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Automatic Adaptive FE analysis of Thin-Walled Structures using 3D Solid Elements

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Abstract

In this study, a new automatic adaptive refinement procedure for thin-walled structures (TWS) using 3D solid elements is suggested. This procedure employs a specially designed superconvergent patch recovery (SPR) procedure for stress recovery, the Zienkiewicz and Zhu (Z-Z) error estimator for the *a posteriori* error estimation, a new refinement strategy for new element size prediction and a special mesh generator for adaptive mesh generation. The proposed procedure is different from other schemes in such a way that the problem domain is separated into two distinct parts: the shell part and the junction part. For stress recovery and error estimation in the shell part, special nodal coordinate systems are used and the stress field is separated into two components. For the refinement strategy, different procedures are employed for the estimation of new element sizes in the shell and the junction parts. Numerical examples are given to validate the effectiveness of the suggested procedure. It is found that by using the suggested refinement procedure, when comparing with uniform refinement, higher convergences rate were achieved and more accurate final solutions were obtained by using fewer of degree of freedoms and less amount of computational time.

Key words: Adaptive FE analysis, Thin-walled Structures, Stress recovery, Error Estimation, Adaptive Refinement

1. Introduction

Stress analysis of thin-walled structures (TWS) such as plates and shells is a frequently encountered problem in structural engineering. In the analysis of plate and shell structures, adaptive refinement procedures using plate and shell finite elements (FE) were successfully developed [1,2]. However, in some applications such as analysis of tubular structures in offshore engineering [3,4], the use of plate and shell FE elements is inadequate and practical design guide [5] indicates explicitly that 3D solid elements *must* be employed for the analysis

of stress concentration near the intersections of TWSs. Despite the rapid advancements in FE technology [6,7] and the development of efficient adaptive algorithms for 3D FE analysis [8,9], the analysis of TWSs using 3D elements appears to be a new and unexplored topic.

Two main reasons why 3D elements are seldom used in the analyses of plates and shells are the unavailability of special automatic mesh generator for TWS analyses and the impractically large number of degrees of freedom (DOFs) needed when uniform meshes are used. To tackle the first difficulty, the author recently had developed a new approach [10] which allows single or multiple layers of element to be generated at different regions of the TWSs. To tackle the second difficulty, a specially designed adaptive refinement scheme is needed to optimize the efficiency of the analysis.

The main objective of this study is to suggest a new adaptive refinement procedure for TWSs using 3D solid elements. In particular, the purposed algorithm is designed for TWSs encountered in structural and offshore engineering where the length to thickness ratio is within the range of 10 to 250. This procedure will employ a modified version of the superconvergent patch recovery procedure [11,12] and the Zienkiewicz and Zhu (Z-Z) error estimator [13] for the *a posteriori* error estimation. A new adaptive refinement strategy is proposed to predict the optimal element size along the surface of the TWS (*the surface direction*) and the optimal numbers of layer of elements in the thickness direction of the TWS. After the new element size distributions are determined, a special mesh generator [10] is employed for adaptive mesh generation. The new adaptive refinement procedure is tested by applying it to solve a few elastostatic problems and its performance is evaluated by comparing the results obtained from uniform refinements.

2. Basic assumptions and notations, the model problem and elements used

2.1 Basic assumptions and notations

In this study, the following basic assumptions regarding the actions of TWSs are adopted:

- (1) Based on the dominating mechanical response of the structures, the whole structure is decomposed into two regions, namely, the *shell part* Ω_{shell} and the *junction part* Ω_{jun} such that $\Omega = \Omega_{shell} \cup \Omega_{jun}$ and $\Omega_{shell} \cap \Omega_{jun} = \emptyset$ (Fig. 1). The TWSs may consist of multiple shell parts that are connected through a numbers of junctions.
- (2) For Ω_{shell} , its length and width are at least one order greater than its thickness.

(3) In Ω_{shell} , the magnitudes and variations (cubic or higher) of the surface (membrane and bending) stress along the shell surface direction are one order higher than that of the normal and shear stress (up to quadratic) in the shell normal direction (Fig 1).

(4) Ω_{jun} is region close to the intersections between two or more shell surfaces. The magnitudes and variations of all stress components are of the same order (Fig. 1).

In this paper, in order to give a systemic description on the adaptive refinement process, a number of basic notations together with many superscripts, subscripts and embellishments are used. Their definitions are summarized in Tables I and II.

2.2 The model problem

The domain of the TWS is denoted as Ω . Since 3D solid elasticity formulation is used, the governing equation over Ω can be expressed as:

$$\mathbf{S}^T \mathbf{D} \mathbf{S} \mathbf{u} - \mathbf{q} = 0 \quad (1)$$

with the strain operator $\mathbf{S}$ defined as

$$\mathbf{S}^T = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \quad (2a)$$

For an isotopic material with Young's modulus E and Poisson's ratio ν , the elastic matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \begin{bmatrix} D_A & D_B & D_B & 0 & 0 & 0 \\ D_B & D_A & D_B & 0 & 0 & 0 \\ D_B & D_B & D_A & 0 & 0 & 0 \\ 0 & 0 & 0 & D_C & 0 & 0 \\ 0 & 0 & 0 & 0 & D_C & 0 \\ 0 & 0 & 0 & 0 & 0 & D_C \end{bmatrix} \quad (2b)$$

where $D_A = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$, $D_B = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $D_C = \frac{E}{2(1+\nu)}$.

The displacement $\mathbf{u}$, strain $\boldsymbol{\varepsilon}$, and stress $\boldsymbol{\sigma}$ are defined as

$$\mathbf{u} = [u, v, w]^T, \quad \boldsymbol{\varepsilon} = \mathbf{S} \mathbf{u} \quad \text{and} \quad \boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \quad (3a)$$

$$\boldsymbol{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}]^T \quad \text{and} \quad \boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}]^T \quad (3b)$$

The FE displacements $\hat{\mathbf{u}}$ and stresses $\hat{\boldsymbol{\sigma}}$ are constructed by using the interpolation function $\mathbf{N}$ and nodal vector $\mathbf{u}^n$ such that

$$\mathbf{u} \approx \hat{\mathbf{u}} = \mathbf{N}\mathbf{u}^n \quad \text{and} \quad \hat{\boldsymbol{\sigma}} = \mathbf{D}\mathbf{S}\mathbf{N}\mathbf{u}^n \quad (4)$$

2.3 3D elements used

In this study, four types of quadratic 3D displacement solid elements, namely, the 10-node tetrahedron (T10), the 13-node pyramid (P13), the 15-node prism (P15) and the 20-node hexahedron (H20) are used (Fig. 2). The 13-node pyramid element (P13) is based on the shape functions presented by Bedrosian [14]. All the element stiffness matrices are fully integrated. Note that the quadratic elements are selected as they are known to be cost effective [8,9] and show no shear and membrane locking in the target range of thickness to length ratio.

3. Coordinate systems and error norms separation

3.1 The Global Coordinate System ($\mathbf{x}, \mathbf{y}, \mathbf{z}$ or $\mathbf{x}_i$), GCS

The GCS (Fig. 3a) defines the nodal coordinates and displacements of the FE model. The three axes of the GCS are denoted as $\mathbf{e}_1$ (0, 0, 1), $\mathbf{e}_2$ (0, 1, 0) and $\mathbf{e}_3$ (0, 0, 1).

3.2 The Patch Coordinate System ($\mathbf{x}^p, \mathbf{y}^p, \mathbf{z}^p$ or $\mathbf{x}_i^p$), PCS

The PCS is used in many stress recovery schemes [1,2,8,9,11,12,15-20] to improve the stability of the recovery procedure. In this study, it is used in the stress recovery and the error estimation of Ω_{jun} . For an element corner node P with global coordinates (x_P, y_P, z_P) , the PCS is a local coordinate system with origin at P defined over all the elements (the patch) connecting to P [11]. The axes $\mathbf{x}^p, \mathbf{y}^p$ and $\mathbf{z}^p$ are parallel to the GCS axes (Fig. 3b). The patch coordinates (x^p, y^p, z^p) of a point Q with global coordinates (x_Q, y_Q, z_Q) are computed as

$$x^p = \frac{x_Q - x_P}{f_x}, \quad y^p = \frac{y_Q - y_P}{f_y}, \quad z^p = \frac{z_Q - z_P}{f_z} \quad (5)$$

In Eqn. 5, f_x, f_y and f_z are scale factors selected based in the size of the patch (see Section 5.2.3 for details) so that the value of (x^p, y^p, z^p) are normalized within the range of [-1,1].

3.3 The Nodal Coordinate System ($\mathbf{x}^l, \mathbf{y}^l, \mathbf{z}^l$ or $\mathbf{x}_i^l$), NCS

The NCS (Fig. 3c) is a set of local coordinate systems established at the corner nodes of elements. In this study, it is employed for the stress recovery and the error estimation of Ω_{shell} .

For an element corner node P , the z^I axis of the NCS is the same as the *nodal normal direction* at P . If P is located at the top, the middle or the bottom surfaces of Ω_{shell} , its nodal normal direction will be defined by the geometrical properties of the TWS [10] and is the same as the true surface normal. If P is an interior node not lying on the mid-surface, then the definition of the true surface normal could not be available. In this case, the *nodal normal vector* for P , $\hat{V}_P$, is computed using the algorithm shown in Box 1 [10].

- (i) Identify the *normal side* AB that contains P and the two end nodes A and B that locate on the mid and top (or bottom) surfaces of the TWS respectively (Fig. 4).
- (ii) Determine V_A and V_B , the normal vectors at A and B respectively from the geometrical properties of the TWS.
- (iii) Compute the *nodal normal vector* $\hat{V}_P$ for P using linear interpolation such that

$$V_P = \frac{V_A|BP| + V_B|AP|}{|AB|} \quad \text{and} \quad \hat{V}_P = \frac{V_P}{|V_P|} \quad (6)$$

Box 1. Definition of the nodal normal vector for an interior element corner node

Note that in steps (i) and (ii) of Box 1, the normal side and normal vectors information are retrieved from the mesh generation scheme during the surface extrusion process [10]. Furthermore, $\hat{V}_P$ will converge to the normal direction of the mid-surface as the thickness of the TWS approaching zero. After the nodal normal vector $\hat{V}_P$ is defined, the unit vectors $\hat{V}_1$, $\hat{V}_2$ and $\hat{V}_3$ corresponding to the axes x^I , y^I , and z^I for the NCS are then computed by using the algorithm shown in Box 2.

- (i) The vector $\hat{V}_3$ is set as $\hat{V}_P$ for the element corner node.
- (ii) If $\hat{V}_3$ is *not* parallel to the global z axis, $\hat{V}_2$ is set perpendicular to $\hat{V}_3$ and parallel to the global y - z plane while $\hat{V}_1$ will be normal to both $\hat{V}_2$ and $\hat{V}_3$. That is

$$\hat{V}_2 = \frac{\hat{V}_3 \times \mathbf{e}_I}{|\hat{V}_3 \times \mathbf{e}_I|} \quad \text{and} \quad \hat{V}_1 = \frac{\hat{V}_2 \times \hat{V}_3}{|\hat{V}_2 \times \hat{V}_3|} \quad (7a)$$

- (iii) If $\hat{V}_3$ is parallel to the global z axis, $\hat{V}_1$ is defined first and $\hat{V}_2$ is obtained by setting it normal to both $\hat{V}_1$ and $\hat{V}_3$ such that

$$\hat{\mathbf{V}}_1 = \frac{\hat{\mathbf{V}}_3 \times \mathbf{e}_2}{|\hat{\mathbf{V}}_3 \times \mathbf{e}_2|} \quad \text{and} \quad \hat{\mathbf{V}}_2 = \frac{\hat{\mathbf{V}}_3 \times \hat{\mathbf{V}}_1}{|\hat{\mathbf{V}}_3 \times \hat{\mathbf{V}}_1|} \quad (7b)$$

Box 2. Definition of the NCS for an element corner node

3.4 Energy norm, error norm and stress components separation

3.4.1 Energy and error norms

Except for some trivial cases, the FE displacement $\hat{\mathbf{u}}$ and stress $\hat{\boldsymbol{\sigma}}$ (Eqn. 4) are inexact and their errors are defined as

$$\mathbf{e}_u = \mathbf{u} - \hat{\mathbf{u}}, \quad \mathbf{e}_\sigma = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \quad (8)$$

The energy norm $\|\mathbf{u}\|_\Omega$ and the error norm $\|\mathbf{e}_u\|_\Omega$ for the whole domain Ω are defined as

$$\|\mathbf{u}\|_\Omega = \sqrt{\int_\Omega (\mathbf{S}\mathbf{u})^T \mathbf{D}(\mathbf{S}\mathbf{u}) d\Omega} = \sqrt{\int_\Omega \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega} \quad (9a)$$

$$\|\mathbf{e}_u\|_\Omega = \sqrt{\int_\Omega (\mathbf{S}\mathbf{e}_u)^T \mathbf{D}(\mathbf{S}\mathbf{e}_u) d\Omega} = \sqrt{\int_\Omega (\mathbf{e}_\sigma)^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega} \quad (9b)$$

The relative error η_Ω of the solution is defined as

$$\eta_\Omega = \frac{\|\mathbf{e}_u\|_\Omega}{\|\mathbf{u}\|_\Omega} \quad (10)$$

The total energy and error norms could be separated according to the contributions from Ω_{jun} and Ω_{shell} :

$$\|\mathbf{u}\|_\Omega^2 = \|\mathbf{u}\|_{\Omega_{jun}}^2 + \|\mathbf{u}\|_{\Omega_{shell}}^2 = \int_{\Omega_{jun}} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega + \int_{\Omega_{shell}} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega \quad (11a)$$

$$\|\mathbf{e}_u\|_\Omega^2 = \|\mathbf{e}_u\|_{\Omega_{jun}}^2 + \|\mathbf{e}_u\|_{\Omega_{shell}}^2 = \int_{\Omega_{jun}} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega + \int_{\Omega_{shell}} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega \quad (11b)$$

The error and energy norms are invariant with respect to the choice of coordinate system. Hence, when evaluating the error and energy norms, one could use two different coordinate systems for Ω_{jun} and Ω_{shell} . Since the variation of all stress components in Ω_{jun} are of similar order, the first integrals in Eqn. 11 could be computed using stress components expressed in the GCS or the PCS. However, as some stress components in Ω_{shell} mainly vary along the surface direction, it is preferable to evaluate the second integrals of Eqn. 11 using stress components expressed in the NCS ($\mathbf{x}^I, \mathbf{y}^I, \mathbf{z}^I$) so that

$$\|\mathbf{u}\|_{\Omega}^2 = \|\mathbf{u}\|_{\Omega_{jun}}^2 + \|\mathbf{u}\|_{\Omega_{shell}}^2 = \int_{\Omega_{jun}} \boldsymbol{\sigma}^T \mathbf{D}^{-1} \boldsymbol{\sigma} d\Omega + \int_{\Omega_{shell}} (\boldsymbol{\sigma}^l)^T \mathbf{D}^{-1} \boldsymbol{\sigma}^l d\Omega \quad (12a)$$

$$\|\mathbf{e}_u\|_{\Omega}^2 = \|\mathbf{e}_u\|_{\Omega_{jun}}^2 + \|\mathbf{e}_u\|_{\Omega_{shell}}^2 = \int_{\Omega_{jun}} (\mathbf{e}_{\sigma})^T \mathbf{D}^{-1} \mathbf{e}_{\sigma} d\Omega + \int_{\Omega_{shell}} (\mathbf{e}_{\sigma^l})^T \mathbf{D}^{-1} \mathbf{e}_{\sigma^l} d\Omega \quad (12b)$$

In Eqn. 12b, $\mathbf{e}_{\sigma^l} = \boldsymbol{\sigma}^l - \hat{\boldsymbol{\sigma}}^l$ with $\boldsymbol{\sigma}^l$ and $\hat{\boldsymbol{\sigma}}^l$ are the exact and the FE stress expressed in the NCS, respectively. The relative errors corresponding to Ω_{jun} ($\eta_{\Omega_{jun}}$) and Ω_{shell} ($\eta_{\Omega_{shell}}$) can now be defined as (c.f. Eqn. 10)

$$\eta_{\Omega_{jun}} = \frac{\|\mathbf{e}_u\|_{\Omega_{jun}}}{\|\mathbf{u}\|_{\Omega_{jun}}} \quad \text{and} \quad \eta_{\Omega_{shell}} = \frac{\|\mathbf{e}_u\|_{\Omega_{shell}}}{\|\mathbf{u}\|_{\Omega_{shell}}} \quad (13)$$

3.4.2 Stress components separation

In this study, the local stress, $\boldsymbol{\sigma}^l = (\sigma_{x^l}, \sigma_{y^l}, \sigma_{z^l}, \tau_{xz^l}, \tau_{yz^l}, \tau_{xy^l})^T$, and strain, $\boldsymbol{\varepsilon}^l = (\varepsilon_{x^l}, \varepsilon_{y^l}, \varepsilon_{z^l}, \gamma_{xz^l}, \gamma_{yz^l}, \gamma_{xy^l})^T$, are separated into the surface ($_{surf} \boldsymbol{\sigma}^l$ and $_{surf} \boldsymbol{\varepsilon}^l$) and the normal components ($_{norm} \boldsymbol{\sigma}^l$ and $_{norm} \boldsymbol{\varepsilon}^l$) such that

$$\boldsymbol{\sigma}^l = \left(\left(_{surf} \boldsymbol{\sigma}^l \right)^T, \left(_{norm} \boldsymbol{\sigma}^l \right)^T \right)^T, \quad \boldsymbol{\varepsilon}^l = \left(\left(_{surf} \boldsymbol{\varepsilon}^l \right)^T, \left(_{norm} \boldsymbol{\varepsilon}^l \right)^T \right)^T \quad \text{and} \quad \begin{bmatrix} _{surf} \boldsymbol{\sigma}^l \\ _{norm} \boldsymbol{\sigma}^l \end{bmatrix} = \mathbf{D}^l \begin{bmatrix} _{surf} \boldsymbol{\varepsilon}^l \\ _{norm} \boldsymbol{\varepsilon}^l \end{bmatrix} \quad (14)$$

where

$$\begin{aligned} _{surf} \boldsymbol{\sigma}^l &= (\sigma_{x^l}, \sigma_{y^l}, \sigma_{z^l}, \tau_{xy^l})^T & \text{and} & \quad _{norm} \boldsymbol{\sigma}^l = (\tau_{xz^l}, \tau_{yz^l})^T \\ _{surf} \boldsymbol{\varepsilon}^l &= (\varepsilon_{x^l}, \varepsilon_{y^l}, \varepsilon_{z^l}, \gamma_{xy^l})^T & \text{and} & \quad _{norm} \boldsymbol{\varepsilon}^l = (\gamma_{xz^l}, \gamma_{yz^l})^T \end{aligned} \quad (15)$$

$\mathbf{D}^l$, the elastic matrix with respects to the NCS, is now partitioned as

$$\mathbf{D}^l = \begin{bmatrix} _{surf} \mathbf{D}^l & 0 \\ 0 & _{norm} \mathbf{D}^l \end{bmatrix} \quad (16)$$

so that for an isotopic material

$$_{surf} \mathbf{D}^l = \begin{bmatrix} D_A & D_B & D_B & 0 \\ D_B & D_A & D_B & 0 \\ D_B & D_B & D_A & 0 \\ 0 & 0 & 0 & D_C \end{bmatrix} \quad \text{and} \quad _{norm} \mathbf{D}^l = \begin{bmatrix} D_C & 0 \\ 0 & D_C \end{bmatrix} \quad (17)$$

Note that the matrices $(_{surf} \mathbf{D}^l)^{-1}$ and $(_{norm} \mathbf{D}^l)^{-1}$ are always positive definite. Therefore,

$\|\mathbf{u}\|_{\Omega_{shell}}$ and $\|\mathbf{e}_u\|_{\Omega_{shell}}$ are separated into two components corresponding to the surface and normal directions.

$$\|\mathbf{u}\|_{\Omega_{shell}}^2 = \text{surf}\|\mathbf{u}\|_{\Omega_{shell}}^2 + \text{norm}\|\mathbf{u}\|_{\Omega_{shell}}^2 \quad \text{and} \quad \|\mathbf{e}_u\|_{\Omega_{shell}}^2 = \text{surf}\|\mathbf{e}_u\|_{\Omega_{shell}}^2 + \text{norm}\|\mathbf{e}_u\|_{\Omega_{shell}}^2 \quad (18a)$$

where

$$\text{surf}\|\mathbf{u}\|_{\Omega_{shell}}^2 = \int_{\Omega_{shell}} (\text{surf}\boldsymbol{\sigma}^l)^T (\text{surf}\mathbf{D}^l)^{-1} \text{surf}\boldsymbol{\sigma}^l d\Omega \quad (18b)$$

$$\text{norm}\|\mathbf{u}\|_{\Omega_{shell}}^2 = \int_{\Omega_{shell}} (\text{norm}\boldsymbol{\sigma}^l)^T (\text{norm}\mathbf{D}^l)^{-1} \text{norm}\boldsymbol{\sigma}^l d\Omega$$

$$\text{surf}\|\mathbf{e}_u\|_{\Omega_{shell}}^2 = \int_{\Omega_{shell}} (\text{surf}\boldsymbol{\sigma}^l - \text{surf}\hat{\boldsymbol{\sigma}}^l)^T (\text{surf}\mathbf{D}^l)^{-1} (\text{surf}\boldsymbol{\sigma}^l - \text{surf}\hat{\boldsymbol{\sigma}}^l) d\Omega \quad (18c)$$

and

$$\text{norm}\|\mathbf{e}_u\|_{\Omega_{shell}}^2 = \int_{\Omega_{shell}} (\text{norm}\boldsymbol{\sigma}^l - \text{norm}\hat{\boldsymbol{\sigma}}^l)^T (\text{norm}\mathbf{D}^l)^{-1} (\text{norm}\boldsymbol{\sigma}^l - \text{norm}\hat{\boldsymbol{\sigma}}^l) d\Omega$$

In Eqn 18a, $\text{surf}\|\mathbf{u}\|_{\Omega_{shell}}$ ($\text{surf}\|\mathbf{e}_u\|_{\Omega_{shell}}$) and $\text{norm}\|\mathbf{u}\|_{\Omega_{shell}}$ ($\text{norm}\|\mathbf{e}_u\|_{\Omega_{shell}}$) are the surface and the normal energy (error) norms for Ω_{shell} , respectively. $\text{surf}\hat{\boldsymbol{\sigma}}^l$ and $\text{norm}\hat{\boldsymbol{\sigma}}^l$ are the FE stress components along the surface and normal directions, respectively. Furthermore, the surface ($\text{surf}\eta_{\Omega_{shell}}$) and normal ($\text{norm}\eta_{\Omega_{shell}}$) relative errors of Ω_{shell} are defined as (c.f. Eqn. 13)

$$\text{surf}\eta_{\Omega_{shell}} = \frac{\text{surf}\|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{surf}\|\mathbf{u}\|_{\Omega_{shell}}} \quad \text{and} \quad \text{norm}\eta_{\Omega_{shell}} = \frac{\text{norm}\|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{norm}\|\mathbf{u}\|_{\Omega_{shell}}} \quad (19)$$

It should be noted that in Eqn. 15, the normal stress component σ_{z^l} is grouped under the surface terms despite that its magnitude is typically one order lower than other surface components. However, this arrangement is justified when a patch loading is applied on the TWS surface such that a jump in σ_{z^l} along the surface direction is resulted.

4. Error estimations

4.1 A priori error estimation for TWSs

Since solid elements are employed, the *a priori* error estimation is cast in a form similar to the case of 3D adaptive FE analysis [8,9]. For any subdomain $\Omega_{sub} \subset \Omega$, the associated error norm $\|\mathbf{e}_u\|_{\Omega_{sub}}$ is related to the *characteristics element size* h_c and the *highest effective polynomial order* p_e in the form

$$\|\mathbf{e}_u\|_{\Omega_{sub}} \approx C(h_c)^{\min(\lambda, p_e)} \quad (20a)$$

where $\lambda < 1.0$ is the strength of singularity [21]. If ND_c is the *characteristics dimension* of the problem, the relationship between h_c and NTD , the total numbers of DOF, can be written as

$$NTD \propto h_c^{-ND_c} \quad (20b)$$

so that

$$\|e_u\|_{\Omega_{sub}} \approx C'(NTD)^{-\min(\lambda, p_e)/ND_c} \quad (20c)$$

The main objective of an h -adaptive refinement procedure is to eliminate the effect of the singularity by constructing a FE mesh such that the total error norm is equally distributed among all the elements [21] so that the convergent rate is improved to

$$\|e_u\|_{\Omega_{sub}} \approx Ch_c^{p_e} \quad \text{or} \quad \|e_u\|_{\Omega_{sub}} \approx C'NTD^{-p_e/ND_c} \quad (20d)$$

For a TWS, due to the domination of different stress components in different parts of the structure and depended on the refinement strategy used, h_c , p_e and ND_c may take different values in the above *a priori* error estimation. In Ω_{jun} , since there is no dominant stress component, $h_c=h$, $p_e=p$, and $ND_c=3$ so that

$$\|e_u\|_{\Omega_{sub}} \approx Ch^p \quad \text{and} \quad \|e_u\|_{\Omega_{sub}} \approx C'NTD^{-p/3} \quad (21)$$

where h is the typical element size and p is the element polynomial order. However, in Ω_{shell} , since $_{surf}\sigma^l$ are the dominating stress components, it is most effective to refine the mesh along the surface direction [1,2]. In this case, $ND_c=2$ and Eqn. 20d becomes

$$_{surf}\|e_u\|_{\Omega_{shell}} \approx Ch_{surf}^{p_{surf}} \quad \text{or} \quad _{surf}\|e_u\|_{\Omega_{shell}} \approx C'NTD^{-p_{surf}/2} \quad (22a)$$

where h_{surf} and p_{surf} are the typical element size and element polynomial order along the surface direction. Finally, if the mesh is sufficiently refined along the surface direction such that $_{surf}\|e_u\|_{\Omega_{shell}}$ is much reduced, it is then more economical to reduce $_{norm}\|e_u\|_{\Omega_{shell}}$ by carrying out refinement along the thickness direction such that $ND_c=1$. That is

$$_{norm}\|e_u\|_{\Omega_{shell}} \approx Ch_{norm}^{p_{norm}} \quad \text{or} \quad _{norm}\|e_u\|_{\Omega_{shell}} \approx C'NTD^{-p_{norm}} \quad (22b)$$

where h_{norm} and p_{norm} are the typical element size and element polynomial order along the normal direction, respectively.

4.2 *A posteriori* error estimation for TWSs

Following the pervious works of the first author [1,2,8,9,22,23], the Zienkiewicz and Zhu (Z^2) error estimator [13] is adopted for the *a posteriori* error estimation. The basic idea of the Z^2 error estimator is to replace the exact stress σ in Eqn. 8 by a recovered stress $\tilde{\sigma}$ such that the estimated pointwise error in stress $\bar{e}_\sigma$ is given by

$$\bar{e}_\sigma = \tilde{\sigma} - \hat{\sigma} \quad (23)$$

The estimated total error norm $\|\overline{\mathbf{e}_u}\|_\Omega$ could now be computed as (c.f. Eqn. 9b)

$$\|\overline{\mathbf{e}_u}\|_\Omega^2 = \int_\Omega \bar{\mathbf{e}}_\sigma^T \mathbf{D}^{-1} \bar{\mathbf{e}}_\sigma d\Omega = \int_\Omega (\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}})^T \mathbf{D}^{-1} (\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) d\Omega \quad (24a)$$

Furthermore, **one can define** the estimated error norm corresponding to different parts of Ω and different stress components as shown below.

Estimated error norms for Ω_{jun} and Ω_{shell} (c.f. Eqn. 11b)

$$\|\overline{\mathbf{e}_u}\|_\Omega^2 = \|\overline{\mathbf{e}_u}\|_{\Omega_{jun}}^2 + \|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 = \int_{\Omega_{jun}} (\bar{\mathbf{e}}_\sigma)^T \mathbf{D}^{-1} \bar{\mathbf{e}}_\sigma d\Omega + \int_{\Omega_{shell}} (\bar{\mathbf{e}}_\sigma)^T \mathbf{D}^{-1} \bar{\mathbf{e}}_\sigma d\Omega \quad (24b)$$

Estimated surface and normal error norms for the shell part (c.f. Eqns. 18a and 18c)

$$\|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 = \text{surf} \|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 + \text{norm} \|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 \quad (25a)$$

$$\begin{aligned} \text{surf} \|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 &= \int_{\Omega_{shell}} \left(\text{surf} \tilde{\boldsymbol{\sigma}}^l - \text{surf} \hat{\boldsymbol{\sigma}}^l \right)^T \left(\text{surf} \mathbf{D}^l \right)^{-1} \left(\text{surf} \tilde{\boldsymbol{\sigma}}^l - \text{surf} \hat{\boldsymbol{\sigma}}^l \right) d\Omega \\ \text{norm} \|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}^2 &= \int_{\Omega_{shell}} \left(\text{norm} \tilde{\boldsymbol{\sigma}}^l - \text{norm} \hat{\boldsymbol{\sigma}}^l \right)^T \left(\text{norm} \mathbf{D}^l \right)^{-1} \left(\text{norm} \tilde{\boldsymbol{\sigma}}^l - \text{norm} \hat{\boldsymbol{\sigma}}^l \right) d\Omega \end{aligned} \quad (25b)$$

In Eqn. 25b, $\text{surf} \tilde{\boldsymbol{\sigma}}^l$ and $\text{norm} \tilde{\boldsymbol{\sigma}}^l$ are, respectively, the surface and normal components of $\tilde{\boldsymbol{\sigma}}^l$.

In order to compute the above the error norms accurately, high order numerical integration rules shown in Table III are used [1,2].

Since the exact energy norm of the problem is not known, in order to compute the estimated relative errors, it is necessary to estimate the energy norms of the solution. In this study, the energy norm of the recovered stress is taken as the estimated energy norm. Hence, the estimated energy norms for Ω_{jun} and Ω_{shell} are given by

$$\|\overline{\mathbf{u}}\|_{\Omega_{jun}}^2 = \int_{\Omega_{jun}} \tilde{\boldsymbol{\sigma}}^T \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} d\Omega \quad \text{and} \quad \|\overline{\mathbf{u}}\|_{\Omega_{shell}}^2 = \int_{\Omega_{shell}} \tilde{\boldsymbol{\sigma}}^T \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} d\Omega \quad (26)$$

$$\|\overline{\mathbf{u}}\|_\Omega^2 = \|\overline{\mathbf{u}}\|_{\Omega_{jun}}^2 + \|\overline{\mathbf{u}}\|_{\Omega_{shell}}^2 \quad (27)$$

Similarly, for the surface and normal components of Ω_{shell}

$$\begin{aligned} \text{surf} \|\overline{\mathbf{u}}\|_{\Omega_{shell}}^2 &= \int_{\Omega_{shell}} \left(\text{surf} \tilde{\boldsymbol{\sigma}}^l \right)^T \left(\text{surf} \mathbf{D}^l \right)^{-1} \text{surf} \tilde{\boldsymbol{\sigma}}^l d\Omega \\ \text{norm} \|\overline{\mathbf{u}}\|_{\Omega_{shell}}^2 &= \int_{\Omega_{shell}} \left(\text{norm} \tilde{\boldsymbol{\sigma}}^l \right)^T \left(\text{norm} \mathbf{D}^l \right)^{-1} \text{norm} \tilde{\boldsymbol{\sigma}}^l d\Omega \end{aligned} \quad (28)$$

Using Eqns. 24, 26 and 28, the estimated relative errors are given by (c.f. Eqns. 13 and 19)

$$\text{For } \Omega: \bar{\eta}_{\Omega} = \frac{\|\mathbf{e}_u\|_{\Omega}}{\|\mathbf{u}\|_{\Omega}}, \quad \text{for } \Omega_{jun}: \bar{\eta}_{\Omega_{jun}} = \frac{\|\mathbf{e}_u\|_{\Omega_{jun}}}{\|\mathbf{u}\|_{\Omega_{jun}}} \quad (29a)$$

$$\text{For } \Omega_{shell}: \bar{\eta}_{\Omega_{shell}} = \frac{\|\mathbf{e}_u\|_{\Omega_{shell}}}{\|\mathbf{u}\|_{\Omega_{shell}}}, \quad \text{surf } \bar{\eta}_{\Omega_{shell}} = \frac{\text{surf } \|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{surf } \|\mathbf{u}\|_{\Omega_{shell}}}, \quad \text{norm } \bar{\eta}_{\Omega_{shell}} = \frac{\text{norm } \|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{norm } \|\mathbf{u}\|_{\Omega_{shell}}} \quad (29b)$$

The performance of an *a posteriori* estimator is assessed by computing its *effectivity index* θ [11,12] which is defined as the ratio between the estimated and the actual error norms. In practice, different effectivity indices corresponding to different subdomain and stress components could be computed as shown below.

$$\text{Global: } \theta_{\Omega} = \frac{\|\mathbf{e}_u\|_{\Omega}}{\|\mathbf{e}_u\|_{\Omega}} \quad \Omega_{jun}: \theta_{\Omega_{jun}} = \frac{\|\mathbf{e}_u\|_{\Omega_{jun}}}{\|\mathbf{e}_u\|_{\Omega_{jun}}} \quad \Omega_{shell}: \theta_{\Omega_{shell}} = \frac{\|\mathbf{e}_u\|_{\Omega_{shell}}}{\|\mathbf{e}_u\|_{\Omega_{shell}}} \quad (30a)$$

$$\text{The } i\text{th elements: } \theta_{\Omega_i} = \frac{\|\mathbf{e}_u\|_{\Omega_i}}{\|\mathbf{e}_u\|_{\Omega_i}} \quad (30b)$$

$$\text{Surface component for } \Omega_{shell}: \text{surf } \theta_{\Omega_{shell}} = \frac{\text{surf } \|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{surf } \|\mathbf{e}_u\|_{\Omega_{shell}}} \quad (30c)$$

$$\text{Normal component for } \Omega_{shell}: \text{norm } \theta_{\Omega_{shell}} = \frac{\text{norm } \|\mathbf{e}_u\|_{\Omega_{shell}}}{\text{norm } \|\mathbf{e}_u\|_{\Omega_{shell}}} \quad (30d)$$

5. Stress recovery procedure for TWSs

As the main idea of the Z^2 error estimator is to construct a more accurate recovered stress field to replace the exact stress for error estimation, the performance of the Z^2 error estimator is highly dependent on the accuracy of the recovered stress. Among the many stress recovery algorithms developed, those based on the *superconvergent patch recovery* (SPR) technique [11,12] appears to be the most popular and reliable [24]. Despite that many different enhancements had been suggested to further improve its performance [15-20] and the method had been extended from 2D elastostatic applications to plate and shell analyses [1,2, 25-28], the basic procedure employed is almost unchanged. In this study, a specially designed SPR technique is proposed for TWS applications.

5.1 The classical SPR technique

In the classical SPR techniques, the recovered stress is constructed by first dividing the FE mesh into a number of overlapping patches, Ω_j , over each of which a continuous stress field σ^* is locally interpolated by a polynomial of the form

$$\sigma^* = Pa \quad (31a)$$

where $\mathbf{P}$ is a vector of polynomial with m terms expressed in the PCS and $\mathbf{a}$ is vector of undetermined coefficients. Ω_J is established as the union of elements that are connecting to the *patch assembly node* which is usually a corner node of an element. A popular choice of $\mathbf{P}$ is the complete polynomial terms corresponding to the order of the element. The value of $\mathbf{a}$ is determined by a local least squares fit over the patch such that a selected functional $\Pi_{SPR}(\mathbf{a})$ is minimized. The basic form of $\Pi_{SPR}(\mathbf{a})$ suggested in reference [11] is given by

$$\Pi_{SPR}(\mathbf{a}) = \sum_{i=1}^{NSP} (\hat{\sigma}(\mathbf{s}_i) - \sigma^*(\mathbf{s}_i))^2 = \sum_{i=1}^{NSP} (\hat{\sigma}(\mathbf{s}_i) - \mathbf{P}\mathbf{a}(\mathbf{s}_i))^2 \quad (31b)$$

where NSP is the numbers of *sampling point* in the patch and $\mathbf{s}_i$ is the coordinates of the i th sampling point. The *superconvergent points* [12] of the elements in the patch, which are usually coincident with the positions of the reduced integration points (Table III), are selected as the sampling points. By minimizing $\Pi_{SPR}(\mathbf{a})$, the recovery equation is given by

$$\sum_{i=1}^{NSP} \mathbf{P}^T(\mathbf{s}_i) \mathbf{P}(\mathbf{s}_i) \mathbf{a} = \sum_{i=1}^{NSP} \mathbf{P}^T(\mathbf{s}_i) \hat{\sigma}(\mathbf{s}_i) \quad \text{or} \quad \mathbf{A}\mathbf{a} = \mathbf{b} \quad (31c)$$

After Eqn. 31c is solved, the recovered stresses are locally defined over Ω_J . The global recovered stress field $\tilde{\sigma}$ is then obtained by simple nodal averaging [11].

5.2 The modified SPR technique for TWSs

As mentioned in Section 2, as the whole TWS is separated into two parts (Ω_{jun} and Ω_{shell}) with distinctive dominating mechanical responses, two different recovery schemes, namely, the *SPR3D scheme* and the *SPRN scheme* are adopted for Ω_{jun} and Ω_{shell} , respectively.

5.2.1 Coordinate system employed

For Ω_{jun} , since the magnitudes and variations of all stress components in any direction are similar, the PCS is employed. Hence, $\mathbf{s}_i$ in Eqn. 31 is the patch coordinates (x_i^p, y_i^p, z_i^p) of the i th sample point. However, for Ω_{shell} , it is preferable to express the stress components in the NCS and then separate them into surface and normal components. The patch assembly node is again taken as the origin and the NCS associates with it is used for stress evaluation at *all* sampling points. Hence, Eqns. 31a and 31b are modified to

$$\sigma^* = \mathbf{P}(\mathbf{s}^l) \mathbf{a} \quad \text{and} \quad \sum_{i=1}^{NSP} \mathbf{P}^T(\mathbf{s}_i^l) \mathbf{P}(\mathbf{s}_i^l) \mathbf{a} = \sum_{i=1}^{NSP} \mathbf{P}^T(\mathbf{s}_i^l) \hat{\sigma}^l(\mathbf{s}_i^l) \quad (31d)$$

where s_i^l is the coordinates of the i th sampling point with respected to the NCS.

5.2.2 Polynomial terms employed

For Ω_{jun} where the SPR3D scheme is used, $\mathbf{P}$ normally contains the same polynomial terms used in N . For example, for a P15 element, $m=15$ and the following 15 terms are used.

$$\left(1, x^p, y^p, z^p, (x^p)^2, (y^p)^2, (z^p)^2, x^p y^p, y^p z^p, x^p z^p, x^p y^p z^p, (x^p)^2 z^p, (y^p)^2 z^p, (z^p)^2 x^p, (z^p)^2 y^p \right) \quad (32a)$$

For Ω_{shell} where the SPRN scheme is used, since the surface stress components dominate the energy norm and their variations along the x^l and y^l directions are higher than that in the z^l direction, it is possible to reduce the highest order term in z^l from $(z^l)^2$ to z^l without affecting the accuracy of σ^* . For example, for a P15 element, significant saving in the computational cost could be achieved by only using the following 12 terms

$$\begin{aligned} \mathbf{P} &= \{1, x^l, y^l, (x^l)^2, x^l y^l, (y^l)^2\} \otimes \{1, z^l\} \\ &= \{1, x^l, y^l, (x^l)^2, x^l y^l, (y^l)^2, z^l, z^l x^l, z^l y^l, z^l (x^l)^2, z^l x^l y^l, z^l (y^l)^2\} \end{aligned} \quad (32b)$$

5.2.3 Other enhancement techniques

Scaling of patch coordinates

In this study, the following scaling procedure is performed in the SPRN schemes to enhance the stability of the recovery procedure.

- (i) Nodes on the boundary of the patch are identified and their number is denoted as NBP .
- (ii) Three scaling factors f_x , f_y and f_z are computed as

$$\begin{aligned} f_x &= \max(|\overrightarrow{PB_i} \cdot \hat{V}_1^l|) \\ f_y &= \max(|\overrightarrow{PB_i} \cdot \hat{V}_2^l|) \quad \text{for } i = 1, \dots, NBP \\ f_z &= \max(|\overrightarrow{PB_i} \cdot \hat{V}_3^l|) \end{aligned} \quad (33)$$

where B_i are the coordinates of the i th boundary node of the patch. $\hat{V}_1^l$, $\hat{V}_2^l$ and $\hat{V}_3^l$ are the bases vectors for the NCS employed.

- (iii) The scaled patch coordinates of the i th sampling point $\tilde{s}_i^l = (\tilde{x}_i^l, \tilde{y}_i^l, \tilde{z}_i^l)$ are computed as

$$\tilde{x}_i^l = \frac{x_i^l}{f_{x^l}}, \quad \tilde{y}_i^l = \frac{y_i^l}{f_{y^l}}, \quad \tilde{z}_i^l = \frac{z_i^l}{f_{z^l}} \quad (34)$$

where $s_i^l = (x_i^l, y_i^l, z_i^l)$ is the unscaled coordinates of the i th sampling point. $\tilde{s}_i^l$ are then employed in Eqn. 31d to improve the condition of the recovery matrix.

Boundary patches without interior nodes

One necessary (but not sufficient) condition for the matrix A to be non-singular is that

$$NSP \geq m \quad (35)$$

In a 3D mesh, the above condition can be satisfied for almost all interior patches. However, for a boundary patch with fewer elements, Eqn. 35 may not be satisfied. To resolve this problem, Zienkiewicz and Zhu [11] suggested that one could extrapolate the recovered stress from interior patches that contain the boundary patch assembly point. However, this method will fail for any boundary node that is not connected to any interior patch (Fig. 5). In this case, the method suggested by Labbé and Garon [20] which uses full integration points as sampling points is adopted.

Boundary patch enhancement

It is well known [1,2,8,9,15-20,22-24] that the recovered stresses obtained from boundary patches are less accurate than those obtained from interior patches. In order to improve the accuracy of the recovered stresses there, different approaches involving impositions of equilibrium and natural boundary conditions had been suggested [15-18]. Although it was shown that these approaches could improve the accuracy of the recovered stresses, one disadvantage of them is that all submatrices of A corresponding to different stress components will be coupled together. In order to improve the accuracy of the recovered stress at a lower cost, an extended 3D implementation of the approach suggested by Ródenas et al. [18] is adopted in this study.

The main idea of Ródenas's approach is to impose the known boundary tractions before solving the recovery equations. From Eqn. 31d, σ_i^* , the i th component of σ^* can be expressed as

$$\sigma_i^* = a_{i1} + a_{i2}x^l + a_{i3}y^l + a_{i4}z^l + a_{i5}(x^l)^2 + \dots \quad (36a)$$

Since the patch assembly point is by default the origin of the NCS, σ_i^* is given by

$$\sigma_i^* = a_{i1} \quad (36b)$$

If the surface traction corresponding to σ_i^* is prescribed and is equal to b_i , one can determine a_{il} before solving the recovery equations so that

$$a_{il} = b_i \quad (36c)$$

In this study, this simple technique is applied to all patch assembly points lying on the top and bottom surfaces with known boundary tractions.

Construction of global smoothed stress field

After the recovery equations are solved for all the patch assembly points and the stress are transformed to the GCS, the global recovered stress field $\tilde{\sigma}$ is constructed by using the *conjoint polynomial technique* [16]. In this technique, $\tilde{\sigma}|_{\Omega_i}$, the recovered stress field within an element Ω_i is constructed by joining the results from all the patch assembly nodes of Ω_i

$$\tilde{\sigma}|_{\Omega_i} = \sum_{j=1}^{NPA} N_j^L(Pa_j) \quad (37)$$

In Eqn. 37, NPA is the number of patch assembly points in Ω_i , a_j are the coefficients corresponding to the j th patch assembly nodes. N_j^L , $j=1, \dots, NPA$ are the linear shape functions corresponding to Ω_i .

5.2.4 Definitions of Ω_{jun} and Ω_{shell} and patch classification near junctions

As different stress recovery and adaptive refinement procedures are applied to patches locating inside Ω_{jun} and Ω_{shell} , elements in the mesh are classified as either *junction part elements* or *shell part elements* based on the normal sides (Section 3.3) connected to them. In the mesh generation scheme used [10], some normal sides of the shell surface near junctions are modified before the formation of the final mesh. An element is classified as a junction part element if one of its corner nodes is connected to a modified normal side. Otherwise, the element is classified as a shell part element. Ω_{jun} and Ω_{shell} are then formed by collecting all the junction and the shell part elements respectively (Fig. 6). If a corner node of an element (e.g. nodes A and B in Fig. 6) is connected to at least one junction element, then a *junction patch* is formed and the SPR3D scheme is employed. If a corner node of an element is connected to shell part elements only (e.g. node C in Fig. 7), a shell patch is formed and the SPRN scheme is employed.

6. The adaptive refinement strategy for TWSs using 3D solid elements

6.1 Target relative errors

In almost all adaptive refinement schemes developed [1,2,8,9,12,22,23,29], their main objectives are to seek a FE solution such that the estimated relative error $\bar{\eta}_{\Omega}$ is less than a user prescribed target $^{tar}\eta_{\Omega}$. That is, the adaptive refinement procedure will be terminated if the following condition is satisfied

$$^{tar}\eta_{\Omega} \geq \bar{\eta}_{\Omega} \quad (38a)$$

However, since in this study the domain is partitioned into Ω_{jun} and Ω_{shell} and in Ω_{shell} stresses are further separated into surface and normal components, it is possible to specify the target relative errors for different domains and stress components. Therefore, besides Eqn. 38a, the following two conditions could also be prescribed by the user.

- (i) Prescribe the target relative errors for Ω , Ω_{jun} and Ω_{shell} . In this case, refinement will be continued until

$$^{tar}\eta_{\Omega} \geq \bar{\eta}_{\Omega} \quad \text{and} \quad ^{tar}\eta_{\Omega_{jun}} \geq \bar{\eta}_{\Omega_{jun}} \quad \text{and} \quad ^{tar}\eta_{\Omega_{shell}} \geq \bar{\eta}_{\Omega_{shell}} \quad (38b)$$

- (ii) Prescribe the target relative errors for all parts and components. In this case, the following additional conditions are imposed as the termination conditions:

$$^{tar}_{surf}\eta_{\Omega_{shell}} \geq ^{surf}\bar{\eta}_{\Omega_{shell}} \quad \text{and} \quad ^{tar}_{norm}\eta_{\Omega_{shell}} \geq ^{norm}\bar{\eta}_{\Omega_{shell}} \quad (38c)$$

If all the above five target relative errors are prescribed by the user, the refinement algorithm will determine the new element size based on the prescribed target relative errors and the current estimated relative errors. However, if either Eqn. 38a or Eqn. 38b is prescribed by the user, it is necessary to compute those unspecified target relative errors.

If Eqn. 38b is used, it is necessary to compute the terms $^{tar}_{surf}\eta_{\Omega_{shell}}$ and $^{tar}_{norm}\eta_{\Omega_{shell}}$. Towards this end, it is found that for a reasonably refinement mesh, good refinement results could often be obtained by equating the ratios $^{tar}_{shell}\eta_{\Omega_{shell}} / ^{tar}\eta_{\Omega_{shell}}$ and $^{tar}_{norm}\eta_{\Omega_{shell}} / ^{tar}\eta_{\Omega_{shell}}$ with their estimated values. That is

$$\frac{^{tar}_{surf}\eta_{\Omega_{shell}}}{^{tar}\eta_{\Omega_{shell}}} \approx \frac{^{surf}\bar{\eta}_{\Omega_{shell}}}{\bar{\eta}_{\Omega_{shell}}} \quad \text{and} \quad \frac{^{tar}_{norm}\eta_{\Omega_{shell}}}{^{tar}\eta_{\Omega_{shell}}} \approx \frac{^{norm}\bar{\eta}_{\Omega_{shell}}}{\bar{\eta}_{\Omega_{shell}}} \quad (39)$$

As the ratios α and β defined as

$$\alpha = \frac{\overline{\|e_u\|_{\Omega_{shell}}^2}}{\overline{\|e_u\|_{\Omega}^2}} \quad \text{and} \quad \beta = \frac{\overline{\|u\|_{\Omega_{shell}}^2}}{\overline{\|u\|_{\Omega}^2}} \quad (40a)$$

could be computed from the results of error estimation, by using Eqn. 25a and 27, one has

$$\begin{aligned} \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} &= \sqrt{\alpha} \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} \quad \text{and} \quad \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} = \sqrt{\beta} \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} \\ \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} &= \sqrt{1-\alpha} \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} \quad \text{and} \quad \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} = \sqrt{1-\beta} \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} \end{aligned} \quad (40b)$$

Hence, the values of $\frac{tar}{shell} \eta_{\Omega_{shell}}$ and $\frac{tar}{norm} \eta_{\Omega_{shell}}$ can be computed as

$$\frac{tar}{surf} \eta_{\Omega_{shell}} \approx \frac{\overline{\eta_{\Omega_{shell}}}}{\overline{\eta_{\Omega_{shell}}}} \cdot \frac{tar}{surf} \eta_{\Omega_{shell}} = \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega_{shell}}}} \cdot \frac{tar}{surf} \eta_{\Omega_{shell}} = \frac{\sqrt{\alpha} \overline{\|e_u\|_{\Omega_{shell}}}}{\sqrt{\beta} \overline{\|u\|_{\Omega_{shell}}}} \cdot \frac{tar}{surf} \eta_{\Omega_{shell}} = \sqrt{\frac{\alpha}{\beta}} \frac{tar}{surf} \eta_{\Omega_{shell}} \quad (41a)$$

$$\frac{tar}{norm} \eta_{\Omega_{shell}} \approx \frac{\overline{\eta_{\Omega_{shell}}}}{\overline{\eta_{\Omega_{shell}}}} \cdot \frac{tar}{norm} \eta_{\Omega_{shell}} = \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega_{shell}}}} \cdot \frac{tar}{norm} \eta_{\Omega_{shell}} = \frac{\sqrt{1-\alpha} \overline{\|e_u\|_{\Omega_{shell}}}}{\sqrt{1-\beta} \overline{\|u\|_{\Omega_{shell}}}} \cdot \frac{tar}{norm} \eta_{\Omega_{shell}} = \sqrt{\frac{1-\alpha}{1-\beta}} \frac{tar}{norm} \eta_{\Omega_{shell}} \quad (41b)$$

Note that due to the thin wall property of Ω_{shell} , $\frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} \approx \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega_{shell}}}}$ so that $\beta \approx 1$ which implies

$\sqrt{1-\beta} \approx 0$. Since $\sqrt{\alpha} < 1.0$, in almost all practical cases it is found that $\sqrt{\frac{\alpha}{\beta}} < 1$ and

$\sqrt{\frac{1-\alpha}{1-\beta}} > 1$. Hence, it can be concluded that $\frac{tar}{surf} \eta_{\Omega_{shell}} \leq \frac{tar}{surf} \eta_{\Omega_{shell}}$ and $\frac{tar}{norm} \eta_{\Omega_{shell}} \geq \frac{tar}{norm} \eta_{\Omega_{shell}}$. Such

results are reasonable since users are often interested to obtain a solution with high accuracy for the surface stress component in Ω_{shell} .

If only Eqn. 38a is specified by the user, it is then necessary to compute $\frac{tar}{jun} \eta_{\Omega_{jun}}$ and $\frac{tar}{shell} \eta_{\Omega_{shell}}$

first. By using a similar assumption as in Eqn. 39, one has

$$\frac{\frac{tar}{jun} \eta_{\Omega_{jun}}}{\frac{tar}{\Omega} \eta_{\Omega}} \approx \frac{\overline{\eta_{\Omega_{jun}}}}{\overline{\eta_{\Omega}}} \quad \text{and} \quad \frac{\frac{tar}{shell} \eta_{\Omega_{shell}}}{\frac{tar}{\Omega} \eta_{\Omega}} \approx \frac{\overline{\eta_{\Omega_{shell}}}}{\overline{\eta_{\Omega}}} \quad (42)$$

Hence, two ratios χ and ψ could be defined such that

$$\chi = \frac{\overline{\|e_u\|_{\Omega_{shell}}^2}}{\overline{\|e_u\|_{\Omega}^2}} \quad \text{and} \quad \psi = \frac{\overline{\|u\|_{\Omega_{shell}}^2}}{\overline{\|u\|_{\Omega}^2}} \quad (43a)$$

$$\begin{aligned} \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} &= \sqrt{\chi} \frac{\overline{\|e_u\|_{\Omega_{shell}}}}{\overline{\|e_u\|_{\Omega}}} \quad \text{and} \quad \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} = \sqrt{\psi} \frac{\overline{\|u\|_{\Omega_{shell}}}}{\overline{\|u\|_{\Omega}}} \\ \frac{\overline{\|e_u\|_{\Omega_{jun}}}}{\overline{\|e_u\|_{\Omega_{jun}}}} &= \sqrt{1-\chi} \frac{\overline{\|e_u\|_{\Omega_{jun}}}}{\overline{\|e_u\|_{\Omega_{jun}}}} \quad \text{and} \quad \frac{\overline{\|u\|_{\Omega_{jun}}}}{\overline{\|u\|_{\Omega_{jun}}}} = \sqrt{1-\psi} \frac{\overline{\|u\|_{\Omega_{jun}}}}{\overline{\|u\|_{\Omega_{jun}}}} \end{aligned} \quad (43b)$$

Eventually, it can be deduced that (c.f. Eqn. 41b)

$${}^{tar}\eta_{\Omega_{shell}} = \sqrt{\frac{\chi}{\psi}} {}^{tar}\eta_{\Omega} \quad \text{and} \quad {}^{tar}\eta_{\Omega_{jun}} = \sqrt{\frac{I-\chi}{I-\psi}} {}^{tar}\eta_{\Omega} \quad (44)$$

After ${}^{tar}\eta_{\Omega_{shell}}$ and ${}^{tar}\eta_{\Omega_{jun}}$ are determined, Eqn. 41 could then be employed to obtain the values of ${}^{tar}_{surf}\eta_{\Omega_{shell}}$ and ${}^{tar}_{notm}\eta_{\Omega_{shell}}$. It should be mentioned that during adaptive refinement, the target relative errors computed from Eqns. 41 and 44 are only used as guidance for the estimation of new elements size but *not* as the termination criteria for the whole adaptive procedure.

6.2 Immediate target relative errors

In an adaptive refinement analysis, a coarse mesh with relative error much larger than the target value (e.g. ${}^{tar}\eta_{\Omega}=5\%$ while $\eta_{\Omega}>50\%$) is frequently employed as an initial mesh. In this case, it is well known that [23] the error estimator **may not be able to estimate** the actual error accurately. If the refinement is carried out based on the target value, the efficiency of the adaptive refinement will be affected. In order to control the refinement speed so that a good balance could be achieved between the effectiveness of individual refinement step and the total numbers of refinement needed [23], an *immediate target relative error* is employed. In this study, during the i th refinement step ($i=1$ for the initial mesh), the immediate target relative error for Ω **is denoted** as ${}^{itar_i}\eta_{\Omega}$. This immediate target shall be adjusted in each refinement step using the algorithm listed in Box 3. Table IV shows a typical example for how the immediate target is adjusted. Note that the algorithm shown in Box 3 is also applied to determine the immediate targets for different subdomains and stress components.

- (i) Compute $\bar{\eta}_{\Omega}$ for the current (i th) mesh
- (ii) If $\bar{\eta}_{\Omega}/2 > {}^{tar}\eta_{\Omega}$, set ${}^{itar_i}\eta_{\Omega} = \bar{\eta}_{\Omega} / 2$, otherwise set ${}^{itar_i}\eta_{\Omega} = {}^{tar}\eta_{\Omega}$
- (iii) ${}^{itar_i}\eta_{\Omega}$ will then be used for the computation of new element sizes for the next mesh

Box 3: Algorithm for adjusting ${}^{itar_i}\eta_{\Omega}$

6.3 Selection of refinement direction

As mention in Section 4.1, a coarse mesh with large relative surface error, it is most effective to refinement the mesh in the surface direction only. However, if the surface element size is sufficiently small, the mesh should be refined in *both* the surface and normal directions. Based on the above argument and after some numerical experiments, refinement in the normal direction for Ω_{shell} will not be considered if the following condition is satisfied

$$\frac{\frac{surf}{tar} \bar{\eta}_{\Omega_{shell}}}{surf \eta_{\Omega_{shell}}} > 5 \quad (45)$$

6.4 New element size computation

In the proposed refinement procedure, in each refinement step, the immediate target errors and the refinement directions are first determined, new element sizes are computed. For Ω_{shell} , new element sizes in both the surface and normal direction are determined. For Ω_{jun} , since elements have similar sizes in all directions, only one new element size parameter is required. By using the mesh generator described in reference [10], a new mesh is constructed by first generating a surface mesh along the TWS mid-surface (Fig. 7a). Volume elements are then extruded from this mid-surface mesh along the normal direction (Fig. 7b). Finally, the volume elements are divided along the thickness direction to form solid FE elements (Fig. 7c). This implies that a mid-surface element could be corresponding to several solid elements in the solid mesh.

6.4.1 New element size calculation for Ω_{jun}

For a 3D solid element Ω_j in Ω_{jun} , only one element size parameter $h_{\Omega_j,old}$ is needed to describe the existing (old) element size, In this study, $h_{\Omega_j,old}$ is defined as

$$h_{\Omega_j,old} = \sqrt{surf A_{\hat{\Omega}}} \quad (46)$$

where $surf A_{\hat{\Omega}}$ is the surface area of the mid-surface element $\hat{\Omega}$ (Fig. 7a) corresponding to Ω_j [10]. Eqn. 46 implies that all elements corresponding to the same mid-surface element share a same value of $h_{\Omega_j,old}$.

If $h_{\Omega_{jun}}$ denotes the typical element size for elements in Ω_{jun} , then the relationship between $h_{\Omega_{jun}}$ and the numbers of element in Ω_{jun} , $NE_{\Omega_{jun}}$, can be written as

$$NE_{\Omega_{jun}} \propto (h_{\Omega_{jun}})^{-3} \quad \text{or} \quad h_{\Omega_{jun}} \propto (NE_{\Omega_{jun}})^{-1/3} \quad (47a)$$

According to the *a priori* convergence rate given Section 4.1, if the elements in the domain are properly refined so that the effect of any singularity in the problem domain is eliminated, the relationship between $h_{\Omega_{jun}}$, $NE_{\Omega_{jun}}$ and $\|\mathbf{e}_u\|_{\Omega_{jun}}$ is given by

$$\|\overline{\mathbf{e}_u}\|_{\Omega_{jun}} \approx \|\mathbf{e}_u\|_{\Omega_{jun}} \approx C(h_{\Omega_{jun}})^p = C'(NE_{\Omega_{jun}})^{-p/3} \quad (47b)$$

where $p=2$ is the order of the elements used. Eqn. 47b implies that

$$\bar{\eta}_{\Omega_{jun}} \approx \eta_{\Omega_{jun}} = C''(NE_{\Omega_{jun}})^{-p/3} \quad (47c)$$

If $^{itar_i} NE_{\Omega_{jun}}$ elements are needed to achieve an immediate target relative error of $^{itar_i} \eta_{\Omega_{jun}}$, then from Eqn. 47c, one could write

$$^{itar_i} \eta_{\Omega_{jun}} = C''(^{itar_i} NE_{\Omega_{jun}})^{-p/3} \quad (47d)$$

$$\text{such that } \frac{\bar{\eta}_{\Omega_{jun}}}{^{itar_i} \eta_{\Omega_{jun}}} = \left(\frac{NE_{\Omega_{jun}}}{^{itar_i} NE_{\Omega_{jun}}} \right)^{-p/3} \quad \text{or} \quad ^{itar_i} NE_{\Omega_{jun}} = NE_{\Omega_{jun}} \left(\frac{\bar{\eta}_{\Omega_{jun}}}{^{itar_i} \eta_{\Omega_{jun}}} \right)^{3/p} \quad (47e)$$

If the mesh is optimally refined so that the total error norm is equally distributed among all $^{itar_i} NE_{\Omega_{jun}}$ elements, one could define an *allowable error norm per element* $^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}}$ for Ω_{jun} . Since under the ideal situation, the sum of the square of the error norms of all elements in Ω_{jun} **should be equal** to the square of the allowable error norm, one could deduce that

$$^{itar_i} NE_{\Omega_{jun}} \left(^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}} \right)^2 = \left(^{itar_i} \eta_{\Omega_{jun}} \right)^2 \|\mathbf{u}\|_{\Omega_{jun}}^2 \approx \left(^{itar_i} \eta_{\Omega_{jun}} \right)^2 \|\overline{\mathbf{u}}\|_{\Omega_{jun}}^2 \quad (48a)$$

$$\text{or} \quad ^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}} = \frac{\left(^{itar_i} \eta_{\Omega_{jun}} \right) \|\overline{\mathbf{u}}\|_{\Omega_{jun}}}{\sqrt{^{itar_i} NE_{\Omega_{jun}}}} \quad (48b)$$

Using Eqns. 47e, the value of $^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}}$ can now be computed as

$$^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}} = \frac{\left(^{itar_i} \eta_{\Omega_{jun}} \right) \|\overline{\mathbf{u}}\|_{\Omega_{jun}}}{\sqrt{NE_{\Omega_{jun}} \left(\frac{\bar{\eta}_{\Omega_{jun}}}{^{itar_i} \eta_{\Omega_{jun}}} \right)^{3/p}}} \quad (48c)$$

Note that all the terms in the RHS of Eqn. 48c are known. Hence, if the estimated error norm for the j th element Ω_j in Ω_{jun} is denoted as $\|\overline{\mathbf{e}_u}\|_{\Omega_j}$, then the *refinement indicator* ς_{Ω_j} for Ω_j could be defined as

$$\varsigma_{\Omega_j} = \frac{\|\overline{\mathbf{e}_u}\|_{\Omega_j}}{^{allow} \|\mathbf{e}_u\|_{\Omega_{jun}}} \quad (49a)$$

A value of $\varsigma_{\Omega_j} > 1$ shall indicate that refinement is needed for the element.

Now consider the error norm for the j th element Ω_j , $\|\mathbf{e}_u\|_{\Omega_j}$, defined as

$$\|\mathbf{e}_u\|_{\Omega_j} = \sqrt{\int_{\Omega_j} (\mathbf{e}_\sigma)^T \mathbf{D}^{-1} \mathbf{e}_\sigma d\Omega} \quad (49b)$$

Since the convergence rate of $\mathbf{e}_\sigma$ is proportional to $h_{\Omega_j}^p$ and the volume of the element is proportional to $h_{\Omega_j}^3$, the convergence rate of $\|\mathbf{e}_u\|_{\Omega_j}$ is given by

$$\|\mathbf{e}_u\|_{\Omega_j} \approx C^* \sqrt{h_{\Omega_j}^p h_{\Omega_j}^p h_{\Omega_j}^3} \approx C^* h_{\Omega_j}^{p+1.5} \quad (49c)$$

Since $h_{\Omega_j,old}$ is the element size corresponding to $\|\mathbf{e}_u\|_{\Omega_j}$ and it is hoped that the new element size $h_{\Omega_j,new}$ for Ω_j could reduce the error norm to $allow\|\mathbf{e}_u\|_{\Omega_{jun}}$, one could write

$$\|\mathbf{e}_u\|_{\Omega_j} \approx C^* h_{\Omega_j,old}^{p+1.5} \quad and \quad allow\|\mathbf{e}_u\|_{\Omega_{jun}} \approx C^* h_{\Omega_j,new}^{p+1.5} \quad or \quad \frac{\|\mathbf{e}_u\|_{\Omega_j}}{allow\|\mathbf{e}_u\|_{\Omega_{jun}}} = \zeta_{\Omega_j} \approx \left(\frac{h_{\Omega_j,old}}{h_{\Omega_j,new}} \right)^{p+1.5} \quad (49d)$$

Hence, $h_{\Omega_j,new}$, can now be computed as

$$h_{\Omega_j,new} = \frac{h_{\Omega_j,old}}{\left(\zeta_{\Omega_j} \right)^{\frac{1}{p+1.5}}} \quad (50)$$

6.4.2 New element size calculation for Ω_{shell}

For a given element Ω_j in Ω_{shell} , two size parameters, $surf h_{\Omega_j,old}$ and $norm h_{\Omega_j,old}$, are needed to define the existing (old) element size of Ω_j in the surface and normal directions, respectively. For $surf h_{\Omega_j,old}$, similar to the case for Ω_{jun} , its value is given by

$$surf h_{\Omega_j,old} = \sqrt{surf A_{\hat{\Omega}}} \quad (51)$$

For $norm h_{\Omega_j,old}$, since the thickness of the TWS may vary within Ω_j and the volume element (Fig. 7b) formed after extrusion may be divided into different numbers of layer of elements at different **surface element corner nodes** (Fig. 7c), the calculation of the element size in the normal direction is more complicated. Firstly, the mid-surface element $\hat{\Omega}$ which is corresponding to Ω_j is identified. Secondly, the nodal normal size parameter $norm h_{\hat{n}_k}$ for the k th surface element corner node of $\hat{\Omega}$ is computed as

$${}_{norm}h_{\bar{n}_k} = \frac{t_k}{NEL_k} \quad (52a)$$

In Eqn. 52a, t_k and NEL_k are, respectively, the thickness of the TWS and the numbers of layer of elements at the k th surface element corner node. Finally, for all elements corresponding to $\hat{\Omega}$, their common old element size ${}_{norm}h_{\Omega_j,old}$ is computed by averaging the nodal parameters from the corner nodes of $\hat{\Omega}$. That is

$${}_{norm}h_{\Omega_j,old} = \frac{1}{NCN} \sum_{k=1}^{NCN} {}_{norm}h_{\bar{n}_k} \quad (52b)$$

where NCN is the numbers of corner node of $\hat{\Omega}$.

Determination of new element size in surface direction

If ${}_{surf}h_{\Omega_{shell}}$ and ${}_{norm}h_{\Omega_{shell}}$ are, respectively, the typical surface and normal element size for elements in Ω_{shell} , then the general relationship between ${}_{surf}h_{\Omega_{shell}}$, ${}_{norm}h_{\Omega_{shell}}$ and $NE_{\Omega_{shell}}$, the numbers of element in Ω_{shell} , can be written as

$$NE_{\Omega_{shell}} \propto ({}_{norm}h_{\Omega_{shell}})^{-1} ({}_{surf}h_{\Omega_{shell}})^{-2} \quad (53a)$$

In this study, it is assumed that the refinement rates in the surface and the normal directions are equal so that Eqn. 53a can be re-written as

$$NE_{\Omega_{shell}} \propto ({}_{norm}h_{\Omega_{shell}})^{-1} ({}_{surf}h_{\Omega_{shell}})^{-2} \approx ({}_{surf}h_{\Omega_{shell}})^{-3} \quad \text{or} \quad {}_{surf}h_{\Omega_{shell}} \propto (NE_{\Omega_{shell}})^{-1/3} \quad (53b)$$

According to the *a priori* convergence rate discussed in Section 4.1, ${}_{surf}\|\mathbf{e}_u\|_{\Omega_{shell}}$ is effectively reduced by refinement along the surface direction. Hence, the relationship between ${}_{surf}h_{\Omega_{shell}}$, $NE_{\Omega_{shell}}$ and ${}_{surf}\|\overline{\mathbf{e}_u}\|_{\Omega_{shell}}$ can be established as (c.f. Eqns. 47b to 47e)

$${}_{surf}\|\overline{\mathbf{e}_u}\|_{\Omega_{shell}} \approx {}_{surf}\|\mathbf{e}_u\|_{\Omega_{shell}} = C'(NE_{\Omega_{shell}})^{-p/3} \quad \text{or} \quad {}_{surf}\bar{\eta}_{\Omega_{shell}} \approx {}_{surf}\eta_{\Omega_{shell}} = C''(NE_{\Omega_{shell}})^{-p/3} \quad (53c)$$

$$\frac{{}_{surf}\bar{\eta}_{\Omega_{shell}}}{{}_{itar_i}{}_{surf}\eta_{\Omega_{shell}}} = \left(\frac{NE_{\Omega_{shell}}}{{}_{itar_i}NE_{\Omega_{shell}}} \right)^{-p/3} \quad \text{or} \quad \frac{{}_{itar_i}NE_{\Omega_{shell}}}{{}_{surf}NE_{\Omega_{shell}}} = \left(\frac{{}_{surf}\bar{\eta}_{\Omega_{shell}}}{{}_{itar_i}\eta_{\Omega_{shell}}} \right)^{3/p} \quad (53d)$$

Using the same argument as in Eqn. 48, once could eventually express the *allowable surface error norm per element* for Ω_{shell} ${}_{surf}^{allow}\|\mathbf{e}_u\|_{\Omega_{shell}}$ in terms of the known terms $NE_{\Omega_{shell}}$,

${}_{surf}\bar{\eta}_{\Omega_{shell}}$, ${}_{itar_i}\eta_{\Omega_{shell}}$ and ${}_{surf}\|\overline{\mathbf{u}}\|_{\Omega_{shell}}$ (c.f. Eqn. 48c) such that

$$\underset{surf}{allow} \|\mathbf{e}_u\|_{\Omega_{shell}} = \frac{\left(\frac{itar_i}{surf} \eta_{\Omega_{shell}} \right) \overline{\|\mathbf{u}\|_{\Omega_{shell}}}}{\sqrt[3/p]{NE_{\Omega_{shell}} \left(\frac{surf \bar{\eta}_{\Omega_{shell}}}{itar_i surf \eta_{\Omega_{jun}}} \right)}} \quad (53e)$$

Hence, the *surface refinement indicator* $\underset{surf}{\varsigma}_{\Omega_j}$ for an element Ω_j in Ω_{shell} can be defined as (c.f. Eqn. 49a)

$$\underset{surf}{\varsigma}_{\Omega_j} = \frac{\underset{surf}{\|\mathbf{e}_u\|_{\Omega_j}}}{\underset{surf}{allow} \|\mathbf{e}_u\|_{\Omega_{shell}}} \quad (54a)$$

As the convergence rate of $\underset{surf}{\mathbf{e}_\sigma}$ is proportional to $\underset{surf}{h}_{\Omega_j}^p$ and the volume of the element is proportional to $\underset{norm}{h}_{\Omega_j} \cdot \underset{surf}{h}_{\Omega_j}^2$, the convergence rate for $\underset{surf}{\|\mathbf{e}_u\|_{\Omega_j}}$ is given by (c.f. Eqn. 49c)

$$\begin{aligned} \underset{surf}{\|\mathbf{e}_u\|_{\Omega_j}} &= \sqrt{\int_{\Omega_j} (\underset{surf}{\mathbf{e}_\sigma})^T \mathbf{D}^{-1} \underset{surf}{\mathbf{e}_\sigma} d\Omega} \\ &\approx C^* \sqrt{\underset{surf}{h}_{\Omega_j}^p \underset{surf}{h}_{\Omega_j}^p \underset{norm}{h}_{\Omega_j} \underset{surf}{h}_{\Omega_j}^2} \\ &= C^* \underset{surf}{h}_{\Omega_j}^{p+1.5} \end{aligned} \quad (54b)$$

Hence, $\underset{surf}{h}_{\Omega_j, new}$ the new surface element size for Ω_j can be computed as

$$\underset{surf}{h}_{\Omega_j, new} = \frac{\underset{surf}{h}_{\Omega_j, old}}{\left(\underset{surf}{\varsigma}_{\Omega_j} \right)^{\frac{1}{p+1.5}}} \quad (55)$$

Determination of new element size in the thickness direction

In order to compute the new element size in the thickness direction, it is again assumed that the refinement rates in the surface and the normal directions are equal so that (c.f. Eqn. 53b)

$$NE_{\Omega_{shell}} \propto (\underset{norm}{h}_{\Omega_{shell}})^{-1} (\underset{surf}{h}_{\Omega_{shell}})^{-2} \approx (\underset{norm}{h}_{\Omega_{shell}})^{-3} \quad \text{or} \quad \underset{norm}{h}_{\Omega_{shell}} \propto (NE_{\Omega_{shell}})^{-1/3} \quad (56a)$$

It is assumed that $\underset{norm}{\|\mathbf{e}_u\|_{\Omega_{shell}}}$ could be effectively reduced by refinement in the thickness direction. Hence, one can write (c.f. Eqns. 53c and 53d)

$$\underset{norm}{\|\mathbf{e}_u\|_{\Omega_{shell}}} \approx \underset{norm}{\|\mathbf{e}_u\|_{\Omega_{shell}}} = C'(NE_{\Omega_{shell}})^{-p/3} \quad \text{or} \quad \underset{norm}{\bar{\eta}_{\Omega_{shell}}} \approx \underset{norm}{\eta_{\Omega_{shell}}} = C''(NE_{\Omega_{shell}})^{-p/3} \quad (56b)$$

$$\frac{\underset{norm}{\bar{\eta}_{\Omega_{shell}}}}{\frac{itar_i}{\underset{norm}{\eta_{\Omega_{shell}}}}} = \left(\frac{NE_{\Omega_{shell}}}{\frac{itar_i}{\underset{norm}{NE_{\Omega_{shell}}}}} \right)^{-p/3} \quad \text{or} \quad \frac{itar_i}{\underset{norm}{NE_{\Omega_{shell}}}} = NE_{\Omega_{shell}} \left(\frac{\underset{norm}{\bar{\eta}_{\Omega_{shell}}}}{\underset{norm}{\eta_{\Omega_{shell}}}} \right)^{3/p} \quad (56c)$$

If $\|e_u\|_{\Omega_{shell}}^{allow_{norm}}$ is the *allowable normal error norm per element* for Ω_{shell} , by reusing the same argument in Eqn. 48, one has (c.f. Eqn. 53e)

$$\|e_u\|_{\Omega_{shell}}^{allow_{norm}} = \frac{\left(\frac{itar_i}{norm} \eta_{\Omega_{shell}}\right)_{norm} \overline{\|u\|_{\Omega_{shell}}}}{\sqrt[3/p]{NE_{\Omega_{shell}} \left(\frac{norm \bar{\eta}_{\Omega_{shell}}}{\frac{itar_i}{norm} \eta_{\Omega_{jun}}}\right)}} \quad (56d)$$

Hence, the *normal refinement indicator* $_{norm} \varsigma_{\Omega_j}$ for Ω_j can be defined as (c.f. Eqn. 54a)

$$_{norm} \varsigma_{\Omega_j} = \frac{_{norm} \overline{\|e_u\|_{\Omega_j}}}{_{norm} \|e_u\|_{\Omega_{shell}}} \quad (57a)$$

Similar to Eqn. 54b, the convergence rate of $_{norm} \|e_u\|_{\Omega_j}$ is given by

$$_{norm} \|e_u\|_{\Omega_j} \approx C^*_{norm} h_{\Omega_j}^{p+1.5} \quad (57b)$$

Finally, $_{norm} h_{\Omega_j, new}$, the new normal element size for Ω_j can now be computed as

$$_{norm} h_{\Omega_j, new} = \frac{_{norm} h_{\Omega_j, old}}{\left(_{norm} \varsigma_{\Omega_j}\right)^{\frac{1}{p+1.5}}} \quad (58)$$

Limits for refinement ratio

Even though immediate target relative errors (Section 6.2) are employed to slow down the initial refinement speed, it is still possible that at some locations (e.g. near a sharp junction), the ratio between the old and new element sizes is still too high. Hence, in the actual implementation, the new element size is limited in such a way that

$$h_{\Omega_j, new} = \max(h_{\Omega_j, new}, \frac{h_{\Omega_j, old}}{6}) \quad (59)$$

$$_{surf} h_{\Omega_j, new} = \max(_{surf} h_{\Omega_j, new}, \frac{_{surf} h_{\Omega_j, old}}{6}), \quad _{norm} h_{\Omega_j, new} = \max(_{norm} h_{\Omega_j, new}, \frac{_{norm} h_{\Omega_j, old}}{2})$$

6.4.3 Nodal based size parameters and aspect ratio adjustment

Nodal based size parameters

As the mesh generator used in this study [10] always discretizes the mid-surface of a TWS into a surface mesh first and then generates the solid mesh by extrusion and divisions, it is necessary to define the new element sizes at all nodal points of the existing mid-surface mesh.

As a mid-surface element $\hat{\Omega}$ may be corresponding to several solid elements, the *solid element based* sizes parameters ($h_{\Omega_j, new}$, $surf h_{\Omega_j, new}$ and $norm h_{\Omega_j, new}$) computed from all those solid elements corresponding to $\hat{\Omega}$ are averaged to form the *mid-surface element based* size parameters for $\hat{\Omega}$. That is

$$\text{For } \hat{\Omega} \text{ in } \Omega_{jun}: h_{\hat{\Omega}, new} = \frac{1}{NSE_{\hat{\Omega}}} \sum_{j=1}^{NSE_{\hat{\Omega}}} h_{\Omega_j, new} \quad (60a)$$

$$\text{For } \hat{\Omega} \text{ in } \Omega_{shell}: surf h_{\hat{\Omega}, new} = \frac{1}{NSE_{\hat{\Omega}}} \sum_{j=1}^{NSE_{\hat{\Omega}}} surf h_{\Omega_j, new} \text{ and } norm h_{\hat{\Omega}, new} = \frac{1}{NSE_{\hat{\Omega}}} \sum_{j=1}^{NSE_{\hat{\Omega}}} norm h_{\Omega_j, new} \quad (60b)$$

In Eqn. 60, $NSE_{\hat{\Omega}}$ is the numbers of solid element corresponding to $\hat{\Omega}$ ($NSE_{\hat{\Omega}}=4$ for the elements shown in Fig. 7c). After the size parameters for all mid-surface elements are computed, the *node based* size parameters $h_{\hat{P}, new}$, $surf h_{\hat{P}, new}$ and $norm h_{\hat{P}, new}$ for **an element corner node** $\hat{P}$ in the mid-surface mesh are then computed as

$$\text{For } \hat{P} \text{ in } \Omega_{jun}: h_{\hat{P}, new} = \frac{1}{NCE_{\hat{P}}} \sum_{i=1}^{NCE_{\hat{P}}} h_{\hat{\Omega}_i, new} \quad (61a)$$

$$\text{For } \hat{P} \text{ in } \Omega_{surf}: surf h_{\hat{P}, new} = \frac{1}{NCE_{\hat{P}}} \sum_{i=1}^{NCE_{\hat{P}}} surf h_{\hat{\Omega}_i, new} \text{ and } norm h_{\hat{P}, new} = \frac{1}{NCE_{\hat{P}}} \sum_{i=1}^{NCE_{\hat{P}}} norm h_{\hat{\Omega}_i, new} \quad (61b)$$

In Eqn. 61, $\hat{\Omega}_i$ and $NCE_{\hat{P}}$ are the i th element and the numbers of surface element connecting to node $\hat{P}$ respectively (Fig. 8).

Aspect ratio adjustments

For elements in Ω_{shell} , before $surf h_{\hat{P}, new}$ and $norm h_{\hat{P}, new}$ are employed for new mesh generation, it is necessary to check their ratio to ensure that extremely thick or thin elements will not be formed. In this study, the value of $norm h_{\hat{P}, new}$ is adjusted in such a way that

$$norm h_{\hat{P}, new} = \begin{cases} 0.05 \times surf h_{\hat{P}, new} & \text{if } \frac{norm h_{\hat{P}, new}}{surf h_{\hat{P}, new}} < 0.05 \\ 20.0 \times surf h_{\hat{P}, new} & \text{if } \frac{norm h_{\hat{P}, new}}{surf h_{\hat{P}, new}} > 20 \\ norm h_{\hat{P}, new} & \text{otherwise} \end{cases} \quad (62)$$

Finally, $NEL_{\hat{P}}$, the numbers of layer of elements at **an element corner node** $\hat{P}$ is calculated as

$$\text{For } \hat{P} \text{ in } \Omega_{jun}: NEL_{\hat{P}} = int\left(\frac{t_{\hat{P}}}{h_{\hat{P}, new}}\right) \quad (63a)$$

For $\hat{P}$ in Ω_{shell} :

$$NEL_{\hat{P}} = \text{int}\left(\frac{t_{\hat{P}}}{norm h_{\hat{P},new}}\right) \quad (63b)$$

where $\text{int}(a)$ is the integer part of the real number a .

A summary for the whole adaptive refinement procedure is shown in Box 4.

- (i) User inputs the initial mesh and prescribed the value of $^{tar}\eta_{\Omega}$ (Eqn. 38a) and/or the values of $^{tar}\eta_{\Omega_{jun}}$, $^{tar}\eta_{\Omega_{shell}}$ (Eqn. 38b), $^{tar}_{surf}\eta_{\Omega_{shell}}$ and $^{tar}_{norm}\eta_{\Omega_{shell}}$ (Eqn. 38c).
- (ii) Carry out FE analysis using the existing mesh.
- (iii) Carry out error estimation.
- (iv) Compute the values of $^{tar}\eta_{\Omega_{jun}}$, $^{tar}\eta_{\Omega_{shell}}$, $^{tar}_{surf}\eta_{\Omega_{shell}}$ and $^{tar}_{norm}\eta_{\Omega_{shell}}$ if they are not defined.
- (v) Check whether the convergence criteria (Eqns. 38a-38c) are satisfied or not.
- (vi) If all the convergence criteria are satisfied, goto step (xiv).
- (vii) Determine the values of $^{itar_i}\eta_{\Omega}$, $^{itar_i}\eta_{\Omega_{jun}}$, $^{itar_i}\eta_{\Omega_{shell}}$, $^{itar_i}_{surf}\eta_{\Omega_{shell}}$ and $^{itar_i}_{norm}\eta_{\Omega_{jun}}$ (Box 3).
- (viii) Select the refinement direction for Ω_{shell} (Eqn. 45).
- (ix) Compute the old and new element sizes for elements in Ω_{jun} and Ω_{shell} (Eqns. 46 to 58).
- (x) Adjust the new element size according to the maximum refinement ratio limits (Eqn. 59).
- (xi) Compute the node based size parameters for all nodes in the mid-surface mesh (Eqn. 61).
- (xii) Perform aspect ratio adjustment and compute the numbers of layer of elements at the normal direction (Eqns. 62 and 63).
- (xiii) Generate new mesh from the new element size information ($h_{\hat{P},new}$ and $NEL_{\hat{P}}$ for nodes in Ω_{jun} , $^{surf}h_{\hat{P},new}$ and $NEL_{\hat{P}}$ for nodes in Ω_{shell}) and goto step (ii).
- (xiv) Quit the adaptive refinement procedure.

Box 4. A summary for the whole adaptive refinement procedure

7. Numerical examples

In this section, four numerical examples are given. In Example 1, a problem with known exact solution was chosen and uniform refinement was carried out to validate the effectiveness of the SPRN scheme. Examples 2 to 4 were selected to demonstrate the effectiveness of the adaptive refinement procedure so both uniform and adaptive refinements were carried out. The target relative errors for Examples 2 to 4 are listed in Table V. All computations were carried out using a low-end PC equipped with an AMD ATHLON 64 3800+ CPU and 4Gb RAM. FE analyses were carried out using the commercial software ABAQUS[30]. As exact 3D solutions for TWS problems are not available for Examples 2 to 4, highly adaptively

refined *reference meshes* with DOFs close to the limit of the hardware used were generated. The recovered stress fields obtained from these highly refined reference meshes were employed for assessing the performance of the adaptive refinement procedure. Note that for Examples 2 to 4, due to the difficulty of the problem and the limited computational power of the low-end PC, it was found that the reference meshes for these problems still have relatively high values of $^{tar}\eta_{\Omega_{jun}}$ and $^{tar}\eta_{\Omega_{shell}}$ (15% to 20 %). Hence, in Table V, higher values of $^{tar}\eta_{\Omega_{jun}}$ and $^{tar}\eta_{\Omega_{shell}}$ (25%-50%) were adopted in these examples.

Example 1: Spherical container subjected to uniform internal pressure

In this example, three spherical shells with an average radius of 10 and thicknesses of $t=0.5, 1$ and 2 subjected to unit uniform internal pressure were used (Fig. 9a). The exact solution for this problem is known [31]. By exploiting symmetry, only one-eighth of the container was modelled (Fig. 9b). For each shell, three uniform meshes were created (Fig. 10). For all the meshes used, refinement was only carried out in the surface direction while 6 layers of element were used in the thickness direction. Both the SPR3D and the SPRN schemes were applied. In this example, the problem domain contains no junction so that $\Omega=\Omega_{shell}$. The results obtained for all the tested cases are summarized in Table VI. It can be seen that the SPRN scheme consistently outperforms the classical SPR3D scheme, even when the shell is relatively thick ($t=2$) and further performance improvements are observed as the shell thickness is reduced. Despite that fewer polynomial terms were used in the SPRN scheme, the recovered stress fields obtained from the SPRN scheme are more accurate than those from the SPR3D scheme.

Example 2: Symmetric quadrant of cylindrical shell roof

In this example, a cylindrical shell roof supported by diaphragms at the two curved ends and is free along the other two straight edges is considered (Fig. 11). By exploiting symmetry, only a quadrant of the roof was analyzed and $\Omega=\Omega_{shell}$. The 2mm thick roof is subjected to loading due to self-weight of $0.208333 \times 10^{-3} \text{ g/mm}^3$. A highly refined mesh (Fig. 12) was used as the reference mesh. In order to test the performance of the SPRN scheme when there is no internal element in the mesh, the four uniform meshes used only consist of two layers of element in the thickness direction. The uniform and adaptive meshes generated and their element effectivity indices distribution are shown in Fig. 13 and 14, respectively. The convergence histories are listed in Table VII and Figs 15 to 17. Four adaptive refinements

were needed to achieve the target accuracy. From the results, it is found that adaptive refinement outperformed uniform refinement with better element effectivity indices distribution. Note that in this problem, a strong boundary layer of shear stress exists along the free edge so that even in the final adaptive mesh, the value of ${}_{norm}\eta_{\Omega_{shell}}$ is still relatively high. However, from Table VII, the corresponding value of ${}_{norm}\theta_{\Omega_{shell}}$ is closed to unity. This can be explained by the fact that overestimation of the error norm at one part of the domain may be compensated by underestimation elsewhere and a reasonably good global effectivity index was obtained despite that the relative error of the solution is still high.

Example 3: Two intersecting plates

In this example, the problem domain is constructed by intersecting two flat plates at 90° with uniform horizontal unit pressure applied to the vertical plate (Fig. 18a). Due to symmetry, only half of the structure was analysed (Fig. 18b). In this example, a singular line EB exists along the intersection. The reference mesh is shown in Fig. 19. The uniform and adaptive meshes generated and their element effectivity indices distributions are shown in Figs. 20 and 21, respectively. Despite the presence of the singular line, four refinements are enough to reach the target accuracy. The convergence histories are shown in Table VIII and Figs. 22 to 26. From the results obtained, it can be concluded that the adaptive refinement outperformed the uniform refinement. When comparing with uniform refinement, a higher convergence rate was achieved so that a more accurate solution was obtained by adaptive refinement with only 25% of DOFs. The zoom in view of the last adaptive mesh near the junction (Fig. 21f) indicates that the aspect ratios of elements in the junction part are closed to unity while single layer of solid elements was employed in almost all other parts of the problem domain.

Example 4: A tubular joint formed by two circular hollow sections (CHS)

In the last examples, a CHS tubular T-joint is employed as shown in Fig. 27. The joint is subjected to a uniform axial loading from the top and is fixed at both ends. Exploiting symmetry, only one quarter of the joint was modelled. The reference mesh is shown in Fig. 28. This example contains a singular curve along the intersection of the joint. The uniform and adaptive meshes used are shown in Figs. 29 and 30 respectively. A zoom in view of the last adaptive mesh near the intersection is shown in Fig. 30f. The convergence histories are shown in Table IX and Figs. 31 to 35. The adaptive refinement reached the target accuracy with five analyses. Again, the adaptive refinement outperformed the uniform refinement in all

means of measurement and a more accurate solution was reached by using 40% DOFs. Finally, Table X lists out the computational times used by different processes for both uniform and adaptive refinements. It can be seen that by using adaptive refinement, considerable saving in computation time is achieved and the adaptive solution were obtained by using a low-end PC within a reasonable time (≈ 30 minutes). It should be noted that the most time consuming process is the error estimation process as high order numerical integrations are employed for the computation of the error norms.

8. Conclusions and future works

In this paper, a new adaptive refinement procedure is developed for the stress analysis of thin-walled structures (TWS) using 3D solid elements. During the adaptive refinement analysis, the whole TWS is divided onto two distinct parts: the *junction part* and the *shell part*. For the junction part, classical stress recovery scheme is used. For the shell part, the stress field is separated into two components, namely, the surface and the normal components. A special nodal coordinate system and a reduction of polynomial terms are employed to increase the accuracy of the recovered stress field. The recovered stress **is then used** in the Zienkiewicz-Zhu error estimator for *a posteriori* error estimation. Specially designed refinement procedures are suggested to define the new element sizes for the shell and the junction parts separately. It is found that by adopting the suggested recovery procedure, the error of the recovered stress is reduced significantly and the resulted error estimator also shows consistent accurate local error estimation throughout the whole problem domain. The numerical examples given demonstrated the advantages of the adaptive refinement procedure over uniform refinement. It is shown that that by using the proposed adaptive refinement scheme, accurate 3D FE solutions could be obtained by using a low-end PC within a reasonable time. With the rapid increases in the computational speed and capacity of small to medium size workstations, it is expected that the CPU time needed by **the proposed method** would be drastically reduced in the near future while the range of applications could be extended to other important areas of structural mechanics such as buckling and dynamic analyses.

References

1. C. K. Lee and R. E. Hobbs (1998), Automatic adaptive refinement for plate bending problems using Reissner-Mindlin plate bending elements, *International Journal for Numerical Methods in Engineering*, Vol. 41, No. 1, pp: 1-63.
2. C. K. Lee and R. E. Hobbs (1997), Automatic adaptive refinement for shell analysis using nine-node assumed strain element, *International Journal for Numerical Methods in Engineering*, Vol. 40, No. 19, pp: 3601-3638.
3. C. K. Lee, S.T. Lie, S. P. Chiew and Y. B. Shao, (2005), Numerical models verification of cracked tubular T, Y and K-joints under combined loads, *Engineering Fracture Mechanics*, Vol. 72, No. 7, pp 983 – 1009
4. C. K. Lee, S.T. Lie, S. P. Chiew and H. L. Ji, (2007), Fatigue behaviors of Square-to-Square Hollow Section T-joint with corner crack. II: Numerical Modeling, *Engineering Fracture Mechanics*. Vol. 74, No. 5, pp 721 – 738
5. Zhao, X.L., Herion, S., Packer, J.A., Puthli, R., Sedlacek, G., Wardenier, J., Weynand, K., van Wingerde, A. and Yeomans, N., (2000). Design Guide for Circular and Rectangular hollow Section Joints under Fatigue Loading. CIDECT, TUV Germany.
6. S. H. Lo and C. Ling (2000), Improvement on the 10-node tetrahedral element for three-dimensional problems, *Computer Methods in Applied Mechanics and Engineering*, Vol. 189, No. 3, pp: 961-974.
7. K. Y. Sze and W. K. Chan (2001), A six-node pentagonal assumed natural strain solid-shell element, *Finite Elements in Analysis and Design*, Vol. 37, No. 8, pp: 639-655.
8. C. K. Lee and S. H. Lo, (1997), Automatic Adaptive 3-D finite element refinement using different-order tetrahedral elements, *International Journal for Numerical Methods in Engineering*, Vol. 40, No. 12, pp 2195 - 2226.
9. C. K. Lee and S. H. Lo, (1997), Automatic adaptive refinement finite element procedure for 3D stress analysis, *Finite Elements in Analysis and Design*, Vol. 25, No. 1, pp 135 - 166.
10. C. K. Lee and Q. X. Xu, (2005), A new automatic adaptive 3D solid mesh generation scheme for thin-walled structures, *International Journal for Numerical Methods in Engineering*, Vol. 62, No. 11, pp: 1519 - 1558.
11. O. C. Zienkiewicz and J. Z. Zhu (1992), The superconvergent patch recovery and a posteriori error estimates. Part 1: The recovery technique”, *International Journal for Numerical Methods in Engineering*, Vol. 33, pp: 1331-1364.
12. O. C. Zienkiewicz and J. Z. Zhu (1992), The superconvergent patch recovery and a posteriori error estimates. Part 2: Error estimates and adaptivity, *International Journal for Numerical Methods in Engineering*, Vol. 33, pp: 1365-1382.

13. O. C. Zienkiewicz and J. Z. Zhu (1987), A simple error estimator and adaptive procedure for practical engineering analysis, *International Journal for Numerical Methods in Engineering*, Vol. 24, 336-357
14. G. Bedrosian (1992), Shape functions and integration formulas for three-dimensional finite element analysis, *International Journal for Numerical Methods in Engineering*, Vol. 35, No. 1, pp: 95-108.
15. N. E. Wiberg and F. Abulwahab (1993), Patch recovery based on superconvergent derivatives and equilibrium, *International Journal for Numerical Methods in Engineering*, Vol. 36, pp: 2703-2724.
16. T. Blacker and T. Belytschko (1994), Superconvergent patch recovery with equilibrium and conjoint interpolant enhancements, *International Journal for Numerical Methods in Engineering*, Vol. 37, pp: 517-536.
17. O. C. Zienkiewicz, J. Z. Zhu and J. Wu (1993), Superconvergent patch recovery techniques – some further tests, *Communications in Numerical Methods in Engineering*, Vol. 9, pp: 251-258.
18. J.J. Rodenas, M.T.F.J. Fuenmayor and A. Vercher, Improvement of the superconvergent patch recovery techniques by the use of constraint equations: The SPR-C technique, in press, *International Journal for Numerical Methods in Engineering*
19. S H Lo and C. K. Lee, (1998), On using different recovery procedures for the construction of smoothed stress in finite element method, *International Journal for Numerical Methods in Engineering*, Vol. 43, pp: 1223 - 1252.
20. P. Labbé and A. Garon, (1995), A robust implementation of Zienkiewicz and Zhu's local patch recovery method, *Communications in Numerical Methods in Engineering*, Vol. 11, pp: 427-434.
21. B. A. Szabo (1990), The use of a priori estimates in engineering computations, *Computer Methods in Applied Mechanics and Engineering*, Vol. 82, pp: 139-154.
22. C. K. Lee, K. Y. Sze and S. H. Lo, On using degenerated solid shell elements in adaptive refinement analysis, *Int. J. Numer. Methods Engrg.*, **45** 627-659 (1999)
23. Lee, C.K., S H Lo, 1999, A full 3d finite element analysis using adaptive refinement and pcg solver with back interpolation, *Computer Methods in Applied Mechanics and Engineering (United States)*, Vol. 170, No. 1, pp 39 - 64.
24. B. Boroomand and F. Mossaiby, (2005), Generalization of robustness test procedure for error estimators. Part II: test results for error estimators using SPR and REP, *International Journal for Numerical Methods in Engineering*, Vol. 64, pp: 461-502.
25. K. M. Okstad, T. Kvamsdal and K. M. Mathisen (1999), Superconvergent Patch Recovery for plate problems using statically admissible stress resultant fields, *International Journal for Numerical Methods in Engineering*, Vol. 44, pp: 697-727.

26. A. A. Yazdani, H. R. Riggs and A. Tessler (2000), Stress recovery and error estimation for shell structures, *International Journal for Numerical Methods in Engineering*, Vol. 47, pp: 1825-1840.
27. B. Boroomand, M. Ghaffarian and O. C. Zienkiewicz (2004), On application of two superconvergent recovery procedures to plate problems, *International Journal for Numerical Methods in Engineering*, Vol. 61, pp: 1644-1673
28. Wenbin Yu, Dewey H. Hodges and Vitali V. Volovoi (2003), Asymptotically accurate 3-D recovery from Reissner-like composite plate finite elements, *Computers & Structures*, Vol. 81, pp: 439-454
29. E. Onate and G. Buggedá, A study of mesh optimality criteria in adaptive finite element analysis, *Engrg. Computations.*, **10**, 307-321 (1993)
30. ABAQUS *Theory Manual and User Manual*, version 6.4, (2004). Hibbit, Karlsson & Sorensen Inc., USA.
31. S. P. Timoshenko and J. N. Goodier (1970), *Theory of elasticity*, Chap. 4, McGraw-Hill, New York, 3rd edition.

Figure 1. Shell part and junction part of a TWS.

(a) 10-node tetrahedron

(b) 13-node pyramid

(c) 15-node prism

(d) 20-node hexahedron

Figure 2. Quadratic 3D elements used.

(a) Global Coordinate System (GCS)

(b) Patch coordinate system (PCS)

(c) Nodal Coordinate System (NCS)

Figure 3. Coordinate systems used.

Figure 4. Definition of nodal normal vector for an interior node.

Figure 5. An example of boundary patch with no interior node.

Figure 6. Definitions of the junction and the shell parts.

Figure 7. 3D solid mesh generation by surface mesh generation, extrusion and division.

Figure 8. Computation of nodal element size parameters for mid-surface mesh.

Figure 9: Example 1, Spherical container subjected to uniform internal pressure
 (a) Problem domain and loading, (b) boundary conditions.

(a) Mesh 1 (NTD=4647) (b) Mesh 2 (NTD=17031) (c) Mesh 3 (NTD=65127)

Figure 10. Uniform meshes used in Example 1.

Figure 11. Example 2, cylindrical shell roof under self-weight.
 (a) Problem domain and loading, (b) boundary conditions.

Figure 12. Reference mesh for Example 2.
 $(NTD=847303, \eta_{\Omega}=0.54\%, \text{surf } \eta_{\Omega_{shell}}=0.28\%, \text{norm } \eta_{\Omega_{shell}}=14.99\%)$

(a) Mesh U1 (initial mesh) and element effectivity index distribution

(b) Mesh U2 and element effectivity index distribution

(c) Mesh U3 and element effectivity index distribution

(d) Mesh U4 and element effectivity index distribution
Figure 13. Uniform refinement mesh for Example 2.

Scale for the element effectivity index distribution for Figs. 13, 14; 20, 21 and 29, 30.

$ \theta-1 \leq 0.05$	$0.05 < \theta-1 \leq 0.25$	$0.25 < \theta-1 \leq 0.50$	$0.50 < \theta-1 \leq 0.75$	$ \theta-1 > 0.75$

(a) Mesh A1 (initial mesh) and element effectivity index distribution

(b) Mesh A2 and element effectivity index distribution

(c) Mesh A3 and element effectivity index distribution

(d) Mesh A4 and element effectivity index distribution

Figure 19. Reference mesh for Example 3.

$NTD=892248$, $\eta_{\Omega}=1.86\%$, $\eta_{\Omega_{jun}}=10.81\%$, $\eta_{\Omega_{shell}}=1.24\%$, $_{surf}\eta_{\Omega_{shell}}=0.61\%$, $_{norm}\eta_{\Omega_{shell}}=22.80\%$

(a) Mesh U1 (initial mesh) and element effectivity index distribution

(b) Mesh U2 and element effectivity index distribution

(c) Mesh U3 and element effectivity index distribution

(d) Mesh U4 and element effectivity index distribution

Figure 20. Uniform refinement meshes for Example 3.

(a) Mesh A1 (initial mesh) and element effectivity index distribution

(b) Mesh A2 and element effectivity index distribution

(c) Mesh A3 and element effectivity index distribution

(d) Mesh A4 and element effectivity index distribution

(e) Mesh A5 and element effectivity index distribution

(f) Zoom in view near the junction for Mesh A5

Figure 21. Adaptive refinement meshes for Example 3.

Figure 22. Result of Example 3, convergence of overall relative error, η_{Ω} .

Figure 23. Result of Example 3, convergence of relative error – Junction Part, $\eta_{\Omega_{jun}}$.

Figure 24. Result of Example 3, convergence of relative error – Shell Part, $\eta_{\Omega_{shell}}$.

Figure 25. Result of Example 3, convergence of surface relative error – Shell Part, $surf \eta_{\Omega_{shell}}$.

Figure 26. Result of Example 2, convergence of normal relative error – Shell Part, $norm \eta_{\Omega_{shell}}$.

Figure 27. Example 4, A tubular circular hollow section (CHS) T-joint under axial loading. (a) Problem domain and loading, (b) boundary conditions.

Figure 28. Reference mesh for Example 4.

$NTD=863958$, $\eta_{\Omega}=1.51\%$, $\eta_{\Omega_{jun}}=9.23\%$, $\eta_{\Omega_{shell}}=1.56\%$, $surf \eta_{\Omega_{shell}}=0.71\%$, $norm \eta_{\Omega_{shell}}=17.53\%$

(a) Mesh U1 (initial mesh) and element effectivity index distribution

(b) Mesh U2 and element effectivity index distribution

(c) Mesh U3 and element effectivity index distribution

(d) Mesh U4 and element effectivity index distribution

Figure 29. Uniform refinement meshes for Example 4.

(a) Mesh A1 (initial mesh) and element effectivity index distribution

(b) Mesh A2 and element effectivity index distribution

(c) Mesh A3 and element effectivity index distribution

(d) Mesh A4 and element effectivity index distribution

(e) Mesh A5 and element effectivity index distribution

(f) Zoom in view near the junction for Mesh A5

Figure 30. Adaptive refinement meshes for Example 4.

Figure 31. Result of Example 4, convergence of overall relative error, η_{Ω} .

Figure 32. Result of Example 4, convergence of relative error – Junction Part, $\eta_{\Omega_{jun}}$.

Figure 33. Result of Example 4, convergence of relative error – Shell Part, $\eta_{\Omega_{shell}}$.

Figure 34. Result of Example 4, convergence of surface relative error – Shell Part, $_{surf} \eta_{\Omega_{shell}}$.

Figure 35. Result of Example 4, convergence of normal relative error – Shell Part, $_{norm}\eta_{\Omega_{shell}}$.

Variables	Notations
constants independent of element size and the order of element	C', C'', C''', C^*
Displacements	$\mathbf{u}$
Strain tensor	$\boldsymbol{\varepsilon}$
Stress tensor	$\boldsymbol{\sigma}$
Point-wise error for displacements	e_u
Point-wise error for stress	e_σ
Sub-domain of the TWS	Ω_{sub}
Material matrix	$\mathbf{D}$
Energy norm over the subdomain Ω_{sub}	$\ \cdot \ _{\Omega_{sub}}$
Element size parameter	h
Effectivity index	θ
Numbers of element	NE
Numbers of node	NN
Total numbers of DOFs	NTD

Table I. Basic notations used.

Element type	NGP/NSP			
	T10	P13	P15	H20
Stiffness Matrix formation	6	3×3×3	9	3×3×3
Stress Recovery	4	2×2×2	2×3	2×2×2
Energy Norm Calculation	15	4×4×4	5×6	5×5×5

Table III. Numbers of integration point and sampling point used.

Mesh	$\bar{\eta}_{\Omega}$ (%)	$^{itar_i}\eta_{\Omega}$ (%)
1 (initial mesh)	22 (22/2>1)	11
2	12 (12/2>1)	6
3	4.4 (4.4/2>1)	2.2
4	1.3 (1.3/2<1)	1
5	0.95	(refinement stopped)

Table IV. A typical example for the values of $^{itar_i}\eta_{\Omega}$ (Note: $^{tar}\eta_{\Omega}=1\%$).

Group	Properties	Sub/superscript and embellishments	Positions of sub/superscripts and embellishments	Examples
Different parts of the TWS	Whole domain	Ω	Lower right hand corner of the variable	$\eta_{\Omega_{shell}}$, $\ \mathbf{u}\ _{\Omega_i}$
	Junction part	Ω_{jun}		
	Shell part	Ω_{shell}		
	The i th element	Ω_i		
Coordinate systems	Global Coordinate System (GCS)	No subscript	Upper right hand side of the variable	σ , x^p , σ^l , ε^l
	Patch Coordinate system (PCS)	p		
	Nodal Coordinate System (NCS)	l		
Stress/strain components	Total energy	No subscript	Lower left hand corner of the variable	$\eta_{\Omega_{shell}}$, $norm \sigma^l$
	Surface direction	$surf$		
	Normal direction	$norm$		
Source of the variables	surface element	frown $\frown$	Top of the variable	$surf \hat{A}_{\Omega_j}$, $\hat{\mathbf{u}}$, $\tilde{\sigma}$, $\overline{\ \mathbf{e}_u\ _{\Omega}}$, σ
	FE solution	hat $\wedge$		
	Recovered stress	Tilde $\sim$		
	Estimated value	over-bar		
	Exact or reference value	No embellishment		
Target, allowable and expected values	Target error	tar	Upper left hand corner of the variable	$tar \eta_{\Omega}$, $itar_i \eta_{\Omega_{shell}}$, $allow \ \mathbf{e}\ _{\Omega_{shell}}$
	Immediate target error for the i th refinement step	$itar_i$		
	allowable error norm per element	$allow$		

Table II. Subscripts, superscripts and embellishments used.

Mesh	NTD	η_{Ω} (%)	$surf \eta_{\Omega_{shell}}$ (%)	$norm \eta_{\Omega_{shell}}$ (%)	$surf \theta_{\Omega_{shell}}$	$norm \theta_{\Omega_{shell}}$	θ_{Ω}
Uniform refinement							
U1	2709	38.32	27.05	883.20	0.91	0.73	1.06
U2	8367	15.85	8.74	429.88	1.20	1.44	1.07
U3	30327	6.18	2.99	176.07	1.08	1.18	1.05
U4	120567	2.44	1.52	62.11	1.18	0.95	1.30
U4	120567	2.44	1.52	62.11	1.18	0.95	1.30
Adaptive refinement							
A1	2709	38.32	27.05	883.20	0.73	1.06	0.91
A2	6999	16.73	9.76	442.11	0.89	0.93	0.92
A3	24687	6.32	3.13	178.49	0.92	0.88	0.89
A4	69621	2.43	1.45	63.23	0.97	1.23	1.16
A5	90753	1.88	1.15	48.28	0.86	1.04	0.98

Table VII. Results of Example 2.

Mesh	FEA			SPR3D		SPRN						$\tilde{\eta}_{\Omega}(SPRN)$
	η_{Ω}	$surf \eta_{\Omega_{shell}}$	$norm \eta_{\Omega_{shell}}$	$\tilde{\eta}_{\Omega}$	θ_{Ω}	$\tilde{\eta}_{\Omega}$	$surf \tilde{\eta}_{shell}$	$norm \tilde{\eta}_{shell}$	θ_{Ω}	$surf \theta_{shell}$	$norm \theta_{shell}$	$\tilde{\eta}_{\Omega}(SPR3D)$
$t=2$												
1	28.24	19.43	20.50	22.64	1.15	7.87	6.14	4.92	0.93	0.93	0.95	0.35
2	7.12	4.21	5.74	3.37	1.07	1.10	0.90	0.64	0.97	0.97	0.98	0.33
3	4.92	0.79	4.86	0.61	1.01	0.39	0.24	0.30	0.99	1.15	0.99	0.64
$t=1$												
1	21.72	14.13	16.50	14.95	0.99	5.12	3.43	3.79	0.94	0.96	0.92	0.34
2	4.65	3.41	3.16	2.71	1.05	0.85	0.71	0.46	0.96	0.95	0.96	0.31
3	1.42	0.74	1.21	0.35	1.01	0.13	0.12	0.07	0.99	0.96	0.99	0.37
$t=0.5$												
1	16.44	10.43	12.71	13.11	0.97	5.92	4.08	4.29	0.87	0.86	0.88	0.45
2	3.51	2.44	2.53	2.07	1.00	0.56	0.41	0.09	0.96	0.97	0.95	0.27
3	0.79	0.60	0.53	0.29	0.99	0.10	0.38	0.05	0.97	0.97	0.98	0.34

Table VI. Results of Example 1.

Legends for Table VI:

FEA = Results from FE analysis; SPR3D = Results from the classic 3D SPR scheme; SPRN = Results from the SPRN scheme.

$$\tilde{\eta}_{\Omega} = \frac{\|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{\Omega}}{\|\mathbf{u}\|_{\Omega}} = \text{Relative error of the recovered stress field, overall.}$$

$$surf \tilde{\eta}_{shell} = \frac{surf \|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{shell}}{surf \|\mathbf{u}\|_{shell}} = \text{Relative error of the recovery stress field, surface components of the shell part.}$$

$$norm \tilde{\eta}_{shell} = \frac{norm \|\tilde{\mathbf{e}}_{\mathbf{u}}\|_{shell}}{norm \|\mathbf{u}\|_{shell}} = \text{Relative error of the recovery stress field, norm components of the shell part.}$$

All relative error shown are in percentage.

Example	$tar \eta_{\Omega} (\%)$	$tar \eta_{\Omega_{jun}} (\%)$	$tar \eta_{\Omega_{shell}} (\%)$	$tar_{surf} \eta_{\Omega_{shell}} (\%)$	$tar_{norm} \eta_{\Omega_{shell}} (\%)$
2	2	---	2	---	---
3	5	25	5	5	50
4	4	25	2	2	25

Table V. Target relative errors for Examples 2 to 4.

Mesh	NTD	η_{Ω} (%)	$\eta_{\Omega_{jun}}$ (%)	$\eta_{\Omega_{shell}}$ (%)	$surf \eta_{\Omega_{shell}}$ (%)	$norm \eta_{\Omega_{shell}}$ (%)
Uniform refinement						
U1	3423	24.38	136.87	16.93	12.82	234.59
U2	12612	15.22	90.63	9.80	8.67	97.27
U3	50217	9.27	55.28	5.96	5.39	54.28
U4	218349	5.50	31.42	3.74	2.93	49.14
Adaptive refinement						
A1	3423	24.38	136.87	16.93	12.82	234.59
A2	6927	15.44	80.82	11.47	9.63	132.17
A3	9309	11.24	49.98	9.27	7.72	109.03
A4	20793	7.57	26.63	6.80	5.61	81.61
A5	58260	4.40	25.09	3.00	1.87	45.49

Table VIII. Results of Example 3.

Mesh	NTD	η_{Ω} (%)	$\eta_{\Omega_{jun}}$ (%)	$\eta_{\Omega_{shell}}$ (%)	$surf \eta_{\Omega_{shell}}$ (%)	$norm \eta_{\Omega_{shell}}$ (%)
Uniform refinement						
U1	7467	25.36	157.48	19.05	16.37	188.87
U2	29976	10.41	78.69	6.14	4.90	71.55
U3	109665	6.22	51.39	2.93	2.15	38.33
U4	379194	4.38	36.54	1.98	1.25	29.67
Adaptive refinement						
A1	7467	25.36	157.48	19.05	16.37	188.87
A2	24813	10.54	77.43	6.54	5.21	76.69
A3	87555	6.22	50.85	3.01	2.24	39.06
A4	115920	4.91	40.14	2.39	1.73	31.70
A5	153357	3.24	23.94	2.00	1.41	27.35

Table IX. Results of Example 4.

Mesh	Mesh generation	FE analysis	Stress recovery	Error estimation	Adaptive refinement	Total
U1	1	1	7	18	---	27
U2	1	7	35	95	---	138
U3	5	27	132	301	---	465
U4	20	129	457	1254	---	1860
Total	27	164	631	1688	---	2490
A1	1	1	7	18	1	28
A2	1	8	25	71	1	106
A3	4	37	103	292	2	438
A4	5	54	139	396	3	597
A5	9	84	188	507	4	792
Total	24	184	462	1284	11	1941

Table X. Example 4, computational time (in seconds) used by different processes.