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New Lower Bounds and Constructions for Binary Codes Correcting Asymmetric Errors

Fang-Wei Fu, San Ling, and Chaoping Xing

Abstract—In this correspondence, we study binary asymmetric error-correcting codes. A general construction for binary asymmetric error-correcting codes is presented. We show that some previously known lower bounds for binary asymmetric error-correcting codes can be obtained from this general construction. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from this general construction. These new lower bounds improve the existing ones.

Index Terms—Asymmetric error-correcting codes, code construction, lower bounds, polynomials.

I. INTRODUCTION

Binary error-correcting codes are usually designed for communication systems modeled by the binary-symmetric channel. However, in certain communication systems, such as optical communications and some computer memory systems, the error probability from 1 to 0 is significantly higher than the error probability from 0 to 1. These communication systems are modeled by the binary asymmetric channel (the Z-channel). Error-correcting codes for the binary asymmetric channel have been studied since the 1950s. There are a number of papers dedicated to the construction of good codes and the derivation of lower and upper bounds for the asymmetric error-correcting codes, see [1]–[33], [35], and references therein. Kløve [19] gave a unified account of error-correcting codes for the binary asymmetric channel.

For two binary n -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \text{and} \quad \mathbf{y} = (y_1, y_2, \dots, y_n)$$

the asymmetric distance between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$d_a(\mathbf{x}, \mathbf{y}) = \max\{N(\mathbf{x}, \mathbf{y}), N(\mathbf{y}, \mathbf{x})\}$$

where

$$N(\mathbf{x}, \mathbf{y}) = |\{i : x_i = 0 \text{ and } y_i = 1\}|.$$

For a binary code $C \subseteq \{0, 1\}^n$, the minimum asymmetric distance of C is defined as

$$\Delta(C) = \min\{d_a(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in C \text{ and } \mathbf{x} \neq \mathbf{y}\}.$$

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It was shown in [24] that a binary code C can correct t or fewer asymmetric errors (1-errors) if and only if $\Delta(C) \geq t + 1$. A binary code of length n and minimum asymmetric distance Δ is called a binary (n, Δ) asymmetric code. Let $\Gamma(n, \Delta)$ denote the maximum number of codewords in a binary code of length n and minimum asymmetric distance Δ . One of the fundamental research problems in the theory of asymmetric error-correcting codes is to determine $\Gamma(n, \Delta)$ or give good lower and upper bounds.

In this correspondence, we give a general construction and some new lower bounds for binary asymmetric error-correcting codes. This correspondence is organized as follows. In Section II, we present a general construction for binary asymmetric error-correcting codes by modifying Xing's construction of binary constant-weight codes (see [34]). In Section III, we first give a general lower bound on the sizes of the binary asymmetric error-correcting codes constructed in Section II. Then, we show that some previously known lower bounds for binary asymmetric error-correcting codes can be obtained from this general construction. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from this general construction. These new lower bounds improve the existing ones.

II. A GENERAL CONSTRUCTION

Xing [34] gave a construction of binary constant-weight codes. By modifying his method, we present a general construction for binary asymmetric error-correcting codes.

Let $\mathbf{F}_q$ be a finite field of q elements, where q is a prime power. For a monic polynomial $f(x) \in \mathbf{F}_q[x]$, consider the residue class ring

$$R = \mathbf{F}_q[x]/(f(x)).$$

Actually, in the isomorphic meaning, here we can consider the residue class ring R as

$$R = \{g(x) \in \mathbf{F}_q[x] : \deg(g(x)) < \deg(f(x))\}.$$

The addition and multiplication operations over R are the polynomial addition and multiplication modulo $f(x)$.

Let $f(x)$ have the factorization

$$f(x) = \prod_{i=1}^k p_i^{e_i}(x)$$

where $p_1(x), \dots, p_k(x)$ are distinct monic irreducible polynomials in $\mathbf{F}_q[x]$ and $e_1, \dots, e_k$ are positive integers. It is known that all invertible polynomials of the ring R form a multiplicative group, denoted by

$$R^* = (\mathbf{F}_q[x]/(f(x)))^*.$$

It is a finite Abelian group and consists of all polynomials in R which are co-prime to $f(x)$, that is,

$$R^* = \{g(x) \in \mathbf{F}_q[x] : \deg(g(x)) < \deg(f(x)) \text{ and } (g(x), f(x)) = 1\}. \quad (1)$$

The multiplication operation $\odot$ over R^* is the polynomial multiplication modulo $f(x)$. Hence, this group contains exactly

$$\Phi(f(x)) \triangleq \prod_{i=1}^k (q^{d_i} - 1) q^{d_i(e_i - 1)}$$

elements, where d_i is the degree of $p_i(x)$. It is obvious that the set $\mathbf{F}_q^*$ of all nonzero elements of $\mathbf{F}_q$ is a subgroup of R^* . The quotient group

$$G = R^* / \mathbf{F}_q^*$$

is a finite Abelian group with

$$\Phi^*(f(x)) \triangleq \frac{1}{(q-1)} \Phi(f(x)) = \frac{1}{(q-1)} \prod_{i=1}^k (q^{d_i} - 1) q^{d_i(e_i-1)}$$

elements. Actually, in the isomorphic meaning, here we can consider G as the set of all monic polynomials of R^* , that is,

$$G = \{g(x) \in \mathbf{F}_q[x] : \deg(g(x)) < \deg(f(x)), g(x) \text{ is monic, and } (g(x), f(x))=1\}. \quad (2)$$

The multiplication operation $\otimes$ over G is given by

$$a(x) \otimes b(x) = M \left(a(x) \odot b(x) \right)$$

where

$$M(h(x)) = h_m^{-1} h(x)$$

for

$$h(x) = h_m x^m + h_{m-1} x^{m-1} + \cdots + h_1 x + h_0 \in \mathbf{F}_q[x].$$

Here $h_m \neq 0$ is the leading coefficient of $h(x)$.

In the following, we use the quotient group G to construct binary asymmetric error-correcting codes. For simplicity, we assume that the finite Abelian groups $(R^*, \odot)$ and $(G, \otimes)$ are given by (1) and (2), respectively.

Construction: Let n and d be two positive integers satisfying $n \leq q$ and $2 \leq d < n$. Let $f(x) \in \mathbf{F}_q[x]$ be a monic polynomial of degree d such that there exist n distinct elements $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{F}_q$ with $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$. Then $(x - \alpha_i)$ is co-prime to $f(x)$ for $i = 1, 2, \dots, n$. Hence,

$$(x - \alpha_i) \in G, \quad i = 1, 2, \dots, n.$$

Consider the map

$$\begin{aligned} \Omega : \{0, 1\}^n &\rightarrow G \\ (c_1, c_2, \dots, c_n) &\mapsto \prod_{i=1}^n \otimes (x - \alpha_i)^{c_i} \in G. \end{aligned}$$

For every $g(x) \in G$, denote

$$C_g = \Omega^{-1}(g(x)).$$

For every $g \in G$, if $C_g \neq \emptyset$, then C_g is a binary $(n, \Delta \geq d)$ asymmetric code.

Proof of the Construction: For every $g \in G$, if $C_g \neq \emptyset$, we want to show that

$$d_a(\mathbf{u}, \mathbf{v}) \geq d, \quad \mathbf{u}, \mathbf{v} \in C_g \text{ and } \mathbf{u} \neq \mathbf{v}.$$

Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. Then

$$\Omega(\mathbf{u}) = \Omega(\mathbf{v}) = g(x) \in G.$$

Hence, the element $\Omega(\mathbf{u})/\Omega(\mathbf{v})$ is the identity in G . This implies that in the group R^* , the element

$$\frac{\Omega(\mathbf{u})}{\Omega(\mathbf{v})} = \frac{\prod_{i=1}^n \odot (x - \alpha_i)^{u_i}}{\prod_{i=1}^n \odot (x - \alpha_i)^{v_i}}$$

is equal to a nonzero element β of $\mathbf{F}_q^*$. Denote

$$S = \{i : u_i = 0 \text{ and } v_i = 1\}$$

and

$$T = \{i : u_i = 1 \text{ and } v_i = 0\}.$$

Then $S \cap T = \emptyset$, and either $S \neq \emptyset$ or $T \neq \emptyset$ since $\mathbf{u} \neq \mathbf{v}$. Furthermore

$$|S| = N(\mathbf{u}, \mathbf{v}), \quad |T| = N(\mathbf{v}, \mathbf{u}).$$

It is easy to see that

$$\frac{\Omega(\mathbf{u})}{\Omega(\mathbf{v})} = \frac{\prod_{i \in T} \odot (x - \alpha_i)}{\prod_{j \in S} \odot (x - \alpha_j)} = \beta$$

in the group R^* . This is equivalent to the fact that $f(x)$ divides the polynomial

$$A(x) \triangleq \prod_{i \in T} (x - \alpha_i) - \beta \prod_{i \in S} (x - \alpha_i) \in \mathbf{F}_q[x].$$

The roots of the polynomial $\prod_{i \in T} (x - \alpha_i)$ are $\alpha_i, i \in T$, and the roots of the polynomial $\beta \prod_{i \in S} (x - \alpha_i)$ are $\alpha_i, i \in S$. Since

$$\{\alpha_i : i \in S\} \cap \{\alpha_i : i \in T\} = \emptyset$$

and either $S \neq \emptyset$ or $T \neq \emptyset$, we have

$$\prod_{i \in T} (x - \alpha_i) \neq \beta \prod_{i \in S} (x - \alpha_i).$$

Hence, $A(x) \neq 0$ and the degree of $A(x)$ is at most

$$\max\{|S|, |T|\} = d_a(\mathbf{u}, \mathbf{v}).$$

Therefore,

$$d_a(\mathbf{u}, \mathbf{v}) \geq \deg(f(x)) = d.$$

This completes the proof. $\square$

For every $g \in G$, if $C_g \neq \emptyset$, C_g is a binary $(n, \Delta \geq d)$ asymmetric code. Hence, C_g can correct $d - 1$ or fewer asymmetric errors (1-errors). Next we design a decoding method for the asymmetric error-correcting code C_g .

Decoding Algorithm: Assume that the received vector is

$$\mathbf{y} = (y_1, y_2, \dots, y_n) \in \{0, 1\}^n.$$

Calculate

$$R_{\mathbf{y}}(x) = \prod_{i=1}^n \otimes (x - \alpha_i)^{y_i} \in G$$

and

$$E(x) = \frac{g(x)}{R_{\mathbf{y}}(x)} \in G \quad (\text{since } g \in G).$$

To find the polynomial $E(x)$, we can use the Euclidean algorithm. Denote $l = \deg(E(x))$.

i) If $l = 0$, that is, $E(x) = 1$, then decode $\mathbf{y}$ into $\mathbf{g}$.

- ii) If $0 < l \leq d-1$ and $E(x)$ has l distinct roots $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_l}$, then decode $\mathbf{y}$ into $\mathbf{c} = (c_1, c_2, \dots, c_n)$ where

$$c_j = \begin{cases} y_j, & j \neq i_1, i_2, \dots, i_l \\ y_j \oplus 1, & j = i_1, i_2, \dots, i_l. \end{cases}$$

- iii) Otherwise, we declare that the decoding has failed.

Proof of the Decoding Algorithm: Suppose the codeword $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is transmitted. Assume that errors occur in positions $i_1, i_2, \dots, i_l$ where $0 \leq l \leq d-1$ and $1 \leq i_1 < i_2 < \dots < i_l \leq n$. Then the received vector $\mathbf{y}$ is given by $\mathbf{y} = \mathbf{c}$ for $l = 0$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ for $1 \leq l \leq d-1$ where

$$\begin{aligned} y_j &= c_j, & \text{for } j \neq i_1, i_2, \dots, i_l \\ y_j &= 0 \text{ and } c_j = 1, & \text{for } j = i_1, i_2, \dots, i_l. \end{aligned}$$

Hence, if $l = 0$, then

$$R_{\mathbf{y}}(x) = R_{\mathbf{c}}(x) = \prod_{j=1}^n \bigotimes (x - \alpha_j)^{c_j} = g(x) \in G$$

and $E(x) = 1$.

If $1 \leq l \leq d-1$, by the fact that $c_j = 1$ for $j = i_1, i_2, \dots, i_l$, we have

$$R_{\mathbf{y}}(x) = \prod_{j=1}^n \bigotimes (x - \alpha_j)^{y_j} = \prod_{j \neq i_1, i_2, \dots, i_l} \bigotimes (x - \alpha_j)^{c_j}$$

and

$$E(x) = \frac{g(x)}{R_{\mathbf{y}}(x)} = (x - \alpha_{i_1})(x - \alpha_{i_2}) \cdots (x - \alpha_{i_l}).$$

Hence, $E(x)$ has l distinct roots $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_l}$. Therefore, we decode $\mathbf{y}$ into $\mathbf{c}$. This completes the proof. $\square$

III. NEW LOWER BOUNDS

From the general construction in Section II, we know that $C_g, g \in G$ form a partition of $\{0, 1\}^n$. Since $|G| = \Phi^*(f(x))$, we can find one element $\pi(x) \in G$ such that

$$|C_{\pi}| \geq \frac{2^n}{\Phi^*(f(x))}.$$

Hence, we obtain the following result.

Theorem 1: Let $\mathbf{F}_q$ be a finite field of q elements, where q is a prime power. Let n and d be two positive integers satisfying $n \leq q$ and $2 \leq d < n$. Let $f(x) \in \mathbf{F}_q[x]$ be a monic polynomial of degree d such that there exist n distinct elements $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{F}_q$ with $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$. Then there exists a binary $(n, \Delta \geq d)$ asymmetric code C with size

$$|C| \geq \frac{2^n}{\Phi^*(f(x))}. \quad (3)$$

From the general construction in Section II, it is easy to see the following.

Corollary 1: With notations as in Section II, we have

$$\Gamma(n, \Delta) \geq \max_{g \in G} |C_g|. \quad (4)$$

Bound (4) is in general stronger than bound (3), but it is less explicit and requires more computation to determine.

Several lower bounds for binary asymmetric error-correcting codes were obtained by a discussion of Varshamov's constructions and their generalizations (see [17, Theorem 6.1] and [11], [12], [17], [29], and [31]). In this section, we first show that these previously known lower bounds for binary asymmetric error-correcting codes can also be

obtained from our general construction and Theorem 1. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from Theorem 1. These new lower bounds improve on the existing ones.

Theorem 2: (see [19, Theorem 6.1])

- i) If n is a prime power, then for $d \geq 2$

$$\Gamma(n, d) \geq \frac{2^n}{n^{d-1} + n^{d-2} + \dots + n + 1}. \quad (5)$$

- ii) If $n+1$ is a prime power, then for $d \geq 3$

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)^{d-1} - 1}. \quad (6)$$

- iii) If q is the least prime power satisfying $q \geq n+2$, then for $d \geq 3$

$$\Gamma(n, d) \geq \frac{2^n}{q^{d-1} - q^{d-2}}. \quad (7)$$

Proof:

- i) Let $q = n$ in Theorem 1 since n is a prime power. Let

$$\mathbf{F}_q = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$$

and let $f(x) \in \mathbf{F}_q[x]$ be a monic irreducible polynomial of degree d ($d \geq 2$). Then

$$\Phi(f(x)) = q^d - 1$$

and

$$\Phi^*(f(x)) = \frac{q^d - 1}{q - 1} = n^{d-1} + n^{d-2} + \dots + n + 1.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$ since $f(x)$ is a monic irreducible polynomial of degree d ($d \geq 2$). Hence, (5) follows from Theorem 1.

- ii) Let $q = n+1$ in Theorem 1 since $n+1$ is a prime power. Let

$$\mathbf{F}_q = \{0, \alpha_1, \alpha_2, \dots, \alpha_n\}$$

and let $f(x) = x f_1(x)$ where $f_1(x) \in \mathbf{F}_q[x]$ is a monic irreducible polynomial of degree $d-1$. Then

$$\Phi(f(x)) = (q-1)(q^{d-1} - 1)$$

and

$$\Phi^*(f(x)) = q^{d-1} - 1 = (n+1)^{d-1} - 1.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$ since $\alpha_i \in \mathbf{F}_q^*$ and $f_1(x)$ is a monic irreducible polynomial with degree $d-1 \geq 2$. Hence, (6) follows from Theorem 1.

- iii) Since q is the least prime power satisfying $q \geq n+2$, we can assume in Theorem 1 that

$$\mathbf{F}_q = \{0, 1, \alpha_1, \alpha_2, \dots, \alpha_n, \dots\}.$$

Let $f(x) = x(x-1)^{d-1}$. Then

$$\Phi(f(x)) = (q-1)^2 q^{d-2}$$

and

$$\Phi^*(f(x)) = (q-1)q^{d-2} = q^{d-1} - q^{d-2}.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$. Hence, (7) follows from Theorem 1. $\square$

Remark 1: As pointed out by one referee, Bose and Cunningham [9] presented a construction of binary asymmetric error-correcting codes if $n + 1$ is a prime power. This construction yields the following lower bound:

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)^{d-1}}. \quad (8)$$

Note that bound (8) is slightly worse than bound (6). The referee observed that after redescribing the construction of Bose and Cunningham [9] in polynomial form it is somewhat similar to our construction here. We note that the construction of Bose and Cunningham is actually a special case of our general construction by taking $f(x) = x^d$ in the proof of Theorem 2 ii). Bound (8) follows from Theorem 1 by noting that $\Phi(x^d) = (q-1)q^{d-1}$ and $\Phi^*(x^d) = q^{d-1} = (n+1)^{d-1}$.

In the following theorem, we show that the lower bounds given by Theorem 2 can be generalized and improved by using Theorem 1. Note that the number of monic quadratic irreducible polynomials in $\mathbf{F}_q[x]$ is $q(q-1)/2$.

Theorem 3:

i) If n is a prime power and $2 \leq d \leq n$, then

$$\Gamma(n, d) \geq \frac{(n-1)2^n}{(n^2-1)^r(n^3-1)^s} \quad (9)$$

where r and s are the two unique nonnegative integers satisfying $d = 2r + 3s$ and $s \in \{0, 1\}$.

ii) If n is not a prime power, denote m as the least positive integer such that $q = n + m$ is a prime power. If $2 \leq d \leq m$, then

$$\Gamma(n, d) \geq \frac{2^n}{(q-1)^{d-1}}. \quad (10)$$

If $d > m$, then

$$\Gamma(n, d) \geq \frac{2^n}{(q-1)^{m-1}q^{s'}(q^2-1)^{r'}} \quad (11)$$

where r' and s' are the two unique nonnegative integers satisfying $d - m = 2r' + s'$ and $s' \in \{0, 1\}$.

Proof:

i) Let $q = n$ in Theorem 1 since n is a prime power. Let

$$\mathbf{F}_q = \{\alpha_1, \alpha_2, \dots, \alpha_q\}.$$

Since

$$r \leq \frac{d}{2} \leq \frac{n}{2} = \frac{q}{2} \leq \frac{q(q-1)}{2}$$

we can choose distinct monic quadratic irreducible polynomials

$$p_1(x), p_2(x), \dots, p_r(x)$$

in $\mathbf{F}_q[x]$ and a monic cubic irreducible polynomials $p(x)$ in $\mathbf{F}_q[x]$. Let

$$f(x) = p^s(x) \prod_{i=1}^r p_i(x).$$

Then $\deg(f(x)) = d$ and

$$\begin{aligned} \Phi(f(x)) &= (q^2-1)^r(q^3-1)^s \\ \Phi^*(f(x)) &= \frac{(q^2-1)^r(q^3-1)^s}{q-1} = \frac{(n^2-1)^r(n^3-1)^s}{n-1}. \end{aligned}$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$. Hence, by Theorem 1

$$\Gamma(n, d) \geq \frac{(n-1)2^n}{(n^2-1)^r(n^3-1)^s}.$$

ii) In Theorem 1, let

$$\mathbf{F}_q = \{\beta_1, \beta_2, \dots, \beta_m, \alpha_1, \alpha_2, \dots, \alpha_n\}.$$

If $2 \leq d \leq m$, let

$$f(x) = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_d).$$

Then

$$\Phi(f(x)) = (q-1)^d, \quad \Phi^*(f(x)) = (q-1)^{d-1}.$$

If $d > m$, by the fact that $d - m = 2r' + s'$, we have

$$r' \leq \frac{d}{2} \leq \frac{n}{2} \leq \frac{q}{2} \leq \frac{q(q-1)}{2}.$$

Hence, we can choose distinct monic quadratic irreducible polynomials

$$p_1(x), p_2(x), \dots, p_{r'}(x)$$

in $\mathbf{F}_q[x]$. Let

$$f(x) = (x - \beta_1)^{1+s'}(x - \beta_2) \cdots (x - \beta_m) \prod_{i=1}^{r'} p_i(x).$$

Then $\deg(f(x)) = d$ and

$$\begin{aligned} \Phi(f(x)) &= (q-1)^m q^{s'}(q^2-1)^{r'} \\ \Phi^*(f(x)) &= (q-1)^{m-1} q^{s'}(q^2-1)^{r'}. \end{aligned}$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \dots, n$. Hence, by Theorem 1, we obtain (10) and (11). $\square$

The lower bound (9) in Theorem 3 is better than the lower bound (5) in Theorem 2. Note that for $d \geq 2$

$$(n^2-1)^r(n^3-1)^s \leq n^{2r+3s} - 1 = n^d - 1$$

and the equality holds if and only if $d = 2$ or 3 . Hence, for $d \geq 2$

$$\frac{(n^2-1)^r(n^3-1)^s}{n-1} \leq n^{d-1} + n^{d-2} + \cdots + n + 1$$

and the equality holds if and only if $d = 2$ or 3 .

The lower bound (9) in Theorem 3 can be rewritten in the following form. If n is a prime power, then

$$\Gamma(n, d) \geq \frac{(n-1)2^n}{(n^2-1)^{\frac{d}{2}}}, \quad d \text{ even and } d \geq 2 \quad (12)$$

$$\Gamma(n, d) \geq \frac{(n-1)2^n}{(n^2-1)^{\frac{(d-3)}{2}}(n^3-1)}, \quad d \text{ odd and } d \geq 3. \quad (13)$$

For two sequences $\{g(n)\}_{n=1}^{\infty}$ and $\{h(n)\}_{n=1}^{\infty}$, we say

$$g(n) = O(h(n)), \quad \text{if } \lim_{n \rightarrow \infty} \frac{g(n)}{h(n)} = 1.$$

By direct computation, it is not hard to see that

$$\text{Bound (12)} - \text{Bound (5)} = O\left(\frac{d2^{n-1}}{n^{d+1}}\right), \quad d \text{ even and } d \geq 4$$

$$\text{Bound (13)} - \text{Bound (5)} = O\left(\frac{(d-3)2^{n-1}}{n^{d+1}}\right), \quad d \text{ odd and } d \geq 5.$$

Let $m = 1$ in Theorem 3 ii), then we obtain the following.

Corollary 2: If $n + 1$ is a prime power, then for $d \geq 2$

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)^s[(n+1)^2-1]^r} \quad (14)$$

where r and s are the two unique nonnegative integers satisfying $d - 1 = 2r + s$ and $s \in \{0, 1\}$.

The lower bound (14) in Corollary 2 is better than the lower bound (6) in Theorem 2. Note that for $d \geq 3$

$$(n+1)^s[(n+1)^2 - 1]^r \leq (n+1)^{2r+s} - 1 = (n+1)^{d-1} - 1$$

and the equality holds if and only if $d = 3$.

The lower bound (14) in Corollary 2 can be rewritten in the following form. If $n+1$ is a prime power, then

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)[(n+1)^2 - 1]^{\frac{(d-2)}{2}}}, \quad d \text{ even and } d \geq 2 \quad (15)$$

$$\Gamma(n, d) \geq \frac{2^n}{[(n+1)^2 - 1]^{\frac{(d-1)}{2}}}, \quad d \text{ odd and } d \geq 3. \quad (16)$$

By direct computation, it is not hard to see that

$$\text{Bound (15)} - \text{Bound (6)} = O\left(\frac{(d-2)2^{n-1}}{n^{d+1}}\right), \quad d \text{ even and } d \geq 4$$

$$\text{Bound (16)} - \text{Bound (6)} = O\left(\frac{(d-1)2^{n-1}}{n^{d+1}}\right), \quad d \text{ odd and } d \geq 5.$$

Let $m = 2$ in Theorem 3 ii), then we obtain the following.

Corollary 3: If $n+2$ is a prime power, then for $d \geq 3$

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)(n+2)^s[(n+2)^2 - 1]^r} \quad (17)$$

where r and s are the two unique nonnegative integers satisfying $d - 2 = 2r + s$ and $s \in \{0, 1\}$.

The lower bound (17) in Corollary 3 is better than the lower bound (7) in Theorem 2. Note that for $d \geq 3$ and $q = n+2$

$$q^s(q^2 - 1)^r \leq q^{2r+s} = q^{d-2}$$

and the equality holds if and only if $d = 3$. Hence, for $d \geq 3$

$$(q-1)q^s(q^2 - 1)^r \leq q^{d-1} - q^{d-2}$$

and the equality holds if and only if $d = 3$.

The lower bound (17) in Corollary 3 can be rewritten in the following form. If $n+2$ is a prime power, then for even d and $d \geq 4$

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)[(n+2)^2 - 1]^{\frac{(d-2)}{2}}} \quad (18)$$

and for odd d and $d \geq 3$,

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)(n+2)[(n+2)^2 - 1]^{\frac{(d-3)}{2}}}. \quad (19)$$

Note that if $n+2$ is a prime power, the lower bound (7) in Theorem 2 is given by

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)(n+2)^{d-2}}, \quad d \geq 3. \quad (20)$$

By direct computation, it is not hard to see that for even d and $d \geq 4$

$$\text{Bound (18)} - \text{Bound (20)} = O\left(\frac{(d-2)2^{n-1}}{n^{d+1}}\right)$$

and for odd d and $d \geq 5$

$$\text{Bound (19)} - \text{Bound (20)} = O\left(\frac{(d-3)2^{n-1}}{n^{d+1}}\right).$$

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REFERENCES

- [1] K. A. S. Abdel-Ghaffar and H. C. Ferreira, "Systematic encoding of the Varshamov-Tenengol's codes and the Constantin-Rao codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 340-345, Jan. 1998.
- [2] S. Al-Bassam and B. Bose, "Asymmetric/unidirectional error correcting and detecting codes," *IEEE Trans. Comput.*, vol. 43, pp. 590-597, May 1994.
- [3] S. Al-Bassam, R. Venkatesan, and S. Al-Muhammadi, "New single asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1619-1623, Sept. 1997.
- [4] S. Al-Bassam and S. Al-Muhammadi, "A single asymmetric error-correcting code with 2^{13} codewords of dimension 17," *IEEE Trans. Inform. Theory*, vol. 46, pp. 269-271, Jan. 2000.
- [5] J. M. Berger, "A note on error detection codes for asymmetric channels," *Inform. Contr.*, vol. 4, pp. 68-73, 1961.
- [6] M. Blaum, *Codes for Detecting and Correcting Unidirectional Errors*. Los Alamitos, CA: IEEE Computer Soc. Press, 1993.
- [7] J. M. Borden, "Optimal asymmetric error detecting codes," *Inform. Contr.*, vol. 53, pp. 66-73, 1982.
- [8] —, "A low-rate bound for asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 600-602, July 1983.
- [9] B. Bose and S. Cunningham, "Asymmetric error correcting codes," in *Sequences II: Methods in Communication, Security, and Computer Science*, R. Capocelli, A. De Santis, and U. Vaccaro, Eds. Berlin, Germany: Springer-Verlag, 1993, pp. 24-35.
- [10] B. Bose and S. Al-Bassam, "On systematic single asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 46, pp. 669-672, Mar. 2000.
- [11] S. D. Constantin and T. R. N. Rao, "On the theory of binary asymmetric error-correcting codes," *Inform. Contr.*, vol. 40, pp. 20-36, 1979.
- [12] P. Delsarte and P. Piret, "Spectral enumerators for certain additive-error-correcting codes over integer alphabets," in *Inform. Contr.*, vol. 48, 1981, pp. 193-210.
- [13] —, "Bounds and constructions for binary asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 125-128, Jan 1981.
- [14] T. Etzion, "Lower bounds for asymmetric and unidirectional codes," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1696-1704, Nov 1991.
- [15] G. Fang and H. C. A. van Tilborg, "Bounds and constructions of asymmetric or unidirectional error-correcting codes," *Appl. Algebra Engrg. Comm. Comput.*, vol. 3, no. 4, pp. 269-300, 1992.
- [16] C. V. Freiman, "Optimal error detection codes for completely asymmetric binary channels," *Inform. Contr.*, vol. 5, pp. 64-71, 1962.
- [17] T. Helleseth and T. Kløve, "On group-theoretic codes for asymmetric channels," *Inform. Contr.*, vol. 49, pp. 1-9, 1981.
- [18] H. Kim and C. Freiman, "Single error-correcting codes for asymmetric binary channels," *IRE Trans. Inform. Theory*, vol. IT-5, pp. 62-66, Jun 1959.
- [19] T. Kløve, "Error correcting codes for the asymmetric channel," Dept. Mathematics, Univ. Bergen, Bergen, Norway, Tech. Rep.18-09-07-81, 1995.
- [20] —, "Upper bounds on codes correcting asymmetric errors," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 128-131, Jan. 1981.
- [21] —, "On Robinson's coding problem," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 450-454, May 1983.
- [22] R. J. McEliece, "Comment on 'A class of codes for asymmetric channels and a problem from the additive theory of numbers'," *IEEE Trans. Inform. Theory*, vol. IT-19, p. 137, Jan. 1973.
- [23] R. J. McEliece and E. R. Rodemich, "The Constantin-Rao construction for asymmetric error-correcting-codes," *Inform. Contr.*, vol. 44, pp. 187-196, 1980.
- [24] T. R. N. Rao and A. S. Chawla, "Asymmetric error codes for some lsi semi-conductor memories," in *Proc. Annu. Southeastern Symp. Systems Theory*, 1975, pp. 170-171.
- [25] T. R. N. Rao and E. Fujiwara, *Error-Control Coding for Computer Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
- [26] J. P. Robinson, "An asymmetric error-correcting ternary code," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 258-261, Mar. 1978.
- [27] Y. Saitoh, K. Yamaguchi, and H. Imai, "Some new binary codes correcting asymmetric/unidirectional errors," *IEEE Trans. Inform. Theory*, vol. 36, pp. 645-647, May 1990.

- [28] A. Shiozaki, "Construction for binary asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 787–789, Sept. 1982.
- [29] R. P. Stanley and M. F. Yoder, "A study of Varshamov codes for asymmetric channels," Jet Propulsion Lab., Tech. Rep. 32–1526, vol. 14, 1973.
- [30] R. R. Varshamov, "A class of codes for asymmetric channels and a problem from the additive theory of numbers," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 92–95, Jan 1973.
- [31] R. R. Varshamov and G. M. Tenen Holtz, "Correction code for single asymmetric error," *Automat. Remote Contr.*, vol. 26, pp. 286–290, 1965.
- [32] J. Weber, C. De Vroedt, and D. Boekee, "New upper bounds on the size of codes correcting asymmetric errors," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 434–437, May 1987.
- [33] ———, "Bounds and constructions for binary codes of length less than 24 and asymmetric distance less than 6," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1321–1331, Sept. 1988.
- [34] C. P. Xing, "Constructions of codes from residue rings of polynomials," *IEEE Trans. Inform. Theory*, vol. 48, pp. 2995–2997, Nov. 2002.
- [35] Z. Zhang and X. Xia, "New lower bounds for binary codes of asymmetric distance two," *IEEE Trans. Inform. Theory*, vol. 38, pp. 1592–1597, Sept 1992.