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ON TRANSITIVE PERMUTATION GROUPS WITH PRIMITIVE SUBCONSTITUENTS

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ABSTRACT

Let G be a transitive permutation group on a set Ω such that, for $\omega \in \Omega$, the stabiliser G_ω induces on each of its orbits in $\Omega \setminus \{\omega\}$ a primitive permutation group (possibly of degree 1). Let N be the normal closure of G_ω in G . Then (Theorem 1) either N factorises as $N = G_\omega G_\delta$ for some $\omega, \delta \in \Omega$, or all unfaithful G_ω -orbits, if any exist, are infinite. This result generalises a theorem of I. M. Isaacs which deals with the case where there is a finite upper bound on the lengths of the G_ω -orbits. Several further results are proved about the structure of G as a permutation group, focussing in particular on the nature of certain G -invariant partitions of Ω .

1. Introduction

This paper reports on an investigation of transitive permutation groups in which a point stabiliser induces on each of its orbits a (possibly trivial) primitive permutation group, that is, each subconstituent is primitive. Thus, for a transitive permutation group G on a set Ω , we shall assume that G satisfies the following condition.

(A) G is a transitive permutation group on a set Ω , and for $\omega \in \Omega$ and each G_ω -orbit $\Delta(\omega)$ in Ω , either $|\Delta(\omega)| = 1$ or G_ω is primitive on $\Delta(\omega)$.

This study was inspired by the ‘Bounded Subdegrees Theorem’ which, in the language of permutation groups, states: for a transitive permutation group G on a set Ω such that, for $\omega \in \Omega$, the length of each orbit of the stabiliser G_ω is at most d , there exists a normal subgroup N of G such that both $|G_\omega : N_\omega|$ and $|N : N_\omega|$ are bounded above by some function of d . This theorem was first proved by G. Schlichting [5], and other versions of it may be found in papers by V. Trofimov [6] and G. M. Bergman and H. W. Lenstra [1]. Moreover, I. M. Isaacs [2] showed that if in addition G satisfies condition (A), then either N contains G_ω with index at most d , or $N = 1$ and G_ω is isomorphic to a subgroup of S_d .

It is known that, in certain special cases, the assumption of a finite upper bound d on the length of the G_ω -orbits is unnecessary for these conclusions to hold. For example, if all the subdegrees $|\Delta(\omega)|$ are either 1 or prime (where no restriction is made on the number of primes occurring), then condition (A) is satisfied; it was shown by the second author [4] (and it also follows from [3, Theorem A]) that at most two primes can occur, and that there is a normal subgroup N with the properties obtained in [2] by Isaacs.

One of the purposes of this paper is to explore the extent to which the assumption of a finite upper bound d on the subdegrees can be removed in Isaacs’ result. Suppose,

then, that G is a transitive permutation group satisfying condition (A). In Section 2 we prove Theorem 1 below, concerning the possible factorisation of the normal closure N in G of a point stabiliser G_ω . We say that a G_ω -orbit $\Delta(\omega)$ is *unfaithful* if the subgroup of G_ω fixing $\Delta(\omega)$ pointwise is nontrivial, and we say that $\Delta(\omega)$ is *nontrivial* if $|\Delta(\omega)| > 1$. (Note that our concept of a nontrivial G_ω -orbit differs from the usual one, which is that $\Delta(\omega)$ is nontrivial if and only if $\Delta(\omega) \neq \{\omega\}$.) Also, a permutation group $G \leq \text{Sym}(\Omega)$ is *regular* on Ω if G is transitive and the only element of G which fixes a point of Ω is the identity element.

THEOREM 1. *Let G be a transitive non-regular permutation group on Ω satisfying condition (A) above, such that, for $\omega \in \Omega$, the stabiliser G_ω is not a cyclic group of prime order. Let N be the normal closure in G of G_ω . Then either $N = G_\gamma G_\delta$ for some $\gamma, \delta \in \Omega$, or each unfaithful, nontrivial G_ω -orbit in Ω (if any exist) is infinite.*

This has as an immediate consequence the following corollary, which is a generalisation of [2, Theorem A].

COROLLARY TO THEOREM 1. *Let G be a transitive permutation group on Ω satisfying condition (A), such that, for $\omega \in \Omega$, all G_ω -orbits are finite. Then either there is a normal subgroup N of G which contains G_ω as a subgroup of finite index (equal to the length of some G_ω -orbit), or G_ω is isomorphic to a subgroup of S_d , where $d > 1$ and d is the length of some G_ω -orbit in Ω .*

Theorem 1 follows from a technical result (Theorem 2.6) which has as its hypotheses two conditions which are weaker than condition (A), and which are similar in spirit to the assumptions of [2, Theorem B].

In Section 3 we analyse the permutation group structure of a group G satisfying the conditions of Theorem 1. We show that the transitive permutation group G on Ω is permutationally isomorphic to a subgroup of an imprimitive wreath product $M \text{ wr } R$ on $\Omega' \times I$, where $R \cong G/N$ and R is faithful and regular on I , and $M \leq \text{Sym}(\Omega')$ is permutationally isomorphic to the transitive permutation group N^{Ω_i} induced by N on one of its orbits Ω_i . Moreover, either there is a prime p such that N is an elementary abelian p -group and has all orbits of length p , or N^{Ω_i} satisfies condition (A). In the latter case, there is a unique system Σ_i of maximal proper blocks of imprimitivity preserved by N in Ω_i , and the primitive permutation group N^{Σ_i} induced by N on Σ_i satisfies condition (A') below. (Condition (A') is stated for a transitive permutation group G on Ω .)

(A') G is a transitive permutation group on a set Ω , and for $\omega \in \Omega$, G_ω has a nontrivial normal subgroup H such that, for each H -orbit $\Delta(\omega)$ in Ω , the permutation group induced by H on $\Delta(\omega)$ is primitive (possibly of degree 1), and H and G_ω have the same set of fixed points in Ω .

Unfortunately, condition (A') is weaker than condition (A). For example, the Frobenius group G of order 20 is 2-transitive on a set Ω of size 5; a point stabiliser G_ω is not primitive on its orbit $\Omega \setminus \{\omega\}$, while the subgroup H of G_ω of order 2 is primitive on each of its two orbits of size 2 in $\Omega \setminus \{\omega\}$.

THEOREM 2. *Let G be a transitive non-regular permutation group on Ω satisfying condition (A). Then G is permutationally isomorphic to a subgroup of the imprimitive*

wreath product $M \text{ wr } R$ acting on $\Omega' \times I$, where $R \leq \text{Sym}(I)$ with R regular on I , and $M \leq \text{Sym}(\Omega')$ such that either

- (i) $M = Z_p$ is regular on $\Omega' = Z_p$ for some prime p , or
- (ii) M is a transitive permutation group on Ω' which satisfies condition (A); moreover, M preserves a unique system Σ of maximal proper blocks of imprimitivity in Ω' , and the primitive permutation group induced by M on Σ satisfies condition (A').

Thus, in searching for an understanding of transitive groups satisfying condition (A), it is unfortunately not sufficient to consider the primitive groups with this condition. Nevertheless, in the special case in which the stabiliser G_α fixes only the point α , case (ii) of Theorem 2 holds with M primitive on Ω' . In this special case, there is a reduction to primitive groups with condition (A). Thus it would be of interest to have a classification of such groups, even just the finite ones.

PROBLEM. Classify all primitive permutation groups satisfying condition (A).

Moreover, it would be of even greater interest to know the primitive permutation groups with condition (A').

PROBLEM. Classify all primitive permutation groups satisfying condition (A').

Finally, in the case where N is intransitive on Ω , we study the action induced by N on the union $\Omega_i \cup \Omega_j$ of two distinct N -orbits, to gain some insight into how large, or small, a subgroup of $M \text{ wr } R$ the group G may be. For a permutation group H on a set Ω , and a subset $\Delta \subseteq \Omega$, H_Δ and $H_{(\Delta)}$ denote the setwise and pointwise stabilisers of Δ in H , respectively, and if H fixes Δ setwise, then H^Δ denotes the permutation group induced by H on Δ .

THEOREM 3. *Let G be a transitive non-regular permutation group on Ω satisfying condition (A), and let N be the normal closure in G of the stabiliser G_ω of a point $\omega \in \Omega$. Suppose that N is intransitive, let Ω_i, Ω_j be distinct N -orbits in Ω , and let $\omega_i \in \Omega_i$ and $\omega_j \in \Omega_j$. Then one of the following holds.*

- (i) *For some prime p , N is an elementary abelian p -group, $|\Omega_i| = |\Omega_j| = p$, and $N^{\Omega_i \cup \Omega_j} \cong Z_p$.*
- (ii) *$N = G_{\omega_i} G_{\omega_j}$, that is, N is transitive on $\Omega_i \times \Omega_j$.*
- (iii) *$G_{\omega_i}^{\Omega_j} \neq 1$, $G_{\omega_j}^{\Omega_i} \neq 1$, and either*
 - (a) *for each nontrivial G_{ω_i} -orbit Δ in Ω_j and each nontrivial G_{ω_j} -orbit Δ' in Ω_i , $G_{(\{\omega_i\} \cup \Delta)}$ and $G_{(\{\omega_j\} \cup \Delta')}$ fix $\Omega_i \cup \Omega_j$ pointwise, or*
 - (b) *each nontrivial G_{ω_i} -orbit in Ω_j and each nontrivial G_{ω_j} -orbit in Ω_i is infinite.*

Theorem 3 will be proved in Section 4. We give examples of groups satisfying condition (A) in each of the cases (i)–(iii). Note, however, that cases (ii) and (iii) (a), and cases (ii) and (iii) (b), are not disjoint.

2. A normal subgroup close to the point stabiliser

In this section we give a proof of Theorem 1. First we introduce some notation which will be used throughout the paper. Then we discuss several properties of the

point stabiliser of a transitive permutation group which are weaker than condition (A), and finally we deduce some consequences of these properties which together yield a proof of Theorem 1.

2.1 Orbitals and suborbits for transitive groups. Let G be a transitive permutation group on a set Ω . Then G induces a natural action on $\Omega \times \Omega$ given by

$$(\alpha, \beta)^g := (\alpha^g, \beta^g) \quad \text{for all } \alpha, \beta \in \Omega, g \in G.$$

The orbits of G in $\Omega \times \Omega$ are called *orbitals*, and there is a bijection between the set of orbitals and the set of G_α -orbits in Ω such that an orbital Δ corresponds to the G_α -orbit $\Delta(\alpha) := \{\beta \mid (\alpha, \beta) \in \Delta\}$. The G_α -orbits are also called *suborbits* of G in Ω , and the permutation group $G_\alpha^{\Delta(\alpha)}$ induced by G_α on a suborbit $\Delta(\alpha)$ is called a *subconstituent* of G . For an orbital Δ , the length of the corresponding suborbit

$$d(\Delta) := |\Delta(\alpha)|$$

is independent of the choice of α , and is called a *subdegree* of G . To each orbital Δ there is a *paired* orbital $\Delta^* := \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$. If G is finite, then Δ and Δ^* correspond to the same subdegree $d(\Delta) = d(\Delta^*) = |\Delta|/|\Omega|$, but this is not necessarily the case when G is infinite.

The *orbital digraph* corresponding to an orbital Δ is the digraph with vertex set Ω and arc set Δ ; we usually refer to this digraph as Δ also. Clearly, the group G is admitted as a vertex-transitive and arc-transitive group of automorphisms of the digraph Δ , and G preserves the partition of Ω given by the connected components of Δ . (By the *connected component* of Δ containing ω we mean the set of all points α such that there is a finite sequence $\omega_0, \dots, \omega_k$ of points with $\omega_0 = \omega$, $\omega_k = \alpha$, and, for each $i = 1, \dots, k$, $(\omega_{i-1}, \omega_i) \in \Delta \cup \Delta^*$.) For orbitals Δ and Γ , we set

$$\Delta \circ \Gamma := \{(\alpha, \beta) \mid \text{for some } \gamma \in \Omega, (\alpha, \gamma) \in \Delta \text{ and } (\gamma, \beta) \in \Gamma\}.$$

The set $\Delta \circ \Gamma$ is a union of orbitals of G , and we write $\Delta \circ \Gamma(\alpha) := \{\beta \mid (\alpha, \beta) \in \Delta \circ \Gamma\}$ for the corresponding union of G_α -orbits.

2.2 Conditions on the point stabiliser. Although our aim is to investigate transitive groups G satisfying condition (A), some of our arguments do not require an assumption as strong as condition (A), and we make weaker assumptions where possible. Condition (A) implies both of the following weaker conditions. A subgroup $G \leq \text{Sym}(\Omega)$ is said to be *quasiprimitive* on Ω if each nontrivial normal subgroup of G is transitive on Ω .

(B1) For $\alpha \in \Omega$ and $\Delta(\alpha)$ a nontrivial G_α -orbit in Ω , $G_\alpha^{\Delta(\alpha)}$ is quasiprimitive.

(B2) For $\alpha, \beta, \gamma \in \Omega$, if $G_{\alpha\beta} \neq G_{\alpha\gamma}$, then $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\gamma} \rangle$.

In [2], Isaacs also replaced the primitivity assumption by two weaker assumptions, one of which was (B1) and the other a weaker version of (B2) in which the condition on the generation of the point stabiliser G_α was required only for points β, γ lying in the same G_α -orbit. In many of our arguments, we wish to exclude the case that G_α has a faithful regular orbit, that is, we assume the following condition (C).

(C) For all $\alpha, \beta \in \Omega$, $G_{\alpha\beta} \neq 1$.

REMARK 2.1. The only primitive permutation groups which are regular are cyclic groups of prime order. Hence if condition (A) holds, and G is not regular on Ω , then condition (C) is equivalent to the assumption that G_α is not finite of prime order.

2.3 *Some consequences of these conditions.* First we prove that under the (very strong) condition (B2), the stabiliser of one point cannot be a proper subgroup of the stabiliser of another point. This result is similar to Lemmas 1 and 2 of [2], but our assumption (B2) allows us to remove the assumption of a finite upper bound on the subdegrees of G .

PROPOSITION 2.2. *Let G be a transitive permutation group on Ω satisfying (B2). Then for $\alpha, \beta \in \Omega$, G_α is not a proper subgroup of G_β .*

Proof. Suppose that we have $G_\alpha < G_\beta$ for some $\alpha, \beta \in \Omega$. Let $g \in G$ be such that $\beta^g = \alpha$, and set $\gamma := \alpha^g$. Then $G_\alpha = G_{\beta^g}^g$, $G_\gamma = G_\alpha^g$, and so $G_\gamma = G_\alpha^g < G_\beta^g = G_\alpha < G_\beta$. Now $G_{\alpha\beta} = G_\alpha$ and $G_{\beta\gamma} = G_\gamma$, and hence we have $G_{\beta\gamma} < G_{\beta\alpha} < G_\beta$. It follows from (B2) that $G_\beta = \langle G_{\beta\gamma}, G_{\beta\alpha} \rangle$, which is a contradiction since $\langle G_{\beta\gamma}, G_{\beta\alpha} \rangle = G_\alpha < G_\beta$.

Next we prove a preliminary result about the normal closure N of G_α in G . The main result about N will be Theorem 2.6 below.

PROPOSITION 2.3. *Let G be a transitive permutation group on Ω satisfying (B2) and (C), and let N be the normal closure in G of a point stabiliser G_α , $\alpha \in \Omega$. If $\alpha, \beta \in \Omega$ are such that $G_\alpha \neq G_\beta$, then $N = \langle G_\alpha, G_\beta \rangle$.*

Proof. Suppose that $\alpha, \beta \in \Omega$ are such that $G_\alpha \neq G_\beta$, and set $T = \langle G_\alpha, G_\beta \rangle$. We claim that $T = N$. Suppose, to the contrary, that there exists $\gamma \in \Omega$ such that $G_\gamma \not\subseteq T$. If $G_{\gamma\alpha} \neq G_{\gamma\beta}$, then by (B2) we would have $G_\gamma = \langle G_{\gamma\alpha}, G_{\gamma\beta} \rangle \subseteq \langle G_\alpha, G_\beta \rangle = T$, which is not the case; hence $G_{\gamma\alpha} = G_{\gamma\beta}$. Therefore $G_{\gamma\alpha} = G_{\gamma\alpha\beta} \subseteq G_{\alpha\beta}$. If this inclusion were proper, then again using (B2) we would have $G_\alpha = \langle G_{\gamma\alpha}, G_{\alpha\beta} \rangle = G_{\alpha\beta} \subseteq G_\beta$, and as $G_\alpha \neq G_\beta$, this would contradict Proposition 2.2. Hence $G_{\alpha\beta} = G_{\gamma\alpha} = G_{\gamma\beta}$.

It follows from condition (C) that there is a point $\delta \in \Omega$ such that $G_{\alpha\beta} \neq G_{\alpha\beta\delta}$. In particular, $G_{\alpha\beta} \neq G_{\alpha\delta}$. Then it follows from the argument of the previous paragraph that $G_\delta \subseteq T$. If $G_{\gamma\delta} \neq G_{\gamma\alpha}$, then by (B2), $G_\gamma = \langle G_{\gamma\delta}, G_{\gamma\alpha} \rangle \subseteq T$, which is not the case. Hence $G_{\gamma\delta} = G_{\gamma\alpha}$, and we showed above that this subgroup was equal to $G_{\alpha\beta}$. This is a contradiction, since $G_{\alpha\beta}$ does not fix δ . Thus $G_\gamma \subseteq T$ for each point γ , and so $T = N$.

Our next step is to use Proposition 2.3 to study a single orbital Δ with subdegree $d(\Delta) > 1$. Theorem 2.4 below analyses the various possibilities. The outcome (ii) is interesting, as it implies that the N -orbit containing δ coincides with $\Delta(\omega)$, where $(\omega, \delta) \in \Delta$. It is this conclusion that we seek for some orbital Δ in the main theorem (Theorem 2.6) of this section.

THEOREM 2.4. *Let G be a transitive permutation group on Ω satisfying (B1), (B2) and (C), let N be the normal closure in G of a point stabiliser G_ω , $\omega \in \Omega$, and let Δ be an orbital with $d(\Delta) > 1$. Then one of the following holds.*

- (i) G_ω is faithful on $\Delta(\omega)$ or on $\Delta^*(\omega)$.
- (ii) $N = G_\omega G_\delta$, where $(\omega, \delta) \in \Delta$.

- (iii) For $(\omega, \delta) \in \Delta$, both $G_{(\{\omega\} \cup \Delta(\omega))}$ and $G_{(\{\delta\} \cup \Delta^*(\delta))}$ are nontrivial, and one is a subgroup of the other. Moreover, the smaller of them is a normal subgroup of N , and fixes pointwise the connected component containing ω of the orbital digraph Δ . Also, $N = G_\omega G_\alpha$ for some $\alpha \in \Omega$.
- (iv) Both $d(\Delta)$ and $d(\Delta^*)$ are infinite, and there exist a pair $(\omega, \alpha) \in \Delta \circ \Delta^*$ such that $\Delta(\omega) \subset \Delta(\alpha)$, and a pair $(\delta, \beta) \in \Delta^* \circ \Delta$ such that $\Delta^*(\delta) \subset \Delta^*(\beta)$.

REMARK 2.5. We show in the proof that if none of (i)–(iii) holds, then $K := G_{(\{\omega\} \cup \Delta(\omega))}$ is transitive on the G_ω -orbit α^{G_ω} , and $L := G_{(\{\delta\} \cup \Delta^*(\delta))}$ is transitive on the G_δ -orbit β^{G_δ} . Also, in this case, we show that (iv) is true for each point α in a G_ω -orbit in $\Delta \circ \Delta^*(\omega)$ on which K acts nontrivially (and hence by (B1) on which K acts transitively), and that (iv) is true for each point β in a G_δ -orbit in $\Delta^* \circ \Delta(\delta)$ on which L acts nontrivially (and hence by (B1) on which L acts transitively).

Proof of Theorem 2.4. Let $\omega \in \Omega$, and set $H := G_\omega$ and $K := H_{(\Delta(\omega))}$. Let $\delta \in \Delta(\omega)$, and set $M := G_\delta$ and $L := M_{(\Delta^*(\delta))}$. Since $d(\Delta) > 1$, $H \neq M$, and hence by Proposition 2.3, $N = \langle H, M \rangle$. Suppose that (i) does not hold, so that both K and L are nontrivial. We claim that each H -orbit $\Sigma(\omega)$ in $\Delta \circ \Delta^*(\omega)$ contains at least one point of $\Delta^*(\delta)$. Let α be a point of such an orbit $\Sigma(\omega)$. By definition, there is some $\delta' \in \Delta(\omega)$ such that $(\delta', \alpha) \in \Delta^*$. Then for some $h \in H$, $(\delta')^h = \delta$, and we have $\alpha^h \in \Sigma(\omega) \cap \Delta^*(\delta)$. This establishes the claim. Similarly, each M -orbit in $\Delta^* \circ \Delta(\delta)$ contains at least one point of $\Delta(\omega)$.

Suppose first that neither $K \subseteq L$ nor $L \subseteq K$. Then there exists a point $\alpha \in \Delta^*(\delta)$ which is not fixed by K , and a point $\beta \in \Delta(\omega)$ which is not fixed by L . By (B1), $\alpha^K = \alpha^H = \Sigma_1(\omega)$ and $\beta^L = \beta^M = \Sigma_2(\delta)$, say. This means that H is transitive on $\Delta(\omega) \times \Sigma_1(\omega)$, whence $H_\alpha = G_{\omega\alpha}$ is transitive on $\Delta(\omega)$. Thus $\Delta(\omega) = \delta^{H_\alpha} \subseteq \delta^{G_\alpha}$, and since $(\alpha, \delta) \in \Delta$, $\delta^{G_\alpha} = \Delta(\alpha)$, so $\Delta(\omega) \subseteq \Delta(\alpha)$. If equality holds, then $\Delta(\omega)$ is an orbit for $\langle H, G_\alpha \rangle$, and $\langle H, G_\alpha \rangle = N$ by Proposition 2.3 since $H \neq G_\alpha$. Hence $N = HM$ and (ii) holds. Thus we may assume that $\Delta(\omega) \subset \Delta(\alpha)$. If $g \in G$ is such that $\alpha^g = \omega$, then $\Delta(\omega^{g^{i+1}}) \subset \Delta(\omega^{g^i})$ for all $i \geq 0$, and it follows that $d(\Delta) = |\Delta(\omega)|$ is infinite. Similarly, $M_\beta = G_{\delta\beta}$ is transitive on $\Delta^*(\delta)$, so $\Delta^*(\delta) = \delta^{M_\beta} \subseteq \Delta^*(\beta)$, and either (ii) holds or the inclusion is proper and (iv) is true. (In fact, if (ii) does not hold, then we have (iv) true for every $\alpha \in \Delta^*(\delta)$ which is not fixed by K and every $\beta \in \Delta(\omega)$ which is not fixed by L .)

Thus either $K \subseteq L$ or $L \subseteq K$. We shall complete the proof for the case where $K \subseteq L$. The proof for $L \subset K$ is analogous and is therefore omitted. So suppose that $K \subseteq L$. We claim that K is normal in $N = \langle H, M \rangle$. To prove this, it is sufficient to prove that $K^m \subseteq K$ for all $m \in M$. So let $m \in M$. We must show that K^m fixes $\{\omega\} \cup \Delta(\omega)$ pointwise. Now $\omega^{m^{-1}}$ belongs to $\omega^M = \Delta^*(\delta)$, which is fixed pointwise by L and hence by K . Thus K^m fixes ω . Now let $\sigma \in \Delta(\omega)$. If σ is fixed by L , then L (a normal subgroup of M) will fix pointwise the M -orbit σ^M , and in particular L will fix $\sigma^{m^{-1}}$. Thus K will fix $\sigma^{m^{-1}}$, and hence K^m will fix σ . On the other hand, if σ is moved by L , then by (B1), σ^L is an M -orbit and hence is fixed setwise by $m^{-1} \in M$. Thus $\sigma^{m^{-1}} \in \sigma^L$, which is contained in $\Delta(\omega)$ (since $L \leq G_\omega$). Hence $\sigma^{m^{-1}}$ is fixed by K , and hence σ is fixed by K^m . Thus K^m fixes $\{\omega\} \cup \Delta(\omega)$ pointwise, as required. So we conclude that K is normal in N . It follows that K fixes pointwise the N -orbit δ^N containing δ .

Suppose that the permutation group induced by N on δ^N is not regular. Then there is a point $\sigma \in \delta^N$ which is moved by $M = G_\delta$. By Proposition 2.3, $N = \langle M, G_\sigma \rangle$. Also, as K is nontrivial, there is a point β which is moved by K . By (B1), since K is normal in both M and G_σ , the K -orbit β^K is an orbit for both M and G_σ , and hence also for

N . Therefore $N = G_\delta G_\beta$ (and if $g \in G$ maps δ to ω , then $N = G_\omega G_\alpha$ where $\alpha = \beta^g$). Thus we have shown that for each N -orbit Ω' , either K is transitive on Ω' or K fixes Ω' pointwise, and in the former case we have $N = G_\delta G_\beta$ for $\beta \in \Omega'$. If K is nontrivial on the connected component containing ω of the orbital digraph Δ , then there must be a pair (α, β) in Δ or Δ^* such that K fixes α but not β . In this situation, we have (by (B1)) that $\beta^N = \beta^K = \Delta(\alpha)$ or $\Delta^*(\alpha)$, and $N = G_\alpha G_\beta$. Thus part (ii) holds. On the other hand, if K fixes pointwise the connected component containing ω of the orbital digraph Δ , then part (iii) holds.

Finally, suppose that the permutation group induced by N on its orbit δ^N is regular. Then we have $K \leq L \leq G_\delta = N_\delta$, so L fixes δ^N pointwise. Thus L fixes $\Delta(\omega)$ (a subset of δ^N) pointwise, so $L = K$. Suppose that K acts nontrivially on $\Delta(\delta)$. Then by (B1), as $K = L$ is normal in M , $\Delta(\delta)$ is an orbit for K . Again by (B1), as K is normal in H , it follows that $\Delta(\delta)$ is also an orbit for H , and hence $\Delta(\delta)$ is an orbit for $N = \langle H, M \rangle$. Therefore also $\Delta(\omega)$ is an N -orbit, and (ii) holds. Similarly, if K acts nontrivially on $\Delta^*(\omega)$, then (ii) holds. Thus we may assume that K fixes pointwise $\Delta(\delta)$ and $\Delta^*(\omega)$, and hence that K fixes pointwise the whole connected component containing ω of the orbital digraph Δ . Since $K \neq 1$, there is a point β moved by K . (Since K is normal in N , K fixes pointwise the N -orbits containing ω and δ , so β does not lie in either of these orbits.) By (B1), $\beta^K = \beta^L$ is an orbit for both H and M , and hence also for N . Therefore $N = G_\omega G_\beta$, and (iii) holds. This completes the proof of Theorem 2.4.

We are now ready to prove the main result of this section.

THEOREM 2.6. *Let G be a transitive permutation group on Ω satisfying (B1), (B2) and (C), and let N be the normal closure in G of the stabiliser G_ω of a point $\omega \in \Omega$. Then either $N = G_\omega G_\delta$ for some $\omega, \delta \in \Omega$, or each nontrivial unfaithful G_ω -orbit (if any exist) is infinite.*

Proof. Suppose that N does not factorise as $N = G_\omega G_\delta$ for any $\omega, \delta \in \Omega$, and suppose that there is an orbital Δ such that $d(\Delta)$ is finite and greater than 1, and $K := G_{(\{\omega\} \cup \Delta(\omega))} \neq 1$. Then it follows from Theorem 2.4 that G_ω is faithful on $\Delta^*(\omega)$. Let $\delta \in \Delta(\omega)$. Since $d(\Delta) > 1$, $G_\delta \neq G_\omega$, and so, by Proposition 2.3, $N = \langle G_\omega, G_\delta \rangle$. Since G_ω is faithful on $\Delta^*(\omega)$, also G_δ is faithful on $\Delta^*(\delta)$, and since $K \neq 1$ and $K \leq G_\delta$, there is a point $\alpha \in \Delta^*(\delta)$ such that $\alpha^K \neq \{\alpha\}$. By (B1), $\alpha^K = \alpha^{G_\omega}$ and $G_\omega = KG_{\omega\alpha}$. Also, $G_\omega \neq G_\alpha$, and hence, by Proposition 2.3, we have $N = \langle G_\omega, G_\alpha \rangle$. Since both ω and α are in $\Delta^*(\delta)$, there is an element $g \in G_\delta$ such that $\omega^g = \alpha$. Now

$$\Delta(\omega) = \delta^{G_\omega} = \delta^{KG_{\omega\alpha}} = \delta^{G_{\omega\alpha}} \subseteq \delta^{G_\alpha},$$

and δ^{G_α} is the orbit of $G_\alpha = (G_\omega)^g$ containing $\delta = \delta^g$, that is, $\delta^{G_\alpha} = \Delta(\omega)^g$. Since $d(\Delta) = |\Delta(\omega)| = |\Delta(\omega)^g|$ is finite, it follows that $\Delta(\omega) = \Delta(\omega)^g$. So $\Delta(\omega)$ is an orbit of $N = \langle G_\omega, G_\alpha \rangle$, and hence $N = G_\delta G_\omega$, contradicting our assumption.

Theorem 1 follows immediately from Theorem 2.6, since condition (A) for a transitive non-regular permutation group G , and the condition that a point stabiliser is not cyclic of prime order, together imply all of the conditions (B1), (B2) and (C).

3. Structure of transitive groups with primitive subconstituents

In this section we describe the permutation group structure of transitive non-regular permutation groups G satisfying condition (A). As in Section 2, we begin by considering the weaker conditions (B1) and (B2). Let N be the normal closure of the point stabilisers of a transitive permutation group G on Ω , and let Ω_i , $i \in I$, be the set of N -orbits in Ω . First we consider the actions of G on the set of N -orbits, and of N on each of its orbits. Recall that for a permutation group H and a subset Δ of points fixed setwise by H , H^Δ denotes the permutation group induced by H on Δ . Also, $\text{fix}_\Delta(H)$ will denote the set of fixed points of H in Δ , and for any subset Φ of points, H_Φ will denote the setwise stabiliser of Φ in H .

LEMMA 3.1. *Let G be a transitive permutation group on Ω , and let Ω_i , $i \in I$, be the set of orbits of the normal closure N in G of the stabiliser G_ω of a point $\omega \in \Omega$. Then G acts regularly on the set $\{\Omega_i \mid i \in I\}$ with kernel N .*

Proof. Choose $i \in I$ and $\omega \in \Omega_i$. As N is transitive on Ω_i , we have $G_{\Omega_i} = NG_\omega$, and since N contains G_ω , it follows that $G_{\Omega_i} = N$. Thus G_{Ω_i} fixes each N -orbit setwise, that is, G acts regularly on the set of N -orbits, and N is the kernel of this action.

THEOREM 3.2. *Let G be a transitive permutation group on Ω satisfying conditions (B2) and (C). Let N be the normal closure in G of the stabiliser G_ω of a point $\omega \in \Omega$, and let Ω_i , $i \in I$, be the set of N -orbits in Ω . Then the following hold.*

- (i) *Either*
 - (a) *for each $i \in I$, N^{Ω_i} is regular, or*
 - (b) *for each $i \in I$ and $\omega \in \Omega_i$, $F := \text{fix}_{\Omega_i}(G_\omega)$ is the unique maximal proper block of imprimitivity for N^{Ω_i} containing ω , and $N_F = N_N(G_\omega)$.*
- (ii) *Let Σ be a set of minimal nontrivial blocks of imprimitivity for G in Ω , and let $B \in \Sigma$. Then either $\Sigma = \{\Omega_i \mid i \in I\}$ and each N^{Ω_i} is primitive, or the setwise stabiliser G_B of B acts regularly on B .*

Proof: Suppose that N^{Ω_i} is not regular. Then $F := \text{fix}_{\Omega_i}(G_\omega)$ is a proper subset of Ω_i and is a block of imprimitivity for N in Ω_i ; its setwise stabiliser is $N_N(G_\omega)$. Let B be a block of imprimitivity for N in Ω_i which contains ω . Then $B \subseteq F$ if and only if $N_B \leq N_N(G_\omega)$. Suppose that N_B is not contained in $N_N(G_\omega)$. Then there is an element $g \in N_B$ such that $G_\omega^g \neq G_\omega$. Set $\delta := \omega^g$, and observe that $G_\delta = G_\omega^g \subseteq \langle G_\omega, g \rangle \subseteq N_B$. Now $G_\omega \neq G_\delta$, and so, by Proposition 2.3, $N = \langle G_\omega, G_\delta \rangle$. It follows that N fixes B setwise, and so $B = \Omega_i$. Thus part (i) is proved.

Now let Σ be a set of minimal nontrivial blocks of imprimitivity for G in Ω , and suppose that B is the block of Σ which contains ω . Suppose that G_B is not regular on B . Then there is a point $\delta \in B$ which is not fixed by G_ω . So $G_\omega \neq G_\delta$ and, by Proposition 2.3, $N = \langle G_\omega, G_\delta \rangle$. It follows that N fixes B setwise, and hence $\Omega_i \subseteq B$. By the minimality of B , $B = \Omega_i$, and so N is primitive on Ω_i .

Now we refine this result for transitive groups with primitive subconstituents. First, we obtain extra information about the action of the normal subgroup N on its orbits.

PROPOSITION 3.3. *Let G be a transitive non-regular permutation group on Ω which satisfies condition (A). Let $\omega \in \Omega$, let N be the normal closure in G of G_ω , and let Ω_i , $i \in I$, be the set of N -orbits in Ω . Let $i \in I$ be such that $\omega \in \Omega_i$.*

- (i) *If N^{Ω_i} is regular, then, for some prime p , all G_ω -orbits have length 1 or p , and N is an elementary abelian p -group with all orbits of length p .*
- (ii) *If N^{Ω_i} is not regular, then $F := \text{fix}_{\Omega_i}(G_\omega)$ is the unique element of the block system $\Sigma_i := \{F^g \mid g \in N\}$ which is fixed by G_ω , and G_ω is primitive on each of its orbits in $\Sigma_i \setminus \{F\}$.*

Proof. Note that $N \neq 1$ since $G_\omega \neq 1$. Suppose first that N^{Ω_i} is regular. Let $\delta \in \Omega \setminus \text{fix}_\Omega(G_\omega)$, and let Ω_j be the N -orbit containing δ . Then $G_\omega \neq G_\delta$, so, by Proposition 2.3, $N = \langle G_\omega, G_\delta \rangle$. Now N^{Ω_j} is also regular, and hence both G_ω and G_δ are normal in N . Therefore $N = G_\omega G_\delta$. Thus $N^{\Omega_j} = G_\omega^{\Omega_j}$, and hence is primitive (since condition (A) holds) and regular (by assumption). By Remark 2.1, $N^{\Omega_j} \cong Z_p$ for some prime p , and (i) holds.

Now suppose that N^{Ω_i} is not regular, so, by Theorem 3.2, F is the unique maximal proper block of imprimitivity for N^{Ω_i} containing ω , and $N_F = N_N(G_\omega)$. Thus N acts primitively on $\Sigma_i = \{F^g \mid g \in N\}$. Let $\delta \in \Omega_i \setminus F$, and let F' be the member of Σ_i containing δ . By condition (A), G_ω is primitive on $\Delta := \delta^{G_\omega}$. Now $\Delta \cap F'$ contains δ and is a block of imprimitivity for G_ω in Δ . Hence either $\Delta \cap F' = \{\delta\}$, or $\Delta \subseteq F'$. Suppose that $\Delta \subseteq F'$. Then G_ω fixes F' setwise. By choice of δ , $G_\omega \neq G_\delta$, and so, by Proposition 2.3, $N = \langle G_\omega, G_\delta \rangle$. Since both G_ω and G_δ fix F' setwise, also N fixes F' setwise. This is a contradiction, since F' is a proper subset of the N -orbit Ω_i . Hence $\Delta \cap F' = \{\delta\}$, so the actions induced by G_ω on Δ and on $(F')^{G_\omega} \subseteq \Sigma_i$ are permutationally isomorphic. In particular, G_ω is primitive on $(F')^{G_\omega}$, and $|(F')^{G_\omega}| > 1$. Thus (ii) holds.

In case (ii) of Proposition 3.3, the group N^{Ω_i} is a transitive permutation group satisfying condition (A), and N^{Ω_i} has a unique primitive quotient N^{Σ_i} . Unfortunately, it is not clear whether N^{Σ_i} satisfies condition (A). It does, however, satisfy the weaker condition (A') on transitive permutation groups with respect to the normal subgroup $G_\omega^{\Sigma_i}$ of $N_F^{\Sigma_i}$.

By Lemma 3.1, the permutation group induced by G on the set I of N -orbits is regular and isomorphic to G/N . Thus the permutation group G on Ω is permutationally isomorphic to a subgroup of an imprimitive wreath product $M \text{ wr } R$ acting on $\Omega' \times I$, where $M \leq \text{Sym}(\Omega')$ is permutationally isomorphic to the transitive permutation group N^{Ω_i} on Ω_i . Theorem 2 therefore follows from Lemma 3.1, Theorem 3.2 and Proposition 3.3.

4. Action of N on the union of two orbits

Finally, in the case where N is intransitive on Ω (in Theorem 2 or in Theorem 3.2), we investigate the permutation group induced by N on the union of two of its orbits. First, we prove a preliminary result in which we consider the action on the second orbit by the stabiliser of a point in the first orbit.

LEMMA 4.1. *Let G be a transitive non-regular permutation group on Ω satisfying conditions (B1), (B2) and (C). Let N be the normal closure in G of the stabiliser G_ω of a point $\omega \in \Omega$. Suppose that N is intransitive on Ω , and let Ω_i, Ω_j be distinct N -orbits. Let $\omega_i \in \Omega_i$. Then one of the following holds.*

- (i) *G_{ω_i} fixes $\Omega_i \cup \Omega_j$ pointwise, that is, N acts semiregularly on $\Omega_i \cup \Omega_j$.*

- (ii) For $\omega_j \in \Omega_j$, $N = G_{\omega_i} G_{\omega_j}$, that is, N is transitive on $\Omega_i \times \Omega_j$.
- (iii) $G_{\omega_i}^{\Omega_j} \neq 1$, and for each nontrivial G_{ω_i} -orbit Δ in Ω_j , the pointwise stabiliser $G_{(\{\omega_i\} \cup \Delta)}^{\Omega_i}$ fixes Ω_i pointwise.
- (iv) $G_{\omega_i}^{\Omega_j} \neq 1$, each nontrivial G_{ω_i} -orbit in Ω_j is infinite, and there is at least one such orbit, say Δ , such that $G_{(\{\omega_i\} \cup \Delta)}$ acts nontrivially on Ω_i .

Proof. Suppose that (i) does not hold. Suppose first that N^{Ω_i} is regular. Then G_{ω_i} is nontrivial on Ω_j , so, for $\omega_j \in \Omega_j$, $G_{\omega_i} \neq G_{\omega_j}$ and hence, by Proposition 2.3, $N = \langle G_{\omega_i}, G_{\omega_j} \rangle$. Since N is also regular on Ω_j , G_{ω_j} fixes Ω_j pointwise, and hence $N^{\Omega_j} = G_{\omega_i}^{\Omega_j}$. Thus G_{ω_i} is transitive on Ω_j , and (ii) holds.

Thus we may assume that N^{Ω_i} is not regular, so there is a point $\delta \in \Omega_i$ such that $G_{\omega_i} \neq G_\delta$, and hence, by Proposition 2.3, such that $N = \langle G_{\omega_i}, G_\delta \rangle$. If G_{ω_i} fixed Ω_j pointwise, so also would G_δ , and hence N would fix Ω_j pointwise, which is not the case. Hence $G_{\omega_i}^{\Omega_j} \neq 1$. Suppose now that (iii) does not hold, so there exists a G_{ω_i} -orbit Δ in Ω_j of length greater than 1 such that $(G_{(\{\omega_i\} \cup \Delta)})^{\Omega_i} \neq 1$. Choose such an orbit Δ having least cardinality subject to $|\Delta| > 1$ and $(G_{(\{\omega_i\} \cup \Delta)})^{\Omega_i} \neq 1$. Suppose that there exists a finite G_{ω_i} -orbit Σ in Ω_j such that $1 < |\Sigma| < |\Delta|$. By the minimality of $|\Delta|$, $(G_{(\{\omega_i\} \cup \Sigma)})^{\Omega_i} = 1$, so $G_{\omega_i}^{\Omega_j}$ is finite. Conjugating by an element of G which maps ω_i to a point $\sigma \in \Sigma$, we see that $G_\sigma^{\Omega_i}$ is finite, whence its subgroup $G_{\omega_i \sigma}^{\Omega_j}$ is also finite. However, since Σ is finite, this means that also $G_{\omega_i}^{\Omega_j}$ is finite, and in particular that Δ is finite. We conclude that if Δ is infinite, then (iv) holds.

Assume now that $\Delta = \Delta(\omega_i)$ is finite, set $K := G_{(\{\omega_i\} \cup \Delta)}$, let $\alpha \in \Omega_i$ be a point moved by K , and set $\Phi := \alpha^K$. By (B1), $\Phi = \alpha^{G_{\omega_i}}$. Let $\omega_j \in \Delta(\omega_i)$, so $\Delta^*(\omega_j) = \omega_i^{G_{\omega_j}}$, and, by Proposition 2.3, $N = \langle G_{\omega_i}, G_{\omega_j} \rangle$. Since $K \leq G_{\omega_i \omega_j} \leq G_{\omega_i}$, we have $\Phi = \alpha^{G_{\omega_i \omega_j}}$, that is, G_{ω_i} is transitive on $\Phi \times \Delta(\omega_i)$. Set $L := G_{(\{\omega_j\} \cup \Delta^*(\omega_j))}$. If L moves α , then, by (B1), $\alpha^L = \alpha^{G_{\omega_j}}$, and since $L \leq G_{\omega_i \omega_j} \leq G_{\omega_j}$, it follows that $\Phi = \alpha^{G_{\omega_i \omega_j}} = \alpha^{G_{\omega_j}}$. Thus Φ is fixed setwise by both G_{ω_i} and G_{ω_j} , and hence by N , which is a contradiction. Hence L fixes α , and in particular $K \not\leq L$. This means that there is a point in $\Delta^*(\omega_j)$ which is moved by K . Thus we may assume that $\alpha \in \Delta^*(\omega_j)$. Since G_{ω_i} is transitive on $\Phi \times \Delta(\omega_i)$, $G_{\omega_i \alpha}$ is transitive on $\Delta(\omega_i)$. Thus $\Delta(\omega_i) \subseteq \omega_j^{G_\alpha} = \Delta(\alpha)$. Since $\Delta(\omega_i)$ is finite, it follows that $\Delta(\omega_i) = \Delta(\alpha)$, and this set is fixed setwise by both G_{ω_i} and G_α . By Proposition 2.3, $N = \langle G_{\omega_i}, G_\alpha \rangle$, so $\Delta(\omega_i)$ is fixed setwise by N , whence $\Delta(\omega_i) = \Omega_j$. Thus G_{ω_i} is transitive on Ω_j , and case (ii) of the lemma holds.

Now we refine this lemma so that the outcomes are symmetric with respect to the two orbits.

THEOREM 4.2. *Let G be a transitive non-regular permutation group on Ω satisfying conditions (B1), (B2) and (C), and let N be the normal closure in G of the stabiliser G_ω of a point $\omega \in \Omega$. Suppose that N is intransitive on Ω , and let Ω_i, Ω_j be distinct N -orbits. Let $\omega_i \in \Omega_i$ and $\omega_j \in \Omega_j$. Then one of the following holds.*

- (i) N acts semiregularly on $\Omega_i \cup \Omega_j$.
- (ii) $N = G_{\omega_i} G_{\omega_j}$, that is, N is transitive on $\Omega_i \times \Omega_j$.
- (iii) $G_{\omega_i}^{\Omega_j} \neq 1$, $G_{\omega_j}^{\Omega_i} \neq 1$, and either
 - (a) for each nontrivial G_{ω_i} -orbit Δ in Ω_j and each nontrivial G_{ω_j} -orbit Δ' in Ω_i , the pointwise stabilisers $G_{(\{\omega_i\} \cup \Delta)}$ and $G_{(\{\omega_j\} \cup \Delta')}$ fix $\Omega_i \cup \Omega_j$ pointwise, or
 - (b) each nontrivial G_{ω_i} -orbit in Ω_j and each nontrivial G_{ω_j} -orbit in Ω_i is infinite.

Proof. Suppose that neither (i) nor (ii) holds. Then one of the outcomes (iii), (iv)

of Lemma 4.1 holds, and also one of these two outcomes holds with Ω_i and Ω_j interchanged.

Suppose first that Lemma 4.1(iv) holds. Then, in particular, $G_{\omega_j}^{\Omega_i} \neq 1$. Suppose that G_{ω_j} has a finite nontrivial orbit Δ' in Ω_i . Then Lemma 4.1(iii) holds with Ω_i and Ω_j interchanged. Hence $G_{\omega_j}^{\Omega_i}$ is finite, and so also $G_{\omega_i}^{\Omega_i}$ is finite. Let $K := N_{(\Omega_i)}$. Then K has finite index in G_{ω_i} . Now, by Lemma 4.1(iv), G_{ω_i} has an infinite orbit in Ω_j , and as $|G_{\omega_i} : K|$ is finite, K also has an infinite orbit, say Δ , in Ω_j . By (B1), Δ is a G_ω -orbit for every point $\omega \in \Omega_i$. Again using Lemma 4.1(iv), there is a point $\omega \in \Omega_i$ for which $G_{\omega_i} \neq G_\omega$, and it follows from Proposition 2.3 that $N = \langle G_{\omega_i}, G_\omega \rangle$. Therefore N fixes Δ setwise, whence $\Omega_j = \Delta$. It follows that $N = G_{\omega_j} G_{\omega_i}$, contradicting our assumptions. Hence all nontrivial G_{ω_j} -orbits in Ω_i are infinite, and case (iii)(b) of the theorem holds. Similarly, if Lemma 4.1(iv) holds with Ω_i and Ω_j interchanged, then case (iii)(b) of the theorem holds.

The only remaining possibility is that Lemma 4.1(iii) holds, and Lemma 4.1(iii) also holds with Ω_i and Ω_j interchanged. It follows that case (iii)(a) of the theorem holds.

Theorem 3 follows immediately from Theorem 4.2 on observing that by Proposition 3.3, if G satisfies condition (A), then the only case where N acts semi-regularly on Ω_i is the case where N is an elementary abelian p -group, and Ω_i has length p . We finish by giving examples of transitive groups satisfying condition (A) and one of the cases (i), (ii), (iii)(a) or (iii)(b) of Theorem 3.

EXAMPLE 4.3. (a) Let p be a prime, and define

$$G := \langle a, b, c \mid a^p = b^p = c^4 = 1, ab = ba, a^c = b, b^c = a \rangle.$$

Then $|G| = 4p^2$. Let $H := \langle a \rangle$, and consider the transitive action of G by right multiplication on the set $\Omega := [G : H]$ of $4p$ right cosets of H in G . The stabiliser H of the point $H \in \Omega$ has $2p$ fixed points and two orbits of length p . Thus condition (A) holds. The normal closure of H in G is $N = \langle a, b \rangle \cong Z_p \times Z_p$, and the N -orbits are the sets $\Omega_i := \{Hb^x c^i \mid 0 \leq x \leq p-1\}$, where $i = 0, 1, 2, 3$. Case (i) of Theorem 3 holds for the orbits Ω_0, Ω_2 , and case (ii) holds for Ω_0, Ω_1 .

(b) The group $A := A_6$ admits an involutory automorphism σ which interchanges the conjugacy classes of subgroups A_5 (stabilisers of points in the natural action of A of degree 6) and $\text{PSL}(2, 5)$ (subgroups which are 2-transitive in the natural action of A). Then A is isomorphic to the subgroup $A^* := \{(g, g^\sigma) \mid g \in A\}$ of the wreath product $A_6 \text{ wr } Z_2$, and the normaliser of A^* in $A_6 \text{ wr } Z_2$ is the group $G := \langle A^*, \tau \rangle$, where $(g_1, g_2)^\tau = (g_2, g_1)$ for $(g_1, g_2) \in A_6 \times A_6$. Let H be the stabiliser in A of the point 1 in the natural action, and set $H^* := \{(g, g^\sigma) \mid g \in H\}$. Then the transitive coset action of G on $[G : H^*]$ has degree 12, the stabiliser H^* has orbits of lengths 1, 5, 6, and condition (A) holds. The subgroup N is equal to A^* , and cases (ii) and (iii)(a) of Theorem 3 hold.

(c) Let X be an infinite set, and let S denote the symmetric group on X . Then $G := S \text{ wr } Z_2$ acts naturally on $X \times Z_2$, condition (A) is satisfied, and cases (ii) and (iii)(b) of Theorem 3 hold.

NOTE ADDED IN PROOF. Condition (A) is a condition on the lattice of subgroups of G . A recent result of F. Wagner (Theorem 4 of F. Wagner, 'Almost invariant families', *Bull. London Math. Soc.* 30 (1998) 235–240) shows that a normal subgroup $N \trianglelefteq G$ 'close' to G_ω can be recovered under lattice-theoretic assumptions more general than in (A).

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