}(C^*) = \frac{\mathbf{F}(C^*)}{\|\mathbf{F}(C^*)\|}$ or $\mathbf{F}(C^*) = \mathbf{0}$. The latter is one of the definitions of level set.

By using this EF flow, we obtain a solution to the Euler-Lagrange equation involving both the normal and tangential components, which is formally stated below.

Proposition 2 *Given an arbitrary initial curve C_0 , the convergent curve $C(p, \infty)$ driven by the following system of flows is the stationary solution of the original curve evolution (11) as well as the corresponding EL equation.*

$$\partial_t C_1^k = \beta(p, t) \mathbf{N}, \quad C_1^k(p, 0) = C_2^{k-1}(p, \infty) \quad (13a)$$

$$\partial_t C_2^k = \langle \mathbf{F}, \mathbf{T} \rangle \mathbf{N}, \quad C_2^k(p, 0) = C_1^k(p, \infty) \quad (13b)$$

$$\begin{cases} k = 1, 2, \dots \\ C_1^0(p, 0) = C_2^0(p, \infty) = C_0 \\ C(\tau, \infty) = \lim_{k \rightarrow \infty} C_1^k(\tau, \infty) = \lim_{k \rightarrow \infty} C_2^k(\tau, \infty) \end{cases}.$$

Proof At the converged solution $C(p, \infty)$ to Eqs. (13a) and (13b), we have

$$\beta(p, t) \mathbf{N} = \langle \mathbf{F}, \mathbf{T} \rangle \mathbf{N} = 0.$$

Substituting the above into the general form of curve evolution equation, i.e. Eq. (11), we have

$$\partial_t C = 0.$$

which completes the proof. $\square$

This proposition tells that the alternation of the two curve evolutions can provide a solution to the original EL equation containing both normal and tangential component, which is previously unknown.

The derived new curve evolution framework can have the following interpretation. Starting from an initial contour curve, the curve evolution of (13a) may converge to a pseudo stationary position, where the PSP occurs. Then the curve evolution of (13b) can help the curve evolution to escape from the pseudo stationary position. At the convergence of (13b), the value of the edge indicator at every contour point of $C(p)$ can be guaranteed unchanged (according to Proposition 3), and the curve evolution of (13a) has been reactivated because $\langle \nabla g, \mathbf{N} \rangle$ tends to be large according to Proposition 1. The process continues until the system of equations in (13) converges.

The curve evolution of (13b) cannot be directly implemented in a level set framework in which the tangent $\mathbf{T}$ cannot be conveniently used. Fortunately, in 2D case, there is a useful relation that $\mathbf{N} = \mathcal{R} \mathbf{T}$, where $\mathcal{R}$ is a matrix for ± 90 degree rotation. Therefore, the EF can be rewritten as follows:

$$\begin{aligned} \partial_t C(p, t) &= \langle \mathbf{F}, \mathbf{T} \rangle \mathbf{N} \\ &= (\mathbf{F}^T \mathbf{T}) \mathbf{N} = (\mathbf{F}^T \mathcal{R} \mathbf{N}) \mathbf{N} \\ &= [(\mathcal{R}^T \mathbf{F})^T \mathbf{N}] \mathbf{N} = \langle \mathcal{R}^T \mathbf{F}, \mathbf{N} \rangle \mathbf{N}. \end{aligned} \quad (14)$$

Fig. 4: The practical effect of rotating the gradient field. The top left is a typical gradient field, visualized by dark blue arrows, in which the green line segment cannot be moved by the local gradients due to the PSP. The top right is the vector field generated by the rotation, visualized by the arrows in red, in which the line segment can now be moved by this vector field. The bottom visualizes both the gradient field and the rotated vector field. It shows that the stationary positions, i.e., the light blue curve, remain stationary after the rotation

The EF (13b) in the system Eqs. (13a) and (13b) can be understood as a repositioning process of the curve. The repositioning process can help the curve to escape from the pseudo stationary positions. The visualization of the idea of EF is presented in Figure 4. The rotation of gradient field can reactivate the curve evolution when it is trapped by PSP, while the true stationary positions remain stationary.

3.4 Weak monotonic energy minimization by Eqs. (13a) and (13b)

We can easily see that the curve evolution equation (13a) can minimize the GeoSnake energy $\mathcal{L}$ as follows.

$$\begin{aligned}
 \frac{d\mathcal{L}}{dt} &= \left\langle \nabla \mathcal{L}, \frac{\partial C_1}{\partial t} \right\rangle \\
 &= \left\langle \langle \nabla g, \mathbf{T} \rangle \mathbf{T} + \langle \nabla g, \mathbf{N} \rangle \mathbf{N} - g\kappa \mathbf{N}, \frac{\partial C_1}{\partial t} \right\rangle \\
 &= \left\langle (\langle \nabla g, \mathbf{N} \rangle - g\kappa) \mathbf{N}, \frac{\partial C_1}{\partial t} \right\rangle + \underbrace{0}_{\mathbf{N} \cdot \mathbf{T} = 0} \\
 &= - \left\| \frac{\partial C_1}{\partial t} \right\|^2 \leq 0
 \end{aligned} \tag{15}$$

However, this does not hold for the Equilibrium Flow (13b). Alternatively, the following proposition holds true.

Proposition 3 *The curve evolution of (12) and the reparametrization $\frac{dp}{dt} = -\frac{\langle \mathbf{F}(p,t), \mathbf{N}(p,t) \rangle}{\|C_p\|}$ where $\mathbf{F} = -\nabla g$, leads to $g(C(p, t)) = g(C(p, t + \tau))$ for any τ before termination of the curve evolution.*

The proof is given in the Appendix. This proposition implies that the edge indicator function g for a chosen parametrization p will not change during the curve evolution driven by EF. Note that (15) holds regardless of the parametrization. Combining (15) and Proposition 3, the entire curve evolution system will thus minimize the value of g on the curve consistently, if the contour length term, $\int \|C_p\| dp$, can be omitted. Formally, we will have the following claim.

Proposition 4 *For $\frac{dp}{dt} = -\frac{\langle \mathbf{F}(p,t), \mathbf{N}(p,t) \rangle}{\|C_p\|}$, the curve evolution defined by the curve evolution equations in Eqs. (13a) and (13b) will follow:*

$$\frac{d}{dt} \int \log \circ g(C(p, t)) dp \leq 0.$$

Comparing the above to the full Geodesic energy (4), we can observe that the contour length term, $\int_C \|C_p\| dp$, is missing in the above. However, if we attach a penalty coefficient to the contour length term and let it approximate zero, the two energies will tend to be equivalent. Accordingly, the global convergence of the alternating curve evolution system Eqs. (13a) and (13b) is guaranteed if the two curve evolutions are convergent separately according to Lyapunov's theory on stability of dynamic system [34]. The above property is weaker than the monotonic minimization of the full GeoSnake energy (4). Therefore, we call it the weak monotonic minimization property for GeoSnake energy.

3.5 The algorithm

The problem we discussed previously is general, but we still focus on analyzing and solving problems with the GeoSnakes where $\beta = g\kappa - \langle \nabla g, \mathbf{N} \rangle$. We adopted the level set method [35] for implementation. In the implementation of the proposed method, we often require the image to be smoothed before the curve evolution. An edge map is then generated by using the smoothed image according to Eq. (2). Simultaneously, an initial contour is to be provided by the user. Afterwards, the curve evolution is run till convergence. The pseudo code is shown in Algorithm 1. Note that the maximum iteration is reached if the contour has little motion. In our experiment, the maximum iteration is set to be 2000 and the maximum cycle is set to be 3. The number of cycles will correspond to the occurrences of PSP. When cycle equals 3, we apply EF once since we foresee only one occurrence of PSP during the curve evolution.

4 Convergence analysis

The convergence of (13a), i.e. the GAC curve evolution, is straightforward. In this section, we elaborate on the convergence analysis for the Equilibrium Flow (13b).

Algorithm 1 Alternating Curve Evolutions

Input: Input Image I , MaximumCycle, MaximumIteration

Output: C_k

```

1:  $I_{sm} \leftarrow I$ , by anisotropic smoothing
2:  $g \leftarrow \nabla I_{sm}$ , by Equation 2
3:  $\mathbf{F} \leftarrow -g$ , by Gradient Vector Flow (GVF)
4:  $c = 0$ 
5: Initialize  $C_0$ 
6: while  $c \leq \text{MaximumCycle}$  do
7:    $k = 0$ 
8:   while  $k \leq \text{MaximumIteration}$  do
9:     if  $\text{mod}(\text{cycle}, 2) = 0$  then
10:       $C_{k+1} \leftarrow C_k$  by the direct gradient decent flow of Eq. (13a)
11:     else
12:       $C_{k+1} \leftarrow C_k$  by the EF, i.e., Eq. (14).
13:     end if
14:      $k = k + 1$ ;
15:   end while
16:    $c = c + 1$ ;
17: end while

```

4.1 Convergence analysis for general curve evolution

Before performing the analysis to our specific Equilibrium Flow, we establish a framework for analyzing the convergence for general curve evolutions. The general framework can then be applied to analyzing our Equilibrium Flow.

The objective is to ascertain whether the convergence is stable. Naturally, we consider the case of unstable as a lack of convergence. Given a curve evolution defined by the following,

$$\frac{\partial C(p, t)}{\partial t} = \langle \mathbf{F}, \mathbf{N} \rangle \mathbf{N}.$$

The convergence of the curve evolution is stated as follows:

$$\lim_{t \rightarrow \tau} \frac{\partial C(p, t)}{\partial t} = \langle \mathbf{F}, \mathbf{N} \rangle \mathbf{N} = 0.$$

The condition of convergence can be found by considering a small perturbation on the converged curve. If the perturbation finally vanishes at that converged curve, the curve evolution has converged to stationary, i.e. achieving stable convergence, otherwise it is unstable. We now examine the convergence of the curve evolutions following this idea.

The curve evolution near the converged curve C^* can be represented as follows.

$$C^* = C(p, t^*) = C(p, t)|_{t=t^*-\tau} + \eta(p, \tau), \quad (16)$$

where η is a general formulation of perturbation, $\tau = t^* - t > 0$ and is small enough and monotonically *decreasing* during the evolution. The fundamental condition of convergence is shown below.

Theorem 1 *If $J_{nn}(C^*) < 0$, the curve evolution is **stable**; If $J_{nn}(C^*) = 0$, the curve evolution is **marginally stable**; If $J_{nn}(C^*) > 0$ the curve evolution is **unstable**, where $J_{nn}(C^*) = [\mathbf{N}^T J[\mathbf{F}]\mathbf{N}](C^*)$, $J[\mathbf{F}]$ is the Jacobian of vector field $\mathbf{F}$.*

Accordingly, we can use $J[\mathbf{F}](C^*)$ to examine (the stability of) the convergence of curve evolution in vector field. The proof, together with some useful differential properties on curve evolutions, is deferred to the Appendix due to its length.

Suppose that the original curve evolution is unstable or marginally stable, a modification on the formulation of the curve evolution equation might be performed. By using the above theorem, it can be verified whether the modified curve evolution has converged to stationary. However, this leads to another question that whether the modified curve evolution maintains the function of the original one. The following theorem could be a fundamental answer to this question.

Theorem 2 *For a fixed p , such that $\frac{\partial S(p,t)}{\partial t} = \frac{\partial C(p,t)}{\partial t} + \mathbf{G}(p,t)$, and $\frac{\partial C(p,t)}{\partial t}$ is Lipschitz continuous with the Lipschitz constant L , Given that $\|\mathbf{G}(p,t)\| \leq \mu, \mu > 0$, then the following holds,*

$$\begin{aligned} \|C(p,t) - S(p,t)\| &\leq \|C(p,0) - S(p,0)\| e^{L(t-t_0)} \\ &\quad + \frac{\mu}{L} \{e^{L(t-t_0)} - 1\} \end{aligned}$$

The above theorem provides an upper bound of the difference between curve evolutions. If the difference is small, the curve evolutions will behave analogously, and we suppose the stationary solutions of them are also similar. We shall not include the proof here, since the proof of a more detailed and general bound for ordinary differential equations, related to Gronwall-Bellman inequality, is clearly stated in Chapter 3 of [34].

4.2 Convergence of Equilibrium Flow

Let us consider an auxiliary curve evolution equation:

$$\frac{\partial C}{\partial t} = \langle -\nabla g, \mathbf{N} \rangle \mathbf{N}, \quad (17)$$

where g is the edge indicator function. The normal projection of the Jacobian of vector field of $-\nabla g$, as required in Theorem 1, is

$$J_{nn}(C^*) = \langle -\nabla^2 g \mathbf{N}(C^*), \mathbf{N}(C^*) \rangle, \quad (18)$$

where $\nabla^2 g$ is the Hessian of g . The following holds true.

Lemma 2 *Given that the curve evolution $C(p,t)$ defined by Equation (17) near the converged curve $C(p,t^*)$, i.e. $C(p,t^* - \delta t)$ can be expanded by Taylor series at t^* , within a radius of convergence $|\delta t| < R$, where t^* is the time of convergence, and given that the initial curve $C(p,0)$ is not on maxima of g , then*

$$\langle -\nabla^2 g \mathbf{N}, \mathbf{N} \rangle \leq 0.$$

The proof can be found in the Appendix.

The above result shows that $J_{nn} \leq 0$ on C^* . According to Theorem 1, it also proves the following.

Proposition 5 *The curve evolution under $\mathbf{F} = \nabla g$ is stable or the worst marginally stable.*

We now turn to the equilibrium flow (13b). The normal projection of the Jacobian of the corresponding vector field, $-\mathcal{R}^T \nabla g$, is

$$J_{nn}(C^*) = \langle -\nabla^2 g \mathbf{N}(C^*), \mathbf{T}(C^*) \rangle. \quad (19)$$

Hence, we have the following equality at C^* :

$$\langle \nabla g, \mathbf{N} \rangle = \langle -\nabla g, \pm \frac{\nabla g}{\|\nabla g\|} \rangle = \pm \|\nabla g\|. \quad (20)$$

This is because the gradient vanishes in the tangential direction at any solution of Eq. (13b). Hence the gradient is in the normal direction. We now differentiate the above to obtain the following,

$$\frac{d}{ds} \pm \|\nabla g(C(s))\| = \pm \frac{\nabla g^T}{\|\nabla g\|} \nabla^2 g C_s = J_{nn}. \quad (21)$$

Note that $\mathbf{T} = C_s$. Therefore, the above proves the following.

Proposition 6 *If we normalize the gradient field, e.g., $\|\nabla g\| = 1$, then $J_{nn}(C^*) = 0$, hence, the EF is marginally stable.*

Let us consider the following modification of the Equilibrium Flow define by Equation (13b):

$$\frac{\partial C(p, t)}{\partial t} = -\langle \mathcal{R}^T \nabla g + \epsilon \nabla g, \mathbf{N} \rangle \mathbf{N}, \quad (22)$$

where $\epsilon > 0$ is chosen to be a constant. It can be shown that the above modified Equilibrium Flow have better property of convergence by the follows.

Proposition 7 *By normalizing $\|\nabla g\|$, e.g. $\|\nabla g\| = 1$, the curve evolution defined by Equation(22) is stable or at least marginally stable.*

The proof is straightforward based on the previous discussions.

Proof The vector field $\mathbf{F}$ is defined below.

$$\mathbf{F} = -(\mathcal{R}^T \nabla g + \epsilon \nabla g)$$

The Jacobian matrix is therefore the following.

$$\begin{aligned} \mathbf{N}^T J[\mathbf{F}] \mathbf{N} &= -\mathbf{N}^T (\mathcal{R}^T J[\nabla g]) \mathbf{N} - \mathbf{N}^T (\epsilon J[\nabla g]) \mathbf{N} \\ &= 0 + \epsilon g_{nn} \leq 0 \end{aligned}$$

which completes the proof. $\square$

We can also show that the slightly revised Equilibrium Flow (22) will not lose its function by showing that Equation (22) behaves analogous to the Equilibrium Flow Equation (13b) using Theorem 2 in which $\mathbf{G} = \epsilon \langle \nabla g, \mathbf{N} \rangle \mathbf{N}$ and ϵ can be small. Therefore, a curve evolution behaving analogously to the Equilibrium Flow but converging more stably is obtained.

5 Experimental results

In this section, we present the experimental results that validate our interpretation and formulation for solving the early termination problem. The proposed method is also compared with other related methods to show its practical usefulness.

5.1 Experiment settings

We describe the details of the parameter settings for implementation of the methods for preprocessing and comparison. The implementation details can be found in the original papers that are cited below.

Edge map[5, 6] The σ of the Gaussian kernel that is used to generate edge map g in Eq. (2) is empirically chosen to be 3 for all images.

GVF[18] The diffusion of the gradients of the edge map is done by GVF as a preprocessing for all the gradient-based methods compared in the experiments. We implement the GVF by using the source code provided in the GVF web site³. We select the parameter $\mu = 0.1$.

Balloon [17, 6] The Balloon used for comparison is chosen to be positive to shrink the curve. The penalty coefficient for the balloon term is chosen to be 1.

Adaptive Balloon[36, 15] For comparison, we adopt the version in [36], which is simpler for analysis comparing to [15], while [36] and [15] are almost equivalent. In our implementation $\lambda = 1$, $\beta = 0.1$.

Chan-Vese model[10] We set $\lambda_1 = \lambda_2 = 1$, $\mu = 0.1$ and $v = 0$ which is a Balloon term. The approximate Dirac delta function δ_ϵ and Heaviside function H_ϵ are chosen as in [10]. The Chan-Vese model can be solved with better quality because of the recent works on convexification, e.g. [37]. We adopt the original level set implementation for fair comparison.

MAC [16] We use Canny edge detection as a preprocessing to specify the current along the boundaries of possible objects, and we set the α in their paper to be 0.5 for all the images.

Regularized edge integration [7] We use $\rho(\cdot) = |\cdot|$ in the edge integration formulation. We use the unified model which is defined by Eq. (53) in [7]. We set β in the unified model to be 1.

GeoCut [26] We use the cut metric $D(p) = \text{diag}(1, g(|\nabla I|))$ where $g(x) = e^{-x^2}$ and I is the image, which is the same as the formulation in their paper. The weights for graph edges are computed by Eq. (6) in [26].

³ <http://iacl.ece.jhu.edu/projects/gvf/>

Fig. 5: Segmentation of the U-shape. (a) is the initialization. (b) is the PSP due to the curve evolution by gradient descent. (c)-(f) is the curve evolution by EF. The curve in (f) is also stationary to the curve evolution by gradient descent.

Total variation regularized active contour [38] We adopt the source code provided by X. Bresson⁴ with the default parameters.

Approximated Dirac delta The approximated delta function δ_ϵ for level set implementation, excluding Chan-Vese model, are chosen to be as,

$$\delta_1(x) = \begin{cases} 0, & |x| > 1 \\ \frac{1}{2}(1 + \cos(\pi x)), & |x| \leq 1 \end{cases}.$$

Re-initialization We adopted the 1st order accurate Essentially Non-Oscillatory (ENO1) based on Baris Sumengen's MATLAB toolbox of the Level Set method⁵.

The algorithms are implemented in MATLAB. The coding of the active contours is largely inspired by Chunming Li's implementation⁶.

5.2 Segmentation results

Before presenting the main results of the segmentation, the proposed method is demonstrated for segmentation of the U-shape in Figure 2. The result is shown in Figure 5.

The other images used for evaluating the proposed method are shown in Figure 7 overlaid with the red initial curves. The same initializations are used in all the comparisons. The initial curves are partially inside the objects and partially outside, which means that solely shrinking or solely expanding the initial curves cannot extract the objects. The image sizes can be found in Table 1. The chosen images are of different characteristics. The image of two rectangles in Figure 7(a) is relatively simple for segmentation since the gray level of both foreground and background are homogeneous and distinct, the edges are strong too. Besides, there are only two objects, hence the topology is simple. In Figure 7(b), the image is also simple except that there are three objects, which is a little more difficult than the former one. In Figure 7(c), there are nine curve segments forming three circles. The edge-based active contours may see

⁴ <https://googledrive.com/host/0B3BTLeCYLunCc1o4YzV1Ui1SeVE/codes.html>

⁵ http://barissumengen.com/level_set_methods/

⁶ <http://www.engr.uconn.edu/~cml/>

three circles of disconnected boundaries in the image, and the region-based method may see the objects in the image as composed of nine narrow and curved regions. Figures 7(d) and 7(e) are MRI medical images of human brain with multiple tumor regions. The original brain scans are shown in Figure 6. From Figures 7(d) and (e), we see that there are many spurious curved-edges in the images. Besides, there are some regions with intensities similar to the targeted tumor regions. These make the region-based active contour to be unsuitable for dealing with such images. However, for edge-based active contours, there also exists the Pseudo Stationary Phenomenon.

Fig. 6: The original real images and the tumor regions (in boxes)

Fig. 7: Inputs and initializations

Fig. 8: Gradient fields, PSP and Saddle points

Figure 8 shows the *normalized* gradient fields of the edge indication functions corresponding to the images in Figures 7 (a), (b) and (c) respectively. The purpose of this figure is to demonstrate the PSP. We also visualize the saddle points by the blue dots on the red lines of which the gradients are in the tangential direction. Subsequently, we study the early termination of the curve evolutions, i.e. the PSP of the GeoSnakes, near those lines.

Fig. 9: Object Extraction by proposed method. The top row is the curve evolution by GAC; The middle is the curve evolution by EF; The bottom row is the final convergence.

The curve evolution processes of the system of alternating curve evolutions following (13), called the GAC+EF, are shown in Figure 9. The objects are correctly segmented. Figure 10 shows the segmentation results produced by other methods, including the GAC, GAC with Balloon, GAC with adaptive Balloon, MAC and the Chan-Vese model, with the same initial curve as shown in 7. It can be observed that none of these compared methods can extract all the objects accurately like GAC+EF. GAC suffers from the early stop of the curve evolution. GAC with Balloon can surpass edges and converge to the interior boundaries or vanish; GAC with adaptive Balloon can be applied for extracting two objects of simple topology as shown in Figure 10, yet it still suffers from the PSP in other images. The gradient field used for the former three methods are extended by Gradient Vector Flow. Interestingly, both the GAC and the GAC with the adaptive Balloon might stop quite close to the pseudo stationary positions as shown in Figure 3, while GAC with adaptive Balloon can escape from some pseudo stationary positions of relatively simple topology. It can also be observed that both MAC and the Chan-Vese model can extract the narrow curved regions in the image of three disconnected circles as shown in Figure 10. It implies that MAC shares certain characteristics of the region-based methods. The edge detection process in MAC does not produce a local gradient field but the indication of intensity homogeneity. The Chan-Vese active contour is not sensitive to initialization

but it often captures other undesired regions. The MAC generally fails to converge to the object boundaries if there are too many edges and corners distributed everywhere in the images.

Fig. 10: Results by other methods. First column from the left is the object extraction by GAC; Second column is the object extraction by GAC with Balloon; Third column is the object extraction by GAC with adaptive Balloon; Fourth column is the object extraction by MAC; Last column is the object extraction by Chan-Vese active contour.

Additionally, we implement the regularized edge integration model [7], GeoCut [26] which is a graph cut based method which conceptually related to GAC, and the total variation regularized active contour (TVAC) model [38]. The three methods are regarded as alternatives to the conventional edge-based active contours. The results of regularized edge integration model, GeoCut and TVAC are shown in Figs. 11, 12 and 13. We observe that the regularized edge integration is able to locating the edges more accurately than GAC, although the early termination problem remains.

The accuracies of the global optimal GeoCut and TVAC are also not very satisfactory, because of false detection of spurious regions and/or missing true regions. The TVAC can produce spurious regions and the labels for the object regions can be either 1 or 0. This is because this method is an automatic object segmentation approach which was not designed to incorporate user's preference of object.

Fig. 11: Segmentation by regularized edge integration [7]

Fig. 12: Segmentation by GeoCut [26]. The first row shows the initial seeds for GeoCut

Fig. 13: Segmentation by TVAC [38]. Note that TVAC does not require initial seeds.

Figure 14 visualizes the 3D tumor model that is obtained by the layer-wise segmentation of the entire MRI sequence of Figure 7(e) using GAC+EF. The cutting plate corresponds to the segmentation result shown in the bottom right corner in Figure 9. The final result has been smoothed by using the `smooth3` function in MATLAB.

Fig. 14: The reconstructed 3D tumor from the MRI sequence of Figure 7(e). The cutting plate corresponds to the segmentation result shown in the right most figure in Figure 9(c)

5.3 The tangential and normal velocities

In the following, a quantitative analysis of the proposed method is provided. From the quantitative results, the stable convergence of the curve evolution algorithms can also be observed. One major contribution in this paper is about the proposed solution, namely the EF, to the Pseudo Stationary Phenomenon that occurs if the tangential velocity on the converged curve does not vanish. The quantitative results demonstrate that the proposed method can reduce the tangential velocity effectively. The tangential velocity in the curve evolutions of the proposed method is shown in the second row from top in Figure 15. The tangential velocity is quite large when GAC converges (0-k1), but due to the EF (k2-k4) the tangential velocity approaches zero, where k3 is the iteration of convergence. The velocities do not reach zero exactly because of the errors in discretization. The normal velocity in the third row from top in Figure 15 shows that the normal velocity drops significantly during the direct curve evolutions of the gradient descent flow. The iterations corresponding to k1, k2, k3, and k4 in Figure 3.12 are given in Table 1. The computational times for the curve evolutions are also presented. The computational cost can be reduced by using fast implementations, e.g. [39–41].

6 Conclusions and future work

This paper addresses the issue of early termination of curve evolution for images containing moderately complex structures and relatively homogeneous regions. This work provides new geometric insights of the problem of early termination of the curve evolution for general edge-based active contours, giving rise to new criteria and solution of edge-based active contours.

The PSP problem identified in this work may not happen only in object segmentation. It may also exist in other computer vision or computer graphics tasks involving optimizing closed curves. Hence, the proposed method may be extended to address the PSP problem in the different contexts. The proposed method is derived based on

Fig. 15: The normal and tangential velocities during curve evolutions. The top row shows the total magnitude of the tangential velocity along the curves; The second row shows the total magnitude of the normal velocity along the curves. k_1, k_2, k_3, k_4 are some key iterations in the curve evolutions. 0- k_2 is the curve evolution by GAC; k_2 - k_4 is by EF; from k_4 onwards is by GAC

geometrical observations and analysis of the PSP problem. One may also approach the PSP problem from the perspective of repositioning a pseudo-stationary curve.

A Appendix

A.1 Proof of Proposition 3

Proof Substituting (12) and $\frac{dp}{dt}$ into $\frac{dC}{dt}$, we obtain the following.

Table 1: Image sizes, key iterations in Figure 15, and time costs of the curve evolutions

Input image	Image size (px)	k1	k2	k3	k4	Total time (s)	GAC time (s)	EF time (s)
80×80	150	300	900	1800	62.44	4.68	18.73
80×80	230	–	800	–	62.33	7.17	15.58
80×80	200	–	920	–	63.34	6.33	19.64
128×128	110	–	1600	–	119.44	6.57	77.64
128×128	120	–	850	–	120.23	7.21	33.06

$$\begin{aligned}
\frac{dC}{dt} &= \frac{\partial C}{\partial t} + \frac{\partial C}{\partial p} \frac{dp}{dt} \\
&= \langle \mathbf{F}, \mathbf{T} \rangle \mathbf{N} - \langle \mathbf{F}, \mathbf{N} \rangle \mathbf{T} \\
&= \langle \mathbf{F}, \mathcal{R}^T \mathbf{N} \rangle \mathbf{N} + \langle \mathbf{F}, \mathcal{R}^T \mathbf{T} \rangle \mathbf{T} \\
&= \langle \mathcal{R} \mathbf{F}, \mathbf{N} \rangle \mathbf{N} + \langle \mathcal{R} \mathbf{F}, \mathbf{T} \rangle \mathbf{T} \\
&= -\mathcal{R} \nabla g(p, t)
\end{aligned} \tag{23}$$

where $\mathcal{R}$ is a 90° rotation matrix of size 2×2 . Thus, taking derivative of g w.r.t. t we obtain the following.

$$\frac{dg}{dt} = \nabla g(p, t) \frac{dp}{dt} = -\langle \nabla g(p, t), \mathcal{R} \nabla g(p, t) \rangle = 0 \tag{24}$$

which completes the proof. $\square$

A.2 The mathematical analysis of the convergence of curve evolution

The following results are basically obtained based on the Lemma of curve evolution, i.e. Lemma 1, and the idea of using the zero level set of a signed distance function to represent closed curves. The main purpose of this appendix is to establish the proof of Theorem 1.

A.2.1 Elementary differential properties of curve evolution

Lemma 3 Given the curve evolution $C(p, t)$ defined as follows,

$$\frac{\partial C}{\partial t} = \epsilon(p, t) \mathbf{N}(p, t),$$

the following holds

$$\frac{\partial^{(k+1)} C}{\partial t^{(k+1)}} = \frac{\partial^k \epsilon(p, t)}{\partial t^k} \mathbf{N}(p, t). \quad (25)$$

To prove the above lemma, we need the lemma below. The proof of Lemma 3 is straightforward by following Lemma 4 and the chain rule of differentiation.

Lemma 4

$$\frac{d\mathbf{N}}{dC}(C) \mathbf{N}(C) = 0. \quad (26)$$

Proof Firstly, we write $\frac{d\mathbf{N}}{dC}(C)$ more explicitly as follows,

$$\frac{d\mathbf{N}}{dC}(C) = \begin{pmatrix} \frac{dN^x}{dC}(C) & \frac{dN^y}{dC}(C) \\ \frac{dN^y}{dy}(C) & \frac{dN^x}{dy}(C) \end{pmatrix} \quad (27)$$

where $\mathbf{N} = [N^x, N^y]^T$.

Using the relation $\frac{\nabla \Phi}{\|\nabla \Phi\|} = \mathbf{N}$ and $\|\nabla \Phi\| = 1$ for signed distance function Φ defined by $C = \{x, y | \Phi(x, y) = 0\}$, we obtain the following.

$$\nabla^2 \Phi = \frac{d\mathbf{N}}{dC}(C) \quad (28)$$

i.e., $\frac{d\mathbf{N}}{dC}(C)$ is the Hessian matrix of Φ at the position C .

Then taking derivative of $\|\nabla \Phi\| = 1$, we have the following.

$$\nabla \|\nabla \Phi\| = (\nabla^2 \Phi) \nabla \Phi = \mathbf{0} \quad (29)$$

which completes the proof. $\square$

Now we prove Theorem 3 as follows.

Proof We prove by induction.

For $k=1$, we have

$$\frac{\partial C}{\partial t} = \epsilon(p, t) \mathbf{N}(p, t) \quad (30)$$

Taking derivative w.r.t t , we obtain the following.

$$\begin{aligned} \frac{\partial^2 C}{\partial t^2} &= \frac{\partial \epsilon(p, t)}{\partial t} \mathbf{N}(p, t) + \epsilon(p, t) \frac{\partial \mathbf{N}}{\partial t} \\ &= \frac{\partial \epsilon(p, t)}{\partial t} \mathbf{N}(p, t) + \epsilon(p, t) \frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t} \\ &= \frac{\partial \epsilon(p, t)}{\partial t} \mathbf{N}(p, t) + \epsilon(p, t) \frac{d\mathbf{N}}{dC} \epsilon(p, t) \mathbf{N}(p, t) \\ &= \frac{\partial \epsilon(p, t)}{\partial t} \mathbf{N}(p, t), \text{ by lemma 4} \end{aligned}$$

Now we set $k = n$, we have,

$$\frac{\partial^{(n)} C}{\partial t^{(n)}} = \frac{\partial \epsilon^{(n-1)}(p, t)}{\partial t^{(n-1)}} \mathbf{N}(p, t)$$

Hence, taking derivative and by lemma 4, we have

$$\frac{\partial^{(n+1)} C}{\partial t^{(n+1)}} = \frac{\partial \epsilon^{(n)}(p, t)}{\partial t^{(n)}} \mathbf{N}(p, t)$$

which completes the proof. $\square$

The above results eventually lead us to a more practical result that is stated as follows, which can be proved by substituting Eq. (25) into the Taylor series.

Lemma 5 *If there exists $R > 0$, such that $\forall |\tau| < R$, such that the Taylor series of $C(p, t + \tau)$ at $C(p, t)$ exists, then there exists a unique $\xi(p, \tau)$, such that,*

$$C(p, t + \tau) = C(p, t) + \tau \xi(p, \tau) \mathbf{N}(p, t), \quad (31)$$

where

$$\xi(p, \tau) = \sum_{k=1}^{\infty} \frac{\tau^{k-1}}{k!} \frac{\partial^{(k-1)} \epsilon(p, t)}{\partial t^{(k-1)}}.$$

By this lemma, we convert the curve evolution in the form of PDE to an algebraic equation. This may facilitate the analysis of curve evolutions.

Proof The proof is constructive.

We expand $C(p, t + \tau)$ at $C(p, t)$ by Taylor series as follows.

$$C(p, t + \tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \frac{\partial^k C(p, t)}{\partial t^k}$$

By Theorem 3, we can rewrite the above as follows.

$$\begin{aligned} C(p, t + \tau) &= C(p, t) + \left\{ \sum_{k=1}^{\infty} \frac{\tau^k}{k!} \frac{\partial^{(k-1)} \epsilon(p, t)}{\partial t^{(k-1)}} \right\} \mathbf{N}(p, t) \\ &= C(p, t) + \tau \xi(p, \tau) \mathbf{N}(p, t) \end{aligned}$$

Since the Taylor expansion exists, hence unique. By setting

$$\xi(p, \tau) = \sum_{k=1}^{\infty} \frac{\tau^{k-1}}{k!} \frac{\partial^{(k-1)} \epsilon(p, t)}{\partial t^{(k-1)}}$$

We complete the proof. $\square$

A.2.2 Proofs of theorems on convergence

Proof (Proof of Theorem 1) We expect to obtain a differential equation that defines a contraction mapping for the perturbation, we then can show that the perturbation contracts to zero.

Firstly, according to Taylor's theorem, the curve evolution equation can be reformulated as follows.

$$\begin{aligned} \frac{\partial C}{\partial t} \Big|_{t=t^*} &= 0 \\ &= \frac{\partial C}{\partial t} \Big|_{t=t^*-\tau} + \tau \frac{\partial}{\partial t} \frac{\partial C}{\partial t} \Big|_{t=t^*-\tau} + O(\tau) \end{aligned} \quad (32)$$

This equation is the basic form for us to construct the contraction mapping. Our idea is to use η to replace the C in the equation to obtain our formulation.

Noting that

$$t = t^* \Leftrightarrow \tau = 0, \text{ and } \frac{dt}{d\tau} = \frac{d\tau}{dt} = -1$$

Differentiating Eq. (16), we can write the following,

$$\frac{\partial \eta(p, \tau)}{\partial \tau} = - \frac{\partial C}{\partial t} \Big|_{t=t^*-\tau} (-1) = f \mathbf{N} \quad (33)$$

where $f\mathbf{N}$ is an auxiliary term that can facilitate the derivation. The form of $f\mathbf{N}$ is from Lemma 1. This relationship allows us to replace C with η for one of the terms of Eq. (32) to construct the contraction mapping defined by a differential equation.

Now we focus on replace the C with η for the other term in Eq. (32). By making use of the Eq. (26), we have

$$\begin{aligned}
& \left. \frac{\partial}{\partial t} \frac{\partial C}{\partial t} \right|_{t=t^*-\tau} = \left. \frac{\partial}{\partial t} \langle \mathbf{F}(C), \mathbf{N}(C) \rangle \mathbf{N}(C) \right|_{t=t^*-\tau} \\
& = \left. \left\langle J[\mathbf{F}](C) \frac{\partial C}{\partial t}, \mathbf{N}(C) \right\rangle \mathbf{N}(C) \right|_{t=t^*-\tau} \\
& + \left. \left\langle \mathbf{F}(C), \underbrace{\frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t}}_{=0} \right\rangle \mathbf{N}(C) \right|_{t=t^*-\tau} \\
& + \left. \langle \mathbf{F}(C), \mathbf{N}(C) \rangle \underbrace{\frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t}}_{=0} \right|_{t=t^*-\tau} \\
& = \left[\mathbf{N}^T \left(J[\mathbf{F}] + \frac{d\mathbf{N}}{dC} \right) f\mathbf{N} \right] (C) \mathbf{N}(C) \\
& = \left[\mathbf{N}^T \left(J[\mathbf{F}] + \frac{d\mathbf{N}}{dC} \right) \mathbf{N} \right] (C) f\mathbf{N}(C) \\
& = \left[\mathbf{N}^T J[\mathbf{F}] \mathbf{N} \right] (C) \frac{\partial \eta}{\partial \tau} (C) \Big|_{\tau=t^*-t}
\end{aligned} \tag{34}$$

where $\frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t} = 0$ is by Lemma 4;

We now need the Taylor expansion of η as follows.

$$\begin{aligned}
\eta(p, 0) &= \eta(p, \tau) - \tau \frac{\partial \eta}{\partial \tau} + O(\tau) \\
&\Leftrightarrow \\
\eta(p, \tau) &= \tau \frac{\partial \eta}{\partial \tau} + O(\tau)
\end{aligned} \tag{35}$$

where $\eta(p, 0) = 0$.

Substituting the above into Eq.(34), we obtain the following.

$$\begin{aligned}
& \left. \frac{\partial C}{\partial t} \right|_{t=t^*-\tau} = -\tau \left. \frac{\partial}{\partial t} \frac{\partial C}{\partial t} \right|_{t=t^*-\tau} + O(\tau) \\
& = -[\mathbf{N}^T J[\mathbf{F}] \mathbf{N}](C) \eta(p, \tau) + O(\tau)
\end{aligned} \tag{36}$$

Combining the above with Eq. (33), we finally arrive at the following.

$$\begin{aligned}
& \left. \frac{\partial \eta}{\partial \tau} \right|_{\tau=t^*-t} = -[\mathbf{N}^T(C) J[\mathbf{F}](C) \mathbf{N}(C)] \eta(p, \tau) \\
& \quad + O(\tau) \\
& = -J_{nn}(C) \eta(p, \tau) + O(\tau)
\end{aligned} \tag{37}$$

where

$$J_{nn}(C) = [\mathbf{N}^T J[\mathbf{F}] \mathbf{N}](C) \tag{38}$$

Now we almost obtain the differential equation in terms of η , but $C = C(p, t^* - \tau)$ can be very arbitrary. We therefore establish the following approximation.

$$\begin{aligned}
Q_{nn}(C) &= [\mathbf{N}^T J[\mathbf{F}]\mathbf{N}](C) \\
&- \tau \left[\underbrace{\frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t}}_{=0} J[\mathbf{F}]\mathbf{N} \right] (C) - \tau \left[\mathbf{N}^T \left[\frac{dJ[\mathbf{F}]}{dC} \otimes \frac{\partial C}{\partial t} \right] \mathbf{N} \right] \\
&- \tau \left[\mathbf{N}^T J[\mathbf{F}] \underbrace{\frac{d\mathbf{N}}{dC} \frac{\partial C}{\partial t}}_{=0} \right] + O(\tau) \\
&= [\mathbf{N}^T J[\mathbf{F}]\mathbf{N}](C^*) - \tau \left[\mathbf{N}^T \left[\frac{dJ[\mathbf{F}]}{dC} \otimes \frac{\partial C}{\partial t} \right] \mathbf{N} \right] + O(\tau) \\
&\approx [\mathbf{N}^T J[\mathbf{F}]\mathbf{N}](C^*)
\end{aligned} \tag{39}$$

where $\frac{dJ[\mathbf{F}]}{dC}$ is the second order derivative of the vector field, which is relatively small comparing to $J[\mathbf{F}]$.

To explain how the differential equation (37) and the approximation Eq. (39) can define a contraction mapping, and thus can be used for testing of convergence, we discretize the differential equation and neglect the higher order terms $O(\tau)$ to obtain the following.

$$\begin{aligned}
\eta^k(p) &= \eta^{k-1}(p) - \Delta \tau J_{nn}(C^*) \eta^{k-1}(p) \\
&= (1 - \Delta \tau J_{nn}(C^*)) \eta^{k-1}(p)
\end{aligned} \tag{40}$$

Then we take norm of the above to obtain the following,

$$\begin{aligned}
\|\eta^k(p)\| &= |1 - \Delta \tau J_{nn}(C^*)| \|\eta^{k-1}(p)\| \\
&= |1 - \Delta \tau J_{nn}(C^*)|^k \|\eta^0(p)\|
\end{aligned} \tag{41}$$

Noting that $\tau = t^* - t$ is decreasing, then $\Delta \tau < 0$. Hence, i.i.f $|1 - \Delta \tau J_{nn}(C^*)| < 1$, we will have a contraction mapping such that $\lim_{t \rightarrow \infty} \|\eta^k(p)\| = 0$, i.e., η will vanish. And i.i.f $J_{nn}(C^*) < 0$, $|1 - \Delta \tau J_{nn}(C^*)| < 1$, which completes the proof. $\square$

Proof (Proof of Lemma 2) Let us consider the function $\mathcal{G}(t) = g(C(p, t))$ for any fixed p , then by substituting in Eq. (17), $d\mathcal{G}(t)/dt = -\|\langle \nabla g, \mathbf{N} \rangle\|^2 \leq 0$. Hence, g is non-increasing w.r.t t . Assuming $C(p, t^*)$ is the converged solution of Eq. (17) where for all $\delta t > 0$, $\langle \nabla g, \mathbf{N} \rangle|_{t=t^*} = 0$, we have that $g(C(p, t^*)) \geq g(C(p, t^* - \delta t))$.

By Theorem 5, there exists $\xi(p)$ such that the following relation between $C(p, t^* - \delta t)$ and $C(p, t^*)$ holds:

$$C(p, t^* - \delta t) = C(p, t^*) + \xi(p) \mathbf{N}(C(p, t^*)).$$

The Taylor expansion of g at the converged solution $C(p, t^*)$, can be written as follows:

$$g(C(p, t^* - \delta t)) = g(C(p, t^*)) + \frac{\xi^2}{2} \langle \nabla^2 g \mathbf{N}, \mathbf{N} \rangle|_{t=t^*} + O(|\xi|^2)|_{t=t^*}. \tag{42}$$

Since $g(C(p, t^*)) \leq g(C(p, t^* - \delta t))$, we have

$$\langle -\nabla^2 g \mathbf{N}, \mathbf{N} \rangle|_{t=t^*} \leq 0,$$

which completes the proof. $\square$

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