The Excluded Minors for Isometric Realizability in the Plane

Fiorini, Samuel; Huynh, Tony; Joret, Gwenaël; Varvitsiotis, Antonios

2017

Fiorini, S., Huynh, T., Joret, G., & Varvitsiotis, A. (2017). The Excluded Minors for Isometric Realizability in the Plane. SIAM Journal on Discrete Mathematics, 31(1), 438-453.

https://hdl.handle.net/10356/81454

https://doi.org/10.1137/16M1064775

SIAM J. DISCRETE MATH.
Vol. 31, No. 1, pp. 438–453

THE EXCLUDED MINORS FOR ISOMETRIC REALIZABILITY IN THE PLANE*

SAMUEL FIORINI†, TONY HUYNH†, GWENAËL JORET‡, AND ANTONIOS VARVITSIOTIS§

Abstract. Let G be a graph and $p \in [1, \infty]$. The parameter $f_p(G)$ is the least integer k such that for all m and all vectors $(r_v)_{v \in V(G)} \subseteq \mathbb{R}^m$, there exist vectors $(q_v)_{v \in V(G)} \subseteq \mathbb{R}^k$ satisfying $\|r_v - r_w\|_p = \|q_v - q_w\|_p$ for all $vw \in E(G)$. It is easy to check that $f_p(G)$ is always finite and that it is minor monotone. By the graph minor theorem of Robertson and Seymour [*J. Combin. Theory Ser. B*, 92 (2004), pp. 325–357], there are a finite number of excluded minors for the property $f_p(G) \leqslant k$. In this paper, we determine the complete set of excluded minors for $f_\infty(G) \leqslant 2$. The two excluded minors are the wheel on five vertices and the graph obtained by gluing two copies of K_4 along an edge and then deleting that edge. We also show that the same two graphs are the complete set of excluded minors for $f_1(G) \leqslant 2$. In addition, we give a family of examples that show that f_∞ is unbounded on the class of planar graphs and f_∞ is not bounded as a function of tree-width.

Key words. finite metric spaces, isometric embeddings, graph minors

AMS subject classification. 05C10

DOI. 10.1137/16M1064775

1. Introduction.

Let X be a finite set and $d : X \times X \to \mathbb{R}_{\geqslant 0}$. We say that (X, d) is a *metric space* if d satisfies the following properties: (i) $d(i, j) = d(j, i)$ for all $i, j \in X$, (ii) $d(i, j) = 0$ if and only if $i = j$, and (iii) $d(i, j) \leqslant d(i, k) + d(k, j)$ for all $i, j, k \in X$. For $x \in \mathbb{R}^m$ define $\|x\|_p := (\sum_{i=1}^m |x_i|^p)^{1/p}$ and $\|x\|_\infty := \max_{i=1}^m |x_i|$. Recall that $\|\cdot\|_p$ is a norm for all $p \in [1, \infty]$. Throughout this paper we denote by ℓ_p^m the metric space $(\mathbb{R}^m, d_p)$ where $d_p(x, y) = \|x - y\|_p$.

A natural way to compare two metric spaces (X, d) and (X', d') is through the use of distance preserving maps from one space to the other. Formally, an *isometric embedding* of (X, d) into (X', d') is a function $\phi : X \to X'$ such that $d(x, y) = d'(\phi(x), \phi(y))$ for all $x, y \in X$.

Typically, the requirement that all pairwise distances be preserved exactly is too restrictive to be useful in practice. To cope with this, a successful theory of embeddings with distortion has been developed, where the requirement that distances be preserved exactly is relaxed to the requirement that no distance shrink or stretch excessively. In this direction, the celebrated theorem of Bourgain [6] asserts that

*Received by the editors March 8, 2016; accepted for publication (in revised form) December 23, 2016; published electronically March 2, 2017.
http://www.siam.org/journals/sidma/31-1/M106477.html
Funding: The work of the first and second authors was supported in part by ERC grant *FOREFRONT* (grant agreement 615640) funded by the European Research Council under the EU's 7th Framework Programme (FP7/2007-2013). The first author also acknowledges support from *ARC* grant AUWB-2012-12/17-ULB2, *COPHYMA* funded by the French community of Belgium. The work of the fourth author was supported in part by the Singapore National Research Foundation under NRF RF Award NRF-NRFF2013-13.
†Mathematics Department, Université Libre de Bruxelles, Brussels B-1050, Belgium (sfiorini@ulb.ac.be, tony.bourbaki@gmail.com).
‡Computer Science Department, Université Libre de Bruxelles, Brussels B-1050, Belgium (gjoret@ulb.ac.be).
§School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, and Centre for Quantum Technologies, National University of Singapore, Singapore (avarvits@gmail.com).

every n-point metric space can be embedded into an $\ell_p^{O(\log^2 n)}$ space with $O(\log n)$ distortion. Moreover, this is best possible up to a constant factor.

Another popular approach is to only require a *subset* of the distances to be preserved exactly. This viewpoint is very graph theoretical and is the approach that we take in this paper.

All graphs in this paper are finite and do not contain loops or parallel edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting some edges. When taking minors, we always suppress parallel edges and loops.

Let G be a graph and $p \in [1, \infty]$. We define $f_p(G)$ to be the least integer k such that for all m and all vectors $(r_v)_{v \in V(G)} \subseteq \mathbb{R}^m$, there exist vectors $(q_v)_{v \in V(G)} \subseteq \mathbb{R}^k$ satisfying

$$\|r_v - r_w\|_p = \|q_v - q_w\|_p \quad \text{for all} \quad vw \in E(G).$$

It is not obvious that this parameter is always finite, but from the conic version of Carathéodory's theorem, it follows that $f_p(G) \leqslant \binom{n}{2}$ for all $p \in [1, \infty]$ and all n-vertex graphs G (see [2] and [7, Proposition 11.2.3]). For $p = 2$, Barvinok [3] showed the better bound $f_2(G) \leqslant (\sqrt{8m+1} - 1)/2$ for graphs G with m edges.

Let K_n denote the complete graph on n vertices. The study of $f_p(K_n)$ for varying values of $p \in [1, \infty]$ is a fundamental problem in the theory of metric embeddings. For the case $p = \infty$, Holsztynski [9] (and subsequently Witsenhausen [19]) showed that

$$\left\lfloor \frac{2n}{3} \right\rfloor \leqslant f_\infty(K_n) \leqslant n - 2 \quad \text{for } n \geqslant 4.$$

Furthermore, Witsenhausen [19] showed that $f_1(K_n) \geqslant n - 2$ for $n \geqslant 3$, which was later improved to

$$f_1(K_n) \geqslant \binom{n-2}{2} \quad \text{for } n \geqslant 3$$

by Ball [2]. Lastly, Ball [2] also showed that

$$f_p(K_n) \geqslant \binom{n-1}{2} \quad \text{for all } 1 < p < 2 \text{ and } n \geqslant 3$$

and that there is a constant c such that

$$f_\infty(K_n) \geqslant n - cn^{3/4} \quad \text{for all } n.$$

The lower bound of $n - cn^{3/4}$ uses the *biclique covering number*, which is the minimum number of complete bipartite subgraphs needed to cover the edges of a graph. Rödl and Ruciński [14] have since shown that there is a constant c such that for every n there exists an n-vertex graph that cannot be covered with $n - c \log n$ complete bipartite subgraphs. This implies that there is a constant c such that

$$f_\infty(K_n) \geqslant n - c \log n \quad \text{for all } n.$$

The parameters $f_p(G)$ are also widely studied in rigidity theory. We refer the interested reader to Kitson [10] and Sitharam and Gao [16] and the references therein.

It is easy to show that for all $p \in [1, \infty]$, the parameter $f_p(G)$ is minor monotone. By the graph minor theorem of Robertson and Seymour [12], there are a finite number of minor-minimal graphs G with $f_p(G) > k$. We call these graphs the *excluded minors* for $f_p(G) \leqslant k$.

The excluded minors for $f_2(G) \leqslant 1$, $f_2(G) \leqslant 2$, and $f_2(G) \leqslant 3$ were determined by Belk and Connelly [4, 5].

THEOREM 1.1 (see [4, 5]). *For every graph G,*
(i) $f_2(G) \leqslant 1$ *if and only if G has no K_3 minor;*
(ii) $f_2(G) \leqslant 2$ *if and only if G has no K_4 minor;*
(iii) $f_2(G) \leqslant 3$ *if and only if G has no K_5 minor and no $K_{2,2,2}$ minor.*

In this paper we mainly focus on the case $p = \infty$. The ℓ_∞-spaces are particularly interesting due to their "universal" nature in terms of isometric embeddings, as illustrated by the following theorem of Fréchet.

THEOREM 1.2 (see [8]). *Every n-point metric space can be isometrically embedded in ℓ_∞^{n-1}.*

Theorem 1.2 allows us to rephrase the condition $f_\infty(G) \leqslant k$ as follows. Let G be a graph and $d : E(G) \to \mathbb{R}_{\geqslant 0}$. The *length* of a path P in G is defined as $\sum_{e \in E(P)} d_e$. Throughout this work we call $d : E(G) \to \mathbb{R}_{\geqslant 0}$ a *distance function on G* if for all edges $xy \in E(G)$, every path from x to y has length at least d_{xy} (in other words, the path consisting of the edge xy is a shortest path). We remark that $d_{xy} = 0$ is allowed in this definition and that d defines a corresponding metric space X on at most $|V(G)|$ points as follows. First contract all edges xy with $d_{xy} = 0$, and then consider the shortest path lengths between pairs of vertices. Hence, by Theorem 1.2, $f_\infty(G) \leqslant k$ if and only if for all distance functions d on G, there exist vectors $(q_v)_{v \in V(G)} \subseteq \mathbb{R}^k$ satisfying

$$\|q_x - q_y\|_\infty = d_{xy} \quad \text{for all} \quad xy \in E(G).$$

Note that for all $p, q \in [1, \infty]$, $\ell_p^1 = \ell_q^1$. Thus, by Theorem 1.1, $f_\infty(G) \leqslant 1$ if and only if G has no K_3 minor. In this paper we determine the complete set of excluded minors for $f_\infty(G) \leqslant 2$. Let W_4 denote the wheel on five vertices and $K_4 +_e K_4$ be the graph obtained by gluing two copies of K_4 along an edge e and then deleting e; see Figure 1. Using techniques from rigidity matroids, Sitharam and Willoughby [17] determined $f_\infty(G)$ for all graphs G with at most five vertices, except for W_4. They conjectured that W_4 is an excluded minor for $f_\infty(G) \leqslant 2$, and that W_4 is the *only* excluded minor for $f_\infty(G) \leqslant 2$. We verify their first conjecture but disprove the second by showing that $K_4 +_e K_4$ is also an excluded minor for $f_\infty(G) \leqslant 2$.

The following is our main result.

THEOREM 1.3 (Main Theorem). *The excluded minors for $f_\infty(G) \leqslant 2$ are W_4 and $K_4 +_e K_4$.*

The proof of Theorem 1.3 is given in section 6. Note that unlike the $p = 2$ case, given points $x, y, x', y' \in \mathbb{R}^m$ with $\|x - y\|_\infty = \|x' - y'\|_\infty$ there does not necessarily exist an isometry of ℓ_∞^m which maps x to x' and y to y'. For example, take $x = x' = (0, 0)$ and $y = (0, 1), y' = (1, 1)$ in ℓ_∞^2. Indeed, the isometries of ℓ_∞^m correspond to signed permutation matrices. Therefore, our proof technique for the $p = \infty$ case is quite different from the $p = 2$ case. For example, we will show that the property $f_\infty(G) \leqslant 2$ is not closed under taking 2-sums.

We also prove the following result, which follows from Theorem 1.3 with a little extra work.

COROLLARY 1.4. *The excluded minors for $f_1(G) \leqslant 2$ are W_4 and $K_4 +_e K_4$.*

Robertson and Seymour [13] proved that testing for a fixed minor can be done in cubic time. Therefore, our results give an explicit cubic-time algorithm to test if $f_1(G) \leqslant 2$ (equivalently, $f_\infty(G) \leqslant 2$). We simply have to test if our input graph contains a W_4 minor or a $K_4 +_e K_4$ minor.

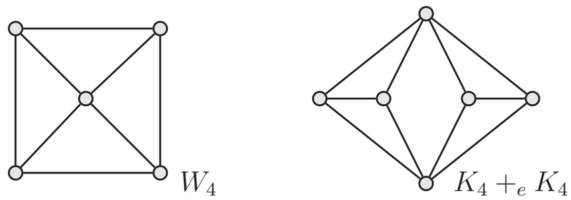


FIG. 1. *The excluded minors for $f_\infty(G) \leqslant 2$.*

In a previous version of this paper, we asked whether f_∞ is bounded on the class of planar graphs. We also asked whether f_∞ is bounded as a function of tree-width. We now have found an example that shows that the answer to both of these questions is negative.

THEOREM 1.5. *For every k there exists a planar graph G with tree-width* 3 *such that $f_\infty(G) \geqslant k$.*

Paper organization. In section 2 we present a few equivalent ways to think about $f_\infty(G)$ and prove some upper and lower bounds. In section 3, we show $f_\infty(K_7) = 5$. In section 4 we show that we can suppress degree-2 vertices when computing $f_\infty(G)$. In section 5 we show that W_4 and $K_4 +_e K_4$ are excluded minors for $f_\infty(G) \leqslant 2$. In section 6 we show that W_4 and $K_4 +_e K_4$ are the *only* excluded minors for $f_\infty(G) \leqslant 2$ and explain how to deduce Corollary 1.4 from the Main Theorem. We conclude the paper in section 7 by proving Theorem 1.5 and discussing some open problems.

2. Potentials and implicit realizations.

In this section we present several equivalent ways to think about the parameter $f_\infty(G)$.

Consider an n-vertex graph G, a distance function d on G, and a realization of (G, d) in ℓ_∞^k, that is, a collection of points $(q_v)_{v \in V(G)} \in \mathbb{R}^k$ such that $||q_v - q_w||_\infty = d_{vw}$ for all $vw \in E(G)$. We can write a $k \times n$ matrix whose columns are the vectors q_v for $v \in V(G)$. In this section we analyze this matrix by looking at its rows, which turn out to be potentials of a natural directed graph associated to (G, d).

Let D be an edge-weighted directed graph, and let $l : A(D) \to \mathbb{R}$ be the length function on the arcs of D. Note that negative lengths are allowed. A function $p : V(D) \to \mathbb{R}$ is called a *potential on D* if $p(v) - p(u) \leqslant l(a)$ for all arcs $a = (u, v) \in A(D)$. We recall the following well-known result characterizing the existence of a potential.

THEOREM 2.1. *A weighted directed graph (D, l) admits a potential if and only if it does not contain any negative length directed cycle.*

Now let $D = D(G, d)$ be the weighted directed graph obtained from (G, d) as follows. First, we bidirect all edges of G. For every edge $uv \in E(G)$, we define the length of both (u, v) and (v, u) to be d_{uv}. That is, the length function l on D is given by

$$l(u, v) = l(v, u) := d_{uv} \quad \text{for all } uv \in E(G). \tag{1}$$

Note that $p : V(D) \to \mathbb{R}$ is a potential on D if and only if $|p(v) - p(u)| \leqslant d_{uv}$ for all $uv \in E(G)$. An edge $uv \in E(G)$ is *tight* for a potential p on D if $|p(v) - p(u)| = d_{uv}$.

Let $(q_v)_{v \in V(G)}$ be a realization of (G, d) in ℓ_∞^k. Clearly, if we define $p_i(v) := q_v(i)$

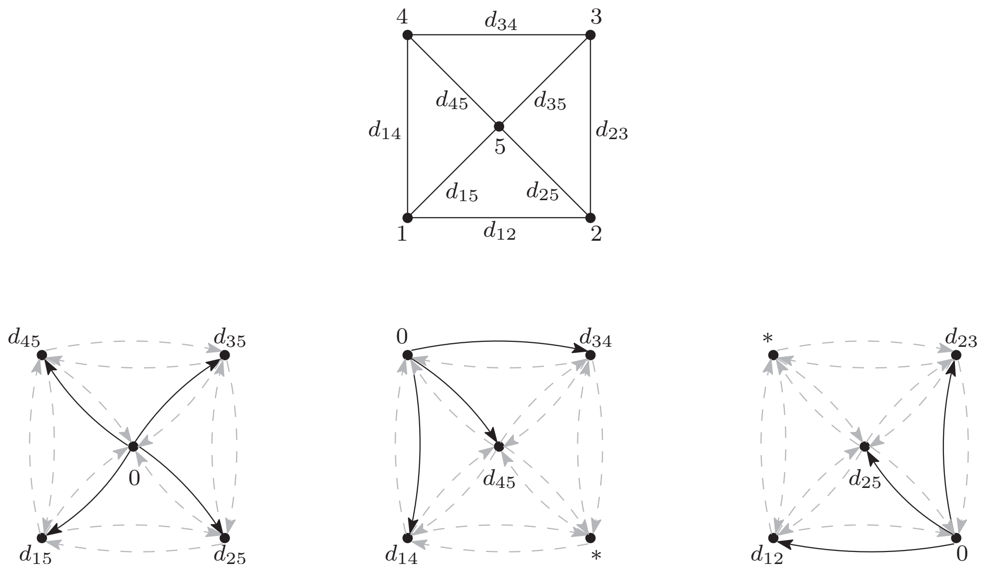


FIG. 2. *If G denotes the 5-vertex wheel W_4, then (G, d) admits a realization in ℓ_∞^3 for all distance functions d, as shown by these three potentials. (The two values labeled $*$ are not used to realize any edge, so they can be set to any value that is feasible.)*

for $i \in [k]$ and $v \in V(G)$, we have that p_i is a potential for all $i \in [k]$. Moreover, every edge of G is tight in some p_i. It is easy to see that the converse also holds.

LEMMA 2.2. *Let G be a graph. A distance function d on G admits a realization $(q_v)_{v \in V(G)}$ in ℓ_∞^k if and only if the directed graph $D = D(G, d)$ with lengths as in* (1) *admits a collection of potentials $(p_i)_{i\in[k]}$ such that every edge $uv \in E(G)$ is tight in some p_i. Moreover, in this equivalence we can take $q_v(i) = p_i(v)$ for all $i \in [k]$ and $v \in V(G)$.*

In view of Lemma 2.2, we get a combinatorial approach to constructing and analyzing realizations. For $F \subseteq E(G)$, let $\overrightarrow{F}$ denote some orientation of F. We say that $\overrightarrow{F}$ is a *feasible orientation* (with respect to d) if there exists a potential p on $D(G, d)$ such that $p(v) - p(u) = d_{uv}$ for all $(u, v) \in \overrightarrow{F}$. See Figure 2 for an illustration. We say that $F \subseteq E(G)$ is *feasible* if it admits a feasible orientation. If a set of edges is not feasible, we say that it is *infeasible*. Notice that $\overrightarrow{F}$ is a feasible orientation if and only if the opposite orientation $\overleftarrow{F}$ is a feasible orientation. Furthermore, note that a subset of a feasible set is also feasible.

The notion of feasible sets allows us to reformulate Lemma 2.2 as follows.

LEMMA 2.3. *Let G be a graph and d be a distance function on G. The pair (G, d) admits a realization in ℓ_∞^k if and only if there exist feasible sets $(F_i)_{i\in[k]}$ such that $\cup_{i=1}^k F_i = E(G)$.*

Given an orientation $\overrightarrow{F}$, we define a modification of the length function $l(d)$ as follows:

$$l(u, v) := \begin{cases} d_{uv} & \text{if } uv \in E(G), (u, v) \notin \overrightarrow{F}, \\ -d_{uv} & \text{if } (u, v) \in \overrightarrow{F}. \end{cases} \tag{2}$$

We denote this length function by $l(d, \overrightarrow{F})$. Note that $\overrightarrow{F}$ is a feasible orientation if and only if $(G, l(d, \overrightarrow{F}))$ admits a potential. By Theorem 2.1, this happens if and only if the weighted digraph $(G, l(d, \overrightarrow{F}))$ does not contain a directed cycle of negative length.

We demonstrate the usefulness of Lemma 2.3 by quickly deriving some nontrivial upper and lower bounds for $f_\infty(G)$.

Note that for every distance function d on G and every vertex v of G, the star centered at v is always feasible with respect to d, as can be seen by orienting all the edges of the star outward (as in Figure 2). From this we obtain the following upper bound.

Lemma 2.4. *For every graph G,*

$$f_\infty(G) \leqslant \tau(G),$$

where $\tau(G)$ denotes the minimum size of a vertex cover of G.

We say that a distance function d is *generic* with respect to G if for every cycle C in G and $S \subseteq E(C)$, we have $\sum_{e\in S} d_e \neq \sum_{e\in E(C)\setminus S} d_e$. Every distance function d on G can be perturbed to a nearby generic distance function d'. Furthermore, we have $f_\infty(G) \leqslant k$ if and only if (G, d) can be realized in ℓ_∞^k for every *generic* distance function d.

Observe that if d is generic, every feasible set is acyclic. Therefore, we immediately obtain the following lemma.

Lemma 2.5. *For every graph G,*

$$f_\infty(G) \geqslant \Upsilon(G),$$

where $\Upsilon(G)$ denotes the minimum number of forests required to partition $E(G)$.

Our next result implies that, if d is generic, every maximal feasible set is a spanning forest.

Lemma 2.6. *Let G be a graph and d be a distance function on G. Then every maximal feasible set $F \subseteq E(G)$ contains a spanning forest.*

Proof. Toward a contradiction, suppose that $F \subseteq E(G)$ is a maximal feasible set that does not contain a spanning forest of G. Let X be the vertex set of a component of $(V(G), F)$ such that G contains at least one edge with exactly one end in X. Let p be any potential that makes all the edges of F tight but no other edges. Let Δ be as large as possible with the property that $p' := p + \Delta \sum_{v\in X} e_v$ is a potential, where e_v denotes the characteristic vector for the vertex v. Then the set of edges that are tight with respect to p' is a proper superset of F, which is a contradiction. ◻

3. $f_\infty(K_7) = 5$. Since $\lfloor \frac{2n}{3} \rfloor = n-2$ for $n \in \{4, 5, 6\}$, it follows that $f_\infty(K_3) = 2$, $f_\infty(K_4) = 2$, $f_\infty(K_5) = 3$, and $f_\infty(K_6) = 4$. Thus, $n = 7$ is the smallest value for which $f_\infty(K_n)$ is unknown. In this section we show that $f_\infty(K_7) = 5$. This result is not needed for our Main Theorem but may be of independent interest.

Proposition 3.1. $f_\infty(K_7) = 5$.

Proof. We already know that $f_\infty(K_7) \leqslant 5$; let us prove that $f_\infty(K_7) \geqslant 5$. To this aim, enumerate the vertices of K_7 as $v_1, \ldots, v_7$, and define a linear ordering L on its edges by letting, for $i < j$ and $k < \ell$,

$$v_i v_j >_L v_k v_\ell$$

if $i < k$, or $i = k$ and $j < \ell$. Let $m := 21$ be the number of edges. Define a distance function d on the graph by letting $d(e) := 2^m + 2^r$ for each edge e, where r is the rank of e in the ordering L. (Thus v_1v_2 has rank m and v_6v_7 has rank 1.) It is easy to check that d is a generic distance function.

We claim that (K_7, d) cannot be realized in ℓ_∞^4. Arguing by contradiction, assume it can. Consider a partition of the edges into four feasible forests $F_1, \dots, F_4$. Before analyzing these, let us note a few properties of a feasible forest F (the easy proofs are left to the reader).

1. A feasible orientation $\overrightarrow{F}$ of F cannot contain a length-2 directed path; hence $\overrightarrow{F}$ is uniquely determined (up to reversing all arcs).
2. If $i < j < k < \ell$, then at most one of the two edges v_iv_j and v_kv_ℓ is in F.
3. If $i < j < k < \ell$, then at most two of the three edges v_iv_k, v_jv_k, v_jv_ℓ are in F.

Now color each edge e of the graph with the index i of the forest F_i in which it is included. By property 2 we may assume without loss of generality that v_1v_2, v_3v_4, and v_5v_6 are colored 1, 2, and 3, respectively. By the same property, none of the two edges v_5v_7, v_6v_7 are colored 1 or 2, and they cannot both be colored 3 (otherwise $v_5v_6v_7$ would be a triangle in F_3); thus there exists $a \in \{5, 6\}$ such that v_av_7 is colored 4.

Next consider the four edges between the sets $\{v_1, v_2\}$ and $\{v_3, v_4\}$. None of these is colored 3 by property 2 (because of the edge v_5v_6) or 4 (because of the edge v_av_7), so each of them is colored 1 or 2. Moreover, in order to avoid monochromatic triangles, the four edges are split into two matchings M_1 and M_2 of size 2, colored 1 and 2, respectively.

Let X be the set of edges v_iv_j with $i, j \geqslant 3$ that are distinct from v_3v_4. (Thus $|X| = 9$.) No edge in X is colored 1 (because of v_1v_2). We claim that no edge in X is colored 2 either. This is clear for those not incident to v_3, thanks to the edge of M_2 that is incident to v_3. Now, suppose for a contradiction that $f \in X$ is incident to v_3 and is colored 2. Then letting e be the edge of M_2 incident to v_4, we see that the edges e, v_3v_4, f are all in F_2, contradicting property 3.

All edges in X are colored 3 or 4, but X has size 9 and spans only five vertices. Therefore, there is a monochromatic cycle in X. This final contradiction concludes the proof. ∎

4. Degree-2 vertices. In this section we show that we can essentially ignore degree-2 vertices when computing $f_\infty(G)$.

Let G_1 and G_2 be graphs that each contain a clique K of size k. A *k-sum* of G_1 and G_2 along K is a graph obtained by gluing G_1 and G_2 along K and then deleting some of the edges of K. In the special case of 2-sums, we use the notation $G_1 \oplus_e G_2$ if we keep the edge e, and $G_1 +_e G_2$ if we delete the edge e.

LEMMA 4.1. *Let H be a graph, and let $e \in E(H)$. If $f_\infty(H) \geqslant 2$, then $f_\infty(H) = f_\infty(H \oplus_e K_3)$.*

Proof. Set $G := H \oplus_e K_3$, let $e = uv$, and let w be the newly added vertex in G. Clearly $f_\infty(G) \geqslant f_\infty(H)$, so it suffices to show that $f_\infty(G) \leqslant f_\infty(H)$. Let d be any distance function on G. The restriction of d to H is also a distance function. Let $(F_i)_{i \in [k]}$ be a collection of $k := f_\infty(H) \geqslant 2$ feasible sets of (H, d) such that $\cup_{i=1}^k F_i = E(H)$.

First, note that each F_i is feasible in G. Indeed, since d is a distance function, and in particular $d_{uw} + d_{wv} \geqslant d_{uv}$, we can extend any potential on $D(H, d)$ to a potential

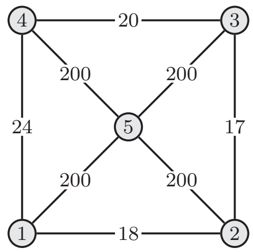


FIG. 3. *W_4 and a distance function that cannot be realized in ℓ_∞^2.*

on $D(G,d)$ by carefully choosing the potential value at w between the value at u and that at v. Without loss of generality, we may assume that $uv \in F_1$. Now extend F_2 to a maximal feasible set $F_2' \subseteq E(G)$. By Lemma 2.6, F_2' contains a spanning forest. Hence, F_2' contains either wu or wv. Without loss of generality, assume that $wu \in F_2'$.

Now let $\overrightarrow{F_1}$ be a feasible orientation of F_1. By reversing all the arcs of $\overrightarrow{F_1}$ if necessary, we may assume that $(u,v) \in \overrightarrow{F_1}$. We claim that $\overrightarrow{F_1'} := \overrightarrow{F_1} \cup \{(w,v)\}$ is a feasible orientation. Indeed, let C be a negative directed cycle in $D = D(G,d)$ with respect to $l := l(d, \overrightarrow{F_1'})$. Since $\overrightarrow{F_1}$ is a feasible orientation, we may assume that $(u,w),(w,v) \in A(C)$. Now $l(u,w) + l(w,v) = d_{uw} - d_{wv} \geqslant -d_{uv} = l(u,v)$, which means that the length of C does not increase if we shortcut it from u to v. Since $\overrightarrow{F_1}$ is a feasible orientation, the length of the shortcut cycle is nonnegative, which contradicts our assumption that C has negative length. Hence, $\overrightarrow{F_1'}$ is a feasible orientation, and the corresponding edge set F_1' is feasible.

We have found k feasible sets F_1', F_2', F_3, …, F_k that cover each edge of G. Thus (G,d) can be realized in ℓ_∞^k. The lemma follows. ∎

We note that the assumption that $f_\infty(H) \geqslant 2$ in Lemma 4.1 is necessary. This can easily be seen by taking $H = K_2$ and $G = K_3$.

We say that G is obtained from H by *subdividing an edge* e if $G = H +_e K_3$.

LEMMA 4.2. *Let G and H be graphs such that G is obtained from H by subdividing an edge. Then $f_\infty(G) = f_\infty(H)$.*

Proof. Clearly $f_\infty(G) \geqslant f_\infty(H)$ since H is a minor of G. It remains to prove $f_\infty(G) \leqslant f_\infty(H)$. If $f_\infty(H) = 1$, then H is a forest, and so is G, implying $f_\infty(G) = 1$. Hence we may assume that $f_\infty(H) \geqslant 2$. Say that G is obtained from H by subdividing an edge uv with a new vertex w. Let $G' := G + uv$. Since G' is obtained from H by adding a new vertex w adjacent to the ends of the edge uv, we have that $f_\infty(G') = f_\infty(H)$ by Lemma 4.1. The graph G being a minor of G', it follows that $f_\infty(G) \leqslant f_\infty(G') = f_\infty(H)$. ∎

5. The graphs W_4 and $K_4 +_e K_4$. In this section we show that W_4 and $K_4 +_e K_4$ are excluded minors for $f_\infty(G) \leqslant 2$.

LEMMA 5.1. *We have that $f_\infty(W_4) = 3$.*

Proof. By Lemma 2.4, $f_\infty(W_4) \leqslant 3$. Toward a contradiction suppose $f_\infty(W_4) \leqslant 2$. Let d be the distance function on W_4 given in Figure 3, and let $q_1, \ldots, q_5$ be an isometric embedding of (G,d) in ℓ_∞^2. Note that $q_1, \ldots, q_4$ all lie on two consecutive sides of a square centered at q_5 with side length 400. By symmetry we may assume that $q_5 = (200, -200)$, that $q_1 = (x, 0)$ where $0 \leqslant x \leqslant 200$, and that $q_i(1) = 0$ or

$q_i(2) = 0$ for $i \in \{2,3,4\}$. We say that $(a,0)$ is *directly right of* $(b,0)$ if $b < a$ (in this case $(b,0)$ is *directly left of* $(a,0)$), $(0,c)$ is *directly below* $(0,d)$ if $c < d$, and $(a,0)$ and $(0,c)$ are *diagonal*.

We first consider the case that q_4 is directly right of q_1. This implies that q_3 must be directly left of q_4 as q_2 would be too far from q_1 (if q_3 is directly right of q_4) or q_3 would be too far from q_4 (if q_3 and q_4 are diagonal). Now, q_2 cannot be directly right of q_3 as q_2 would be too far from q_1, and q_2 cannot be directly left of q_3 as q_2 would be too close to q_1. Thus, q_2 and q_3 are diagonal. But now $\|q_1 - q_2\|_\infty \leqslant \|q_3 - q_2\|_\infty$, which is a contradiction.

We next consider the case that q_4 is directly left of q_1. Again, q_3 cannot be directly left of q_4. Suppose that q_3 is directly right of q_4. Again, q_2 cannot be directly right of or left of q_3. Thus, q_2 and q_3 are diagonal. But now $\|q_2 - q_3\|_\infty \geqslant 20$, which is a contradiction. Thus, q_3 and q_4 must be diagonal. If q_2 is directly above or directly below q_3, then $\|q_2 - q_1\|_\infty \geqslant 24$, which is a contradiction. Thus, q_2 and q_3 are diagonal. Since $d_{3,4} = 20$, we must have $q_3 = (-20,0)$ or $q_4 = (0,20)$. In the first case, $\|q_2 - q_3\|_\infty \geqslant 20$, and in the second case $\|q_2 - q_1\|_\infty \geqslant 27$, both of which are contradictions.

The remaining case is that q_1 and q_4 are diagonal. Thus, $q_1 = (24,0)$ or $q_4 = (0,-24)$. Suppose $q_1 = (24,0)$. If q_2 and q_1 are diagonal, then $\|q_1 - q_2\|_\infty \geqslant 24$, which is a contradiction. If q_2 is directly right of q_1, then q_3 is too far away from q_4. Thus, $q_2 = (6,0)$. Evidently, q_3 cannot be directly left of q_2. If q_3 is directly right of q_2, we have $\|q_3 - q_4\|_\infty \geqslant 23$, which is a contradiction. If q_3 and q_2 are diagonal, then q_3 and q_4 are too close. We finish with the subcase that $q_4 = (0,-24)$. Again, we must have $q_3 = (0,-4)$. If q_2 is directly below q_3, then $\|q_2 - q_1\|_\infty \geqslant 21$, which is a contradiction. If q_2 and q_3 are diagonal, then $q_2 = (17,0)$ and is too close to q_1. This completes the subcase and the proof. ∎

Lemma 5.2. *The graph W_4 is an excluded minor for $f_\infty(G) \leqslant 2$. Moreover, W_4 is the only excluded minor for $f_\infty(G) \leqslant 2$ among all graphs with at most five vertices.*

Proof. By the previous lemma, $f_\infty(W_4) = 3$, so to prove that W_4 is an excluded minor it suffices to show that every proper minor H of W_4 satisfies $f_\infty(H) \leqslant 2$. If $|V(H)| \leqslant 4$, then $f_\infty(H) \leqslant 2$ since $f_\infty(K_4) \leqslant 2$. Now, say H is obtained from W_4 by only deleting edges. Deleting an edge yields a degree-2 vertex, which we can suppress by either Lemma 4.2 or Lemma 4.1. Again, we get a graph with at most four vertices, so we are done.

For the second part, let H be an excluded minor for $f_\infty(G) \leqslant 2$ with $|V(H)| \leqslant 5$. If H has a W_4 minor, then $H = W_4$. So we may assume that H has no W_4 minor. Let $e = ab$ and $f = ac$ be edges of K_5. By Lemma 4.1 we have that $f_\infty(K_5 - \{e,f\}) = f_\infty(K_4) = 2$. Since H has no W_4 minor, this implies that H is a minor of $K_5 - \{e,f\}$. But then, $f_\infty(H) \leqslant f_\infty(K_5 - \{e,f\}) = 2$, which is a contradiction. ∎

Lemma 5.3. *We have that $f_\infty(K_4 +_e K_4) = 3$.*

Proof. To simplify notation, throughout this proof we set $G := K_4 +_e K_4$. Furthermore, we use the labeling of the nodes of G given in Figure 4.

We first show that $f_\infty(G) \leqslant 3$. Let d be an arbitrary distance function on G. Note that $F_0 = \{02, 03, 04, 05\}$ and $F_1 = \{12, 13, 14, 15\}$ are feasible sets because they are stars. Thus, if $\{23, 45\}$ is feasible, then (G,d) can be realized in ℓ_∞^3 by Lemma 2.3. To conclude the proof, assume that $\{23, 45\}$ is not feasible. Note that $F_3 = \{30, 31, 32\}$ and $F_5 = \{50, 51, 54\}$ are feasible because they are stars. Let F_3'

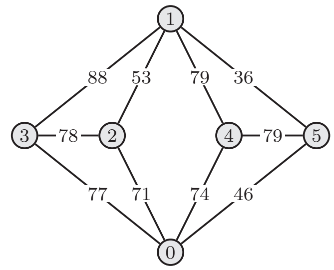


FIG. 4. $K_4 +_e K_4$ *and a distance function that cannot be realized in* ℓ^2_∞.

and F'_5 be maximal feasible sets containing F_3 and F_5, respectively. By Lemma 2.6, F'_3 and F'_5 each span all the vertices of G. Therefore, since $\{23, 45\}$ is not feasible, we must have $\{02, 12\} \cap F'_5 \neq \emptyset$ and $\{04, 14\} \cap F'_3 \neq \emptyset$. Let $F := E(G) \setminus (F'_3 \cup F'_5)$. Thus, F is a subset of $\{02, 04\}$, $\{12, 14\}$, $\{02, 14\}$, or $\{12, 04\}$. In the first two cases, F is feasible since it is a subset of a star. In the last two cases, note that $\{(0, 2), (4, 1)\}$ and $\{(1, 2), (4, 0)\}$ are feasible orientations of $\{02, 14\}$ and $\{12, 04\}$, respectively. Hence, F is also feasible in the last two cases. Since F'_3, F'_5, and F are feasible sets covering all the edges of G, Lemma 2.3 yields $f_\infty(G) \leqslant 3$.

To show that $f_\infty(G) = 3$, it remains to exhibit a distance function d on G such that (G, d) is not realizable in ℓ^2_∞. We exhibit such a distance function in Figure 4. Toward a contradiction, suppose that $E(G)$ can be partitioned into two feasible sets T_1 and T_2. It is easy to check that d is a generic distance function, and so T_1 and T_2 are both forests.[1] Thus, $|T_1|, |T_2| \leqslant |V(G)| - 1 = 5$ edges. Since $|E(G)| = 10$, we conclude that T_1 and T_2 are both spanning trees. Let T_L and T_R be the subgraphs of T_1 induced by $\{0, 1, 2, 3\}$ and $\{0, 1, 4, 5\}$, respectively. By interchanging T_1 and T_2, we may assume that $|E(T_L)| = 3$. Therefore, there are six possibilities for each of T_L and T_R, and these are shown in Figure 6. The six possibilities for T_L are shown in the first column of the table, and the six possibilities for T_R are shown in the first row.

We rule out each of the 36 possibilities for T_1 by showing that at least one of T_1 or T_2 is infeasible. To do this, we show that for all orientations $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ of T_1 and T_2, at least one of $\overrightarrow{T_1}$ or $\overrightarrow{T_2}$ contains an infeasible orientation.

If abc forms a triangle in G, note that $\{(a, b), (b, c)\}$ is an infeasible orientation. Indeed, the triangle inequality combined with the fact that d is generic implies that the directed cycle (a, b, c) is negative. We denote this infeasible orientation as $\Delta(a, b, c)$. In Figure 5, we list more infeasible orientations that do not come from triangles. These infeasible orientations consist only of the oriented arcs in each picture. However, for the benefit of the reader, we have included dashed edges to indicate the negative cycle in $D(G, d)$.

The remainder of the proof is summarized in Figure 6. Each entry in the table gives the infeasible orientations to apply in order to obtain a contradiction. For example, consider the fourth row of the table. For this entire row, it suffices to only consider the edges in $E(T_L)$. By symmetry, we may assume that $(0, 2) \in \overrightarrow{T_L}$. Next, $\Delta(3, 0, 2)$ implies that $(0, 3) \in \overrightarrow{T_L}$. Then, $A2$ implies $(1, 3) \in \overrightarrow{T_L}$. Since $(1, 3), (0, 2) \in \overrightarrow{T_L}$, we contradict $A1$. Thus, $\Delta(3, 0, 2)$, $A1$, and $A2$ are sufficient to derive a contradiction. Sometimes the infeasible orientations need to be applied to

[1] If one does not want to check genericity, simply perturb d to a nearby generic distance function.

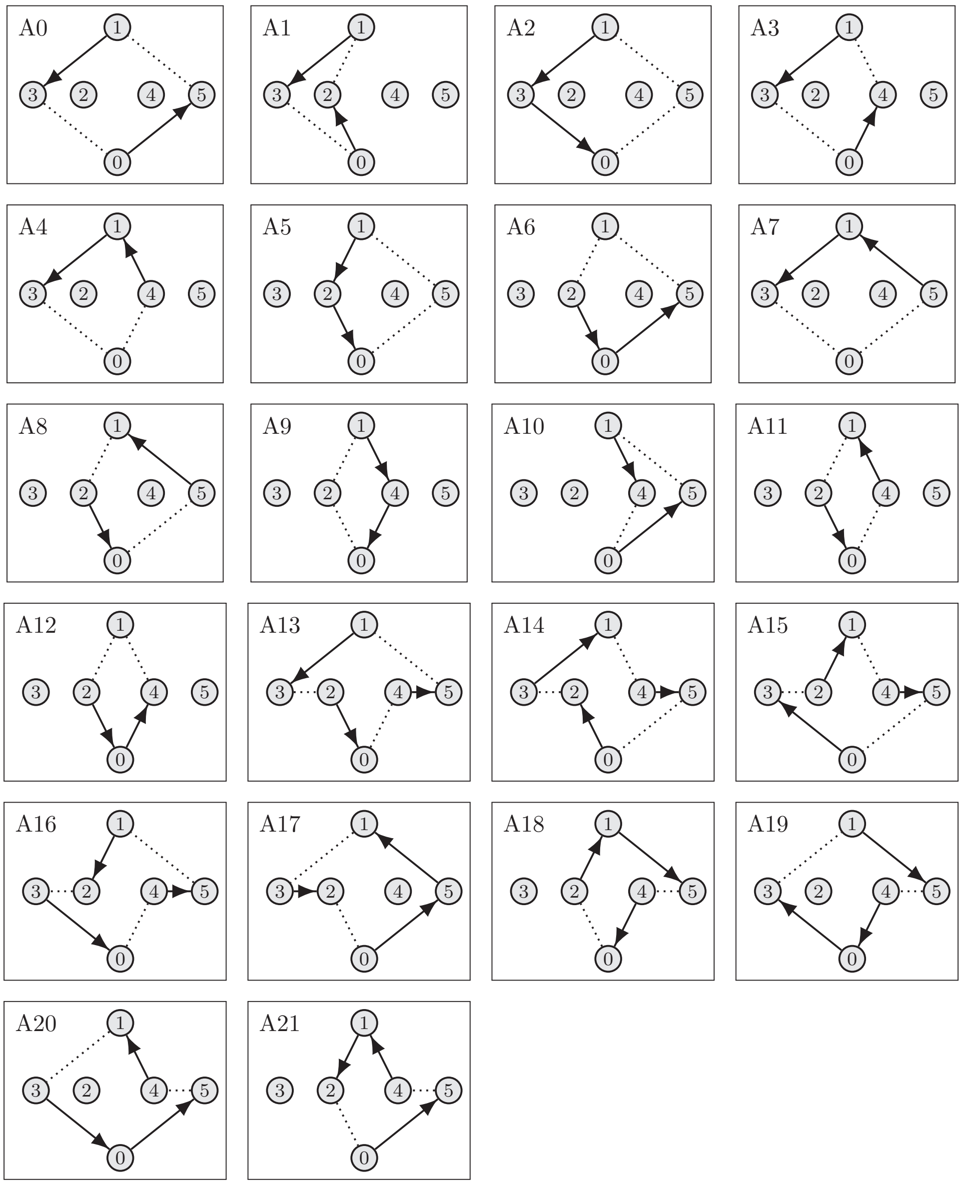


Fig. 5. *Infeasible orientations A0–A21.*

T_2 instead of to T_1, in which case we have specified so. ∎

Lemma 5.4. *The graph $K_4 +_e K_4$ is an excluded minor for $f_\infty(G) \leqslant 2$.*

Proof. By the previous lemma, $f_\infty(K_4 +_e K_4) = 3$, so it suffices to show that every proper minor H of $K_4 +_e K_4$ satisfies $f_\infty(H) \leqslant 2$. Contracting an edge of $K_4 +_e K_4$ yields a five-vertex graph which is not 3-connected. In particular, the latter graph does not have W_4 as a minor. We are done in this case, since by Lemma 5.2, W_4 is the only excluded minor for $f_\infty(G) \leqslant 2$ among graphs on at most five vertices.

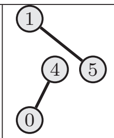
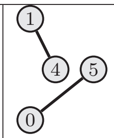
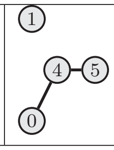
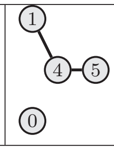
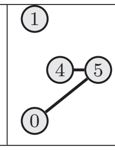
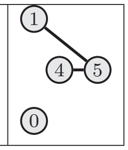
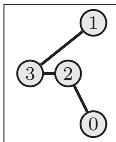
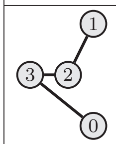
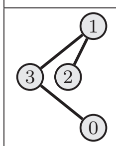
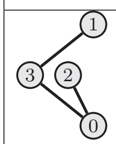
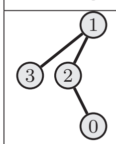
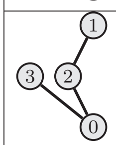
						
	$\Delta(0,5,4)$ $\Delta(5,4,1)$ A20, A21 A15 (in T_2)	$\Delta(0,4,5)$ $\Delta(4,5,1)$ A7, A19 A16 (in T_2)	$\Delta(0,2,3)$ $\Delta(1,3,2)$ A13, A14	$\Delta(0,2,3)$ $\Delta(1,3,2)$ A13, A14	$\Delta(0,2,3)$ $\Delta(1,3,2)$ A13, A14	$\Delta(0,2,3)$ $\Delta(1,3,2)$ A13, A14
	$\Delta(0,5,4)$ $\Delta(5,4,1)$ A0, A6 A14 (in T_2)	$\Delta(0,5,4)$ $\Delta(5,4,1)$ A7, A6 A14 (in T_2)	$\Delta(0,3,2)$ $\Delta(1,2,3)$ A15, A16	$\Delta(0,3,2)$ $\Delta(1,2,3)$ A15, A16	$\Delta(0,3,2)$ $\Delta(1,2,3)$ A15, A16	$\Delta(0,3,2)$ $\Delta(1,2,3)$ A15, A16
	$\Delta(2,1,3)$ A2, A7 A3, A19	$\Delta(2,1,3)$ A2, A4 A20, A10	$\Delta(0,2,3)$ $\Delta(4,1,5)$ A11, A6 A7 (in T_2)	$\Delta(0,2,3)$ $\Delta(0,5,1)$ $\Delta(4,0,5)$ A8, A6 (in T_2)	$\Delta(0,2,3)$ $\Delta(4,1,5)$ A8, A9 A12 (in T_2)	$\Delta(4,0,5)$ A9,A10 (in T_2)
	$\Delta(3,0,2)$ A1, A2	$\Delta(3,0,2)$ A1, A2	$\Delta(3,0,2)$ A1, A2	$\Delta(3,0,2)$ A1, A2	$\Delta(3,0,2)$ A1, A2	$\Delta(3,0,2)$ A1, A2
	A5, A8 A12, A18	A5, A6 A10, A20	A1, A13 A14	A1, A13 A14	A1, A13 A14	A1, A13 A14
	A5, A8 A12, A18	A5, A6 A10, A20	$\Delta(1,3,2)$ A0, A7 A17 (in T_2)	$\Delta(1,3,2)$ A0, A7 A17 (in T_2)	A3, A4 A9 (in T_2)	A3, A4 A9 (in T_2)

FIG. 6. *Proofs for all* 36 *possibilities for* T_1.

Deleting an edge from $K_4 +_e K_4$ creates a degree-2 vertex, which we can suppress by either Lemma 4.2 or Lemma 4.1. We then conclude as above, since the resulting five-vertex graph is not 3-connected and thus does not contain a W_4 minor. ∎

6. Proof of the Main Theorem.

The *wheel* on $n+1$ vertices, denoted by W_n, is the graph obtained by adding a universal vertex to an n-cycle. If G and G' are graphs such that $G = G' \backslash e$, we say that G' is obtained from G by *adding an edge*. Let $v \in V(G)$ with $\deg_G(v) \geqslant 4$. By *splitting* v, we mean the operation of first deleting v and then adding two new adjacent vertices v_1 and v_2, where each neighbor of v in G is adjacent to exactly one of v_1 and v_2, and v_1 and v_2 have degree at least three in the new graph.

We require the following classic theorem of Tutte [18].

THEOREM 6.1 (Tutte's wheel theorem). *Every* 3*-connected graph is obtained from a wheel by adding edges and splitting vertices.*

The following characterization of graphs without a W_4 minor is well known. For the convenience of the reader, we give a quick proof via Theorem 6.1.

THEOREM 6.2. *The only* 3*-connected graph with no* W_4 *minor is* K_4.

Proof. Let G be a 3-connected graph with no W_4 minor. By Tutte's wheel theorem, G is obtained from some W_n by adding edges and splitting vertices. Since G has no W_4 minor, we must have $n = 3$. If $G \neq W_3$, then we get a contradiction, since there is no way to add an edge to W_3 and stay simple, and there is no way to split a vertex (W_3 is cubic). Thus, $G = W_3 = K_4$, as required. ∎

We also need the following two technical lemmas.

LEMMA 6.3. *Let* G *be a* 2*-connected graph and* u *and* v *be distinct vertices of* G. *If* G *has a* K_4 *minor, then* G *has a* K_4 *minor* K *where* u *and* v *are contracted to distinct vertices of* K.

Proof. Let u and v be distinct vertices of G. Since G has a K_4 minor and K_4 is cubic, G also has a subgraph H which is a subdivision of K_4. By Menger's theorem, there are two disjoint paths from $\{u, v\}$ to $V(H)$. By contracting these paths onto $V(H)$, we may assume that $u, v \in V(H)$. But now in H we can contract u and v onto distinct branch vertices of K_4. ∎

We let $K_4 - e$ denote the graph obtained from K_4 by removing an edge e.

LEMMA 6.4. *Let* G *be a* 2*-connected graph with distinct vertices* u *and* v *such that* $\deg(w) \geqslant 3$ *for all* $w \in V(G) \setminus \{u, v\}$. *Then* G *has a* $K_4 - e$ *minor where* u *and* v *are contracted to the endpoints of* e.

Proof. Note that $G + uv$ has a K_4 minor since it has minimum degree 3. Thus, the result follows by applying Lemma 6.3 to $G + uv$. ∎

Note that for all $p \in [1, \infty]$ and $m \in \mathbb{N}$, the property $f_p(G) \leqslant m$ is closed under 0- and 1-sums. However, the graph $K_4 +_e K_4$ shows that the property $f_\infty(G) \leqslant 2$ is not closed under taking 2-sums.

We are now ready to prove our main result.

THEOREM 1.3. *The excluded minors for* $f_\infty(G) \leqslant 2$ *are* W_4 *and* $K_4 +_e K_4$.

Proof. Let G be a minor-minimal graph with $f_\infty(G) \geqslant 3$. By minimality and the preceding discussion, G is 2-connected. By Lemmas 4.1 and 4.2 we may assume that G has minimum degree 3. By Lemmas 5.2 and 5.4 we may assume that G does not have a W_4 or $K_4 +_e K_4$ minor. If G is 3-connected, then by Theorem 6.2, $G = K_4$, which is a contradiction since $f_\infty(K_4) = 2$. Thus, $G = G_1 +_f G_2$ or $G = G_1 \oplus_f G_2$ for some graphs G_1 and G_2 with $f := ab \in E(G_1) \cap E(G_2)$ and $|E(G_1)|, |E(G_2)| > 1$. Since $f \in E(G_1) \cap E(G_2)$ and G is 2-connected, it follows that G_1 and G_2 are both 2-connected. By Lemma 6.4, G_1 has a $K_4 - e$ minor where a and b are contracted to the endpoints of e, and G_2 has a $K_4 - e$ minor where a and b are contracted to the endpoints of e. Combining these two minors, we get a $K_4 +_f K_4$ minor in G, which is a contradiction. ∎

Finally, we prove Corollary 1.4.

COROLLARY 1.4. *The excluded minors for* $f_1(G) \leqslant 2$ *are* W_4 *and* $K_4 +_e K_4$.

Proof. Note that the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \to (\frac{x-y}{2}, \frac{x+y}{2})$ is an isometry between the metric spaces ℓ_∞^2 and ℓ_1^2. Thus for every graph G and distance function d on G, (G, d) is realizable in ℓ_∞^2 if and only if it is realizable in ℓ_1^2. Therefore, $f_\infty(G) \leqslant 2$ implies $f_1(G) \leqslant 2$.

Moreover, it follows from the equivalence between ℓ_1-embeddability and membership in the cut cone [1] and Seymour's linear description of the cut cone of K_5-minor free graphs [15] that *every* distance function d on a graph G can be realized in some ℓ_1^m if G is K_5-minor free. Hence for all K_5-minor free graphs G, we have $f_\infty(G) \leqslant 2$ if and only if $f_1(G) \leqslant 2$.

We claim that in fact $f_\infty(G) \leqslant 2$ if and only if $f_1(G) \leqslant 2$ for *all* graphs G. Indeed, otherwise there would exist a graph G such that $f_\infty(G) > 2$ and $f_1(G) \leqslant 2$. Then G would have a K_5 minor, and thus $f_1(G) \geqslant f_1(K_5) \geqslant 3$ (the last inequality is proved in [19]), which is a contradiction. The result follows. ∎

7. The example and some open problems. A *tree-decomposition* of a graph G is a pair $(T, \mathcal{B})$ where T is a tree and $\mathcal{B} := \{B_t \mid t \in V(T)\}$ is a collection of subsets of vertices of G satisfying the following:

- $G = \bigcup_{t \in V(T)} G[B_t]$, and
- for each $v \in V(G)$, the set of all $w \in V(T)$ such that $v \in B_w$ induces a connected subtree of T.

The *width* of $(T, \mathcal{B})$ is $\max\{|B_t| - 1 \mid t \in V(T)\}$. The *tree-width* of G is the minimum width taken over all tree-decompositions of G. The *path-width* of G is defined analogously, except we insist that T is a path instead of an arbitrary tree.

Fix any tree T with at least two vertices. Let $V^+ := \{v^+ \mid v \in V(T)\}$ and $V^- := \{v^- \mid v \in V(T)\}$ be two disjoint copies of $V(T)$. We construct a planar graph $T \circ K_4$ from T by replacing each vertex v of T by a pair of vertices v^+, v^- in $T \circ K_4$ and each edge vw of T by the 4-clique $\{v^+, v^-, w^+, w^-\}$ in $T \circ K_4$. Formally, $V(T \circ K_4) = \{v^+ \mid v \in V(T)\} \cup \{v^- \mid v \in V(T)\}$ and $E(T \circ K_4) = \{v^+v^- \mid v \in V(T)\} \cup \{v^+w^-, v^+w^+, v^-w^+, v^-w^- \mid vw \in E(T)\}$. We now prove the following strengthened form of Theorem 1.5.

THEOREM 1.5. *For every tree T with at least two vertices, $T \circ K_4$ is planar with tree-width* 3 *and $f_\infty(T \circ K_4) \geqslant |V(T)|$.*

Proof. Clearly, $T \circ K_4$ is planar since K_4 is planar and planarity is closed under taking 2-sums. It is also easy to see that $T \circ K_4$ has tree-width 3. For the last part, we order the edges of T arbitrarily and define a function $d : E(T \circ K_4) \to \mathbb{R}_{\geqslant 0}$ by letting $d_{v^+v^-} := 1$ for $v \in V(T)$, and $d_{v^+w^+} = d_{v^-w^-} := 2^{-i}$, $d_{v^+w^-} = d_{v^-w^+} := 1 - 2^{-i}$ for the ith edge $vw \in E(T)$.

CLAIM 7.1. *The function $d : E(T \circ K_4) \to \mathbb{R}_{\geqslant 0}$ is a distance function on $T \circ K_4$.*

Proof. We have to check that $d(P) \geqslant d_e$ for all edges e and all paths P between the endpoints of e, where $d(P) := \sum_{f \in E(P)} d_f$. Clearly, the inequality is satisfied if P contains the edge e. Similarly, if P contains the edge v^+v^- for some $v \in V(T)$, then $d(P) \geqslant d_{v^+v^-} = 1 \geqslant d_e$. Thus we may assume that P is a path in $T \circ K_4 - (\{e\} \cup \{v^+v^- \mid v \in V(T)\})$.

Every edge f in the cut $\delta(V^+)$ has $d_f \geqslant 1 - 2^{-1} = \frac{1}{2}$. Hence, if P contains at least two edges in the cut $\delta(V^+)$, then $d(P) \geqslant \frac{1}{2} + \frac{1}{2} = 1 \geqslant d_e$. So we may further assume that P contains at most one edge in $\delta(V^+)$.

Since P does not contain the edge e, and $T \circ K_4[V^+]$ and $T \circ K_4[V^-]$ are both isomorphic to the tree T, the path P cannot be completely contained in either of these induced subgraphs. Thus P crosses $\delta(V^+)$ exactly once, and $e = u^+z^-$ for some u^+ and z^-. Let $f = v^+w^-$ denote the unique edge of P in $\delta(V^+)$, where $vw \in E(T)$. Then P consists of a path in $T \circ K_4[V^+]$ from u^+ to v^+, followed by the edge v^+w^-, followed by a path in $T \circ K_4[V^-]$ from w^- to z^-. Thus P contains v^+w^+ or v^-w^-. Without loss of generality, P contains v^+w^+ and $d(P) \geqslant d_{v^+w^+} + d_{v^+w^-} = 2^{-i} + 1 - 2^{-i} = 1 \geqslant d_e$, where i is the index of the edge $vw \in E(T)$. ◻

Claim 7.2. *For all distinct $v, w \in V(T)$, no feasible set of $(T \circ K_4, d)$ can contain both v^+v^- and w^+w^-.*

Proof. Let $e := v^+v^-$ and $f := w^+w^-$. There are only two possible feasible orientations of $\{e, f\}$ (up to reversing both edges). Therefore, to prove the claim, it suffices to exhibit paths P_1, P_2, Q_1, Q_2 such that

- P_1 has ends v^+ and w^+ and P_2 has ends v^- and w^-,
- Q_1 has ends v^+ and w^- and Q_2 has ends v^- and w^+, and
- $d(P_1) + d(P_2) < d_e + d_f = 2$ and $d(Q_1) + d(Q_2) < d_e + d_f = 2$.

Consider the unique path $P = u_1 \cdots u_k$ in T from $u_1 := v$ to $u_k := w$.

We take $P_1 := u_1^+ \cdots u_k^+$ and $P_2 := u_1^- \cdots u_k^-$. Then $d(P_1) = d(P_2)$ is a sum of distinct powers of two of the form 2^{-i}, where $i \geqslant 1$ is an integer. Thus $d(P_1) = d(P_2) < \sum_{i=1}^{\infty} 2^{-i} = 1$ and in particular $d(P_1) + d(P_2) < 1 + 1 = 2$.

Pick j in $\{1, \dots, k-1\}$ such that in the ordering of $E(T)$, $u_j u_{j+1} \in E(T)$ is the minimum. We take $Q_1 := u_1^+ \cdots u_j^+ u_{j+1}^- \cdots u_k^-$ and $Q_2 := u_1^- \cdots u_j^- u_{j+1}^+ \cdots u_k^+$. Then

$$d(Q_1) = d(Q_2) < 1 - 2^{-i} + \sum_{\ell=i+1}^{\infty} 2^{-\ell} = 1 - 2^{-i} + 2^{-i} = 1.$$

Thus $d(Q_1) + d(Q_2) < 1 + 1 = 2$, as required. ◻

Any realization of $(T \circ K_4, d)$ into ℓ_∞^m implies a partition of the edges of $T \circ K_4$ into m feasible sets. By the previous claim, no two of the edges of the form v^+v^-, where $v \in V(T)$, can be put in the same feasible set. Thus we have $f_\infty(T \circ K_4) \geqslant |V(T)|$. ◻

Note that by a classic result of Nash-Williams [11], every planar graph can be partitioned into three forests. Thus, Theorem 1.5 shows that $f_\infty(G) - \Upsilon(G)$ can be arbitrarily large. Furthermore, by taking T to be a path or a star in Theorem 1.5, we see that f_∞ is not bounded as a function of path-width or as a function of diameter.

As promised, we finish the paper with a couple of open problems. One natural question is to try to extend Theorem 1.3 to higher dimensions.

Question 7.3. *What are the excluded minors for $f_\infty(G) \leqslant 3$?*

Let P_4 be a path with four vertices and S_3 be a star with three leaves. By Theorem 1.5, $f_\infty(P_4 \circ K_4) \geqslant 4$ and $f_\infty(S_3 \circ K_4) \geqslant 4$. Thus, $P_4 \circ K_4$ and $S_3 \circ K_4$ each contain an excluded minor for $f_\infty(G) \leqslant 3$.

Finally, it is also interesting to ask how the excluded minors for $f_p(G) \leqslant k$ change for $p \in [1, \infty]$. Let $\mathcal{G}$ be the set of all finite graphs, and define $\mathrm{ex} : [1, \infty] \times \mathbb{N} \to 2^{\mathcal{G}}$ by letting $\mathrm{ex}(p, k)$ be the set of excluded minors for $f_p(G) \leqslant k$. Fix k, and define $p_1 \equiv_k p_2$ if $\mathrm{ex}(p_1, k) = \mathrm{ex}(p_2, k)$. Note that $\equiv_k$ is an equivalence relation on $[1, \infty]$. It may be possible to prove something about the structure of the equivalence classes of $\equiv_k$ without knowing the function $\mathrm{ex}(p, k)$. For example, by the graph minor theorem, there are only countably many minor-closed properties. Thus, some equivalence class of $\equiv_k$ is necessarily uncountable.

Question 7.4. *If C is an equivalence class of $\equiv_k$ such that $|C|$ is uncountable, does C necessarily contain an interval?*

Acknowledgment. We thank the two anonymous referees for their helpful comments and in particular for pointing out a gap in the proof of Corollary 1.4 in a previous version of the paper.

REFERENCES

[1] D. Avis, *Some Polyhedral Cones Related to Metric Spaces*, Ph.D. thesis, Stanford University, Stanford, CA, 1977.
[2] K. Ball, *Isometric embedding in l_p-spaces*, European J. Combin., 11 (1990), pp. 305–311.
[3] A. I. Barvinok, *Problems of distance geometry and convex properties of quadratic maps*, Discrete Comput. Geom., 13 (1995), pp. 189–202, https://doi.org/10.1007/BF02574037.
[4] M. Belk, *Realizability of graphs in three dimensions*, Discrete Comput. Geom., 37 (2007), pp. 139–162.
[5] M. Belk and R. Connelly, *Realizability of graphs*, Discrete Comput. Geom., 37 (2007), pp. 125–137.
[6] J. Bourgain, *On Lipschitz embeddings of finite metric spaces in Hilbert space*, Israel J. Math., 52 (1985), pp. 46–52.
[7] M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer, New York, 1997.
[8] M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann., 68 (1910), pp. 145–168.
[9] W. Holsztynski, $\mathbb{R}^n$ *as a universal metric space*, Notices Amer. Math. Soc., 25 (1978), pp. A–367.
[10] D. Kitson, *Finite and infinitesimal rigidity with polyhedral norms*, Discrete Comput. Geom., 54 (2015), pp. 390–411, https://doi.org/10.1007/s00454-015-9706-x.
[11] C. St. J. A. Nash-Williams, *Decomposition of finite graphs into forests*, J. London Math. Soc., 39 (1964), 12.
[12] N. Robertson and P. Seymour, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B, 92 (2004), pp. 325–357.
[13] N. Robertson and P. D. Seymour, *Graph minors. XIII. The disjoint paths problem*, J. Combin. Theory Ser. B, 63 (1995), pp. 65–110, https://doi.org/10.1006/jctb.1995.1006.
[14] V. Rödl and A. Ruciński, *Bipartite coverings of graphs*, Combin. Probab. Comput., 6 (1997), pp. 349–352, https://doi.org/10.1017/S0963548397003064.
[15] P. D. Seymour, *Matroids and multicommodity flows*, European J. Combin., 2 (1981), pp. 257–290, https://doi.org/10.1016/S0195-6698(81)80033-9.
[16] M. Sitharam and H. Gao, *Characterizing graphs with convex and connected Cayley configuration spaces*, Discrete Comput. Geom., 43 (2010), pp. 594–625, https://doi.org/10.1007/s00454-009-9160-8.
[17] M. Sitharam and J. Willoughby, *On Flattenability of Graphs*, preprint, https://arxiv.org/abs/1503.01489, 2015.
[18] W. T. Tutte, *A theory of* 3*-connected graphs*, Indag. Math., 23 (1961), pp. 441–455.
[19] H. S. Witsenhausen, *Minimum dimension embedding of finite metric spaces*, J. Combin. Theory Ser. A, 42 (1986), pp. 184–199.