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COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE IN SEVERAL VARIABLES

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ABSTRACT. We characterize all selfadjoint as well as all unitary anti-linear weighted composition operators acting on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$. Then we obtain a complete description of anti-linear weighted composition operators that are conjugations. These results allow us to determine which bounded linear weighted composition operators can be complex symmetric on $\mathcal{F}^2(\mathbb{C}^n)$.

1. INTRODUCTION

1.1. Basic Definitions and Notations. Let $\mathcal{H}$ be a complex separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a corresponding norm $\| \cdot \|$.

Definition 1.1. A mapping T acting on $\mathcal{H}$ is said to be *anti-linear* (also *conjugate-linear*), if

$$T(ax + by) = \bar{a} T(x) + \bar{b} T(y), \quad \forall x, y \in \mathcal{H}, \forall a, b \in \mathbb{C}.$$

The adjoint T^* of a bounded anti-linear operator T is defined by

$$\langle Tx, y \rangle = \langle T^*y, x \rangle, \quad \forall x, y \in \mathcal{H}.$$

By Riesz's lemma, T^* exists uniquely, and is also bounded anti-linear. We say that T is *selfadjoint* if $T = T^*$, *unitary* if $T^{-1} = T^*$, and *invertible* if there exists a bounded anti-linear operator S such that $ST = TS = I$, the identity operator.

Definition 1.2. An anti-linear mapping $\mathcal{C}: \mathcal{H} \rightarrow \mathcal{H}$ is called a *conjugation*, if it is

- (1) involutive: $\mathcal{C}^2 = I$;
- (2) isometric: $\|\mathcal{C}x\| = \|x\|$, for all $x \in \mathcal{H}$.

Note that an anti-linear operator $\mathcal{C}$ is a conjugation if and only if $\mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^*$, that is, it is both unitary and selfadjoint. This fact is essentially used in the sequel.

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Definition 1.3. A bounded linear operator T on $\mathcal{H}$ is called *complex symmetric*, if there exists a conjugation $\mathcal{C}$ on $\mathcal{H}$, such that $T = \mathcal{C}T^*\mathcal{C}$. In this case, T is often called a $\mathcal{C}$ -*symmetric operator*, or *CSO*, for short.

The theory of complex symmetric operators, which has recently received a lot of attention, was initiated by Garcia and Putinar in [7, 8] with the motivation coming from function theory and matrix analysis. The class of complex symmetric operators is unexpectedly large. It includes, in particular, the Volterra integration operators, normal operators, Hankel operators, compressed Toeplitz operators, etc.

1.2. Fock spaces of entire functions. Let $n \geq 1$ be a natural number. The Fock space $\mathcal{F}^2(\mathbb{C}^n)$ consists of all entire functions in $\mathbb{C}^n$ for which

$$\|f\| = \left(\frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dV(z) \right)^{1/2} < \infty,$$

where dV is the Lebesgue measure on $\mathbb{C}^n$. It is a reproducing kernel Hilbert space, with the inner product $\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} dV(z)$ and kernel $K_z(u) = e^{\langle u, z \rangle}$. Note that the norm in the definition of Fock space is equivalent to the norm induced by the inner product above.

Since $\langle f, K_z \rangle = f(z)$, we always have

$$(1.1) \quad |f(z)| \leq \|f\| \cdot \|K_z\| = e^{\frac{|z|^2}{2}} \|f\|, \quad z \in \mathbb{C}^n.$$

We also use the normalized kernel

$$(1.2) \quad k_u(z) = \frac{K_u(z)}{\|K_u\|} = e^{\langle z, u \rangle - \frac{|u|^2}{2}}, \quad u, z \in \mathbb{C}^n.$$

1.3. Weighted Composition Operators. Let $\psi \not\equiv 0$ be a holomorphic function in some domain $G \subseteq \mathbb{C}^n$ and φ a holomorphic self-mapping on G . A *linear weighted composition operator* $W_{\psi, \varphi}$ induced by ψ and φ is defined as follows

$$W_{\psi, \varphi} f = \psi \cdot f \circ \varphi.$$

The interaction between the operator theoretic properties of $W_{\psi, \varphi}$ and the function theoretic properties of ψ and φ (such as boundedness, compactness, compact difference, essential norm, etc.) has attracted researchers. There are many papers and books devoted to this topic.

What concerns complex symmetric weighted composition operators, so far as we know, there seems to be very few articles. Garcia and Hammond for the first time raised the question about whether there exist weighted composition operators that are complex symmetric. The answer is affirmative: these authors in [6] and Jung et al. in [12] discovered independently a structure of complex symmetric linear weighted

composition operators on Hardy spaces in the unit disk, for a specific conjugation

$$(1.3) \quad \mathcal{J}f(z) = \overline{f(\bar{z})}.$$

In [16] Wang and Yao extended some results to Hardy spaces in the unit ball, with the same conjugation $\mathcal{J}$. For general conjugations, the complex symmetry of weighted composition operators was investigated in [1, 4, 5, 15] when the symbol φ is the automorphic function or the linear fractional function.

In our recent paper [9], we studied the complex symmetry of bounded linear weighted composition operators on the Fock space $\mathcal{F}^2(\mathbb{C})$ of entire functions. Inspired by the conjugation $\mathcal{J}$, we introduced *general anti-linear* weighted composition operators of the form

$$(\mathcal{A}_{\xi,\eta}f)(z) = \xi(z)\overline{f(\eta(z))},$$

where $\xi \not\equiv 0$ and η are entire functions on $\mathbb{C}$. This contains, with $\xi(z) \equiv 1$ and $\eta(z) = z$, the conjugation $\mathcal{J}$ above as a very special case.

We have succeeded to characterize all *isometric* as well as all *involution* anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C})$, which together give all the so-called *weighted composition conjugations*. These results allowed us to obtain a complete characterization of the bounded linear weighted composition operators that are complex symmetric with respect to weighted composition conjugations.

It is worth to note that the results in [9] were essentially used in [10], where we introduced the new concept in dynamical systems: the complex symmetric C_0 -semigroups on Hilbert spaces, and studied these semigroups on the Fock space $\mathcal{F}^2(\mathbb{C})$.

The aim of the present paper is to investigate complex symmetry of linear weighted composition operators on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ of entire functions of several variables with $n \geq 2$.

It should be noted that our approach for the multi-dimensional case is *different* from that used previously in [9]. Namely, our solution is based on the notice that an anti-linear operator $\mathcal{C}$ is a conjugation if and only if it is both unitary and selfadjoint. We are able to characterize all *unitary* as well as all *selfadjoint* anti-linear weighted composition operators to obtain the structure of weighted composition conjugations on $\mathcal{F}^2(\mathbb{C}^n)$. These results allow us to determine which of bounded linear weighted composition operators are complex symmetric on $\mathcal{F}^2(\mathbb{C}^n)$. In comparison with the paper [9], the method used in the present paper is more useful for studying a complex symmetry on those function spaces, including Hardy spaces, where characterizations for unitariness were studied successfully.

We note that some of our results were announced in [11].

2. PRELIMINARIES

In a study of complex symmetric operators, boundedness as well as unitariness are among the most important features. In [13] Le investigated a bounded property of $W_{\psi,\varphi}$ on the Fock space $\mathcal{F}^2(\mathbb{C})$ and a unitary property on some Hilbert spaces in the unit ball of $\mathbb{C}^n$. Later, following Le's ideas, Zhao [17, 18] extended some of those results to $\mathcal{F}^2(\mathbb{C}^n)$, under the *additional assumption* that the symbol function $\varphi(z) = Az + b$ is injective. This assumption is equivalent to invertibility of a matrix A .

To state these results, which are summarized in the proposition below, for convenience, we put

$$M(\psi, \varphi) = \sup_{z \in \mathbb{C}^n} M_z(\psi, \varphi) \quad \text{and} \quad m(\psi, \varphi) = \inf_{z \in \mathbb{C}^n} M_z(\psi, \varphi),$$

where $M_z(\psi, \varphi) = |\psi(z)|^2 e^{|\varphi(z)|^2 - |z|^2}$, $z \in \mathbb{C}^n$.

Proposition 2.1 ([17, 18]). *Let $\psi \not\equiv 0$ be an entire functions on $\mathbb{C}^n$, and let φ be a holomorphic self-mapping on $\mathbb{C}^n$.*

- (1) *If $M(\psi, \varphi) < \infty$, then $\varphi(z) = Az + b$, where A is a linear operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$ and $b \in \mathbb{C}^n$. When A is unitary, there exists a constant $c \in \mathbb{C}$ such that $\psi(z) = ce^{-\langle Az, b \rangle}$.*
- (2) *If $W_{\psi,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then there exists a linear operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$ and $b \in \mathbb{C}^n$, such that $\varphi(z) = Az + b$ and $M(\psi, \varphi) < \infty$. In particular, when A is invertible, these two conditions together are sufficient also.*
- (3) *$W_{\psi,\varphi}$ is bounded invertible on $\mathcal{F}^2(\mathbb{C}^n)$, if and only if there exists an invertible linear operator A on $\mathbb{C}^n$ with $\|A\| = 1$, $b \in \mathbb{C}^n$, such that $\varphi(z) = Az + b$ and $0 < m(\psi, \varphi) \leq M(\psi, \varphi) < \infty$.*
- (4) *$W_{\psi,\varphi}$ is unitary on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if there exists an unitary operator A on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C} \setminus \{0\}$, such that $\varphi(z) = Az + b$ and $\psi(z) = ce^{-\langle Az, b \rangle}$, $z \in \mathbb{C}^n$, with $|c|^2 e^{|b|^2} = 1$.*

We notice that an invertible operator A in Proposition 2.1(3) satisfies $\|A^{-1}\| = 1$. Also such a square matrix must necessarily be unitary. This phenomenon is easily generalized to a general situation.

Lemma 2.2. *A bounded linear operator U , acting on a Hilbert space $\mathcal{H}$, is unitary if and only if U is invertible with $\|U\| = \|U^{-1}\| = 1$.*

Proof. The necessity is clear, because U is isometric: $\|Ux\| = \|x\|$, for all $x \in \mathcal{H}$. For the sufficiency, suppose that U is invertible with $\|U\| = \|U^{-1}\| = 1$. If $x \in \mathcal{H}$, then

$$\|Ux\| \leq \|U\| \cdot \|x\| = \|x\| = \|U^{-1}Ux\| \leq \|U^{-1}\| \cdot \|Ux\| = \|Ux\|,$$

which implies that U is isometric, and so it is unitary. $\square$

Lemma 2.2 allows us to precise the important property of the matrix A in the characterization of bounded invertible $W_{\psi,\varphi}$.

Proposition 2.3. *A weighted composition operator $W_{\psi,\varphi}$ induced by the symbols ψ and φ , is bounded invertible on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if*

$$(2.1) \quad \varphi(z) = Az + b, \quad \text{and} \quad \psi(z) = ce^{-\langle Az, b \rangle}, \quad z \in \mathbb{C}^n,$$

where A is a linear unitary operator on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C} \setminus \{0\}$.

Proof. Suppose $W_{\psi,\varphi}$ is bounded invertible on $\mathcal{F}^2(\mathbb{C}^n)$. By Proposition 2.1(3) and Lemma 2.2, $\varphi(z) = Az + b$, where A is a linear unitary operator on $\mathbb{C}^n$ and $b \in \mathbb{C}^n$. By Proposition 2.1(1), we have (2.1).

Conversely, suppose φ and ψ are of the form (2.1). We have

$$M_z(\psi, \varphi) = |ce^{-\langle Az, b \rangle}|^2 e^{|Az+b|^2 - |z|^2} = |c|^2 e^{|b|^2},$$

which shows that $0 < m(\psi, \varphi) = M(\psi, \varphi) < \infty$. By Proposition 2.1(3), $W_{\psi,\varphi}$ is bounded and invertible on $\mathcal{F}^2(\mathbb{C}^n)$. $\square$

Obviously, Proposition 2.1(4) is an immediate consequence of Proposition 2.3.

Remark 2.4. Since involution is a particular type of invertible operators, functions ψ, φ that induce an involutive bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{F}^2(\mathbb{C}^n)$, should have the form (2.1). So Proposition 2.3 is useful in characterizing involutive property of both linear and anti-linear weighted composition operators.

3. ANTI-LINEAR WEIGHTED COMPOSITION OPERATORS

In this section we consider anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C}^n)$ and study their properties.

The symbol $\kappa\eta$ is used to indicate the composition $\kappa \circ \eta$, where η is a mapping from $\mathbb{C}^n$ into itself and κ is the standard conjugation on $\mathbb{C}^n$ defined by

$$(3.1) \quad \kappa(z_1, z_2, \dots, z_n) = (\overline{z_1}, \overline{z_2}, \dots, \overline{z_n}).$$

We start the section with a direct generalization of [9, Lemma 3.5].

Lemma 3.1. *The operator $\mathcal{K}f(z) = \overline{f(\kappa(z))}$ is a conjugation on $\mathcal{F}^2(\mathbb{C}^n)$.*

We define an anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ induced by an entire function $\xi \not\equiv 0$ in $\mathbb{C}^n$ and a holomorphic self-mapping η on $\mathbb{C}^n$ as follows:

$$(\mathcal{A}_{\xi,\eta}f)(z) = \xi(z) \overline{f(\kappa\eta(z))}.$$

A question to ask is: Does there exist any relationship between a linear operator $W_{\xi,\eta}$ and an anti-linear operator $\mathcal{A}_{\xi,\eta}$? The answer is affirmative, and more than that, as we can see below, the conjugation $\mathcal{K}$ plays an essential role in the relationship.

Proposition 3.2. *We have $\mathcal{A}_{\xi,\eta} = W_{\xi,\eta}\mathcal{K}$, or the same $W_{\xi,\eta} = \mathcal{A}_{\xi,\eta}\mathcal{K}$.*

Proof. Take arbitrarily $f(z) \in \mathcal{F}^2(\mathbb{C}^n)$. Since $\mathcal{K}f(z) = \overline{f(\kappa(z))}$, for all $z \in \mathbb{C}^n$, we have

$$(W_{\xi,\eta}\mathcal{K})f(z) = \xi(z)\mathcal{K}f(\eta(z)) = \xi(z)\overline{f(\kappa\eta(z))} = (\mathcal{A}_{\xi,\eta}f)(z).$$

Furthermore, the formula $\mathcal{A}_{\xi,\eta} = W_{\xi,\eta}\mathcal{K}$ is equivalent to $W_{\xi,\eta} = \mathcal{A}_{\xi,\eta}\mathcal{K}$, because $\mathcal{K}$ is involutive. $\square$

Proposition 3.2 allows us to show that $W_{\xi,\eta}$ and $\mathcal{A}_{\xi,\eta}$ have several common important properties.

Theorem 3.3. *Let $\xi \not\equiv 0$ be an entire function in $\mathbb{C}^n$ and let η be a holomorphic self-mapping on $\mathbb{C}^n$. The following assertions hold:*

- (1) $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if so is $W_{\xi,\eta}$.
- (2) $\mathcal{A}_{\xi,\eta}$ is invertible on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if so is $W_{\xi,\eta}$.
- (3) $\mathcal{A}_{\xi,\eta}$ is unitary on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if so is $W_{\xi,\eta}$.

Proof. (1) Since $\mathcal{K}$ is bounded, $\mathcal{A}_{\xi,\eta}$ and $W_{\xi,\eta}$ are bounded on $\mathcal{F}^2(\mathbb{C}^n)$ simultaneously on $\mathcal{F}^2(\mathbb{C}^n)$.

(2) It is clear that if $\mathcal{A}_{\xi,\eta}$ is invertible, then so is $W_{\xi,\eta}$, with $W_{\xi,\eta}^{-1} = \mathcal{K}^{-1}\mathcal{A}_{\xi,\eta}^{-1}$; also if $W_{\xi,\eta}$ is invertible, then so is $\mathcal{A}_{\xi,\eta}$, with $\mathcal{A}_{\xi,\eta}^{-1} = \mathcal{K}^{-1}W_{\xi,\eta}^{-1}$.

(3) We have

$$\mathcal{A}_{\xi,\eta}^* = \mathcal{K}^*W_{\xi,\eta}^* = \mathcal{K}W_{\xi,\eta}^*, \quad \mathcal{A}_{\xi,\eta}^{-1} = \mathcal{K}^{-1}W_{\xi,\eta}^{-1} = \mathcal{K}W_{\xi,\eta}^{-1}.$$

Thus, the operator $\mathcal{A}_{\xi,\eta}$ is unitary if and only if $\mathcal{K}W_{\xi,\eta}^* = \mathcal{K}W_{\xi,\eta}^{-1}$, and hence since $\mathcal{K}$ is involutive, $W_{\xi,\eta}$ is unitary. $\square$

The following result shows that a square of an anti-linear weighted composition operator is a linear weighted composition operator, with appropriate symbols.

Lemma 3.4. *We have $\mathcal{A}_{\xi,\eta}^2 = W_{\zeta,\vartheta}$, where $\zeta(z) = \xi(z)\overline{\xi(\kappa\eta(z))}$ and $\vartheta(z) = (\kappa\eta)^2(z)$.*

Proof. For every $f(z) \in \mathcal{F}^2(\mathbb{C}^n)$, we have

$$\mathcal{A}_{\xi,\eta}^2 f(z) = \xi(z)\overline{\mathcal{A}_{\xi,\eta}f(\kappa\eta(z))} = \xi(z)\overline{\xi(\kappa\eta(z))}f((\kappa\eta)^2(z)) = W_{\zeta,\vartheta}f(z).$$

$\square$

3.1. Bounded, invertible, and unitary properties. First we characterize a boundedness of anti-linear weighted composition operators.

By Theorem 3.3(1) and Proposition 2.1, we get immediately the following result for boundedness of an anti-linear operator $\mathcal{A}_{\xi,\eta}$.

Proposition 3.5. (1) *If $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then $\xi \in \mathcal{F}^2(\mathbb{C}^n)$ and $M(\xi, \eta) < \infty$. Moreover, in this case, $\eta(z) = Az + b$, where A is a linear operator on $\mathbb{C}^n$ with $\|A\| \leq 1$ and $b \in \mathbb{C}^n$.*

(2) *Conversely, if $M(\xi, \eta) < \infty$, and in addition, η is injective, then $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.*

The following result, in case when $\xi(z)$ has an exponential form (while $\eta(z)$ is affine), provides an important equivalent relationship between a bounded linear weighted composition operator $W_{\xi,\eta}$ and a bounded anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$.

Theorem 3.6. *Let $\eta(z) = Az + b$ and $\xi(z) = e^{\langle z, \beta \rangle}$, where A is a linear operator on $\mathbb{C}^n$, $b, \beta \in \mathbb{C}^n$. The following assertions are equivalent:*

- (1) $W_{\xi,\eta}: \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is bounded.
- (2) $\mathcal{A}_{\xi,\eta}: \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is bounded.
- (3) $W_{1,\eta\beta}: \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is bounded.
- (4) $\|A\| \leq 1$ and $\langle Ax, b + A\beta \rangle = 0$, whenever $|Ax| = |x|$, $x \in \mathbb{C}^n$.

Proof. Since (1) $\iff$ (2) has been proved in Theorem 3.3 above, and (3) $\iff$ (4) is due to [2], it suffices to prove (2) $\iff$ (3).

For each $\beta \in \mathbb{C}^n$, we put $\eta_\beta(z) = \eta(z + \beta)$, $z \in \mathbb{C}^n$, and $U_\beta f(z) = k_\beta(z)f(z - \beta)$. Note that by [17], U_β is unitary on $\mathcal{F}^2(\mathbb{C}^n)$. If $f(z) \in \mathcal{F}^2(\mathbb{C}^n)$, then we have

$$\begin{aligned} U_{-\beta}\mathcal{A}_{\xi,\eta}f(z) &= k_{-\beta}(z)\mathcal{A}_{\xi,\eta}f(z + \beta) = k_{-\beta}(z)\xi(z + \beta)\overline{f(\kappa\eta_\beta(z))} \\ &= e^{|\beta|^2/2}\mathcal{A}_{1,\eta\beta}f(z) = e^{|\beta|^2/2}W_{1,\eta\beta}\mathcal{K}f(z). \end{aligned}$$

Since $U_{-\beta}$ is unitary and $\mathcal{K}$ is a conjugation, the above relation shows that $\mathcal{A}_{\xi,\eta}$ is bounded if and only if so is $W_{1,\eta\beta}$. $\square$

Recall that if a linear weighted composition operator $W_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$ then

$$(3.2) \quad W_{\xi,\eta}^*K_z = \overline{\xi(z)}K_{\eta(z)}, \quad z \in \mathbb{C}^n.$$

The following result is a similar version for the case of anti-linear weighted composition operators.

Lemma 3.7. *If $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then*

$$\mathcal{A}_{\xi,\eta}^*K_z = \xi(z)K_{\kappa\eta(z)}, \quad z \in \mathbb{C}^n.$$

Proof. By Lemma 3.2, we have

$$\mathcal{A}_{\xi,\eta}^*K_z = \mathcal{K}W_{\xi,\eta}^*K_z = \mathcal{K}(\overline{\xi(z)}K_{\eta(z)}) = \xi(z)\mathcal{K}(K_{\eta(z)}).$$

Moreover, for each $u \in \mathbb{C}^n$,

$$\mathcal{K}(K_{\eta(z)})(u) = \overline{K_{\eta(z)}(\kappa(u))} = e^{\langle \eta(z), \kappa(u) \rangle} = e^{\langle u, \kappa\eta(z) \rangle} = K_{\kappa\eta(z)}(u),$$

because κ is a conjugation on $\mathbb{C}^n$. Hence $\mathcal{A}_{\xi,\eta}^*K_z = \xi(z)K_{\kappa\eta(z)}$. $\square$

Next, we investigate an invertibility of bounded anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C}^n)$.

Theorem 3.8. *The operator $\mathcal{A}_{\xi,\eta}$ is invertible on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if*

$$(3.3) \quad \eta(z) = Az + b, \quad \text{and} \quad \xi(z) = ce^{-(Az, b)}, \quad z \in \mathbb{C}^n,$$

where A is a linear unitary operator on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C} \setminus \{0\}$.

Proof. By Theorem 3.3, $\mathcal{A}_{\xi,\eta}$ is invertible if and only if $W_{\xi,\eta}$ is invertible. Then the desired result follows from Proposition 2.3. $\square$

The last task of this part is to study a unitary property of bounded anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C}^n)$.

Theorem 3.9. *The operator $\mathcal{A}_{\xi,\eta} : \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is unitary if and only if condition (3.3) holds and*

$$(3.4) \quad |c|^2 e^{|b|^2} = 1.$$

Proof. By Theorem 3.3, $\mathcal{A}_{\xi,\eta} : \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is unitary if and only if $W_{\xi,\eta} : \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is unitary. Then, by Proposition 2.1(4), we obtain (3.3)–(3.4). $\square$

As an immediate consequence of Theorems 3.8–3.9, we get the following result.

Corollary 3.10. *An anti-linear operator $\mathcal{A}_{\xi,\eta}$ is unitary on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if it is invertible and condition (3.4) holds.*

3.2. Selfadjoint property. We characterize selfadjointness of bounded anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C}^n)$. Since the span of kernel functions is dense in $\mathcal{F}^2(\mathbb{C}^n)$, our problem is reduced to finding ξ and η such that

$$\mathcal{A}_{\xi,\eta}^* K_z = \mathcal{A}_{\xi,\eta} K_z, \quad \forall z \in \mathbb{C}^n.$$

It turns out that in this case, the matrix A appeared in the formula of the function η in Proposition 3.5, is complex symmetric.

Theorem 3.11. *The following statements are equivalent:*

- (1) *The linear operator $W_{\xi,\eta} = \mathcal{A}_{\xi,\eta} \mathcal{K}$ is $\mathcal{C}$ -symmetric on $\mathcal{F}^2(\mathbb{C}^n)$.*
- (2) *The anti-linear operator $\mathcal{A}_{\xi,\eta}$ is selfadjoint on $\mathcal{F}^2(\mathbb{C}^n)$.*
- (3) *The functions ξ and η are of the forms*

$$(3.5) \quad \eta(z) = Az + b, \quad \text{and} \quad \xi(z) = ce^{\langle z, \kappa b \rangle}, \quad z \in \mathbb{C}^n,$$

where A is a linear operator on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C} \setminus \{0\}$, such that

$$(3.6) \quad \|A\| \leq 1, \quad \langle Ax, b + A\kappa b \rangle = 0, \quad \text{whenever } |Ax| = |x|, \quad x \in \mathbb{C}^n,$$

and

$$(3.7) \quad \kappa A \kappa = A^*.$$

Proof. By Proposition 3.2, we have

$$\mathcal{A}_{\xi,\eta} = W_{\xi,\eta} \mathcal{K}, \quad \mathcal{A}_{\xi,\eta}^* = \mathcal{K}^* W_{\xi,\eta}^* = \mathcal{K} W_{\xi,\eta}^*.$$

These equations clearly imply that (1) $\iff$ (2).

We now prove (2) $\iff$ (3). Note that a boundedness of $\mathcal{A}_{\xi,\eta}$ is equivalent to (3.6), by Theorem 3.6.

Suppose that $\mathcal{A}_{\xi,\eta}$ is selfadjoint, that is, $\mathcal{A}_{\xi,\eta}^* = \mathcal{A}_{\xi,\eta}$. For arbitrary $z, u \in \mathbb{C}^n$, by Lemma 3.7, we have

$$\xi(z)K_{\kappa\eta(z)}(u) = \mathcal{A}_{\xi,\eta}^*K_z(u) = \mathcal{A}_{\xi,\eta}K_z(u) = \xi(u)\overline{K_z(\kappa\eta(u))},$$

which gives

$$\xi(z)e^{\langle u, \kappa\eta(z) \rangle} = \xi(u)e^{\langle z, \kappa\eta(u) \rangle}.$$

Since $\eta(z) = Az + b$, the above equality means

$$(3.8) \quad \xi(z)e^{\langle u, \kappa Az + \kappa b \rangle} = \xi(u)e^{\langle z, \kappa Au + \kappa b \rangle}.$$

Letting $u = 0$, we get the form of $\xi(z)$ in (3.5), with $c = \xi(0)$. Substituting $\xi(z)$ back into (3.8), we have

$$\langle u, \kappa Az \rangle = \langle z, \kappa Au \rangle.$$

Since κ is a conjugation on $\mathbb{C}^n$, the above equality can be rewritten as

$$\langle Az, \kappa u \rangle = \langle u, \kappa Az \rangle = \langle z, \kappa Au \rangle = \langle z, (\kappa A \kappa) \kappa u \rangle.$$

Consequently, for every $z, w \in \mathbb{C}^n$, we have

$$\langle Az, w \rangle = \langle z, (\kappa A \kappa) w \rangle,$$

which means that $(\kappa A \kappa)w = A^*w$, $\forall w \in \mathbb{C}^n$, and so we get (3.7).

Conversely, suppose that (3.5)–(3.7) hold. In this case, a direct computation gives

$$\mathcal{A}_{\xi,\eta}^*K_z = \mathcal{A}_{\xi,\eta}K_z, \quad \forall z \in \mathbb{C}^n.$$

Since the span of the kernel functions is dense in $\mathcal{F}^2(\mathbb{C}^n)$, also $\mathcal{A}_{\xi,\eta}^*$ and $\mathcal{A}_{\xi,\eta}$ are bounded, we have

$$\mathcal{A}_{\xi,\eta}^*f = \mathcal{A}_{\xi,\eta}f, \quad \forall f \in \mathcal{F}^2(\mathbb{C}^n).$$

This shows that $\mathcal{A}_{\xi,\eta}$ is selfadjoint. □

Remark 3.12. Theorem 3.11 provides the structure of bounded linear weighted composition operators which are $\mathcal{C}$ -symmetric on $\mathcal{F}^2(\mathbb{C}^n)$. In Section 4, we give another class of complex symmetric bounded linear weighted composition operators, which contains properly $\mathcal{C}$ -symmetric operators in Theorem 3.11.

3.3. Conjugation. As noted in Introduction, an anti-linear operator $\mathcal{C}$ is a conjugation if and only if it is both unitary and selfadjoint. So combining Theorem 3.9, Corollary 3.10, and Theorem 3.11 yields the following important criterion for an anti-linear weighted composition operator to be a conjugation.

Theorem 3.13. *The operator $\mathcal{A}_{\xi,\eta} : \mathcal{F}^2(\mathbb{C}^n) \rightarrow \mathcal{F}^2(\mathbb{C}^n)$ is a conjugation if and only if*

$$(3.9) \quad \eta(z) = Az + b, \quad \text{and} \quad \xi(z) = ce^{-\langle Az, b \rangle}, \quad z \in \mathbb{C},$$

where A is a linear operator on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C}$ that satisfy the following conditions

$$(3.10) \quad A^{-1} = A^* = \kappa A \kappa,$$

$$(3.11) \quad A\kappa b + b = 0,$$

$$(3.12) \quad |c|^2 e^{|b|^2} = 1.$$

3.4. Involutive property. The final study in this section is to investigate involutivity of anti-linear weighted composition operators.

It turns out that the characterization of involution ensures that anti-linear weighted composition operators are conjugations.

Theorem 3.14. *Suppose that $\mathcal{A}_{\xi,\eta}$ is a bounded anti-linear weighted composition operator. The following statements are equivalent:*

- (1) $\mathcal{A}_{\xi,\eta}$ is involutive.
- (2) $\mathcal{A}_{\xi,\eta}$ is a conjugation.
- (3) The functions ξ, η satisfy conditions (3.10)–(3.12).

Proof. By Theorem 3.13, (2) $\iff$ (3). Now we prove (1) $\iff$ (3).

Suppose that $\mathcal{A}_{\xi,\eta}$ is involutive. Note that an involutive bounded operator is also invertible. Then by Theorem 3.8, we have (3.9), where the linear operator A is unitary on $\mathbb{C}^n$, i.e. $A^{-1} = A^*$.

Since $\mathcal{A}_{\xi,\eta}^2 = I$, the identity operator on $\mathcal{F}^2(\mathbb{C}^n)$, we have

$$(3.13) \quad \xi(z) \overline{\xi(\kappa\eta(z))} f((\kappa\eta)^2(z)) = f(z), \quad \forall f \in \mathcal{F}^2(\mathbb{C}^n).$$

In particular, for $f(z) \equiv 1 \in \mathcal{F}^2(\mathbb{C}^n)$, we get

$$(3.14) \quad \xi(z) \overline{\xi(\kappa\eta(z))} = 1.$$

Then equation (3.13) is reduced to

$$(3.15) \quad f((\kappa\eta)^2(z)) = f(z), \quad \forall f \in \mathcal{F}^2(\mathbb{C}^n).$$

Now taking $f(z) \equiv z$ in (3.15), we have

$$(\kappa\eta)^2(z) = z, \quad \forall z \in \mathbb{C}^n,$$

which means, as $\eta(z) = Az + b$, that

$$(\kappa A)^2 z + \kappa A \kappa b + \kappa b = z, \quad \forall z \in \mathbb{C}^n,$$

from which (3.10) and (3.11) follow.

Since $\xi(z) = ce^{-\langle Az, b \rangle}$, by (3.10)–(3.11), we get

$$\xi(\kappa\eta(z)) = ce^{-\langle A\kappa\eta(z), b \rangle} = ce^{|b|^2} e^{-\langle \kappa z, b \rangle}.$$

Consequently,

$$(3.16) \quad \xi(z) \overline{\xi(\kappa\eta(z))} = ce^{-\langle Az, b \rangle} \overline{ce^{|b|^2} e^{-\langle \kappa z, b \rangle}} = |c|^2 e^{|b|^2} e^{-\langle Az, b \rangle - \langle b, \kappa z \rangle},$$

which, by (3.14), implies (3.12).

Conversely, suppose conditions (3.9)–(3.12) hold. Taking into account Theorem 3.6, from conditions (3.10) and (3.11) it follows that the operator $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. So the proof is complete if we show that $\mathcal{A}_{\xi,\eta}$ is involutive.

Indeed, take arbitrarily $f(z) \in \mathcal{F}^2(\mathbb{C}^n)$. By (3.16), we have

$$\xi(z) \overline{\xi(\kappa\eta(z))} = |c|^2 e^{|b|^2} e^{-\langle Az, b \rangle - \langle b, \kappa z \rangle} = e^{-\langle z, A^*(b + A\kappa b) \rangle} = 1.$$

On the other hand, by (3.10)–(3.11), we have

$$(\kappa\eta)^2(z) = (\kappa A)^2 z + \kappa(A\kappa b + b) = z, \quad \forall z \in \mathbb{C}^n.$$

Thus,

$$\mathcal{A}_{\xi,\eta}^2 f(z) = \xi(z) \overline{\xi(\kappa\eta(z))} f((\kappa\eta)^2(z)) = f(z), \quad \forall z \in \mathbb{C}^n,$$

which show that $\mathcal{A}_{\xi,\eta}$ is involutive. $\square$

Remark 3.15. It should be noted that for the case $n = 1$, we have succeeded to characterize an isometry of $\mathcal{A}_{\xi,\eta}$ on the space $\mathcal{F}^2(\mathbb{C})$. It also showed that isometric anti-linear weighted composition operators form a proper subclass of involutive anti-linear weighted composition operators.

In the higher dimensional case $n \geq 2$ the situation is not clear, and so we pose the following question.

Question 3.16. *Characterize the class of isometric anti-linear weighted composition operators on the space $\mathcal{F}^2(\mathbb{C}^n)$, $n \geq 2$.*

4. LINEAR WEIGHTED COMPOSITION OPERATORS

4.1. $\mathcal{C}_{A,b,c}$ -symmetry. In what follows, the weighted composition conjugation on $\mathcal{F}^2(\mathbb{C}^n)$ is denoted by $\mathcal{C}_{A,b,c}$, that is

$$(4.1) \quad \mathcal{C}_{A,b,c} f(z) := ce^{\langle z, \kappa b \rangle} \overline{f(\kappa(Az + b))},$$

where A, b, c satisfy conditions of Theorem 3.13. Clearly, for the case $n = 1$, with $A = 1, b = 0, c = 1$ we get the classical conjugation (1.3).

We consider bounded linear weighted composition operators, and study under what conditions, these operators are $\mathcal{C}_{A,b,c}$ -symmetric.

We note that since the conjugation is involutive, the equation $T = \mathcal{C}T^*\mathcal{C}$ in Definition 1.3 can be rewritten as $T\mathcal{C} = \mathcal{C}T^*$, or $T^* = \mathcal{C}T\mathcal{C}$.

We have the following important criterion.

Theorem 4.1. *Let $\mathcal{C}_{A,b,c}$ be a weighted composition conjugation on $\mathcal{F}^2(\mathbb{C}^n)$, defined by (4.1) (that is $\eta(z) = Az + b$ and $\xi(z) = ce^{-\langle Az, b \rangle}$, with conditions (3.10) – (3.12)). Let further, $\psi \neq 0$ be an entire function in $\mathbb{C}^n$, and φ a holomorphic self-mapping on $\mathbb{C}^n$. Then the operator $W_{\psi,\varphi}$ is $\mathcal{C}_{A,b,c}$ -symmetric on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if the following conditions hold:*

$$(4.2) \quad \varphi(z) = Qz + p, \text{ and } \psi(z) = \psi(0)e^{\langle z, q \rangle},$$

where Q is a linear operator on $\mathbb{C}^n$, $p, q \in \mathbb{C}^n$, with $q = \kappa Ap + \kappa b - Q^* \kappa b$, such that

$$(4.3) \quad (AQ)^* = \kappa A Q \kappa,$$

$$(4.4) \quad \|Q\| \leq 1, \text{ and } \langle Qx, p + Qq \rangle = 0 \text{ whenever } |Qx| = |x|, x \in \mathbb{C}^n.$$

Proof. Suppose $W_{\psi,\varphi}$ is $\mathcal{C}_{A,b,c}$ -symmetric on $\mathcal{F}^2(\mathbb{C}^n)$. This implies, in particular, that

$$(4.5) \quad \mathcal{C}_{A,b,c}W_{\psi,\varphi}^*K_z(u) = W_{\psi,\varphi}\mathcal{C}_{A,b,c}K_z(u), \quad \forall z, u \in \mathbb{C}^n.$$

Since $W_{\psi,\varphi}$ is bounded, by Proposition 2.1(1), $\varphi(z) = Qz + p$, where Q is a linear operator on $\mathbb{C}^n$ with $\|Q\| \leq 1$ and $p \in \mathbb{C}^n$. Also by (3.2), $W_{\psi,\varphi}^*K_z = \overline{\psi(z)}K_{\varphi(z)}$. Then the left-hand side of equation (4.5) gives

$$\mathcal{C}_{A,b,c}W_{\psi,\varphi}^*K_z(u) = c\psi(z)e^{\langle u, \kappa b \rangle + \langle Qz+p, \kappa(Au+b) \rangle}.$$

On the other hand, since $\mathcal{C}_{A,b,c}K_z(u) = ce^{\langle u, \kappa b \rangle} \overline{K_z(\kappa(Au+b))}$, the right-hand side of equation (4.5) shows

$$W_{\psi,\varphi}\mathcal{C}_{A,b,c}K_z(u) = \psi(u)ce^{\langle Qu+p, \kappa b \rangle + \langle z, \kappa AQu + \kappa Ap + \kappa b \rangle}.$$

Consequently,

$$(4.6) \quad \psi(z)e^{\langle u, \kappa b \rangle + \langle Qz+p, \kappa(Au+b) \rangle} = \psi(u)e^{\langle Qu+p, \kappa b \rangle + \langle z, \kappa AQu + \kappa Ap + \kappa b \rangle}.$$

Letting $u = 0$ in this equation, we get

$$\psi(z)e^{\langle Qz+p, \kappa b \rangle} = \psi(0)e^{\langle p, \kappa b \rangle + \langle z, \kappa Ap + \kappa b \rangle},$$

and the second relation in (4.2) follows

$$\psi(z) = \psi(0)e^{\langle z, q \rangle}, \quad q = \kappa Ap + \kappa b - Q^*\kappa b.$$

Now substituting ψ back in both sides of (4.6), we obtain

$$\langle Qz, \kappa Au \rangle + \langle p, \kappa Au \rangle = \langle z, \kappa AQu \rangle + \langle u, \kappa Ap \rangle, \quad \forall z, u \in \mathbb{C}^n.$$

Furthermore, since κ is the standard conjugation on $\mathbb{C}^n$ defined by (3.1) and A is κ -symmetric, we have

$$\langle p, \kappa Au \rangle = \langle p, A^*\kappa u \rangle = \langle Ap, \kappa u \rangle = \langle u, \kappa Ap \rangle.$$

Then the equation above is reduced to

$$\langle Qz, \kappa Au \rangle = \langle z, \kappa AQu \rangle, \quad \forall z, u \in \mathbb{C}^n,$$

which implies that $Q^*\kappa Au = \kappa AQu$, $\forall u \in \mathbb{C}^n$, that is $Q^*\kappa A = \kappa AQ$. But $Q^*\kappa A = Q^*A^*\kappa = (AQ)^*\kappa$, and so $(AQ)^*\kappa = \kappa AQ$. Applying κ to this equation yields (4.3)

$$(AQ)^* = \kappa AQ\kappa,$$

which also means that AQ is κ -symmetric on $\mathbb{C}^n$.

Finally, (4.4) follows from boundedness of $W_{\psi,\varphi}$, by Theorem 3.6.

Conversely, suppose that conditions (4.2) – (4.4) hold. In this case, by Theorem 3.6, the weighted composition operator $W_{\psi,\varphi}$ induced by ψ and φ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, and its adjoint operator $W_{\psi,\varphi}^*$ is also bounded on $\mathcal{F}^2(\mathbb{C}^n)$. By direct computations, we have

$$\mathcal{C}_{A,b,c}W_{\psi,\varphi}^*K_z = W_{\psi,\varphi}\mathcal{C}_{A,b,c}K_z, \quad \forall z \in \mathbb{C}^n.$$

Since the span of the kernel functions is dense in $\mathcal{F}^2(\mathbb{C}^n)$, we have

$$\mathcal{C}_{A,b,c}W_{\psi,\varphi}^*f = W_{\psi,\varphi}\mathcal{C}_{A,b,c}f, \quad \forall f \in \mathcal{F}^2(\mathbb{C}^n),$$

which shows that $W_{\psi,\varphi}$ is $\mathcal{C}_{A,b,c}$ -symmetric. $\square$

4.2. Normality. As mentioned in the introduction, every normal operator is complex symmetric with respect to some conjugation. We recall that a bounded linear operator T on a complex Hilbert space is called *normal* if it commutes with its adjoint T^* , that is $TT^* = T^*T$.

A natural question to ask is whether $\mathcal{C}_{a,b,c}$ -symmetric weighted composition operators are normal. To give the answer to this question, we first need the following result.

Lemma 4.2. *Let $W_{\psi,\varphi}$ be a $\mathcal{C}_{A,b,c}$ -symmetric operator on $\mathcal{F}^2(\mathbb{C}^n)$, induced by φ, ψ (that is, Theorem 4.1 holds). We have*

$$\begin{aligned} (1) \quad & W_{\psi,\varphi} W_{\psi,\varphi}^* K_z(u) = |\psi(0)|^2 e^{\langle Qu, Qz \rangle + \langle u, q + Q^*p \rangle + \langle q + Q^*p, z \rangle + |p|^2}. \\ (2) \quad & W_{\psi,\varphi}^* W_{\psi,\varphi} K_z(u) = |\psi(0)|^2 e^{\langle Q^*u, Q^*z \rangle + \langle u, Qq + p \rangle + \langle p + Qq, z \rangle + |q|^2}. \end{aligned}$$

Proof. We use formula (3.2) and the form of ψ in Theorem 4.1.

1) We have

$$\begin{aligned} W_{\psi,\varphi} W_{\psi,\varphi}^* K_z(u) &= W_{\psi,\varphi} (\overline{\psi(z)} K_{\varphi(z)})(u) = \overline{\psi(z)} \psi(u) e^{\langle \varphi(u), \varphi(z) \rangle} \\ &= |\psi(0)|^2 e^{\langle Qu, Qz \rangle + \langle u, q + Q^*p \rangle + \langle q + Q^*p, z \rangle + |p|^2}. \end{aligned}$$

2) Since

$$\begin{aligned} W_{\psi,\varphi} K_z(u) &= \psi(u) K_z(\varphi(u)) = \psi(0) e^{\langle u, q \rangle + \langle \varphi(u), z \rangle} \\ &= \psi(0) e^{\langle p, z \rangle} e^{\langle u, q + Q^*z \rangle} = \psi(0) e^{\langle p, z \rangle} K_{q + Q^*z}(u), \end{aligned}$$

we have

$$\begin{aligned} W_{\psi,\varphi}^* W_{\psi,\varphi} K_z(u) &= \psi(0) e^{\langle p, z \rangle} \overline{\psi(q + Q^*z)} K_{\varphi(q + Q^*z)}(u) \\ &= |\psi(0)|^2 e^{\langle p, z \rangle + \langle q, q + Q^*z \rangle + \langle u, Q(q + Q^*z) + p \rangle} \\ &= |\psi(0)|^2 e^{\langle Q^*u, Q^*z \rangle + \langle u, Qq + p \rangle + \langle p + Qq, z \rangle + |q|^2}. \end{aligned}$$

$\square$

We have the following criterion for normality of such operators.

Proposition 4.3. *Let $W_{\psi,\varphi}$ be a $\mathcal{C}_{A,b,c}$ -symmetric operator on $\mathcal{F}^2(\mathbb{C}^n)$, induced by φ, ψ (that is Theorem 4.1 holds). Then $W_{\psi,\varphi}$ is normal if and only if Q is normal (that is $Q^*Q = QQ^*$) and*

$$(4.7) \quad (Q^* - I_n)p = (Q - I_n)q, \quad |p| = |q|.$$

Proof. Since $W_{\psi,\varphi}$ is bounded, and the span of the kernel functions is dense in $\mathcal{F}^2(\mathbb{C}^n)$, the normality of $W_{\psi,\varphi}$ is equivalent to

$$W_{\psi,\varphi} W_{\psi,\varphi}^* K_u(z) = W_{\psi,\varphi}^* W_{\psi,\varphi} K_u(z), \quad \text{for all } u, z \in \mathbb{C}^n.$$

By Lemma 4.2, the above equation is reduced to

$$\begin{aligned} & \langle Qu, Qz \rangle + \langle u, q + Q^*p \rangle + \langle q + Q^*p, z \rangle + |p|^2 \\ &= \langle Q^*u, Q^*z \rangle + \langle u, Qq + p \rangle + \langle p + Qq, z \rangle + |q|^2, \quad \forall u, z \in \mathbb{C}^n, \end{aligned}$$

This equation is equivalent to

$$\begin{cases} \langle Qu, Qz \rangle = \langle Q^*u, Q^*z \rangle, \quad \forall z, u \in \mathbb{C}^n \\ Q^*p + q = Qq + p \quad \text{and} \quad |p| = |q|. \end{cases}$$

The first equation shows that $Q^*Q = QQ^*$, that is Q is normal, while the last two equations are (4.7). $\square$

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Note added in the proof-reading. A recent work by L. Zhao [19] appears to have a characterization for isometric linear weighted composition operators on $\mathcal{F}(\mathbb{C}^n)$.

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