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Product Damage and Free Sampling: A Newsvendor Model with Passive and Proactive Self-Consumption

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A retailer sells a single product for a single period. During transportation and storage, some of these products are consumed by the retailer either (1) due to unavoidable damages (*passive self-consumption*), or (2) distributed for free to the customers (*proactive self-consumption*). This creates a mismatch between the amount purchased by the retailer and the amount available for sale. We study passive self-consumption with (i) fixed and (ii) proportional consumption, and proactive self-consumption with (iii) additive and (iv) multiplicative demand. Under proactive self-consumption, the retailer holds more inventory and receives a higher profit; the reverse is true under passive self-consumption. Yet, (i), (iii) and (iv) result in a higher order quantity and same fill rate compared to no self-consumption, (ii) may result in a higher or lower order quantity with a lower fill rate. When both types of self-consumption co-exist, the optimal policy can be complicated. We characterize the optimal policy and show through numerical studies that the optimal policy can take at most three formats: sell to the market with positive proactive self-consumption, sell to the market with zero proactive self-consumption and do not sell to the market. Interestingly, the optimal order quantity is not smooth in the fraction of the proportional self-consumption. Further we find that when the market adoption rate is uncertain, the optimal strategy preserves a similar structure. The retailer benefits from expediting if the difference between the high and the low market adoption rates is high and the probability of a high market adoption rate is low.

Keywords: Newsvendor model, self-consumption, optimal order quantity, additive and multiplication demand

1 Introduction

In many industries, retailers typically consume a certain amount of product prior to demand realization (e.g. see Lee et al. 1997). We refer to this practice by the retailers as *self-consumption*. There are two types of self-consumption: *passive self-consumption* and *proactive self-consumption*. Under passive self-consumption, some of the goods are damaged and cannot be sold to the market due to the retailer's inefficiency in logistics or storage. For example, in the agricultural industry, according to Kader (2005), post-harvest losses are significant and unavoidable. It is estimated that “In the United States, the losses of fresh fruits and vegetables are estimated to range from 2% to

23%” and that “*post-harvest losses in developing countries vary greatly from 1% to 50% or even higher*”. Alternatively, under proactive self-consumption, the retailer uses a certain amount of the product to increase demand. For example, many companies in the food, appliance and cosmetic sectors offer free samples to attract more customers (see Lammers 1991). For instance, Sensodyne offers free samples of toothpaste to dental patients to increase the visibility of its product and Costco offers free samples of frozen food and ice cream to introduce these new products to customers.

An example with both types of self-consumption is the sales of a popular mandarin fruit, Lukan, during Chinese New Year. The fruit is grown in Yong Chun, a town in Guangzhou province in China, and sold in Singapore. During the delivery and storage process, Lukan is highly perishable and requires strict temperature and humidity conditions to maintain its quality. However, these conditions are difficult to meet, thereby resulting in high passive self-consumption. In order to increase the number of buyers, many grocery stores offer free samples of Lukan to the shoppers, which contributes to the proactive self-consumption. Other holiday products such as the New Year’s cookies, rice cakes and mooncakes also incur both types of self-consumption. The two types of self-consumption complicate the inventory management process, and the optimal ordering policy cannot be directly derived from the classical newsvendor model. On the one hand, compared to the classical newsvendor model, the retailer should order less and have fewer goods for sale because she incurs a higher ordering cost per unit sold due to the passive self-consumption. On the other hand, the retailer should order more because the proactive self-consumption increases the market size. In this paper, we study the effect of passive self-consumption and proactive self-consumption on the retailer’s optimal ordering policies, fill rates and proactive self-consumption levels, and compare our results to the classical newsvendor model.

In section 2, we provide a detailed literature review. In section 3, we set up models for passive and proactive self-consumption. We consider passive self-consumption when the self-consumption quantity is (i) a fixed amount, and (ii) proportional to the order quantity; and proactive self-consumption with (iii) additive, and (iv) multiplicative demand. In section 4, we solve for the optimal order quantities under each of these models. We find that proactive self-consumption leads to a higher demand, and thus the retailer provides more inventory to the market and receives a higher profit. However, passive self-consumption leads to higher per unit cost. The retailer thus provides less inventory to the market and receives lower profits. (i), (iii) and (iv) result in a larger order quantity and same fill rate compared to no self-consumption, (ii) may result in a larger or smaller order quantity with a smaller fill rate. When both types of self-consumption co-exist, the optimal policy can be complicated. We show through numerical studies that the optimal policy

can take at most three formats: sell to the market with positive proactive self-consumption, sell to the market with zero proactive self-consumption and do not sell to the market. Interestingly, the optimal order quantity is not smooth in the fraction of the proportional self-consumption. In section 5, we discuss the retailer’s optimal policy when the market adoption rate is uncertain and show that the optimal policy has a similar structure to that with known market adoption rate. When the retailer has the option to expedite, he benefits from learning the adoption rate over time if the two conditions are satisfied: (a) the difference between the high and the low market adoption rates is high, and (2) the probability of incurring a high market adoption rate is low. In section 6, we conclude the paper.

2 Literature review

In the marketing literature, sampling has been studied by many researchers. For example, Lammers (1991) finds empirical evidence for the positive effect of sampling on immediate purchase. The author provided free samples of chocolates in a shop and found that sampling increased immediate sales. In particular, 84% of the sampled group purchased chocolates on this visit versus 59% of the non-sampled group, which implies that sampling increases immediate sales. Bawa and Shoemaker (2004) conducted two field experiments on free samples. They show that “*unlike other consumer promotions such as coupons, free samples can produce measurable long term effects on sales that can be observed as much as 12 months after the promotion.*” Heiman et al. (2001) model both immediate sampling effects and long term sampling effects and derive the optimal dynamic sampling policy. However, in all of these papers free sampling is studied only from the perspective of marketing, where price is the main decision the retailers make. In our paper, we focus on inventory decisions and assume that the selling price is exogenous, i.e. the retailer is selling to a perfect competitive market where no single firm has the power to set the product price. Indeed, across different grocery stores, the Lukans, the New year cookies and the rice cakes are sold at prices with little variations, and none of them are deviating from these prices.

Product damage, defined as the damage on the product so that its value decreases and must be sold at a lower price, has been studied extensively in economics. Many papers study the pricing decisions of the damaged goods (see Deneckere and McAfee 1996 and Varian 1995). This paper considers a special case of the product damage where price is not a decision: a product is either completely damaged so that we can no longer put it for sale (i.e., price is zero), or it is not damaged (i.e., price is exogenously given). Instead, we study the effect of product damage on the retailer’s inventory decisions.

In operations management literature, the newsvendor model has been studied in many research papers, see Khouja (1999) for a detailed review. There are two streams of research in newsvendor models that are the most related to this paper: (1) models with a random yield and (2) models with pricing decisions.

In the random yield model, the retailer receives a random replenishment quantity that he has partial or no control over and only pays for the quantity it received. Yano and Lee (1995) provide a detailed review of this problem, which is also covered by Gerchak et al. (1988), Henig and Gerchak 1990, Dada et al. (2003). However, unlike the random yield model, in this paper we assume that the retailer decides (controls) the amount of product he consumes and pays for its self-consumption. (The self-consumption quantity is determined by the total order quantity under passive self-consumption, and is decided by the retailer under pro-active self-consumption)

Researchers have also studied newsvendor models in which demand is price dependent (see Dana Jr. and Petruzzi 2001, Petruzzi and Dada 1999, Lau and Lau 1988, Polatoglu 1991 and Whitin 1955). In particular, Polatoglu (1991) and Petruzzi and Dada (1999) study pricing problems under (1) an additive demand model (in which the mean of the demand is a function of price and the uncertainty is an additive term to the demand) and (2) a multiplicative demand model (in which the uncertainty is multiplied by the mean demand). Unlike these models, we assume that the demand is a function of the proactive self-consumption quantity. In particular, we study the effects of proactive self-consumption with additive and multiplicative demand, i.e. that the mean demand is a function of the self-consumption quantity and the uncertainty is either added to (additive demand) or multiplied by (multiplicative demand) the mean demand. Dana Jr. and Petruzzi (2001) consider demand to be a function of inventory levels and show that ‘*the retailer holds more inventory, provides a higher fill rate and earns a higher profit*’. We consider demand to be a function of the self-consumed quantity under proactive self-consumption. We show that under this condition the retailer orders more provides more inventory and earns a higher profit. However, under passive self-consumption, the retailer orders more provides less inventory and earns less profit.

On top of the above literature, inventory models with perishable products are also related to this paper (see for example Balcer 1983, Deniz 2004, Federgruen et al., Feng and Xiao 2006). However, in the perishable product models, the newly purchased inventories do not perish immediately and they are lost only after some time during which demand could occur and consume some inventories. In this paper, the self-consumption occurs prior to the demand realization that the self-consumed inventories cannot be used to satisfy demand.

3 The model

The following table summarizes the list of notations used in this paper.

Consider a retailer that sells a single product over a single period. The unit purchase cost of this product is c and the sales price is p , $p > c > 0$. The retailer faces a random demand D with a mean of μ and a continuous and positive support on¹ $[0, \bar{d}]$ with a cumulative distribution function (c.d.f) $F_D(\cdot)$ and a probability density function (p.d.f) $f_D(\cdot)$. Let $\bar{F}_D(\cdot) = 1 - F_D(\cdot)$ and define its inverse by² $(\bar{F}_D)^{-1}$.

At the beginning of the period, the retailer places an order for Q units. The order arrives immediately and the retailer consumes an amount q , $Q \geq q \geq 0$. Then the random demand D is realized and the retailer fulfills this demand with its on hand inventory $Q - q$. The retailer's objective is to maximize his expected profit. If $q = 0$ then the retailer incurs no self-consumption and faces a classical newsvendor problem. In this case, the retailer's optimal order quantity is $Q_n = (\bar{F}_D)^{-1}(c/p)$, his corresponding profit is $\pi_n = p \left(\mu - E[(\bar{F}_D)^{-1}(c/p) - D]^+ \right) - c(\bar{F}_D)^{-1}(c/p)$ and the fill rate is $1 - c/p$. If $q \geq 0$ then we have the following cases.

- **Passive self-consumption.** When the retailer incurs passive self-consumption, the self-consumption quantity q is determined by external factors, such as inefficiencies in logistics and storage. However, the demand distribution is independent of the self-consumption. We consider two types of passive self-consumption.
 - *Fixed-quantity self-consumption (FSC)* The retailer consumes a fixed quantity $q_0 \geq 0$ if $Q \geq q_0$. This occurs when the self-consumption level is independent of the order quantity.
 - *Proportional self-consumption (PSC)* The retailer consumes a fixed proportion of his order quantity, $q = \alpha Q$, $\alpha \in [0, 1]$.
- **Proactive self-consumption.** When the retailer incurs proactive self-consumption, he can influence the demand through his self-consumption and the self-consumption quantity q becomes endogenous. Denote $D(q)$ as the demand for a given self-consumption quantity, we consider the following two cases.

- *Additive demand (PRA).* The demand is additive if $D(q) = d(q) + \epsilon$, where $d(q) \geq 0$ and is increasing and concave in q . The demand uncertainty is captured by ϵ , which is

¹If $E[D^2] < \infty$, then $\bar{d}$ can take the value of infinity, otherwise $0 < \bar{d} < \infty$.

²We similarly define $\bar{F}_\epsilon(\cdot) ((\bar{F}_\epsilon)^{-1})$ and $\bar{F}_\nu(\cdot) ((\bar{F}_\nu)^{-1})$ for the latter part of the paper.

Notation	Definition
α	the fixed proportion of self-consumption under proportional self-consumption
ϵ	demand uncertainty under additive demand and proactive self-consumption
λ	market adoption rate
λ_H	possible high market adoption rate
λ_L	possible low market adoption rate
ρ_H	probability of high market adoption rate
ρ_L	probability of low market adoption rate
$\bar{\lambda}$	expected market adoption rate
$\bar{\lambda}(q)$	a function of the market adoption rate given q
ν	demand uncertainty under multiplicative demand and proactive self-consumption
μ	demand mean
π^*	retailer's expected profit under optimal policy
π_0	retailer's optimal expected profit when demand is ν
π_α	retailer's optimal expected profit when demand is ν and selling price is $(1 - \alpha)p$
π_λ	retailer's expected profit with market adoption rate $\bar{\lambda}(q^*)$ and self-consumption level q^*
π_n	retailer's expected profit under optimal order quantity without self-consumption
$\pi(c, p, q Q)$	retailer's expected profit given all parameters
$\bar{\pi}(Q_e \lambda, Q, q)$	retailer's expected profit with expediting option
c	unit purchase cost
c_e	unit purchase cost under expediting
D	retailer's random demand
$D(q)$	retailer's demand function under proactive self-consumption
$d(q)$	retailer's expected demand under proactive self-consumption
$\bar{d}$	upper bound of the retailer's demand
$F_D(\cdot), F_\epsilon(\cdot)$	c.d.f. of retailer's demand $D, \epsilon, \lambda, \nu$
$F_\lambda(\cdot), F_\nu(\cdot)$	
$\bar{F}_D(\cdot), \bar{F}_\epsilon(\cdot)$	$1 - F_D(\cdot), 1 - F_\epsilon(\cdot), 1 - F_\lambda(\cdot), 1 - F_\nu(\cdot)$
$\bar{F}_\lambda(\cdot), \bar{F}_\nu(\cdot)$	
$(\bar{F}_D)^{-1}, (\bar{F}_\epsilon)^{-1}$	inverse of $\bar{F}_D(\cdot), \bar{F}_\epsilon(\cdot), \bar{F}_\lambda(\cdot), \bar{F}_\nu(\cdot)$
$(\bar{F}_\lambda)^{-1}, (\bar{F}_\nu)^{-1}$	
$f_D(\cdot), f_\epsilon(\cdot)$	p.d.f. of retailer's demand $D, \epsilon, \lambda, \nu$
$f_\lambda(\cdot), f_\nu(\cdot)$	
N	total market size
p	unit sales price
Q	retailer's order quantity
$\tilde{Q}$	retailer's optimal order quantity under no expediting option
Q^*	optimal order quantity
Q_e	expediting quantity
Q_e^*	optimal expediting quantity
$Q_{e,H}^*$	optimal expediting quantity under high market adoption rate
$Q_{e,L}^*$	optimal expediting quantity under low market adoption rate
Q_n	retailer's optimal order quantity without self-consumption
$\mathcal{Q}$	feasible set of retailer's order quantity and self-consumption quantity
q	retailer's self-consumption quantity
$\tilde{q}$	retailer's optimal self-consumption quantity under no expediting option
q_0	retailer's self-consumption quantity under fixed-quantity self-consumption
q^0	retailer's optimal self-consumption quantity when expediting is optimal only under λ_H

a random variable with an expected value of zero and has a continuous support with a c.d.f $F_\epsilon(\cdot)$ and a p.d.f $f_\epsilon(\cdot)$.

- *Multiplicative demand (PRM)*. The demand is multiplicative if $D(q) = d(q)\nu$, where $d(q) \geq 0$ and is increasing and concave in q . The demand uncertainty is captured by ν , which is a non-negative random variable with a mean equal to one and that has a continuous support with a c.d.f $F_\nu(\cdot)$ and a p.d.f $f_\nu(\cdot)$.

We use a specific mean demand $d(q) = N(1 - e^{-\lambda q})$ as an example for both additive and multiplicative demand in the analysis later to demonstrate the impact of proactive self-consumption. In $d(q)$, N is the total market size, and $\lambda > 0$ is the market adoption rate. $1 - e^{-\lambda q}$ is the fraction of the market that would purchase the product given that q units are consumed.³

Let $\pi(c, p, q|Q) = pE[(Q - q) \wedge D] - cQ$ denote the retailer's expected profit. Under passive self-consumption, the retailer's optimization problem is given by

$$\max_{Q \in \mathcal{Q}} \pi(c, p, q|Q),$$

where $\mathcal{Q} = \{Q, 0|Q \geq q_0\}$ under fixed self-consumption, and $\mathcal{Q} = \{Q|Q \geq 0\}$ under proportional self-consumption. Similarly, under proactive self-consumption, the retailer's optimization problem is given by

$$\max_{(Q, q) \in \mathcal{Q}} \pi(c, p, q|Q),$$

where $\mathcal{Q} = \{(Q, q)|Q \geq q \geq 0\}$ under both additive and multiplicative demand.

4 The optimal order quantities

In this section, we solve for the optimal order quantities and discuss the differences between and effects of both passive and proactive self-consumption on the retailer's decisions and profits.

4.1 Passive self-consumption

When self-consumption is unavoidable and does not have any effect on demand, the retailer passively consumes the product and bears all of the costs. We focus on the effects of passive self-consumption on the retailer's optimal order quantities and profits in the rest of this subsection.

³The total number of adoptions as a function of distributed quantity has been studied in Iyengar and Berger (2014), they find that *the number of influence doses within a given period has a non-linear and sub-additive impact on adoption: More doses are associated with increased adoption, but with sharply diminishing marginal impact*. In our paper, we choose to use exponential functions to approximate the increasing and sub-additive properties found in Iyengar and Berger (2014).

Fixed-quantity self-consumption. When self-consumption is fixed, the retailer either orders no less than his consumption quantity or does not enter the market and orders zero. Recall that $q = q_0$. The following proposition characterizes the retailer's optimal order quantity and profit.

Proposition 1 *The retailer's profit $\pi(c, p, q_0|Q)$ is concave in Q . The optimal order quantity is $Q^* = q_0 + \bar{F}_D^{-1}(c/p)$ and the corresponding fill rate is $1 - c/p$ if and only if q_0 satisfies*

$$q_0 \leq (p/c) \left(\mu - E \left[(\bar{F}_D)^{-1}(c/p) - D \right]^- \right) - (\bar{F}_D)^{-1}(c/p). \quad (1)$$

Otherwise, $Q^ = 0$, and the fill rate is 0. The retailer's profit under Q^* is $\pi^* = [\pi_n - cq_0]^+$.*

Proposition 1 has the following implications. First, the retailer orders the self-consumption quantity plus the optimal order quantity with no self-consumption if inequality (1) holds. Under optimality, the number of units available, $Q^* - q_0$, is independent of the self-consumption quantity, q_0 . Second, the retailer sells a positive amount, if q_0 is below a certain threshold. The threshold is always positive when $p > c$ and equals zero when $p = c$. It is also increasing in p and decreasing in c ,⁴ i.e. the retailer is more willing to sell the product if the price is high or the cost is low. Third, the retailer's profit under self-consumption is always less than when there is no self-consumption and the difference between the two profit is the cost of self-consumption cq_0 if inequality (1) holds.

Proportional self-consumption. When the self-consumption quantity is α fraction of the order quantity, the retailer can order an arbitrary amount of the product. The following proposition characterizes the retailer's optimal order quantity and profit.

Proposition 2 *The retailer's expected profit $\pi(c, p, \alpha Q|Q)$ is concave in Q . If $(1 - \alpha)p > c$, then the optimal order quantity is $Q^* = \frac{1}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right)$. The corresponding profit is*

$$\pi^* = p \left(\mu - E \left[(\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right) - D \right]^- \right) - \frac{c}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right),$$

which is the optimal profit under no self-consumption with purchase cost $c/(1 - \alpha)$. The fill rate is $1 - \frac{c}{(1-\alpha)p}$. If $(1 - \alpha)p \leq c$, then $Q^ = \pi^* = 0$.*

We make the following observations from Proposition 2. First, the retailer's optimal profit is decreasing in α and c , and increasing in p . Thus, it is lower than the retailer's profit with no self-consumption ($\alpha = 0$). Second, because of the loss due to self-consumption, the optimal order

⁴To verify that the threshold is increasing in p , we take derivative of the threshold over p . After some simplification, the expression of the derivative is $\left[(\bar{F}_D)^{-1}(c/p) - E \left[(\bar{F}_D)^{-1}(c/p) - D \right]^+ \right] / c > 0$. This hence proves the increasing property of the threshold. The threshold is decreasing in c can be shown similarly.

quantity is larger compared to that with no self-consumption under purchase cost $c/(1-\alpha)$, i.e. $Q^* \geq \bar{F}_D^{-1}\left(\frac{c}{(1-\alpha)p}\right)$. When the demand distribution has increasing failure rate (IFR) (see Barlow and Proschan 1975), the optimal order quantity is unimodal in α , i.e. Q^* first increases in α and then decreases in α .^{5,6} Note that when $(1-\alpha)p \leq c$, we have $Q^* = 0$. Therefore, the optimal order quantity Q^* is larger than that under no self-consumption when α is small, and less when α is large. Third, the retailer is willing to sell the product only if $(1-\alpha)p > c$, because for every unit purchased, only $1-\alpha$ units are available for sale, which generate a revenue of $(1-\alpha)p$ if they are sold. Finally, the fill rate is $1 - \frac{c}{(1-\alpha)p}$, which is less than that with no self-consumption ($\alpha = 0$)

4.2 Proactive self-consumption

Under proactive self-consumption, the retailer decides both the order quantity and self-consumption quantity, and the latter has a positive effect on the market and hence increases the mean demand. We study the behavior of the retailer's optimal order quantities, self-consumption quantities and profit under additive and multiplicative demand.

Additive demand. When demand is additive, the mean demand is independent of the market uncertainty. Hence the self-consumption quantity only affects the demand through its expected value. Denote q^* as the optimal self-consumption quantity. The retailer's optimal order quantity and self-consumption quantity are characterized in the following proposition.

Proposition 3 *The retailer's profit $\pi(c, p|q, Q)$ is concave in Q , and (q^*, Q^*) satisfies $d'(q^*) = c/(p-c)$, and $Q^* = q^* + d(q^*) + (\bar{F}_\epsilon)^{-1}(c/p)$. If $d'(q)$ is strictly decreasing, then (q^*, Q^*) is the unique solution to the above equations, moreover, if $d'(0) < c/(p-c)$, then $q^* = 0$. If $q^* > 0$, the fill rate, $1 - c/p$, is the same as that under no self-consumption. Further, if $D(q) = N(1 - e^{-\lambda q}) + \epsilon$, note that $Q_n = (\bar{F}_\epsilon)^{-1}(c/p)$. then (q^*, Q^*) are given by*

$$q^* = \left[\frac{1}{\lambda} \ln \frac{(p-c)\lambda N}{c} \right]^+, \quad Q^* = q^* + N(1 - e^{-\lambda q^*}) + Q_n.$$

When $\lambda N \leq \frac{c}{p-c}$, we have $q^* = 0$, and $Q^* = Q_n$.

Proposition 3 implies that under general additive demand, the optimal order quantity is the sum of the optimal order quantity with no self-consumption, $d(q^*) + (\bar{F}_\epsilon)^{-1}(c/p)$, and the optimal

⁵To show that Q^* is unimodal in α , first note that $\beta = 1/(1-\alpha)$ is strictly decreasing in α . Thus, it is sufficient to show Q^* is unimodal in β . Taking derivative of the optimal order quantity over β and multiplying by $f_D((\bar{F}_D)^{-1}(\beta c/p))$, we have the modified derivative equals $(\bar{F}_D)^{-1}(\beta c/p) f_D((\bar{F}_D)^{-1}(\beta c/p)) - \beta c/p$. Note that $\beta c/p = \bar{F}_D((\bar{F}_D)^{-1}(\beta c/p))$, the modified derivative is in the form of $xf_D(x) - \bar{F}_D(x)$ where $x = (\bar{F}_D)^{-1}(\beta c/p)$. Thus, the statement holds because of the IFR property and the fact that $1/x$ is strictly decreasing in x .

⁶In general, the optimal order quantity is neither increasing nor decreasing in the consumption percentage α because $\frac{1}{1-\alpha}$ is increasing in α but $\bar{F}_D^{-1}\left(\frac{c}{(1-\alpha)p}\right)$ is decreasing in α .

self-consumption quantity q^* . The optimal self-consumption quantity, q^* , is chosen such that the marginal increment of the demand equals the ratio of cost to profit margin. Moreover, the optimal self-consumption quantity, q^* , optimal order quantity, Q^* , and quantity available for sale, $Q^* - q^*$ are increasing in the profit margin, $p - c$, and decreasing in the purchase cost, c . This is because when the profit margin is larger, the retailer is willing to increase his self-consumption for a larger demand, as a result, he orders a larger quantity and provides more product for sale. When the purchase cost is higher, the retailer decreases his self-consumption to reduce cost, as a result, he orders a smaller quantity and provides less product for sale.

When $D(q) = N(1 - e^{-\lambda q}) + \epsilon$, the optimal self-consumption quantity, q^* , optimal order quantity, Q^* , and quantity available for sale, $Q^* - q^*$ are increasing in the market size N , and $Q^* - q^* \geq Q_n$ is linear in N . Note that λN is the marginal increment of mean demand when the retailer starts to self-consume, and $c/(p - c)$ is the marginal cost of self-consumption. Thus, when⁷ $\lambda N \leq \frac{c}{p - c}$, no self-consumption is optimal; when $\lambda N > \frac{c}{p - c}$, the retailer consumes a positive amount.

Moreover, the self-consumption quantity q^* first increases in the market adoption rate λ when $\lambda \in \left[\frac{c}{(p - c)N}, \frac{ce}{(p - c)N}\right]$, then decreases when $\lambda \geq \frac{ce}{(p - c)N}$. This is because as the market adoption rate increases, the marginal mean demand in self-consumption quantity first increases then decreases. Thus, the retailer is optimal to first increase self-consumption quantity to increase mean demand then decrease self-consumption quantity to reduce cost.⁸ Similarly, we express $Q^* = q^* + N - \frac{c}{(p - c)\lambda} + Q_n$, and it is first increasing in the market adoption rate, λ , when $\lambda \in \left[\frac{c}{(p - c)N}, \frac{c}{(p - c)N} e^{\frac{p}{p - c}}\right]$, then decreasing in λ when $\lambda \geq \frac{c}{(p - c)N} e^{\frac{p}{p - c}}$. The amount available for sale, $Q^* - q^*$, is increasing and concave in the market adoption rate λ .

The retailer's optimal profit can be expressed as follows

$$\pi^* = pN(1 - e^{-\lambda q^*}) - c\left(q^* + N(1 - e^{-\lambda q^*})\right) + \pi_n,$$

which is the sum of the profit under mean demand and the profit with no self-consumption.

Multiplicative demand. When demand is multiplicative, the self-consumption quantity affects not only the mean demand but also demand uncertainty. Denote $\wedge$ as the minimum of two numbers. The corresponding optimal quantities are given in the following proposition.

Proposition 4 *The profit $\pi(c, p|q, Q)$ is concave in Q . The optimal quantities (q^*, Q^*) satisfies*

$$d'(q^*) = \frac{c}{\pi_0}, \quad Q^* = q^* + d(q^*) (\bar{F}_\nu)^{-1}(c/p),$$

⁷Either the market size N or the adoption rate λ is small, or the profit margin $p - c$ is small

⁸It is easy to show that q^* is unimodal in λ , and solving for the first order condition we can compute the critical points of λ .

where $\pi_0 = pE \left[(\bar{F}_\nu)^{-1} (c/p) \wedge \nu \right] - c (\bar{F}_\nu)^{-1} (c/p)$ is the retailer's profit when demand is ν . If $d'(q)$ is strictly decreasing, then (q^*, Q^*) is the unique solution to the above equations, moreover, if $d'(0) < c/\pi_0$, then $q^* = 0$. If $q^* > 0$, the fill rate is $1 - c/p$, which is the same as that under no self-consumption. If $D(q) = N (1 - e^{-\lambda q}) \nu$, then

$$q^* = \left[\frac{1}{\lambda} \ln \frac{\pi_0 \lambda N}{c} \right]^+, \quad Q^* = q^* + N (1 - e^{-\lambda q^*}) (\bar{F}_\nu)^{-1} (c/p).$$

When $\pi_0 \lambda N \leq c$, $q^* = Q^* = 0$ and the retailer's profit is zero.

Proposition 4 implies that the optimal order quantity is the sum of the optimal self-consumption quantity and the optimal order quantity if demand is $d(q^*)\nu$. The optimal self-consumption quantity, q^* , is chosen such that the marginal revenue with respect to self-consumption quantity is equal to the purchase cost. Moreover, the optimal self-consumption quantity, order quantity and the quantity available for sale are decreasing in purchase cost, c , and increasing in the profit margin, $p - c$.

When $D(q) = N (1 - e^{-\lambda q}) \nu$, similar to additive demand, the optimal self-consumption quantity is first increasing in the adoption rate, λ , when $\lambda \in \left[\frac{c}{\pi_0 N}, \frac{ce}{\pi_0 N} \right]$, and then decreasing in the adoption rate when $\lambda \geq \frac{ce}{\pi_0 N}$. The optimal order quantity is first increasing in the adoption rate λ when $\lambda \in \left[\frac{c}{\pi_0 N}, \frac{c}{\pi_0 N} e^{1 + \frac{c}{\pi_0 N} (\bar{F}_\nu)^{-1} (c/p)} \right]$; then decreasing in λ when $\lambda \geq \frac{c}{\pi_0 N} e^{1 + \frac{c}{\pi_0 N} (\bar{F}_\nu)^{-1} (c/p)}$.

The retailer's optimal profit under multiplicative demand is

$$\pi^* = N (1 - e^{-\lambda q^*}) \pi_0 - c q^* = \pi_0 N - \frac{c}{\lambda} \left(1 + \ln \frac{\pi_0 \lambda N}{c} \right).$$

This implies that the retailer's profit is increasing and convex in the market size, N and linear in π_0 . Moreover, the means demand under optimality $N (1 - e^{-\lambda q^*})$ and the quantity available for sale $Q^* - q^*$ are also linear in the market size N .

4.3 Discussion

In this section, we compare the optimal self-consumption quantities, quantities available for sale, and retailer's optimal profit to understand the differences between passive and proactive self-consumption.

Optimal self-consumption quantity. For passive self-consumption, the self-consumption quantity under PSC is $\frac{\alpha}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right)$, and

$$\lim_{\alpha \rightarrow 0} \frac{\alpha}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right) = \lim_{\alpha \rightarrow 1} \frac{\alpha}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right) = 0.$$

Therefore, it is less than q_0 (the self-consumption quantity under FSC) when α is close to 0 or 1. For intermediate values of α , $\frac{\alpha}{1-\alpha}(\bar{F}_D)^{-1}\left(\frac{c}{(1-\alpha)p}\right)$ (the self-consumption quantity under PSC) could be higher than q_0 due to a large order quantity.

Under proactive self-consumption, if $\pi_0 > p - c$, then the self-consumption quantity under PRM is larger than that under PRA, i.e. $\left[\frac{1}{\lambda} \ln \frac{\pi_0 \lambda N}{c}\right]^+ > \left[\frac{1}{\lambda} \ln \frac{(p-c)\lambda N}{c}\right]^+$. Otherwise, if $\pi_0 < p - c$ then $\left[\frac{1}{\lambda} \ln \frac{\pi_0 \lambda N}{c}\right]^+ < \left[\frac{1}{\lambda} \ln \frac{(p-c)\lambda N}{c}\right]^+$. Note that for all distributions of ν such that $(\bar{F}_\nu)^{-1}(c/p)$ is fixed, π_0 is larger if ν is larger in stochastic convex order.⁹ If ν is more uncertain, the retailer is more likely to choose a larger self-consumption quantity under PRM than under PRA.

Quantity available for sale Under passive self-consumption, the quantity available for sale is always less compared to no self-consumption, i.e. $Q^* - q^* \leq Q_n$. This is because the retailer incurs a purchase cost for both $Q^* - q^*$ and the quantity he consumes, i.e. q^* , which makes the actual purchase cost per unit available for sale higher, and hence results in a lower $Q^* - q^*$.

However, under proactive self-consumption, the quantity available for sale is higher under both PRA and PRM, i.e. $Q^* - q^* \geq Q_n$. This is because proactive self-consumption can increase the market size and hence lead to larger quantity available for sale under optimality. It is a fundamental difference between passive and proactive self-consumption.

Retailer's optimal profit By using a rationale similar to the one used for analyzing the quantity available for sale, we find that the retailer's optimal profit decreases under passive self-consumption and increases under proactive self-consumption.

Optimal Ordering Quantities with Two Types of Self-Consumption

When the retailer experiences both passive and proactive self-consumption, the optimal ordering quantities and self-consumption levels reflect the effect of both types of self-consumption.

In the following paragraphs, we discuss the joint effect of passive and proactive self-consumption on the retailer's ordering strategies. Starting with fixed passive self-consumption, the retailer passively consumes a fixed quantity of q_0 if he decides to sell a positive amount of product to the market. Under both types of proactive self-consumption, we find that if the optimal order quantities Q^* and the optimal proactive self-consumption quantities q^* given in Table 1 satisfy $0 < pE[(Q^* - q_0 - q^*) \wedge D(q^*)] - cQ^*$, then the retailer is willing to order Q^* and sell $Q^* - q^* - q_0 > 0$ amount of product to the market. Otherwise, it is not profitable for the retailer to sell in the market, and hence $Q^* = q^* = 0$. Table 1 lists the optimal ordering quantities Q^* and the optimal proactive self-consumption quantities q^* when it is profitable for the retailer to sell to the market.

⁹See Shaked and Shanthikumar (2006) for definition of stochastic convex orders.

Fixed Passive Self-Consumption	Additive Demand (TFA)	Multiplicative Demand (TFM)
Optimal order quantity Q^*	$q_0 + q^* + d(q^*) + (\bar{F}_\epsilon)^{-1}(c/p)$	$q_0 + q^* + d(q^*) (\bar{F}_\nu)^{-1}(c/p)$
Optimal proactive self-consumption quantity q^*	$d'(q^*) = c/(p - c)$	$d'(q^*) = c/\pi_0$

Table 1: Optimal order quantities and proactive self-consumption quantities under fixed passive self-consumption

In particular, under both TFA and TFM, the optimal proactive self-consumption quantity q^* is independent of the fixed passive self-consumption quantity q_0 , and equals the optimal proactive self-consumption quantity when there is no passive self-consumption. Therefore, similar to the results when there is only proactive self-consumption, the proactive self-consumption quantity q^* equals zero if $d'(0) < c/(p - c)$ under TFA and $d'(0) < c/\pi_0$ under TFM respectively. However, the optimal ordering quantity Q^* is linear and increasing in q_0 . It equals the optimal order quantity with no passive self-consumption plus the fixed passive self-consumption quantity q_0 . The fill rate for both types of proactive self-consumption is $1 - c/p$, which is the same as when there is no self-consumption. These results imply that when both fixed passive self-consumption and proactive self-consumption exist, the retailer can find the optimal policies following the sequence: (1) find the optimal proactive self-consumption level q^* by ignoring the fixed passive self-consumption quantity q_0 ; (2) find the optimal ordering quantity Q^* by first finding the optimal ordering quantity when there is no passive consumption and then adding the fixed passive self-consumption quantity q_0 , and (3) decide whether it is profitable to sell to the market.

Similar to the case where there is only fixed passive consumption, the fixed passive self-consumption quantity q_0 under two types of self-consumption only affects the optimal ordering quantity Q^* linearly. However, the criteria for the retailer to enter the market is different from the fixed passive consumption model. To see the difference, $0 < pE[(Q^* - q_0 - q^*) \wedge D(q^*)] - cQ^*$ implies that the retailer's profit is non-negative if he orders Q^* and proactively consumes q^* , this hence implies a less restrictive constraint compared to inequality (1), which can be obtained from $0 < pE[(Q^* - q_0 - q^*) \wedge D(q^*)] - cQ^*$ by simply setting $q^* = 0$. Thus, with proactive self-consumption, the retailer can increase his chance of entering the market by optimally modifying the market through q^* and choose the optimal order quantity Q^* . Nonetheless, the fill rate of the retailer remains the same and it is independent of whether there is proactive self-consumption or not.

Under both TPA and TPM, we find that if the consumed fraction α , the unit selling price p

and the unit purchasing cost c satisfy $(1 - \alpha)p > c$, then the retailer is willing to order Q^* and sell $(1 - \alpha)Q^* - q^* > 0$ amount of product to the market (given in Table 2). Otherwise, it is not profitable for the retailer to sell to the market, i.e. $Q^* = q^* = 0$. Unlike the market entering condition under TFA and TFM, here whether the retailer enters the market does not depend on the optimal order quantity Q^* and the proactive self-consumption quantity q^* .

Proportional Self-Consumption	Additive Demand (TPA)	Multiplicative Demand (TPM)
Optimal order quantity Q^*	$\frac{q^* + d(q^*) + (\bar{F}_\epsilon)^{-1}(c/((1-\alpha)p))}{1-\alpha}$	$\frac{q^* + d(q^*)(\bar{F}_\nu)^{-1}(c/((1-\alpha)p))}{1-\alpha}$
Optimal proactive self-consumption quantity q^*	$d'(q^*) = c/((1-\alpha)p - c)$	$d'(q^*) = c/\pi_\alpha^*$

$$*\pi_\alpha = (1 - \alpha)pE \left[(\bar{F}_\nu)^{-1}(c/((1-\alpha)p)) \wedge \nu \right] - c(\bar{F}_\nu)^{-1}(c/((1-\alpha)p)).$$

Table 2: Optimal order quantities and self-consumption quantities under proportional passive self-consumption

The proportional self-consumption affects the retailer's ordering and self-consumption policies by incurring a discount on the unit selling price. This effect is similar to the case where only proportional self-consumption occurs. However, unlike under TFA and TFM, the proportional self-consumption does affect the optimal proactive self-consumption quantity q^* . Specifically, under both TPA and TPM, the optimal proactive self-consumption quantity q^* decreases in the fraction of the passive self-consumption α . Interestingly, the optimal proactive self-consumption quantity q^* equals zero if $d'(0) < c/((1-\alpha)p - c)$ under TPA and $d'(0) < c/\pi_\alpha$ under TPM respectively. This implies that as α increases, q^* decreases to zero and remains at zero when α further increases. The retailer's optimal ordering policy then degenerates to that under proportional passive consumption. Moreover, q^* is always less than when there is no passive self-consumption if $d'(q)$ is strictly decreasing. The optimal ordering quantity Q^* may increase or decrease over α and the fill rate is $1 - c/((1-\alpha)p)$, which is less than when there is no self-consumption.

We conduct the following numerical analysis to gain more insights on how the demand and cost parameters affect the optimal ordering quantity Q^* and the optimal proactive self-consumption quantity q^* when proportional passive self-consumption exists. The patterns of the optimal policy and optimal expected profit with the fixed passive consumption are obvious, so we do not provide the numerical study on these settings.

Assume the demand takes the form of $d(q) = N(1 - e^{-\lambda q})$. Both the additive uncertainty ϵ and the multiplicative uncertainty ν follow a uniform distribution on $[0, 1]$ (we use this distribution

Figure 1: Optimal order quantity and proactive self-consumption quantity under TPA

for computational simplicity). They can be converted to satisfies the assumptions in the model by subtracting a constant term $0.5p$ in the additive demand model and multiplying 2 to $d(q)$ in the multiplicative demand model. We study the effect of (1) α on Q^* , q^* , optimal total self-consumption and the optimal expected profit, and (2) λ on Q^* and q^* and the optimal expected profit.

(1) Fix $N = 1$, $c = 1$, $p = 10$, and $\lambda = 1$. The following graphs show the optimal ordering quantity Q^* , the optimal proactive self-consumption q^* and the optimal total self-consumption when α changes from 0 to 1 with a step size of 0.01.

We have the following observations from Figure 1: (1) As α increases, the fraction of passive self-consumption increases and the optimal policy changes from selling to the market with positive proactive self-consumption to zero proactive self-consumption and then to not entering the market. (2) The optimal order quantity Q^* and the total self-consumption quantity are unimodal with kinks in the fraction of the passive self-consumption α . Moreover, within each region that has the same type of optimal policy, both Q^* and the total self-consumption quantity are unimodal in α . (3) The total inventory for sale is decreasing in α .

Figure 2 shows that the expected profit under the optimal policy is in general decreasing in α but it is neither convex nor concave in α . When $\alpha < 0.8$, the expected profit under the optimal policy consists of the profit from the uncertain market as well as the profit from the proactive self-consumption. When $0.8 \leq \alpha < 0.9$, $q^* = 0$, the retailer no longer invests to enlarge the market and the retailer's problem degenerates to the problem with only the proportional self-consumption. The expected profit under the optimal policy consists only the profit from the uncertain market. When

Figure 2: Expected profit under the optimal policy under TPA

$\alpha \geq 0.9$, $Q^* = 0$, the retailer no longer enters the market and hence gains a profit of zero.

Figure 3 shows the optimal policy under TPM. When α increases, unlike the optimal policy under TPA, the optimal policy here changes from the retailer entering the market with positive proactive self-consumption directly to him not entering the market. This is because under TPM, the expected market size is a multiple of the proactive consumption quantity. Thus, if the retailer does not proactively consume any quantity, then the expected market size will be zero. Because there are only two types of optimal policy, the optimal order quantity Q^* and the total self-consumption quantity are unimodal in the fraction of the passive self-consumption α . Similar to additive demand setting, the total inventory for sale under the multiplicative demand is decreasing in α . Similar to additive demand setting, Figure 4 shows that the expected profit under the optimal policy is in general decreasing in α but it is neither convex nor concave in α .

(2) Fix $N = 1$, $c = 1$, $p = 10$, $\alpha = 0.5$. The following graphs show the optimal ordering quantity Q^* , the optimal proactive self-consumption q^* and the optimal total self-consumption when λ changes from 0 to 2. Figure 5 shows the following: (1) As λ increases, the market adopts the product more easily through proactive self-consumption. Therefore, when λ moves above a certain threshold (here it is 0.28), the optimal policy changes from selling to the market without positive proactive self-consumption to positive proactive self-consumption. (2) When $q^* > 0$, the optimal order quantity Q^* , the total self-consumption quantity, and the optimal proactive self-consumption quantity q^* first increase and then decrease in λ . This is because as λ increases, the marginal increment in market size from proactive self-consumption first increases and then decreases

Figure 3: Optimal order quantity and proactive self-consumption quantity under TPM

Figure 4: Expected profit under the optimal policy under TPM

Figure 5: Optimal order quantity and proactive self-consumption quantity under TPA

while the marginal cost of proactive self-consumption stays as a constant. Therefore, it is profitable for the retailer to first increase q^* to gain a larger market size then decrease q^* to reduce the cost of proactive self-consumption.

Figure 6 shows the following: (1) As λ increases, different from Figure 5, when λ moves above a certain threshold (here 0.62), the optimal policy changes from not selling to the market to selling to the market with positive proactive self-consumption. (2) When $q^* > 0$, the optimal order quantity Q^* , the total self-consumption quantity, and the optimal proactive self-consumption quantity q^* first increase and then decrease in λ for the same reason as in Figure 5.

The profit functions under both additive and multiplicative demand are increasing in the market adoption rate λ , however, similar to Figure 2 and Figure 4 the expected profit is neither concave nor convex in λ .

Summary In summary, the retailer orders more under both passive and proactive self-consumption than when there is no self-consumption for different reasons. (1) Under passive self-consumption, the retailer orders more inventory to cover the amount he consumes, but the quantity available for sale decreases due to the self-consumption and therefore the retailer's profit decreases. In practice, companies try to eliminate or reduce the amount of passive self-consumption by increasing the efficiency of logistics (proportional self-consumption) or reducing the wastage that occurs during the product design phase (fixed self-consumption). Under fixed self-consumption, the retailer sells to the market if market size is larger than a certain threshold. In contrast, under proportional self-consumption, the retailer's decision is independent of the market size and the retailer instead

Figure 6: Optimal order quantity and proactive self-consumption quantity under TPM

Figure 7: Expected profit under the optimal policy under TPA

Figure 8: Expected profit under the optimal policy under TPM

sells to the market only when the profit margin is higher than a certain threshold. As a result, in practice we observe fixed self-consumption only in large markets and proportional self-consumption in high profit margin markets. (2) Under proactive self-consumption, the retailer orders more inventory to increase his market size and therefore the quantity available for sale increases and so does the retailer's profit. The retailer benefits from proactive self-consumption because the additional revenue from a larger market size outweighs the costs of self-consumption. In practice, proactive self-consumption is commonly observed in the high profit margin sectors (such as high end food, i.e. chocolate, coffee, wine and high end restaurants). This observation is linked to the fact that the self-consumption quantity increases in the profit margin but it is independent of the potential market size. In particular, the optimal self-consumption quantity is a function of the profit margin, $p - c$, under additive demand and it is a function of the expected profit margin π_0 under multiplicative demand. (3) When both types of self-consumption exist, the optimal policy can be very complicated. We show through numerical studies that the optimal policy can take different formats and the optimal order quantities can have multiple kinks under TPA.

5 Proactive Self-Consumption with Uncertain Market Adoption Rate

In this section, we demonstrate the effect of proactive self-consumption when the market adoption rate λ is uncertain. In practice, this happens when the retailer sells a new product that he has little information about the market adoption rate. In particular, the retailer knows the distribution of the

market adoption rate initially and learns its realization after his self-consumption.¹⁰ We consider additive demand $D(q) = N(1 - e^{-\lambda q}) + \epsilon$ in the following analysis.¹¹

Remark The demand function, $D(q) = N(1 - e^{-\lambda q}) + \epsilon$ and $D(q) = N(1 - e^{-\lambda q})\nu$ are quite general and representative. In Iyengar and Berger (2014), the authors showed with data and experiments that the total adoption (expected market size) is an increasing and concave function of the doses that are given for free (self-consumption quantity). Indeed, Iyengar and Berger (2014) states that '*More doses (self-consumption quantity) are associated with increased adoption, but with sharply diminishing marginal impact*'. This finding implies that the expected market size (adoption) is an exponential function of the self-consumption quantity (doses), which has the increasing and concave property as well as a sharply diminishing marginal impact when the self-consumption quantity increases.

5.1 Market Adoption Rate with General Distribution

Assume λ has positive support on $[0, \infty)$, with mean $\bar{\lambda}$ and finite variance, and let $F_\lambda(\cdot)$ be its c.d.f.. Following the same steps in the proof of Proposition 3, we can prove the following lemma and characterize the optimal self-consumption quantity and optimal order quantity when the market adoption rate is uncertain.

Lemma 1 *The optimal self-consumption quantity and order quantity satisfies*

$$c/p = E \left[\bar{F}_\epsilon \left(Q^* - q^* - N(1 - e^{-\lambda q^*}) \right) \right]. \quad (2)$$

Under optimality, the retailer provides a fill rate of c/p , which equals that with no self-consumption.

Denote $\bar{\lambda}(q)$ as the market adoption rate such that $Q - q - N(1 - e^{-\bar{\lambda}(q)q}) = (\bar{F}_\epsilon)^{-1}(c/p)$ provided that (q, Q) satisfies equation (2). Then Q^* can be expressed as $Q^* = q^* + N(1 - e^{-\bar{\lambda}(q^*)q^*}) + (\bar{F}_\epsilon)^{-1}(c/p)$, and $\bar{\lambda}(q)q$ is increasing in q . Thus, (q^*, Q^*) satisfies similar properties as described in Proposition 3, i.e. the available quantity for sale equals the mean demand with market adoption rate $\bar{\lambda}(q^*)$ plus the safety stock under no self-consumption. The retailer's profit under optimality is

$$\begin{aligned} \pi^* = & (p - c)N(1 - e^{-\bar{\lambda}(q^*)q^*}) - cq^* - c(\bar{F}_\epsilon)^{-1}(c/p) \\ & + pE \left[(\bar{F}_\epsilon)^{-1}(c/p) \wedge \left(\epsilon + N(e^{-\bar{\lambda}(q^*)q^*} - e^{-\lambda q^*}) \right) \right]. \end{aligned}$$

¹⁰For well known research work on demand information updating, see e.g., Li and Zhang (2013), Gurnani and Tang (1999), Yan et al. (2003), etc.

¹¹For multiplicative demand, the analysis and insights are similar, so we omit them in this paper.

Let $\pi_\lambda = (p - c)N \left(1 - e^{-\bar{\lambda}(q^*)q^*}\right) - cq^* + \pi_n$ be the retailer's profit with market rate $\bar{\lambda}(q^*)$ and self-consumption quantity q^* . The following proposition shows the relationship between the retailer's optimal profit and π_λ .

Proposition 5 *If $\bar{F}_\epsilon(\cdot)$ is convex, then $\pi^* \leq \pi_\lambda$. The equality holds if and only if the market adoption rate equals $\bar{\lambda}(q^*)$ with probability 1.*

Maximizing the retailer's profit over $\lambda(q^*)$, Proposition 5 implies that the retailer's profit under optimality is less when the market adoption rate is stochastic than when the market adoption rate is deterministic. Therefore, the information on the market adoption rate is valuable to the retailer. As a result, in practice, the retailer has incentive to learn about the market adoption rate and then modify his ordering policy accordingly. We look at this possibility in the next subsection.

5.2 Self-Consumption with Expediting Options

In this section we study the effect of proactive self-consumption if the retailer has the option to expedite. In Bawa and Shoemaker (2004) and Heiman (2001), the authors have shown that the effect of free sampling has long term effects on the sales, therefore, the retailer has the incentive to expedite to accommodate the enlarged market if necessary. Prior to the self-consumption, assume that λ has two possible outcomes λ_H and λ_L ($\lambda_H > \lambda_L > 0$). They occur with corresponding probabilities $\rho_H(\geq 0)$ and $\rho_L(\geq 0)$, and $\rho_H + \rho_L = 1$. Similar to the basic model on additive proactive self-consumption, the retailer first places an initial order of Q units at a price of c per unit, and then self-consumes q . Then the retailer learns the market adoption rate after he consumes a positive amount and decides whether or not to expedite an additional quantity, $Q_e(\geq 0)$, at a higher per unit price c_e ($c < c_e < p$). Alternatively, he just sells $Q_n = (\bar{F}_\epsilon)^{-1}(c/p)$ units to the market.¹² Then the demand is realized, and the retailer fulfills the demand and earns p per unit sold.

The retailer first solves the expediting problem given the market adoption rate λ , initial order quantity Q and self-consumption quantity q . Recall that $D(q) = N(1 - e^{-\lambda q}) + \epsilon$. Let $\bar{\pi}(Q_e|\lambda, Q, q)$ be the retailer's expected profit, then

$$\bar{\pi}(Q_e|\lambda, Q, q) = pE[(Q - q + Q_e) \wedge D(q)] - c_e Q_e.$$

The retailer's expediting policy is found by solving the following optimization problem:

$$\max_{Q_e \geq 0} \bar{\pi}(Q_e|\lambda, Q, q)$$

¹²Note that if the retailer consumes zero, then the demand becomes ϵ , which is independent of the market adoption rate, and hence the retailer's problem reduces to the classical newsvendor problem, and the retailer will not expedite.

Figure 9: The sequence of events on self-consumption with expediting options

The following lemma shows the solution to the retailer's expediting problem.

Lemma 2 $\bar{\pi}(Q_e|\lambda, Q, q)$ is concave in Q_e . The optimal expediting quantity satisfies

$$Q_e^* = Q_e^*(\lambda) = \left[q - Q + N(1 - e^{-\lambda q}) + (\bar{F}_\epsilon)^{-1}(c_e/p) \right]^+,$$

and $Q_{e,H}^* = Q_e^*(\lambda_H) \geq Q_e^*(\lambda_L) = Q_{e,L}^*$ for all Q, q . The retailer's expected profit after expediting is

$$\bar{\pi}(Q_e^*|\lambda, Q, q) = \begin{cases} (p - c_e)N(1 - e^{-\lambda q}) - c_e(q - Q) \\ + pE \left[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon \right] - c_e(\bar{F}_\epsilon)^{-1}(c_e/p) & \text{if } Q_e^*(\lambda) > 0, \\ pE[(Q - q) \wedge D(q)] & \text{if } Q_e^*(\lambda) = 0. \end{cases}$$

Lemma 2 implies that when $Q - q < N(1 - e^{-\lambda q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$, the retailer should expedite a positive amount. Since $N(1 - e^{-\lambda q})$ is an increasing function of λ , given Q and q , there are only three possible cases after the retailer observes λ : (1) the retailer expedites under both λ_H and λ_L , i.e. $Q - q < N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$; (2) the retailer expedites only under λ_H , i.e. $N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p) \leq Q - q < N(1 - e^{-\lambda_H q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$; and (3) the retailer does not expedite, i.e. $N(1 - e^{-\lambda_H q}) + (\bar{F}_\epsilon)^{-1}(c_e/p) \leq Q - q$. Therefore, the retailer's expected profit given his initial order quantity and self-consumption quantity,

$$\pi(Q, q) = \rho_H \bar{\pi}(Q_{e,H}^*|\lambda_H, Q, q) + \rho_L \bar{\pi}(Q_{e,L}^*|\lambda_L, Q, q) - cQ,$$

Figure 10: The optimal expediting policy under the three possible cases.

should be considered separately under the three cases

$$\pi(Q, q) = \begin{cases} (p - c_e)N(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) + (c_e - c)Q - c_e q \\ \quad + pE\left[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon\right] - c_e(\bar{F}_\epsilon)^{-1}(c_e/p), & \text{if case (1) occurs,} \\ pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - \rho_H c_e N(1 - e^{-\lambda_H q}) + (\rho_H c_e - c)Q \\ \quad - \rho_H c_e q + \rho_L pE\left[(Q - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon\right] \\ \quad + \rho_H\left(pE\left[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon\right] - c_e(\bar{F}_\epsilon)^{-1}(c_e/p)\right), & \text{if case (2) occurs,} \\ pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) \\ \quad + \rho_H pE\left[(Q - q - N(1 - e^{-\lambda_H q})) \wedge \epsilon\right] \\ \quad + \rho_L pE\left[(Q - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon\right] - cQ, & \text{if case (3) occurs.} \end{cases}$$

Proposition 6 For a given q , $\pi(Q, q)$ is continuous and concave in Q . Moreover, case (1) is never optimal, and we have $Q^* - q^* \geq N(1 - e^{-\lambda_L q^*}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$.

Proposition 6 implies that given the self-consumption quantity q , the optimal order quantity can be determined by comparing the first order derivative of $\pi(Q, q)$ over Q under case (2) and (3).¹³ Moreover, it is always suboptimal to expedite under both market adoption rates, which implies that the retailer orders enough quantity so that under the low market adoption rate λ_L , he does not need to expedite.¹⁴

Note that the first order condition under case (2) gives an order quantity of

$$Q = q + (\bar{F}_\epsilon)^{-1}\left(\frac{c - \rho_H c_e}{\rho_L p}\right) + N(1 - e^{-\lambda_L q}). \quad (3)$$

This expression of Q is valid only when $c - \rho_H c_e > 0$. If $c - \rho_H c_e \leq 0$, then $\pi(Q, q)$ is always increasing in Q under case (2), thus, case (3) is optimal, i.e. the retailer always orders enough so that he does not expedite, and his profit equals to that without the expediting option.¹⁵ Note that

¹³Because $\pi(Q, q)$ is continuous and concave in Q , the optimal order quantity satisfies the first order condition. Thus, if either the solution to the first order condition under case (2) or (3) is feasible, then that solution is optimal. Otherwise, the boundary of case (2) and (3) $N(1 - e^{-\lambda_H q}) + (\bar{F}_\epsilon)^{-1}(c_e/p) = Q - q$ gives the optimal order quantity.

¹⁴Indeed, the quantity that is available to the market should exceed the optimal order quantity with no self-consumption.

¹⁵Note that $c - \rho_H c_e > \rho_L p$, will never happen because $c_e > c$ and $p > c$.

equation (3) is optimal if (Q^*, q^*) is under case 2, thus, the optimal order quantity for any q satisfies equation (3) if and only if:

$$q + (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right) + N \left(1 - e^{-\lambda_L q} \right) \leq q + (\bar{F}_\epsilon)^{-1} (c_e/p) + N \left(1 - e^{-\lambda_H q} \right).$$

Simplifying the above equation, we have:

$$N \left(e^{-\lambda_L q} - e^{-\lambda_H q} \right) \geq -(\bar{F}_\epsilon)^{-1} (c_e/p) + (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right). \quad (4)$$

Note that the left hand side of equation (4) is first increasing then decreasing in q ,¹⁶ and $(c - \rho_H c_e)/(\rho_L p) \leq c_e/p$, so that the right hand side is a positive constant. Thus, there exists $0 \leq q_1 \leq q_2$, such that

$$N \left(e^{-\lambda_L q} - e^{-\lambda_H q} \right) = -(\bar{F}_\epsilon)^{-1} (c_e/p) + (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right),$$

and hence equation (4) holds if and only if $q \in [q_1, q_2]$.¹⁷ Thus, the optimal order quantity satisfies

$$\begin{aligned} Q &= q + (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right) + N \left(1 - e^{-\lambda_L q} \right) & \text{if } q \in [q_1, q_2], \\ \rho_H p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_H q})) + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})) &= c & \text{if } q \notin [q_1, q_2], \end{aligned}$$

and¹⁸ the corresponding retailer's profit as a function of the self-consumption quantity is

$$\pi^*(q) = \begin{cases} -N(p - c_e)\rho_H e^{-\lambda_H q} - N(\rho_L p - (c - \rho_H c_e))e^{-\lambda_L q} - cq + (p - c)N \\ \quad + (\rho_H c_e - c) (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right) + \rho_L p E \left[(\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_e}{\rho_L p} \right) \wedge \epsilon \right] \\ \quad + \rho_H \left(p E \left[(\bar{F}_\epsilon)^{-1} (c_e/p) \wedge \epsilon \right] - c_e (\bar{F}_\epsilon)^{-1} (c_e/p) \right) & \text{if } q \in [q_1, q_2], \\ pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) + \rho_H p E \left[(Q^*(q) - q - N(1 - e^{-\lambda_H q})) \wedge \epsilon \right] \\ \quad + \rho_L p E \left[(Q^*(q) - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon \right] - cQ^*(q) & \text{if } q \notin [q_1, q_2], \end{cases}$$

where $Q^*(q)$ is the optimal order quantity under case (3). Let $\tilde{q}$ and $\tilde{Q}$ be the optimal self-consumption quantity and order quantity if the retailer does not have the option to expedite. Denote q^0 as the optimal self-consumption quantity under case (2) if we relax the constraint in equation (4), then it satisfies the following equation

$$\lambda_H N(p - c_e)\rho_H e^{-\lambda_H q^0} + \lambda_L N(\rho_L p - (c - \rho_H c_e))e^{-\lambda_L q^0} - c = 0.$$

Proposition 7 *The optimal self-consumption quantity and order quantity are given by*

$$q^* = \begin{cases} q^0 & q^0 \in [q_1, q_2], \\ \max\{\tilde{q}, q_2\} & q^0 > q_2, \\ \min\{\tilde{q}, q_1\} & q^0 < q_1. \end{cases}$$

¹⁶This is because (i) the left hand side converges to zero if $q \rightarrow 0$ or $q \rightarrow \infty$, (ii) the left hand side is always non-negative, and (iii) there is only one q such that the derivative of the left hand side equals zero.

¹⁷If equation (4) never holds, then we pick any $q_1 > q_2 > 0$.

¹⁸Note that in both cases, $Q \geq q$ is a feasible solution.

and

$$Q^* = \begin{cases} q^* + (\bar{F}_\epsilon)^{-1} \left(\frac{c - \rho_H c_\epsilon}{\rho_L p} \right) + N(1 - e^{-\lambda_L q^*}) & q^* \in [q_1, q_2], \\ \tilde{Q} & q^* \notin [q_1, q_2]. \end{cases}$$

Thus, if $q^* \in [q_1, q_2]$, the retailer expedites only if $\lambda = \lambda_H$; when q^* is either larger than q_2 or smaller than q_1 , the retailer does not expedite under both market adoption rates. Moreover, when $N(\rho_H \lambda_H + \rho_L \lambda_L) \leq c/(p - c)$, $q^* = \tilde{q} = 0$, i.e. the retailer does not self-consume, and the retailer's problem reduces to the classical newsvendor problem..

Proposition 7 implies that when the retailer has the option to learn the market adoption rate through self-consumption, he can benefit from the additional information if case (2) is optimal. In particular, the retailer can increase his profit through expediting if the market adoption rate is high. Note that the right hand side of equation (4) is decreasing in ρ_L , thus q_1 decreases in ρ_L and q_2 increases in ρ_L , and the range where case (2) is optimal is wider. As a result, the retailer is more likely to expedite under high adoption rate if ρ_L is larger. The left hand side of equation (4) is increasing in λ_H and decreasing in λ_L . This implies that if λ_H is large and λ_L is small, i.e. the difference between high adoption rate and low adoption rate is large, the retailer is also more likely to expedite under high adoption rate. In sum, the retailer benefits from learning the market adoption rate when the probability of high market adoption rate is low but the uncertainty in market scale is high. Otherwise, if the retailer knows the market is large with high probability or both λ_L and λ_H are large with small difference, then he will choose a larger q^* , and Q^* to accommodate the large market instead of expediting; if both λ_L and λ_H are small, then he will choose a small q^* and Q^* , which again is enough to accommodate the small market under both market adoption rates without expediting.

6 Conclusion and Future Research

In this paper, we study a new type of newsvendor model with two types of self-consumption: passive self-consumption and proactive self-consumption. We find that both passive and proactive self-consumption cause the retailer to order more compared to no self-consumption, but for different reasons. Under passive self-consumption, even though the retailer orders more inventory than that under no self-consumption, the per unit cost is higher and the amount of inventory for sale is lower, which results in a lower profit. Under proactive self-consumption, the market size is larger compared to no self-consumption, which leads to a larger amount of inventory for sale and more profit. When both types of self-consumption exist, the optimal policy has three possible formats:

(1) the retailer enters the market with positive proactive self-consumption (2) the retailer enters the market with zero proactive self-consumption (3) the retailer does not enter the market. When demand is multiplicative only (1) and (3) are possible. Numerical study shows that when demand is additive, the optimal order quantity can have multiple kinks in the fraction of passive self-consumption. If the market adoption rate can be learned over time, proactive self-consumption can also benefit the retailer when the probability of a high market adoption rate is low but the scale of the market is high.

One interesting direction for further research would be to investigate proactive self-consumption in a dynamic setting, in which the market size changes with the total amount of product sold. It would also be interesting to look at multi-period models and how proactive and passive self-consumption affect the optimal ordering policy and profit over time. The concept of self-consumption can also be applied to study sustainability-related issues and humanitarian operations because many companies consume their own products for daily operations and the amount of product lost during delivery is closely related to the delivery time and conditions.

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Appendix

Proof (Proof of Proposition 1) First we show that $\pi(c, p, q_0|Q)$ is concave in Q . Notice that the first term of $\pi(c, p, q_0|Q)$ is concave because taking minimum of an increasing linear function with a constant is concave, and taking expectation over a concave function is also concave. The second term is linear in Q , thus it is concave in Q . Therefore, $\pi(c, p, q_0|Q)$ is concave in Q .

To solve for Q^* , we first drop the indicator function in the profit function and take the derivative of the profit function over Q ,

$$\frac{\partial \pi(c, p, q_0|Q)}{\partial Q} = p \text{Prob}(Q - q_0 < D) - c.$$

Solving for the first order condition, we have the expression for $Q^* = q_0 + \bar{F}_D^{-1}(c/p)$. Substituting Q^* into the profit function we have $\pi^* = \left[p \left(\mu - E[(\bar{F}_D)^{-1}(c/p) - D]^- \right) - c((\bar{F}_D)^{-1}(c/p) + q_0) \right]$. Next we solve for the unique q_0 such that the value of the above profit function is greater or equal to zero. Note that $\left[p \left(\mu - E[(\bar{F}_D)^{-1}(c/p) - D]^- \right) - c((\bar{F}_D)^{-1}(c/p) + q_0) \right] \geq 0$ if and only if $(p/c) \left(\mu - E[(\bar{F}_D)^{-1}(c/p) - D]^- \right) - (\bar{F}_D)^{-1}(c/p) \geq q_0$. This thus completes the proof. ■

Proof (Proof of Proposition 2) Similar to the proof of Proposition 1, $\pi(c, p, \alpha|Q)$ is concave in Q . We take the derivative of π over Q ,

$$\frac{\partial \pi(c, p, \alpha|Q)}{\partial Q} = p \text{Prob}(Q(1 - \alpha) < D) - c.$$

Solving for the first order condition, we have the expression for $Q^* = \frac{1}{1-\alpha}(\bar{F}_D)^{-1}\left(\frac{c}{(1-\alpha)p}\right)$. If $\frac{c}{(1-\alpha)p} < 1$ then the solution to Q^* is feasible and hence $\frac{1}{1-\alpha}(\bar{F}_D)^{-1}\left(\frac{c}{(1-\alpha)p}\right)$ is the unique solution

for Q^* . Now substitute Q^* into the profit function we have

$$\pi^* = p \left(\mu - E \left[(\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right) - D \right]^- \right) - \frac{c}{1-\alpha} (\bar{F}_D)^{-1} \left(\frac{c}{(1-\alpha)p} \right).$$

If $\frac{c}{(1-\alpha)p} \geq 1$, then it is not profitable for the retailer to sell to the market and hence $Q^* = 0$ and $\pi(c, p, \alpha | Q^*) = 0$. This thus completes the proof. $\blacksquare$

Proof (Proof of Proposition 3) Note that the retailer's expected profit given (q, Q) is

$$\pi(c, p, q | Q) = p (d(q) - E [Q - q - d(q) - \epsilon]^-) - cQ.$$

For any given q , similar to Proposition 1, we have $\pi(c, p | q, Q)$ is concave in Q . Solving for the first order condition for fixed q we have $Q = q + d(q) + (\bar{F}_\epsilon)^{-1} (c/p)$. Because $D(q) \geq 0$ a.e., we have $d(q) + (\bar{F}_\epsilon)^{-1} (c/p) \geq 0$ and hence $Q \geq q$. Thus, for any given q , the Q that satisfies the first order condition is a feasible solution, and hence it is the maximizer of the profit. Therefore, since Q^* is the maximizer if $q = q^*$, it also satisfies the first order condition.

Substitute $Q = q + d(q) + (\bar{F}_\epsilon)^{-1} (c/p)$ into $\pi(c, p, q | Q)$, we obtain the retailer's profit as a function of q :

$$\begin{aligned} & p \left(d(q) - E \left[(\bar{F}_\epsilon)^{-1} (c/p) - \epsilon \right]^- \right) - c \left(q + d(q) + (\bar{F}_\epsilon)^{-1} (c/p) \right) \\ &= (p - c)d(q) - cq - pE \left[(\bar{F}_\epsilon)^{-1} (c/p) - \epsilon \right]^- - c(\bar{F}_\epsilon)^{-1} (c/p). \end{aligned}$$

Note that the first term is increasing and concave in q and the second term is decreasing and linear in q . Thus, the above expression of the retailer's profit is concave in q . Solving for the first order derivative, we have $d'(q) = c/(p - c)$. Thus, the results under general additive demand hold. Following the same logic, we obtain the results for $D(q) = N(1 - e^{-\lambda q}) + \epsilon$. $\blacksquare$

Proof (Proof of Proposition 4) Note that the retailer's expected profit given (q, Q) is

$$\pi(c, p, q | Q) = p (d(q)E[\nu] - E [Q - q - d(q)\nu]^-) - cQ.$$

For any given q , similar to Proposition 3, we have $\pi(c, p, q | Q)$ is concave in Q . Solving for the first order condition for fixed q we have $Q = q + d(q) (\bar{F}_\nu)^{-1} (c/p)$. Because $\nu \geq 0$ a.e., we have $Q \geq q$. Thus, for any given q , the Q that satisfies the first order condition is a feasible solution, and hence it is the maximizer of the profit. If $q = q^*$, then Q^* is the maximizer, thus it satisfies the first order condition. Substitute $Q = q + d(q) (\bar{F}_\nu)^{-1} (c/p)$ into $\pi(c, p, q | Q)$, we obtain the retailer's profit:

$$\begin{aligned} & pd(q)E \left[(\bar{F}_\nu)^{-1} (c/p) \wedge \nu \right] - c \left(q + d(q) (\bar{F}_\nu)^{-1} (c/p) \right) \\ &= \left(pE \left[(\bar{F}_\nu)^{-1} (c/p) \wedge \nu \right] - c(\bar{F}_\nu)^{-1} (c/p) \right) d(q) - cq. \end{aligned}$$

Note that the first term is increasing and concave in q and the section term is decreasing and linear in q . Thus, the above expression of the retailer's profit is concave in q . Solving for the first order derivative, we have $d'(q) = c/\pi_0$, where $\pi_0 = pE\left[(\bar{F}_\nu)^{-1}(c/p) \wedge \nu\right] - c(\bar{F}_\nu)^{-1}(c/p)$ is the retailer's optimal profit when demand is ν . This thus proves the results for general multiplicative demand. Substituting $d(q) = N(1 - e^{-\lambda q})$ in the the general demand function and following the same steps, we can show that the corresponding results hold. ■

Proof (Proof of Proposition 5) Note that from the definition of $\bar{\lambda}(q^*)$, we have

$$c/p = \bar{F}_\epsilon(Q^* - q^* - N(1 - e^{-\bar{\lambda}(q^*)q^*})) = E\left[\bar{F}_\epsilon\left(Q^* - q^* - N\left(1 - e^{-\lambda q^*}\right)\right)\right].$$

Because $\bar{F}_\epsilon(\cdot)$ is decreasing and convex, we can apply Jensen's inequality and we have $E[e^{-\lambda q^*}] \geq e^{-\bar{\lambda}(q^*)q^*}$. Moreover, the last term in π^* is decreasing and concave in $e^{-\lambda q^*}$. Thus,

$$\begin{aligned} E\left[(\bar{F}_\epsilon)^{-1}(c/p) \wedge \left(\epsilon + N(e^{-\bar{\lambda}(q^*)q^*} - e^{-\lambda q^*})\right)\right] &\leq (\bar{F}_\epsilon)^{-1}(c/p) \wedge \left(\epsilon + N(e^{-\bar{\lambda}(q^*)q^*} - E[e^{-\lambda q^*}])\right) \\ &\leq (\bar{F}_\epsilon)^{-1}(c/p) \wedge \left(\epsilon + N(e^{-\bar{\lambda}(q^*)q^*} - e^{-\bar{\lambda}(q^*)q^*})\right) \\ &= (\bar{F}_\epsilon)^{-1}(c/p) \wedge \epsilon, \end{aligned}$$

where the first inequality follows from Jensen's inequality and the last term in π^* is decreasing and concave in $e^{-\lambda q^*}$, and the second inequality follows from $E[e^{-\lambda q^*}] \geq e^{-\bar{\lambda}(q^*)q^*}$. Substituting the left hand side of the above inequality into the retailer's profit, we have our results. ■

Proof (Proof of Lemma 2) The concavity result follows from the same proof as in Proposition 1. Then solve for the first order condition taken into account of the no-negativity of Q_e . We have the expression for $Q_e^*(\lambda)$. Also note that $Q_e^*(\lambda, Q, q)$ is increasing in λ , we have $Q_e^*(\lambda_H) \geq Q_e^*(\lambda_L)$. Substitute $Q_e^*(\lambda)$ into $\bar{\pi}(Q_e|\lambda, Q, q)$ we have the expression for the retailer's profit under optimal expediting. ■

Proof (Proof of Proposition 6) For a given q , we first verify that $\pi(Q, q)$ is continuous in Q .

Substitute $Q - q = N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$ into case (1). We have

$$\begin{aligned}
& (p - c_e)N(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) + (c_e - c)Q - c_e q + pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] - c_e (\bar{F}_\epsilon)^{-1}(c_e/p) \\
&= (p - c_e)N(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - cQ + c_e(\rho_H + \rho_L)(N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)) \\
&+ pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] - c_e (\bar{F}_\epsilon)^{-1}(c_e/p) \\
&= pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - \rho_H c_e N(1 - e^{-\lambda_H q}) - cQ + \rho_H c_e (N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)) \\
&+ \rho_H (pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] - c_e (\bar{F}_\epsilon)^{-1}(c_e/p)) + \rho_L pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] \\
&= pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - \rho_H c_e N(1 - e^{-\lambda_H q}) - cQ + \rho_H c_e (Q - q) \\
&+ \rho_H (pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] - c_e (\bar{F}_\epsilon)^{-1}(c_e/p)) + \rho_L pE[(Q - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon]
\end{aligned}$$

which is the expression in case (2). Similarly, substitute $Q - q = N(1 - e^{-\lambda_H q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$ into case (2), we have

$$\begin{aligned}
& pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - \rho_H c_e N(1 - e^{-\lambda_H q}) - cQ + \rho_H c_e (Q - q) \\
&+ \rho_H (pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon] - c_e (\bar{F}_\epsilon)^{-1}(c_e/p)) + \rho_L pE[(Q - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon] \\
&= pN(1 - \rho_H e^{-\lambda_H q} - \rho_L e^{-\lambda_L q}) - cQ + \rho_L pE[(Q - q - N(1 - e^{-\lambda_L q})) \wedge \epsilon] \\
&+ \rho_H pE[(\bar{F}_\epsilon)^{-1}(c_e/p) \wedge \epsilon],
\end{aligned}$$

which is the expression of case (3). Thus, $\pi(Q, q)$ is continuous in Q for any given q . Next we take derivative of $\pi(Q, q)$ over Q , and we have the following expression for $\partial\pi(Q, q)/\partial Q$:

$$\frac{\partial\pi(Q, q)}{\partial Q} = \begin{cases} c_e - c, & \text{if case (1) occurs,} \\ \rho_H c_e - c + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})), & \text{if case (2) occurs,} \\ -c + \rho_H p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_H q})) + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})), & \text{if case (3) occurs.} \end{cases}$$

Note that when $Q - q = N(1 - e^{-\lambda_L q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$, the derivative under case (2) satisfies:

$$\rho_H c_e - c + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})) \rho_H c_e - c + \rho_L p \bar{F}_\epsilon((\bar{F}_\epsilon)^{-1}(c_e/p)) = c_e - c \geq 0.$$

Similarly, when $Q - q = N(1 - e^{-\lambda_H q}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$, the derivative under case (2) satisfies:

$$\begin{aligned}
& \rho_H c_e - c + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})) \\
&= -c + \rho_H p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_H q})) + \rho_L p \bar{F}_\epsilon(Q - q - N(1 - e^{-\lambda_L q})).
\end{aligned}$$

Note that $\pi(Q, q)$ is piecewise concave in Q for any given q . Its derivative over Q is piecewise decreasing in Q . Also from the above boundary analysis, we can conclude that $\partial\pi(Q, q)/\partial Q$ is decreasing in Q . Thus, $\pi(Q, q)$ is concave in Q . Because $\pi(Q, q)$ is increasing in Q under case (1),

and hence case (1) is suboptimal, thus, under optimality, we have $Q^* - q^* \geq N(1 - e^{-\lambda L q^*}) + (\bar{F}_\epsilon)^{-1}(c_e/p)$. ■

Proof (Proof of Proposition 7) Note that $\bar{\pi}(Q_e|\lambda, Q, q)$ is jointly concave in (Q_e, Q, q) , this is because $Q - q + Q_e$ is linear in (Q_e, Q, q) and $N(1 - e^{-\lambda q})$ is concave in q . Thus, taking the minimum of the two functions results in a concave function, which leads to the concavity of $\bar{\pi}(Q_e|\lambda, Q, q)$. Thus, maximizing over $Q_e \geq 0$ results in a concave function, and taking expectation of that concave function over λ , we have $\pi(Q, q)$ is jointly concave in (Q, q) . For any give q , maximizing over Q , $Q \geq q$ leads to a concave function of q , because of the concavity of $\pi(Q, q)$ and the linear constraint, i.e. $\pi^*(q)$ is concave in q .

Therefore, maximizing over q , we have if $q^0 \in [q_1, q_2]$, then due to the concavity of $\pi^*(q)$, $q^* = q^0$, and case (2) is optimal. If $q^0 \notin [q_1, q_2]$, then case (2) is suboptimal and case (3) must be optimal. thus, in this case, if $q^0 > q_2$, then $\pi^*(q)$ is increasing in $[q_1, q_2]$, which implies that $q^* \geq q_2$, thus, $q^* = \max\{\tilde{q}, q_2\}$. Similarly, if $q^0 < q_1$, then $\pi^*(q)$ is decreasing in $[q_1, q_2]$, which implies that $q^* \leq q_1$, thus, $q^* = \min\{\tilde{q}, q_1\}$.

$q^* = 0$ if and only if $\pi^{*'}(0) \leq 0$. Thus, substitute $Q^*(0) = Q_n$ into $\pi^{*'}(0)$ we have the condition under which $q^* = 0$. ■