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Output-Feedback Protocols without Controller Interaction for Consensus of Homogeneous Multi-Agent Systems: A Unified Robust Control View[☆]

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Abstract

This paper investigates the consensus problem of homogeneous linear multi-agent systems using output feedback. An observer-type protocol is formulated, which only requires the relative output information of neighbours and does *not* require information exchange between controllers. It is shown that the protocol bridges some existing ones of the same nature. A robust control approach is presented for consensus protocol design, which requires one to solve a Riccati equation and a linear matrix inequality. Dual results are also discussed. By virtue of low- and high-gain techniques, it is shown that certain solvability conditions are to be satisfied to achieve consensus for agents that are not exponentially unstable or are minimum-phase, leading to a unified point of view for some existing results. A numerical example is provided to illustrate the advantages of the proposed design method.

Keywords: output feedback protocol, consensus, multi-agent system, robust control, low gain, high gain.

1. Introduction

In recent years, coordination control of multi-agent systems undoubtedly has been one of the active research themes in the systems and control community because of its wide variety of applications in formation control, flocking, oscillator synchronization, space rendezvous and so on (Olfati-Saber et al., 2007; Kopeikin et al., 2013). A fundamental task of controlling a network of multi-agent agents is to design a protocol such that all agents work cooperatively and finally reach an agreement, which is the consensus problem. A protocol determines how agents interact with each other and exchange information over the network. There have been a lot of studies on the design of various consensus protocols for multi-agent systems, see Ren and Beard (2008); Tang et al. (2014); Tuna (2008); Li et al. (2010); You and Xie (2011); Qin and Yu (2014); Ma and Zhang (2010). For static protocols with relative state feedback, Riccati equation based methods (Zhang et al., 2011; Tuna, 2008) or linear matrix inequality based methods (Li et al., 2010) have been derived for protocol design.

It is known that state feedback sometimes is not practical because full state information is not always available, which motivates the development of output feedback based controller design methods. For this reason, considerable attention has been paid to the design of consensus protocols without agent state information but with agent output information instead, see Hong et al. (2008); Scardovi and Sepulchre (2009);

Wieland et al. (2011); Zhang et al. (2011); Li et al. (2010); Trentelman et al. (2013); Hengster-Movric and Lewis (2013); Hengster-Movric et al. (2015); Zhou and Lin (2014). In particular, the distributed observer-type protocols presented in Li et al. (2010); Zhang et al. (2011); Trentelman et al. (2013) still satisfy the fascinating separation principle such that the two gain matrices in the protocols can be designed separately.

It should be pointed out that most of the aforementioned observer-type protocols require extra information exchange between the controller/observer of each agent and that of its neighbours. Note that the information exchange we mention here would be completely different from the output information collection from neighbours: output information in fact indicates some measured physical quantities, while the information exchanged between controllers/observers is artificial. This feature may incur technical difficulties and/or costs to implement these observer-type protocols. Therefore, the study of protocols without any extra information exchange except the output information is of practical significance and has been one of the recent hotspots (Seo et al., 2009; Zhao et al., 2013; Grip et al., 2015; Zhou and Lin, 2014). A consensus problem of this nature for homogeneous linear multi-agent systems was investigated in Seo et al. (2009), where the low-gain feedback control technique was shown to be useful for designing dynamic output feedback protocols without controller information exchange. Other low-gain-type approaches were also derived in Zhao et al. (2013); Zhou and Lin (2014) but different techniques were employed. A common assumption for these low-gain approaches is that each agent is not exponentially unstable. On the other hand, the authors in Grip et al. (2015) proposed a high-gain method for designing dynamic output feedback protocols for homogeneous and heterogeneous multi-agent systems that satisfy some kinds of minimum-phase assumptions. Regarding these results, what draws our interest is the fact that the aforementioned methods actually discuss quite similar protocols, but are established in the low-gain and high-gain frameworks that appear to be sharply distinct from each other. Therefore we are naturally led to the following questions: Is it possible to formulate these protocols in a general framework? And how

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to design and interpret such protocols from a unified point of view? In this paper we attempt to provide an answer to these questions.

In this paper, we will investigate the output feedback based consensus problem of generic homogeneous linear multi-agent systems. Specifically, we focus on designing an observer-type dynamic protocol that only makes use of the relative output information of neighbouring nodes but does not involve any extra information exchange between controllers/observers. A unified robust control method will be presented for designing such dynamic protocols (Section 2.2). Furthermore, we will discuss the low-gain and high-gain aspects of the protocol design method (Sections 2.3 and 2.4), respectively, so as to uncover the conditions under which a required output feedback based protocol must exist and can be found by the presented design method. Dual results will also be addressed (Section 2.5). Preliminary results in this paper was presented in Li et al. (2016). The main contributions of the paper are stated as follows.

1) A general dynamic observer-type protocol is composed, which involves an additional parameter to bridge the commonly used output feedback based protocols that deal with similar problems. Moreover, the meaning and benefit of this parameter are clarified in a robust control framework.

2) An H_∞ control approach is presented for protocol design. By virtue of the low-gain and high-gain techniques, it is shown that this approach must be feasible for agents that are either not exponentially unstable or minimum-phase. Since the presented approach can take advantage of both low-gain and high-gain techniques, it provides a unified point of view on how to design output feedback protocols without information exchange between controllers/observers. In addition, dual results are still discussed in the same framework.

Notation: $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ are the sets of all $m \times n$ real and complex matrices, respectively. $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ are the sets of complex numbers on the closed right and left half complex plane, respectively. $\mathbf{I}$ denotes an identity matrix with appropriate dimension. The notation $P > 0$ (≥ 0) means that matrix P is positive definite (semi-definite). For a complex number, $\text{Re}(\cdot)$ represents its real part. For two matrices A and B , $A \otimes B$ is the Kronecker product. For a transfer function matrix $G(s) \in RH_\infty$ with RH_∞ as the set of all stable rational transfer function matrices, $\|G\|_\infty$ denotes its H_∞ norm.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denote a directed graph consisting of a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subseteq \mathcal{V}^2$. Suppose that $\mathcal{G}(\mathcal{V}, \mathcal{E})$ has N nodes. We index the node set by $\mathcal{V} = \{1, 2, \dots, N\}$. The ordered pair $(i, j) \in \mathcal{E}$ means that there is a link from node i to node j , for which, node i is said to be a neighbour of node j . Denote by $\mathcal{N}_i$ the neighbouring node set of node i . In the context of this paper, $(i, j) \in \mathcal{E}$ actually indicates that information can be transmitted from node i to node j . In this paper, suppose that nodes are not self-connected, that is, $(i, i) \notin \mathcal{E}$ for $i = 1, 2, \dots, N$. Define the adjacency matrix $A = [a_{ij}]_{N \times N}$ as $a_{ij} > 0$ for $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Further define the (normalized) Laplacian matrix $\mathcal{L} = [l_{ij}]_{N \times N}$ as $\mathcal{L}_{ii} = 1$ and $\mathcal{L}_{ij} = -a_{ij} / (\sum_{k \in \mathcal{N}_i} a_{ik})$ for $i, j = 1, 2, \dots, N$ and $i \neq j$. If the graph has a node from which every other node can be reached through any directed path, we say this graph has a directed spanning tree.

Lemma 1 (Fax and Murray (2004)). *All eigenvalues of the Laplacian matrix $\mathcal{L}$ lie in a closed unit disk centered at the point (1, 0) on the complex plane.*

2. Main Results

2.1. Problem statement

Consider a distributed network of N identical agents. The dynamics of each agent is given by a linear time-invariant system described by the following state-space equations:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), y_i(t) = Cx_i(t), i = 1, 2, \dots, N \quad (1)$$

where $x_i(t) \in \mathbb{R}^{n_x}$, $u_i(t) \in \mathbb{R}^{n_u}$, $y_i(t) \in \mathbb{R}^{n_y}$ are the local state, control input and measurement output, respectively, and A, B, C are real, constant matrices with appropriate dimensions. The matrix pair (A, B) is stabilizable and (A, C) is detectable.

Let the communication topology that describes the information flow among agents be represented by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, and the associated adjacency matrix and Laplacian matrix by $[a_{ij}]_{N \times N}$ and $\mathcal{L} = [l_{ij}]_{N \times N}$, respectively. Suppose that each agent can collect the relative measurement output between itself and its neighbouring nodes. That is, the signal obtained by agent i is given by

$$\tilde{y}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(y_i(t) - y_j(t)) = \sum_{j=1}^N l_{ij}y_j(t). \quad (2)$$

Throughout the paper, the following assumption is made for the communication graph $\mathcal{G}$.

Assumption 1. The graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ contains a directed spanning tree.

Lemma 2 (Ren and Beard (2005)). *Zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ if and only if the associated graph has a spanning tree.*

To make use of $\tilde{y}_i(t)$ to attain consensus, in this paper, we are interested in a dynamic protocol of the following form:

$$\dot{\tilde{x}}_i(t) = (A + \theta BK + FC) \tilde{x}_i(t) - F \tilde{y}_i(t), u_i(t) = K \tilde{x}_i(t) \quad (3)$$

where $\tilde{x}_i(t) \in \mathbb{R}^{n_x}$, $\theta \in \mathbb{R}$, and K, F are appropriately dimensioned real matrices. Scalar θ and matrices K, F are protocol parameters to be determined.

Remark 1. Most of the existing results on observer-type output feedback protocols (e.g., Li et al. (2010); Zhang et al. (2011); Trentelman et al. (2013)) require each observer to have access to the relative observer states of the neighbouring nodes. The protocol in (3) only requires the relative measurement $\tilde{y}_i(t)$ between a local node and its neighbours, which is simpler and easier to implement. Moreover, compared with the protocols in Seo et al. (2009); Zhao et al. (2013); Zhou and Lin (2014), our protocol in (3) is more general which unifies the existing ones. Specifically, the protocol in (3), when $\theta = 0$, reduces to (Zhou and Lin, 2014, (63)) and, when $\theta = 1$, to (Seo et al., 2009, (18)) or (Zhao et al., 2013, (18)). The additional scalar θ provides flexibility for adjusting the protocols to achieve better consensus performance.

The consensus problem of multi-agent systems to be addressed in the paper is formally stated as follows.

Problem 1. For a multi-agent system with agent dynamics represented by (1), find a dynamic protocol (3) such that the states of the closed-loop system satisfy

$$\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0, \lim_{t \rightarrow \infty} (\tilde{x}_i(t) - \tilde{x}_j(t)) = 0, \forall i, j = 1, 2, \dots, N.$$

2.2. Consensus protocol design

Connecting the protocol (3) with the agent (1), we can get the local closed-loop dynamics of agent i given by

$$\dot{\xi}_i(t) = \underbrace{\begin{bmatrix} A & BK \\ 0 & A + \theta BK + FC \end{bmatrix}}_{\mathcal{A}} \xi_i(t) - \sum_{j=1}^N \underbrace{l_{ij}}_{\mathcal{B}} \begin{bmatrix} 0 & 0 \\ FC & 0 \end{bmatrix} \xi_j(t) \quad (4)$$

where $\xi_i(t) = \text{col}\{x_i(t), \tilde{x}_i(t)\}$. By stacking all states together and defining an augmented state vector $\xi \triangleq \text{col}\{\xi_1, \xi_2, \dots, \xi_N\}$, the overall closed-loop network dynamics of the multi-agent system can be written as $\dot{\xi}(t) = (\mathbf{I}_N \otimes \mathcal{A} - \mathcal{L} \otimes \mathcal{B}) \xi(t)$. Let $v \in \mathbb{R}^N$ be the vector satisfying $v^T \mathcal{L} = 0$ (that is, v^T is a left eigenvector of the Laplacian matrix L corresponding to the zero eigenvalue) and $v^T \mathbf{1}_N = 1$ with $\mathbf{1}_N$ being the $N \times 1$ vector of ones. Consider a new variable $e(t)$ as $e(t) = ((\mathbf{I}_N - \mathbf{1}_N v^T) \otimes \mathbf{I}_{2n_x}) \xi(t)$. Following Li et al. (2010), it is known that $e(t) = 0$ if and only if $\xi_1(t) = \dots = \xi_N(t)$. Hence, the error dynamics of the overall network can be characterized by $\dot{e}(t) = (\mathbf{I}_N \otimes \mathcal{A} - \mathcal{L} \otimes \mathcal{B}) e(t)$.

Under Assumption 1, it follows that there exists a nonsingular matrix $T \in \mathbb{R}^{N \times N}$ such that $T = \begin{bmatrix} v^T \\ U \end{bmatrix}$, $T^{-1} = \begin{bmatrix} \mathbf{1}_N & V \end{bmatrix}$ and the Laplacian matrix $\mathcal{L}$ can be decomposed into

$$\mathcal{L} = T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & J_{\mathcal{L}} \end{bmatrix} T = T^{-1} J T \quad (5)$$

where $J_{\mathcal{L}} \in \mathbb{R}^{(N-1) \times (N-1)}$ is a Jordan matrix with the diagonal entries being the $N-1$ nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$. By the state transformation $\tilde{e} = (T \otimes \mathbf{I}_{2n_x}) e$, the error dynamics can be rewritten as $\dot{\tilde{e}}(t) = (\mathbf{I}_N \otimes \mathcal{A} - J \otimes \mathcal{B}) \tilde{e}(t)$, the stability of which, due to the special structure of J and the fact that

$$\tilde{e}_1(t) = (v^T \otimes \mathbf{I}_{2n_x}) e = ((v^T \otimes \mathbf{I}_{2n_x}) (\mathbf{I}_N - \mathbf{1}_N v^T) \otimes \mathbf{I}_{2n_x}) \xi(t) \equiv 0,$$

is equivalent to that of the following $N-1$ subsystems:

$$\dot{\tilde{e}}_i(t) = (\mathcal{A} - \lambda_i \mathcal{B}) \tilde{e}_i(t), \quad i = 2, 3, \dots, N$$

where λ_i , $i = 2, 3, \dots, N$, are the nonzero eigenvalues of $\mathcal{L}$. Consequently we have the following result.

Lemma 3. *Under Assumption 1, the multi-agent system (1) with the protocol (3) reaches consensus if and only if all matrices $\mathcal{A} - \lambda_i \mathcal{B}$, $i = 2, 3, \dots, N$, are Hurwitz.*

In light of Lemma 3, we next pay attention to the Hurwitzness of $\mathcal{A} - \lambda_i \mathcal{B}$, which can be expressed as

$$\det(s\mathbf{I} - (\mathcal{A} - \lambda_i \mathcal{B})) \neq 0, \quad \forall s \in \bar{\mathbb{C}}^+, \quad i = 2, 3, \dots, N. \quad (6)$$

To make the above condition more tractable for protocol design, define three matrices as

$$\bar{A}_i \triangleq \begin{bmatrix} A + \lambda_i BK & \lambda_i BK \\ 0 & A + FC \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} 0_{n_x \times n_u} \\ B \end{bmatrix}, \quad \bar{K} \triangleq \begin{bmatrix} K & K \end{bmatrix}. \quad (7)$$

With the matrix $\bar{A}_i$ assumed to be Hurwitz, performing some algebraic manipulations leads to

$$\begin{aligned} \det(s\mathbf{I} - (\mathcal{A} - \lambda_i \mathcal{B})) &= \det(V(s\mathbf{I} - (\mathcal{A} - \lambda_i \mathcal{B}))V^{-1}) \\ &= \det(s\mathbf{I} - \bar{A}_i + (\lambda_i - \theta)\bar{B}\bar{K}) \end{aligned}$$

$$\begin{aligned} &= \det(s\mathbf{I} - \bar{A}_i) \det(\mathbf{I}_{2n_x} + (\lambda_i - \theta)(s\mathbf{I} - \bar{A}_i)^{-1} \bar{B}\bar{K}) \\ &= \det(s\mathbf{I} - \bar{A}_i) \det(\mathbf{I}_{n_u} + (\lambda_i - \theta)\bar{K}(s\mathbf{I} - \bar{A}_i)^{-1} \bar{B}) \end{aligned} \quad (8)$$

where $V = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{I} & \lambda_i^{-1} \mathbf{I} \end{bmatrix}$ and the equality in (8) is due to Sylvester's determinant identity. Thus we have the following result on the solvability of the consensus problem.

Theorem 1. *Under Assumption 1, the multi-agent system (1) with the protocol (3) reaches consensus if, for $i = 2, 3, \dots, N$, the transfer function $\bar{T}_i(s) \triangleq \bar{K}(s\mathbf{I} - \bar{A}_i)^{-1} \bar{B} \in RH_{\infty}$ satisfies*

$$|\lambda_i - \theta| \cdot \|\bar{T}_i\|_{\infty} < 1. \quad (9)$$

Proof 1. $\bar{T}_i(s) \in RH_{\infty}$ implies $\det(s\mathbf{I} - \bar{A}_i) \neq 0$ for all $s \in \bar{\mathbb{C}}^+$, $i = 2, 3, \dots, N$. Moreover, $\|(\lambda_i - \theta)\bar{T}_i\|_{\infty} = |\lambda_i - \theta| \cdot \|\bar{T}_i\|_{\infty} < 1$ guarantees $\det(\mathbf{I}_{n_u} + (\lambda_i - \theta)\bar{T}_i(s)) \neq 0$ for all $s \in \bar{\mathbb{C}}^+$, $i = 2, 3, \dots, N$. Therefore, the conditions in (6) hold. Then one can complete the proof by applying Lemma 3.

Remark 2. Theorem 1 is more general than that in Zhao et al. (2013). It follows from Lemma 1 that for $\theta = 1$, there always holds $|\lambda_i - \theta| = |\lambda_i - 1| \leq 1$, which implies that $\|\bar{T}_i\|_{\infty} < 1$ is a sufficient condition, which is exactly the one in (Zhao et al., 2013, Theorem 1). For a given communication graph, the inequality $|\lambda_i - 1| \leq 1$ is the most conservative estimation of the bound of $|\lambda_i - \theta|$ for $\theta = 1$, which is useful when the communication graph is completely *unknown* and $\theta = 1$ is *fixed*. However, if the spectra of the graph are unknown, it would be difficult to design K and F . Moreover, if the graph spectra are known or can be better estimated, requiring $\|\bar{T}_i\|_{\infty} < 1$ in general is more restrictive than the condition in (9), or if θ is user-defined (for instance, $\theta = 0$ is used in Zhou and Lin (2014)), $\|\bar{T}_i\|_{\infty} < 1$ might not be sufficient to ensure consensus.

Remark 3. Theorem 1 manifests that, if we can find a scalar θ and matrices K, F such that (9) is satisfied, then the consensus problem is solved. Obviously, the smaller the value of $|\lambda_i - \theta|$ is, the larger the H_{∞} norm $\|\bar{T}_i\|_{\infty}$ is allowed, leading to a more relaxed requirement on $\|\bar{T}_i\|_{\infty}$. Define

$$\begin{aligned} \gamma_1^* &\triangleq \min_{\theta \in [0, 2]} \min_{\gamma \in [0, 1]} \{\gamma \in \mathbb{R} \mid |\lambda_i - \theta| \leq \gamma, i = 2, 3, \dots, N\} \\ \theta^* &\triangleq \arg \min_{\theta \in [0, 2]} \min_{\gamma \in [0, 1]} \{\gamma \in \mathbb{R} \mid |\lambda_i - \theta| \leq \gamma, i = 2, 3, \dots, N\}. \end{aligned} \quad (10)$$

Lemma 1 implies that restricting $\gamma \in [0, 1]$ and $\theta \in [0, 2]$ does not lose generality. Intuitively, γ_1^* as defined above is the shortest radius of the closed disk that contains all the nonzero eigenvalues of the Laplacian on the complex plane, and θ^* is the corresponding x -coordinate of the center of that disk.

The following transfer functions and technical lemma are introduced for later use:

$$G_{i,K}(s) \triangleq \lambda_i K(s\mathbf{I} - (A + \lambda_i BK))^{-1} BK + K \quad (11)$$

$$G_F(s) \triangleq (s\mathbf{I} - (A + FC))^{-1} B. \quad (12)$$

Lemma 4. *Let a scalar $\gamma > 0$ be given. Consider an n th-order system $\dot{x}(t) = Ax(t) + Bw(t) + B_u u(t)$, $z = Cx(t)$ with (A, B_u) being stabilizable. The following statements are equivalent.*

- 1) *There exists a state feedback controller $u = Kx$ such that the resulting closed-loop system $T(s) = C(s\mathbf{I} - (A + B_u K))^{-1} B$ is stable and satisfies $\|T\|_{\infty} < \gamma$.*

2) There exists a matrix $P = P^T \in \mathbb{R}^{n \times n}$ and a scalar $\mu > 0$ such that

$$\begin{bmatrix} PA^T + AP - \mu B_u B_u^T + BB^T & PC^T \\ CP & -\gamma^2 \mathbf{I} \end{bmatrix} < 0, P > 0. \quad (13)$$

Moreover, the gain K can be obtained as $K = -\frac{\mu}{2} B_u^T P^{-1}$.

Proof 2. Using the bounded real lemma (see (Skelton et al., 1997, Lemma 7.1.1)), it is known that Statement 1 holds if and only if there exist matrices $P = P^T > 0$ and K such that

$$\begin{bmatrix} P(A + B_u K)^T + (A + B_u K)P + BB^T & PC^T \\ CP & -\gamma^2 \mathbf{I} \end{bmatrix} < 0. \quad (14)$$

With P and μ satisfying (13), the implication of 2) $\Rightarrow$ 1) can be proved by substituting $K = -\frac{\mu}{2} B_u^T P^{-1}$ into (14). The converse can be proved by eliminating KP as a whole via Finsler's Lemma (see (Boyd et al., 1994, Section 2.6.2)), which transforms (14) into (13) with an extra scalar μ .

Based on Theorem 1 and motivated by Zhao et al. (2013), the following steps (Algorithm I) are proposed for designing a dynamic consensus protocol (3).

1. Choose a scalar $\varepsilon > 0$, and solve the equation

$$A^T P + PA + \varepsilon \mathbf{I} - PBB^T P = 0 \quad (15)$$

for a $P > 0$. Compute the gain matrix K as $K = -\frac{\alpha}{2} B^T P$, where α is a scalar satisfying $\alpha \geq \frac{1}{\min_{i=2, \dots, N} \text{Re}(\lambda_i)}$.

2. Choose two scalars γ_1 and γ_2 such that $\gamma_1 \geq \gamma_1^*$ and $\gamma_2 \geq \gamma_2^*$, respectively, where γ_1^* is defined in (10) and

$$\gamma_2^* \triangleq \max_{i=2, \dots, N} \|G_{i,K}\|_\infty. \quad (16)$$

3. Find a matrix $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and a scalar $\mu > 0$, if exist, satisfying the following linear matrix inequalities:

$$\begin{bmatrix} A^T Q + QA - \mu C^T C + \mathbf{I} & QB \\ B^T Q & -\frac{1}{\gamma_1^2 \gamma_2^2} \mathbf{I} \end{bmatrix} < 0, Q > 0. \quad (17)$$

The gain matrix F can be obtained as $F = -\frac{\mu}{2} Q^{-1} C^T$.

Regarding the above design procedures, we have the following result.

Theorem 2. Under Assumption 1 and with matrices K and F found by Algorithm I, the multi-agent system (1) with protocol (3) reaches consensus.

Proof 3. First we show that the gain matrix K obtained in Step 1 guarantees the Hurwitzness of $A + \lambda_i BK$ for $i = 2, 3, \dots, N$. The proof is similar to that of (Zhang et al., 2011, Theorem 1). Since the matrix pair (A, B) is stabilizable, the equation (15) has a unique solution $P > 0$. From $K = -\frac{\alpha}{2} B^T P$, $\alpha \geq \frac{1}{\min_{i=2, \dots, N} \text{Re}(\lambda_i)}$ and $\text{Re}(\lambda_i) > 0$ for $i = 2, 3, \dots, N$, it follows that

$$\begin{aligned} (A + \lambda_i BK)^* P + P(A + \lambda_i BK) &= A^T P + PA - \frac{\alpha}{2} (\lambda_i^* + \lambda_i) PBB^T P \\ &= -\varepsilon \mathbf{I} - (\alpha \text{Re}(\lambda_i) - 1) PBB^T P \\ &\leq -\varepsilon \mathbf{I} < 0 \end{aligned} \quad (18)$$

which, according to the Lyapunov stability condition, implies that $A + \lambda_i BK$ with $K = -\frac{\alpha}{2} B^T P$ is Hurwitz for $i = 2, 3, \dots, N$.

In view of Lemma 4, it follows that the feasibility of the linear matrix inequalities in (17) guarantees

$$\frac{1}{\gamma_1 \gamma_2} > \|B^T (s\mathbf{I} - (A^T + C^T F^T))^{-1}\|_\infty = \|G_F^T\|_\infty$$

with $F = -\frac{\mu}{2} Q^{-1} C^T$. Therefore, $\|G_F\|_\infty = \|G_F^T\|_\infty < \frac{1}{\gamma_1 \gamma_2}$.

By calculation, the following relation can be verified:

$$\tilde{T}_i(s) = \underbrace{(\lambda_i K (s\mathbf{I} - (A + \lambda_i BK))^{-1} BK + K)}_{G_{i,K}(s)} \underbrace{(s\mathbf{I} - (A + FC))^{-1} B}_{G_F(s)}. \quad (19)$$

Note that scalars γ_1 and γ_2 prescribed in Step 2 satisfy $\gamma_1 \geq \gamma_1^*$ and $\gamma_2 \geq \gamma_2^*$. From the above proof, we have

$$|\lambda_i - \theta| \cdot \|\tilde{T}_i\|_\infty \leq |\lambda_i - \theta| \cdot \|G_{i,K}\|_\infty \cdot \|G_F\|_\infty < 1.$$

Finally, according to Theorem 1, it is shown that the multi-agent system (1) controlled by the protocol (3) with gain matrices K and F obtained by Algorithm I reaches consensus.

From the proof, the meaning of each step of Algorithm I is obvious: Step 1 is to find a stabilizing controller gain K such that $A + \lambda_i BK$ is Hurwitz for $i = 2, 3, \dots, N$; Step 2 is to compute the upper bounds of $|\lambda_i - \theta|$ and $\|G_{i,K}\|_\infty$, respectively; and Step 3 is to compute the observer gain F such that $A + FC$ is Hurwitz and $\|G_F\|_\infty < \frac{1}{\gamma_1 \gamma_2}$. In Step 1 and Step 3, one only needs to solve an algebraic equation or linear matrix inequality problem associated with a system of the same order as that of the agents.

There are three places in Algorithm I that require the global spectral information of the network, which are associated with γ_1 , γ_2 and α . The scalar γ_1 is related to the graph spectra in a straightforward way that has been discussed in Remarks 2 and 3, while γ_2 is related to the graph spectra through a transfer function $G_{i,K}(s)$. The scalar α is a factor that is commonly used for consensus protocol design in most literature. The following result shows the possibility to estimate γ_2 via the value of α .

Proposition 1. Consider the multi-agent system (1) and suppose that matrix P and scalar α are obtained according to Step 1 of Algorithm I. Define two scalars $\check{\gamma}_2$ and $\hat{\gamma}_2$ as

$$\check{\gamma}_2 \triangleq \frac{\alpha}{2} \sqrt{\lambda_{\max}(\alpha^2 \Phi (2\varepsilon \alpha \mathbf{I} - \alpha^2 \Phi)^{-1} \Phi + \Phi)} \quad (20)$$

$$\hat{\gamma}_2 \triangleq \inf \left\{ \gamma \in \mathbb{R} \mid \exists p \in \mathbb{R}, \begin{bmatrix} -4\varepsilon p \mathbf{I} + \alpha^2 \Phi & (4p\alpha - \alpha^2) \Phi \\ (4p\alpha - \alpha^2) \Phi & \alpha^2 \Phi - 4\gamma^2 \mathbf{I} \end{bmatrix} < 0 \right\} \quad (21)$$

where $\Phi = PBB^T P$ and $\lambda_{\max}(\cdot)$ is the largest eigenvalue of a symmetric matrix. We have the following statements.

- 1) If $\alpha \lambda_{\max}(\Phi) < 2\varepsilon$, then γ_2 can be chosen as $\gamma_2 = \min\{\check{\gamma}_2, \hat{\gamma}_2\}$;
- 2) If $\alpha \lambda_{\max}(\Phi) \geq 2\varepsilon$, then γ_2 can be chosen as $\gamma_2 = \hat{\gamma}_2$.

Proof 4. It is known from the bounded real lemma (see (Skelton et al., 1997, Lemma 7.1.1)) that $\gamma_2^* < \gamma_2$ holds if and only if there exists a matrix $S = S^* > 0$ such that

$$\begin{bmatrix} (A + \lambda_i BK)^* S + S(A + \lambda_i BK) + K^T K & \lambda_i SBK + K^T K \\ \lambda_i^* K^T B^T S^* + K^T K & K^T K - \gamma_2^2 \mathbf{I} \end{bmatrix} < 0. \quad (22)$$

By substituting $S = pP$ and $K = -\frac{\alpha}{2} B^T P$ and noting the inequality in (18), it is seen that (22) holds if

$$\begin{bmatrix} -\varepsilon p \mathbf{I} + \frac{\alpha^2}{4} \Phi & \left(\frac{\alpha^2}{4} - \frac{\alpha p \lambda_i}{2} \right) \Phi \\ \left(\frac{\alpha^2}{4} - \frac{\alpha p \lambda_i^*}{2} \right) \Phi & \frac{\alpha^2}{4} \Phi - \gamma_2^2 \mathbf{I} \end{bmatrix} < 0$$

or equivalently,

$$\begin{bmatrix} -4\epsilon p\mathbf{I} + \alpha^2\Phi & \alpha|\alpha - 2p\lambda_i|\Phi \\ \alpha|\alpha - 2p\lambda_i|\Phi & \alpha^2\Phi - 4\gamma_2^2\mathbf{I} \end{bmatrix} < 0 \quad (23)$$

where Φ is defined in (20).

Lemma 1 indicates that λ_i lies in the closed unit disk centered at $(1, 0)$ on the complex plane. Therefore, for $i = 2, 3, \dots, N$, we have $|\alpha - 2p\lambda_i| \leq \alpha$ when $p \in (0, \frac{\alpha}{2}]$ and $|\alpha - 2p\lambda_i| \leq 4p - \alpha$ when $p \in [\frac{\alpha}{2}, \infty)$. Note that a necessary condition for (23) is $p > \frac{\alpha^2}{4\epsilon} \lambda_{\max}(\Phi) \triangleq p^*$. First we consider the case, $p^* < \frac{\alpha}{2}$ (i.e., $\alpha \lambda_{\max}(\Phi) < 2\epsilon$). For $p \in (p^*, \frac{\alpha}{2}]$, due to $|\alpha - 2p\lambda_i| \leq \alpha$, a sufficient condition for (23) is $\begin{bmatrix} -4\epsilon p\mathbf{I} + \alpha^2\Phi & \alpha^2\Phi \\ \alpha^2\Phi & \alpha^2\Phi - 4\gamma_2^2\mathbf{I} \end{bmatrix} < 0$ which, by the Schur Complements, is equivalent to $\alpha^2\Phi - 4\gamma_2^2\mathbf{I} + \alpha^4\Phi(4\epsilon p\mathbf{I} - \alpha^2\Phi)^{-1}\Phi < 0$. When $p \in (p^*, \frac{\alpha}{2}]$, the infimum of γ_2 in the above inequality, which is attained at $p = \frac{\alpha}{2}$, is given by $\hat{\gamma}_2$ that is defined in (20). When $p \in [\frac{\alpha}{2}, \infty)$, due to $|\alpha - 2p\lambda_i| \leq 4p - \alpha$, a sufficient condition for (23) is given by the inequality that is included in (21). Thus, the infimum of γ_2 for this case is $\hat{\gamma}_2$ defined in (21). Now consider the case, $p^* \geq \frac{\alpha}{2}$ (i.e., $\alpha \lambda_{\max}(\Phi) \geq 2\epsilon$). For this case, p belongs to $[\frac{\alpha}{2}, \infty)$ and therefore, the minimum of γ_2 in this case is also given by $\hat{\gamma}_2$.

In the next two subsections, some insights into Algorithm I will be provided. Specifically, sufficient conditions will be presented to illuminate that the proposed method is guaranteed to find certain solution for the consensus problem, leading to low-gain and high-gain interpretations, respectively.

2.3. A low-gain perspective for protocol design

From the proof of Theorem 2, it is seen that in the design process we actually decompose $\bar{T}_i(s)$ as $\bar{T}_i(s) = G_{i,K}(s)G_F(s)$ and then look for K and F such that $|\lambda_i - \theta| \cdot \|G_{i,K}\|_\infty \cdot \|G_F\|_\infty < 1$. If there exists a K or F rendering $\|G_{i,K}\|_\infty$ and/or $\|G_F\|_\infty$ arbitrarily small, then the inequality $|\lambda_i - \theta| \cdot \|G_{i,K}\|_\infty \cdot \|G_F\|_\infty < 1$ can always be satisfied. So it is of practical interest to understand under what conditions these scenarios must occur. In this subsection, we investigate the system $G_{i,K}(s)$ and show that a low-gain approach can be used to interpret the mentioned scenario.

The following assumption and technical lemma are introduced for later use.

Assumption 2. All the poles of the matrix A in (1) belong to $\bar{\mathbb{C}}^-$.

Lemma 5. Suppose that the transfer function $G_{i,K}(s)$ defined in (11) satisfies Assumption 2 and the matrices $A + \lambda_i BK$ are Hurwitz for $i = 2, 3, \dots, N$. Then $\|G_{i,K}\|_\infty \rightarrow 0$ as $K \rightarrow 0$.

Proof 5. The result is dual from (Zhao et al., 2013, Lemma 6). Rewrite $G_{i,K}(s)$ as $G_{i,K}(s) = K(s\mathbf{I} - (A + \lambda_i BK))^{-1}(s\mathbf{I} - A)$. When A is Hurwitz, then as $K \rightarrow 0$, obviously $G_{i,K}(s) \rightarrow K(s\mathbf{I} - A)^{-1}(s\mathbf{I} - A) = K \rightarrow 0$. When A is not Hurwitz but satisfies Assumption 2, there exists a scalar $\rho^* > 0$ such that the matrix $A - \rho\mathbf{I}$ is Hurwitz for all $\rho \in (0^+, \rho^*)$. Introduce $G_{i,K}(s, \rho) = K(s\mathbf{I} - (A - \rho\mathbf{I} + \lambda_i BK))^{-1}(s\mathbf{I} - (A - \rho\mathbf{I}))$. Note that $G_{i,K}(s, \rho) \rightarrow G_{i,K}(s)$ as $\rho \rightarrow 0^+$, and for $\rho \in (0^+, \rho^*)$ and $K \rightarrow 0$, we have $G_{i,K}(s, \rho) \rightarrow K(s\mathbf{I} - (A - \rho\mathbf{I}))^{-1}(s\mathbf{I} - (A - \rho\mathbf{I})) = K \rightarrow 0$. Therefore as $\rho \rightarrow 0^+$ and $K \rightarrow 0$, we obtain $G_{i,K}(s, \rho) \rightarrow G_{i,K}(s)$ and $G_{i,K}(s, \rho) \rightarrow 0$, implying that $G_{i,K}(s) \rightarrow 0$ as $K \rightarrow 0$. In summary, we have $\|G_{i,K}\|_\infty \rightarrow 0$ as $K \rightarrow 0$.

Combining Assumptions 1 and 2 and Lemma 5, we have the following result concerning a low-gain interpretation of the consensus protocol design method.

Theorem 3. Under Assumptions 1 and 2, there always exists a protocol (3) such that the multi-agent system (1) reaches consensus. Moreover, Algorithms I with sufficiently small $\epsilon > 0$ provides a solution of the gain matrices K and F .

Proof 6. On the one hand, under Assumption 2, it is known (Lin, 1998, Lemma 2.2.6) that the solution P of the Riccati equation (15) bears $P \rightarrow 0$ as $\epsilon \rightarrow 0$. Consequently, the gain matrix $K = -\frac{\alpha}{2}B^T P \rightarrow 0$ as $\epsilon \rightarrow 0$. So, according to Lemma 5, for any scalar $\gamma_2^* > 0$, there always exists a scalar $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, there holds $\|G_{i,K}\|_\infty < \gamma_2^*$. On the other hand, since the matrix pair (A, C) is detectable (i.e., (A^T, C^T) is stabilizable), it follows from Lemma 4 that there always exists a $\gamma_3^* > 0$ such that the linear matrix inequality (17) gives a stabilizing solution $F = -\frac{\mu}{2}Q^{-1}C^T$ that guarantees $\|G_F(s)\|_\infty < \gamma_3^*$. Taking sufficiently small γ_2^* can ensure $\gamma_1^* \cdot \gamma_2^* \cdot \gamma_3^* < 1$, that is, the resulting closed-loop system satisfies $|\lambda_i - \theta| \cdot \|\bar{T}_i\|_\infty \leq 1$. Finally one can complete the proof by applying Theorem 1.

Remark 4. Assumption 2, though seemingly restrictive, indeed covers some important kinds of agent dynamics that are commonly investigated in the field of multi-agent systems, especially integrator-type systems (Ren and Beard, 2008). From the proof, it is seen that a smaller ϵ would be more likely to produce a required consensus controller, entailing a *low-gain* matrix K . Using methods different from the one in this paper, the authors in Seo et al. (2009); Zhou and Lin (2014) also derived low-gain approaches to designing similar consensus protocols. The authors in Zhao et al. (2013) employed a similar method to obtain a special case of the protocol (3) (exactly, the results in Zhao et al. (2013) are only a special case of the dual version of those in this subsection, for which please refer to Section 2.5). Therefore, Theorem 3 in this paper provides a unified low-gain perspective for the design of output feedback protocols without controller/observer interaction.

2.4. A high-gain perspective for protocol design

In this subsection, we turn to the transfer function $G_F(s)$, based on which, we shall provide a high-gain perspective for the proposed design method. For facilitating our statement, consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bw(t), \quad y(t) = Cx(t), \quad z(t) = Ex(t) \quad (24)$$

where $x(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$, and A , B , C and E are appropriately dimensioned real matrices. It is assumed that 1) (A, C) is detectable and (A, B) is stabilizable, 2) B and C are of full rank, and 3) there holds

$$\text{rank} \begin{bmatrix} A - s\mathbf{I} & B \\ C & 0 \end{bmatrix} = n_x + n_u, \quad \forall s \in \bar{\mathbb{C}}^+. \quad (25)$$

The following technical lemma (see (Petersen and Hollot, 1988, Lemma 3.1)) will be useful for deriving our results.

Lemma 6. Consider the system in (24). Let R be a given positive definite matrix. Then, given any $\gamma > 0$, there exists a $\epsilon^* > 0$ such that the algebraic Riccati equation

$$A\Sigma + \Sigma A^T + R + \epsilon\gamma^{-1}BB^T - \Sigma C^T C\Sigma + \epsilon^{-1}\gamma^{-1}\Sigma E^T E\Sigma = 0$$

has a positive definite solution of Σ for all $\epsilon \geq \epsilon^*$. Furthermore, with $L = \Sigma C^T$, the transfer function $G(s) = E(s\mathbf{I} - A + LC)^{-1}B$ is stable and satisfies $\|G\|_\infty \leq \gamma$.

For the multi-agent system in (1), in this subsection we further make the following assumption.

Assumption 3. For the multi-agent system in (1), matrices B and C are of full rank, and the triple (A, B, C) satisfies (25).

The full rank assumption for B and C can be made without loss of generality. The assumption for the triple (A, B, C) satisfying (25) implies that each agent of the multi-agent system in (1), denoted by the transfer function $C(s\mathbf{I} - A)^{-1}B$, is left invertible and minimum-phase. By combining Lemma 6 and Assumption 3, the main result of this subsection is presented as follows.

Theorem 4. Under Assumptions 1 and 3, there always exists a protocol (3) such that the multi-agent system (1) reaches consensus. Moreover, Algorithm I provides a solution of the gain matrices K and F .

Proof 7. Because the pair (A, B) is stabilizable, it is always possible to find a stabilizing controller gain K in Step 1 of Algorithm I. Suppose that γ_1 and γ_2 are obtained as in Step 2 of Algorithm I. In view of Lemma 6 with $E = \mathbf{I}$ and $\gamma = \gamma_1^{-1}\gamma_2^{-1}$, Assumption 3 guarantees that, for a given matrix $R > 0$ and for some $\epsilon > 0$, there always exists a positive definite Σ to the equation

$$A\Sigma + \Sigma A^T + R + \epsilon\gamma_1\gamma_2 BB^T - \Sigma C^T C \Sigma + \epsilon^{-1}\gamma_1\gamma_2 \Sigma^2 = 0 \quad (26)$$

leading to $A\Sigma + \Sigma A^T + \epsilon\gamma_1\gamma_2 BB^T - \Sigma C^T C \Sigma + \epsilon^{-1}\gamma_1\gamma_2 \Sigma^2 < 0$. Pre- and post-multiplying this inequality by $\sqrt{\frac{\epsilon}{\gamma_1}}\gamma_2 \Sigma^{-1}$, we have $\frac{\epsilon}{\gamma_1}\gamma_2(\Sigma^{-1}A + A^T\Sigma^{-1}) + \epsilon^2\Sigma^{-1}BB^T\Sigma^{-1} - \frac{\epsilon}{\gamma_1}\gamma_2 C^T C + \mathbf{I} < 0$, which, by the Schur Complements, is equivalent to

$$\begin{bmatrix} \frac{\epsilon}{\gamma_1}\gamma_2 \Sigma^{-1}A + \frac{\epsilon}{\gamma_1}\gamma_2 A^T \Sigma^{-1} - \frac{\epsilon}{\gamma_1}\gamma_2 C^T C + \mathbf{I} & \frac{\epsilon}{\gamma_1}\gamma_2 \Sigma^{-1}B \\ \frac{\epsilon}{\gamma_1}\gamma_2 B^T \Sigma^{-1} & -\frac{1}{\gamma_1\gamma_2}\mathbf{I} \end{bmatrix} < 0.$$

The above inequality is the one in (17) with $Q = \frac{\epsilon}{\gamma_1}\gamma_2 \Sigma^{-1} > 0$ and $\mu = \frac{\epsilon}{\gamma_1}\gamma_2 > 0$. Therefore, Algorithm I under Assumptions 1 and 3 always can produce gain matrices K and F that, according to Theorem 2, solves the considered consensus problem.

The low- and high-gain interpretations of Algorithm I presented in these two subsections are summarized in Figure 1.

Figure 1: Summary of Algorithm I.

Remark 5. Note that the Riccati equation in (26), if feasible for some ϵ^* , is also solvable for sufficiently high ϵ (by Lemma 6) and that the solution of Σ to this equation satisfies $\frac{1}{\sqrt{\epsilon}}\Sigma C^T \rightarrow BM$ as $\epsilon \rightarrow \infty$ (see (Petersen and Hollot, 1988, Lemma 3.3)), where M is some constant matrix. Moreover, from the proof of Theorem 4, it is seen that an F can be obtained as $F = -\frac{\mu}{2}Q^{-1}C^T =$

$-\frac{\epsilon}{\gamma_1}\gamma_2\left(\frac{\epsilon}{\gamma_1}\gamma_2\Sigma^{-1}\right)^{-1}C^T = -\frac{1}{2}\Sigma C^T$. Therefore, there holds $\Sigma C^T \rightarrow \infty$ as $\epsilon \rightarrow \infty$ and the existence of any F must imply that of high-gain ones. Such an explanation is consistent with the high-gain observer idea used in observer-based output feedback control (see Petersen and Hollot (1988)).

Remark 6. Under similar assumptions, the high-gain observer idea was also utilized to solve synchronization problems of networks in Grip et al. (2015). However, there are a few obvious differences between the results in the reference and in this paper. Robust control techniques utilized to obtain our results are different from those in the reference. Moreover, only single-input-single-output agents were considered in the reference while we investigate networks of generic linear agents.

Remark 7. Except for the differences and similarities pointed out in Remarks 1, 2 and 4, this paper and Zhou and Lin (2014) actually focus on tackling different design challenges. The former gives more detailed analysis of consensus protocols in (3), providing a unified low- and high-gain point of view for the existence of such protocols, while the latter systematically copes with predictive protocols with input delay. Note that the high-gain aspect was not revealed in Zhou and Lin (2014). Technically, the design method in the reference is based on a parametric Lyapunov equation, while the one in this paper depends on the Riccati equation (15).

2.5. Dual results

In the subsection, we discuss the dual aspects of the results previously presented. For the multi-agent system in (1) and (2), in this subsection, we consider the following protocol instead,

$$\dot{\tilde{x}}_i(t) = (A + BK + \theta FC) \tilde{x}_i(t) - F \tilde{y}_i(t), \quad u_i(t) = K \tilde{x}_i(t) \quad (27)$$

where K and F are appropriately dimensioned real matrices to be designed and $\theta \in \mathbb{R}$ is a parameter to be clarified. The protocol in (27) also bears the merit that, compared with the observer-based protocols in Li et al. (2010); Zhang et al. (2011); Trentelman et al. (2013); Zhou and Lin (2014), there is no extra information exchange between controllers/observers, which would help reduce communication costs and implementation complexity. A consensus problem can be formulated similarly as Problem 1 but with the protocol (27) to be considered.

Following similar arguments, one can show that the consensus error $\tilde{e}(t)$ is governed by $\dot{\tilde{e}}(t) = (\mathbf{I}_N \otimes \tilde{\mathcal{A}} - J \otimes \mathcal{B}) \tilde{e}(t)$, where J is defined in (5), $\mathcal{B}$ is defined in (4) and

$$\tilde{\mathcal{A}} = \begin{bmatrix} A & BK \\ 0 & A + BK + \theta FC \end{bmatrix}. \quad (28)$$

Due to the special structure of J , we only need to investigate the $N - 1$ subsystems, $\tilde{e}_i(t) = (\tilde{\mathcal{A}} - \lambda_i \mathcal{B}) \tilde{e}_i(t)$, $i = 2, 3, \dots, N$, and consequently arrive at the following lemma (which can be proved by following similar arguments as that of Lemma 3).

Lemma 7. Under Assumption 1, the multi-agent system (1) with the protocol (27) reaches consensus if and only if all matrices $\tilde{\mathcal{A}} - \lambda_i \mathcal{B}$, $i = 2, 3, \dots, N$, are Hurwitz.

The Hurwitzness of $\tilde{\mathcal{A}} - \lambda_i \mathcal{B}$ can be characterized by

$$\det(s\mathbf{I} - (\tilde{\mathcal{A}} - \lambda_i \mathcal{B})) \neq 0, \quad \forall s \in \bar{\mathbb{C}}^+, \quad i = 2, 3, \dots, N.$$

Define three matrices as

$$\tilde{A}_i \triangleq \begin{bmatrix} A+BK & BK \\ 0 & A+\lambda_i FC \end{bmatrix}, \tilde{F} \triangleq \begin{bmatrix} 0_{n_x \times n_y} \\ F \end{bmatrix}, \tilde{C} \triangleq [C \quad C] \quad (29)$$

and we have

$$\begin{aligned} & \det(s\mathbf{I} - (\tilde{A}_i - \lambda_i \tilde{B})) \\ &= \det \left(\begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{I} & \mathbf{I} \end{bmatrix} \left(s\mathbf{I} - \begin{bmatrix} A & BK \\ -\lambda_i FC & A+BK+\theta FC \end{bmatrix} \right) \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \right) \\ &= \det(s\mathbf{I} - \tilde{A}_i + (\lambda_i - \theta) \tilde{F} \tilde{C}) \\ &= \det(s\mathbf{I} - \tilde{A}_i) \det(\mathbf{I}_{2n_x} + (\lambda_i - \theta)(s\mathbf{I} - \tilde{A}_i)^{-1} \tilde{F} \tilde{C}) \\ &= \det(s\mathbf{I} - \tilde{A}_i) \det(\mathbf{I}_{n_y} + (\lambda_i - \theta) \tilde{C} (s\mathbf{I} - \tilde{A}_i)^{-1} \tilde{F}). \end{aligned} \quad (30)$$

Then the following result can be obtained, which is the counterpart of Theorem 1 for the protocol (27).

Theorem 5. *Under Assumption 1, the multi-agent system (1) with the protocol (27) reaches consensus if, for $i = 2, 3, \dots, N$, the transfer function $\tilde{T}_i(s) \triangleq \tilde{C}(s\mathbf{I} - \tilde{A}_i)^{-1} \tilde{F} \in RH_\infty$ satisfies*

$$|\lambda_i - \theta| \cdot \|\tilde{T}_i\|_\infty < 1. \quad (31)$$

Similar to Theorem 1, Theorem 5 can be utilized to convert the consensus problem into an H_∞ one. So the next task is to find K and F such that (31) is satisfied. Moreover, it is clear from Theorem 5 that the scalar θ in (31) plays the same role as that in (9), thus Remarks 2 and 3 are also applicable for the θ in the protocol (27). Especially, when we take $\theta = 1$ and thus have $|\lambda_i - \theta| \leq 1$, the inequality (31) can be replaced by a stronger one, $\|\tilde{T}_i\|_\infty < 1$, just reducing to the one in Zhao et al. (2013).

To illuminate how the results in this subsection can be interpreted as the “dual” of those in the previous ones, note that the transfer function $\tilde{T}_i(s)$ can be further decomposed as

$$\tilde{T}_i(s) = \underbrace{C(s\mathbf{I} - (A+BK))^{-1}}_{\tilde{G}_K(s)} \underbrace{(\lambda_i FC(s\mathbf{I} - (A+\lambda_i FC))^{-1} F + F)}_{\tilde{G}_{i,F}(s)}. \quad (32)$$

By transpose, we have $\tilde{T}_i^T(s) = \tilde{G}_{i,F}^T(s) \tilde{G}_K^T(s)$, where $\tilde{G}_{i,F}^T(s)$ and $\tilde{G}_K^T(s)$ are in similar form of $G_{i,K}(s)$ in (11) and $G_F(s)$ in (12), respectively. Therefore, the “dual” sense can be interpreted in a way that Algorithm I for designing the controller gain K in $G_{i,K}(s)$ can be applied to the design of the observer gain F in $\tilde{G}_{i,F}^T(s)$ in this subsection, while the one for the observer gain F in $G_F(s)$ can be applied to the controller gain K in $\tilde{G}_K^T(s)$. For saving space, more details on the dual protocol are omitted. Finally note that although the decomposition in (32) was also employed in Zhao et al. (2013), arguments therein were only restricted to the low-gain aspect of F under Assumption 2, but never covered the high-gain aspect.

Remark 8. The protocol in (3) and the dual form in (27) would be further unified by the following control protocol:

$$\dot{\tilde{x}}_i(t) = (A + \theta_1 BK + \theta_2 FC) \tilde{x}_i(t) - F \tilde{y}_i(t), u_i(t) = K \tilde{x}_i(t) \quad (33)$$

where θ_1 and θ_2 are two scalars. For the protocols in (3) and (27), the key step to realize the presented analysis and synthesis paradigm is the transformations in (8) and (30), which, when applied to the protocol in (33), lead to

$$\det(s\mathbf{I} - \tilde{A}_i) = \det(s\mathbf{I} - \tilde{A}_i + (\lambda_i - \theta_1) \tilde{B} \tilde{K} + (1 - \theta_2) \tilde{F} \tilde{C}) \quad (34)$$

$$= \det(s\mathbf{I} - \tilde{A}_i + (1 - \theta_1) \tilde{B} \tilde{K} + (\lambda_i - \theta_2) \tilde{F} \tilde{C}) \quad (35)$$

where $\tilde{A}_i = \begin{bmatrix} A & BK \\ -\lambda_i FC & A + \theta_1 BK + \theta_2 FC \end{bmatrix}$. Compared with (8) (respectively, (30)), the extra term $(1 - \theta_2) \tilde{F} \tilde{C}$ in (34) (respectively, $(1 - \theta_1) \tilde{B} \tilde{K}$ in (35)) with $\theta_2 \neq 1$ (respectively, $\theta_1 \neq 1$) entails difficulties in realizing a design procedure. For instance, by Sylvester’s determinant identity, we have

$$\begin{aligned} \det(s\mathbf{I} - \tilde{A}_i) &= \det(s\mathbf{I} - \tilde{A}_i + (1 - \theta_2) \tilde{F} \tilde{C}) \\ &\quad \times \det(\mathbf{I} + (\lambda_i - \theta_1) \tilde{K} (s\mathbf{I} - \tilde{A}_i + (1 - \theta_2) \tilde{F} \tilde{C})^{-1} \tilde{B}) \end{aligned}$$

which results in a sufficient H_∞ condition for the protocol (33): $|\lambda_i - \theta_1| \cdot \|\tilde{K} (s\mathbf{I} - \tilde{A}_i + (1 - \theta_2) \tilde{F} \tilde{C})^{-1} \tilde{B}\|_\infty < 1$. Although this inequality is more general than (9), one cannot easily deal with it by separating the gains K and F as in (19). Moreover, θ_2 does not have the same meaning as θ_1 or θ pointed out in Remark 2.

Remark 9. Suppose that the matrices $A(\omega)$, $B(\omega)$ and $C(\omega)$ in (1) depend on uncertain parameters $\omega \in \mathbb{R}^{n_\omega}$ and are continuous with respect to ω . Let Ω be a neighbourhood of ω at the nominal value $\bar{\omega}$. Following similar arguments, it can be shown that consensus is guaranteed for all $\omega \in \Omega$ if and only if $\Phi_i(\omega) = \begin{bmatrix} A(\omega) & B(\omega)K \\ -\lambda_i FC(\omega) & A + \theta BK + FC \end{bmatrix}$, $i = 2, \dots, N$ are Hurwitz for all $\omega \in \Omega$, where $A = A(\bar{\omega})$, $B = B(\bar{\omega})$ and $C = C(\bar{\omega})$. Let K and F be any gains found by Algorithm I such that consensus is reached for $\omega = \bar{\omega}$, or equivalently, $\text{Re}(\text{spec}(\Phi_i(\bar{\omega}))) < 0$. Since the eigenvalues of a matrix are continuous with respect to the elements of the matrix, there always exists a constant scalar $\eta > 0$ such that $\text{Re}(\text{spec}(\Phi_i(\omega))) < 0$, $\forall \omega \in \{\omega \in \mathbb{R}^{n_\omega} | |\omega - \bar{\omega}| \leq \eta\}$, that is, state consensus is maintained for all $\omega \in \{\omega \in \mathbb{R}^{n_\omega} | |\omega - \bar{\omega}| \leq \eta\}$. Thus, the protocol in (3) possesses some degree of robustness against parameter uncertainties of the multi-agent system.

Finally we present a numerical example to illustrate the advantages of the proposed protocol design method.

Example 1. Consider a 5-agent system in (1) with matrices A , B and C given by

$$A = \begin{bmatrix} 0.0023 & -0.1549 & 0.0880 \\ 0.2043 & -0.0931 & 0.0909 \\ -0.1348 & -0.1843 & 0.1108 \end{bmatrix}, B = \begin{bmatrix} 0.8 \\ 0.4 \\ 2 \end{bmatrix} \\ C = [5.32 \quad 1.56 \quad -2.44]$$

and the connection graph given in Figure 2. The nonzero eigenvalues of the Laplacian $\mathcal{L}$ are $\lambda_{2,3} = 1.4312 \pm 0.7196j$ and $\lambda_{4,5} = 1.0688 \pm 0.4158j$. So $\alpha \geq \frac{1}{\min_{i=2,\dots,N} \text{Re}(\lambda_i)} = 0.9356$ is required for the design method. Moreover, the values of θ^* and γ_1^* in (10) are $\theta^* = 1.4312$ and $\gamma_1^* = 0.7196$, respectively.

Figure 2: The connection topology of the multi-agent system in Example 1.

The agents have three unstable poles, 0.02 and $\pm 0.2236j$, and a single invariant zero, 0.09. Hence, neither Assumption 2 nor Assumption 3 is met. We check the solvability of the consensus problem with Algorithm I under different (θ, γ_1) . It is found that,

when $\theta = 1$, $\gamma_1 = 1$ (corresponding to Zhao et al. (2013)) or $\theta = 1$, $\gamma_1 = \min_i |\lambda_i - 1| = 0.8389$, the algorithm *cannot* provide a feasible solution for this example. Alternatively, let $\varepsilon = 0.01$ and $\alpha = 0.9356$. A feedback gain K can be first obtained as

$$K = \begin{bmatrix} -0.0094 & -0.0200 & -0.0623 \end{bmatrix}.$$

Then the value of γ_2^* in (16) is given by $\gamma_2^* = 0.0812$. Let $\theta = \theta^*$, $\gamma_1 = \gamma_1^*$ and $\gamma_2 = \gamma_2^*$. A feedback gain F can be further obtained as

$$F = \begin{bmatrix} 412.56 & 2650.9 & 2630.0 \end{bmatrix}^T.$$

Figure 3 shows the closed-loop state response of the multi-agent system under the designed controller gains. Clearly the closed-loop network reaches consensus.

Figure 3: State response of the multi-agent system. A: agents; B: controllers.

3. Conclusion

In this paper, we have investigated the problem of designing output feedback protocols for consensus of multi-agent systems with generic linear dynamics. A new observer-type protocol has been established, which covers some existing ones as special cases. This protocol requires no information exchange between controllers, thus saving extra communicating costs. By transforming the protocol design problem into a robust control problem, an H_∞ approach has been constructed for protocol design. Through discussing the low- and high-gain aspects, respectively, we have shown that certain solvability conditions are to be met in order to design a suitable protocol for consensus of agents that are not exponentially unstable or are minimum-phase. In this sense, our approach provides a unified framework for designing output feedback protocols without information exchange between controllers. Finally, a numerical example has been provided to demonstrate the effectiveness of the proposed design method.

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