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Complex Composite Derivative and Its Application to Edge Detection*

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Abstract. In this paper, a detailed study on a composite derivative is performed. The composite derivative, which is formed from the combination of fractional integration and derivative and performs a 90° phase shift as the traditional first derivative does, is applied to edge detection and the results are analyzed, emphasizing the compromise ability between selectivity and noise suppression. Both objective and subjective comparisons with other edge detectors are carried out, including evaluations through the use of the benchmark Berkeley Segmentation Dataset (BSDS500). In contrast with the classical first-order derivative, the composite derivative is order-steerable; one can adjust the orders of fractional integration and derivative to tune magnitude characteristic and reach a compromise between sensitivity to noise and detection accuracy.

Key words. fractional derivative, fractional integral, composite derivative, robustness to noise, edge detection

AMS subject classifications. 68U10, 26A33

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1. Introduction. Edge detection is an important technique in image processing that refers to the process of identifying and locating sharp discontinuities in an image. Image edges contain important information about the contours of an object which provide sound resources for future feature extraction and target detection [24, 3]. The result of edge detection will directly affect subsequent image analysis, understanding, and restoration [43, 11, 41]. Therefore, edge detection plays a key role in image preprocessing [22, 40].

Edge pixels, exerting the most significant influence on brightness changes, are essentially some local parts of an image, that is, those points (called singular points) change dramatically in neighborhood (local) intensity. Therefore, edges are the collection of singularities in a

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two-dimensional (2D) image. Traditional edge detecting methods generally employ enhancement algorithms to highlight singular points (edge pixels). Since edge enhancement can be accomplished by calculating the gradient magnitude, differential operations naturally become the main means of edge detection and extraction [35, 33, 9, 1].

Early approaches to edge detection aim at quantifying the presence of a boundary at a given image location through local measurements. Differential operation is a basic mathematical operation and has been widely used in signal analysis and processing, especially in the area of signal singular points detection and extraction. Most traditional edge detection methods adopt the integer-order differential. Several popular differential operators are presented: *Roberts* operator [35], *Sobel* operator [39], *Prewitt* operator [33], *Canny* operator [4, 36, 45, 10], and *LoG* operator [8, 6, 44, 42]. They can be categorized into first derivative based (Roberts, Sobel, Prewitt, and Canny) and second derivative based (LoG). Based on Canny's criteria, Demigny [10] gives the criteria for discrete sampled signals and derives optimal filters for each of the criteria and for any combination of them. By using the derivative of *Gaussian* function, Jacob and Unser [19] propose a general approach for the design of 2D feature detectors from a class of steerable functions based on the optimization of a Canny-like criterion. Their operators have a better orientation selectivity than the classical gradient or Hessian-based detectors. Mahmoodi [22] presents a "Bessel integral filter" that is scale invariant by enhancing Canny's optimality criteria and derives a noise-free localization index to account for the detection accuracy of sharp corners in the absence of noise. A richer description can be obtained considering the response of an image to a family of filters of different scales and orientations [41]. An example is the oriented energy approach [29, 15], which uses quadrature pairs of even and odd symmetric filters. Lindeberg [21] proposes a filter-based method with an automatic-scale selection mechanism. More recent local approaches take into account color and texture information and make use of learning techniques for cue combination [13, 25]. Dollar, Tu, and Belongie [13] introduce a boosted edge learning algorithm which attempts to learn an edge classifier in the form of a probabilistic boosting tree from thousands of simple features computed on image patches. Martin, Fowlkes, and Malik [25] define gradient operators for brightness, color, and texture channels and use them as input to a logistic regression classifier for predicting edge strength. The large range of scales at which objects may appear in an image remains a concern for these modern local approaches. The class of edge detection methods including nonlocal differential operators is also getting more and more diverse recently; see [30, 18, 17, 16, 23, 27, 2, 14] and the references therein. An orthogonal line of work in contour detection focuses primarily on another level of processing, globalization, which utilizes the output of a local detector [30, 2].

The classical first derivative has the important advantage of being directional and is broadly used in the above methods. However, the classical first derivative, whose magnitude characteristic is not steerable, is not stable in the presence of noise. Since image edges and noises both belong to high-frequency components, derivative operation will amplify noises as well [4, 37, 19]. To suppress noise, smoothing filters are commonly introduced before applying the derivative operation. The detection accuracy increases along with the derivative order, while the robustness to noise diminishes along with that. Because sensitivity to noise and detection accuracy are contradictory, improving one performance would inevitably sacrifice the other. How to balance these two performance requirements has long been a research topic of

the traditional derivative approaches [4, 19, 16, 23, 2, 22, 14]. Chun et al. initially introduced a composite derivative in [5]. Yet only basic concepts and preliminary results were provided, and the paper is written in Chinese. In this work, a comprehensive study is performed on the algorithm from [5], including mathematical basis, characteristics analysis, design motivation, and extensive experiments.

Section 2 covers the development of the composite derivative. The composite derivative, formed from the combination of fractional integration and derivative, is defined rigorously by the Riemann–Liouville (RL) operator. The composite derivative, which performs a 90° phase shift as the traditional first derivative does, provides a new approach to calculating the gradient of an image. The major differences from the classical first-order derivative are that the composite derivative is order-steerable, and one can adjust the orders of fractional integration and derivative ($\alpha + \beta = 1$ and $0 < \beta < \alpha < 1$, where α denotes fractional derivative order and β denotes fractional integration order) to tune the magnitude characteristic and reach a compromise between sensitivity to noise and detection accuracy. Section 3 presents the masks of composite derivative in spatial domain. The composite derivative includes two parts. One part is a fractional integration, and the other part is a fractional differentiation, which will be realized by filter convolution. The detailed theoretical deduction of the fractional integral is presented, and the fractional derivative mask is reiterated in section 3. In section 4, extensive quantitative and qualitative experiments are performed on a large annotated database, using the precision-recall framework, which has found widespread use for evaluating contour detection.

2. Design and analysis of composite derivative. In signal processing, a causal filter is a linear and time-invariant causal system. The word causal indicates that the output depends only on past and present inputs [12, 34]. Apart from these inputs, noncausal filters are under the influence of future inputs as well [38]. On the other hand, a filter whose output depends only on future inputs is anticausal [20]. Systems (including filters) that are realizable (i.e., that can operate in real time) must be causal because such systems cannot act on future inputs. Replacing the time dimension by the spacial dimension, such as in 2D image processing, noncausal filters are realizable. A noncausal filter has the freedom of letting its output depend on the past, present, and future inputs. A noncausal composite filter is initially proposed in [5], using fractional-order calculus. The composite derivative is formed from the combination of forward and backward fractional calculus. The forward and backward fractional calculus correspond to causal and anticausal filters, respectively. Hence, the composite derivative is noncausal. In this section, the composite derivative is defined rigorously by the Riemann–Liouville operator as follows.

2.1. Composite derivative. The RL definitions for forward and backward fractional integral of order $\tau \in \mathbb{C}$ of function $f(x)$ [7, 28] are, respectively,

$$(2.1) \quad {}_0I_x^\tau f(x) = \frac{1}{\Gamma(\tau)} \int_0^x (x-u)^{\tau-1} f(u) du$$

and

$$(2.2) \quad {}_xI_0^\tau f(x) = \frac{1}{\Gamma(\tau)} \int_x^0 (u-x)^{\tau-1} f(u) du,$$

where $\tau \in \mathbb{C}$, $\Re(\tau) > 0$, $\Gamma(\tau) = \int_0^\infty e^{-t} t^{\tau-1} dt$.

Hence, we have

$$\begin{aligned}
 \mathcal{L}\{{}_x I_0^\tau f(x)\} &= \mathcal{L}\left\{\frac{1}{\Gamma(\tau)} \int_x^0 (u-x)^{\tau-1} f(u) du\right\} \\
 &= \frac{(-1)^\tau}{\Gamma(\tau)} \mathcal{L}\left\{\frac{1}{\Gamma(\tau)} \int_0^x (x-u)^{\tau-1} f(u) du\right\} \\
 &= \frac{(-1)^\tau}{\Gamma(\tau)} \mathcal{L}\{x^{\tau-1} * f(x)\} \\
 &= \frac{(-1)^\tau}{\Gamma(\tau)} \mathcal{L}\{x^{\tau-1}\} \mathcal{L}\{f(x)\} \\
 &= \frac{(-1)^\tau}{\Gamma(\tau)} s^{-1-\tau+1} \Gamma(\tau-1+1) \mathcal{L}\{f(x)\} \\
 &= (-s)^{-\tau} \mathcal{L}\{f(x)\}
 \end{aligned}
 \tag{2.3}$$

and

$$\mathcal{L}\{{}_0 I_x^\tau f(x)\} = s^{-\tau} \mathcal{L}\{f(x)\},
 \tag{2.4}$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.

Let $n \in \mathbb{N}$; then by the theory of the Laplace transform, we know that

$$\mathcal{L}\left\{\frac{d^n}{dx^n} f(x)\right\} = s^n \mathcal{L}\{f(x)\} - \sum_{r=0}^{n-1} s^r \frac{d^{n-r-1}}{dx^{n-r-1}} f(0_+).
 \tag{2.5}$$

The RL definition for the forward fractional derivative of order ν [7, 28] of function $f(x)$ is defined by

$$\begin{aligned}
 {}_0 D_x^\nu f(x) &= \frac{d^n}{dx^n} {}_0 I_x^{n-\nu} f(x) \\
 &= \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \int_0^x (x-u)^{n-1-\nu} f(u) du,
 \end{aligned}
 \tag{2.6}$$

where $\nu \in \mathcal{C}$, $\mathbf{R}(\nu) > 0$, $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, and $n-1 \leq \nu < n$.

Therefore,

$$\begin{aligned}
 \mathcal{L}\{{}_0 D_x^\nu f(x)\} &= \mathcal{L}\left\{\frac{d^n}{dx^n} {}_0 I_x^{n-\nu} f(x)\right\} \\
 &= s^n \mathcal{L}\{{}_0 I_x^{n-\nu} f(x)\} - \sum_{r=0}^{n-1} s^r \frac{d^{n-r-1}}{dx^{n-r-1}} {}_0 I_x^{n-\nu} f(0_+) \\
 &= s^\nu \mathcal{L}\{f(x)\} - \sum_{r=0}^{n-1} s^r \frac{d^{v-r-1}}{dx^{v-r-1}} f(0_+).
 \end{aligned}
 \tag{2.7}$$

Then the Laplace transform of a fractional derivative for zero initial conditions has the form [31, 32]

$$(2.8) \quad \mathcal{L}\{ {}_0D_x^\nu f(x) \} = s^\nu \mathcal{L}\{ f(x) \}.$$

The composite derivative, defined by $(-s)^{-\beta} \cdot s^\alpha$, can be achieved by the following two steps:

$$(2.9) \quad \begin{cases} \text{step 1: } (-s)^{-\beta}, \\ \text{step 2: } s^\alpha, \end{cases}$$

where $0 < \beta < \alpha < 1$ and $\alpha + \beta = 1$.

$$(2.10) \quad \begin{aligned} & (-s)^{-\beta} \cdot s^\alpha \\ &= (-j\omega)^{-\beta} \cdot (j\omega)^\alpha \\ &= \omega^{-\beta} e^{j(-\frac{\pi}{2})(-\beta)} \cdot \omega^\alpha e^{j\frac{\pi}{2}\alpha} \\ &= \omega^{\alpha-\beta} e^{j\frac{\pi}{2}(\alpha+\beta)} \\ &= \omega^{\alpha-\beta} e^{j\frac{\pi}{2}}. \end{aligned}$$

In step 1 (anticausal), the image is filtered by $(-s)^{-\beta}$, where $0 < \beta < 1$, and $-\beta$ indicates that the step is an integral process. $-s$ indicates that the integral process runs along the reverse timeline, that is, the sequence of the input signal is reversed at first, then the sequence of integration result is reversed. In step 2 (causal), the result of step 1 is filtered by s^α , where $0 < \alpha < 1$, $0 < \beta < \alpha < 1$, and $\alpha + \beta = 1$. The final result is a composite derivative, $s^\alpha/(-s)^\beta$ (noncausal). Because $\alpha + \beta = 1$, the phase characteristic of the composite derivative is always 90° (as seen in (2.10)): step 1 and step 2 contribute $\frac{\pi}{2}\beta$ and $\frac{\pi}{2}\alpha$, respectively, which is equivalent to that of the traditional first derivative. Note that we will use the negative of this composite derivative, $(-s)^\alpha/s^\beta$. The phase characteristic of the negative of the composite derivative is always -90° (as seen in (2.11)):

$$(2.11) \quad \begin{aligned} & s^{-\beta} \cdot (-s)^\alpha \\ &= (j\omega)^{-\beta} \cdot (-j\omega)^\alpha \\ &= \omega^{\alpha-\beta} e^{-j\frac{\pi}{2}}. \end{aligned}$$

Thereby, following ideas from the traditional first-order derivative theory, the composite derivative can be applied to locate the edge. When the above operator is applied to edge detection, an accurate phase shift theoretically ensures correct localization of edge points. As a model for the edge, we choose the parabolic step-type transition

$$(2.12) \quad f(x) = \begin{cases} 0, & x \leq 0, \\ ax^2, & 0 < x \leq x_0, \\ -ax^2 + 4ax_0x - 2ax_0^2, & x_0 < x \leq 2x_0, \\ 2ax_0^2, & x > 2x_0, \end{cases}$$

where $a = 1/(2x_0^2)$ and $x_0 = 200$.

Figure 1. Parabolic step-type transition, where $a = 1/(2x_0^2)$.

Figure 2. Normalized results of the response of s^α for various α to $f(x)$.

As seen in Figure 1 the abscissa of the inflexion point is 200, i.e., the abscissa $x_0 = 200$ is an edge point. According to the property of the first-order derivative gradient operator, the abscissa of the composite derivative maximum should correspond to the abscissa $x_0 = 200$. Figure 2 shows the normalized results of the response of s^α for various α to $f(x)$. It can be seen that the abscissa of s^α maximum no longer corresponds to that of the transition inflexion point. The larger α is, the greater the shift to the right of the abscissa $x_0 = 200$. Figures 3 and 4 are the normalized results of the response of $s^\alpha/(-s)^\beta$ and $(-s)^\alpha/s^\beta$ for various α to

Figure 3. Normalized results of the response of $(-s)^{-\beta} \cdot s^\alpha$ for various α to $f(x)$.

Figure 4. Normalized results of the response of $s^{-\beta} \cdot (-s)^\alpha$ for various α to $f(x)$.

$f(x)$, respectively. One can see that the abscissas of the composite derivative (for various α) maximum are accurately located at the abscissa $x_0 = 200$.

2.2. The improvement of detection selectivity. Note that the localization is a measure of the width of the peak. The detection selectivity of an inflexion point, S_N , can be defined as the inverse of the bandwidth at $N\%$ of the detector response maximum Δx (Figure 5), i.e.,

$$(2.13) \quad S_N = \frac{1}{\Delta x}.$$

The drift in position of the maximum will decrease as the response becomes sharper. Hence, in order to further improve the localization of the edge point, the response of the

Figure 5. The definition of detection selectivity.

Figure 6. Selectivity improvement by creating a cusp.

designed operator to a step-type parabolic transition should present a sharper cusp on the abscissa of the inflexion (edge) point (Figure 6).

Figures 3 and 4 show the normalized graph of the response of (2.10) and (2.11) to the step-type transition of Figure 1. As can be seen in Figures 3 and 4, the abscissa of the composite derivative maximum and its negative version are accurately located at the transition inflexion point, i.e., the composite derivative is optimized with the localization constraint. However, it has a high bandwidth (Δx), and the expected cusp at the abscissa of the inflexion point $x_0 = 200$ is not perfectly achieved; hence, the response will have many false maximums.

Figure 7. Normalized results of $f(x)$, $f'(x)$, and $f^{(n)}(x)$ for various α .

If (2.11) is subtracted from (2.10), we get the final composite derivative, defined by

$$\begin{aligned}
 & (-s)^{-\beta} \cdot s^{\alpha} - s^{-\beta} \cdot (-s)^{\alpha} \\
 &= \omega^{\alpha-\beta} e^{j\frac{\pi}{2}} - \omega^{\alpha-\beta} e^{-j\frac{\pi}{2}} \\
 &= \omega^{\alpha-\beta} (e^{j\frac{\pi}{2}} - e^{-j\frac{\pi}{2}}) \\
 &= \omega^{\alpha-\beta} \left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2} - \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} \right) \\
 &= 2\omega^{\alpha-\beta} \sin \frac{\pi}{2} \\
 &= 2\omega^{\alpha-\beta} e^{j\frac{\pi}{2}}.
 \end{aligned}
 \tag{2.14}$$

One can see that the magnitude of (2.14) is the sum of the magnitudes of (2.10) and (2.11), and the phase characteristic of (2.14) is always 90° . The normalized graph of the response of (2.14) to the step-type transition of Figure 1 is shown in Figure 7. In Figure 7, the solid line, the dashed line, the dash-dot line, and the dotted line indicate the normalized result of the approximate differentiation when α equals to 0.6, 0.7, 0.8, and 0.9, respectively (β equals to 0.4, 0.3, 0.2, and 0.1, respectively), and $f'(x)$ is the first derivative of $f(x)$. It can be seen that the cusp in Figure 7 is sharper than that in Figure 3, and the larger α is, the higher S_N will be.

2.3. Robustness to noise. There will be more or less noise in real images. Therefore, the stability in the presence of noise is a crucial indicator when one chooses and evaluates edge detection algorithms. The classical first derivative, whose magnitude characteristic is not steerable, is not stable in the presence of noise. The sensitivity to noise diminishes along with derivative order. The fractional-order integral parts in the composite derivative are employed

Figure 8. The representative graph of the frequency spectrum of the composite derivative.

as a special low-pass filter to suppress noise in the new algorithm, while contributing phase. Hence, one can adjust the robustness to noise by changing the value of integral order β .

The magnitude characteristic of the composite derivative is $s^\alpha/(-s)^\beta$ is $2\omega^{(\alpha-\beta)}$ (as seen in (2.14)), where ω denotes frequency. From Figure 8, we can see that the stability of the composite derivative to noise diminishes along with α . In other words, the immunity to noise increases along with β .

Figure 9 shows that $f(x)$ is contaminated by random noise with normal distribution (zero mean and 0.05^2 variance). The detection results (results 1–3) are shown in Figure 10(a) when $f(x)$ is added with three different random noises, respectively. The parameters of the composite derivative are as follows: $\beta = 0.2$ ($\alpha = 0.8$), the mask length is 3, and the width of the filtering function is 2. Result 1 is not smooth on the left side of the edge point ($x = 200$), while results 2 and 3 fluctuate significantly around the edge point. However, all the results obtain the maximum (the edge point) at abscissa $x = 200$. Figure 10(b) shows the detection results of the composite when $\beta = 0.1, 0.2, 0.3$, respectively. First, when β is 0.3 (α is 0.7), the detection result has the best smoothing effect. While β decreases to 0.1, the smoothing effect tends to be relatively poor and the resulted curve oscillates noticeably around the edge point. Figure 10(b) also shows that β will affect detection accuracy as well. When β is 0.1 (α is 0.9), the peak of the curve is at $x = 201$, obviously leading to edge positioning error. However, the composite derivative with $\beta = 0.2, 0.3$ ($\alpha = 0.8, 0.7$) can accurately locate the edge point at $x = 200$. These figures demonstrate that the composite derivative bears good noise immunity performance. The fractional-order integral part in the composite derivative is employed as a special low-pass filter to suppress noise. The stability of the composite derivative to noise diminishes along with α (derivative order). In other words, the immunity to noise increases along with β (integral order). Therefore the stability of the composite derivative in presence of noise, which can minimize the impact of noise factors, can be tuned by adjusting the value of $\alpha - \beta$.

Figure 9. $f(x)$ contaminated by noise.

Figure 10. Normalized detection results of noisy $f(x)$ by the compound derivative: (a) with different noise and (b) with different β .

The composite derivative, which can simulate the first derivative effectively, provides a new approach to calculating the gradient of an image, denoted by ∇f , and defined as the vector in (2.15):

$$(2.15) \quad \nabla f = \begin{bmatrix} g_x \\ g_y \end{bmatrix},$$

where g_x and g_y are the composite derivatives in the x and y directions, respectively.

Different from the classical first-order derivative, the composite derivative is order-steerable, i.e., one can adjust $\alpha - \beta$ to tune the magnitude characteristic and to achieve a trade-off between sensitivity to noise and detection accuracy.

3. Fractional integral and fractional derivative mask. The composite derivative includes two parts. One part is a fractional integration, and the other part is a fractional differentiation, which will be realized by filter convolution. The specific realization is detailed in the following two subsections. First, the detailed theoretical deduction of the fractional integral is presented. Then the fractional derivative mask is reiterated.

3.1. Fractional integral. The Weyl's definitions for forward and backward fractional integrals of order $\tau \in C$, $\Re(\tau) \geq 0$ of function $f(x)$ [7, 28] are defined as, respectively,

$$(3.1) \quad {}_{-\infty}W_x^\tau f(x) = \frac{1}{\Gamma(\tau)} \int_{-\infty}^x (x-u)^{\tau-1} f(u) du$$

and

$$(3.2) \quad {}_xW_\infty^\tau f(x) = \frac{1}{\Gamma(\tau)} \int_x^\infty (u-x)^{\tau-1} f(u) du.$$

It can be demonstrated that the unitary Dirac's impulse function $\delta(x)$ with $b \leq c \leq d$ has the property of

$$(3.3) \quad \int_b^d \delta(x-c) f(x) dx = f(c).$$

According to (3.1) and (3.3), the forward fractional integral with order τ of $\delta(x)$ can be inferred as follows:

$$(3.4) \quad \begin{aligned} I_+^\tau \delta(x) &= \frac{1}{\Gamma(\tau)} \int_{-\infty}^x (x-t)^{\tau-1} \delta(t) dt \\ &= \frac{1}{\Gamma(\tau)} \int_{-\infty}^x \delta(t-0) (x-t)^{\tau-1} dt \\ &= \frac{1}{\Gamma(\tau)} x^{\tau-1}, \quad x > 0, \end{aligned}$$

where $\tau \in C$, $\Re(\tau) \geq 0$.

According to (3.2) and (3.3), the backward fractional integral with order τ of $\delta(x)$ can be inferred as follows:

$$(3.5) \quad \begin{aligned} I_-^\tau \delta(x) &= \frac{1}{\Gamma(\tau)} \int_x^\infty (t-x)^{\tau-1} \delta(t) dt \\ &= \frac{1}{\Gamma(\tau)} \int_x^\infty \delta(t-0) (t-x)^{\tau-1} dt \\ &= \frac{1}{\Gamma(\tau)} (-x)^{\tau-1}, \quad x < 0, \end{aligned}$$

where $\tau \in C$, $\Re(\tau) \geq 0$.

Figure 11. The representative graph of the noncausal fractional-order integrator for $\tau = -0.4$.

Hence, $I_+^\tau \delta(x)$ and $I_-^\tau \delta(x)$ can be written as, respectively,

$$(3.6) \quad I_+^\tau \delta(x) = \begin{cases} x^{\tau-1}/\Gamma(\tau), & x > 0, \\ 0, & x \leq 0, \end{cases}$$

and

$$(3.7) \quad I_-^\tau \delta(x) = \begin{cases} 0, & x \geq 0, \\ (-x)^{\tau-1}/\Gamma(\tau), & x < 0, \end{cases}$$

where $\tau \in C$, $\Re(\tau) \geq 0$, and

$$(3.8) \quad \Gamma(\tau) = \int_0^\infty e^{-t} t^{\tau-1} dt.$$

Adding (3.7) to (3.6), a novel τ -order fractional integrator can be obtained as

$$(3.9) \quad \begin{aligned} I^\tau \delta(x) &= I_+^\tau \delta(x) + I_-^\tau \delta(x) \\ &= \begin{cases} x^{\tau-1}/\Gamma(\tau), & x > 0, \\ 0, & x = 0, \\ (-x)^{\tau-1}/\Gamma(\tau), & x < 0. \end{cases} \end{aligned}$$

The representative graph of the fractional-order integrator for $\tau = 0.4$ is shown in Figure 11.

Thereby, the τ -order fractional integral of $f(x)$ will be written as a convolution between image $f(x)$ and the fractional-order integrator $D^{-\tau} \delta(x)$, which can be represented by

$$(3.10) \quad I^\tau f(x) = f(x) * I^\tau \delta(x),$$

where $\tau \in C$ and $\Re(\tau) \geq 0$.

3.2. Fractional derivative mask. The first-order derivative of $f(x)$ for increasing x can be written as

$$(3.11) \quad D_+ f(x) = \frac{f(x) - f(x-h)}{h},$$

where h is infinitely small.

A shift operator q defined by

$$(3.12) \quad q^{-1}f(x) = f(x-h)$$

gives

$$(3.13) \quad D_+ f(x) = \frac{1 - q^{-1}}{h} f(x).$$

Hence the ν -order derivative of $f(x)$ for increasing x can be written as

$$(3.14) \quad D_+^\nu f(x) = \left(\frac{1 - q^{-1}}{h} \right)^\nu f(x).$$

Additionally, the first derivative of $f(x)$ for increasing x , $D_+ f(x)$, can be written as

$$(3.15) \quad D_+ f(x) = \frac{f(x+h) - f(x)}{h},$$

where h is infinitely small.

A shift operator q defined by

$$(3.16) \quad qf(x) = f(x+h)$$

gives

$$(3.17) \quad D_+ f(x) = \frac{q - 1}{h} f(x).$$

So the ν -order derivative of $f(x)$ for increasing x can be written as

$$(3.18) \quad D_+^\nu f(x) = \left(\frac{q - 1}{h} \right)^\nu f(x).$$

Averaging (3.14) and (3.18), it can be obtained that

$$(3.19) \quad D_+^\nu f(x) = \frac{1}{2h^\nu} [(1 - q^{-1})^\nu + (q - 1)^\nu] f(x).$$

And then, by expanding (3.19) using Newton's binomial formula, $D_+^\nu f(x)$ can be written as

$$(3.20) \quad \begin{aligned} D_+^\nu f(x) &= \frac{1}{2h^\nu} \sum_{k=0}^{\infty} z_k [(-1)^k q^{-k} + (-1)^{v-k} q^k] f(x) \\ &= \frac{1}{2h^\nu} \sum_{k=0}^{\infty} z_k [(-1)^k f(x - kh) + (-1)^{v-k} f(x + kh)], \end{aligned}$$

where

$$(3.21) \quad z_k = \frac{v(v-1)\dots(v-k+1)}{k!}.$$

Similarly, the first derivative of $f(x)$ for decreasing x , $D_-f(x)$, can be written as in (3.22), as well as (3.23):

$$(3.22) \quad D_-f(x) = \frac{f(x) - f(x+h)}{h}$$

and

$$(3.23) \quad D_-f(x) = \frac{f(x-h) - f(x)}{h},$$

where h is infinitely small.

By using the former deduction processing, the v -order fractional derivative of $f(x)$ for decreasing x can be written as

$$(3.24) \quad \begin{aligned} D_-^v f(x) &= \frac{1}{2h^v} \sum_{k=0}^{\infty} z_k [(-1)^k q^k + (-1)^{v-k} q^{-k}] f(x) \\ &= \frac{1}{2h^v} \sum_{k=0}^{\infty} z_k [(-1)^k f(x+kh) + (-1)^{v-k} f(x-kh)]. \end{aligned}$$

Similar to [26], in order to improve detection accuracy, (3.24) should be subtracted from (3.20), i.e., we have a new derivative operator,

$$(3.25) \quad \begin{aligned} D^v f(x) &= D_+^v f(x) - D_-^v f(x) \\ &= \frac{1}{h^v} \sum_{k=0}^{\infty} [a_k f(x-kh) - a_k f(x+kh)], \end{aligned}$$

where

$$(3.26) \quad a_k = \frac{1}{2} [(-1)^k - (-1)^{v-k}] z_k.$$

Hence, the v -order fractional derivative corresponding to the horizontal component can be written as

$$(3.27) \quad X^v = [+a_m \dots + a_k \dots + a_1 \quad 0 \quad -a_1 \dots -a_k \dots -a_m].$$

Since v is a fraction, the component $(-1)^{v-k}$ of a_k is complex. That is, the proposed mask is a complex mask, which is one major difference among the other existing derivative masks and

the proposed mask. For example, when $v = 0.5$, $m = 1$, X_{mask} is $[-0.25 + 0.25i \ 0 \ 0.25 - 0.25i]$. Thus the modulus of the complex result of the mask convolution should be calculated to obtain the gradient magnitude.

4. Experiments and analysis. The gradient of an image reflects the rate of grayscale change and reaches the maximum at edge points. The composite derivative proposed in this paper, which performs a 90° phase shift as the traditional first derivative does, provides a new approach to calculating the gradient of an image. Convolution is an efficient method for rapid calculation of gradient. As presented in section 3, two ways of convolution are involved. One is the convolution between an image and the impulse response of fractional integral (fractional integrator); the other is the convolution between a filtered image and the fractional-order differentiator. The composite derivative in section 2 uses the combination of fractional integration and derivative to performing a first-order derivative and then to calculate an image gradient.

After computing the gradient magnitude, threshold is used to determine edges. Similar to the *Canny* operator, this paper applies the double-threshold method for edge detection and connection [4] with non-maxima suppression on the gradient magnitude. Non-maxima suppression and the double-threshold method can help to detect real edges and suppress the response of fake edges [4].

To verify the effectiveness of the new algorithm, experiments are performed on a large annotated database (Berkeley Segmentation Dataset, BSDS500), which consists of 500 images with contours annotated by more than one human, leading to 1500 ground-truth contour images. Simultaneously, we present the evaluation of multiple edge detection algorithms on the BSDS500, using the precision-recall framework, which has found widespread use for evaluating contour detection [25, 2].

Figure 12 presents part of this benchmark. Figure 13 shows the precision-recall comparison between the composite fractional-order differentiator and integer-order detectors on the original BSDS500 (without noise). Figure 14 presents the detection results of multiple edge detectors on some images of the original BSDS500 (without noise), which have been shown in Figure 12. It can be seen that our composite derivative performs comparably to the *Canny* operator. Figure 15 presents the precision-recall comparison between the composite fractional-order differentiator and integer-order detectors on the BSDS500, which is artificially degraded by an additive *Gaussian* noise with mean 0.3 and variance 0.1, and Figure 16 presents the detection results of multiple edge detectors on some images of BSDS500 with Gaussian noise. It can be seen that both the composite derivative and the *Canny* operator have better immunity to noise than the other operators. One can adjust fractional order to achieve a trade-off between sensitivity to noise and detection accuracy. In Figures 13 and 15, leading edge detection approaches are ranked according to their maximum F-measure ($F = \frac{2 \cdot \text{Precision} \cdot \text{Recall}}{\text{Precision} + \text{Recall}}$) [2]. Iso-F curves (isohypse curves of F) are shown in green. Average agreement between human subjects is indicated by the green dot. The thresholds selected in Figures 14 and 16 correspond to the points of maximal F-measure on the curves in Figures 13 and 15, respectively.

Additionally, Figure 17 presents the precision-recall comparison between the composite derivative with different α and the methods derived in [19] on the BSDS500, which is artificially degraded by an additional noise causing 5 dB signal-to-noise (SNR), and Figures 18 and 19 present the detection results (the thresholds selected correspond to the points of maximal F-

Figure 12. *BSDS500 [2]. Top to bottom: Image and ground-truth segment boundaries handdrawn by different human subjects. The BSDS500 consists of 300 training and 200 test images, each with multiple ground-truth boundaries.*

measure on the curves in Figure 17). Jacob and Unser [19] propose a general approach for the design of 2D feature detectors from a class of steerable functions based on the optimization of a Canny-like criterion. The operators (denoted by “Jacob-Unser”) derived in [19] have a better orientation selectivity than the classical gradient or Hessian-based detectors. It can be seen that the sensitivity to noise increases along with derivative order α , i.e., diminishes along with integral order β , and one can adjust $\alpha - \beta$ to employ a trade-off between robustness to noise and detection accuracy.

Figure 13. Evaluation of edge detectors on BSDS500. Composite derivative with $\beta = 0.25$, $\alpha = 0.75$. Iso-F curves are shown in green. Average agreement between human subjects is indicated by the green dot.

The class of edge detection methods involving nonlocal differential operators is also getting more and more diverse [30, 2]. The gPb (globalized probability of boundary) detector derived in [2] combines multiple local cues into a globalization framework based on spectral graph theory. We consider substituting our composite derivative for the classical differential operator in gPb, termed “CD-gPb.” The precision-recall comparison between CD-gPb and gPb on the BSDS500 (BSDS500 is artificially degraded by an additional noise causing 5 dB SNR) and the detection results are presented in Figures 17, 18, and 19. It can be seen that CD-gPb, providing better precision for most choices of recall, obtains the highest F-measure and performs comparably to gPb.

Quantitatively, the composite derivative provides better precision for most choices of recall, as shown in Figures 13, 15, and 17. Qualitatively, the use of a composite derivative translates into the improvement of stability in the presence of noise and completion of edges in the output, as shown in Figures 14, 16, 18, and 19. The experiments indicate that the proposed composite derivative performs a better selectivity/immunity-to-noise compromise.

5. Conclusion. The composite derivative is obtained through the combination of fractional integration and differentiation. The differentiation and the integration are realized by the fractional complex mask and the fractional integrator, respectively. The successful application of the complex mask proves that it is feasible and effective for introducing a complex calculation into signal singularity detection.

The composite derivative, which is formed from the combination of fractional integration and derivative and performs a 90° phase shift as the traditional first derivative does, provides a

Figure 14. Top to bottom: Image, Roberts, Sobel, Prewitt, Canny, and composite derivative. The thresholds selected correspond to the points of maximal F -measure on the curves in Figure 13.

new approach to calculating the gradient of an image. The major differences from the classical first-order derivative are that the composite derivative is order-steerable, and one can adjust the orders of fractional integration and derivative to tune magnitude characteristics and reach a compromise between sensitivity to noise and detection accuracy. The images containing sufficient and abrupt changes in intensity benefit the most from using the composite derivative.

The comparison of the composite derivative relative to integer-order differentiators (Roberts, Sobel, Prewitt, and Canny) and the state-of-the-art methods is presented using a precision-recall curve. Performances are evaluated using a wide variety of images (BSDS500)

Figure 15. Evaluation of edge detectors on BSDS500, artificially degraded by an additional Gaussian noise with mean 0.3 and variance 0.1. Composite derivative with $\beta = 0.35$, $\alpha = 0.65$. Iso-F curves are shown in green. Average agreement between human subjects is indicated by the green dot.

in the absence of and presence of noise. Performance levels reveal that the composite differentiator is robust to noise and provides a better capacity for selecting edges.

Figure 16. Top to bottom: Image, Roberts, Sobel, Prewitt, Canny, and composite derivative on the BSDS500, artificially degraded by an additional Gaussian noise with mean 0.3 and variance 0.1. The thresholds selected correspond to the points of maximal F-measure on the curves in Figure 15.

Figure 17. Evaluation of edge detectors on BSDS500, artificially degraded by an additional noise causing 5 dB SNR. Composite derivative with $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$. Iso- F curves are shown in green. Average agreement between human subjects is indicated by the green dot.

Figure 18. Top to bottom and left to right: Image, image degraded by an additional noise causing 5 dB SNR, CD-gPb, $\alpha = 0.85$, gPb, composite derivative $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, Jacob-Unser, $M = 1$. The thresholds selected correspond to the points of maximal F-measure on the curves in Figure 17.

Figure 19. Top to bottom and left to right: Image, image degraded by an additional noise causing 5 dB SNR, CD-gPb, $\alpha = 0.85$, gPb, composite derivative $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$, Jacob-Unser, $M = 1$. The thresholds selected correspond to the points of maximal F-measure on the curves in Figure 17.

REFERENCES

- [1] M.S.C. ALMEIDA AND L.B. ALMEIDA, *Blind and semi-blind deblurring of natural images*, IEEE Trans. Image Process., 19 (2010), pp. 36–52.
- [2] P. ARBELAEZ, M. MAIRE, C. FOWLKES, AND J. MALIK, *Contour detection and hierarchical image segmentation*, IEEE Trans. Pattern Anal. Mach. Intelligence, 33 (2011), pp. 898–916.
- [3] A. BELYAEV, *Implicit image differentiation and filtering with applications to image sharpening*, SIAM J. Imaging Sci., 6 (2013), pp. 660–679.
- [4] J. CANNY, *A computational approach to edge detection*, IEEE Trans. Pattern Anal. Mach. Intell., 8 (1986), pp. 679–698.
- [5] H. CHUN, Y. YE, J. BIN, AND X. ZHOU, *A novel edge detection method based on fractional-order calculus mask*, Acta Automat. Sinica, 38 (2012), pp. 776–787.
- [6] S.A. COLEMAN, B.W. SCOTNEY, AND S. SUGANTHAN, *Edge detecting for range data using Laplacian operators*, IEEE Trans. Image Process., 19 (2010), pp. 2814–2824.
- [7] S. DAS, *Functional Fractional Calculus*, Springer-Verlag, Berlin, 2011.
- [8] E.R. DAVIES, *Design of optimal Gaussian operators in small neighbourhoods*, Image Vision Computing, 5 (1987), pp. 199–205.
- [9] L.S. DAVIS, *A survey of edge detection techniques*, Computer Graphics Image Process., 4 (1975), pp. 248–270.
- [10] D. DEMIGNY, *On optimal linear filtering for edge detection*, IEEE Trans. Image Process., 11 (2002), pp. 728–737.
- [11] J.A. DOBROSOTSKAYA AND A.L. BERTOZZI, *Analysis of the wavelet Ginzburg–Landau energy in image applications with edges*, SIAM J. Imaging Sci., 6 (2013), pp. 698–729.
- [12] N. DOKUCHAEV, *On sub-ideal causal smoothing filters*, Signal Process., 92 (2012), pp. 219–223.
- [13] P. DOLLAR, Z. TU, AND S. BELONGIE, *Supervised learning of edges and object boundaries*, in Proceedings of IEEE Computer Society Conference on Computer Vision and Pattern Recognition, vol. 2, 2006, pp. 1964–1971.
- [14] Q. DU, M. GUNZBURGER, R.B. LEHOUCQ, AND K. ZHOU, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Math. Models Methods Appl. Sci., 23 (2013), pp. 493–540.
- [15] W.T. FREEMAN AND E.H. ADELSON, *The design and use of steerable filters*, IEEE Trans. Pattern Anal. Mach. Intell., 13 (1991), pp. 891–906.
- [16] G. GILBOA, *Nonlinear scale space with spatially varying stopping time*, IEEE Trans. Pattern Anal. Mach. Intell., 30 (2008), pp. 2175–2187.
- [17] G. GILBOA AND S. OSHER, *Nonlocal linear image regularization and supervised segmentation*, Multiscale Model. Simul., 6 (2007), pp. 595–630.
- [18] G. GILBOA, N. SOCHEN, AND Y.Y. ZEEVI, *Image enhancement and denoising by complex diffusion processes*, IEEE Trans. Pattern Anal. Mach. Intell., 26 (2004), pp. 1020–1036.
- [19] M. JACOB AND M. UNSER, *Design of steerable filters for feature detection using canny-like criteria*, IEEE Trans. Pattern Anal. Mach. Intell., 26 (2004), pp. 1007–1019.
- [20] F.I. KARAHANOGU, I. BAYRAM, AND D. VAN DE VILLE, *A signal processing approach to generalized 1-D total variation*, IEEE Trans. Signal Process., 59 (2011), pp. 5265–5274.
- [21] T. LINDBERG, *Edge detection and ridge detection with automatic scale selection*, Internat. J. Comput. Vision, 30 (1998), pp. 117–156.
- [22] S. MAHMOODI, *Scale-invariant filtering design and analysis for edge detection*, Proc. R. Soc. Ser. A, 467 (2011), pp. 1719–1738.
- [23] S.G. MALLAT, *A Wavelet Tour of Signal Processing*, 3rd ed., Academic Press, Salt Lake City, UT, 2008.
- [24] D. MARR AND E. HILDRETH, *Theory of edge detection*, Proc. R. Soc. Lond. Ser. B Biol. Sci., 207 (1980), pp. 187–217.
- [25] D.R. MARTIN, C.C. FOWLKES, AND J. MALIK, *Learning to detect natural image boundaries using local brightness, color, and texture cues*, IEEE Trans. Pattern Anal. Mach. Intell., 26 (2004), pp. 530–549.
- [26] B. MATHIEU, P. MELCHIOR, A. OUSTALOUP, AND C. CEYRAL, *Fractional differentiation for edge detection*, Signal Process., 83 (2003), pp. 2421–2432.

- [27] F. MEYER AND J. STAWIASKI, *A Stochastic Evaluation of the Contour Strength*, in Pattern Recognition, Lecture Notes in Comput. Sci. 6376, Springer, New York, 2010, pp. 513–522.
- [28] K.S. MILLER AND B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.
- [29] M.C. MORRONE AND R.A. OWENS, *Feature detection from local energy*, Pattern Recognition Lett., 6 (1987), pp. 303–313.
- [30] P. PARENT AND S.W. ZUCKER, *Trace inference, curvature consistency, and curve detection*, IEEE Trans. Pattern Anal. Mach. Intell., 11 (1989), pp. 823–839.
- [31] I. PETRÁŠ, *Stability of fractional-order systems with rational orders: A survey*, Fract. Calc. Appl. Anal., 12 (2009), pp. 269–298.
- [32] I. PODLUBNY, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, New York, 1998.
- [33] J.M.S. PREWITT, *Object Enhancement and Extraction*, Academic Press, New York, 1970.
- [34] R.D. RETHERFORD AND M.K. CHOE, *Statistical Models for Causal Analysis*, Wiley-Interscience, New York, 2011.
- [35] L.G. ROBERTS, *Machine Perception of Three-Dimensional Solids*, Ph.D. thesis, MIT, Cambridge, MA, 1963.
- [36] E. SABER, A.M. TEKALP, AND G. BOZDAGI, *Fusion of color and edge information for improved segmentation and edge linking*, Image Vision Computing, 15 (1997), pp. 769–780.
- [37] R.J. SCHALKOFF, *Digital Image Processing and Computer Vision*, Wiley, New York, 1989.
- [38] P.F. SNOWDON, *Perceptual Concepts as Non-Causal Concepts*, Oxford University Press, New York, 2011.
- [39] I.E. SOBEL, *Camera Models and Machine Perception*, Technical report, Stanford University, 1970.
- [40] W. STEFAN, R.A. RENAUT, AND A. GELB, *Improved total variation-type regularization using higher order edge detectors*, SIAM J. Imaging Sci., 3 (2010), pp. 232–251.
- [41] Z. SU, X. LUO, Z. DENG, Y. LIANG, AND Z. JI, *Edge-preserving texture suppression filter based on joint filtering schemes*, IEEE Trans. Multimedia, 15 (2013), pp. 535–548.
- [42] R. SUNDARAM, *Algorithms for adaptive transform edge detection*, IEEE Trans. Signal Process., 47 (1999), pp. 2313–2317.
- [43] L. WANG, S. XIANG, G. MENG, H.-Y. WU, AND C. PAN, *Edge-directed single image super-resolution via adaptive gradient magnitude self-interpolation*, IEEE Trans. Circuits Systems Video Tech., 23 (2013), pp. 1289–1299.
- [44] X. WANG, *Laplacian operator-based edge detectors*, IEEE Trans. Pattern Anal. Mach. Intell., 29 (2007), pp. 886–890.
- [45] Y. ZHANG AND P.I. ROCKETT, *The Bayesian operating point of the Canny edge detector*, IEEE Trans. Image Process., 15 (2006), pp. 3409–3416.