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HILBERT SPACES OF ENTIRE FUNCTIONS AND COMPOSITION OPERATORS

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Abstract. The aim of this paper is to study composition operators on Hilbert spaces of entire functions in the complex plane $\mathbb{C}$. The following results are obtained: criteria for invariance and boundedness; estimates for essential norm, which give criteria for compactness of such operators; criteria for compact differences. Our results contain the results of boundedness and compactness by Chacóns and Giménez (Proc. Amer. Math. Soc., 2007) as particular cases.

1. Introduction

The Hilbert spaces of entire functions in the complex plane $\mathbb{C}$ and composition operators acting on them have been studied in numerous works (see, e.g., [1–4]). Amongst those cases, there are two types of spaces that are noteworthy.

The first type is the family of spaces E^2_γ. Let (γ_n) be a sequence of positive real numbers such that $\gamma_{n+1}/\gamma_n \downarrow 0$, define

$$(1) \qquad E^2_\gamma = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ entire: } \|f\|_\gamma := \left(\sum_{n=0}^{\infty} |a_n|^2 \gamma_n^{-2} \right)^{1/2} < +\infty \right\}.$$

This Hilbert space E^2_γ was first introduced in [3], when the authors studied the cyclicity of the translation operators acting on some Hilbert spaces of entire functions. One of the notable results of [3] states that if $n\gamma_n/\gamma_{n-1} \downarrow \tau$ for some $\tau > 0$, then there is $f \in E^2_\gamma$ of order 1 and type $\sigma < \tau$. Moreover, any translation operator $T_b : \ f(z) \mapsto f(z+b)$ $(b \in \mathbb{C})$ acting on this space is bounded.

From this observation, the authors of [2] defined composition operators C_φ acting on the space E^2_γ with $n\gamma_n/\gamma_{n-1} \downarrow \tau > 0$ (denoted by $E^2_\gamma(T,\tau)$), and derived the criteria for boundedness and compactness of these operators.

Another familiar type of Hilbert spaces of entire functions is the family of Fock spaces, also known as Segal–Bargmann spaces. Let $\alpha > 0$ and dA be the area measure on $\mathbb{C}$, the Fock space F^2_α is the set of all entire functions f such that

$$\|f\|_\alpha := \left(\frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dA(z) \right)^{1/2} < +\infty.$$

The boundedness and compactness of composition operators C_φ acting on F^2_α was studied comprehensively in [1].

An interesting similarity between the two papers [1] and [2] about the boundedness of C_φ is that the inducing function φ must necessarily take the form $az+b$, with $|a| \le 1$ and $b \in \mathbb{C}$. However, while this condition is also sufficient for the boundedness of C_φ in $E^2_\gamma(T,\tau)$, in case of F^2_α, it requires additionally that $b = 0$

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when $|a| = 1$. This suggests us that $E^2_\gamma(T,\tau)$ and F^2_α may belong to the same class of Hilbert spaces of entire functions, but to different subclasses.

In the present paper, we define some classes of Hilbert spaces of entire functions, which both $E^2_\gamma(T,\tau)$ and F^2_α belong to, namely the family of spaces $\mathcal{H}(\beta^+_\rho)$. The necessary conditions for boundedness and compactness of composition operator C_φ on $\mathcal{H}(\beta^+_\rho)$ are provided and consistent with the results discovered in both [1] and [2]. The subclass $\mathcal{H}(\beta^+_\rho, T)$ of $\mathcal{H}(\beta^+_\rho)$ is introduced. The criteria for boundedness, estimates for essential norms, criteria for compactness and compact differences of composition operators acting on any $\mathcal{H}(\beta^+_\rho, T)$ space are developed.

We note that some results (of boundedness and compactness) were announced in [5].

2. Various types of Hilbert spaces of entire functions

Let $\beta = (\beta_n)$ be a sequence of positive real numbers. It is well-known that the set of complex sequences

$$\ell^2_\beta = \left\{ \mathbf{a} = (a_n) : \ \|\mathbf{a}\|_\beta := \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\} \tag{2}$$

is a Hilbert space with the inner product

$$\langle \mathbf{a}, \mathbf{c} \rangle = \sum_{n=0}^{\infty} a_n \overline{c_n} \beta_n^2, \quad \mathbf{a} = (a_n), \mathbf{c} = (c_n) \in \ell^2_\beta.$$

This type of sequence spaces has many important applications in studying operators on function spaces. We refer the reader to [4] for more detailed information.

On the other hand, a complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ represents an entire function if and only if

$$\lim_{n\to\infty} |a_n|^{1/n} = 0. \tag{3}$$

Let us denote by E the set of all complex sequences (a_n) that satisfy condition (3). Clearly, E is a vector space over $\mathbb{C}$. It is natural to consider all coefficient sequences (a_n) in the intersection $\ell^2_\beta \cap E$. However, depending on (β_n), the space $\ell^2_\beta \cap E$ may not be complete in norm (2). Therefore, the function space

$$\mathcal{H}(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n, \ (a_n) \in \ell^2_\beta \cap E \right\} \tag{4}$$

equipped with norm $\|f\| := \|(a_n)\|_\beta$ is not necessarily complete, and thus not a Hilbert space.

The following result, whose proof is easy, characterizes all possible inclusion–exclusion relations between ℓ^2_β and E.

Theorem 2.1. *Let $\beta_* = \liminf_{n\to\infty} (\beta_n)^{1/n}$ and $\beta^* = \limsup_{n\to\infty} (\beta_n)^{1/n}$. Exactly one of the following alternative cases can happen:*

- *(i)* $\ell^2_\beta \subsetneq E$ *if and only if* $\beta_* = +\infty$.
- *(ii)* $E \subsetneq \ell^2_\beta$ *if and only if* $\beta^* < +\infty$.
- *(iii)* $E \setminus \ell^2_\beta \neq \varnothing$ *and* $\ell^2_\beta \setminus E \neq \varnothing$ *if and only if* $\beta_* < \beta^* = +\infty$.

From Theorem 2.1, one can easily prove that (β_n) satisfying $\beta_* = +\infty$ is the necessary and sufficient condition for the set $\mathcal{H}(\beta)$ defined in (4) to be a complete normed space, whence we have the following formal definition of a Hilbert space of entire functions.

Definition 2.2. Let (β_n) be a sequence of positive real numbers such that $\beta_* = +\infty$. The Hilbert space of entire functions induced by (β_n) is defined as

$$\mathcal{H}(\beta_E) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n,\ \|f\| := \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\}.$$

For any positive sequence $\beta = (\beta_n)$, we define the number $\beta_\rho = \liminf_{n\to\infty} \dfrac{\log \beta_n}{n \log n}$, and the function $\mu_\beta \colon \mathbb{N} \to \mathbb{R}^+$, $\mu_\beta(n) = n\beta_{n-1}/\beta_n$.

We also introduce the following sets of sequences $\beta = (\beta_n)$ that are of great importance in the sequel.

$$\mathbf{A} = \{\beta \ :\ \beta_* = +\infty\} \qquad \mathbf{B} = \{\beta \ :\ \beta_\rho > 0\} \qquad \mathbf{C} = \{\beta \ :\ \beta_\rho < +\infty\}$$
$$\mathbf{D} = \{\beta \ :\ \beta_n/\beta_{n+1} \downarrow 0\} \qquad \mathbf{G} = \{\beta \ :\ \mu_\beta \text{ is bounded}\}$$
$$\mathbf{H} = \{\beta \ :\ \exists\, \tau = \tau(\beta) > 0 \text{ such that } \mu_\beta(n) \downarrow \tau\}.$$

We have the following result.

Proposition 2.3. *The following are true.*

(*i*) $\mathbf{D} \subsetneq \mathbf{A}$ *but* $\mathbf{C} \nsubseteq \mathbf{A}$.
(*ii*) $\mathbf{D} \setminus \mathbf{G} \neq \varnothing$ *and* $\mathbf{G} \setminus \mathbf{D} \neq \varnothing$.
(*iii*) $\mathbf{G} \subsetneq \mathbf{B} \subsetneq \mathbf{A}$, *so* $\mathbf{B} \setminus \mathbf{D} \neq \varnothing$ *and* $\mathbf{D} \setminus \mathbf{B} \neq \varnothing$.
(*iv*) $\mathbf{H} \subsetneq \mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} \cap \mathbf{G} = \mathbf{C} \cap \mathbf{D} \cap \mathbf{G}$.
(*v*) $(\mathbf{C} \cap \mathbf{G}) \setminus (\mathbf{D} \cap \mathbf{G}) \neq \varnothing$ *and* $(\mathbf{D} \cap \mathbf{G}) \setminus (\mathbf{C} \cap \mathbf{G}) \neq \varnothing$.

Proof. • (*i*) Let $\beta_n \equiv 1$ for all n, clearly $\beta_\rho = 0 < +\infty$ but $\beta_* = 1 < +\infty$. This shows $\mathbf{C} \nsubseteq \mathbf{A}$.

To show $\mathbf{D} \subseteq \mathbf{A}$, let $\beta = (\beta_n) \in \mathbf{D}$. Since $\beta_{n+1}/\beta_n \uparrow +\infty$, there is some N_1 such that for $n > N_1$, $\beta_n/\beta_{n-1} > 2$. So $\beta_n - \beta_{n-1} > \beta_{n-1} \geq \beta_{N_1}$ $(n > N_1)$. Therefore, $\beta_n \uparrow +\infty$ for $n > N_1$. Choose $N_2 > N_1$ such that $\beta_n > 1$ for $n > N_2$.

For any $S > 0$ arbitrarily large, choose N_3 such that whenever $n \geq N_3$, $\beta_{n+1}/\beta_n > S^2$. Let $N = \max\{N_2, N_3\}$, then $\beta \in \mathbf{A}$ because for $n > 2N$,

$$\beta_n = \beta_N \prod_{p=N}^{n-1} \frac{\beta_{p+1}}{\beta_p} > 1 \cdot S^{2(n-N)} = S^{n+(n-2N)} > S^n.$$

To see why $\mathbf{D} \neq \mathbf{A}$, choose the following $\beta^{(1)}$, which is in $\mathbf{A}$ but not $\mathbf{D}$:

$$\beta_n^{(1)} = \begin{cases} n^n & \text{if } n \text{ is odd,} \\ n^{2n} & \text{if } n \text{ is even.} \end{cases}$$

• (*ii*) Choose $\beta^{(2)}$: $\beta_n^{(2)} = \sqrt{n!}$, then $\beta_n^{(2)}/\beta_{n+1}^{(2)} = (\sqrt{n+1})^{-1} \downarrow 0$, but $\mu_\beta^{(2)}(n) = \sqrt{n} \uparrow \infty$. This shows $\mathbf{D} \setminus \mathbf{G} \neq \varnothing$. Choose $\beta^{(3)} = (\beta_n^{(3)})$:

$$\beta_n^{(3)} = \begin{cases} n! & \text{if } n \text{ is odd,} \\ 2n! & \text{if } n \text{ is even} \end{cases}$$

$$\Longrightarrow\ \mu_\beta^{(3)}(n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases} \quad \text{and} \quad \frac{\beta_n^{(3)}}{\beta_{n+1}^{(3)}} = \begin{cases} \frac{1}{2(n+1)} & \text{if } n \text{ is odd,} \\ \frac{2}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

Therefore, $\beta^{(3)} \in \mathbf{G}$ but $\beta^{(3)} \notin \mathbf{D}$, which implies $\mathbf{G} \setminus \mathbf{D} \neq \varnothing$.

• (*iii*) Let $\beta \in \mathbf{B}$. There is $c > 0$ such that $\frac{\log \beta_n}{n \log n} > c$ for all n, so $\lim_{n\to\infty} \sqrt[n]{\beta_n} \geq \lim_{n\to\infty} n^c \to \infty$. This shows $\mathbf{B} \subseteq \mathbf{A}$.

Choose $\beta^{(4)}$: $\beta_0^{(4)} = \beta_1^{(4)} = \beta_2^{(4)} = 1$ and $\beta_n^{(4)} = (\log n)^n$ for $n \geq 3$. Then $\beta_*^{(4)} = +\infty$, but $\beta_\rho^{(4)} = 0$, so $\mathbf{B} \subsetneq \mathbf{A}$. Note that $\beta^{(4)} \in \mathbf{D}$, so $\mathbf{D} \nsubseteq \mathbf{B}$.

To prove $\mathbf{G} \subseteq \mathbf{B}$, let $\beta \in \mathbf{G}$. There is $S > 0$ such that $\mu_\beta(n) < S$ for all n, so

$$\beta_n > \frac{n\beta_{n-1}}{S} > \frac{n(n-1)\beta_{n-2}}{S^2} > \cdots > \frac{n!\beta_0}{S^n}.$$

Since $n! \geq n^{n/2}$, we have

$$\frac{\log \beta_n}{n \log n} > \frac{\log \beta_0 + \log n! - n \log S}{n \log n} \geq \frac{\log \beta_0 + \frac{n}{2}\log n - n \log S}{n \log n} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Hence, $\beta_\rho > 0$ as we need. Now choose $\beta = \beta^{(1)}$ as above, clearly $\beta_\rho^{(1)} = 1$, but $\mu_\beta^{(1)}(2n+1) = \left(\frac{4n^2}{2n+1}\right)^{2n}$ is not bounded, so $\beta \notin \mathbf{G}$ and $\mathbf{G} \subsetneq \mathbf{B}$. This together with $\mathbf{G} \not\subseteq \mathbf{D}$ also imply $\mathbf{B} \not\subseteq \mathbf{D}$.

Therefore, $\mathbf{G} \subsetneq \mathbf{B} \subsetneq \mathbf{A}$; $\mathbf{B} \setminus \mathbf{D} \neq \varnothing$ and $\mathbf{D} \setminus \mathbf{B} \neq \varnothing$.

• (iv) Let $\beta \in \mathbf{H}$. Since $(\mu_\beta(n))$ is convergent, it is bounded and clearly $\beta \in \mathbf{G}$. From the hypothesis $\frac{n\beta_{n-1}}{\beta_n} \downarrow \tau > 0$, we have

$$\frac{\beta_{n-1}}{\beta_n} > \frac{(n+1)\beta_n}{n\beta_{n+1}} > \frac{\beta_n}{\beta_{n+1}} \text{ and } \frac{\beta_n}{\beta_{n+1}} < \frac{n\beta_{n-1}}{(n+1)\beta_n} < \cdots < \frac{\beta_0}{(n+1)\beta_1} \to 0,$$

so $\beta \in \mathbf{D}$. Also, as $n! \leq n^n$,

$$\beta_n < \frac{n\beta_{n-1}}{\tau} < \cdots < \frac{n!\beta_0}{\tau^n},$$

which implies

$$\frac{\log \beta_n}{n \log n} < \frac{\log n! + \log \beta_0 - n \log \tau}{n \log n} \leq \frac{n \log n + \log \beta_0 - n \log \tau}{n \log n} \to 1 \quad \text{as } n \to \infty,$$

so $\beta \in \mathbf{C}$. Therefore, $\mathbf{H} \subseteq \mathbf{C} \cap \mathbf{D} \cap \mathbf{G}$. Finally, choose $\beta^{(5)}: \ \beta_0^{(5)} = 1, \ \beta_n^{(5)} = n^{2n}$ $(n > 0)$, then it is clear that $\beta^{(5)} \in \mathbf{C} \cap \mathbf{D} \cap \mathbf{G}$, but $\mu_\beta^{(5)}(n) \downarrow 0$, so $\beta \notin \mathbf{H}$. Hence, $\mathbf{H} \subsetneq \mathbf{C} \cap \mathbf{D} \cap \mathbf{G}$.

• (v) Consider $\beta = \beta^{(3)}$, then $\beta \in (\mathbf{C} \cap \mathbf{G}) \setminus (\mathbf{D} \cap \mathbf{G})$.

Also, for $\beta = \beta^{(6)}$ where $\beta_n^{(6)} = n^{n^2}$, then $\beta \in (\mathbf{D} \cap \mathbf{G}) \setminus (\mathbf{C} \cap \mathbf{G})$. □

Using Proposition 2.3, we are ready to define different types of Hilbert spaces of entire functions. Clearly, E_γ^2 is a space $\mathcal{H}(\beta_E)$ with $\beta \in \mathbf{D}$. Depending on the weights β, we denote several subclasses of $\mathcal{H}(\beta_E)$ as follows:

$$\mathcal{H}(\beta_\rho), \text{ if } \beta \in \mathbf{B}; \qquad \mathcal{H}(\beta_\rho^+), \text{ if } \beta \in \mathbf{B} \cap \mathbf{C}; \qquad \mathcal{H}(\beta_\rho^+, T), \text{ if } \beta \in \mathbf{C} \cap \mathbf{G}$$
$$E_\gamma^2(T), \text{ if } \beta \in \mathbf{D} \cap \mathbf{G}; \qquad E_\gamma^2(T, \tau), \text{ if } \beta \in \mathbf{H}.$$

Be reminded that the condition $\beta \in \mathbf{H}$ and the class $E_\gamma^2(T, \tau)$ are the main scopes of the paper [2], but $E_\gamma^2(T, \tau)$ is the smallest subclass of $\mathcal{H}(\beta_E)$ we consider in this paper. The spaces $E_\gamma^2(T, \tau)$ have been introduced in Section 1. We study spaces $\mathcal{H}(\beta_\rho)$ and $\mathcal{H}(\beta_\rho^+)$ in Section 2.1, spaces $\mathcal{H}(\beta_\rho^+, T)$ and $E_\gamma^2(T)$ in Section 2.2.

2.1. **Spaces $\mathcal{H}(\beta_\rho)$ and $\mathcal{H}(\beta_\rho^+)$.** One of the main characteristics of an entire function is its growth order. Recall (see, e.g., [6]) a power series $f(z) = \sum_{n=0}^\infty a_n z^n$ has finite order ρ if and only if $\rho = \limsup\limits_{n \to \infty} \frac{-n \log n}{\log |a_n|} < +\infty$.

Consider two properties:

(P_1) Every function $f \in \mathcal{H}(\beta_E)$ has a finite order

(P_2) There exists $g \in \mathcal{H}(\beta_E)$ that has a nonzero order.

We have the following results.

Theorem 2.4. *A Hilbert space $\mathcal{H}(\beta_E)$ has property (P_1) if and only if it is induced by $\beta \in \mathbf{B}$.*

Proof. • **Necessity:** Assume that every function in $\mathcal{H}(\beta_E)$ has a finite order but $\beta_\rho = 0$. There thus exists an increasing sequence (n_k) such that

$$\forall k, \frac{\log \beta_{n_k}}{n_k \log n_k} < \frac{1}{k} \implies \log \beta_{n_k} < \frac{n_k \log n_k}{k}.$$

Choose the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with (a_n) as follow:

$$a_n = \begin{cases} \frac{1}{n\beta_n} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Clearly, $\|f\|^2 \leq \pi^2/6 < +\infty$, and since $\lim_{k\to\infty} (\beta_{n_k})^{1/n_k} = +\infty$ (Theorem 2.1(i)),

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{k\to\infty} \frac{1}{n_k^{1/n_k} \beta_{n_k}^{1/n_k}} = \limsup_{k\to\infty} \frac{1}{\beta_{n_k}^{1/n_k}} = \lim_{k\to\infty} \frac{1}{\beta_{n_k}^{1/n_k}} = 0,$$

so $f \in \mathcal{H}(\beta_E)$. But then,

$$\limsup_{n\to\infty} \frac{-n \log n}{\log |a_n|} = \limsup_{k\to\infty} \frac{n_k \log n_k}{\log(n_k \beta_{n_k})} \geq \limsup_{k\to\infty} \frac{n_k \log n_k}{\log n_k + \frac{n_k \log n_k}{k}} = +\infty,$$

a contradiction to f having finite order.

• **Sufficiency:** Since $\beta \in \mathbf{B} \subsetneq \mathbf{A}$ (Theorem 2.3(iii)), it induces a $\mathcal{H}(\beta_E)$ space. Assume $\beta_\rho > 0$ but there is some $f(z) = \sum_{n=0}^{\infty} a_n z^n$ not having finite order in $\mathcal{H}(\beta_E)$. We can thus choose an increasing sequence (n_k) such that $|a_{n_k}| < 1$ and

$$\frac{-n_k \log n_k}{\log |a_{n_k}|} > k \implies |a_{n_k}| > n_k^{-n_k/k}.$$

From the hypothesis $\beta_\rho > 0$, there are numbers $c > 0$ and $N \in \mathbb{N}$ such that $\forall n > N$, $\frac{\log \beta_n}{n \log n} > c$, so for all $n > N$, $\beta_n > n^{cn}$.

There is a K such that $kc > 1$, $\forall k > K$, so

$$|a_{n_k}| \beta_{n_k} > n_k^{cn_k - n_k/k} = n_k^{n_k(c-1/k)} > 1.$$

This contradicts $\|f\| < +\infty$. □

Theorem 2.5. *A Hilbert space $\mathcal{H}(\beta_E)$ has property (P_2) if and only if it is induced by $\beta \in \mathbf{A} \cap \mathbf{C}$.*

Proof. Note that β induces a $\mathcal{H}(\beta_E)$ space if and only if $\beta \in \mathbf{A}$ (Theorem 2.1(i)), we only need to prove that $\mathcal{H}(\beta_E)$ has property (P_2) if and only if $\beta \in \mathbf{C}$.

• **Necessity:** Suppose $\beta_\rho = +\infty$. For any $(a_n) \in \ell^2_\beta$ that induces the power series $f \in \mathcal{H}(\beta_E)$, there is some N such that $|a_n|\beta_n < 1$ for all $n > N$. Hence,

$$\limsup_{n\to\infty} \frac{n \log n}{\log 1/|a_n|} \leq \limsup_{n\to\infty} \frac{n \log n}{\log \beta_n} = 0,$$

which shows that any $f \in \mathcal{H}(\beta_E)$ has order 0.

• **Sufficiency:** Suppose $\beta_\rho < +\infty$. Then there exist $c > 0$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $\frac{\log \beta_{n_k}}{n_k \log n_k} < c$.

Choose (a_n) as in (5), then $f \in \mathcal{H}(\beta_E)$. Moreover,

$$\limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|} = \limsup_{k\to\infty} \frac{n_k \log n_k}{\log n_k \beta_{n_k}} \geq \limsup_{k\to\infty} \frac{n_k \log n_k}{(cn_k + 1) \log n_k} = \frac{1}{c} > 0,$$

so the power series induced by this sequence is of nonzero order. □

As a consequence of Theorems 2.4 and 2.5, we have the following corollary.

Corollary 2.6. *Exactly one of the following alternative cases happens to a Hilbert space $\mathcal{H}(\beta_E)$ induced by β.*

(*i*) *Every function* $f \in \mathcal{H}(\beta_E)$ *has order* 0 *if and only if* $\beta_\rho = +\infty$, *i.e.,* $\beta \in \mathbf{B} \setminus \mathbf{C}$.
(*ii*) *There exists a function* $f \in \mathcal{H}(\beta_E)$ *that does not have finite order if and only if* $\beta_\rho = 0$, *i.e.,* $\beta \in \mathbf{C} \setminus \mathbf{B}$.
(*iii*) *Every* $f \in \mathcal{H}(\beta_E)$ *has finite order and there exists* $g \in \mathcal{H}(\beta_E)$ *having positive order if and only if* $\beta_\rho \in (0, +\infty)$, *i.e.* $\beta \in \mathbf{B} \cap \mathbf{C}$.

By Corollary 2.6, a space $\mathcal{H}(\beta_\rho)$ is a Hilbert space of entire functions of finite orders, while a space $\mathcal{H}(\beta_\rho^+)$ is a space $\mathcal{H}(\beta_\rho)$ having at least one function of a positive order. Therefore, $\mathcal{H}(\beta_\rho)$ is a special case of $\mathcal{H}(\beta_E)$, $\mathcal{H}(\beta_\rho^+)$ is a special case of $\mathcal{H}(\beta_\rho)$ and $\mathcal{H}(\beta_\rho^+, T)$ is a special case of $\mathcal{H}(\beta_\rho^+)$.

Remark 2.7. It is clear that any space $E_\gamma^2(T, \tau)$ is a space $\mathcal{H}(\beta_\rho^+)$ (Proposition 2.3 (*iv*)), and any space F_α^2 is a space $\mathcal{H}(\beta_\rho^+)$ with $\beta_n = \sqrt{\alpha^{-n} n!}$. Hence both $E_\gamma^2(T, \tau)$ and F_α^2 belong to the same class $\mathcal{H}(\beta_\rho^+)$. However, while $E_\gamma^2(T, \tau)$ also belongs to the subclass $\mathcal{H}(\beta_\rho^+, T)$, it is not true for F_α^2 (as $\mu_\beta(n) = \sqrt{\alpha n} \uparrow +\infty$).

2.2. **Spaces** $E_\gamma^2(T)$ **and** $\mathcal{H}(\beta_\rho^+, T)$**.** We have the following important remark.

Remark 2.8. Provided $\beta \in \mathbf{A}$, for any $(a_n) \in \ell_\beta^2$, the series $f(z) = \sum_{n=0}^\infty a_n z^n$ is absolutely convergent for any $z \in \mathbb{C}$ by Theorem 2.1. Hence, $\forall\ w \in \mathbb{C}$,

$$\begin{aligned} f(z+w) &= \sum_{n=0}^\infty a_n (z+w)^n = \sum_{n=0}^\infty a_n \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} \\ &= \sum_{k=0}^\infty \left[\sum_{n=k}^\infty \binom{n}{k} a_n w^{n-k} \right] z^k = \sum_{k=0}^\infty \frac{z^k}{k!} \sum_{n=k}^\infty \frac{n!}{(n-k)!} a_n w^{n-k} = \sum_{k=0}^\infty \frac{z^k}{k!} f^{(k)}(w). \end{aligned}$$

For each $n \in \mathbb{N}$, define $h_n(z) = \beta_n^{-1} z^n$. Clearly, $h_n \in \mathcal{H}(\beta_E)$ and has norm $\|h_n\| = 1$ and growth order 0. The sequence (h_n) forms an orthonormal basis for $\mathcal{H}(\beta_E)$ and is crucial for the study of $\mathcal{H}(\beta_E)$ and its composition operators.

The class $E_\gamma^2(T)$ was introduced in [3], when the authors investigated the cyclic behavior of the translation operators. It is noteworthy that for an $E_\gamma^2(T)$ space, any operator T_b ($b \in \mathbb{C}$) acting on it is bounded. With an appropriate modification, the condition $\beta \in \mathbf{D}$ can be omitted and the following results, which generalize the corresponding results in [3], still hold.

Proposition 2.9. *Let* D *be the derivative operator acting on a space* $\mathcal{H}(\beta_E)$, *i.e.* $D(f) = f'$, $f \in \mathcal{H}(\beta_E)$ *and* $b \in \mathbb{C} \setminus \{0\}$. *The following are equivalent*
(*i*) D *is bounded on* $\mathcal{H}(\beta_E)$.
(*ii*) *The translation operator* T_b *is bounded on* $\mathcal{H}(\beta_E)$.
(*iii*) *The function* μ_β *is bounded.*

Proof. • (*i*) $\Longrightarrow$ (*ii*): Suppose D is bounded. There is $M > 0$ such that $\|D(f)\| \le M\|f\|$, $f \in \mathcal{H}(\beta_E)$. Remark 2.8 shows that

$$f(z+b) = \sum_{k=0}^\infty \frac{b^k}{k!} f^{(k)}(z), \quad \text{and so} \quad T_b(f) = \sum_{k=0}^\infty \frac{b^k}{k!} D^k(f).$$

Therefore,

$$\|T_b(f)\| \le \sum_{n=0}^\infty \frac{|b|^k}{k!} \|D^k(f)\| \le \|f\| \sum_{n=0}^\infty \frac{|b|^k}{k!} M^k = \|f\| e^{M|b|},$$

which shows that T_b is bounded.

• $(ii) \implies (iii)$: Suppose (ii) holds. Then there is $M > 0 : \|T_b(f)\| \le M\|f\|$, for all $f \in \mathcal{H}(\beta_E)$. In particular, for $f = h_n$, we have $\|T_b(h_n)\| \le M$. Hence,

$$|b|^2\mu_\beta(n)^2 = |b|^2\frac{n^2\beta_{n-1}^2}{\beta_n^2} \le \|T_b(h_n)\|^2 = \sum_{k=0}^{n}|b|^{2k}\binom{n}{k}^2\frac{\beta_{n-k}^2}{\beta_n^2} \le M^2,$$

so μ_β is bounded by $M/|b|$.
• $(iii) \implies (i)$: Suppose there is $M > 0$ such that $\mu_\beta(n) \le M$ for all $n \ge 0$. For any $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}(\beta_E)$, we have

$$\|D(f)\|^2 = \sum_{n=1}^\infty n^2|a_n|^2\beta_{n-1}^2 = \sum_{n=1}^\infty \mu_\beta(n)^2|a_n|^2\beta_n^2 \le M^2\|f\|^2.$$

Therefore, D is also bounded. □

Note that from Proposition 2.9, it follows that $\|D\|_{op} = \sup\{\mu_\beta(n),\ n \in \mathbb{N}\}$, where the notation $\|\cdot\|_{op}$ denotes the operator norm.

Proposition 2.10. *The derivative operator D acting on a Hilbert space $\mathcal{H}(\beta_E)$ is compact if and only if $\lim_{n\to\infty}\mu_\beta(n) = 0$.*

Proof. The proof makes use of the following well-known result (see, e.g. [8]): A linear operator L acting on a Hilbert space H is compact if and only if for every (x_n) that converges to 0 weakly, $(L(x_n))$ converges to 0 strongly.

• **Necessity:** Suppose D is compact on $\mathcal{H}(\beta_E)$, let $g(z) = \sum_{n=0}^\infty b_n z^n \in \mathcal{H}(\beta_E)$, then $\sum_{n=0}^\infty |b_n|^2\beta_n^2 < \infty$ and so $\lim_{n\to\infty}|b_n|\beta_n = 0$.

We have $\lim_{n\to\infty}\langle h_n, g\rangle = \lim_{n\to\infty}\overline{b_n}\beta_n = 0$, so h_n converges weakly to 0. The compactness of D implies

$$\lim_{n\to\infty}\|D(h_n)\| = \lim_{n\to\infty}\|nz^{n-1}/\beta_n\| = \lim_{n\to\infty}\mu_\beta(n) = 0.$$

• **Sufficiency:** Suppose $\mu_\beta(n) \to 0$ as $n \to \infty$. Let (f_m) be an arbitrary sequence in $\mathcal{H}(\beta_E)$ that converges weakly to 0, where $f_m(z) = \sum_{n=0}^\infty a_n^{(m)} z^n$. By Principle of uniform boundedness, there exists $A > 0$ such that $\|f_m\| \le A$. We prove that $D(f_m)$ converges strongly to 0.

For any $\varepsilon > 0$, there is $N \in \mathbb{N}$: for $n > N$, $\mu_\beta(n) < (A\sqrt{2})^{-1}\varepsilon$. We have

$$\begin{aligned}\|D(f_m)\|^2 &= \sum_{n=1}^\infty \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 = \sum_{n=1}^N \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 + \sum_{n=N+1}^\infty \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 \\ &\le \sum_{n=1}^N \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 + \frac{\varepsilon^2}{2A^2}\|f_m\|^2 \le \sum_{n=1}^N \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 + \frac{\varepsilon^2}{2}. \qquad (6)\end{aligned}$$

Let U be an upper bound of μ_β. Since (f_m) is weakly convergent to 0, we have $\lim_{m\to\infty}\langle f_m, h_n\rangle = \lim_{m\to\infty} a_n^{(m)}\beta_n = 0$ for each n. Thus, for each $n \in \{1, \dots, N\}$ there exists M_n such that

$$\forall m > M_n,\ |a_n^{(m)}|\beta_n < \frac{\varepsilon}{U\sqrt{2N}}.$$

Hence, for $m > K := \max\{M_1, \dots, M_N\}$,

$$\sum_{n=1}^N \mu_\beta(n)^2|a_n^{(m)}|^2\beta_n^2 < \sum_{n=1}^N U^2\frac{\varepsilon^2}{2NU^2} = \frac{\varepsilon^2}{2}. \qquad (7)$$

From (6) and (7), for $m > K$, we have $\|D(f_m)\| < \varepsilon$. □

Since $\mathcal{H}(\beta_\rho^+, T)$ is a space $\mathcal{H}(\beta_E)$ with $\beta \in \mathbf{C} \cap \mathbf{G}$, as a consequence of Proposition 2.9, any translation operator acting on it is invariant and bounded. Moreover, $E^2_\gamma(T, \tau)$ is a special case of $\mathcal{H}(\beta_\rho^+, T)$.

Denote by $\{\mathcal{H}(\beta_E)\}$ the set of all Hilbert spaces $\mathcal{H}(\beta_E)$, and similarly to the notations $\{\mathcal{H}(\beta_\rho)\}$, $\{\mathcal{H}(\beta_\rho^+)\}$, $\{\mathcal{H}(\beta_\rho^+, T)\}$, $\{E^2_\gamma\}$, $\{E^2_\gamma(T)\}$ and $\{E^2_\gamma(T, \tau)\}$. Note that by Proposition 2.3 (v), the two classes $\{E^2_\gamma(T)\}$ and $\{\mathcal{H}(\beta_\rho^+, T)\}$ do not contain each other. The following inclusion–exclusion chains, as a corollary of Proposition 2.3, summarize what we have discussed in Section 2.

Corollary 2.11. *The following exclusion–inclusion diagram is true.*

$$\begin{array}{ccccccc} \{E^2_\gamma(T,\tau)\} & & & \subsetneq & \{E^2_\gamma(T)\} & \subsetneq & \{E^2_\gamma\} \\ \subsetneq & & & & \subsetneq & & \subsetneq \\ \{\mathcal{H}(\beta_\rho^+, T)\} & \subsetneq & \{\mathcal{H}(\beta_\rho^+)\} & \subsetneq & \{\mathcal{H}(\beta_\rho)\} & \subsetneq & \{\mathcal{H}(\beta_E)\}. \end{array}$$

3. Composition Operators on $\mathcal{H}(\beta_\rho^+)$ and $\mathcal{H}(\beta_\rho^+, T)$

Let $\mathcal{H}(\beta_E)$ be a Hilbert space induced by $\beta = (\beta_n)$ and φ be an entire function. Define the composition operator C_φ, induced by φ, acting on $\mathcal{H}(\beta_E)$ as

$$C_\varphi(f) = f \circ \varphi, \ f \in \mathcal{H}(\beta_E).$$

Properties such as invariance, boundedness, compactness, closed range, essential norms, etc. are usual topics in study of composition operators (see, e.g., [4]). Although the criteria for such properties in the most general case $\beta \in \mathbf{A}$ are still open problems, there have been studies for the specific case $\beta \in \mathbf{H}$.

For any space $E^2_\gamma(T, \tau)$, i.e. space $\mathcal{H}(\beta_E)$ with $\beta \in \mathbf{H}$, the criteria for the boundedness and compactness of C_φ have been discovered in [2]. Note that with the techniques of proof used in that paper, one must utilize the hypothesis $\mu_\beta(n) \downarrow \tau > 0$.

A natural question arises: how about a more general case when β is in some superset of $\mathbf{H}$?

This section generalizes these results of [2] above to the spaces $\mathcal{H}(\beta_\rho^+)$ (i.e., $\beta \in \mathbf{B} \cap \mathbf{C}$) and $\mathcal{H}(\beta_\rho^+, T)$ (i.e., $\beta \in \mathbf{C} \cap \mathbf{G}$). Our approach to the problem undoubtedly do not consider the existence of τ.

3.1. Invariance and boundedness.

We first note that the concepts of invariance and boundedness are different. A linear operator L acting on a normed space X is invariant if L is a self-map, i.e. $L(X) \subseteq X$. Thus while boundedness implies invariance, the reverse is not always true. However, in case X is a functional Banach space, these two properties are equivalent (see, e.g., [4]).

Since any space $\mathcal{H}(\beta_E)$ is a functional Banach space, C_φ is invariant if and only if it is bounded on $\mathcal{H}(\beta_E)$.

The following key result about composition of entire functions is due to Pólya.

Lemma 3.1 ([7])**.** *Let g and h be entire functions such that $f = g \circ h$ is an entire function of finite order. Then either*
(1) h is a polynomial and g is of finite order, or
(2) h is not a polynomial, but a function of finite order, and g is of order 0.

We have the following result.

Theorem 3.2. *If a composition operator C_φ, induced by an entire function φ, is bounded on $\mathcal{H}(\beta_\rho^+)$, then $\varphi(z) = az + b$ $(a, b \in \mathbb{C})$ and $|a| \le 1$.*

Proof. Let C_φ be bounded. By Corollary 2.6 there is some $f \in \mathcal{H}(\beta_\rho^+)$ with positive growth order, so Case (2) of Lemma 3.1 cannot occur, φ must be a polynomial. Hence $\varphi(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0$, where $b_m \neq 0$.

Since C_φ is bounded, there exists some $M>0$ such that, for all $n\in\mathbb{N}$

$$\|C_\varphi(h_n)\|\le M\|h_n\|=M \implies \left\|\frac{1}{\beta_n}(b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0)^n\right\|\le M.$$

The leading coefficient of $(b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0)^n$ is b_m^n (of the term z^{mn}). Hence, for all $n\in\mathbb{N}$,

$$\frac{|b_m|^n\beta_{mn}}{\beta_n}=\sqrt{\frac{|b_m|^{2n}\beta_{mn}^2}{\beta_n^2}}\le\left\|\frac{1}{\beta_n}(b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0)^n\right\|\le M. \tag{8}$$

We prove by contradiction that $m\le 1$. Assume $m\ge 2$, (8) is equivalent to

$$\sqrt[n]{\frac{\beta_{mn}}{\beta_n}}\le\frac{\sqrt[n]{M}}{|b_m|},\ \forall n\in\mathbb{N}.$$

Since $\lim_{n\to\infty}\sqrt[n]{M}=1$, $\left(\sqrt[n]{\beta_{mn}/\beta_n}\right)_{n=1}^\infty$ is bounded, by $S=\sup\{\sqrt[n]{M}/|b_m|,\ n\in\mathbb{N}\}$. We can assume $S>1$. Then,

$$\sqrt[n]{\frac{\beta_{mn}}{\beta_n}}\le S,\ \sqrt[mn]{\frac{\beta_{m^2n}}{\beta_{mn}}}\le S,\ \sqrt[m^2n]{\frac{\beta_{m^3n}}{\beta_{m^2n}}}\le S,\dots$$

Thus for any $n,k\in\mathbb{N}$,

$$\frac{\beta_{m^kn}}{\beta_n}=\prod_{p=1}^k\frac{\beta_{m^pn}}{\beta_{m^{p-1}n}}\le\prod_{p=1}^kS^{m^{p-1}n}=S^{n\frac{m^k-1}{m-1}}. \tag{9}$$

Since $\lim_{n\to\infty}\sqrt[n]{\beta_n}=+\infty$, we can find N sufficiently large such that $\sqrt[N]{\beta_N}>S$. Fix N and regard $\sqrt[N]{\beta_N}$ as a constant, we can find N' such that $\sqrt[n]{\beta_n}>\sqrt[N]{\beta_N}$ for any $n>N'$. Choose $K: m^KN>N'$, then

$$\sqrt[m^KN]{\beta_{m^KN}}>\sqrt[N]{\beta_N}.$$

Therefore,

$$\frac{\beta_{m^KN}}{\beta_N}>\frac{\beta_N^{m^K}}{\beta_N}=\beta_N^{m^K-1}>S^{N(m^K-1)}\ge S^{N\frac{m^K-1}{m-1}}\quad(\text{since } m\ge 2),$$

which clearly contradicts (9). Hence, $m=1$ and the statement (8) simply becomes "for some $M>0$, $|b_1|^n\le M$ for all n", but this happens if and only if $|b_1|\le 1$. Therefore, $\varphi(z)=az+b$ for some $a\in\mathbb{C}$ with $|a|\le 1$. □

Remark 3.3. In the proof of the necessity for boundedness of composition operators on $E^2_\gamma(T,\tau)$ obtained in [2], the authors essentially followed the hypothesis $\mu_\beta(n)\downarrow\tau>0$, i.e. $\beta\in\mathbf{H}$, to prove that $\varphi(z)=az+b$. This result is similar to our Theorem 3.2, but our method of proof can be applied to any space $\mathcal{H}(\beta_\rho^+)$.

Example 3.4. Recall $\beta=\beta^{(3)}$ in the proof of Proposition 2.3. We have $\beta_\rho\le 1$, $\mu_\beta(n)\le 2$ and β is not decreasing. Thus $\beta\in(\mathbf{C}\cap\mathbf{G})\setminus\mathbf{H}$, so it is a $\mathcal{H}(\beta_\rho^+)$ space (in fact, a $\mathcal{H}(\beta_\rho^+,T)$ space), but not an $E^2_\gamma(T,\tau)$ space. While the results from [2] cannot be applied, Theorem 3.2 still holds, i.e. any operator C_φ bounded on this $\mathcal{H}(\beta_\rho^+)$ space must be induced by some $\varphi(z)=az+b$ with $|a|\le 1$.

We have obtained the necessity for boundedness of C_φ on any space $\mathcal{H}(\beta_\rho^+)$. Particularly for the case $\mathcal{H}(\beta_\rho^+,T)$, the reverse is also true.

Theorem 3.5. *Let C_φ be a composition operator induced by some entire function φ acting on $\mathcal{H}(\beta_\rho^+,T)$. If $\varphi(z)=az+b$ $(a,b\in\mathbb{C})$ where $|a|\le 1$, then C_φ is bounded on $\mathcal{H}(\beta_\rho^+,T)$.*

Proof. Suppose first $b = 0$, then $\varphi(z) = az$, with $|a| \le 1$. For any function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta_\rho^+, T)$, we have $\|C_\varphi(f)\|^2 = \sum_{n=0}^{\infty} |a_n|^2 |a|^{2n} \beta_n^2 \le \|f\|^2$, and so C_{az} is bounded.

In the case $\varphi(z) = az + b$ with $b \ne 0$, Proposition 2.9 shows that T_b is bounded, while C_{az} is bounded as proven above. The boundedness of C_φ is then obvious since $C_\varphi = C_{az} \circ T_b$. □

3.2. **Essential norm and compactness.** The following necessary condition for the compactness of composition operators on $\mathcal{H}(\beta_\rho^+)$ is easy to verify.

Theorem 3.6. *If a composition operator C_φ induced by an entire function φ is compact on $\mathcal{H}(\beta_\rho^+)$, then $\varphi(z) = az + b$, with $|a| < 1$.*

Proof. Since C_φ is compact, it is necessarily bounded, and hence $\varphi(z) = az + b$ with $|a| \le 1$ by Proposition 3.2.

Since (h_n) converges weakly to 0, $C_\varphi(h_n)$ converges strongly to 0. Moreover,

$$|a|^{2n} \le \sum_{k=0}^{n} \binom{n}{k}^2 |b|^{2(n-k)} |a|^{2k} \frac{\beta_k^2}{\beta_n^2} = \|C_\varphi(h_n)\|^2.$$

Consequently, $\lim_{n\to\infty} |a|^{2n} = 0$, which implies that $|a| < 1$. □

Similarly to the result of boundedness, particularly for the space $\mathcal{H}(\beta_\rho^+, T)$, the necessity for the compactness of an operator C_φ is also its sufficiency. This can be derived from a study of essential norm of those operators.

Let X be a Banach space and $\mathcal{K}(X)$ be the set of all compact operators on X. The essential norm of a bounded linear operator L on X, denoted as $\|L\|_e$, is defined as

$$\|L\|_e = \inf\{\|L - K\|_{op} : K \in \mathcal{K}(X)\}.$$

Clearly, L is compact if and only if $\|L\|_e = 0$.

The following simple lemma is needed.

Lemma 3.7. *For each $p \in \mathbb{N}$, the linear operator*

$$K_p : f(z) = \sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{p} a_n z^n$$

is compact on $\mathcal{H}(\beta_E)$.

Put $0 \le \mathfrak{D}_* := \inf\{\mu_\beta(n),\ n \in \mathbb{N}\} \le \mathfrak{D}^* := \sup\{\mu_\beta(n),\ n \in \mathbb{N}\} < \infty$. We have the following theorem.

Theorem 3.8. *Let C_φ be a bounded composition operator on $\mathcal{H}(\beta_\rho^+, T)$ induced by $\varphi(z) = az + b$. Then*

$$\|C_\varphi\|_e \simeq \lim_{n\to\infty} |a|^n.$$

More precisely,

$$\sqrt{|a|^2 + |b|^2 \mathfrak{D}_*^2} \lim_{n\to\infty} |a|^n \le \|C_\varphi\|_e \le e^{|b|\mathfrak{D}^*} \lim_{n\to\infty} |a|^n.$$

Proof. We note that $\|D\|_{op} = \mathfrak{D}^*$. Also, since $|a| \le 1$,

$$\inf_{n\in\mathbb{N}} |a|^n = \lim_{n\to\infty} |a|^n = \begin{cases} 0 & \text{if } |a| < 1, \\ 1 & \text{if } |a| = 1. \end{cases}$$

• **Lower bound:** Take an arbitrary $\varepsilon > 0$. Consider the sequence (h_n) in Section 2.2, with the note that $\|h_n\| = 1$ and (h_n) weakly converges to 0. By definition of essential norm, there exists a compact operator K such that for all $n \in \mathbb{N}$,

$$\|C_\varphi\|_e \ge \|C_\varphi - K\|_{op} - \frac{\varepsilon}{2} \ge \|(C_\varphi - K)(h_n)\| - \frac{\varepsilon}{2} \ge \|C_\varphi(h_n)\| - \|K(h_n)\| - \frac{\varepsilon}{2}.$$

Moreover, since (h_n) weakly converges to 0 and K is compact, $\lim_{n\to\infty} \|K(h_n)\| = 0$. Hence, there is $N \in \mathbb{N}$ such that $\|K(h_n)\| \leq \frac{\varepsilon}{2}$, $\forall n \geq N$. Thus, for $n \geq N$,

$$\|C_\varphi\|_e \geq \|C_\varphi(h_n)\| - \varepsilon.$$

We also have

$$\begin{aligned}\|C_\varphi(h_n)\| &= \sqrt{\frac{1}{\beta_n^2}\sum_{k=0}^{n}\binom{n}{k}^2 |a|^{2k}|b|^{2(n-k)}\beta_k^2} \geq \sqrt{|a|^{2n} + |a|^{2(n-1)}|b|^2\frac{n^2\beta_{n-1}^2}{\beta_n^2}} \\ &= |a|^{n-1}\sqrt{|a|^2 + |b|^2\mu_\beta(n)^2} \geq |a|^{n-1}\sqrt{|a|^2 + |b|^2\mathfrak{D}_*^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|C_\varphi\|_e &\geq \inf_{n\geq N}\|C_\varphi(h_n)\| - \varepsilon \geq \inf_{n\geq N}(|a|^{n-1}\sqrt{|a|^2 + |b|^2\mathfrak{D}_*^2}) - \varepsilon \\ &= \sqrt{|a|^2 + |b|^2\mathfrak{D}_*^2}\inf_{n\geq N}|a|^{n-1} - \varepsilon = \sqrt{|a|^2 + |b|^2\mathfrak{D}_*^2}\lim_{n\to\infty}|a|^n - \varepsilon.\end{aligned}$$

Since ε is arbitrarily small, we obtain the desired lower bound

$$\|C_\varphi\|_e \geq \sqrt{|a|^2 + |b|^2\mathfrak{D}_*^2}\lim_{n\to\infty}|a|^n.$$

• **Upper bound:** By Lemma 3.7, each operator K_p is compact and so is $K_p \circ C_\varphi$. Therefore,

$$\|C_\varphi\|_e \leq \inf_{p\in\mathbb{N}}\|C_\varphi - K_p \circ C_\varphi\|.$$

For any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\beta_\rho^+, T)$, by Remark 2.8, we have

$$\begin{aligned}\|(C_\varphi - K_p \circ C_\varphi)(f)\| &= \Big\|\sum_{k=p+1}^{\infty} a^k z^k\Big(\sum_{n=k}^{\infty}\binom{n}{k}a_n b^{n-k}\Big)\Big\| \\ &\leq |a|^{p+1}\Big\|\sum_{k=p+1}^{\infty} z^k\sum_{n=k}^{\infty}\binom{n}{k}a_n b^{n-k}\Big\| = |a|^{p+1}\Big\|\sum_{m=0}^{\infty}\frac{b^m}{m!}\sum_{k=p+1}^{\infty}\frac{(m+k)!}{k!}a_{m+k}z^k\Big\| \\ &\leq |a|^{p+1}\sum_{m=0}^{\infty}\frac{|b|^m}{m!}\Big\|\sum_{k=p+1}^{\infty}\frac{(m+k)!}{k!}a_{m+k}z^k\Big\| \\ &\leq |a|^{p+1}\sum_{m=0}^{\infty}\frac{|b|^m}{m!}\Big\|\sum_{k=0}^{\infty}\frac{(m+k)!}{k!}a_{m+k}z^k\Big\| = |a|^{p+1}\sum_{m=0}^{\infty}\frac{|b|^m}{m!}\|D^m(f)\| \\ &\leq |a|^{p+1}\|f\|e^{|b|\mathfrak{D}^*}\end{aligned}$$

Hence, $\forall p \in \mathbb{N}$, $\|C_\varphi - K_p \circ C_\varphi\| \leq |a|^{p+1}e^{|b|\mathfrak{D}^*}$. □

Corollary 3.9. *Let C_φ be a composition operator on $\mathcal{H}(\beta_\rho^+, T)$ induced by some $\varphi(z) = az + b$. Then C_φ is compact if and only if $|a| < 1$.*

Remark 3.10. A comparable result in [2] states that if $\varphi(z) = az + b$ with $|a| < 1$, then the composition operator C_φ acting on $E_\gamma^2(T, \tau)$ is compact. However, in the proof of this statement, the authors relied on the hypothesis $\mu_\beta(n) \downarrow \tau > 0$. They introduced the Hilbert space $\mathfrak{H}_{W_\tau}$ of all entire functions f satisfying

$$\|f\|_W^2 = \int_C |f(z)|^2 W_\tau(|z|)dA(z) < \infty,$$

where $W_\tau(z) = e^{-2\tau z}$. The proof is then complicated and is only applicable to the case $\beta \in \mathbf{H}$. Theorem 3.8 and Proposition 3.9, however, can be derived from fundamental results in functional analysis, without taking τ into consideration.

3.3. **Compact difference of two composition operators on $\mathcal{H}(\beta_\rho^+, T)$.** In this section, we study the compactness of the difference of two bounded composition operators acting on $\mathcal{H}(\beta_\rho^+, T)$.

Let C_φ and C_ψ be induced by $\varphi(z) = az + b$ and $\psi(z) = cz + d$, respectively ($|a| \le 1$, $|c| \le 1$). Since the translation operator T_d is bounded on $\mathcal{H}(\beta_\rho^+, T)$ and

$$C_\varphi - C_\psi = (C_{az+b-d} - C_{cz}) \circ T_d,$$

the difference $C_\varphi - C_\psi$ is compact if and only if $C_{az+b-d} - C_{cz}$ is compact (because T_d, which is precisely C_{z+d}, is not compact, by Corollary 3.9). Therefore, we can assume, without the loss of generality, that $d = 0$.

We have the following result.

Proposition 3.11. *Let C_φ and C_ψ be bounded composition operators on $\mathcal{H}(\beta_\rho^+, T)$ induced by $\varphi(z) = az + b$ and $\psi(z) = cz$ respectively. If the difference $C_\varphi - C_\psi$ is compact, then exactly one of the following holds:*

(*i*) *both C_φ and C_ψ are compact (i.e., $|a| < 1, |c| < 1$);*
(*ii*) *both C_φ and C_ψ are not compact (i.e., $|a| = |c| = 1$) and $a = c$.*

Proof. Suppose that $C_\varphi - C_\psi$ is compact. Then either both C_φ and C_ψ are compact, or both C_φ and C_ψ are not compact.

For the latter case, we have $|a| = |c| = 1$. Assume on the contrary that $a \neq c$. Consider the sequence (h_n) in Section 2.2, which is weakly convergent to 0, then $(C_\varphi - C_\psi)(h_n)$ strongly converges to 0 as $n \to \infty$. Since

$$\begin{aligned}\|(C_\varphi - C_\psi)(h_n)\|^2 &= \frac{1}{\beta_n^2}\left\|(a^n - c^n)z^n + \sum_{k=0}^{n-1}\binom{n}{k}(az)^k b^{n-k}\right\|^2 \\ &= |a^n - c^n|^2 + \sum_{k=0}^{n-1}\binom{n}{k}^2 |a|^{2k}|b|^{2(n-k)}\frac{\beta_k^2}{\beta_n^2} \ge |a^n - c^n|^2,\end{aligned}$$

we have,

$$\lim_{n\to\infty} |a^n - c^n| = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} |1 - y^n| = 0, \ y = \frac{c}{a} \neq 1.$$

Hence, for $\varepsilon = |1 - y|/2 > 0$, there is $N > 0 : |1 - y^n| < \varepsilon/2, \ \forall \ n \ge N$. Then,

$$2\varepsilon = |1 - y| = |y^{N+1} - y^N| \le |1 - y^{N+1}| + |1 - y^N| < \varepsilon, \quad \text{(since } |y| = 1)$$

which is impossible. The proof is completed. □

The case (ii) in Proposition 3.11 leads to the following result.

Proposition 3.12. *Suppose $|a| = 1$ and $b \neq 0$, the difference $C_{az+b} - C_{az}$ is compact if and only if $\lim_{n\to\infty} \mu_\beta(n) = 0$.*

Proof. • **Necessity:** Suppose that $C_{az+b} - C_{az}$ is compact. As in the proof of Proposition 3.11, since $|a| = 1$,

$$\|(C_{az+b} - C_{az})(h_n)\|^2 = \sum_{k=0}^{n-1}\binom{n}{k}^2 |a|^{2k}|b|^{2(n-k)}\frac{\beta_k^2}{\beta_n^2} \ge (|b|\mu_\beta(n))^2.$$

Since $(C_{az+b} - C_{az})(h_n)$ converges strongly to 0, $\lim_{n\to\infty} |b|\mu_\beta(n) = 0$, which gives $\lim_{n\to\infty} \mu_\beta(n) = 0$.

• **Sufficiency:** Suppose $\mu_\beta(n)$ converges to 0, the derivative operator D is then compact (Proposition 2.10). Thus, for any sequence (f_m) converging weakly to 0, $\|D(f_m)\|$ converges to 0.

Let $f_m(z)=\sum_{n=0}^{\infty}a_n^{(m)}z^n$ $(m\in\mathbb{N})$. From Remark 2.8, since $|a|=1$, we have

$$\begin{aligned}
&\|(C_{az+b}-C_{az})(f_m)\| \\
&=\Big\|\sum_{n=0}^{\infty}a_n^{(m)}\sum_{k=0}^{n}\binom{n}{k}(az)^{n-k}b^k-\sum_{n=0}^{\infty}a_n^{(m)}(az)^n\Big\| \\
&=\Big\|\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}a_n^{(m)}(az)^{n-k}b^k\Big\|=\Big\|\sum_{k=1}^{\infty}\frac{b^k}{k!}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n^{(m)}(az)^{n-k}\Big\| \\
&\le\sum_{k=1}^{\infty}\frac{|b|^k}{k!}\Big\|\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n^{(m)}(az)^{n-k}\Big\|=\sum_{k=1}^{\infty}\frac{|b|^k}{k!}\sqrt{\sum_{n=k}^{\infty}\frac{n!^2}{(n-k)!^2}|a_n^{(m)}|^2\beta_{n-k}^2} \\
&=\sum_{k=1}^{\infty}\frac{|b|^k}{k!}\|D^k(f_m)\|\le\|D(f_m)\|\sum_{k=1}^{\infty}\frac{|b|^k}{k!}\|D\|_{op}^{k-1}\le\|D(f_m)\|\frac{e^{|b|\mathfrak{D}^*}}{\mathfrak{D}^*},
\end{aligned}$$

where $\mathfrak{D}^*=\|D\|_{op}=\sup\{\mu_\beta(n),\ n\in\mathbb{N}\}$.

Consequently, $\lim\limits_{m\to\infty}\|(C_{az+b}-C_a)(f_m)\|=0$, so $C_{az+b}-C_{az}$ is compact. □

Finally, as consequences of Propositions 3.11 – 3.12, we obtain the following criteria about the compactness of the difference of two bounded composition operators C_φ and C_ψ on $\mathcal{H}(\beta_\rho^+,T)$.

Theorem 3.13. *Let C_φ and C_ψ be two bounded composition operators on $\mathcal{H}(\beta_\rho^+,T)$ induced by $\varphi(z)=az+b$ and $\psi(z)=cz+d$ respectively. The difference $C_\varphi-C_\psi$ is compact if and only if either of the following conditions holds:*

(i) both C_φ and C_ψ are compact (i.e. $|a|<1,\ |c|<1$);

(ii) $|a|=|c|=1$ *and* $\begin{cases}a=c, & \text{if } \lim\limits_{n\to\infty}\mu_\beta(n)=0\\ a=c,\ b=d, & \text{otherwise.}\end{cases}$

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References

[1] B. J. Carswell, B. D. MacCluer, and A. Schuster, *Composition operators on the Fock space*, Acta Sci. Math. (Szeged) **69** (2003), no. 3–4, 871–887.

[2] G. A. Chacón, G. R. Chacón, and J. Giménez, *Composition operators on spaces of entire functions*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2205–2218.

[3] K. C. Chan and J. H. Shapiro, *The cyclic behaviour of translation operators on Hilbert spaces of entire functions*, Indiana Univ. Math. J. **40** (1991), no. 4, 1421–1449.

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[5] M. L. Doan and L. H. Khoi, *Composition operators on Hilbert spaces of entire functions*, C. R. Math. Acad. Sci. Paris. Ser I **353** (2015), no. 6, 495–499.

[6] B. Ya. Levin, *Lectures on Entire Functions*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI., 1996.

[7] G. Pólya, *On an integral function of an integral function*, J. Lond. Math. Soc. **1** (1926), 12–15.

[8] K. Zhu, *Operator Theory in Function Spaces*, Amer. Math. Soc., Providence, RI., 1990.

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