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New Lower Bounds and Constructions for Binary Codes Correcting Asymmetric Errors

Fang-Wei Fu, San Ling, and Chaoping Xing

***Abstract*—In this correspondence, we study binary asymmetric error-correcting codes. A general construction for binary asymmetric error-correcting codes is presented. We show that some previously known lower bounds for binary asymmetric error-correcting codes can be obtained from this general construction. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from this general construction. These new lower bounds improve the existing ones.**

***Index Terms*—Asymmetric error-correcting codes, code construction, lower bounds, polynomials.**

I. Introduction

Binary error-correcting codes are usually designed for communication systems modeled by the binary-symmetric channel. However, in certain communication systems, such as optical communications and some computer memory systems, the error probability from 1 to 0 is significantly higher than the error probability from 0 to 1. These communication systems are modeled by the binary asymmetric channel (the Z-channel). Error-correcting codes for the binary asymmetric channel have been studied since the 1950s. There are a number of papers dedicated to the construction of good codes and the derivation of lower and upper bounds for the asymmetric error-correcting codes, see [1]–[33], [35], and references therein. Kløve [19] gave a unified account of error-correcting codes for the binary asymmetric channel.

For two binary n-tuples

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \boldsymbol{y} = (y_1, y_2, \ldots, y_n)$$

the asymmetric distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$d_a(\boldsymbol{x}, \boldsymbol{y}) = \max\{N(\boldsymbol{x}, \boldsymbol{y}), N(\boldsymbol{y}, \boldsymbol{x})\}$$

where

$$N(\boldsymbol{x}, \boldsymbol{y}) = \mid \{i : \ x_i = 0 \text{ and } y_i = 1\} \mid .$$

For a binary code $C \subseteq \{0,1\}^n$, the minimum asymmetric distance of C is defined as

$$\Delta(C) = \min\{d_a(\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{x}, \boldsymbol{y} \in C \text{ and } \boldsymbol{x} \neq \boldsymbol{y}\}.$$

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It was shown in [24] that a binary code C can correct t or fewer asymmetric errors (1-errors) if and only if $\Delta(C) \geq t+1$. A binary code of length n and minimum asymmetric distance Δ is called a binary (n, Δ) asymmetric code. Let $\Gamma(n, \Delta)$ denote the maximum number of codewords in a binary code of length n and minimum asymmetric distance Δ. One of the fundamental research problems in the theory of asymmetric error-correcting codes is to determine $\Gamma(n, \Delta)$ or give good lower and upper bounds.

In this correspondence, we give a general construction and some new lower bounds for binary asymmetric error-correcting codes. This correspondence is organized as follows. In Section II, we present a general construction for binary asymmetric error-correcting codes by modifying Xing's construction of binary constant-weight codes (see [34]). In Section III, we first give a general lower bound on the sizes of the binary asymmetric error-correcting codes constructed in Section II. Then, we show that some previously known lower bounds for binary asymmetric error-correcting codes can be obtained from this general construction. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from this general construction. These new lower bounds improve the existing ones.

II. A General Construction

Xing [34] gave a construction of binary constant-weight codes. By modifying his method, we present a general construction for binary asymmetric error-correcting codes.

Let $\boldsymbol{F}_q$ be a finite field of q elements, where q is a prime power. For a monic polynomial $f(x) \in \boldsymbol{F}_q[x]$, consider the residue class ring

$$R = \boldsymbol{F}_q[x]/(f(x)).$$

Actually, in the isomorphic meaning, here we can consider the residue class ring R as

$$R = \{g(x) \in \boldsymbol{F}_q[x] : \ \deg(g(x)) < \deg(f(x))\}.$$

The addition and multiplication operations over R are the polynomial addition and multiplication modulo $f(x)$.

Let $f(x)$ have the factorization

$$f(x) = \prod_{i=1}^{k} p_i^{e_i}(x)$$

where $p_1(x), \ldots, p_k(x)$ are distinct monic irreducible polynomials in $\boldsymbol{F}_q[x]$ and $e_1, \ldots, e_k$ are positive integers. It is known that all invertible polynomials of the ring R form a multiplicative group, denoted by

$$R^* = (\boldsymbol{F}_q[x]/(f(x)))^*.$$

It is a finite Abelian group and consists of all polynomials in R which are co-prime to $f(x)$, that is,

$$R^* = \{g(x) \in \boldsymbol{F}_q[x] : \quad \deg(g(x)) < \deg(f(x)) \text{ and } (g(x), f(x))=1\}. \quad (1)$$

The multiplication operation $\bigodot$ over R^* is the polynomial multiplication modulo $f(x)$. Hence, this group contains exactly

$$\Phi\left(f(x)\right) \triangleq \prod_{i=1}^{k} (q^{d_i} - 1) q^{d_i(e_i - 1)}$$

elements, where d_i is the degree of $p_i(x)$. It is obvious that the set $\boldsymbol{F}_q^*$ of all nonzero elements of $\boldsymbol{F}_q$ is a subgroup of R^*. The quotient group

$$G = R^*/\boldsymbol{F}_q^*$$

is a finite Abelian group with

$$\Phi^*(f(x)) \triangleq \frac{1}{(q-1)}\Phi(f(x)) = \frac{1}{(q-1)}\prod_{i=1}^{k}(q^{d_i}-1)q^{d_i(e_i-1)}$$

elements. Actually, in the isomorphic meaning, here we can consider G as the set of all monic polynomials of R^*, that is,

$$G = \{g(x) \in \boldsymbol{F}_q[x]: \quad \deg(g(x)) < \deg(f(x)),\ g(x) \text{ is monic, and } (g(x), f(x))=1\}. \quad (2)$$

The multiplication operation $\bigotimes$ over G is given by

$$a(x)\bigotimes b(x) = M\left(a(x)\bigodot b(x)\right)$$

where

$$M(h(x)) = h_m^{-1}h(x)$$

for

$$h(x) = h_m x^m + h_{m-1}x^{m-1} + \cdots + h_1 x + h_0 \in \boldsymbol{F}_q[x].$$

Here $h_m \neq 0$ is the leading coefficient of $h(x)$.

In the following, we use the quotient group G to construct binary asymmetric error-correcting codes. For simplicity, we assume that the finite Abelian groups $(R^*, \bigodot)$ and $(G, \bigotimes)$ are given by (1) and (2), respectively.

Construction: Let n and d be two positive integers satisfying $n \leq q$ and $2 \leq d < n$. Let $f(x) \in \boldsymbol{F}_q[x]$ be a monic polynomial of degree d such that there exist n distinct elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in \boldsymbol{F}_q$ with $f(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Then $(x-\alpha_i)$ is co-prime to $f(x)$ for $i = 1, 2, \ldots, n$. Hence,

$$(x-\alpha_i) \in G, \qquad i = 1, 2, \ldots, n.$$

Consider the map

$$\Omega: \{0,1\}^n \to G$$
$$(c_1, c_2, \ldots, c_n) \mapsto \prod_{i=1}^{n}\bigotimes(x-\alpha_i)^{c_i} \in G.$$

For every $g(x) \in G$, denote

$$C_g = \Omega^{-1}(g(x)).$$

For every $g \in G$, if $C_g \neq \emptyset$, then C_g is a binary $(n, \Delta \geq d)$ asymmetric code.

Proof of the Construction: For every $g \in G$, if $C_g \neq \emptyset$, we want to show that

$$d_a(\boldsymbol{u}, \boldsymbol{v}) \geq d, \qquad \boldsymbol{u}, \boldsymbol{v} \in C_g \text{ and } \boldsymbol{u} \neq \boldsymbol{v}.$$

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$. Then

$$\Omega(\boldsymbol{u}) = \Omega(\boldsymbol{v}) = g(x) \in G.$$

Hence, the element $\Omega(\boldsymbol{u})/\Omega(\boldsymbol{v})$ is the identity in G. This implies that in the group R^*, the element

$$\frac{\Omega(\boldsymbol{u})}{\Omega(\boldsymbol{v})} = \frac{\prod_{i=1}^{n}\bigodot(x-\alpha_i)^{u_i}}{\prod_{i=1}^{n}\bigodot(x-\alpha_i)^{v_i}}$$

is equal to a nonzero element β of $\boldsymbol{F}_q^*$. Denote

$$S = \{i: \ u_i = 0 \text{ and } v_i = 1\}$$

and

$$T = \{i: \ u_i = 1 \text{ and } v_i = 0\}.$$

Then $S \bigcap T = \emptyset$, and either $S \neq \emptyset$ or $T \neq \emptyset$ since $\boldsymbol{u} \neq \boldsymbol{v}$. Furthermore

$$|\ S\ | = N(\boldsymbol{u}, \boldsymbol{v}), \quad |\ T\ | = N(\boldsymbol{v}, \boldsymbol{u}).$$

It is easy to see that

$$\frac{\Omega(\boldsymbol{u})}{\Omega(\boldsymbol{v})} = \frac{\prod_{i\in T}\bigodot(x-\alpha_i)}{\prod_{j\in S}\bigodot(x-\alpha_j)} = \beta$$

in the group R^*. This is equivalent to the fact that $f(x)$ divides the polynomial

$$A(x) \triangleq \prod_{i\in T}(x-\alpha_i) - \beta\prod_{i\in S}(x-\alpha_i) \in \boldsymbol{F}_q[x].$$

The roots of the polynomial $\prod_{i\in T}(x-\alpha_i)$ are $\alpha_i, i \in T$, and the roots of the polynomial $\beta\prod_{i\in S}(x-\alpha_i)$ are $\alpha_i, i \in S$. Since

$$\{\alpha_i : i \in S\}\bigcap\{\alpha_i : i \in T\} = \emptyset$$

and either $S \neq \emptyset$ or $T \neq \emptyset$, we have

$$\prod_{i\in T}(x-\alpha_i) \neq \beta\prod_{i\in S}(x-\alpha_i).$$

Hence, $A(x) \neq 0$ and the degree of $A(x)$ is at most

$$\max\{|\ S\ |, |\ T\ |\} = d_a(\boldsymbol{u}, \boldsymbol{v}).$$

Therefore,

$$d_a(\boldsymbol{u}, \boldsymbol{v}) \geq \deg(f(x)) = d.$$

This completes the proof. □

For every $g \in G$, if $C_g \neq \emptyset$, C_g is a binary $(n, \Delta \geq d)$ asymmetric code. Hence, C_g can correct $d-1$ or fewer asymmetric errors (1-errors). Next we design a decoding method for the asymmetric error-correcting code C_g.

Decoding Algorithm: Assume that the received vector is

$$\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \{0,1\}^n.$$

Calculate

$$R_{\boldsymbol{y}}(x) = \prod_{i=1}^{n}\bigotimes(x-\alpha_i)^{y_i} \in G$$

and

$$E(x) = \frac{g(x)}{R_{\boldsymbol{y}}(x)} \in G \qquad (\text{since } g \in G).$$

To find the polynomial $E(x)$, we can use the Euclidean algorithm. Denote $l = \deg(E(x))$.

i) If $l = 0$, that is, $E(x) = 1$, then decode $\boldsymbol{y}$ into $\boldsymbol{y}$.

ii) If $0 < l \leq d-1$ and $E(x)$ has l distinct roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_l}$, then decode $\boldsymbol{y}$ into $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ where

$$c_j = \begin{cases} y_j, & j \neq i_1, i_2, \ldots, i_l \\ y_j \oplus 1, & j = i_1, i_2, \ldots, i_l. \end{cases}$$

iii) Otherwise, we declare that the decoding has failed.

Proof of the Decoding Algorithm: Suppose the codeword $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ is transmitted. Assume that errors occur in positions $i_1, i_2, \ldots, i_l$ where $0 \leq l \leq d-1$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Then the received vector $\boldsymbol{y}$ is given by $\boldsymbol{y} = \boldsymbol{c}$ for $l = 0$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ for $1 \leq l \leq d-1$ where

$$\begin{aligned} &y_j = c_j, && \text{for } j \neq i_1, i_2, \ldots, i_l \\ &y_j = 0 \text{ and } c_j = 1, && \text{for } j = i_1, i_2, \ldots, i_l. \end{aligned}$$

Hence, if $l = 0$, then

$$R_{\boldsymbol{y}}(x) = R_{\boldsymbol{c}}(x) = \prod_{j=1}^{n} \bigotimes (x - \alpha_j)^{c_j} = g(x) \in G$$

and $E(x) = 1$.

If $1 \leq l \leq d-1$, by the fact that $c_j = 1$ for $j = i_1, i_2, \ldots, i_l$, we have

$$R_{\boldsymbol{y}}(x) = \prod_{j=1}^{n} \bigotimes (x - \alpha_j)^{y_j} = \prod_{j \neq i_1, i_2, \ldots, i_l} \bigotimes (x - \alpha_j)^{c_j}$$

and

$$E(x) = \frac{g(x)}{R_{\boldsymbol{y}}(x)} = (x - \alpha_{i_1})(x - \alpha_{i_2}) \cdots (x - \alpha_{i_l}).$$

Hence, $E(x)$ has l distinct roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_l}$. Therefore, we decode $\boldsymbol{y}$ into $\boldsymbol{c}$. This completes the proof. □

III. New Lower Bounds

From the general construction in Section II, we know that $C_g,\ g \in G$ form a partition of $\{0,1\}^n$. Since $\mid G \mid = \Phi^*(f(x))$, we can find one element $\pi(x) \in G$ such that

$$\mid C_\pi \mid \geq \frac{2^n}{\Phi^*(f(x))}.$$

Hence, we obtain the following result.

Theorem 1: Let $\boldsymbol{F}_q$ be a finite field of q elements, where q is a prime power. Let n and d be two positive integers satisfying $n \leq q$ and $2 \leq d < n$. Let $f(x) \in \boldsymbol{F}_q[x]$ be a monic polynomial of degree d such that there exist n distinct elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in \boldsymbol{F}_q$ with $f(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Then there exists a binary $(n, \Delta \geq d)$ asymmetric code C with size

$$\mid C \mid \geq \frac{2^n}{\Phi^*(f(x))}. \tag{3}$$

From the general construction in Section II, it is easy to see the following.

Corollary 1: With notations as in Section II, we have

$$\Gamma(n, \Delta) \geq \max_{g \in G} \mid C_g \mid . \tag{4}$$

Bound (4) is in general stronger than bound (3), but it is less explicit and requires more computation to determine.

Several lower bounds for binary asymmetric error-correcting codes were obtained by a discussion of Varshamov's constructions and their generalizations (see [17, Theorem 6.1] and [11], [12], [17], [29], and [31]). In this section, we first show that these previously known lower bounds for binary asymmetric error-correcting codes can also be obtained from our general construction and Theorem 1. Furthermore, some new lower bounds for binary asymmetric error-correcting codes are obtained from Theorem 1. These new lower bounds improve on the existing ones.

Theorem 2: (see [19, Theorem 6.1])

i) If n is a prime power, then for $d \geq 2$

$$\Gamma(n, d) \geq \frac{2^n}{n^{d-1} + n^{d-2} + \cdots + n + 1}. \tag{5}$$

ii) If $n+1$ is a prime power, then for $d \geq 3$

$$\Gamma(n, d) \geq \frac{2^n}{(n+1)^{d-1} - 1}. \tag{6}$$

iii) If q is the least prime power satisfying $q \geq n+2$, then for $d \geq 3$

$$\Gamma(n, d) \geq \frac{2^n}{q^{d-1} - q^{d-2}}. \tag{7}$$

Proof:

i) Let $q = n$ in Theorem 1 since n is a prime power. Let

$$\boldsymbol{F}_q = \{\alpha_1, \alpha_2, \ldots, \alpha_q\}$$

and let $f(x) \in \boldsymbol{F}_q[x]$ be a monic irreducible polynomial of degree d $(d \geq 2)$. Then

$$\Phi(f(x)) = q^d - 1$$

and

$$\Phi^*(f(x)) = \frac{q^d - 1}{q - 1} = n^{d-1} + n^{d-2} + \cdots + n + 1.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$ since $f(x)$ is a monic irreducible polynomial of degree d $(d \geq 2)$. Hence, (5) follows from Theorem 1.

ii) Let $q = n+1$ in Theorem 1 since $n+1$ is a prime power. Let

$$\boldsymbol{F}_q = \{0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$$

and let $f(x) = x f_1(x)$ where $f_1(x) \in \boldsymbol{F}_q[x]$ is a monic irreducible polynomial of degree $d-1$. Then

$$\Phi(f(x)) = (q-1)(q^{d-1} - 1)$$

and

$$\Phi^*(f(x)) = q^{d-1} - 1 = (n+1)^{d-1} - 1.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$ since $\alpha_i \in \boldsymbol{F}_q^*$ and $f_1(x)$ is a monic irreducible polynomial with degree $d - 1 \geq 2$. Hence, (6) follows from Theorem 1.

iii) Since q is the least prime power satisfying $q \geq n+2$, we can assume in Theorem 1 that

$$\boldsymbol{F}_q = \{0, 1, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}.$$

Let $f(x) = x(x-1)^{d-1}$. Then

$$\Phi(f(x)) = (q-1)^2 q^{d-2}$$

and

$$\Phi^*(f(x)) = (q-1)q^{d-2} = q^{d-1} - q^{d-2}.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Hence, (7) follows from Theorem 1. □

Remark 1: As pointed out by one referee, Bose and Cunningham [9] presented a construction of binary asymmetric error-correcting codes if $n+1$ is a prime power. This construction yields the following lower bound:

$$\Gamma(n,d) \geq \frac{2^n}{(n+1)^{d-1}}. \tag{8}$$

Note that bound (8) is slightly worse than bound (6). The referee observed that after redescribing the construction of Bose and Cunningham [9] in polynomial form it is somewhat similar to our construction here. We note that the construction of Bose and Cunningham is actually a special case of our general construction by taking $f(x) = x^d$ in the proof of Theorem 2 ii). Bound (8) follows from Theorem 1 by noting that $\Phi(x^d) = (q-1)q^{d-1}$ and $\Phi^*(x^d) = q^{d-1} = (n+1)^{d-1}$.

In the following theorem, we show that the lower bounds given by Theorem 2 can be generalized and improved by using Theorem 1. Note that the number of monic quadratic irreducible polynomials in $\boldsymbol{F}_q[x]$ is $q(q-1)/2$.

Theorem 3:

i) If n is a prime power and $2 \leq d \leq n$, then

$$\Gamma(n,d) \geq \frac{(n-1)2^n}{(n^2-1)^r(n^3-1)^s} \tag{9}$$

where r and s are the two unique nonnegative integers satisfying $d = 2r+3s$ and $s \in \{0,1\}$.

ii) If n is not a prime power, denote m as the least positive integer such that $q = n+m$ is a prime power. If $2 \leq d \leq m$, then

$$\Gamma(n,d) \geq \frac{2^n}{(q-1)^{d-1}}. \tag{10}$$

If $d > m$, then

$$\Gamma(n,d) \geq \frac{2^n}{(q-1)^{m-1}q^{s'}(q^2-1)^{r'}} \tag{11}$$

where r' and s' are the two unique nonnegative integers satisfying $d-m = 2r'+s'$ and $s' \in \{0,1\}$.

Proof:

i) Let $q = n$ in Theorem 1 since n is a prime power. Let

$$\boldsymbol{F}_q = \{\alpha_1, \alpha_2, \ldots, \alpha_q\}.$$

Since

$$r \leq \frac{d}{2} \leq \frac{n}{2} = \frac{q}{2} \leq \frac{q(q-1)}{2}$$

we can choose distinct monic quadratic irreducible polynomials

$$p_1(x), p_2(x), \ldots, p_r(x)$$

in $\boldsymbol{F}_q[x]$ and a monic cubic irreducible polynomials $p(x)$ in $\boldsymbol{F}_q[x]$. Let

$$f(x) = p^s(x)\prod_{i=1}^{r} p_i(x).$$

Then $\deg(f(x)) = d$ and

$$\Phi(f(x)) = (q^2-1)^r(q^3-1)^s$$

$$\Phi^*(f(x)) = \frac{(q^2-1)^r(q^3-1)^s}{q-1} = \frac{(n^2-1)^r(n^3-1)^s}{n-1}.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1,2,\ldots,n$. Hence, by Theorem 1

$$\Gamma(n,d) \geq \frac{(n-1)2^n}{(n^2-1)^r(n^3-1)^s}.$$

ii) In Theorem 1, let

$$\boldsymbol{F}_q = \{\beta_1, \beta_2, \ldots, \beta_m, \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

If $2 \leq d \leq m$, let

$$f(x) = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_d).$$

Then

$$\Phi(f(x)) = (q-1)^d, \qquad \Phi^*(f(x)) = (q-1)^{d-1}.$$

If $d > m$, by the fact that $d-m = 2r'+s'$, we have

$$r' \leq \frac{d}{2} \leq \frac{n}{2} \leq \frac{q}{2} \leq \frac{q(q-1)}{2}.$$

Hence, we can choose distinct monic quadratic irreducible polynomials

$$p_1(x), p_2(x), \ldots, p_{r'}(x)$$

in $\boldsymbol{F}_q[x]$. Let

$$f(x) = (x-\beta_1)^{1+s'}(x-\beta_2)\cdots(x-\beta_m)\prod_{i=1}^{r'} p_i(x).$$

Then $\deg(f(x)) = d$ and

$$\Phi(f(x)) = (q-1)^m q^{s'}(q^2-1)^{r'}$$

$$\Phi^*(f(x)) = (q-1)^{m-1}q^{s'}(q^2-1)^{r'}.$$

It is easy to see that $f(\alpha_i) \neq 0$ for all $i = 1,2,\ldots,n$. Hence, by Theorem 1, we obtain (10) and (11). □

The lower bound (9) in Theorem 3 is better than the lower bound (5) in Theorem 2. Note that for $d \geq 2$

$$(n^2-1)^r(n^3-1)^s \leq n^{2r+3s} - 1 = n^d - 1$$

and the equality holds if and only if $d = 2$ or 3. Hence, for $d \geq 2$

$$\frac{(n^2-1)^r(n^3-1)^s}{n-1} \leq n^{d-1} + n^{d-2} + \cdots + n + 1$$

and the equality holds if and only if $d = 2$ or 3.

The lower bound (9) in Theorem 3 can be rewritten in the following form. If n is a prime power, then

$$\Gamma(n,d) \geq \frac{(n-1)2^n}{(n^2-1)^{\frac{d}{2}}}, \qquad d \text{ even and } d \geq 2 \tag{12}$$

$$\Gamma(n,d) \geq \frac{(n-1)2^n}{(n^2-1)^{\frac{(d-3)}{2}}(n^3-1)}, \qquad d \text{ odd and } d \geq 3. \tag{13}$$

For two sequences $\{g(n)\}_{n=1}^{\infty}$ and $\{h(n)\}_{n=1}^{\infty}$, we say

$$g(n) = O(h(n)), \qquad \text{if } \lim_{n\to\infty}\frac{g(n)}{h(n)} = 1.$$

By direct computation, it is not hard to see that

$$\text{Bound (12)} - \text{Bound (5)} = O\left(\frac{d2^{n-1}}{n^{d+1}}\right), \qquad d \text{ even and } d \geq 4$$

$$\text{Bound (13)} - \text{Bound (5)} = O\left(\frac{(d-3)2^{n-1}}{n^{d+1}}\right), \quad d \text{ odd and } d \geq 5.$$

Let $m = 1$ in Theorem 3 ii), then we obtain the following.

Corollary 2: If $n+1$ is a prime power, then for $d \geq 2$

$$\Gamma(n,d) \geq \frac{2^n}{(n+1)^s[(n+1)^2-1]^r} \tag{14}$$

where r and s are the two unique nonnegative integers satisfying $d-1=2r+s$ and $s\in\{0,1\}$.

The lower bound (14) in Corollary 2 is better than the lower bound (6) in Theorem 2. Note that for $d\geq 3$

$$(n+1)^s[(n+1)^2-1]^r \leq (n+1)^{2r+s}-1=(n+1)^{d-1}-1$$

and the equality holds if and only if $d=3$.

The lower bound (14) in Corollary 2 can be rewritten in the following form. If $n+1$ is a prime power, then

$$\Gamma(n,d)\geq \frac{2^n}{(n+1)[(n+1)^2-1]^{\frac{(d-2)}{2}}}, \quad d \text{ even and } d\geq 2 \quad (15)$$

$$\Gamma(n,d)\geq \frac{2^n}{[(n+1)^2-1]^{\frac{(d-1)}{2}}}, \quad d \text{ odd and } d\geq 3. \quad (16)$$

By direct computation, it is not hard to see that

$$\text{Bound (15)} - \text{Bound (6)} = O\left(\frac{(d-2)2^{n-1}}{n^{d+1}}\right), \quad d \text{ even and } d\geq 4$$

$$\text{Bound (16)} - \text{Bound (6)} = O\left(\frac{(d-1)2^{n-1}}{n^{d+1}}\right), \quad d \text{ odd and } d\geq 5.$$

Let $m=2$ in Theorem 3 ii), then we obtain the following.

Corollary 3: If $n+2$ is a prime power, then for $d\geq 3$

$$\Gamma(n,d)\geq \frac{2^n}{(n+1)(n+2)^s[(n+2)^2-1]^r} \quad (17)$$

where r and s are the two unique nonnegative integers satisfying $d-2=2r+s$ and $s\in\{0,1\}$.

The lower bound (17) in Corollary 3 is better than the lower bound (7) in Theorem 2. Note that for $d\geq 3$ and $q=n+2$

$$q^s(q^2-1)^r \leq q^{2r+s}=q^{d-2}$$

and the equality holds if and only if $d=3$. Hence, for $d\geq 3$

$$(q-1)q^s(q^2-1)^r \leq q^{d-1}-q^{d-2}$$

and the equality holds if and only if $d=3$.

The lower bound (17) in Corollary 3 can be rewritten in the following form. If $n+2$ is a prime power, then for even d and $d\geq 4$

$$\Gamma(n,d)\geq \frac{2^n}{(n+1)[(n+2)^2-1]^{\frac{(d-2)}{2}}} \quad (18)$$

and for odd d and $d\geq 3$,

$$\Gamma(n,d)\geq \frac{2^n}{(n+1)(n+2)[(n+2)^2-1]^{\frac{(d-3)}{2}}}. \quad (19)$$

Note that if $n+2$ is a prime power, the lower bound (7) in Theorem 2 is given by

$$\Gamma(n,d)\geq \frac{2^n}{(n+1)(n+2)^{d-2}}, \quad d\geq 3. \quad (20)$$

By direct computation, it is not hard to see that for even d and $d\geq 4$

$$\text{Bound (18)} - \text{Bound (20)} = O\left(\frac{(d-2)2^{n-1}}{n^{d+1}}\right)$$

and for odd d and $d\geq 5$

$$\text{Bound (19)} - \text{Bound (20)} = O\left(\frac{(d-3)2^{n-1}}{n^{d+1}}\right).$$

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