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MAP/G/1 Queue Under Workload Control and Postprocessing

Ho Woo Lee,¹ Sahng Hoon Cheon,²
Se Won Lee,¹ and Jung Woo Back¹

¹Department of Systems Management Engineering,
Sungkyunkwan University, Suwon, Korea

²u-Biz Team, Dongbu CNI, Seoul, Korea

Abstract: This article studies the performance of the MAP/G/1 queue under workload control and post processing. We first obtain the probability generating function of the queue length distribution. Then we derive the mean queue length. We present computational experiences and compare the MAP queue with the Poisson queue.

Keywords: *D*-policy; MAP/G/1; Single-vacation.

Mathematics Subject Classification: 60K25.

1. INTRODUCTION

Careful controls of system maintenance and work-in-process inventory are among the most critical factors in the cost-effective operation of a production system. In most studies concerning production systems, the Poisson process has been assumed as the feed process and the *N*-policy

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Address correspondence to Ho Woo Lee, Department of Systems Management Engineering, Sungkyunkwan University, Suwon, 440-746, South Korea; E-mail: hwlee@skku.edu

of Yadin and Naor [32] was used to control the system start-up time. But in real production settings, independent and identically distributed (iid) exponential interarrival times are hardly found. Moreover it is sometimes easier to control the start-up time by the amount of work instead of the number of waiting parts. Examples can be found in chemical production systems.

In this article, we consider a very general production system with correlated interarrival times and maintenance period in which the start-up time is controlled by the amount of work. We model this system by the MAP/G/1 queue under the D -policy and single vacation.

It is widely known that the MAP (Markovian arrival process) can represent a variety of arrival processes which includes, as special cases, the Poisson process, the phase-type renewal process, the MMPP (Markov modulated Poisson process) and superpositions of these. Readers are advised to see Lucantoni et al. [19], Lucantoni [20, 21], Neuts [22, 23], and Ramaswami [24] for the formal definition of MAP and early analyses of MAP-related queueing systems.

In our system, the server leaves for a vacation (post-processing period) of random length V as soon as there is no work to process (see Figure 1). In many real world systems, maintenance operations or cleaning operations are performed during this post-processing period. After the vacation, if the accumulated workload (i.e., the sum of the service times of the waiting customers) exceeds D , the server immediately begins to serve the customers. If not, he waits in the system until the workload exceeds D .

The behavioral complexity of our system is in the “uncommon” relationship between the service times of the customers that arrive during the idle period. For example, assume that the threshold D is crossed by the third customer as in Figure 1. Obviously the three service times $S_1, S_2,$

Figure 1. The workload process.

and S_3 are not independent because one larger S_i would mean smaller S_j . Moreover, they are stochastically different from the ordinary service time random variable S because S_1 and S_2 are smaller than D . The server must spend $U_D = S_1 + S_2 + S_3$ amount of time on serving the three “special” customers, during each of which, it is necessary to keep track of the arrivals of the “ordinary” customers who have iid ordinary service times.

Due to the conditional dependency of the service times of the customers who arrive during the idle period, it is impossible to apply the well-known factorization property of the MAP/G/1 queue with generalized vacations to our system [8, 12].

Studies on the D -policy queueing systems were pioneered by Balachandran [3], Balachandran and Tijms [4], Boxma [5], and Tijms [29] for the M/G/1 queue. Their primary concern was in the optimal control of D under a linear cost structure. While these authors used the mean workload, Chae and Park [6] used the mean queue length to determine the optimal D .

Boxma [5] showed that the optimal D -policy is superior over the optimal N -policy for all service time distributions if the mean workload is used under a linear cost function. But, Artalejo [2] showed that the same is no longer true if the mean queue length is used in the cost function. Gakis et al. [11] considered the lengths of the idle and busy periods. Sivazlian [27] provided an approximate formula for optimal D in terms of the first three moments of the service time. Dshalalow [9] carried out the first extensive study on the queue length process of the batch-arrival modified- D -policy queues with vacations. Chae and Park [7] derived the probability generating function of the queue length of the M/G/1/ D -policy queue. Artalejo [1] derived the complete queue length distribution.

The MAP/G/1/ D -policy queue without vacations was studied by Lee and Song [14] and Lee et al. [13]. The MAP/G/1/ D -policy queue under the multiple vacations was studied by Lee et al. [16]. For more works on D -policy queueing systems, readers are referenced to Li and Niu [17], Rubin and Zhang [26], Tijms [30, 31], Lillo and Martin [18], and Feinberg and Kella [10].

2. THE SYSTEM AND NOTATION

In this article, we consider the queueing system with the following specifications:

- (1) Customers arrive according to the MAP (Markovian arrival process) with parameter matrices (C, D) . At their arrivals, they take a random sample from the service time distribution function $S(x)$ and use it as their service times.

(2) The idle server leaves for a vacation as soon as the system becomes empty. After the vacation, if the workload of the waiting customers is greater than D , he begins to serve the customers right away. If not, he waits in the system for the total sum of the workloads of the arriving customers to exceed D (dormant period).

(3) The service times and vacation times are iid respectively and are independent of the arrival process. We assume that the service times and vacation times are continuous. But the results of this article can be applied to the cases of non-continuous service times with minor modifications.

This article is objective is to derive the distribution of the queue length and the mean value, and investigate the effects of the MAP arrivals.

Throughout the article, we will use the notation as follows.

$S, S(x), s(x), S^*(\theta), E(S)$: service time random variable, its distribution function (DF), its probability density function (pdf), its Laplace-Stieltjes transform (LST) and its mean,

$S^{(n)}(x) = \Pr(S_1 + S_2 + \dots + S_n \leq x)$, $(S^{(0)}(x) = 1)$, $s^{(n)}(x)$: DF and pdf of the n -fold convolution of S with itself,

$V, V(x), V^*(\theta)$: vacation time random variable, its DF and its LST,

N_D : number of customers at the start of a busy period,

U_D : workload at the start of the busy period ($U_D > D$),

$(E)_{ij}$: (i, j) -element of a matrix E ,

$(h)_j$: j th element of a vector h ,

(C, D) : parameter matrices of the underlying Markov chain (UMC) of the MAP,

m : dimension of the phase of the UMC,

$J(t)$: phase of the UMC at time t ,

$\pi_i = \lim_{t \rightarrow \infty} \Pr[J(t) = i]$, $(i = 1, 2, \dots, m)$,

$\pi = (\pi_1, \pi_2, \dots, \pi_m)$,

e : $(m \times 1)$ vector of 1's,

$\lambda = \pi D e$: customer arrival rate,

$\rho = \lambda E(S)$: traffic intensity.

3. ANALYSIS OF THE IDLE PERIOD

Let us call the customers who arrive during the idle period the Special Customers (SC) and the customers who arrive during the busy period the Ordinary Customers (OC).

Deriving the queue length (i.e., the total number of customers in the system including the one in service) distribution of the queueing

system with MAP arrivals starts with the derivation of the queue length distribution at an arbitrary departure. Consider an arbitrary SC (we will call this customer the test-SC) who departs the system. The customers left behind by this test-SC are the SCs who arrive during the remaining idle period since his arrival and the OCs who arrive until his service is finished. The number of these customers again depends on the amount of work at the arrival instance of the test-SC. Thus, the analysis of the idle period is a necessity to the analysis of the queue length process of the whole system.

Let us define $(V_n)_{ij}$ as the joint probability that n customers arrive during a vacation and the UMC phase at the end of the vacation is j under the condition that the UMC phase at the start of the vacation is i . Let V_n be the matrix of $(V_n)_{ij}$. Denoting $V(z)$ as the matrix generating function (GF) of $\{V_n\}$, we have [20]

$$V(z) = \sum_{n=0}^{\infty} V_n z^n = \int_0^{\infty} e^{(C+Dz)x} dV(x). \quad (3.1)$$

Let $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_m)$ be the phase probability vector at the end of a busy period. Then, to derive the mean number $E(N_V)$ of customers that arrive during an arbitrary vacation, we use $e^{(C+zD)x} = I + \sum_{n=1}^{\infty} \frac{x^n}{n!} (C+zD)^n$. Then, from (3.1), we get

$$E(N_V) = \kappa \left[\frac{d}{dz} V(z) \Big|_{z=1} \right] \mathbf{e} = \kappa \int_0^{\infty} \frac{d}{dz} \left[\sum_{n=0}^{\infty} \frac{x^n}{n!} (C+zD)^n \mathbf{e} dV(x) \right] \Big|_{z=1}. \quad (3.2a)$$

To find $\frac{d}{dz} (C+zD)^n$, we use, for an arbitrary matrix $H(z)$,

$$\frac{d}{dz} [H(z)]^n = \sum_{k=0}^{n-1} [H(z)]^k \left[\frac{d}{dz} H(z) \right] [H(z)]^{n-k-1}.$$

Then we get

$$\frac{d}{dz} \left[\sum_{n=0}^{\infty} \frac{x^n}{n!} (C+zD)^n \right] = \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n-1} (C+zD)^k D (C+zD)^{n-k-1}. \quad (3.2b)$$

Since $(C+D)$ is the infinitesimal generator of the stationary probability vector π of the UMC, we have $(C+D)^{n-k-1} \mathbf{e} = \mathbf{0}$ for $k \leq n-2$. Thus, (3.2a) becomes

$$E(N_V) = \kappa \int_0^{\infty} \sum_{n=1}^{\infty} \frac{x^n}{n!} (C+D)^{n-1} D \mathbf{e} dV(x). \quad (3.2c)$$

Let us write (3.2c) as follows.

$$E(N_V) = \kappa(\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1} \int_0^\infty \sum_{n=1}^\infty \frac{x^n}{n!} (\mathbf{e}\pi + \mathbf{C} + \mathbf{D})(\mathbf{C} + \mathbf{D})^{n-1} \mathbf{D}\mathbf{e} dV(x). \quad (3.2d)$$

Then using $\pi(\mathbf{C} + \mathbf{D})^{n-1} = \mathbf{0}$ for $n \geq 2$, we get

$$\begin{aligned} E(N_V) &= \kappa(\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1} \int_0^\infty \left[x\mathbf{e}\pi\mathbf{D}\mathbf{e} + \sum_{n=1}^\infty \frac{x^n}{n!} (\mathbf{C} + \mathbf{D})^n \mathbf{D}\mathbf{e} \right] dV(x) \\ &= \kappa(\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1} \int_0^\infty [x\mathbf{e}\pi\mathbf{D}\mathbf{e} + (e^{(\mathbf{C}+\mathbf{D})x} - \mathbf{I})\mathbf{D}\mathbf{e}] dV(x) \\ &= \lambda E(V) + \kappa(\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1} (\mathbf{V} - \mathbf{I})\mathbf{D}\mathbf{e}, \end{aligned} \quad (3.2e)$$

where we used $\pi\mathbf{D}\mathbf{e} = \lambda$, $(\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1}\mathbf{e} = \mathbf{e}$ and $\int_0^\infty e^{(\mathbf{C}+\mathbf{D})x} dV(x) = \mathbf{V}(z)|_{z=1} = \mathbf{V}$ is the transition probability matrix of the UMC phases during a vacation.

π is the stationary probability vector of the UMC and can be obtained from

$$\pi(\mathbf{C} + \mathbf{D}) = \mathbf{0}, \quad \pi\mathbf{e} = 1. \quad (3.2f)$$

Theorem 3.1. *Let I_D be the length of the idle period. Then its mean value becomes,*

$$\begin{aligned} E(I_D) &= E(V) + \kappa \left\{ \mathbf{V}_0 \sum_{n=0}^\infty [(-\mathbf{C})^{-1}\mathbf{D}]^n (-\mathbf{C})^{-1}\mathbf{e} S^{(n)}(D) + \int_{w=0}^D \sum_{k=1}^\infty \mathbf{V}_k S^{(k)}(w) \right. \\ &\quad \left. \times \sum_{n=0}^\infty [(-\mathbf{C})^{-1}\mathbf{D}]^n (-\mathbf{C})^{-1}\mathbf{e} S^{(n)}(D-w) dw \right\}. \end{aligned} \quad (3.3)$$

Proof. We first note that the threshold D is crossed by the k th customer with probability $S^{(k-1)}(D) - S^{(k)}(D)$ [25]. Thus the LST of the length of the idle period of the MAP/G/1 queue under the D -policy without vacation is given by

$$\mathbf{I}_{NV}^*(\theta) = \sum_{k=1}^\infty [(\theta\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}]^k [S^{(k-1)}(D) - S^{(k)}(D)]. \quad (3.4)$$

Let us define $[\mathbf{V}_n(x)]_{ij}$ as the joint conditional probability that the length of the vacation is less than or equal to x , n customers arrive during the vacation and the UMC phase is j at the end of the vacation under the condition that the UMC phase is i at the start of the vacation. Let $\mathbf{V}_n(x)$

be the matrix of $[V_n(x)]_{ij}$. If no customers arrive during the vacation, the matrix LST of the idle period becomes

$$\sum_{k=1}^{\infty} V_0^*(\theta) [(\theta \mathbf{I} - \mathbf{C})^{-1} \mathbf{D}]^k [S^{(k-1)}(D) - S^{(k)}(D)], \quad (3.5)$$

where $V_0^*(\theta) = \int_0^{\infty} e^{-\theta x} dV_0(x)$ is the LST of the length of the vacation during which no customers arrive.

If n customers arrive during the vacation and the workload at the end of the vacation is $y(>D)$, the LST of the length of the idle period becomes

$$\int_{x=0}^{\infty} \int_{y=D}^{\infty} e^{-\theta x} \sum_{n=1}^{\infty} dV_n(x) s^{(n)}(y) dy. \quad (3.6)$$

If k customers arrive during the vacation and the workload at the end of the vacation is $y(<D)$, the LST of the length of the idle period becomes

$$\begin{aligned} & \int_{x=0}^{\infty} e^{-\theta x} \int_{w=0}^D \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} dV_k(x) s^{(k)}(w) [(\theta \mathbf{I} - \mathbf{C})^{-1} \mathbf{D}]^{n-k} \\ & \times [S^{(n-k-1)}(D-w) - S^{(n-k)}(D-w)] dw. \end{aligned} \quad (3.7)$$

Combining (3.5)–(3.7) yields the matrix LST $\mathbf{I}_D^*(\theta)$ of the length of the idle period. Then (3.3) can be obtained from $E(I_D) = -\boldsymbol{\kappa} \frac{d}{d\theta} \mathbf{I}_D^*(\theta)|_{\theta=0} \mathbf{e}$. $\square$

Now, let us derive the joint distribution of the queue length, the workload and the UMC phase at the start of the busy period. In the simple MAP/G/1 queue under the D -policy (without vacation), there are k customers with workload x at the start of the busy period with matrix probability,

$$\Phi_D(k, x) = \begin{cases} (-\mathbf{C})^{-1} \mathbf{D} dS(x), & (x > D, k = 1), \\ [(-\mathbf{C})^{-1} \mathbf{D}]^k \int_{y=0}^D s(x-y) dS^{(k-1)}(y), & (x > D, k \geq 2). \end{cases} \quad (3.8)$$

Let us define the joint probability $\Omega_{ij}(n, x)$ of the queue length $N_D = n$, the workload $U_D = x$ and the UMC phase $J_B = j$ at the start of the busy period given the UMC phase $J_0 = i$ at the end of the previous busy period. Then we have the following theorem.

Theorem 3.2. *The matrix $\Omega(n, x)$ of $\Omega_{ij}(n, x)$ becomes*

$$\Omega(n, x) = \begin{cases} V_0 \Phi_D(1, x) + V_1 s(x), & (n = 1, x > D) \\ V_0 \Phi_D(n, x) + V_n s^{(n)}(x) \\ + \int_{w=0}^D \sum_{l=1}^{n-1} V_l s^{(l)}(w) \Phi_{D-w}(n-l, x-w) dw, & (n \geq 2, x > D), \end{cases} \quad (3.9)$$

where $\sum_{l=1}^0 (\cdot)$ is interpreted as zero.

Proof. (First equation) If no customers arrive during the vacation, the busy period starts with the next customer if his service time is greater than D which accounts for the first term. The second term represents the case in which one customer arrives during the vacation and his service time is greater than D .

(Second equation) If no customers arrive during the vacation and D is exceeded by the n th next customer, the busy period starts with those n customers which accounts for the first term. If n customers arrive during the vacation and sum of the service times of the n customers exceeds D , the busy period starts with those n customers which accounts for the second term. If l customers arrive during the vacation with total service time w , we need $n-l$ more customers with workload $x-w$ until the busy period starts, which accounts for the last term. $\square$

4. QUEUE LENGTH AT AN ARBITRARY DEPARTURE

In this section, we derive the queue length distribution at an arbitrary departure point. Let us define the joint probabilities as follows:

$x_{k,i} = \Pr$ (at an arbitrary departure, the queue length (i.e., the total number of customers regardless of their types) is k and UMC phase is i),

$x_{k,i}^{sc} = \Pr$ (at an arbitrary departure, the queue length is k , UMC phase is i and the departing customer is a SC),

$x_{k,i}^{oc} = \Pr$ (at an arbitrary departure, the queue length is k , UMC phase is i and the departing customer is a OC).

Let us define the vectors $\mathbf{x}_k = (x_{k,1}, x_{k,2}, \dots, x_{k,m})$, $\mathbf{x}_k^{sc} = (x_{k,1}^{sc}, x_{k,2}^{sc}, \dots, x_{k,m}^{sc})$, and $\mathbf{x}_k^{oc} = (x_{k,1}^{oc}, x_{k,2}^{oc}, \dots, x_{k,m}^{oc})$. Then, we have

$$\mathbf{x}_k = \mathbf{x}_k^{sc} + \mathbf{x}_k^{oc}, \quad (k \geq 0). \quad (4.1a)$$

Defining vector GFs $X(z) = \sum_{k=0}^{\infty} z^k \mathbf{x}_k$, $X_{sc}(z) = \sum_{k=0}^{\infty} z^k \mathbf{x}_k^{sc}$ and $X_{oc}(z) = \sum_{k=0}^{\infty} z^k \mathbf{x}_k^{oc}$, we have

$$X(z) = X_{sc}(z) + X_{oc}(z). \quad (4.1b)$$

4.1. Queue Length at an Arbitrary SC Departure

To derive the vector GF $X_{sc}(z)$ of the queue length at the departure of an arbitrary SC (test-SC), we need the information of the workload at the arrival instance of the test-SC.

The test-SC may arrive during the vacation or during the dormant period. This situation can be classified into the five disjoint cases as follows:

Case-(1) The test-SC arrives during the vacation:

- (a) The workload just after the arrival of the test-SC (including the service time of the test-SC itself) is less than or equal to D , but the workload at the end of the vacation exceeds D .
- (b) The threshold D is exceeded before the arrival of the test-SC or by the test-SC itself.
- (c) The workload at the end of the vacation does not exceed D .

Case-(2) The test-SC arrives during the dormant period:

- (a) No customers arrive during the vacation.
- (b) One or more customers arrive during the vacation but the workload at the end of the vacation does not exceed D .

Let us consider the Case-(1) first. In this case, the number of customers left behind by the departing test-SC is the sum of the following two:

- (i) The number N_{RI} of SCs that arrive after the arrival of The test-SC, and
- (ii) The number of OCs that arrive during the sum U_{TC} of the service times of the SCs up to the test-SC (including the service time of the test-SC).

Obviously N_{RI} and U_{TC} are not independent. Let us define the joint probability as follows,

$$\alpha(w, n, j)dw = \Pr(w < U_{TC} \leq w + dw, N_{RI} = n, J_B = j). \quad (4.2)$$

To find $\alpha(w, n, j)$, we need to find the probability $V_+(k, n, j)$ that the UMC phase is j at the end of the vacation during which the test-SC arrives, the test-SC is the k th customer during the vacation ($k \geq 1$) and n more customers arrive during the remaining vacation ($n \geq 0$). We have the following theorem.

Theorem 4.1. *We have*

$$V_+(k, n, j) = \frac{1}{1 - v_0} \cdot \left(\frac{\kappa V_{k+n}}{E(A_V)} \right)_j = \left(\frac{\kappa V_{k+n}}{E(N_V)} \right)_j, \quad (4.3)$$

where

$$E(A_V) = \frac{1}{1-v_0} \kappa \left[\frac{d}{dz} V(z) \Big|_{z=1} \right]$$

$$\mathbf{e} = \frac{1}{1-v_0} \kappa [\lambda E(V) + (\mathbf{e}\pi + \mathbf{C} + \mathbf{D})^{-1}(\mathbf{V} - \mathbf{I})\mathbf{D}]\mathbf{e} \quad (4.4)$$

is the mean number of customers that arrive during the vacation under the condition that one or more customers arrive, and v_0 is the probability that no customers arrive during a vacation which is given by $v_0 = \kappa V_0 \mathbf{e} = \kappa \int_0^\infty e^{Cx} dV(x) \mathbf{e}$.

Proof. First we have $\Pr(A_V = n, J_B = j) = \frac{1}{1-v_0} (\kappa V_n)_j$, ($n \geq 1$). Thus, the PGF of A_V becomes $A_V(z) = \sum_{n=1}^\infty \frac{1}{1-v_0} \kappa V_n z^n \mathbf{e} = \frac{1}{1-v_0} \kappa [V(z) - V_0] \mathbf{e}$ which leads to (4.4) (note (3.2a)).

To prove (4.3), consider the discrete-renewal process generated by iid A_V . Then the probability $a_{k+n,j}$ that the test-SC belongs to a group of size $k+n$ and the phase is j at the end of the vacation becomes $a_{k+n,j} = \frac{(k+n)\Pr(A_V=k+n, J_B=j)}{E(A_V)}$. The test-SC is the k th customer with probability $\frac{1}{k+n}$. Thus, the probability that the test-SC is the k th customer during the vacation, there are n additional customers arriving during the vacation and the UMC phase at the end of the vacation is j becomes $a_{k+n,j} \cdot \frac{1}{k+n} = \left(\frac{1-v_0}{E(A_V)} \kappa V_{k+n} \mathbf{e} \right)_j$. The second equality of (4.3) comes from (3.2a). $\square$

Now we are ready to express those $\alpha(w, n, j)$'s defined in (4.2) for those five different cases mentioned earlier.

Case-(1)-(a) (See Figure 2) With probability $\frac{\kappa V_{k+n}}{E(N_V)} s^{(k)}(w) s^{(n)}(u)$, the workload after the arrival of the test-SC is $w(\leq D)$ and the sum of the service times of the SCs who arrive after the test-SC during the remaining vacation is u . Thus in this case we have

$$\alpha(w, n, j) = \left[\int_{u=D-w}^\infty \sum_{k=1}^\infty \frac{\kappa V_{k+n}}{E(N_V)} s^{(k)}(w) s^{(n)}(u) du \right]_j, \quad (n \geq 1, w \leq D). \quad (4.5)$$

The matrix GF of the number of customers who arrive during w is given by $e^{(C+Dz)w}$. Thus, the queue length vector GF at the departure of the test-SC under Case-(1)-(a) becomes

$$\alpha_{1a}(z) = \phi_V \left[\frac{1}{1-v_0} \int_{w=0}^D \int_{u=D-w}^\infty \sum_{n=1}^\infty \sum_{k=1}^\infty z^n \frac{\kappa V_{k+n}}{E(A_V)} s^{(k)}(w) s^{(n)}(u) e^{(C+Dz)w} du dw \right]$$

$$= \frac{1}{E(N_{cycle})} \left[\int_{w=0}^D \int_{u=D-w}^\infty \sum_{n=1}^\infty \sum_{k=1}^\infty z^n \kappa V_{k+n} s^{(k)}(w) s^{(n)}(u) e^{(C+Dz)w} du dw \right]. \quad (4.6)$$

Figure 2. Case-(1)-(a).

where, in the second equality, we used

$$\phi_V = \frac{E(N_V)}{E(N_{cycle})}, \quad (4.7)$$

which is the probability that the test-SC is a SC that arrives during a vacation.

In (4.7), $E(N_V)$ is the mean number of customers that arrive during a vacation which was given in (3.2a) and $E(N_{cycle})$ is the mean number of customers that are served during a cycle. $E(N_{cycle})$ can be obtained from

$$E(N_{cycle}) = \kappa \kappa^*, \quad (4.8)$$

where κ^* is the vector of the mean number of customers that are served during a cycle given the phase of the UMC at the start of the cycle. κ and κ^* will be derived later in (4.19) and (4.20).

In the analogous way, for the other four cases, we get the followings:

Case-(1)-(b)

$$\alpha_{1b}(z) = \frac{1}{E(N_{cycle})} \left[\int_{w=D}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} z^n \kappa V_{k+n} s^{(k)}(w) e^{(C+Dz)w} dw \right]. \quad (4.9)$$

Case-(1)-(c)

$$\begin{aligned} \alpha_{1c}(z) = \frac{1}{E(N_{cycle})} & \left[\int_{w=D}^{\infty} \int_{u=0}^{D-w} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} z^n \kappa V_{k+n} s^{(k)}(w) s^{(l)}(u) \right. \\ & \left. \times \Phi_{D-w-u}(z) e^{(C+Dz)w} du dw \right] \end{aligned} \quad (4.10)$$

where $\Phi_D(z) = \sum_{k=1}^{\infty} [\int_D^{\infty} \Phi_D(k, x) dx] z^k$ is the matrix GF of the queue length at the busy period starting point under the D -policy without vacation.

The queue length GF under Case-(2) can be obtained by using the results of MAP/G/1 queue under the D -policy without vacation [14] and becomes as follows:

Case-(2)-(a) We have

$$\begin{aligned} \alpha_{2a}(z) = & \frac{\kappa V_0}{E(N_{cycle})} \left\{ [(-C)^{-1} D] \int_{w=D}^{\infty} e^{(C+Dz)w} dS(w) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} z^k [(-C)^{-1} D]^n \right. \\ & \times \int_{w=0}^D e^{(C+Dz)w} [S^{(k-1)}(D-x) - S^{(k)}(D-x)] dS^{(n-k)}(w) \\ & \left. + \sum_{n=2}^{\infty} [(-C)^{-1} D]^n \int_{w=D}^{\infty} e^{(C+Dz)w} \int_{y=0}^D s(x-y) dS^{(n-1)}(y) dw \right\}. \quad (4.11) \end{aligned}$$

Case-(2)-(b)

$$\begin{aligned} \alpha_{2b}(z) = & \int_{x=0}^D \frac{\kappa \sum_{m=1}^{\infty} V_m s^{(m)}(x)}{E(N_{cycle})} \\ & \times \left\{ [(-C)^{-1} D] \int_{w=D-x}^{\infty} e^{(C+Dz)(w+x)} dS(w) + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} z^k [(-C)^{-1} D]^n \right. \\ & \times \int_{w=0}^{D-x} e^{(C+Dz)(w+x)} [S^{(k-1)}(D-x-w) - S^{(k)}(D-x-w)] dS^{(n-k)}(w) \\ & \left. + \sum_{n=2}^{\infty} [(-C)^{-1} D]^n \int_{w=D-x}^{\infty} e^{(C+Dz)(w+x)} \int_{y=0}^{D-x} s(x-y) dS^{(n-1)}(y) dw \right\} dx. \quad (4.12) \end{aligned}$$

Now, finally the vector GF $X_{sc}(z)$ of the queue length at the end of the departure of an arbitrary SC (including the probability that the departure point is a SC departure) can be obtained from

$$X_{sc}(z) = \{\alpha_{1a}(z) + \alpha_{1b}(z) + \alpha_{1c}(z) + \alpha_{2a}(z) + \alpha_{2b}(z)\}. \quad (4.13)$$

4.2. Queue Length at an Arbitrary OC Departure

In this section, we derive the vector queue length GF $X_{oc}(z)$ at an arbitrary OC-departure. Note that the services of OCs begin only after the services of all SCs are finished.

To obtain the probability vector $\mathbf{x}_k^{oc}$ of the queue length just after the ‘current’ OC-departure, we consider the queue length at the ‘previous’ departure. Let $\mathbf{x}_{j,last}^{sc}$ be the vector probability that the previous departure

is a SC-departure (this is the last departure among the N_D SC departures) and the queue length is j . Let $\mathbf{X}_{sc}^{last}(z)$ be its vector GF. Then, we have [14]

$$\mathbf{X}_{oc}(z) = [\mathbf{X}_{sc}^{last}(z) - \mathbf{x}_0] \mathbf{A}(z) [z\mathbf{I} - \mathbf{A}(z)]^{-1}, \quad (4.14)$$

where $\mathbf{A}(z) = \int_0^\infty e^{(C+Dz)x} dS(x)$ is the matrix GF of the number of customers that arrive during a service time.

Now, we need to obtain the vector generating function $\mathbf{X}_{sc}^{last}(z)$ and the vector $\mathbf{x}_0$ contained in (4.14). To derive $\mathbf{X}_{sc}^{last}(z)$, we note that the customers that are left by the departing last SC are the ones that arrive during U_D , which is the workload at the start of the busy period. Thus we get

$$\begin{aligned} \mathbf{X}_{sc}^{last}(z) = & \frac{1}{E(N_{cycle})} \left[\int_{w=D}^\infty \sum_{k=1}^\infty \boldsymbol{\kappa} \mathbf{V}_k s^{(k)}(w) e^{(C+Dz)w} dw \right] \\ & + \frac{\boldsymbol{\kappa} \mathbf{V}_0}{E(N_{cycle})} \left[[(-C)^{-1} \mathbf{D}] \int_{t=D}^\infty e^{(C+D)t} dS(t) \right. \\ & + \sum_{n=2}^\infty [(-C)^{-1} \mathbf{D}]^n \int_{t=D}^\infty e^{(C+Dz)t} \int_{y=0}^D s(t-y) dS^{(n-1)}(y) dt \left. \right] \\ & + \int_{x=0}^D \frac{\boldsymbol{\kappa} \sum_{m=1}^\infty \mathbf{V}_m s^{(m)}(x)}{E(N_{cycle})} \left[[(-C)^{-1} \mathbf{D}] \int_{t=D-x}^\infty e^{(C+Dz)(t+x)} dS(t) \right. \\ & + \sum_{n=2}^\infty [(-C)^{-1} \mathbf{D}]^n \int_{t=D-x}^\infty e^{(C+Dz)(t+x)} \int_{y=0}^{D-x} s(t-y) dS^{(n-1)}(y) dt \left. \right] dx. \end{aligned} \quad (4.15)$$

Now, we are ready to obtain $\mathbf{x}_0$. To this end, we first need to obtain $\mathbf{K}(z)$ which is the matrix GF of the number of customers that are served during a cycle (see Fig. 1). We have the following theorem.

Theorem 4.2. *We have*

$$\begin{aligned} \mathbf{K}(z) = & \int_{x=D}^\infty z \mathbf{V}_0 [(-C)^{-1} \mathbf{D}] e^{[C+D\mathbf{G}(z)]x} dS(x) + \int_{x=D}^\infty \sum_{n=1}^\infty z^n \mathbf{V}_n e^{[C+D\mathbf{G}(z)]x} dS^{(n)}(x) \\ & + \int_{x=D}^\infty \sum_{n=2}^\infty z^n \left\{ \mathbf{V}_0 [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^D s(x-y) dS^{(n-1)}(y) \right. \\ & + \int_{w=0}^D \mathbf{V}_{n-1} [(-C)^{-1} \mathbf{D}] dS(x-w) dS^{(n-1)}(w) \\ & + \int_{w=0}^D \sum_{l=1}^\infty z^l \mathbf{V}_l [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^{D-w} s(x-w-y) dS^{(n-1)}(y) dS^{(l)}(w) \left. \right\} \\ & \times e^{[C+D\mathbf{G}(z)]x} dx, \end{aligned} \quad (4.16)$$

where $\mathbf{G}(z) = z \int_0^\infty e^{[C+D\mathbf{G}(z)]x} dS(x)$ is the matrix GF that represents the number of customers that are served during a fundamental period [22, 23].

Proof. Let us define $Q_{i,j}(k_1, k_2)$ as the probability that there are k_1 SCs at the busy period starting point and there are k_2 SCs at the end of the service of the last SC with UMC phase j under the condition that the UMC phase is i at the start of the idle period. Let $\mathbf{Q}_{(k_1, k_2)}$ be the matrix of $Q_{i,j}(k_1, k_2)$. Then, using (3.9), we get $\mathbf{Q}_{(k_1, k_2)} = \int_D^\infty \mathbf{\Omega}(k_1, x) \cdot \mathbf{P}(k_2, x) dx$ where $\mathbf{P}(n, x)$ is the matrix probability that n customers arrive during x . Let us define the matrix GF $\mathbf{Q}(z_1, z_2) = \sum_{k_1=1}^\infty \sum_{k_2=0}^\infty \mathbf{Q}_{(k_1, k_2)} z_1^{k_1} z_2^{k_2}$. Then, (4.16) can be obtained from

$$\mathbf{K}(z) = \mathbf{Q}(z_1, z_2)|_{z_1=z, z_2=G(z)}, \quad (4.17)$$

where we used $\sum_{k_2=0}^\infty z_2^{k_2} \cdot \mathbf{P}(k_2, x) = e^{(C+Dz_2)x}$. $\square$

The matrix $\mathbf{K}$ that denotes the phase shift probability during a cycle becomes

$$\begin{aligned} \mathbf{K} = \mathbf{K}(z)|_{z=1} &= \int_{x=D}^\infty \mathbf{V}_0 [(-C)^{-1} \mathbf{D}] e^{(C+D\mathbf{G})x} dS(x) \\ &+ \int_{x=D}^\infty \sum_{n=1}^\infty \mathbf{V}_n e^{(C+D\mathbf{G})x} dS^{(n)}(x) \\ &+ \int_{x=D}^\infty \sum_{n=2}^\infty \left\{ \mathbf{V}_0 [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^D s(x-y) dS^{(n-1)}(y) \right. \\ &+ \int_{w=0}^D \mathbf{V}_{n-1} [(-C)^{-1} \mathbf{D}] dS(x-w) dS^{(n-1)}(w) \\ &+ \left. \int_{w=0}^D \sum_{l=1}^\infty \mathbf{V}_l [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^{D-w} s(x-w-y) dS^{(n-1)}(y) dS^{(l)}(w) \right\} \\ &\times e^{(C+D\mathbf{G})x} dx. \end{aligned} \quad (4.18)$$

The phase probability vector $\boldsymbol{\kappa}$ at the start of the cycle can be obtained from

$$\boldsymbol{\kappa} = \boldsymbol{\kappa} \mathbf{K}, \quad \boldsymbol{\kappa} \mathbf{e} = 1. \quad (4.19)$$

The vector $\boldsymbol{\kappa}^*$ that denotes the mean number of customers that are served during a cycle becomes

$$\begin{aligned} \boldsymbol{\kappa}^* &= \frac{d}{dz} \mathbf{K}(z)|_{z=1} \mathbf{e} = \int_{x=D}^\infty \mathbf{V}_0 [(-C)^{-1} \mathbf{D}] \\ &\times \left\{ e^{(C+D\mathbf{G})x} + \left[\frac{d}{dz} e^{[C+D\mathbf{G}(z)]x} \right]_{z=1} \right\} \mathbf{e} dS(x) \end{aligned}$$

$$\begin{aligned}
& + \int_{x=D}^{\infty} \sum_{n=1}^{\infty} \left\{ n V_n e^{(C+DG)x} + V_n \left[\frac{d}{dz} e^{[C+DG(z)]x} \Big|_{z=1} \right] \right\} \mathbf{e} dS^{(n)}(x) \\
& + \int_{x=D}^{\infty} \sum_{n=2}^{\infty} \left\{ \left[V_0 [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^D s(x-y) dS^{(n-1)}(y) \right. \right. \\
& + \int_{w=0}^D V_{n-1} [(-C)^{-1} \mathbf{D}] dS(x-w) dS^{(n-1)}(w) \\
& + \left. \left. \int_{w=0}^D \sum_{l=1}^{\infty} V_l [(-C)^{-1} \mathbf{D}]^n \int_{y=0}^{D-w} s(x-w-y) dS^{(n-1)}(y) dS^{(l)}(w) \right] \right. \\
& \times \left. \left\{ n e^{(C+DG)x} + \left[\frac{d}{dz} e^{[C+DG(z)]x} \Big|_{z=1} \right] \right\} \mathbf{e} \right\} dx \\
& + \int_{x=D}^{\infty} \sum_{n=2}^{\infty} \left\{ \int_{w=0}^D \sum_{l=1}^{\infty} l V_l [(-C)^{-1} \mathbf{D}]^n \right. \\
& \times \left. \int_{y=0}^{D-w} s(x-w-y) dS^{(n-1)}(y) dS^{(l)}(w) \right\} e^{(C+DG)x} \mathbf{e} dx, \quad (4.20)
\end{aligned}$$

where we used $\frac{d}{dz} e^{[C+DG(z)]x} \Big|_{z=1} \mathbf{e} = \sum_{k=1}^{\infty} \frac{x^k}{k!} (C + DG)^{k-1} \mathbf{D} \boldsymbol{\mu}$ in which $\boldsymbol{\mu}$ is the mean number of customers that are served during a fundamental period which is given by [20],

$$\boldsymbol{\mu} = \frac{d}{dz} \mathbf{G}(z) \Big|_{z=1} \mathbf{e} = (\mathbf{I} - \mathbf{G} + \mathbf{e}\mathbf{g}) [\mathbf{I} - \mathbf{A} + (\mathbf{e} - \boldsymbol{\beta})\mathbf{g}]^{-1} \mathbf{e}. \quad (4.21)$$

In (4.21), $\mathbf{A} = \int_0^{\infty} e^{(C+D)x} dS(x)$ is the phase change probability during a service time, $\mathbf{G} = \mathbf{G}(z) \Big|_{z=1}$ is the phase change probability during a fundamental period and $\mathbf{g} = (g_1, g_2, \dots, g_m)$ is the stationary vector of $\mathbf{G}$ which satisfies

$$\mathbf{g} = \mathbf{g}\mathbf{G}, \quad \mathbf{g}\mathbf{e} = 1. \quad (4.22)$$

$\boldsymbol{\beta}$ in (4.21) is the mean number of customers that arrive during a service time and is given by [20]

$$\boldsymbol{\beta} = \frac{d}{dz} \mathbf{A}(z) \Big|_{z=1} \mathbf{e} = \rho \mathbf{e} + (\mathbf{e}\boldsymbol{\pi} + \mathbf{C} + \mathbf{D})^{-1} (\mathbf{A} - \mathbf{I}) \mathbf{D} \mathbf{e}. \quad (4.23)$$

Now, finally $\mathbf{x}_0$ can be obtained from [22]

$$\mathbf{x}_0 = \frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa}\boldsymbol{\kappa}^*}. \quad (4.24)$$

Using (4.15) and (4.24) in (4.14) allows us to obtain the complete $X_{oc}(z)$.

Now, at last, the vector GF $X(z)$ of the queue length at an arbitrary departure can be obtained by using (4.13)–(4.15) in (4.1b).

5. THE QUEUE LENGTH AT AN ARBITRARY TIME

Let $y_{k,i}$ be the probability that there are k customers in the system and the UMC phase is i at an arbitrary time in steady-state. Let $\mathbf{y}_k$ be the vector $\mathbf{y}_k = (y_{k,1}, \dots, y_{k,m})$. If we define the vector GF $\mathbf{Y}(z) = \sum_{k=0}^{\infty} \mathbf{y}_k z^k$, we have (Takine and Takahashi [28]).

$$\mathbf{Y}(z)(\mathbf{C} + \mathbf{D}z) = \lambda(z-1)\mathbf{X}(z) = \lambda(z-1)[\mathbf{X}_{sc}(z) + \mathbf{X}_{oc}(z)]. \quad (5.1)$$

6. THE MEAN QUEUE LENGTHS

Using $\mathbf{M}$ and $\mathbf{M}^{(n)}$ for a matrix GF $\mathbf{M}(z)$ to mean $\mathbf{M} = \mathbf{M}(z)|_{z=1}$ and $\mathbf{M}^{(n)} = \frac{d^n}{dz^n} \mathbf{M}(z)|_{z=1}$, the departure point mean queue length L_d can be obtained from (4.1b) as follows,

$$L_d = \frac{d}{dz} \mathbf{X}(z) \Big|_{z=1} \mathbf{e} = \mathbf{X}^{(1)} \mathbf{e} = (\mathbf{X}_{sc}^{(1)} + \mathbf{X}_{oc}^{(1)}) \mathbf{e}. \quad (6.1)$$

From (4.13), we get, after a laborious manipulation,

$$\mathbf{X}_{sc}^{(1)} \mathbf{e} = \{\mathbf{A} + \mathbf{B} + \mathbf{C}\}, \quad (6.2a)$$

where

$$\begin{aligned} \mathbf{A} = \frac{1}{E(N_{cycle})} & \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n \cdot \boldsymbol{\kappa} \mathbf{V}_{k+n} \mathbf{e} \right. \\ & + \int_{w=0}^D \int_{u=D-w}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \boldsymbol{\kappa} \mathbf{V}_{k+n} s^{(k)}(w) s^{(n)}(u) \left[\frac{d}{dz} e^{(\mathbf{C} + \mathbf{D}z)w} \Big|_{z=1} \mathbf{e} \right] du dw \\ & + \int_{w=D}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \boldsymbol{\kappa} \mathbf{V}_{k+n} s^{(k)}(w) \left[\frac{d}{dz} e^{(\mathbf{C} + \mathbf{D}z)w} \Big|_{z=1} \mathbf{e} \right] dw \\ & + \int_{w=0}^D \int_{u=0}^{D-w} \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \boldsymbol{\kappa} \mathbf{V}_{k+l} s^{(k)}(w) s^{(l)}(u) \boldsymbol{\Phi}_{D-w-u}^{(1)} \mathbf{e} du dw \\ & \left. + \int_{w=0}^D \int_{u=0}^{D-w} \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \boldsymbol{\kappa} \mathbf{V}_{k+l} s^{(k)}(w) s^{(l)}(u) \boldsymbol{\Phi}_{D-w-u} \left[\frac{d}{dz} e^{(\mathbf{C} + \mathbf{D}z)w} \Big|_{z=1} \mathbf{e} \right] du dw \right\} \quad (6.2b) \end{aligned}$$

$$\begin{aligned} \mathbf{B} = \frac{\boldsymbol{\kappa} \mathbf{V}_0}{E(N_{cycle})} & \left\{ \int_{w=D}^{\infty} [(-\mathbf{C})^{-1} \mathbf{D}] \left[\frac{d}{dz} e^{(\mathbf{C} + \mathbf{D}z)w} \Big|_{z=1} \mathbf{e} \right] dS(w) \right. \\ & + \int_{w=0}^D [1 + \mathbf{M}(D-w)] dM(w) \mathbf{e} \\ & \left. + \int_{w=0}^D \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} [(-\mathbf{C})^{-1} \mathbf{D}]^n \right. \end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{d}{dz} e^{(C+Dz)w} \Big|_{z=1} \mathbf{e} \right] \left[S^{(k-1)}(D-x) - S^{(k)}(D-x) \right] dS^{(n-k)}(w) \\
& + \int_{w=D}^{\infty} \sum_{n=2}^{\infty} [(-C)^{-1} \mathbf{D}]^n \left[\frac{d}{dz} e^{(C+Dz)w} \Big|_{z=1} \mathbf{e} \right] \int_{y=0}^D s(x-y) dS^{(n-1)}(y) dw \Big\}
\end{aligned} \tag{6.2c}$$

$$\begin{aligned}
C = & \int_{x=0}^D \frac{\kappa \sum_{m=1}^{\infty} V_m S^{(m)}(x)}{E(N_{cycle})} \left\{ \int_{w=D-x}^{\infty} [(-C)^{-1} \mathbf{D}] \left[\frac{d}{dz} e^{(C+Dz)(w+x)} \Big|_{z=1} \mathbf{e} \right] dS(w) \right. \\
& + \int_{w=0}^{D-x} [1 + M(D-x-w)] dM(w) \mathbf{e} \\
& + \int_{w=0}^{D-x} \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} [(-C)^{-1} \mathbf{D}]^n \left[\frac{d}{dz} e^{(C+Dz)(w+x)} \Big|_{z=1} \mathbf{e} \right] \\
& \times [S^{(k-1)}(D-x-w) - S^{(k)}(D-x-w)] dS^{(n-k)}(w) \\
& + \int_{w=D-x}^{\infty} \sum_{n=2}^{\infty} [(-C)^{-1} \mathbf{D}]^n \left[\frac{d}{dz} e^{(C+Dz)(w+x)} \Big|_{z=1} \mathbf{e} \right] \\
& \times \int_{y=0}^{D-x} s(w-y) dS^{(n-1)}(y) dw \Big\} dx
\end{aligned} \tag{6.2d}$$

$\mathbf{X}_{oc}^{(1)} \mathbf{e}$ can be derived by employing the well-known standard technique of [19] to (4.14) and we state only the final result here:

$$\begin{aligned}
\mathbf{X}_{oc}^{(1)} \mathbf{e} = & \frac{1}{2(1-\rho)} \{ \mathbf{B}^{(2)} \mathbf{e} + \mathbf{X}_{oc} \mathbf{A}^{(2)} \mathbf{e} \\
& + 2[\mathbf{B}^{(1)} - \mathbf{X}_{oc}(\mathbf{I} - \mathbf{A}^{(1)})](\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1} \boldsymbol{\beta} \}.
\end{aligned} \tag{6.3}$$

Using (6.2a) and (6.3) in (6.1) yields the departure point mean queue length L_d .

The mean queue length L at an arbitrary time can be obtained from (5.1) and we get

$$L = \mathbf{Y}^{(1)} \mathbf{e} = L_d - \left(\mathbf{X} - \frac{\pi \mathbf{D}}{\lambda} \right) (\mathbf{e}\boldsymbol{\pi} + \mathbf{C} + \mathbf{D})^{-1} \mathbf{D} \mathbf{e}, \tag{6.4}$$

where $\mathbf{X}$ can be obtained from

$$\mathbf{X} = \mathbf{X}_{sc} + \mathbf{X}_{oc}. \tag{6.5}$$

$\mathbf{X}_{sc}$ in (6.5) can be obtained from (4.13) just by using $z = 1$. $\mathbf{X}_{oc}$ can be obtained from (4.14) and becomes

$$\mathbf{X}_{oc} = \boldsymbol{\pi} - \mathbf{X}_{sc} \mathbf{e}\boldsymbol{\pi} + (\mathbf{X}_{oc}^{last} - \mathbf{x}_0) \mathbf{A}(\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1}. \tag{6.6}$$

7. COMPUTATIONAL EXPERIENCE AND COMPARISON WITH THE POISSON QUEUE

7.1. Verification Through Simulation

To verify our analytic results through simulation, we use SIMSCRIPT II.5 simulation language. For the service time and vacation time distributions, we assume exponential distributions. Our simulation time is 12,000,000 time units. As a comparison criterion, we use the absolute percentage error that is defined by

$$\frac{|\text{theoretical value} - \text{simulation estimates}|}{\text{theoretical value}} \times 100.$$

The parameter matrices are assumed to be $\mathbf{C} = \begin{bmatrix} -10 & 1 \\ 0.4 & -0.8 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 9 & 0 \\ 0 & 0.4 \end{bmatrix}$, and we further assume $E(S) = \frac{63}{200}$, $E(V) = 1$, $D = 2$.

From $\boldsymbol{\pi}(\mathbf{C} + \mathbf{D}) = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{e} = 1$, the stationary vector probability of the UMC becomes $\boldsymbol{\pi} = (\pi_1, \pi_2) = (2/7, 5/7)$. Then, the arrival rate becomes $\lambda = \boldsymbol{\pi}\mathbf{D}\mathbf{e} = 20/7$ and we get $\rho = \lambda E(S) = 0.9$.

As can be seen in Table 1 and many other parameter combinations that are not shown here, the percentage errors were consistently within 0.5%, which shows that our analytical results are correct.

7.2. Comparison with the Poisson Queue

In this section, we present our computational experiences by comparing the MAP/G/1 system with the M/G/1 queue under the same threshold values, mean arrival rates and mean service times. We will assume the same parameter matrices as in Subsection 7.1.

For simplicity, we assume that the vacation time follows the exponential distribution with mean $E(V) = 1.0$. We also assume that the service times follows the exponential distribution. We will vary mean service time $E(S)$ so that we have different values of traffic intensity $\rho = \lambda E(S)$.

Table 2 shows the comparisons of the mean queue lengths of the two different systems when $D = 2.0$. The third column shows the ratio of the two mean queue lengths.

Table 1. Comparison with simulation estimates

	Theoretical value	Simulation	Absolute percentage error
L_d	48.1191	47.9581	0.335
L	44.4330	44.2074	0.328

Table 2. Comparison with the Poisson queue ($D = 0.5$)

ρ	$L(\text{MAP})$	$L(\text{Poisson})$	$\frac{L(\text{MAP})}{L(\text{Poisson})}$
0.1	6.8880	7.6526	0.9001
0.2	3.9035	4.3074	0.9062
0.3	3.5190	3.4777	1.0119
0.4	4.0708	3.3069	1.2310
0.5	5.3288	3.4478	1.5456
0.6	7.5180	3.8497	1.9529
0.7	11.3661	4.6308	2.4545
0.8	19.0558	6.2694	3.0395
0.9	42.9500	11.2548	3.8162
0.95	91.2229	21.2507	4.2917

Table 3 shows the comparisons of the mean queue lengths of the two different systems when $D = 2.0$.

Figures 3 and 4 are the graphs of Tables 2 and 3. We see in the figures that when the traffic intensity ρ is very low, higher traffic intensity does not necessarily mean larger mean queue length. This peculiar phenomenon is typical in the D -policy queueing systems. This occurs when the mean service time is very small compared to the threshold value D . To be more specific, when ρ is extremely low, it takes many arrivals to surpass the threshold and we have larger mean queue lengths during the idle period, which, in turn, increases the overall mean queue length.

It is seen in the figures that when the traffic is heavy, the naive Poisson assumptions may result in a severe underestimation of the mean queue length.

Table 3. Comparison with the Poisson queue ($D = 2.0$)

ρ	$L(\text{MAP})$	$L(\text{Poisson})$	$\frac{L(\text{MAP})}{L(\text{Poisson})}$
0.1	27.5227	29.0757	0.9466
0.2	13.6842	14.8261	0.9230
0.3	9.6898	10.1498	0.9230
0.4	8.4316	7.9325	1.0629
0.5	8.6202	6.7848	1.2705
0.6	10.0964	6.3007	1.6024
0.7	13.4197	6.4393	2.0840
0.8	20.8914	7.5952	2.7506
0.9	44.4330	12.2084	3.6395
0.95	92.0176	22.0495	4.1732

Figure 3. Comparison with the Poisson queue ($D = 0.5$).

8. SUMMARY

In this article, we analyzed the MAP/G/1 queue under the D -policy and single vacation. We first derived the queue length distribution at an arbitrary departure and then obtained the queue length distribution at an arbitrary time.

Figure 4. Comparison with the Poisson queue ($D = 2.0$).

We derived the mean queue lengths and verified our analytical results through simulation. We lastly presented computational experiences and compared the MAP queue with the Poisson queue. Our computation shows that naive Poisson assumptions may lead to a severe underestimation of the mean queue length.

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