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A three-dimensional continuum model incorporating static and kinetic effects for granular flows with applications to collapse of a two-dimensional granular column

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This work extends a one-dimensional continuum model for granular flows down inclined planes [C. H. Lee and C. J. Huang, “Kinetic-theory-based model of dense granular flows down inclined planes,” *Phys. Fluids* **24**, 073303 (2012)] to solve three-dimensional problems involving both static and flow states. The new model decomposes the shear stress and pressure into enduring-contact and kinetic components. One novelty of the present model is the determination of the enduring-contact component of pressure, which is a composition of a pressure depending only on the volume fraction and a pressure derived from the dilatancy law together with the equation of state from the kinetic theory. Another novelty of this study is a new numerical scheme that can avoid numerical instability caused by large volume fractions. To demonstrate its capability, the present model is applied to simulate the collapse of a granular column with various aspect ratios. The evolution of the column shape, the flow field, the final height, and the run-out predicted by the present model agree well with those provided by discrete element methods and experiments. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4935626>]

I. INTRODUCTION

Flows of granular materials occur in various problems related to geophysical hazards such as avalanches, landslides, and debris flows. Several studies^{1–18} have been devoted to develop models for granular materials using continuum description to study such problems. However, existing models^{1–18} are limited to specific flow conditions due to the complex rheological characteristics of granular materials, and thus a general-purpose model is still lacking so far, as will be discussed in detail below.

Granular materials behave like a solid when they are statically packed.^{19,20} A granular material in its solid state can sustain a certain amount of shear stress, which can be simply described by Mohr-Coulomb failure criterion.²¹ The volume fraction of a granular material in its solid state, ν , is related to the applied pressure, p , and the microscopic geometric arrangement of particles. By assuming that the effect of microscopic geometric arrangement on p can be neglected, several simple $p - \nu$ relations^{10,11,15,22,23} have been proposed.

If the shear stress exerted on a granular material exceeds its critical value, the granular material flows.^{19,20} A flowing granular material has several states, and previous studies suggest that its states can be characterized by either (1) the strain rate and the total pressure,²⁴ or (2) the strain rate and

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the volume fraction.^{25,26} For a problem involving relatively high volume fractions and when the effect of diffusion of fluctuation energy on rheological characteristics is not significant, the states of a granular material can be roughly characterized by the inertial number,²⁷ $I \equiv 2dD/\sqrt{p/\rho_d}$, where d is the diameter of particles, D the second invariant of the strain rate tensor $\mathbf{D}$ (i.e., $D \equiv \sqrt{\mathbf{D} : \mathbf{D}/2}$), and ρ_d the density of particles. From small to large values of I , the states of a granular material are quasi-static state, dense state, and inertial state.²⁷ In the quasi-static state, which is also called the elastic-quasistatic state,²⁴ each particle contacts with others enduringly, and the particles creep very slowly; the microscopic enduring-contact effect leads to a rate-independent shear stress, and the Coulomb relationship (i.e., the ratio of shear stress to pressure is a constant) can be used to compute shear stresses if the pressure is known.¹ In the inertial state, also known as the inertial-collision state,²⁴ grains fly freely and they instantaneously collide with each other, and thus both the microscopic inertia and collision effects on momentum transfer govern the flow. Because grains in the inertial state behave analogously to molecules, various kinetic-theory-based hydrodynamic theories (KTBTs) have been developed based on granular kinetic theory,^{3-5,14,16-18} KTBTs involve constitutive relations for variables such as shear stress and pressure. The quasi-static and inertial states are two extreme states for flows of granular material. The dense state, also called the elastic-inertial state,²⁴ is a state between the two extremes; for this state, the microscopic enduring-contact effect, the collision effect, and the inertia effect on momentum transfer all are important, resulting in macroscopic visco-plasticity, which is measured by the ratio of the shear stress to pressure. When the diffusion of fluctuation energy has an insignificant effect on the rheological characteristics, the macroscopic visco-plasticity can be described by the effective friction $\eta(I)$, with I being the inertial number I .^{20,27-29} Various constitutive relations^{2,27} have been proposed in the literature to describe the function $\eta(I)$, and $\eta(I)$ -based constitutive relations have been successfully applied to simulate granular flows on a pile,^{2,30,31} the collapse of a granular column,^{32,33} and granular flows in a silo.³⁴ We remark that Refs. 32 and 34 did not compare their numerical results with experiments. All these studies indicated that the divergence of the energy flux did not have significant effects on the global behaviors of the granular flows. Since various states of granular material may appear simultaneously in a process, a model applicable to various states and capable of describing the transition between different states is required. As I approaches zero, $\eta(I)$ reduces to Coulomb relationship.^{2,27} $\eta(I)$ -based constitutive relations^{2,27} are also applicable to the quasi-static state although some interesting phenomena such as hysteretic behavior³⁵ cannot be captured. To simulate flows involving solid, quasi-static, and dense states, the $\eta(I)$ -based constitutive relations have been modified using regularization techniques,^{32,36,37} which consider a granular material to be a very viscous fluid.⁸¹ Another method based on $\eta(I)$ -based constitutive relations to simulate such flows is to use the augmented Lagrangian method,³³ which explicitly deals with the static/flowing transition related to the yield stress. However, $\eta(I)$ -based numerical models are still not able to deal with the inertial state since $\eta(I)$ is obtained for dense state. In real world problems such as pouring beads on a pile²⁰ and collapse of granular columns, inertial state may co-exist with other states.

In order to develop a model applicable to all four states of granular material (the solid, quasi-static, dense, and inertial states), KTBTs can be extended to consider the enduring-contact effects on shear stresses by one of the following four approaches:

- (1) modifying the viscous coefficient used in the kinetic theory when a granular material is near the jamming state,^{38,39}
- (2) replacing the particle diameter in the term describing energy dissipation by the length of particle chain,^{6,9}
- (3) incorporating a new component into the constitutive relations for the shear stress originating from the enduring-contact effect, which is related to an order parameter identifying the states of granular material.^{35,40-43} or
- (4) incorporating a rate-independent stress into the constitutive relations for shear stresses.^{7,8,10-12,15,44}

Among the four approaches, the third³⁵ and fourth⁷ approaches have been applied to simulate one-dimensional, steady, surface-driven flows under gravity, which are characterized by the coexistence of all four states. The third approach, which relates the shear stress to the thickness of

granular layer, allows one to compute shear stress in a simple way if the thickness of granular layer is given, and has been applied to simulate avalanches.⁴³ However, the third approach uses a highly nonlinear equation to describe the evolution of the order parameter,³⁵ which is difficult to solve numerically; furthermore, the shear stresses need to be given as model parameters in the simulation³⁵ or computed through some simplifications.⁴³ The model of Ref. 45, which is based on the second approach, was used for frictionless beads. It is worth mentioning that in Ref. 45 the length scale in the term describing energy dissipation is regarded as a correlation length instead of the length of particle chain. However, when $\nu \rightarrow \nu_{\max}$, Fig. 9 of Ref. 45 shows that $\eta \ll 0.1$, while the DEM simulations of Refs. 27 and 46 show that η falls in the range of 0.1–0.2. It is not clear if the model of Ref. 45 can still be used when $\nu \rightarrow \nu_{\max}$.

The model proposed in this study is based on the fourth approach, where the rate-independent stress increases linearly with an enduring-contact component of pressure, p^E , originating from the enduring contact between particles. For the inertial state ($p^E = 0$), existing models based on the fourth approach^{7,8,10–12,15,44} can be reduced to KTBHTs; whereas for the quasi-static state, such models reduce to the Coulomb relationship. A key point in the fourth approach is how to determine p^E appropriately so that the rheological characteristics of the dense state can be reproduced. Lee and Huang⁷ suggested calculating p^E by using $p^E = p - p^K$, where p = total pressure (determined by analyzing the balance of forces) and p^K = the kinetic component of pressure originating from the inertia and collision effects on momentum transfer. Note that p^K is given by the constitutive relations for pressure in KTBHT.⁵ Lee and Huang's scheme,⁷ however, is only applicable to one-dimensional and steady problems. Lee and Huang⁸ also suggested calculating p^E by using $p^E = \max(p^D - p^K, 0)$, with p^D being the pressure predicted by dilatancy laws⁴⁷ ($p^D - D - \nu$ relationships), which involve both the enduring-contact and kinematic components of pressure. Although Lee and Huang's model⁸ can reproduce the rheological characteristics of the dense state, it is not applicable to the solid state due to the inapplicability of the dilatancy laws to the solid state. Furthermore, p^D is sensitive to variations in ν when ν closes to the critical value ν_o , which is slightly lower than $\nu_{\max}$ (the maximum value of ν), and $\nu = \nu_o$ occurs when $D \rightarrow 0$. Other models based on the aforementioned fourth approach calculate p^E by using $p^E - \nu$ relationships.^{10,11,15} However, one shortcoming arising from using $p^E - \nu$ relationships is that p^E is sensitive to ν when ν closes to $\nu_{\max}$. Moreover, previous discrete element simulations⁴⁸ suggested that p^E should depend on strain rate, implying that further improvement is needed for the $p^E - \nu$ relationships to incorporate effects of strain rate. We remark that no existing studies have applied the fourth approach together with the $p^E - \nu$ relationships to study problems where all four states co-exist.

Based on the literature review given above, there are two challenges in developing a general-purpose model for granular flow problems based on the aforementioned fourth approach. The first challenge is to find a proper constitutive relation to determine the enduring-contact pressure (p^E) so that the model can predict rheological behaviors accurately. To address the first challenge, this study combines a $p^E - \nu$ relationship, which is applicable to the solid state, with $p^E = \max(p^D - p^K, 0)$, which is applicable to the quasi-static, dense, and inertial states. The other challenge is how to ensure numerical stability. When ν increases rapidly toward ν_o , numerical instability may occur because p^E becomes very sensitive to ν . To address this challenge, this study introduces a new numerical scheme to prevent ν from increasing rapidly toward ν_o .

To demonstrate the applicability of the present model to granular flow problems involving all four states, the model is applied to simulate the collapse of a granular column, which involves all four states. Computed results are compared with published experimental results⁴⁹ and the results obtained by using a discrete element method (DEM).⁵⁰

II. MATHEMATICAL MODEL

A. Governing equations

Although granular materials have discrete nature from a microscopic view, their macroscopic behaviors can still be described by governing equations similar to those used in continuum mechanics.^{51,52} The local state of a granular material is characterized by its mean mass, momentum,

and fluctuating kinetic energy. Governing equations describing conservations of mean mass, momentum, and fluctuating kinetic energy are

$$\frac{\partial \rho_d v}{\partial t} + \nabla \cdot (\rho_d v \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial \rho_d v \mathbf{u}}{\partial t} + \nabla \cdot (\rho_d v \mathbf{u} \mathbf{u}) = -\nabla p + \nabla \cdot (\boldsymbol{\tau}) + \rho_d v \mathbf{g}, \quad (2)$$

$$\frac{3}{2} \left[\frac{\partial \rho_d v T}{\partial t} + \nabla \cdot (\rho_d v \mathbf{u} T) \right] = -\nabla \cdot \mathbf{q} + \Theta - \varepsilon, \quad (3)$$

where ρ_d = density of particles, v = volume fraction, $\mathbf{u}$ = mean velocity, $\boldsymbol{\tau}$ = shear stress, p = pressure, $\mathbf{g}$ = acceleration due to gravity, T = granular temperature defined by $T \equiv \langle \mathbf{u}' \cdot \mathbf{u}' \rangle / 3$ with $\mathbf{u}'$ being the instantaneous deviation from the mean velocity and $\langle \cdot \rangle$ an ensemble average operator, $\mathbf{q}$ = flux of mean fluctuating kinetic energy, Θ = production of mean fluctuating kinetic energy, and ε = dissipation of mean fluctuating kinetic energy. Here, we assume that particles are incompressible (i.e., ρ_d is a constant). Equations (1)–(3) have 20 unknowns (v , $\mathbf{u}$, p , $\boldsymbol{\tau}$, T , $\mathbf{q}$, Θ , and ε), and proper constitutive relations for p , $\boldsymbol{\tau}$, $\mathbf{q}$, Θ , and ε are needed to solve the problem defined by Eqs. (1)–(3).

B. Constitutive relations

As mentioned in the Introduction, different microscopic behaviors lead to different macroscopic rheological characteristics. To consider both the enduring-contact effect and the kinetic effect (the inertia and collision effects) on momentum transfer, $\boldsymbol{\tau}$ and p each is decomposed into an enduring-contact part and a kinetic part (see, e.g., Refs. 7 and 8), i.e.,

$$\boldsymbol{\tau} = \boldsymbol{\tau}^K + \boldsymbol{\tau}^E \quad (4)$$

and

$$p = p^K + p^E, \quad (5)$$

where the superscripts “ K ” and “ E ” represent the kinetic and enduring-contact components, respectively. For $\mathbf{q}$ and ε , this study neglects their enduring-contact components.^{7,8,12,44} Our numerical results will show that this treatment can yield acceptable results.

It is worth noting that determinations of $\boldsymbol{\tau}^K$, p^K , $\boldsymbol{\tau}^E$, and p^E depend on how we view them. Berzi *et al.*¹⁵ argued that the enduring-contact components should be independent of strain rate, and they computed the kinetic components by modifying KTBHT. From a microscopic view, however, Volfson *et al.*³⁵ argued that (1) the enduring-contact components should be computed based on the enduring contact, which is the contact between two particles in a stuck state and has a duration longer than the typical collision time, and (2) the kinetic components should be computed based on non-enduring contact. Reference 35 has shown that the enduring-contact components depend on strain rate, and that KTBHT can yield results of $\boldsymbol{\tau}^K$ and p^K in good agreement with those obtained by DEM.^{35,48} The present study follows these studies^{7,35,48} to compute the kinetic components directly using KTBHT. The constitutive relations for $\boldsymbol{\tau}^K$ and p^K in KTBHT, which were developed by Jenkins³ and used by Lee and Huang,⁸ are listed below. $\boldsymbol{\tau}^K$ is modeled by

$$\boldsymbol{\tau}^K = \left[\left(\lambda^K - \frac{2}{3} \mu^K \right) \nabla \cdot \mathbf{u} \right] \mathbf{I} + 2 \mu^K \mathbf{D}, \quad (6)$$

where

$$\lambda^K = \frac{4 \rho_d d v^2 g_o (1 + e) T^{1/2}}{3 \pi^{1/2}} \quad (7)$$

and

$$\mu^K = \frac{5 \rho_d \pi^{1/2} d T^{1/2}}{24 g_o (1 + e) (3 - e)} \left[1 + \frac{2}{5} v g_o (3e - 1) (1 + e) \right] \left[1 + \frac{4}{5} v g_o (1 + e) \right] + \frac{3}{5} \lambda^K \quad (8)$$

with g_o = pair distribution function and e = coefficient of restitution. p^K is modeled by

$$p^K = \rho_d \nu T [1 + 2(1 + e)\nu g_o]. \quad (9)$$

Based on the same KTBHT,³ $\mathbf{q}$ can be computed by

$$\mathbf{q} = -\kappa \nabla T + \delta \nabla \nu, \quad (10)$$

where

$$\kappa = \frac{25\rho_d \pi^{1/2} d T^{1/2}}{4g_o(1+e)(49-33e)} \left[1 + \frac{3}{5} \nu g_o(1+e)^2(2e-1) \right] \left[1 + \frac{6}{5} \nu g_o(1+e) \right] + \frac{3}{2} \lambda^K \quad (11)$$

and

$$\delta = \frac{15\rho_d e(1-e)\pi^{1/2} d T^{3/2}}{4(49-33e)\nu g_o} \frac{d(\nu^2 g_o)}{d\nu} \left[1 + \frac{6}{5} \nu g_o(1+e) \right]. \quad (12)$$

The dissipation of mean fluctuating kinetic energy, ε , is given by

$$\varepsilon = 9 \frac{\lambda^K(1-e)}{d^2} T. \quad (13)$$

The parameters λ^K , μ^K , p^K , and κ involve the radial distribution function g_o , which represents the spatial particle-particle correlation. Following Lee and Huang,⁸ this study adopts the radial distribution function proposed by Savage,⁵³

$$g_o = \frac{1 - 7/16}{(1 - \nu/\nu_{\max})^2}, \quad (14)$$

where $\nu_{\max} = 0.63$ and there is a singularity at $\nu = \nu_{\max}$. Using Eq. (14) yields rheological characteristics of dense granular material comprised by glass beads that have been observed experimentally.⁸ We remark that a radial distribution function, considering the effects of inter-particle friction and particle stiffness, has just been reported in two recent papers,^{54,55} where the $\nu_{\max}$ is as a function of inter-particle friction and particle stiffness. The constitutive relations given in Eqs. (6)–(13) have neglected the effects of the tangential contact during collisions. To consider such effects, Refs. 56 and 57 suggested that the restitution coefficient e should be replaced by an effective restitution coefficient $e_{\text{eff}} \leq e$ in those constitutive relations, and that e_{eff} should depend on both e and the friction coefficient of particles.^{56,57}

The production of mean fluctuating kinetic energy (Θ) is due to the work done by shear stress. This study considers only the work done by the kinetic component of the shear stress for simplicity, that is,

$$\Theta = 2\boldsymbol{\tau}^K : \mathbf{D}. \quad (15)$$

Several studies^{7,8,12} have suggested including the work done by $\boldsymbol{\tau}^E$ in Θ to allow their models to predict hysteretic flow thresholds; however, doing so increases the numerical difficulty in solving the present continuum model.

Lee and Huang^{7,8} suggested applying the Coulomb relationship to calculate $\boldsymbol{\tau}^E$ but their studies are limited to one-dimensional problems. To express the Coulomb relationship in a form that is applicable to three dimensional problems, we follow Ref. 58 to write $\boldsymbol{\tau}^E$ as

$$\boldsymbol{\tau}^E = p^E \eta^S \frac{\mathbf{D}}{D}, \quad (16)$$

where $\eta^S = \tan \theta^S$ with θ^S being the static internal friction angle. We remark that the term $p^E \eta^S$ is considered to be a yield stress and Eq. (16) is valid only when $\tau^E > p^E \eta^S$, implying that the granular material is in its solid state when $\tau^E \leq p^E \eta^S$.

The constitutive relations given in Eqs. (4)–(16) are very similar to those suggested by Lee and Huang,⁸ who expressed these relations in their one-dimensional forms and considered the work done by $\boldsymbol{\tau}^E$. The constitutive relations we will introduce below differ from those in Lee and Huang⁸ since the new constitutive relations can be applied to the solid state as well.

In order to solve flow problems where the solid state also occurs, we use Papanastasiou's regularization technique⁵⁸ to rewrite τ^E as

$$\tau^E = 2\mu^E \mathbf{D} \quad (17)$$

with

$$\mu^E = p^E \eta^s \frac{[1 - \exp(-nD)]}{2D}, \quad (18)$$

where n is a model parameter. Equations (17) and (18) regard a granular material in its solid state as a very viscous fluid. When n approaches infinity, Eq. (17) together with Eq. (18) reduces to Eq. (16). This study takes $n = 10^5$ as suggested by Ref. 7.

As mentioned in the Introduction, the determination of p^E is one of the key points in this study. Previous studies^{7,35,48} suggest that the p^K should be computed by using both KTBHT and the instantaneous contact defined by Volfson *et al.*³⁵ Since the total pressure in the dense state can be computed using the dilatancy law obtained from experiments or DEM simulations, Ref. 8 suggests that p^E should be computed by using $p^E = \max(p^D - p^K, 0)$, which has been validated in Ref. 7. However, $p^E = \max(p^D - p^K, 0)$, which depends on both D and T , is applicable only to granular materials in fluid state. To calculate p^E for solid state, certain $p^E - \nu$ relationships need to be adopted. Consequently, we propose that p^E consists of two parts: p^{ES} for the solid state and p^{EF} for the fluid state, namely,

$$p^E = p^{ES} + p^{EF}. \quad (19)$$

In this study, we adopt the $p - \nu$ relationship proposed by Hsu *et al.*²² for p^{ES} , i.e.,

$$p^{ES} = K\nu[\max(\nu - \nu_{\min}, 0)]^\chi \left\{ 1 + \sin \left[\max \left(\frac{\nu - \nu_{\min}}{\nu_{\max} - \nu_{\min}}, 0 \right) \pi - \frac{\pi}{2} \right] \right\}, \quad (20)$$

where $\nu_{\min}$ = random loose packing fraction, and K and χ are two model parameters. Obviously, K is related to the Young's modulus of the granular material and the other terms are associated with the deformation of the material. As stated in Ref. 22, Eq. (20) is obtained through the following steps: (1) performing a nonlinear contact analysis of a homogeneously packed granular material, (2) assuming that the average compressive volume strain equals $(\nu - \nu_{\min})^{2\chi/3}$, and (3) obtaining an empirical expression for the coordination number (i.e., the average number of particles touching a given particle) by DEM simulations. In Ref. 22,

$$K = \frac{E}{3\sqrt{3}\pi(1 - \sigma^2)}, \quad (21)$$

with E = Young's modulus of an individual particle and σ = Poisson's ratio. Reference 22 suggested $\chi = 5.5$, which was, however, obtained according to sediment concentration in sheet flow regimes. Accordingly, it may be more appropriate to regard χ as a free parameter in this study. When χ is larger than its critical value, numerical stability may become an issue. In this study, $\chi = 1.5$ is adopted. Our sensitivity analysis has showed that for the problem studied in Section V C 2, the computational results obtained with different values of χ in the range of 1.5–3.5 are similar. This study also adopts $\nu_{\min} = 0.57$ as suggested by Ref. 22. The reason why we adopt Eq. (20) is because p^{ES} and its first derivative are continuous everywhere. It has been reported in a recent paper⁵⁴ that p^{ES} can be calculated by using $p^{ES} = 0.15\pi E(\nu + \nu_{\max})$, where $\nu_{\max}$ depends on the inter-particle friction,⁵⁵ and thus considers the effects of both inter-particle friction and stiffness. The present study does not consider the effect of inter-particle friction on p^{ES} , and the results in Refs. 54 and 55 provide a possibility of further development of the present work.

We use the following equation to calculate p^{EF} :

$$p^{EF} = \max(p^D - p^K, 0) \quad (22)$$

and use the dilatancy law suggested by Ref. 46 to compute p^D , which consists of an enduring-contact component and a kinetic component. Note that the dilatancy law,^{31,46,47} is obtained for small values of I (i.e., the quasi-static and dense states). In Eq. (22) we assume $p^{EF} = p^D - p^K$; however,

because p^{EF} is always non-negative, the limiter $p^{EF} \geq 0$ has to be used. If $p^{ES} = 0$, the condition of $p^{EF} = 0$ ($p^D \leq p^K$) means that the granular material is in its inertial state, and the change of p^{EF} from nonzero to zero means a transition from the dense state to the inertial state, which has been verified in Ref. 7 for granular materials consisting of two-dimensional circles. By assuming that the diffusion of T is insignificant, Ref. 46 expressed the dilatancy law as

$$\frac{1}{\nu} = \frac{1}{\nu_o} + aI^\delta, \quad (23)$$

where ν_o , a , and δ are model parameters. The values of ν_o , a , and δ have been provided by Ref. 46 for frictionless beads. We adopted $\nu_o = 0.6$ as suggested by Ref. 31, and took $a = 0.28$ and $\delta = 0.58$ for a better fit to the numerical results for frictional particles.⁴⁷ Equation (23) states that p^D is a function of ν and D due to $I \equiv 2dD^{1/2}/\sqrt{p/\rho_d}$. For simple plane shear flows, D can be expressed using T . Accordingly, p^D can be given by

$$p^D = \rho_d \nu T \left[9 \left(\frac{(1-e)\lambda'}{\mu'} \right) \nu^{2/\delta-1} \left(\frac{a\nu_o}{\nu_o - \nu} \right)^{2/\delta} \right], \quad (24)$$

where $\lambda' = \lambda^K/\rho_d dT^{1/2}$ and $\mu' = \mu^K/\rho_d dT^{1/2}$. Compared to either Eq. (23) or the dilatancy law used in Ref. 8, Eq. (24) can ensure the numerical stability of the scheme. Therefore, this study uses Eq. (24) to calculate p^D . As mentioned above, this study neglects the work done by $\boldsymbol{\tau}^K$ in Θ ; this is because if we consider it in our model, the D - T relationship and Eq. (24) will become very complicated (see Ref. 8).

Different from the study of Ref. 8, the present study incorporates an additional enduring-contact component into the expression for pressure (Eq. (19)). A constitutive relation for p^E , applicable to the solid, quasi-static, dense, and inertial states, should at least satisfy the following four conditions:

- (1) It can describe the $p^E - \nu$ relationship for granular materials in solid state if the effect of microscopic geometric arrangement on p^E is neglected.
- (2) It can predict $p^E = 0$ for large values of I (i.e., the inertial state) so that $\boldsymbol{\tau}$ can reduce to $\boldsymbol{\tau}^K$.
- (3) It can reproduce the $\eta - I$ relationship observed in experiments involving both quasi-static and dense states when the diffusion of T is insignificant.
- (4) It can reproduce the $\nu - I$ relationship described by Eq. (23) for granular materials in dense state when the diffusion of T is insignificant.

If $p^{ES} = 0$, the present model reduces to the model of Ref. 8, and the second to fourth conditions can be satisfied. However, in the present model, $p^{ES} \neq 0$ for $\nu > \nu_{min}$, and thus we can only ensure that p^E obtained by Eqs. (19)–(24) can satisfy the first condition: p^E is a function of ν when $\nu > \nu_{min}$ as well as $T = 0$ (i.e., the solid state). At the end of Section IV, we will discuss how the present model can satisfy the other three conditions as well.

III. NUMERICAL SOLUTION PROCEDURE

To solve the governing equations and constitutive relations presented in Section II, this study uses an open source computational-fluid-dynamics library: OpenFOAM[®], which was developed based on finite volume methods and supports unstructured grid systems (for more details, please refer to Ref. 59). The most important feature of OpenFOAM is that it can mimic the form of partial differential equations in the code, and provide an easy, quick, and reliable way to solve the equations presented in Section II.

As suggested by Rusche,⁶⁰ we solve the momentum equation (Eq. (2)) in the “phase-intensive” form, which is obtained by dividing the momentum equation by $\rho_d \nu$,

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{u} \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{u} = -\frac{\nabla p}{\nu \rho_d} + \frac{1}{\nu \rho_d} \nabla \cdot \boldsymbol{\tau} + \mathbf{g}. \quad (25)$$

Rusche⁶⁰ indicated that solving the momentum equation in the phase-intensive form could avoid numerical noises occurring when $\nu \rightarrow 0$. The solution to Eq. (25) can be expressed in a semi-discretized form as

$$\mathbf{u} = \frac{(\mathbf{A})_H}{(\mathbf{A})_D} - \frac{\nabla p}{\nu \rho_d (\mathbf{A})_D} + \frac{\mathbf{g}}{(\mathbf{A})_D}, \quad (26)$$

where $\mathbf{A}$ denotes the system of linear algebraic equations arising from the discretization of Eq. (25), $(\mathbf{A})_D$ = diagonal coefficients in $\mathbf{A}$, and $(\mathbf{A})_H = \mathbf{b} - (\mathbf{A})_O \mathbf{u}$ with $\mathbf{b} = \mathbf{A} \mathbf{u} + \nabla p / \nu \rho_d - \mathbf{g}$ and $(\mathbf{A})_O$ = off-diagonal coefficients in $\mathbf{A}$. We remark that both $(\mathbf{A})_D$ and $(\mathbf{A})_H$ depend on the numerical schemes used to discretize the left-hand side of Eq. (25) and the second term on the right-hand side of Eq. (25). OpenFOAM provides functions to access $(\mathbf{A})_D$ and $(\mathbf{A})_H$. If we directly compute $\mathbf{u}$ by using Eq. (26) and then compute the volume fraction ν by solving Eq. (1), ν may increase rapidly toward ν_o , leading to an infinite p^D (see Eq. (24)).

To avoid a rapid increase of ν toward ν_o , we rewrite ∇p as $\nu \nabla p' + p' \nabla \nu$, where $p' = p / \nu$ and propose the following predictor-corrector solution procedure to solve Eq. (26):

Predictor:

$$\mathbf{u}^* = \frac{\mathbf{A}_H}{\mathbf{A}_D} - \frac{\nabla p'}{\rho_d \mathbf{A}_D} + \frac{\mathbf{g}}{\mathbf{A}_D}, \quad (27)$$

corrector:

$$\mathbf{u} = \mathbf{u}^* - \frac{p'}{\nu \rho_d \mathbf{A}_D} \nabla \nu. \quad (28)$$

Combining Eqs. (1) and (28) results in the following equation describing the evolution of ν :

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{u}^*) = \nabla \cdot (\Gamma \nabla \nu), \quad (29)$$

where $\Gamma = p' / \rho_d \mathbf{A}_D$. The procedure for numerically solving the governing equations is outlined as follows:

- (1) Solve Eq. (3) for T and compute the coefficients.
- (2) Solve Eq. (27) for $\mathbf{u}^*$.
- (3) Solve Eq. (29) for ν .
- (4) Compute $\mathbf{u}$ from Eq. (28).
- (5) Repeat (1)-(4) for n_L times,

where n_L is determined at the run time. Although we solve the equations governing $\mathbf{u}$ and T in their phase-intensive forms, numerical noises can still occur in the region where $\nu \approx 0$. To avoid the numerical noise, we force $\mathbf{u} = 0$ and $T = 0$ on grids with $\nu < 10^{-4}$; but when a grid with $\nu < 10^{-4}$ is adjacent to another grid with $\nu > 10^{-4}$, both $\mathbf{u}$ and T are determined according to $\nabla \mathbf{u} = 0$ and $\nabla T = 0$. Other trivial details about the numerical treatments used in this study are similar to those used for the particle phase of “twoPhaseEulerFoam,” a two-phase-flow solver provided by OpenFOAM, and can be found in Ref. 60; for example, the calculation of $\mathbf{u}$ at the grid faces and the replacement of $1/\nu$ by $1/(\nu + 0.001)$ to avoid numerical instability resulting from $\nu \rightarrow 0$. We remark that “twoPhaseEulerFoam” directly solves $\mathbf{u}$ by using Eq. (26) and the continuum equation solved in “twoPhaseEulerFoam” is an advection equation, which differs from the one used in the present study. Comments on applications of “twoPhaseEulerFoam” to granular flows are given in Appendix B.

OpenFOAM provides various discretization schemes, which can be selected at the run time. Various discretization schemes provided by OpenFOAM can be found in Ref. 61. For the simulations performed in this study, a second-order time-implicit scheme is used, and a second-order linear interpolation in space is used for all variables except for velocity. To interpolate velocities, the total-variation-diminishing (TVD) limited linear interpolation scheme is adopted here.

FIG. 1. (a) Comparison of the η – I relationships predicted by the present model (lines) with those deduced from the experimental data²⁸ (open circles). (b) Comparison of the ν – I relationships predicted by the present model and Eq. (23) (dashed lines). The predicted η – I relationships were obtained by using $\eta^S = 0.38$ and $e_{\text{eff}} = 0.46, 0.58$, and 0.7 for glass beads with $d = 0.5$ mm. The solid lines represent the predicted η – I relationships for $p^E > 0$, and the dotted lines for $p^E = 0$.

IV. RHEOLOGICAL CHARACTERISTICS

We now use the present model to examine the rheological characteristics of granular materials, described by η (the ratio of shear stress to pressure) and ν , for the quasi-static, dense, or inertial states. The solid state ($I = 0$) is not discussed here. Plane shear flows are the conceptually simplest problems used to study rheological properties.²⁸ The approach to extract η – I and ν – I relationships using the present model for plane shear flows can be found in Appendix A.

As an example, three η – I relationships predicted by the present model are shown in Fig. 1(a), where the η – I relationship, derived by GDR MiDi²⁸ from the experimental results for granular flows down a rough inclined slope,²⁹ is also shown for comparison. In the experiment of Ref. 29, glass beads with $\rho_d = 2500$ kg/m³ and $d = 0.5 \pm 0.04$ mm were used. When computing η , the values of e_{eff} , η^S , and K/p must be given since e_{eff} , η^S , and K all are material properties. Three values of $e_{\text{eff}} = 0.46, 0.58$, and 0.7 were used. The corresponding value of $\eta^S = 0.38$ was estimated by using $\eta^S = \tan \theta^S$, where θ^S is the angle of repose.²⁹ We remark that the values of I (Eq. (A4)) and η (Eq. (A6)) are not sensitive to K/p : varying K/p from 10^{-1} to 10^5 only yields a relative change of $1.2 \times 10^{-3}\%$ in η and 1.6% in I .

As shown in Fig. 1(a), for the quasi-static state ($I \rightarrow 0$), the predicted η seems independent of I and remains a constant, say η^S , implying that the present model is applicable for the quasi-static state. For the dense state (i.e., $I > 0$ and $p^S > 0$), the predicted η increases with increasing I and the predicted η – I relationships agree well with the experimental results; this comparison validates the present model for the dense state. For the inertial state (i.e., $I > 0$ and $p^S = 0$), the predicted η decreases with increasing I , and the present model reduces to KTBHT. It can be seen from Fig. 1(a) that η reaches a maximum value at the point where the dense state transits to the inertial state. The maximum value of η , which corresponds to the maximum inclination angle under which the granular material steadily flows down an inclined plane,⁸ is influenced by e_{eff} . For $p^E > 0$, the η – I relationships for different values of e_{eff} are similar. The η – I relationship shown in Fig. 1(a) is similar to that in Ref. 8; however, the η – I relationship in Ref. 8 increases with decreasing I when $I \rightarrow 0$, a trend not present in our results. Therefore, the present continuum model is not capable of predicting hysteretic flow thresholds, possibly because the work done by τ^E is not considered in the present model (see the discussion about Eq. (15)).

The ν – I relationships predicted by the present model are presented in Fig. 1(b), where Eq. (23) is also superposed for comparison. Figure 1(b) shows that ν decreases with increasing I and it suddenly drops as the granular material enters into the inertial regime ($p^E = 0$). Although p^{ES} plays a role when $\nu > 0.57$ (see Eq. (20)), the ν – I relationships predicted by the present model still agree well with those predicted by Eq. (23) and are in agreement with the ν – I relationships

presented in Ref. 46. This means that for granular materials in flow states, p^{ES} has an insignificant contribution to either p ($=p^K + p^{ES} + p^{EF}$) or I ($\equiv 2dD^{1/2}/\sqrt{p/\rho_d}$). As a result, including p^{ES} in the expression for p affects only the behavior of the granular material in solid state.

It has been pointed out in Section II B that four conditions must be satisfied by p^E so that the model can be applicable to problems involving all four states (the solid, quasi-static, dense, and inertial states). It can be concluded from Fig. 1 that the present model described in Section II can reduce to KTBHT (e.g., $p^E = 0$ for large I , which is the second condition for the constitutive relation of p^E presented in Section II B) and reproduce rheological characteristics for the quasi-static and dense state as observed in the experiment (the third condition). Additionally, the predicted $\nu - I$ relationship for the dense state is identical to the dilatancy law obtained experimentally (the fourth condition). Consequently, all the four conditions listed in Section II B can be satisfied (see Section II B for the first condition) and the present model is applicable to problems involving all four states. When using the dilatancy law, we have to assume that the diffusion of T is insignificant. Therefore, the present model may not be applicable to problems where the diffusion of T is significant.

V. APPLICATION TO COLLAPSE OF A GRANULAR COLUMN

To demonstrate the capability of the present model in simulating a general granular flow problem, the model is applied to simulate the collapse of a granular column. Collapsing process of a granular column involves not only four states of granular materials but also different types of problems: compact static granular column problem, surface flow problem, and deposition problem. Furthermore, the collapsing process of a granular column involves large values of ν , which usually can be a source of numerical instability as discussed in Section III. Therefore, the collapse of a granular column is a good example to demonstrate the capability of the present model. There has been no attempt in the literature to simulate this problem with compressibility (i.e., a variable ν) being considered.

The results obtained by using the present model are compared with the experiments reported in Ref. 49. Since it is difficult to fully mimic experimental conditions in our numerical simulations and since the detailed flow fields were not reported in Ref. 49, we also performed a DEM simulation to provide another numerical dataset for comparison.

A. Summary of the experimental results of Lajeunesse *et al.*

Lajeunesse *et al.*⁴⁹ reported a series of experiments studying dynamics, kinematics, and deposit morphology of granular column collapse. The granular material adopted in Ref. 49 (glass beads of density $\rho_d = 2500 \text{ kg/m}^3$ and diameter $d = 1.15 \text{ mm}$) is very similar to that for System 3 in Ref. 29 (see Section II B); its corresponding values of η^S and e_{eff} have been estimated in Ref. 8 so that those two model parameters are available as input parameters to our model.

Lajeunesse *et al.*⁴⁹ conducted the experiment in a rectangular channel lined with sandpaper. A vertical gate was equipped at one end of the channel to create a reservoir. The reservoir was partially filled with glass beads to form a granular column. Collapsing of the granular column occurred after the gate was quickly lifted open. Key experimental results of Ref. 49 are summarized below.

The flow induced by the collapse of a granular column, which is localized near the free surface, is initiated by a Coulomb-like failure and eventually the granular material will come to rest. The results of Lajeunesse *et al.*⁴⁹ indicate that the flow dynamics and the deposit morphologies depend primarily on the initial aspect ratio of the granular column, H_i/L_i , where H_i = initial height and L_i = initial length. When $H_i/L_i < 3$, the granular column spreads in the form of avalanching, leading to a truncated cone-shaped deposit ($H_i/L_i < 0.74$) or a conical-shaped deposit ($H_i/L_i > 0.74$). When $H_i/L_i > 3$, the granular column forms a pile; its upper part descends first and then spreads, leading to a conical-shaped deposit. The deposit morphology can be described by the following two scaling laws:

$$\frac{\Delta L}{L_i} \propto \begin{cases} H_i/L_i, & H_i/L_i < a_L \\ (H_i/L_i)^\alpha, & H_i/L_i > a_L \end{cases} \quad (30)$$

and

$$\frac{H_f}{L_i} \propto \begin{cases} H_i/L_i, & H_i/L_i < a_H \\ (H_i/L_i)^\beta, & H_i/L_i > a_H \end{cases}, \quad (31)$$

where $\Delta L = L_f - L_i$ with L_f = length of the final deposit and H_f = height of the final deposit. ΔL is called “run-out distance” in the literature. The empirical parameters in Eqs. (30) and (31) are $\alpha = 2/3$, $\beta = 1/3$, $a_L = 3$, and $a_H = 0.7$. Lube *et al.*⁶² found that $1.8 \leq a_L \leq 2.8$ and $\beta = 0.4$ for sand, rice, and sugar. The kinematics of the flow front can be well described using the characteristic time scale $\tau_c = \sqrt{H_i/g}$. Theoretically, the flow front will undergo three phases: an acceleration phase, a deceleration phase, and a nearly constant speed phase in between. The duration of each phase varies with problems. The experimental results of Lajeunesse *et al.*⁴⁹ revealed that the flow front accelerated when $t < 0.8\tau_c$, decelerated when $2.6\tau_c < t < 3.2\tau_c$, and moved with a nearly constant speed within a time window which should be narrower than $0.8\tau_c < t < 2.6\tau_c$.

B. Numerical setup

1. DEM

DEM has been proven to be able to accurately reproduce the collapsing process of granular columns.⁶³ This study uses an open source DEM software, called “LAMMPS Improved for General Granular and Granular Heat Transfer Simulations” (LIGGGHTS).⁵⁰ The computational domain is schematically shown in Fig. 2. A granular column, with height = H_i and length = L_i and width = W_i , is located at one end of the channel, which has a length of 0.3 m. Referring to Fig. 2, the periodic boundary condition is used in the x_3 -direction, and $W_i = 5d$ is taken to minimize the effects of the side walls on the flow induced by the collapse of the granular column. The bed is covered by the beads of $d = 1.15$ mm, and the glass beads with $d = 1.15 \pm 0.1$ mm are placed in the reservoir. A linear spring-dashpot model is adopted here to calculate the normal contact forces with the stiffness coefficient $k = 10^4$ N/m and the damping coefficient $\gamma = 23\,000$ s⁻¹, resulting in a value of $e \approx 0.7$ ($\gamma = \sqrt{8k/m\{1 + [\pi/\ln(e)]^2\}}$ with m = mass of the particle). The same model parameters are adopted here to calculate the tangential contact forces, which are, however, limited by the Coulomb friction forces with friction coefficient $\mu_p = 0.5$, as suggested in Ref. 63. Girolami *et al.*⁶³ found that using different values of μ_p in the range of 0.2-0.6 yielded only a small change in the length of the final deposit (less than 8%). In addition to the normal and tangential contact forces, Girolami *et al.*⁶³ also considered rolling friction. Since $k = Ed/6(1 - \sigma^2)$ with σ being the Poisson’s ratio,⁶³ $k = 10^4$ N/m corresponds to a Young’s modulus of $O(10^6)$ Pa. Although $E = O(10^6)$ Pa is less than the actual value of E for glasses ($O(10^{10})$ Pa), Ref. 63 suggests that using $E = O(10^6)$ Pa can yield accurate results and allow one to use larger time steps in simulations than using $E = O(10^{10})$ Pa. The time step $\Delta t = 10^{-5}$ s is taken, which ensures that each contact duration exceeds $10\Delta t$. In our DEM simulations, the gate is lifted up instantaneously.

FIG. 2. Computational domain for collapse of a granular column.

2. The present continuum model

To use the present continuum model to simulate the problem shown in Fig. 2, both the length and height of the channel vary in different runs to reduce the computational time, but it is ensured that the granular flow will not touch both the far end and the top cover of the channel. In this study, we simulate a two-dimensional problem without considering sidewall effects. We remark that the present continuum model is applicable to three-dimensional problems. The no-slip boundary condition is applied on the bed, but a slip boundary condition is applied at $x_1 = 0$. The upper surface of the granular material is assumed to be free of any shear stresses. At all boundaries, the gradients of both v and T are set to zero. The boundary condition for v , together with the no-slip boundary condition, ensures no new granular material can enter into the computational domain. The boundary condition for T means that the boundary does not supply or absorb T . The level of T absorbed or supplied by the boundary may depend on the types of materials used to construct the boundary.⁶⁴ The boundary condition for T used here is supported by the results of Lube *et al.*,⁶⁵ who used several types of materials to construct the bed in their experiment: smooth wooden plane, smooth transparent Perspex plane, and rough plane made of sandpaper. The results of Lube *et al.*⁶⁵ for run-out showed that the collapse of a granular column did not depend on materials used to construct the bed, suggesting that the results are insensitive to small amount of T supplied or absorbed by the boundary. Ionescu *et al.*³³ used an $\eta(I)$ -based constitutive relation, which was obtained by neglecting the flux of T at the boundary, to simulate collapsing granular column and their run-out results showed quantitative agreement with the experiment. Therefore, we believe using a zero flux of T as the boundary condition for T is reasonable. A grid size of $dx = 0.0025$ m and a time step $\Delta t = 10^{-5}$ s are used in the simulation. The loop used to solve this model (see Section III) is repeated five times in each time step (i.e., $n_L = 5$). The parameters used in our simulations are summarized in Table I. The values of ρ_d and d are the same as those used in the experiment⁴⁹ and the DEM simulations described in Section V B 1. The value of e_{eff} was suggested by Ref. 8, who obtained the value of e_{eff} by fitting the maximum η predicted by a numerical model⁸ to the experimental results for glass beads with $d = 1.15 \pm 0.1$ mm. The value of η^S was simply given by the tangent of the angle of repose obtained from the experiment.²⁹ According to Ref. 66, v_{max} ranges from 0.625 to 0.641 and v_{min} from 0.56 to 0.6; we take $v_{\text{max}} = 0.63$ and $v_{\text{min}} = 0.57$ in this study. $v_o = 0.6$ was suggested by Ref. 31. The values of a and δ were obtained by fitting Eq. (23) (the $v - I$ relationship) to the DEM results of Ref. 47. For K and χ , which are associated with the pressure in the solid state, the results of our sensitivity analysis have shown that the run-out distance (ΔL) and the final height (H_f) are insensitive to both K and χ (see Section V C 3). Following Ref. 7, we adopt $n = 10^5$: a larger value of n yields a higher viscosity when the granular material approaches the solid state (see Eq. (18)). The initial values of the volume fraction v_i are within 0.61 and 0.63 in the experiment of Ref. 49. Due to the limitation of the dilatancy law ($v < v_o$, see Eq. (23)) and the use of the simple $p^{ES} - v$ relationship (Eq. (20)), $v_i = 0.575$ is simply taken, which exceeds the random loose packing fraction (v_{min}). Effects of v_i on the collapse process will be discussed in Sec. V C. We use “setFields,” an OpenFOAM built-in function, to set up v_i .

C. Results

1. Flow field

As mentioned in Section V A, previous studies have shown that both the collapsing process of a granular column and the deposit morphology depend on the initial aspect ratio (H_i/L_i). To show this using our numerical results, we examine here three cases: $H_i = 6.12$ cm and $L_i = 10.2$ cm (case 1),

TABLE I. The values of all model parameters for all simulations. (Units: ρ_d in kg/m³; d in mm; K in Pa).

ρ_d	d	e_{eff}	η^S	v_{max}	v_{min}	v_o	a	δ	K	χ	N
2500	1.15	0.58	0.42	0.63	0.57	0.6	0.28	0.58	10^8	1.5	10^5

FIG. 3. Sequences of the collapsing process at $t = 0.5\tau_c$, τ_c , $2\tau_c$, and $3\tau_c$ for case 1. (a) Flow evolution: the color points are the centers of particles predicted by DEM, whereas the solid lines represent contours computed using the present method for $\nu = 0.1, 0.3$, and 0.5 . The volume fraction ν decreases outward. (b) Velocity fields: the color maps are the contours of the volume-averaged velocity predicted by DEM, and the solid lines represent the computed results using the present continuum method; the intervals of the contour levels represented by the solid lines are the same as those in the color bar. (c) The white pixels indicate the flowing layer observed in the experiment.⁴⁹

$H_i = 13.44$ cm and $L_i = 5.6$ cm (case 2), and $H_i = 16.5$ cm and $L_i = 1$ cm (case 3). The reason we chose these three cases is because they have different deposit morphologies, as mentioned in Section V A. Figures 3–5 show four snapshots of our numerical results obtained by using the present continuum method (CM in short hereafter) and the DEM. These four snapshots show different stages of the flow front: speeding up ($t = 0.5\tau_c$), moving with a nearly constant speed ($t = 1\tau_c$ and $2\tau_c$), and slowing down ($t = 3\tau_c$). Experimental results for these three cases⁴⁹ are also shown in Figs. 3–5 for comparison. We remark that our CM simulations typically take a few hours of CPU (central processing unit) time for 1 s of physical time on a computer with one thread of Intel® Core™ i7 CUP 870; for example, case 2 took 3.48 h of CPU time for 1 s of physical time. Our DEM simulations typically take more CPU time than CM simulations; for example, using the same CPU but two threads, case 2 took 7.77 h of CPU time for 1 s of physical time.

Before discussing the evolution of the shape of the collapsing granular column, we need to define the free surface of the granular material for the results computed by using CM. Here, the contour line of $\nu = 0.1$ obtained by CM is considered to be the free surface, which matches the shape of granular column given by DEM for case 1 at $t = 0.5\tau_c$ (see Fig. 3(a)). For $\nu > 0.1$, the distance between the edges of neighboring particles is less than a particle diameter.⁶⁷ In Figs. 3(c), 4(c), and 5(c), we only presented the experimental results corresponding to the moving layer (white pixels);⁴⁹ this is because the moving layer is localized near the free surface.

For cases 1 (Figs. 3(a) and 3(c)) and 2 (Figs. 4(a) and 4(c)), the granular column spreads through avalanching on its front side; for case 3 (Figs. 5(a) and 5(c)), the granular column forms a pile after its upper part descends, and then spreads. The shapes of granular column obtained by CM and DEM agree well with each other, which validates CM. However, it is observed that the flow front obtained by CM moves slower than that obtained by DEM at $t = 2\tau_c$ and $3\tau_c$; this

FIG. 4. Same as Fig. 3 but for case 2.

is possibly because the no-slip boundary condition, imposed at the bed, cannot fully reflect the complex interaction between the bed and the particles.³³

As shown in Figs. 3 and 4, the shapes of the granular column simulated by CM and DEM differ from the experiment at $t = 0.5\tau_c$ and τ_c , but they eventually become similar at $t = 2\tau_c$ and $3\tau_c$. At

FIG. 5. Same as Fig. 3 but for case 3.

$t = 0.5\tau_c$ and τ_c , the foot of the granular column moves forward more faster in the experiment than in both CM and DEM simulations. The measured upper surface of the granular column remains more or less horizontal in the region immediately behind the flow front but the simulated upper surfaces show noticeable inclination, which were also observed in the simulations of Refs. 33 and 65. In Fig. 5, the simulated upper surface of the granular column descends rapidly than that in the experiment. According to the numerical study of Ionescu *et al.*,³³ the difference at $t = 0.5\tau_c$ and τ_c may be due to the different released mechanisms used in the simulations and the experiment: in the experiment, the granular flow is released from the reservoir by lifting up the gate at a finite speed, while in the numerical simulations, the gate is lifted up instantaneously. Previous studies³³ suggest that lifting up the gate with a finite speed can make the front propagate slower even though the final run-out distance can be the same as that obtained for lifting up the gate instantaneously.

Figures 3(a) and 4(a) show that ν decreases rapidly toward the free surface for cases 1 and 2, but the decrease is mild in the upper part of the granular column for case 3 (see Fig. 5(a)). It can be seen from the results of DEM for case 2 (Fig. 4(a)) that a saltating region,⁶⁸ a cloud of bouncing particles, exists at the flow front for $t = 2\tau_c$. In the vicinity of the flow front, even though our continuum model captures a phenomenon similar to saltation, we are not able to confirm if the glass gains are really saltating. The no-slip boundary condition imposed at the bed and the low volume fraction in the vicinity of the flow front might have contributed to the observed phenomena. It is known that kinetic theory is not suitable for very diluted flows where the volume fraction is close to zero. In our simulations, the volume fraction is less than 0.3 in the vicinity of the flow front.

After having discussed the evolution of the column shape simulated by the present continuum method, we now examine the flow field. Figures 3(b) and 4(b) show that the velocity increases toward the flow front, whereas Fig. 5(b) shows that the velocity in the upper part of the granular column is nearly uniform for $t \leq \tau_c$. The velocity distributions computed by CM and DEM are also in agreement with the experiment.⁴⁹ As shown in Figs. 3(b)-5(b), both CM and DEM predict nearly identical flow fields (except in the vicinity of the flow front), which validates the present continuum model again. Comparing the flow fields computed by the CM and DEM with the experiment (the white pixels in Figs. 3(c)-5(c)),⁴⁹ it can be found that the zones where $|\mathbf{u}| > 0.2$ m/s approximately match the moving layers in the experiment. Therefore, we adopt $|\mathbf{u}| = 0.2$ m/s as a criterion to distinguish the moving layer from the non-moving layer below. We remark that the contour line for $|\mathbf{u}| = 0.2$ m/s obtained by the CM is close to that for $|\mathbf{u}| = 0.1$ m/s when $t > \tau_c$, meaning that the flow velocity decreases quickly in the direction opposite to the flow for $|\mathbf{u}| < 0.2$ m/s, and thus using $|\mathbf{u}| = 0.2$ m/s as the criterion is acceptable.

Figures 3(b)-5(b) show that the flow is initiated ($t = 0.5\tau_c$) by the rupture along a well-defined failure line, above which the granular material flows down and below which the granular material is not moving. Here, we regard the contour line for $|\mathbf{u}| = 0.2$ m/s as the failure line. The failure angles (θ_f) in the numerical results of both the CM and DEM are approximately 54° (See Fig. 4(b)), which is close to the measurement ($\theta_f = 50^\circ$ – 55°) and to that predicted by Mohr-Coulomb failure criterion ($\theta_f = 45^\circ + \theta^S/2 = 56^\circ$).⁴⁹ For case 1, since $H_i/L_i = 0.6$, which is much smaller than $\tan 54^\circ (=1.37)$, the failure line extends to the upper surface, and the flow is localized near the frontal side of the collapsing column. Since $H_i/L_i = 5.6$ and 16.5 for cases 2 and 3, respectively, which are larger than $\tan 52^\circ$, the failure line goes through the entire granular column and the whole granular column is moving.

As shown in Figs. 3(b)-5(b), there is always a non-moving layer below the moving layer except in the vicinity of the flow front where the moving layer is in direct contact with the bed. Accordingly, it can be anticipated that the effect of the bed is significant only in the vicinity of the flow front, and the incorrect boundary condition imposed at the bed may only induce local errors there. This may explain why the CM underestimates the speed of the front compared to the DEM, but is able to predict flow field correctly away from the flow front.

In the present study, only the work done by the kinetic component of the shear stress is included in Eq. (15) for simplicity. However, this treatment prevents one from predicting the hysteric flow threshold at which the angle of repose equals the static internal friction angle. Including the enduring-contact component of the shear stress in Eq. (15) produces an angle of repose slightly smaller than the static internal friction angle (around 2° – 4°).⁸ If η^S is fixed, excluding the

FIG. 6. Profiles of a collapsing granular column at $t = \tau_c$, $2\tau_c$, and $3\tau_c$ for case 2 with $\nu_i = 0.575$ and $dx = 2.5$ mm (solid line), $\nu_i = 0.58$ and $dx = 2.5$ mm (dotted line), and $\nu_i = 0.575$ and $dx = 1.25$ mm (dashed line).

enduring-contact component from Eq. (15) may result in a final surface profile slightly steeper than that obtained by including it.

2. Influence of numerical setup and model parameters on the simulated collapsing process of column collapse

When simulating the collapse of a granular column using the present continuum method, we used $dx = 2.5$ mm and $\nu_i = 0.575$, which slightly exceeds $\nu_{min} = 0.57$. To examine effects of ν_i and dx on the simulated collapsing process, the profiles of the granular column at $t = \tau_c$, $2\tau_c$, and $3\tau_c$ for case 2 are shown in Fig. 6 for various values of ν_i and dx . Three combinations of ν_i and dx were examined: $\nu_i = 0.575$ with $dx = 2.5$ mm, $\nu_i = 0.58$ with $dx = 2.5$ mm, and $\nu_i = 0.575$ with $dx = 1.25$ mm. The results agree well with each other, showing that the simulated collapsing process is not sensitive to a small change in ν_i and the numerical simulations converge with $dx = 2.5$ mm. However, it can be seen from Fig. 6 that the foot of the column for $\nu_i = 0.58$ spreads slightly faster than that for $\nu_i = 0.575$ when $t < 0.5\tau_c$, but they tend to be consistent after $t > \tau_c$, possibly because using $\nu_i = 0.58$ has yielded a slightly large pressure gradient (see Eq. (20)) in the vicinity of the flow front. In contrast, the foot for $dx = 1.25$ mm is similar to that for $dx = 2.5$ mm, but the former spreads slightly faster than the latter when $t > 2\tau_c$. Since the flow front ($x_1 > 0.2$ m) is about 10 mm in thickness, using $dx = 2.5$ mm may induce a slightly larger numerical error than using $dx = 1.25$ mm near the flow front.

Table II summarizes our results on how sensitive the run-out distance (ΔL) and the final height (H_f) are to the model parameters. In this sensitivity analysis, the base values for the model parameters listed in Table I were used, which give us $I_{tran} = 1.182$, $\eta_{tran} = 0.539$, $\Delta L = 0.207$, and

TABLE II. Summary of the sensitivity analysis.

Varying parameter	Varying ranges	I_{tran}	$\eta_{tran} (10^{-1})$	ΔL (cm)	H_f (cm)
e_{eff}	0.46 $\rightarrow$ 0.70	1.492 $\rightarrow$ 0.882 (69%)	5.84 $\rightarrow$ 4.79 (22%)	20.4 $\rightarrow$ 21.5 (5%)	6.53 $\rightarrow$ 6.45 (1%)
η^S	0.30 $\rightarrow$ 0.51	1.182	5.39	23.5 $\rightarrow$ 19.2 (22%)	5.32 $\rightarrow$ 7.32 (38%)
v_{max}	0.62 $\rightarrow$ 0.64	1.182	5.39	20.5 $\rightarrow$ 20.9 (2%)	6.46 $\rightarrow$ 6.50 (1%)
v_{min}	0.565 $\rightarrow$ 0.575	1.182	5.39	20.7–21.0 (1%)	6.50 $\rightarrow$ 6.45 (1%)
v_o	0.59 $\rightarrow$ 0.61	1.226 $\rightarrow$ 1.136 (7%)	5.37 $\rightarrow$ 5.41 (1%)	21.0 $\rightarrow$ 20.6 (2%)	6.50 $\rightarrow$ 6.47 (<1%)
a	0.18 $\rightarrow$ 0.38	0.989 $\rightarrow$ 1.346 (15%)	5.49 $\rightarrow$ 5.31 (3%)	21.0 $\rightarrow$ 20.7 (1%)	6.45 $\rightarrow$ 6.53 (1%)
δ	0.39 $\rightarrow$ 0.77	1.168 $\rightarrow$ 1.199 (3%)	5.40 $\rightarrow$ 5.38 (<1%)	20.7–21.1 (2%)	6.52 $\rightarrow$ 6.45 (1%)
K	$10^5 \rightarrow 10^9$	1.182	5.39	20.5–20.7 (1%)	6.33 $\rightarrow$ 6.53 (3%)
χ	1, 1.5 $\rightarrow$ 3.5	1.182	5.39	19.5, 20.7 $\rightarrow$ 20.6 (<1%)	6.58 $\rightarrow$ 6.29 (5%)
n	$10^3 \rightarrow 10^6$	1.182	5.39	20.9 $\rightarrow$ 20.7 (<1%)	6.48 (<1%)

$H_f = 0.0648$. Here, I_{tran} and η_{tran} are critical values of I and η for the incipient transition between the dense state ($p^E > 0$) and the inertial state ($p^E = 0$) (see Fig. 1). The ranges of these model parameters used in the sensitivity analysis and the corresponding ranges of I_{tran} , η_{tran} , ΔL , and H_f are listed in Table II, where the symbol “ $\rightarrow$ ” indicates the direction of the change of I_{tran} , η_{tran} , ΔL , or H_f by increasing a model parameter in its respective range. The symbol “ $-$ ” indicates that there is no specific varying trend. For comparison, the rheological characteristics that affect the collapsing process are also included in Table II as η_{tran} and I_{tran} . We remark that η^S is also a parameter describing the rheology (see Fig. 1). The ranges of e_{eff} , η^S , a , and δ cover the values reported in the literature: $e_{eff} = 0.64$ in Ref. 12, $0.30 \leq \eta^S \leq 0.51$ in Ref. 69, $a = 0.18$ and 0.20 in Ref. 46, and $\delta = 0.39$ and 0.42 in Ref. 46. $v_{max} = 0.63 \pm 0.1$ covers the range of 0.625 – 0.641 reported in Ref. 64. $v_o = 0.6 \pm 0.1$ and $v_{min} = 0.57 \pm 0.05$ are chosen in this study; this is because when either $v_o \leq 0.58$ or $v_{min} \leq 0.56$ is used, the granular column will expand quickly at the initial state due to a very large pressure caused by using these values (see Eqs. (20) and (24)). Reference 63 suggests that using Young’s modulus of $E = O(10^6)$ Pa and $E = O(10^{10})$ Pa in DEM may yield very similar results. According to Eq. (21), $K = O(10^5)$ – $O(10^9)$ Pa corresponds to $E = O(10^6)$ – $O(10^{10})$ Pa; therefore, this range of K is adopted in the present study. Hsu *et al.*²² suggested $\chi = 5.5$ for the failure concentration close to c_{max} in sheet flows where the granular material transits from solid state to flow state. However, according to the dilatancy law (Eq. (23)) and the $p^{ES} - v$ relationship (Eq. (20)), the failure concentration should be less than c_o but larger than c_{min} . Therefore, we take $\chi = 1 - 3.5$ in this study. We remark that using $\chi \geq 4$ may yield numerical instability and using $\chi = 1$ leads to an expansion of granular column at the initial stage. It can be seen from Table II that $\eta^S (= \tan \theta^S)$ has strong effects on both ΔL and H_f : a smaller value of η^S produces a milder profile of granular column (i.e., a larger ΔL and a smaller H_f), which is consistent with the numerical results obtained by solving an $\eta(I)$ -based model.^{32,33} Although e_{eff} strongly affects the rheological characteristics, it has insignificant effect on ΔL . As shown in Table II, both K and χ have only minor influence on H_f : a larger value of K leads to a slightly larger H_f and a larger value of χ leads to a slightly smaller H_f because both K and χ reflect compressibility.

3. Scaling law for the final height and run-out distance

To examine the values of H_f/L_i and $\Delta L/L_i$ computed using the present continuum model, four numerical simulations were performed. For each case, H_i or L_i was fixed but H_i/L_i varied in a range. The values of H_f/L_i and $\Delta L/L_i$ computed using the present model are presented in Fig. 7, where the experimental results reported in Refs. 49 and 70 and the DEM results reported in Ref. 63, as well as the scaling laws (Eqs. (30) and (31)), are also included for comparison. The particle diameters and channel widths reported in the literature^{49,63,70} are summarized in Table III.

FIG. 7. Normalized final height (a) and run-out (b) against the column aspect ratio. The symbols in (b) are the same as those in (a).

TABLE III. Key experimental conditions in relevant studies reported in the literature.

Reference	Channel width (W)	Particle diameter (d)	W/d
Lajeunesse <i>et al.</i> ⁴⁹	45 mm	1.5 and 3 mm	30 and 15
Girolami <i>et al.</i> ⁶³	45 mm	3 and 5 mm	15 and 9
Balmforth and Kerswell ⁷⁰	10 mm (narrow)	0.8 mm	12.5
Balmforth and Kerswell ⁷⁰	200 mm (wide)	0.8 mm	250

We remark that the results plotted in Fig. 7 were obtained using glass beads. It can be seen from Fig. 7 that the values of H_f/L_i and $\Delta L/L_i$ computed using the present continuum model agree well with the two scaling laws and those reported in the literature.^{49,63} The computed $\Delta L/L_i$ follows the trend of the measured $\Delta L/L_i$ reported in Ref. 70, but our values of $\Delta L/L_i$ fall between those for wider and narrower channels.

It is worth mentioning that the channel width has influence on $\Delta L/L_i$ in the experiment. A wider channel leads to a larger $\Delta L/L_i$, as shown in Fig. 7. Physically, a wide channel has less averaged friction force per unit width on granular flows than a narrow channel, and the values of $\Delta L/L_i$ computed using the present continuum model should be close to those obtained from a wide channel. The difference between our results and the measurement for the wide channel may be due to the use of the no-slip boundary condition at the bed. Another possible reason is the three-dimensional features associated with a wide channel in the experiment. For slow flows, Ref. 71 suggested to use a simple Coulomb model to account for the friction forces from the side walls. However, this simple treatment may not be applicable for fast flows because the collisions between particles and the walls become predominant in the vicinity of the flow front. Consequently, possible sidewall effects were not considered in this example.

4. States of the granular column

As mentioned in Section II, when $p^E = 0$, the present model reduces to KTBHT (see Eqs. (4), (5), and (18)) and the granular material is in its inertial state. Furthermore, when $p \approx p^E \neq 0$ and $p^K \approx 0$, the particle inertia and the collision between particles play insignificant role in the momentum transfer, and the granular material is in its quasi-static or solid state. Accordingly, p^E/p can be regarded as an index to approximately distinguish among different states of a granular material: the solid and quasi-static states exist when $p^E/p > 0.99$, the dense state when $0.01 \leq p^E/p \leq 0.99$, and the inertial state when $p^E/p < 0.01$. The state maps for cases 1-3 at $t = \tau_c$ and $2\tau_c$ are presented in Fig. 8, which shows that the solid and quasi-static states ($p^E/p < 0.01$), the dense state ($0.01 \leq p^E/p \leq 0.99$), and the inertial state ($p^E/p > 0.99$) coexist. This is the reason why we have used the collapse of a granular column to test the present model. After comparing Fig. 8

FIG. 8. Distributions of p^E/p at $t = \tau_c$ (top row) and $2\tau_c$ (bottom row) for case 1 (a), case 2 (b), and case 3 (c).

with Figs. 3(b)-5(b), it can be found that the regions of solid and quasi-static states in Fig. 8 match the non-moving regions in Figs. 3(b)-5(b), and that the granular flows in the moving layer are in the dense state except in the regions near the flow fronts for cases 1-3 and near the upper part of the granular column for case 3 where the granular material is in its inertial state. When the flow front is in inertial state, the flow front consists of a cloud of bouncing particles, a phenomenon reported also in the study of Lagrée *et al.*,³² who used a contact mechanic method.

VI. DISCUSSION

The results presented in Sections IV and V demonstrate that the present continuum model is applicable to general granular flow problems involving solid, quasi-static, dense, and inertial states, while existing models based on KTBHTs are applicable only to simple one-dimensional problems such as surface-driven granular flows³⁵ and granular flows down an inclined plane.^{9,12} This section compares the present continuum model with the existing CM models that have been used to study the collapse of a granular column.

There are two groups of continuum models for simulating the collapse of a granular column: the models based on the Saint-Venant/shallow-layer approximation (e.g., Refs. 72–75), and the models that do not use this assumption. Here, we only compare the present model with the latter^{32,33,76} because we did not use Saint-Venant/shallow-layer approximation. The model of Lagrée *et al.*³² was developed based on an incompressible Navier-Stokes solver and the $\eta(I)$ -rheology (i.e., $\eta(I) = \eta^S + \Delta\eta/(I_o/I + 1)$ with $\Delta\eta$ and I_o being two model parameters).^{2,27} The model of Ionescu *et al.*³³ was developed based on the $\eta(I)$ -rheology and an iterative decomposition-coordination formulation coupled with an augmented Lagrangian method. These two models^{32,33} are applicable to solid, quasi-static, and dense states because both use the $\eta(I)$ -rheology. Crosta *et al.*⁷⁶ developed a numerical model based on a Lagrangian approach (a computational solid dynamics approach) and an elasto-plastic model; their model is limited to solid (elastic) and quasi-static (plastic) states. Although the model of Lagrée *et al.*³² can reproduce well the collapsing process of a granular column obtained by using DEM, Lagrée *et al.*³² did not compare the numerical results with physical experiments. The models of Crosta *et al.*⁷⁶ and Ionescu *et al.*³³ can reproduce quantitatively the experimental results of a collapsing granular column. These models^{32,33,76} did not capture the flow front moving in inertial state, implying that the inertial state of the flow front may have insignificant effects on the flow dynamics of collapsing granular columns. Since the present model can also capture the flow front moving in inertial state, it is suitable to study problems involving all four states of granular material. Comparing with these three models,^{32,33,76} the main advantage of the present continuum model is its generality and extensibility. For example, the present continuum model considers the effects of volume fraction ν on the constitutive relations, and thus is applicable to problems with values of ν varying from high to low. Additionally, the present continuum model can be easily extended to include fluid phase under the two-phase flow (a continuum phase and a dispersed phase) framework.^{77–80} However, solving the present continuum model needs more computational resources than the other three models;^{32,33,76} this is because achieving generality and extensibility requires more equations (e.g., Eq. (3)) to be solved. Currently, the present continuum model is applicable only to spherical particles. To make the present continuum model applicable to other particle shapes, further improvement on KTBHT is required.

VII. CONCLUSIONS

This study developed a three-dimensional continuum model for flows of granular materials (spherical beads) by adding new enduring-contact components to the shear stress and pressure in KTBHT. The enduring-contact component of shear stress was taken to be proportional to the enduring-contact component of pressure. A novelty of this study is the determination of the enduring-contact component of pressure. We combined two approaches to determine the enduring-contact component of pressure: one for solid state and the other for fluid state. The combined approach makes the present continuum model applicable to problems involving both solid and

fluid states. The rheological characteristics revealed by the present continuum model for small as well as large inertial numbers agree with those observed in quasi-static, dense, and inertial states, suggesting that the present model is applicable to those three fluid states when the diffusion effect of the granular temperature is insignificant.

A new numerical scheme, based on a predictor-corrector method, was developed to solve the present model. This new numerical scheme can avoid possible numerical instability that normally occurs when ν is large. This scheme solves the mass balance equation in an advection-diffusion form, but not in an advection form, and the diffusion term prevents numerical instability.

The present continuum model was applied to simulate the collapse of a two-dimensional granular column. Our results showed that the present continuum model could reproduce the column shape and flow field in good agreement with the results of DEM. The simulated non-flowing zone and final deposit were consistent with the experimental results reported in the literature. Comparing the predicted final height and run-out with the results obtained experimentally and by DEM, the predicted final heights agreed well with these results, whereas the run-out was slightly underestimated, perhaps because of the non-slip boundary imposed at the bed. Consequently, the dynamics of a collapsing granular column is not perfectly reproduced in this study. Even though the present model is far more general than previous models, it does not reproduce the detailed dynamics of a collapsing granular column as good as Ionescu *et al.*³³ Finally, the state maps revealed that the granular material in the following zone was in the dense state except near the flow front and the upper part of the granular column when the aspect ratio is large. One important observation is that the present continuum model can capture the flow front in the inertial state, a difficulty currently facing other models.

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APPENDIX A: PLANE SHEAR FLOW

A plane shear flow can be established between two moving parallel rough plates. Here, we assume that the plane shear flow is steady ($\partial/\partial t = 0$) and uniform, that is, u_1 (in the direction of the moving plate) is the only non-zero velocity component and varies only with x_2 (normal to the plate). For this plane shear flow, τ_{12} is the only non-zero shear stress. After simplification and substitution of Eqs. (6)–(8), (13), and (15) into Eq. (3), we have

$$\left(\frac{\partial u_1}{\partial x_2}\right)^2 - \frac{3(1-e)\lambda'}{4\mu'd^2}(12T) = 0, \quad (\text{A1})$$

where $\lambda' = \lambda^K/\rho_d d T^{1/2}$ and $\mu' = \mu^K/\rho_d d T^{1/2}$. Equation (A1) indicates that $\partial u_1/\partial x_2$ can be expressed by $T^{1/2}$ as

$$\frac{\partial u_1}{\partial x_2} = \frac{3}{d} \left(\frac{(1-e)\lambda'}{\mu'} \right)^{1/2} T^{1/2}. \quad (\text{A2})$$

Substituting Eqs. (9), (19), (20), (22), and (24) into Eq. (5) and invoking the relationship between $\partial u_1/\partial x_2$ and T (Eq. (A2)) yields

$$p = K f_1 + \rho_d \nu \max(f_2, f_3) f_4^2 d^2 \left(\frac{\partial u_1}{\partial x_2} \right)^2, \quad (\text{A3})$$

where $f_1 = p^{ES}/K$, $f_2 = p^D/\rho_d \nu T$, $f_3 = p^K/\rho_d \nu T$, and $f_4 = [9(1-e)\lambda'/\mu']^{-1/2}$. Since $I = d \partial u_1/\partial x_2/\sqrt{p/\rho_d}$, one can obtain

$$I = \sqrt{\frac{1 - K f_1/p}{\nu \max(f_2, f_3) f_4^2}}. \quad (\text{A4})$$

If values of K/p , ν , and e are given, one can obtain I from Eq. (A4). Similarly, substituting Eqs. (6), (9), (17)–(20), (22), and (24) into Eq. (4) and invoking Eq. (A2) leads to

$$\tau_{12} = K f_1 \eta^S + \rho_d d^2 [f_4 \mu' + \nu \max(f_2 - f_3, 0) \eta^S f_4^2] \left(\frac{\partial u_1}{\partial x_2} \right)^2. \quad (\text{A5})$$

Since $\eta = \tau_{12}/p$, one can obtain

$$\eta = \frac{K}{p} f_1 \eta^S + [f_4 \mu' + \nu \max(f_2 - f_3, 0) \eta^S f_4^2] I^2. \quad (\text{A6})$$

Once values of K/p , e (or e_{eff}), and ν are given, η can be obtained from Eqs. (A4) and (A6).

APPENDIX B: COMMENTS ON APPLYING A TWO-PHASE EULER SOLVER IN OpenFOAM TO GRANULAR COLUMN COLLAPSE

OpenFOAM provides a solver “twoPhaseEulerFoam” to simulate particle-fluid flows. If the density and viscosity of fluid are set to be low enough that the inertial force (depending on the density of fluid) and drag force (depending on the viscosity of fluid) between two phases are insignificant, “twoPhaseEulerFoam” can be used to simulate granular flows. The stress for the granular phase in “twoPhaseEulerFoam” is similar to our proposition in Section II B, but the enduring-contact component of pressure (p^E) is only given by a $p^E - \nu$ relation. We have tried to simulate the collapse of a granular column using “twoPhaseEulerFoam”; however, the computed flow never stopped, possibly because “twoPhaseEulerFoam” overestimates the pore fluid pressure but underestimates the enduring-contact component of the pressure for granular phase, even when using the constitutive equations listed in Section II B. We have also tried to remove the fluid part in “twoPhaseEulerFoam” to bypass the problem arising from overestimating the pore fluid pressure, but it yielded numerical instability.

- ¹ R. Nedderman, *Statics and Kinematics of Granular Materials* (Cambridge University Press, Cambridge, 1992).
- ² P. Jop, Y. Forterre, and O. Pouliquen, “A constitutive law for dense granular flows,” *Nature* **441**, 727 (2006).
- ³ J. T. Jenkins and M. W. Richman, “Grad’s 13-moment system for a dense gas of inelastic spheres,” *Arch. Ration. Mech. Anal.* **87**, 355 (1985).
- ⁴ C. K. K. Lun, “Kinetic-theory for granular flow of dense, slightly inelastic, slightly rough spheres,” *J. Fluid Mech.* **233**, 539 (1991).
- ⁵ J. T. Jenkins and M. W. Richman, “Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks,” *Phys. Fluids* **28**, 3485 (1985).
- ⁶ J. T. Jenkins, “Dense shearing flows of inelastic disks,” *Phys. Fluids* **18**, 103307 (2006).
- ⁷ C. H. Lee and C. J. Huang, “Model of sheared granular material and application to surface-driven granular flows under gravity,” *Phys. Fluids* **22**, 043307 (2010).
- ⁸ C. H. Lee and C. J. Huang, “Kinetic-theory-based model of dense granular flows down inclined planes,” *Phys. Fluids* **24**, 073303 (2012).
- ⁹ J. T. Jenkins, “Dense inclined flows of inelastic spheres,” *Granular Matter* **10**, 47 (2007).
- ¹⁰ P. C. Johnson and R. Jackson, “Frictional–collisional constitutive relations for granular materials, with application to plane shearing,” *J. Fluid Mech.* **176**, 67 (1987).
- ¹¹ P. C. Johnson, P. Nott, and R. Jackson, “Frictional–collisional equations of motion for particulate flows and their application to chutes,” *J. Fluid Mech.* **210**, 501 (1990).
- ¹² M. Louge, “Model for dense granular flows down bumpy inclines,” *Phys. Rev. E* **67**, 061303 (2003).
- ¹³ M. Bisi, G. Spiga, and G. Toscani, “Grad’s equations and hydrodynamics for weakly inelastic granular flows,” *Phys. Fluids* **16**, 4235 (2004).
- ¹⁴ J. J. Brey and J. W. Dufty, “Hydrodynamic modes for a granular gas from kinetic theory,” *Phys. Rev. E* **72**, 011303 (2005).
- ¹⁵ D. Berzi, C. G. di Prisco, and D. Vescovi, “Constitutive relations for steady, dense granular flows,” *Phys. Rev. E* **84**, 031301 (2011).
- ¹⁶ J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, “Hydrodynamics for granular flow at low density,” *Phys. Rev. E* **58**, 4638 (1998).
- ¹⁷ B. C. Eu and H. Farhat, “Kinetic theory of fluidized granular matter,” *Phys. Rev. E* **55**, 4187 (1997).
- ¹⁸ V. Garzo and J. W. Dufty, “Dense fluid transport for inelastic hard spheres,” *Phys. Rev. E* **59**, 5895 (1999).
- ¹⁹ H. Jaeger, S. Nagel, and R. Behringer, “Granular solids, liquids, and gases,” *Rev. Mod. Phys.* **68**, 1259 (1996).
- ²⁰ Y. Forterre and O. Pouliquen, “Flows of dense granular media,” *Annu. Rev. Fluid Mech.* **40**, 1 (2008).
- ²¹ Y. Zhang and C. S. Campbell, “The interface between fluid-like and solid-like behaviour in two-dimensional granular flows,” *J. Fluid Mech.* **237**, 541 (1992).
- ²² T. J. Hsu, J. T. Jenkins, and P. L. F. Liu, “On two-phase sediment transport: Sheet flow of massive particles,” *Proc. R. Soc. A* **460**, 2223 (2004).

- ²³ C. Jossierand, P. Lagree, and D. Lhuillier, "Granular pressure and the thickness of a layer jamming on a rough incline," *Europhys. Lett.* **73**, 363 (2006).
- ²⁴ C. S. Campbell, "Stress-controlled elastic granular shear flows," *J. Fluid Mech.* **539**, 273 (2005).
- ²⁵ S. Chialvo, J. Sun, and S. Sundaresan, "Bridging the rheology of granular flows in three regimes," *Phys. Rev. E* **85**, 021305 (2012).
- ²⁶ C. S. Campbell, "Granular shear flows at the elastic limit," *J. Fluid Mech.* **465**, 261 (2002).
- ²⁷ F. da Cruz, S. Emam, M. Prochnow, J. N. Roux, and F. Chevoir, "Rheophysics of dense granular materials: Discrete simulation of plane shear flows," *Phys. Rev. E* **72**, 021309 (2005).
- ²⁸ G. D. R. MiDi, "On dense granular flows," *Eur. Phys. J. E* **14**, 341 (2004).
- ²⁹ O. Pouliquen, "Scaling laws in granular flows down rough inclined planes," *Phys. Fluids* **11**, 542 (1999).
- ³⁰ P. Jop, Y. Forterre, and O. Pouliquen, "Initiation of granular surface flows in a narrow channel," *Phys. Fluids* **19**, 088102 (2007).
- ³¹ O. Pouliquen, C. Cassar, P. Jop, Y. Forterre, and M. Nicolas, "Flow of dense granular material: Towards simple constitutive laws," *J. Stat. Mech.: Theory Exp.* **2006**, P07020.
- ³² P.-Y. Lagre, L. Staron, and S. Popinet, "The granular column collapse as a continuum: Validity of a two-dimensional Navier–Stokes model with a $\mu(I)$ -rheology," *J. Fluid Mech.* **686**, 378 (2011).
- ³³ I. Ionescu, A. Mangeney, F. Bouchut, and O. Roche, "Viscoplastic modelling of granular column collapse with pressure dependent rheology," *J. Non-Newtonian Fluid Mech.* **219**, 1 (2015).
- ³⁴ L. Staron, P.-Y. Lagre, and S. Popinet, "Continuum simulation of the discharge of the granular silo: A validation test for the $\mu(I)$ visco-plastic flow law," *Eur. Phys. J. E* **37**, 5 (2014).
- ³⁵ D. Volfson, L. S. Tsimring, and I. S. Aranson, "Partially fluidized shear granular flows: Continuum theory and molecular dynamics simulations," *Phys. Rev. E* **68**, 021301 (2003).
- ³⁶ J. Chauchat and M. Mdale, "A three-dimensional numerical model for dense granular flows based on the rheology," *J. Comput. Phys.* **256**, 696 (2014).
- ³⁷ L. Staron, P.-Y. Lagre, and S. Popinet, "The granular silo as a continuum plastic flow: The hour-glass vs the clepsydra," *Phys. Fluids* **24**, 103301 (2012).
- ³⁸ L. Bocquet, W. Losert, D. Schalk, T. C. Lubensky, and J. P. Gollub, "Granular shear flow dynamics and forces: Experiment and continuum theory," *Phys. Rev. E* **65**, 011307 (2001).
- ³⁹ L. Bocquet, J. Errami, and T. C. Lubensky, "Hydrodynamic model for a dynamical jammed-to-flowing transition in gravity driven granular media," *Phys. Rev. Lett.* **89**, 184301 (2002).
- ⁴⁰ D. Volfson, L. S. Tsimring, and I. S. Aranson, "Stick-slip dynamics of a granular layer under shear," *Phys. Rev. E* **69**, 031302 (2004).
- ⁴¹ D. Volfson, L. S. Tsimring, and I. S. Aranson, "Order parameter description of stationary partially fluidized shear granular flows," *Phys. Rev. Lett.* **90**, 254301 (2003).
- ⁴² I. S. Aranson and L. S. Tsimring, "Continuum theory of partially fluidized granular flows," *Phys. Rev. E* **65**, 061303 (2002).
- ⁴³ A. Mangeney, L. Tsimring, D. Volfson, I. Aranson, and F. Bouchut, "Avalanche mobility induced by the presence of an erodible bed and associated entrainment," *Geophys. Res. Lett.* **34**, L22401, doi:10.1029/2007GL031348 (2007).
- ⁴⁴ M. Louge and S. Keast, "On dense granular flows down flat frictional inclines," *Phys. Fluids* **13**, 1213 (2001).
- ⁴⁵ D. Vescovi, D. Berzi, P. Richard, and N. Brodu, "Plane shear flows of frictionless spheres: Kinetic theory and 3D soft-sphere discrete element method simulations," *Phys. Fluids* **26**, 053305 (2014).
- ⁴⁶ P.-E. Peyneau and J.-N. Roux, "Frictionless bead packs have macroscopic friction, but no dilatancy," *Phys. Rev. E* **78**, 011307 (2008).
- ⁴⁷ T. Hatano, "Power-law friction in closely packed granular materials," *Phys. Rev. E* **75**, 060301 (2007).
- ⁴⁸ V. Vidyapati and S. Subramaniam, "Granular rheology and phase transition: DEM simulations and order-parameter based constitutive model," *Chem. Eng. Sci.* **72**, 20 (2012).
- ⁴⁹ E. Lajeunesse, J. B. Monnier, and G. M. Homsy, "Granular slumping on a horizontal surface," *Phys. Fluids* **17**, 103302 (2005).
- ⁵⁰ C. Kloss, C. Goniva, A. Hager, S. Amberger, and S. Pirker, "Models, algorithms and validation for opensource DEM and CFD-DEM," *Prog. Comput. Fluid Dyn.* **12**, 140 (2012).
- ⁵¹ M. Babi, "Average balance equations for granular materials," *Int. J. Eng. Sci.* **35**, 523 (1997).
- ⁵² H. P. Zhu and A. B. Yu, "Averaging method of granular materials," *Phys. Rev. E* **66**, 021302 (2002).
- ⁵³ S. Savage, "Analyses of slow high-concentration flows of granular materials," *J. Fluid Mech.* **377**, 1 (1998).
- ⁵⁴ D. Berzi and J. T. Jenkins, "Steady shearing flows of deformable, inelastic spheres," *Soft Matter* **11**, 4799 (2015).
- ⁵⁵ D. Berzi and D. Vescovi, "Different singularities in the functions of extended kinetic theory at the origin of the yield stress in granular flows," *Phys. Fluids* **27**, 013302 (2015).
- ⁵⁶ D. K. Yoon and J. T. Jenkins, "Kinetic theory for identical, frictional, nearly elastic disks," *Phys. Fluids* **17**, 083301 (2005).
- ⁵⁷ J. T. Jenkins and C. Zhang, "Kinetic theory for identical, frictional, nearly elastic spheres," *Phys. Fluids* **14**, 1228 (2002).
- ⁵⁸ T. C. Papanastasiou, "Flows of materials with yield," *J. Rheol.* **31**, 385 (1987).
- ⁵⁹ H. Jasak, "OpenFOAM: Open source CFD in research and industry," *Int. J. Nav. Archit. Ocean Eng.* **1**, 89 (2009).
- ⁶⁰ H. Rusche, "Computational fluid dynamics of dispersed two-phase flows at high phase fractions," Ph.D. thesis, University of London, London, 2003.
- ⁶¹ H. Jasak, "Error analysis and estimation for the finite volume method with applications to fluid flows," Ph.D. thesis, University of London, London, 1996.
- ⁶² G. Lube, H. E. Huppert, R. S. J. Sparks, and A. Freundt, "Collapses of two-dimensional granular columns," *Phys. Rev. E* **72**, 041301 (2005).
- ⁶³ L. Girolami, V. Hergault, G. Vinay, and A. Wachs, "A three-dimensional discrete-grain model for the simulation of dam-break rectangular collapses: Comparison between numerical results and experiments," *Granular Matter* **14**, 381 (2012).
- ⁶⁴ M. Richman, "Boundary conditions based upon a modified Maxwellian velocity distribution for flows of identical, smooth, nearly elastic spheres," *Acta Mech.* **75**, 227 (1988).

- ⁶⁵ G. Lube, H. E. Huppert, R. S. J. Sparks, and M. A. Hallworth, "Axisymmetric collapses of granular columns," *J. Fluid Mech.* **508**, 175 (2004).
- ⁶⁶ F. A. Dullien, *Porous Media: Fluid Transport and Pore Structure* (Academic Press, New York, 1991).
- ⁶⁷ S. Torquato, "Nearest-neighbor statistics for packings of hard spheres and disks," *Phys. Rev. E* **51**, 3170 (1995).
- ⁶⁸ G. Lube, H. E. Huppert, R. S. J. Sparks, and A. Freundt, "Granular column collapses down rough, inclined channels," *J. Fluid Mech.* **675**, 347 (2011).
- ⁶⁹ R. Albert, I. Albert, D. Hornbaker, P. Schiffer, and A.-L. Barabási, "Maximum angle of stability in wet and dry spherical granular media," *Phys. Rev. E* **56**, R6271 (1997).
- ⁷⁰ N. Balmforth and R. Kerswell, "Granular collapse in two dimensions," *J. Fluid Mech.* **538**, 399 (2005).
- ⁷¹ P. Jop, Y. Forterre, and O. Pouliquen, "Crucial role of sidewalls in granular surface flows: Consequences for the rheology," *J. Fluid Mech.* **541**, 167 (2005).
- ⁷² E. Larrieu, L. Staron, and E. Hinch, "Raining into shallow water as a description of the collapse of a column of grains," *J. Fluid Mech.* **554**, 259 (2006).
- ⁷³ A. Mangeney-Castelnau, F. Bouchut, J. Vilotte, E. Lajeunesse, A. Aubertin, and M. Pirulli, "On the use of Saint Venant equations to simulate the spreading of a granular mass," *J. Geophys. Res.* **110**, B09103, doi:10.1029/2004jb003161doi:10.1029/2004jb003161 (2005).
- ⁷⁴ E. E. Doyle, H. E. Huppert, G. Lube, H. M. Mader, and R. S. J. Sparks, "Static and flowing regions in granular collapses down channels: Insights from a sedimenting shallow water model," *Phys. Fluids* **19**, 106601 (2007).
- ⁷⁵ R. Kerswell, "Dam break with Coulomb friction: A model for granular slumping?," *Phys. Fluids* **17**, 057101 (2005).
- ⁷⁶ G. Crosta, S. Imposimato, and D. Roddeman, "Numerical modeling of 2-D granular step collapse on erodible and nonerodible surface," *J. Geophys. Res.* **114**, F03020, doi:10.1029/2008jf001186 (2009).
- ⁷⁷ G. J. Hwang and H. H. Shen, "Modeling the solid-phase stress in a fluid solid mixture," *Int. J. Multiphase Flow* **15**, 257 (1989).
- ⁷⁸ G. J. Hwang and H. H. Shen, "Modeling the phase interaction in the momentum equations of a fluid solid mixture," *Int. J. Multiphase Flow* **17**, 45 (1991).
- ⁷⁹ G.-J. Hwang and H. H. Shen, "Fluctuation energy equations for turbulent fluid solid flows," *Int. J. Multiphase Flow* **19**, 887 (1993).
- ⁸⁰ C.-H. Lee, Z. Huang, and Y.-M. Chiew, "A multi-scale turbulent dispersion model for dilute flows with suspended sediment," *Adv. Water Resour.* **79**, 18 (2015).
- ⁸¹ See also, C. Lusso, A. Ern, F. Bouchut, A. Mangeney, M. Farin, and O. Roche, "Two-dimensional simulation by regularization of free surface viscoplastic flows with Drucker-Prager yield stress and application to granular collapse," <hal-01133786> (2015).