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2013

Deslauriers-Gauthier, S., & Marziliano, P. (2013). Sampling Signals With a Finite Rate of Innovation on the Sphere. *IEEE Transactions on Signal Processing*, 61(18), 4552-4561.

<https://hdl.handle.net/10356/82375>

<https://doi.org/10.1109/TSP.2013.2272289>

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Sampling Signals with a Finite Rate of Innovation on the Sphere

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Abstract—The state of the art in sampling theory now contains several theorems for signals that are non-bandlimited. For signals on the sphere however, most theorems still require the assumptions of bandlimitedness. In this work we show that a particular class of non-bandlimited signals, which have a finite rate of innovation, can be exactly recovered using a finite number of samples. We consider a sampling scheme where K weighted Diracs are convolved with a kernel on the rotation group. We prove that if the sampling kernel has a bandlimit $L = 2K$ then $(2K - 1)(4K - 1) + 1$ equiangular samples are sufficient for exact reconstruction. If the samples are uniformly distributed on the sphere, we argue that the signal can be accurately reconstructed using $4K^2$ samples and validate our claim through numerical simulations. To further reduce the number of samples required, we design an optimal sampling kernel that achieves accurate reconstruction of the signal using only $3K$ samples, the number of parameters of the weighted Diracs. In addition to weighted Diracs, we show that our results can be extended to sample Diracs integrated along the azimuth. Finally, we consider kernels with antipodal symmetry which are common in applications such as diffusion magnetic resonance imaging.

Index Terms—sampling theorem, spherical convolution, annihilating filter, spherical harmonic, finite rate of innovation.

I. INTRODUCTION

MANIPULATING the spectrum of a signal through filtering is arguably the most popular way of enforcing desirable properties, for example smoothness. Reciprocally, a common way to gain insight on the properties of a signal is to observe its spectrum. Signals on the sphere are no exception to this rule. When the structure of brain tissues is investigated in diffusion magnetic resonance imaging, harmonic analysis plays a central role for both reconstruction [1], [2] and regularization [3], [4]. Other applications where signals on the sphere are common include lighting in image rendering [5], acoustics [6], and source localization [7], to name a few.

The spectrum of a signal on the sphere is computed using the spherical harmonic transform, the natural extension of the Fourier transform. It is common to assume that the spherical harmonics transform of a signal will yield a finite number of non-zero coefficients. When this assumption is satisfied, we say that the signal is bandlimited or has a bandlimit (a strict definition will be provided in a subsequent section). The existence of a bandlimit is a powerful assumption that

has been used to devise sampling theorems for signals on the sphere, for example [8] and [9]. More recently, McEwen *et al.* [10] proposed a new sampling theorem for bandlimited signals on the sphere that reduces by approximately half the number of samples required for the exact reconstruction. Another assumption, popularized by the advances in compressed sensing [11], [12], is the existence of a sparse representation for the signal of interest. For signals on the sphere, sparse reconstruction has been considered in [13]–[16]. While these algorithms reduce the number of samples necessary for good approximation, they do not provide exact reconstruction results.

In the non-sphere domain, the situation is a bit different since we now have algorithms to sample and reconstruct non-bandlimited functions, for example functions that live in a shift-invariant space [17]. Vetterli *et al.* [18], [19] showed that a certain class of signals which have a finite number of degrees of freedom, called finite rate of innovation (FRI) signals, can be sampled and reconstructed perfectly at their rate of innovation. In this case, it is the fact that FRI signals have a parametric representation that is exploited. The typical approach is to compute the inner product of a signal of interest $f(t)$ with a known sampling kernel $h(t)$ to yield the observed signal $s(t)$. Given an appropriate sampling kernel, originally a sinc or Gaussian, the parameters of $f(t)$ can be recovered from the samples of $s(t)$. These results were extended to other kernels including polynomial and exponential reproducing kernels [20] and to 2D signals in [21]–[24]. While signals on the sphere are essentially 2D, they rely on different definitions of inner product, convolution, and harmonic transform, making the current theory inapplicable. To illustrate this idea, consider the case where $s(t)$ is a stream of K Diracs. The observed signal $s(t)$ is then a sum of K scaled and *translated* versions of $h(t)$. This is appropriate in Cartesian space but not on the sphere where the convolution involves a rotation. Indeed, we want s to be obtained by summing K scaled and *rotated* versions of h .

In this paper, we extend on our previous work [25] and show that it is possible to reconstruct certain non-bandlimited signals on the sphere, namely Diracs and integrated Diracs, using a finite number of samples. The typical stream of K Diracs is adapted to the sphere yielding a model with $3K$ degrees of freedom. In contrast to our preliminary work which focused on diffusion magnetic resonance imaging, we provide general proofs that explicitly state the minimum number of samples required for the exact reconstruction of the signal of interest. We also highlight the dependence of these proofs on the type of kernel and sampling scheme used. Furthermore, we develop an optimal sampling kernel that allows the accurate recovery

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S. Deslauriers-Gauthier is funded by the Singapore International Graduate Award of the Agency for Science, Technology and Research (A*STAR), Singapore.

of Diracs using the critical number of samples $3K$. Finally, we consider variations of our model in the form of integrated Diracs and the special case of signals with antipodal symmetry.

The remainder of the article is divided as follows. In Section II, we review the essential concepts of harmonic analysis on the sphere. Section III contains the main results of this article, namely the reconstruction theorems for two types of sampling schemes and the specific requirements on the sampling kernels. In Section IV we show that our results can be extended to include integrated Diracs. We also consider sampling kernels that exhibit antipodal symmetry, which are not covered in the theorems of Section III. To validate the results of the previous sections, we perform several numerical simulations and evaluate the impact of noise in Section V. Finally, Section VI concludes the paper.

II. BACKGROUND

A bandlimited signal on the sphere is, by definition, uniquely characterized by a finite number of coefficients in the frequency domain. Like their Cartesian counterparts, sampling theorems on the sphere exploit this property to state the minimum number of samples needed to exactly recover the signal. Central to these theorems is the spherical harmonic transform, the spherical equivalent of the Fourier transform. For non-bandlimited signals the reconstruction process is challenging because their spherical harmonic expansion involves an infinite sum. The typical signal processing approach is to filter the signal and recover its approximation. This requires an appropriate definition of the convolution operation on the sphere, which involves rotations rather than translations. In this section, we review the definition of the spherical harmonics, the spherical Fourier transform, and the rotational harmonics. We also recall the main results of McEwen *et al.* [10] on sampling bandlimited functions.

A. Spherical harmonics

A point on the unit sphere in Euclidean 3D space $\mathbb{S}^2$ can be identified using two angles $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$ or using a unit norm 3×1 vector $\mathbf{u}$. To simplify the notation, we will interchangeably write $f(\theta, \varphi)$ and $f(\mathbf{u})$ where we assume that both functions are equal if (θ, φ) and $\mathbf{u}$ are related by

$$\mathbf{u}(\theta, \varphi) = [\cos \varphi \sin \theta \quad \sin \varphi \sin \theta \quad \cos \theta]^t.$$

The spherical harmonic of degree ℓ and order m is defined as

$$Y_\ell^m(\theta, \varphi) = N_\ell^m P_\ell^m(\cos \theta) e^{jm\varphi} \quad (1)$$

where N_ℓ^m is a normalization coefficient and $P_\ell^m(\cos \theta)$ is the associated Legendre polynomial of degree ℓ and order m . In this work, we select N_ℓ^m such that the spherical harmonics are orthonormal but our results are independent of this choice. The associated Legendre polynomials are given by

$$P_\ell^m(\cos \theta) = (-1)^m \sin^m \theta \frac{d^m}{d(\cos \theta)^m} P_\ell(\cos \theta)$$

with

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell, \quad (2)$$

$\ell = 0, 1, \dots$, and $0 \leq m \leq \ell$. The set of all spherical harmonics is a basis for the space of square integrable functions on the sphere. As a direct consequence, any function $f(\mathbf{u}) \in L^2(\mathbb{S}^2)$ can be expanded as a sum of spherical harmonics

$$f(\mathbf{u}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_\ell^m Y_\ell^m(\mathbf{u}) \quad (3)$$

where the coefficients $\hat{f}_\ell^m$ are the spherical harmonic coefficients of f . These coefficients are obtained by computing the spherical Fourier transform of f , given by [26]

$$\hat{f}_\ell^m = \int_{\mathbb{S}^2} f(\mathbf{u}) \overline{Y_\ell^m(\mathbf{u})} d\mathbf{u}$$

where the over bar denotes conjugation. A special class of functions on the sphere is the class of bandlimited functions which can be exactly represented using a finite sum of spherical harmonics. Formally, we say a signal defined on $\mathbb{S}^2$ is bandlimited at L or has a bandlimit L if $\hat{f}_\ell^m = 0$ for all $\ell \geq L$. Note that a signal with a bandlimit L has exactly L^2 degrees of freedom¹.

B. Rotational harmonics

Elements of the 3D rotation group $\mathbb{SO}(3)$ are identified by three angles $0 \leq \varphi, \psi < 2\pi$ and $0 \leq \theta < \pi$ or by a 3×3 real orthogonal matrix $\mathbf{G}$ with a determinant of one. To simplify the notation, we will interchangeably write $h(\varphi, \theta, \psi)$ and $h(\mathbf{G})$ where we assume that both functions are equal if (φ, θ, ψ) and $\mathbf{G}$ are related by

$$\mathbf{G} = \mathbf{A}(\varphi) \mathbf{B}(\theta) \mathbf{A}(\psi)$$

where

$$\mathbf{A}(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{B}(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}.$$

The rotational harmonics D_ℓ^{mn} with $\ell = 0, 1, \dots$ and $-\ell \leq m, n \leq \ell$, also known as Wigner functions, are given by

$$D_\ell^{mn}(\varphi, \theta, \psi) = e^{-jm\varphi} d_\ell^{mn}(\cos \theta) e^{-jn\psi}$$

with

$$d_\ell^{mn}(\cos \theta) = j^{m-n} \frac{\sin^{n-m} \theta (1 + \cos \theta)^m}{2^\ell [(\ell + m)! (\ell - m)!]^{1/2}} \left[\frac{(\ell - n)!}{(\ell + n)!} \right]^{1/2} \\ \times \frac{d^{\ell+n}}{d(\cos \theta)^{\ell+n}} (\cos \theta - 1)^{\ell+m} (\cos \theta + 1)^{\ell-m}.$$

Because the Wigner functions are orthogonal and complete, any $h(\mathbf{G}) \in L^2(\mathbb{SO}(3))$ can be expanded as a sum of rotational harmonics

$$h(\mathbf{G}) = \sum_{\ell=0}^{\infty} \frac{(2\ell + 1)}{8\pi^2} \sum_{m=-\ell}^{-\ell} \sum_{n=-\ell}^{\ell} \hat{h}_\ell^{mn} \overline{D_\ell^{mn}}(\mathbf{G}).$$

¹If the infinite sum in Eq. (3) is truncated at $L-1$, the number of spherical harmonic coefficients is $\sum_{\ell=0}^{L-1} (2\ell + 1) = L + 2 \sum_{\ell=0}^{L-1} \ell = L^2$.

The $\hat{h}_\ell^{mn}$ are the rotational harmonics coefficients of $h(\mathbf{G})$ and they are obtained by computing the rotational Fourier transform

$$\hat{h}_\ell^{mn} = \int_{\mathbb{SO}(3)} h(\mathbf{G}) D_\ell^{mn}(\mathbf{G}) d\mathbf{G}.$$

Similarly to spherical harmonics, we say that a signal $h(\mathbf{G}) \in \mathbb{SO}(3)$ has a bandlimit L if $\hat{h}_\ell^{mn} = 0$ for any $\ell \geq L$.

The spherical harmonics are strongly related to the rotational harmonics. A specific relation, which we will use in our proofs is that [27]

$$Y_\ell^m(\mathbf{G}^{-1}\mathbf{u}) = \sum_{i=-\ell}^{\ell} Y_\ell^i(\mathbf{u}) D_\ell^{im}(\mathbf{G}). \quad (4)$$

C. Sampling band-limited signals on the sphere

Current sampling theorems on the sphere deal with bandlimited signals because their spherical harmonic transform has a finite number of non-zero terms. As was stated earlier, a signal with a bandlimit L has L^2 degrees of freedom. It is therefore reasonable to assume that L^2 samples in the space domain will be sufficient to characterize a bandlimited function. To our knowledge, no sampling theorem reaches this lower bound. The recent sampling theorem of McEwen *et al.* [10] requires approximately $2L^2$ samples which improves on previous results by a factor of two. They make use of equiangular samples whose positions are given by

$$\theta_t = \frac{\pi(2t+1)}{2L-1}$$

where $t = 0, \dots, L-1$ and

$$\varphi_p = \frac{2\pi p}{2L-1}$$

where $p = 0, \dots, 2L-2$. Note that this type of sampling requires a number of samples $N = (n-1)(2n-1) + 1$ for some $n \in \mathbb{N}$. Since the recovery of bandlimited signals will be central to our discussion, we summarize their results in the following Lemma.

Lemma 1: Let $s(\theta, \varphi) \in L^2(\mathbb{S}^2)$ be a signal on the sphere with a bandlimit L . Then $(L-1)(2L-1) + 1$ equiangular samples are sufficient to recover $s(\theta, \varphi)$ exactly.

Although the algorithm derived by McEwen *et al.* is not optimal in terms of number of samples, it is stable, has a low complexity, and does not require pre-computations. These advantages are significant if we deal with large bandlimits (say more than 1024). For low bandlimits, the simple linear algebra approach commonly used in diffusion magnetic resonance imaging may be more appropriate. Let (θ_n, φ_n) with $n = 1, \dots, N$ be the sample locations and let $s_n = s(\theta_n, \varphi_n)$. To simplify the notation, we also define a new index $i = \ell^2 + \ell + m + 1$ to write $Y_i = Y_\ell^m$ and $\hat{s}_i = \hat{s}_\ell^m$. The spherical harmonic expansion of a bandlimited signal is then given by

$$\mathbf{s} = \mathbf{Y} \hat{\mathbf{s}}$$

Fig. 1. Comparison of two sampling schemes using 128 samples. The left column illustrates equiangular sampling viewed from the top (a) and bottom (b). The right column illustrates samples locations obtained using the electrostatic repulsion model viewed from the top (c) and bottom (d).

where $\mathbf{s} = [s_1 \ \dots \ s_N]^t$, $\hat{\mathbf{s}} = [\hat{s}_1 \ \dots \ \hat{s}_{L^2}]^t$, and

$$\mathbf{Y} = \begin{bmatrix} Y_1(\theta_1, \varphi_1) & \dots & Y_{L^2}(\theta_1, \varphi_1) \\ \vdots & \ddots & \vdots \\ Y_1(\theta_N, \varphi_N) & \dots & Y_{L^2}(\theta_N, \varphi_N) \end{bmatrix}. \quad (5)$$

Given the samples, we recover the spherical harmonic coefficients by computing $\hat{\mathbf{s}} = \mathbf{Y}^+ \mathbf{s}$ where $\cdot^+$ stands for the Moore-Penrose pseudo-inverse. If we assume $N \geq L^2$ then the exact recovery of the signal is guaranteed if $\mathbf{Y}$ has full column rank, a property that depends on the sample locations. Instinctively, we would like to distribute the samples uniformly on the surface of the sphere. This is a difficult problem to solve analytically for an arbitrary number of measurements. Nonetheless, numerical approaches that produce almost uniform sampling, e.g. the electrostatic repulsion model [28], are available. While the full column rank condition on $\mathbf{Y}$ is not guaranteed in theory, in practice it is fairly easy to generate samples locations that will produce a well conditioned matrix using those methods. A comparison between equiangular samples and typical location obtained using the electrostatic repulsion model is illustrated in Fig. 1.

III. SAMPLING FRI SIGNALS ON THE SPHERE

Signals with a finite rate of innovation have parametric representation and a finite number of degrees of freedom. The typical example is the stream of K weighted Diracs which is modelled using K weights and K locations. Extending this type of signal on the sphere, we obtain a signal $f(\mathbf{u}) \in L^2(\mathbb{S}^2)$ of the form

$$f(\theta, \varphi) = \sum_{k=1}^K a_k \delta(\cos \theta - \cos \theta_k) \delta(\varphi - \varphi_k) \quad (6)$$

where $a_k \in \mathbb{R}$, $0 \leq \theta, \theta_k < \pi$, and $0 \leq \varphi, \varphi_k < 2\pi$. We refer to signals that have the form of Eq. (6) as K orientations. Like their Cartesian counterparts, orientations are non bandlimited signals and therefore cannot be exactly recovered using the current sampling theorems. We will address this problem by proposing new sampling theorems that allows the exact reconstruction of orientations.

We consider a sampling scheme, illustrated in Fig. 2, where the signal of interest $f(\mathbf{u})$ is convolved with a sampling kernel $h(\mathbf{G}) \in L^2(\mathbb{SO}(3))$, that is

$$s(\mathbf{u}) = (f * h)(\mathbf{u}) = \int_{\mathbb{SO}(3)} f(\mathbf{G}^{-1}\mathbf{u}) h(\mathbf{G}) d\mathbf{G}. \quad (7)$$

The observed signal $s(\mathbf{u})$ is then sampled at the positions $\{\mathbf{u}_n\}$ with $n = 1, \dots, N$. Our objective is to recover $f(\mathbf{u})$ given the knowledge of $h(\mathbf{G})$ and K .

The convolution of the signal of interest $f(\mathbf{u})$ with the sampling kernel is difficult to implement as part of an acquisition system. The problem is that $f(\mathbf{u})$ is non bandlimited and has a very small support in the space domain. Therefore, the convolution cannot be implemented efficiently and accurately in the space or harmonic domains. Nonetheless, the signal model of Eq. (7) is relevant because it occurs naturally in applications, notably in diffusion magnetic resonance imaging. In this context, $f(\mathbf{u})$ corresponds to the distribution of fibers in a voxel and $h(\mathbf{G})$ corresponds to the response of a single fiber. More generally, the model of Eq. (7) fits signals that can be modeled as a the sum of K rotations of a single function.

The reconstruction algorithm we propose proceeds in two steps. First, the spherical harmonic coefficients of $f(\mathbf{u})$ are recovered from the samples $s_n = (f * h)(\mathbf{u}_n)$. Second, the parameters a_k, θ_k , and φ_k of the orientations are computed from the spherical harmonic coefficients. Because the first step depends on the number of coefficients needed in the second step, it is more intuitive to present the second step first. In Section III-A, we show that a small set of spherical harmonic coefficients of $f(\mathbf{u})$ are sufficient to find its parameters. We then show that these coefficients can be obtained from the samples s_n with $n = 1, \dots, N$ where N depends on the type of sampling and on $h(\mathbf{G})$. In Section III-B, we derive and illustrate results for a bandlimited kernel with equiangular sampling. In Section III-C, we use uniform sampling with with a bandlimited kernel and devise an optimal sampling kernel.

A. Finding the signal parameters from the spherical harmonics coefficients

In this section, we assume the spherical harmonic coefficients $\hat{f}_\ell^m$ of the orientations are available. We will derive an algorithm to recover the parameters of the orientation using the coefficients.

Consider the spherical harmonics expansion of the orientations

$$\begin{aligned} \hat{f}_\ell^m &= \int_{\mathbb{S}^2} f(\mathbf{u}) \overline{Y_\ell^m(\mathbf{u})} d\mathbf{u} \\ &= \int_0^{2\pi} \int_0^\pi f(\theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin \theta d\theta d\varphi \end{aligned}$$

Fig. 2. The sampling scheme considered in this work. The orientations $f(\mathbf{u})$ are convolved with a sampling kernel $h(\mathbf{G})$ to yield the observed signal $s(\mathbf{u})$ which is then sampled. The samples s_n are used to recover the spherical harmonic (SH) coefficients $\hat{f}_\ell^m$ which are in turn used to find the signal parameters θ_k, φ_k , and a_k .

$$\begin{aligned} &\stackrel{(a)}{=} N_\ell^m \sum_{k=1}^K a_k \int_{-1}^1 \delta(\cos \theta - \cos \theta_k) P_\ell^m(\cos \theta) d(\cos \theta) \\ &\quad \times \int_0^{2\pi} \delta(\varphi - \varphi_k) e^{-jm\varphi} d\varphi \\ &\stackrel{(b)}{=} N_\ell^m \sum_{k=1}^K a_k P_\ell^m(\cos \theta_k) e^{-jm\varphi_k} \end{aligned} \quad (8)$$

where (a) is obtained by replacing $f(\theta, \varphi)$ and $Y_\ell^m(\theta, \varphi)$ by their respective definitions of Eq. (6) and Eq. (1). We then obtain (b) by using the sifting property of the Dirac.

We are specifically interested in the associated Legendre polynomial with equal degree and order, i.e. $m = \ell$,

$$\begin{aligned} P_\ell^\ell(\cos \theta) &= (-1)^\ell \sin^\ell \theta \frac{d^\ell}{d(\cos \theta)^\ell} P_\ell(\cos \theta) \\ &= N_\ell \sin^\ell \theta \end{aligned}$$

where N_ℓ is a constant. This simplification is obtained by noting that P_ℓ given in Eq. (2) is a polynomial of degree ℓ so its ℓ^{th} derivative is a constant. Substituting this associated Legendre polynomial into Eq. (8) we get

$$\begin{aligned} \hat{f}_\ell^\ell &= N_\ell^\ell N_\ell \sum_{k=1}^K a_k \sin^\ell \theta_k e^{-j\ell\varphi_k} \\ &= N_\ell^\ell N_\ell \sum_{k=1}^K a_k s_k^\ell u_k^\ell \\ &= N_\ell^\ell N_\ell \sum_{k=1}^K a_k r_k^\ell \end{aligned}$$

where $u_k = e^{-j\varphi_k}$, $s_k = \sin \theta_k$, and $r_k = u_k s_k$. Finally, we define the normalized samples $z_\ell = \hat{f}_\ell^\ell / (N_\ell^\ell N_\ell)$ which are given by

$$z_\ell = \sum_{k=1}^K a_k r_k^\ell. \quad (9)$$

The normalized samples are of special interest because they are a linear combination of exponentials and thus the annihilating filter method [18] can be employed to find the values of a_k and r_k .

Consider a filter $\mathbf{h} = [h_0 \ \dots \ h_K]^t$ whose z -transform is

$$H(z) = \prod_{k=1}^K (1 - r_k z^{-1}) = \sum_{i=0}^K h_i z^{-i}.$$

In words, the roots of this filter are r_k with $k = 1, \dots, K$. The convolution of $\mathbf{h}$ with $\mathbf{z} = [z_1 \ \dots \ z_N]^t$ gives

$$\begin{aligned} (\mathbf{h} * \mathbf{z})_n &= \sum_{i=0}^K h_i z_{n-i} \\ &= \sum_{i=0}^K h_i \left(\sum_{k=1}^K a_k r_k^{n-i} \right) \\ &= \sum_{k=1}^K a_k r_k^n \sum_{i=0}^K h_i r_k^{-i} = 0 \end{aligned} \quad (10)$$

where the equality to zero is obtained because the right hand sum is the filter being evaluated at one of its roots. The linear system of equations in Eq. (10) with $n = K, \dots, 2K - 1$ can be written in matrix form as

$$\begin{bmatrix} z_K & z_{K-1} & \cdots & z_0 \\ z_{K+1} & z_K & \ddots & z_1 \\ \vdots & \ddots & \ddots & \vdots \\ z_{2K-1} & z_{2K-2} & \cdots & z_{K-1} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_K \end{bmatrix} = \mathbf{Z}\mathbf{h} = \mathbf{0}. \quad (11)$$

This Toeplitz system requires $2K$ contiguous values of z_ℓ to be built. It provides a unique solution, provided that the r_k 's are distinct. Once we find the coefficients of $\mathbf{h}$ using Eq. (11), we can recover the r_k 's by computing its roots. Recall that $r_k = s_k e^{-j\varphi_k}$ meaning its magnitude and phase corresponds to $s_k = \sin \theta_k$ and $-\varphi_k$, respectively. Note that the knowledge of s_k does not uniquely define θ_k .

Now that the value of the r_k 's are known, we can recover the a_k 's using the linear Vandermonde system in Eq. (9) with $\ell = 1, \dots, K$,

$$\begin{bmatrix} 1 & \cdots & 1 \\ r_1 & \cdots & r_K \\ \vdots & \ddots & \vdots \\ r_1^K & \cdots & r_K^K \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_K \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_K \end{bmatrix}.$$

It also yields a unique solution for the a_k 's provided that the r_k 's are distinct.

The final step to recover all the parameters of $f(\theta, \varphi)$ is to disambiguate the value of θ_k . To achieve this, consider the Legendre polynomial whose order is 1 less than its degree

$$\begin{aligned} P_{\ell-1}(\cos \theta) &= (-1)^{\ell-1} \sin^{\ell-1} \theta \frac{d^{\ell-1}}{d(\cos \theta)^{\ell-1}} P_{\ell}(\cos \theta) \\ &= M_{\ell} \sin^{\ell-1} \theta \cos \theta \end{aligned}$$

where M_{ℓ} is a constant. This simplification is obtained by noting that $P_{\ell}(\cos \theta)$ is a polynomial of degree ℓ so its $(\ell - 1)^{\text{th}}$ derivative is proportional to $\cos \theta$. Substituting this

polynomial into Eq. (9) we get

$$\begin{aligned} \hat{f}_{\ell}^{\ell-1} &= N_{\ell}^{\ell-1} M_{\ell} \sum_{k=1}^K a_k \cos \theta_k \sin^{\ell-1} \theta_k e^{-j(\ell-1)\varphi_k} \\ &= N_{\ell}^{\ell-1} M_{\ell} \sum_{k=1}^K a_k c_k r_k^{\ell-1} \end{aligned}$$

with $c_k = \cos \theta_k$. Again defining the normalized measurements $w_{\ell} = \hat{f}_{\ell}^{\ell-1} / (N_{\ell}^{\ell-1} M_{\ell})$, and rewriting in matrix form we get

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ r_1^{K-1} & \cdots & r_K^{K-1} \end{bmatrix} \begin{bmatrix} c_1 a_1 \\ \vdots \\ c_K a_K \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_K \end{bmatrix}.$$

The Vandermonde system above yields a unique solution for the c_k 's which in turn uniquely define the values of θ_k . These results allow us to postulate the following conclusion.

Theorem 1: Let $f(\theta, \varphi)$ be a set of K orientations defined as

$$f(\theta, \varphi) = \sum_{k=1}^K a_k \delta(\cos \theta - \cos \theta_k) \delta(\varphi - \varphi_k)$$

where $a_k \in \mathbb{R}$, $0 \leq \theta, \theta_k < \pi$, and $0 \leq \varphi, \varphi_k < 2\pi$. The spherical harmonics $\hat{f}_{\ell}^{\ell}$ with $\ell = 0, \dots, 2K - 1$ and $\hat{f}_{\ell}^{\ell-1}$ with $\ell = 1, \dots, K$ are sufficient to recover $f(\theta, \varphi)$ exactly.

B. Bandlimited sampling kernel and equiangular sampling

From Theorem 1, we know that weighted orientations can be recovered from their spherical harmonic coefficients. We now turn to the problem of finding the coefficients from the observed samples s_n with $n = 1, \dots, N$. We consider equiangular sampling and an ideal bandlimited kernel.

Definition 1: An ideal bandlimited sampling kernel is a kernel $h(\mathbf{G})$ whose rotational harmonics are given by

$$\hat{h}_{\ell}^{mn} = \begin{cases} c_{\ell m} & \ell < L \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

with the constants $c_{\ell m} \neq 0$ and $L > 0$.

Consider again the observed signal given by

$$s(\mathbf{u}) = \int_{\mathbb{SO}(3)} f(\mathbf{G}^{-1}\mathbf{u}) h(\mathbf{G}) d\mathbf{G}.$$

Replacing the orientations by their spherical harmonic expansion yields

$$s(\mathbf{u}) = \int_{\mathbb{SO}(3)} \left(\sum_{\ell=0}^{\ell=\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m Y_{\ell}^m(\mathbf{G}^{-1}\mathbf{u}) \right) h(\mathbf{G}) d\mathbf{G}.$$

Substituting the spherical harmonics by the relation of Eq. (4) we get

$$\begin{aligned}
s(\mathbf{u}) &= \int_{\mathbb{SO}(3)} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m \sum_{i=-\ell}^{\ell} Y_{\ell}^i(\mathbf{u}) D_{\ell}^{im}(\mathbf{G}) h(\mathbf{G}) d\mathbf{G} \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m \sum_{i=-\ell}^{\ell} Y_{\ell}^i(\mathbf{u}) \int_{\mathbb{SO}(3)} D_{\ell}^{im}(\mathbf{G}) h(\mathbf{G}) d\mathbf{G} \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m \sum_{i=-\ell}^{\ell} Y_{\ell}^i(\mathbf{u}) \hat{h}_{\ell}^{im} \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m J_{\ell}^m(\mathbf{u}) \tag{12}
\end{aligned}$$

with

$$J_{\ell}^m(\mathbf{u}) = \sum_{i=-\ell}^{\ell} Y_{\ell}^i(\mathbf{u}) \hat{h}_{\ell}^{im}. \tag{13}$$

Noting the definition of the ideal bandlimited kernel and assuming without loss of generality that $c_{\ell m} = 1$ for any ℓ and m , we find that $J_{\ell}^m(\mathbf{u}) = Y_{\ell}^m(\mathbf{u})$. Therefore, Eq. (12) reduces to

$$s(\mathbf{u}) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell}^m Y_{\ell}^m(\mathbf{u}).$$

Clearly, the signal $s(\mathbf{u})$ has a bandlimit L and has the same spherical harmonics coefficients as $f(\mathbf{u})$ for $\ell < L$. From Theorem 1, we know that the spherical harmonics $\hat{f}_{\ell}^{\ell}$ with $\ell = 0, \dots, 2K-1$ and $\hat{f}_{\ell}^{\ell-1}$ with $\ell = 1, \dots, K$ are sufficient to recover $f(\theta, \varphi)$ exactly. We conclude that the bandlimit of the sampling kernel must be $L = 2K$. Finally, Lemma 1 tells us that the number of samples required to find all spherical harmonic coefficients is $(2K-1)(4K-1)+1$. This leads us to the following result.

Corollary 1: (Equiangular samples and ideal kernel) Let $f(\theta, \varphi) \in L^2(\mathbb{S}^2)$ be K orientations as defined in Eq. (6) and let $h \in L^2(\mathbb{SO}(3))$ be a sampling kernel with a bandlimit $L = 2K$. The equiangular samples $s_n = (f * h)(\theta_n, \varphi_n)$ with $n = 1, \dots, (2K-1)(4K-1)+1$ are sufficient to recover $f(\theta, \varphi)$ exactly.

The reconstruction process for orientations with $K = 5$ is illustrated in Fig. 3. We would like to highlight that our definition of the ideal bandlimited kernel encompasses simpler kernels found in practice. For example, kernels in diffusion magnetic resonance imaging are assumed to be axisymmetric (in addition to antipodally symmetric as discussed in Section IV-B). For axisymmetric kernels, the rotational harmonics are $\hat{h}_{\ell}^{mn} = c_{\ell}$ for $m = n$ and 0 otherwise. Inserting an axisymmetric ideal bandlimited kernel in Eq. (13) and then Eq. (12) yields $\hat{s}_{\ell}^m = \hat{f}_{\ell}^m c_{\ell}$. Clearly the coefficients of $\hat{f}_{\ell}^m$ can be recovered from the samples of $s(\mathbf{u})$ and we reach results similar to those of Corollary 1.

C. Bandlimited sampling kernel and uniform sampling

Assuming sample locations that produce a full column rank matrix $\mathbf{Y}$ in Eq. (5), the spherical harmonic coefficients of a bandlimited signal can be recovered from $\hat{\mathbf{s}} = \mathbf{Y}^+ \mathbf{s}$. We

also showed in Section III-B that $s(\mathbf{u}) = (f * h)(\mathbf{u})$ has the same spherical harmonic coefficients as f for $\ell < L$ if h has a bandlimit L . Assuming that $L = 2K$, this leads us to $\hat{\mathbf{f}} = \mathbf{Y}^+ \mathbf{s}$ which indicates that a minimum of $N = L^2 = 4K^2$ samples are required to recover f . The accurate recovery of the coefficients $\hat{\mathbf{f}}$ depends on the rank of $\mathbf{Y}$ which in turn relies on the number and locations of the samples, as we discussed in Section II-C. To our knowledge there are no theoretical guarantees that, for an arbitrary number of samples $N = L^2$, it is possible to find locations that yield a full column rank matrix $\mathbf{Y}$. In practice however, the sample locations obtained from the electron repulsion model yields well conditioned matrices $\mathbf{Y}$ suggesting that $4K^2$ samples are sufficient to accurately recover $f(\mathbf{u})$. To reinforce this claim, we perform numerical simulations in Section V.

The linear system described above requires $4K^2$ measurements to compute $4K^2$ spherical harmonic coefficients, but only $3K$ are used to find the parameters of $f(\mathbf{u})$. To further reduce the number of samples used, we design an optimal filter that sets to zero all superfluous spherical harmonics coefficients.

Definition 2: An optimal sampling kernel is a kernel whose rotational harmonics are given by

$$\hat{h}_{\ell}^{mn} = \begin{cases} 1 & \ell < L \text{ and } m = n = \ell \\ 1 & 0 < \ell \leq L/2 \text{ and } m = n = \ell - 1 \\ 0 & \text{otherwise} \end{cases}$$

for $L > 0$.

Selecting the bandlimit $L = 2K$ and inserting rotational harmonics of the optimal sampling kernel into Eq. (13) and Eq. (12) we find that

$$s(\mathbf{u}) = \sum_{\ell=0}^{2K-1} \hat{f}_{\ell}^{\ell} Y_{\ell}^{\ell}(\mathbf{u}) + \sum_{\ell=1}^K \hat{f}_{\ell}^{\ell-1} Y_{\ell}^{\ell-1}(\mathbf{u}).$$

We can also write this system as $\mathbf{s} = \tilde{\mathbf{Y}} \hat{\mathbf{s}}$ where $\tilde{\mathbf{Y}}$ is a $N \times 3K$ matrix that contains only the columns of $\mathbf{Y}$ where $m = \ell$ or $m = \ell - 1$. Given N samples with $N \geq 3K$, we can recover the spherical harmonics coefficients accurately from $\hat{\mathbf{s}} = \tilde{\mathbf{Y}}^+ \mathbf{s}$, provided that $\tilde{\mathbf{Y}}$ has full column rank. Once again there are no theoretical guarantees that $3K$ samples will be sufficient for the exact recovery of $f(\mathbf{u})$, but the numerical simulations in Section V show that very accurate recovery is nonetheless achieved at the critical rate.

IV. VARIATIONS ON A THEME

A. Integrated Diracs

Similar to FRI results on streams of Diracs which can be extended to piecewise polynomials, our results on the sphere can also be extended to other non bandlimited signals. Consider signals of the form

$$\frac{\partial}{\partial \varphi} F(\theta, \varphi) = f(\theta, \varphi) \tag{14}$$

where $f(\theta, \varphi)$ is given by Eq. (6). Note that F and f have the same parameters and number of degrees of freedom. By

Fig. 3. Illustration of the reconstruction process using the ideal filter for $K = 5$. The radius in a direction represents the magnitude of the real component of the signal in that direction. Positive values are displayed in red and negative values are displayed in blue. (a) The 5 orientation $f(\mathbf{u})$ we wish to sample and recover; (b) the sampling kernel $h(\mathbf{G})$ (convolved with a single orientations pointing up for illustration purposes); (c) the observed signal $s(\mathbf{u}) = (f * h)(\mathbf{u})$; (d) the $N = 172$ equiangular samples required for exact reconstruction. The recovered signal is not shown, but is identical to the signal in (a).

replacing $F(\theta, \varphi)$ by its spherical harmonics expansion and noting that

$$\frac{d}{d\varphi} Y_\ell^m(\theta, \varphi) = jm Y_\ell^m(\theta, \varphi)$$

we get

$$\begin{aligned} f(\theta, \varphi) &= \frac{d}{d\varphi} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{F}_\ell^m Y_\ell^m(\theta, \varphi) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{F}_\ell^m \frac{d}{d\varphi} Y_\ell^m(\theta, \varphi) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} jm \hat{F}_\ell^m Y_\ell^m(\theta, \varphi). \end{aligned} \quad (15)$$

By comparing Eq. (15) and Eq. (3) we observe that $\hat{f}_\ell^m = jm \hat{F}_\ell^m$ for $m \neq 0$. More generally, if we have

$$\frac{\partial^n}{\partial \varphi^n} F(\theta, \varphi) = f(\theta, \varphi)$$

then $\hat{f}_\ell^m = (jm)^n \hat{F}_\ell^m$ for $m \neq 0$. Now if we consider the filtered signal $s(\mathbf{u}) = (F * h)(\mathbf{u})$ where the kernel $h(\mathbf{G})$ has a bandlimit L we find that

$$s(\theta, \varphi) = \sum_{\ell=0}^{L-1} \sum_{m=-\ell}^{\ell} \hat{F}_\ell^m Y_\ell^m(\theta, \varphi).$$

The spherical harmonics coefficients $\hat{F}_\ell^m$ and thus $\hat{f}_\ell^m$ can be computed using Lemma 1. Because the computation of $\hat{f}_\ell^m$

for $m = 0$ involves a division by 0, we use the spherical harmonics coefficients $\hat{f}_\ell^\ell$ with $\ell = 1, \dots, 2K$ and $\hat{f}_\ell^{\ell-1}$ with $\ell = 2, \dots, K+1$ and Theorem 1 to recover all the signal parameters a_k, θ_k , and φ_k .

Corollary 2: (Integrated orientations) Let $f(\theta, \varphi) \in L^2(\mathbb{S}^2)$ be K orientations as defined in Eq. (6) and define $F(\theta, \varphi)$ such that

$$\frac{\partial^n}{\partial \varphi^n} F(\theta, \varphi) = f(\theta, \varphi).$$

Let $h(\mathbf{G}) \in L^2(\mathbb{SO}(3))$ be an ideal sampling kernel with a bandlimit $L = 2K + 1$. Then the equiangular samples $s_n = (F * h)(\theta_n, \varphi_n)$ with $n = 1, \dots, 8K^2$ are sufficient to recover $F(\theta, \varphi)$ exactly.

While Corollary 2 considers equiangular samples, uniformly distributed samples could also be used as discussed in the previous sections. The reconstruction process for integrated orientations with $K = 2$ is illustrated in Fig. 4.

B. Antipodal symmetry

In the previous sections, our definitions of the sampling kernels implied they did not exhibit antipodal symmetry. When a function is symmetric, its rotational harmonics are 0 for odd degrees ℓ . There are many situations where the kernel is dictated by the process and is symmetric, see for example [1]. Although the orientations cannot be retrieved, we would still like to recover the directions² using a minimal number of samples.

Consider the kernel whose rotational harmonics are

$$\hat{h}_\ell^{mn} = \begin{cases} c_{\ell m} & \ell \text{ even}, \ell < L, \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

where $c_{\ell m} \neq 0$ and $L > 0$. Assuming without loss of generality that $c_{\ell m} = 1$ and inserting this kernel into Eq. (13), then Eq. (12) becomes

$$s(\mathbf{u}) = \sum_{\ell=0}^{L-1} \sum_{m=-2\ell}^{2\ell} \hat{f}_{2\ell}^m Y_{2\ell}^m(\mathbf{u})$$

which is also symmetric. We observe that this signal could also have been obtained by assuming symmetric orientations, that is

$$\hat{f}_\ell^m = \begin{cases} N_\ell^m \sum_{k=1}^K a_k P_\ell^m(\cos \theta_k) e^{-jm\varphi_k} & \ell \text{ even} \\ 0 & \ell \text{ odd} \end{cases}.$$

This definition of the spherical harmonics leads us to the normalized samples

$$z_\ell = \begin{cases} \sum_{k=1}^K a_k r_k^\ell & \ell \text{ even} \\ 0 & \ell \text{ odd.} \end{cases}$$

Now consider the annihilating filter $\mathbf{h} = [h_0 \dots h_{2K}]^t$ whose z -transform is

$$H(z) = \prod_{k=1}^K (1 - r_k z^{-1})(1 + r_k z^{-1}) = \sum_{i=0}^K h_{2i} z^{-2i}.$$

²To clarify, we consider that two antipodal points on the sphere have the same direction, but different orientations.

Notice that this filter has the coefficient $h_i = 0$ for odd i . The convolution of $\mathbf{h}$ with $\mathbf{z}$ is

$$\begin{aligned} (\mathbf{h} * \mathbf{z})_n &= \sum_{i=0}^K h_{2i} z_{n-2i} \\ &= \sum_{i=0}^K h_{2i} \left(\sum_{k=1}^K a_k r_k^{n-2i} \right) \\ &= \sum_{k=1}^K a_k r_k^n \sum_{i=0}^K h_{2i} r_k^{-2i} = 0 \end{aligned}$$

where the equality to zero is again obtained because the right hand sum is the filter being evaluated at one of its roots. Using $n = 2K, \dots, 4K - 2$ which requires z_ℓ for $\ell = 0, \dots, 4K - 2$ we can build a linear system as in Eq. (11) to recover the annihilating filter coefficients. Once again, the roots of $\mathbf{h}$ uniquely define the r_k 's and an additional set of roots which correspond to the $-r_k$'s. Because we assume antipodal symmetry of f we can discard all $-r_k$'s. To recover the a_k 's and φ_k 's, we proceed as described in Section III-A.

The procedure described above requires the spherical harmonics $\hat{f}_\ell^\ell$ with $\ell = 0, 2, \dots, 4K - 2$ and $\hat{f}_\ell^{\ell-1}$ with $\ell = 2, 4, \dots, 2K$. This means that the bandwidth of the kernel must be adjusted to satisfy $L \geq 4K - 1$. Finally the number of samples required is $(4K - 2)(8K - 3) + 1$ and at least $(4K - 1)^2$ using equiangular and uniform sampling, respectively.

V. NUMERICAL SIMULATIONS

In this section, we validate the algorithms presented in the previous sections through numerical simulations. We evaluate the performance of our algorithm using the angular error defined as

$$\alpha = \frac{1}{K} \sum_{k=1}^K \cos^{-1}(\mathbf{u}_k^t \mathbf{u}'_k)$$

where $\mathbf{u}_k = [\cos \varphi_k \sin \theta_k \quad \sin \varphi_k \sin \theta_k \quad \cos \theta_k]^t$ is the k^{th} orientation and $\mathbf{u}'_k$ is the recovered k^{th} orientation.

Artificial signals were created by randomly selecting the parameters $a_k \in [-1, 1]$, $\theta_k \in [0, \pi]$, and $\varphi_k \in [0, 2\pi]$. The spherical harmonic coefficients were computed using Eq. (8) and the samples were generated with Eq. (12).

Fig. 5 illustrates the angular error obtained when using samples distributed with the electrostatic repulsion model and an ideal bandlimited sampling kernel. The angular error is plotted against the number of measurements N for values of K ranging from 1 to 5. We observe that the error falls below 10^{-3} as soon as the number of measurements satisfies $N \geq 4K^2$ which is consistent with the observations in Section III-C. The spikes in angular error observed when $N \geq 4K^2$ are caused by orientations that are randomly generated very close to each other (< 1 degree). When this occurs, the annihilating filter has near identical roots which causes numerical problems when recovering the signal parameters. However, in all cases observed, the resulting angular error never exceeded 0.01 degree.

Fig. 6 illustrates the angular error obtained using the optimal filter and samples distributed with the electrostatic repulsion

Fig. 4. Illustration of the reconstruction process using the ideal filter for $K = 2$ integrated orientations. The original orientation in (a) are integrated to produce the signal of interest in (b). The signal of interest is then filtered to produce the observed signal presented in (c). By multiplying the spherical harmonics in (c) by j_m , we obtain in (d) a bandlimited version of the spherical harmonic of the orientations.

Fig. 5. Relation of the angular error α as a function of the number of measurements N obtained in numerical simulations using an ideal bandlimited kernel. For all value of K illustrated, the sharp drop in the angular error occurs at $4K^2$.

model. The angular error is plotted against the number of measurements N for values of K ranging from 1 to 5. As suggested in Section III-C, the error drops below 10^{-6} when $N \geq 3K$.

To evaluate the resilience of our method to noise, we performed additional simulations using noisy samples. The artificial signals were generated as described earlier, but complex white Gaussian noise was added to the samples. The standard deviation of the noise was selected to achieve signal to noise ratios (SNR) of 10, 20, and 30 dB. The mean angular error as a function of the number of samples used is illustrated in Fig. 7. Each point is the average computed over 100 simulations. For all SNRs, a sharp decrease at $N = 16$ is observed which corresponds to the expected minimum number of samples described in Section III-C. As the number of samples is increased, the angular error is reduced to approximately 5,

10 and 15 degrees for SNRs of 30, 20, and 10dB respectively. These results should be understood as worst case scenarios, since no special steps were taken to mitigate the effects of the additive noise. In practical situations, the results could probably be improved by regularizing the computation of the spherical harmonics of $s(\mathbf{u})$ and using denoising methods used in FRI [19], [29], [30]. However, the choice of the method will be influenced by the number of samples available, the expected SNR, the type of kernel and the number of orientations. Selecting the optimal approach is out of the scope of this paper.

VI. CONCLUSION

In this work, we have developed and proved several new sampling theorems for non-bandlimited functions on the sphere. In the case of a kernel with a bandlimit $L = 2K$

Fig. 6. Relation of the angular error α as a function of the number of measurements N obtained in numerical simulations using an ideal bandlimited kernel. For all value of K illustrated, the sharp drop in the angular error occurs at $3K$.

Fig. 7. (a) Relation of the angular error α as a function of the number of measurements N and the SNR, obtained in numerical simulations using an ideal bandlimited kernel and $K = 2$. (b) Example of orientations recovered for $N = 64$ and $\text{SNR} = 20\text{dB}$. Each dot is the result of a simulation.

and an equiangular sampling scheme, we have proved that $(2K - 1)(4K - 1) + 1$ samples are sufficient to exactly recover K orientations. While this sampling scheme is not optimal in terms of number of samples, it allows the use of the sampling theorem developed by McEwen *et al.* which can handle very large bandlimits. When the sampling kernel has a bandlimit $L = 2K$ and the samples are uniformly distributed on the sphere, we argue that the number of samples necessary

to accurately find the signal parameters can be reduced to $4K^2$. In this case our argument hinges on the proper conditioning of the spherical harmonic matrix $\mathbf{Y}$. The numerical experiments performed in Section V emphasize that this condition is easy to satisfy, for example using samples distributed with the electrostatic repulsion model. Because our goal is to reduce the number of samples required for exact recovery, we also developed an optimal sampling kernel. The optimal kernel

allows the accurate recovery of K orientations using just $3K$ samples assuming full column rank of the spherical harmonic matrix.

Ideally, we would like sampling theorems to be applicable to the broadest signal class as possible. To enhance our results, we showed that our method can also be applied for the reconstruction of orientation integrated along φ , the azimuth. Due to the structure of the spherical harmonics however, our results do not apply to orientation integrated along θ , the colatitude. Generalizing our results to a broader class of signal is the subject of our current research.

REFERENCES

- [1] J. Tournier, F. Calamante, D. Gadian, and A. Connelly, "Direct estimation of the fiber orientation density function from diffusion-weighted mri data using spherical deconvolution," *NeuroImage*, pp. 1176–1185, 2004.
- [2] C. Hess, P. Mukherjee, E. Han, D. Xu, and D. Vigneron, "Q-ball reconstruction of multimodal fiber orientations using the spherical harmonic basis," *Magnetic Resonance in Medicine*, pp. 104–117, 2006.
- [3] M. Descoteaux, E. Angelino, S. Fitzgibbons, and D. Deriche, "Regularized, fast, and robust analytical q-ball imaging," *Magnetic Resonance in Medicine*, pp. 497–510, 2007.
- [4] J. Tournier, F. Calamante, and A. Connelly, "Robust determination of the fibre orientation distribution in diffusion mri: Non-negativity constrained super-resolved spherical deconvolution," *NeuroImage*, pp. 1459–1472, 2007.
- [5] J. Kronander, D. Jonsson, J. Low, P. Ljung, A. Ynnerman, and J. Unger, "Efficient visibility encoding for dynamic illumination in direct volume rendering," *IEEE Transactions on Visualization and Computer Graphics*, pp. 447–462, 2012.
- [6] T. Abhayapala and A. Gupta, "Spherical harmonic analysis of wavefields using multiple circular sensor arrays," *IEEE Transactions on Audio, Speech, and Language Processing*, pp. 1655–1666, 2010.
- [7] S. Argentieri, P. Danes, and P. Soueres, "Modal analysis based beamforming for nearfield or farfield speaker localization in robotics," in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2006, pp. 866–871.
- [8] J. Driscoll and D. M. Healy, "Computing fourier transforms and convolutions on the 2-sphere," *Advances in Applied Mathematics*, pp. 202–250, 1994.
- [9] D. Healy, D. Rockmore, P. Kostelec, and S. Moore, "FFTs for the 2-sphere - improvements and variations," *The Journal of Fourier Analysis and Applications*, pp. 341–385, 2003.
- [10] J. McEwen and Y. Wiaux, "A novel sampling theorem on the sphere," *IEEE Transactions on Signal Processing*, vol. 59, pp. 5876–5887, 2011.
- [11] D. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, pp. 1289–1306, 2006.
- [12] E. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, pp. 489–509, 2006.
- [13] J. McEwen, G. Puy, J. P. Thiran, P. Vandergheynst, D. V. D. Ville, and Y. Wiaux, "Sampling theorems and compressive sensing on the sphere," in *Wavelets and Sparsity XIV*, 2011.
- [14] J. McEwen, G. Puy, J. Thiran, P. Vandergheynst, D. V. D. Ville, and Y. Wiaux, "Sparse signal reconstruction on the sphere: implications of a new sampling theorem," *CoRR*, vol. abs/1205.1013, 2012.
- [15] O. Michailovich, Y. Rath, and S. Dolui, "On approximation of orientation distributions by means of spherical ridgelets," *IEEE Transactions on Image Processing*, pp. 461–477, 2010.
- [16] —, "Spatially regularized compressed sensing for high angular resolution diffusion imaging," *IEEE Transactions on Medical Imaging*, pp. 1100–1115, 2011.
- [17] M. Unser, "Sampling - 50 years after Shannon," *Proceedings of the IEEE*, pp. 569–587, 2000.
- [18] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Transactions on Signal Processing*, vol. 50, pp. 1417–1428, 2002.
- [19] T. Blu, P. L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot, "Sparse sampling of signal innovations," *IEEE Signal Processing Magazine*, vol. 25, pp. 31–40, 2008.
- [20] P. Dragotti, M. Vetterli, and T. Blu, "Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets strang-fix," *IEEE Transactions on Signal Processing*, pp. 1741–1757, 2007.
- [21] L. Baboulaz and P. Dragotti, "Exact feature extraction using finite rate of innovation principles with an application to image super-resolution," *IEEE Transactions on Image Processing*, pp. 281–298, 2009.
- [22] C. Chen, P. Marziliano, and A. Kot, "2D finite rate of innovation reconstruction method for step edge and polygon signals in the presence of noise," *IEEE Transactions on Signal Processing*, pp. 2851–2859, 2012.
- [23] I. Maravic and M. Vetterli, "Exact sampling results for some classes of parametric nonbandlimited 2d signals," *IEEE Transactions on Signal Processing*, vol. 52, no. 1, pp. 175–189, 2004.
- [24] P. Shukla and P. L. Dragotti, "Sampling schemes for multidimensional signals with finite rate of innovation," *IEEE Transactions on Signal Processing*, vol. 55, no. 7, pp. 3670–3686, 2007.
- [25] S. Deslauriers-Gauthier and P. Marziliano, "Spherical finite rate of innovation theory for the recovery of fiber orientations," in *Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, 2012.
- [26] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum theory of angular momentum*. World Scientific Publishing, 1988.
- [27] D. M. Brink and G. R. Satchler, *Angular momentum*. Oxford University Press, 1993.
- [28] D. K. Jones, M. A. Horsfield, and A. Simmons, "Optimal strategies for measuring diffusion in anisotropic systems by magnetic resonance imaging," *Magnetic resonance in medicine*, vol. 42, pp. 515–525, 1999.
- [29] P. L. Dragotti and F. Homann, "Sampling signals with finite rate of innovation in the presence of noise," in *IEEE International Conference on Acoustics, Speech and Signal Processing*, 2009.
- [30] S. Deslauriers-Gauthier and P. Marziliano, "Noisy finite rate of innovation beyond cadzow," *Sampling Theory in Signal and Image Processing Special Issue*, in press, 2012.