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Distributed Average Consensus based on Structural Weight-Balanceability

R. Haghighi and C. C. Cheah

Abstract

This paper presents the concept of structural weight-balanceability of network systems as a useful method to examine the average consensus of the systems. We introduce the notion of structurally unobservable node based on structural observability concept to determine the structural weight-balanceability of a network. We propose a simple technique to identify structurally unobservable nodes, and consequently infer on structural weight-balanceability of the network. A mathematical method based on principal minors of the associated Laplacian matrix of the network is developed to modify a structurally weight-balanceable network into a weight-balanced one. A distributed method is presented to compute normalized principal minors in a large-scale network system with directed communication topology. This is the first work that provides a rigorous mathematical formulation to check the structural weight-balanceability of network systems and make a structurally weight-balanceable network, weight-balanced. Finally, simulation results are presented to illustrate the performance of the proposed method in dealing with networks with a large number of nodes.

I. INTRODUCTION

Rapid growing of network technologies has led to a great demand for development of networked control systems [1]-[6], thus opening up new opportunities and challenges in understanding such control systems. Information consensus is one of the important problems in networked control systems that has wide applications [7]. Consensus and agreement problems have been studied in computer science [8], and received increasing attentions in control community in the past years [9]. The consensus problem for a network of agents with undirected communication topology was addressed in [10],[11]. Later, Moreau [12], Ren and Beard [13] extended the

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results to a network of agents with directed communication topology. In a recent work, Ma and Zhang [14] introduced the notion of consensusability, and presented the necessary and sufficient conditions for consensusability of multi-agent systems.

A special class of the consensus problem is called distributed average consensus, where the consensus value is the average of the initial values [15]. Average consensus is achievable for networks with undirected information flow if and only if the networks are connected. Olfati-Saber and Murray [11] studied average consensus problem for networks with directed graph, and showed that the weight-balanced and strong connectivity are necessary and sufficient conditions for reaching the average consensus. Hence, the weight-balanced condition is a crucial factor in reaching average consensus. Networks with weight-balanced communication digraphs have been frequently used in average consensus problem [16]-[20]. Cortes [21] proposed a distributed consensus algorithm for weight-balanced networks. Recently, Ghareisifard and Cortes [22] developed an algorithm for generating weight-balanced digraphs, however, the proposed algorithm is based on the assumption that each node is aware of its outgoing neighbors and is able to adjust the weights on the outgoing edges which requires a feasible method for identification of the outgoing neighbors and the computation of the out-degree. Although considerable research has been devoted to the distributed consensus problem, a rigorous mathematical formulation on identifying average consensusable and weight-balancing of networks has not been presented. This problem emerges as a crucially complicated issue when dealing with networks with large number of nodes.

Controllability and observability represent fundamental concepts of dynamic systems. However, uncertainty in the parameters of a system, may lead to uncertainty in identifying the controllability and observability of the system. Nevertheless, controllability and observability are inherent properties of the systems. Lin [23] examined the controllability of the linear systems using graph-theoretic techniques and introduced the concept of structural controllability, which was further developed in [24]. Similar conditions have been addressed for the observability of structured systems, by means of duality theorem [25]. In recent years, the controllability and observability problems of network systems have received considerable attention [26]-[28]. Rahmani et al. [29] proposed the controllability of multi-agent systems with multiple leaders.

Parlangeli and Notarstefano [30] provided necessary and sufficient conditions for the observability of path and cycle graphs. An application of network observability in intelligent transportation systems was examined by Castillo et al. [31]. However, as networks become more complex, the above mentioned methods do not provide satisfactory performance due to computational issues. To alleviate the problem, considerable investigations have focused on the understanding of the characteristics of complex networks [32] recently. A notable work in this area is carried out by Liu et al. [33] which addressed the controllability of complex networks.

In this paper, we present the concept of structural weight-balanceability to examine the average consensus in network systems. Here, the structural weight-balanceability refers to the ability of making a network, weight-balanced.¹ We introduce the notion of structurally unobservable node based on the structural observability concept. We present a simple tool using principal minors of the associated Laplacian matrix of the network to identify the structurally unobservable nodes in networks even with a large number of nodes. The relationship between the existence of the structurally unobservable nodes and the structural weight-balanceability of networks is also explored. A simple mathematical method is proposed to make a structurally weight-balanceable network, weight-balanced. In addition, a distributed method is provided to compute the normalized principal minors of the Laplacian matrix in large-scale networks. To the best of our knowledge, this is the first work that provides a rigorous mathematical formulation to check the structural weight-balanceability of networks and make a structurally weight-balanceable network, weight-balanced. The proposed methods are useful in examining the average consensusability, and specifically identifying the globally reachable nodes which has not been addressed previously. The contribution of the paper is threefold. First, the concept of structural weight-balanceability is presented as a useful framework to investigate average consensus in network systems. In the previous results on distributed average consensus problem, the mathematical examination of the strong connectivity in a network has not been presented, therefore it is difficult to

¹It should be pointed out that, the notion of structural balance has been introduced in social networks [34] with a definition that suits signed networks, and it is based on the signs of interconnecting links without considering the links weights. Here, we consider the weight-balancing problem in unsigned networks. We introduce the notion of structural weight-balance for unsigned networks as the existence of a set of weights that make a network weight-balanced. Consequently, we propose a mathematical method to distributively check the structural weight-balanceability.

examine structural weight-balanceability for networks with large number of nodes. We propose a simple technique to determine structural weight-balanceability in network systems. Second, a mathematical method based on principal minors of the Laplacian matrix is presented to make the structurally weight-balanceable network, weight-balanced. Third, a distributed method is proposed to compute normalized principal minors of the Laplacian matrix associated with a large-scale network system with directed communication topology. The result thus also allows the identification of globally reachable node(s) in a distributed way.

II. PRELIMINARIES

The communication between nodes can be expressed by a weighted directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, such that $\mathcal{V} = \{v_1, \dots, v_N\}$ represents the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrix where $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. A graph $\mathcal{H}$ is said to be a subgraph of a graph $\mathcal{G}$ if $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{G})$. A directed spanning tree or arborescence is a digraph such that there is a unique directed path from a designated root node to every other node. The (in-degree) graph Laplacian matrix of a digraph $\mathcal{G}$, denoted by $\mathcal{L}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}\{a_{11}, \dots, a_{NN}\}$ is the degree matrix such that $a_{ii} = - \sum_{(j=1) \& (j \neq i)}^N a_{ij}$. A graph Laplacian matrix $\mathcal{L}$ is called weight-balanced or balanced Laplacian matrix if $\sum_{\substack{j=1 \\ j \neq i}}^N a_{ji} = \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}$ for $i = 1, 2, \dots, N$. A digraph $\mathcal{G}$ is called an arborescence diverging from node u , if there is only one directed path between root u and any other node of $\mathcal{G}$. If there is an arborescence sub-digraph diverging from an arbitrary node u , then u is called a globally reachable node.

Lemma 1. *Let $\mathcal{G}'$ be a sub-digraph of a digraph $\mathcal{G}$ which contains an arborescence, the associate Laplacian matrix $\mathcal{L}'$ of $\mathcal{G}'$ has rank $N - 1$.*

Definition 1. *A matrix A_s is said to be a structured matrix if its elements are either fixed zeros or independent free parameters [35]. Two matrices are structurally equivalent if their zero entries coincide.*

Definition 2. *The structural rank (srnk) of a matrix is the maximum rank of all structurally equivalent matrices [35].*

III. NETWORK OBSERVABILITY

According to classical control theory, a dynamical system is observable if the knowledge of the input and output over a finite time is sufficient to determine the state of the system at any time. Even though a system with a pair of (A, C) might be unobservable, it can be observable for another pair of (A_1, C_1) with the same structure [23]. Consequently, a system is said to be *structurally observable* if there exists at least one admissible realization which is observable [37]. As stated in the classical observability theories, a system is unobservable if it has unobservable state, therefore a structured system is *structurally unobservable* if there exists a state x^* which is unobservable for any values of the system parameters. In the following, we examine the *structural observability* of network systems, and we express the condition that leads to *structural unobservability* of network systems.

Consider a network of nodes that consensus protocol is applied. The closed loop system can be expressed as follows:

$$\begin{cases} \dot{x} = -\mathcal{L}x \\ y = C_i x \end{cases}, \quad (1)$$

where $x = [x_1, x_2, \dots, x_N]^T$, $C_i = \begin{bmatrix} 0, 0, \dots, 0, 1, 0, \dots, 0 \end{bmatrix}$, and y represents the output node. To examine the observability, we form the following zero-input response:

$$y_z = C_i e^{-\mathcal{L}t} x_0, \quad (2)$$

Therefore, the term $C_i e^{-\mathcal{L}t}$ refers to the i^{th} row of the matrix $e^{-\mathcal{L}t}$. Using Cayley-Hamilton theorem [38], $e^{-\mathcal{L}t}$ can be expanded as follows:

$$e^{-\mathcal{L}t} = \sum_{i=0}^{N-1} \alpha_i(t) \mathcal{L}^i, \quad (3)$$

where $\alpha_i(t)$ are scalar functions of time. According to lemma A1 in the Appendix, the $(i, j)^{th}$ element of the matrix series (3) is zero, if there is no path from node j to node i . In this case, the j^{th} element of vector $C_i e^{-\mathcal{L}t}$ is zero for all values of the link weights. Therefore,

$x^* = \begin{bmatrix} 0, 0, \dots, 0, \alpha, 0, \dots, 0 \end{bmatrix}^T$ is an unobservable state of the system. Consequently, We can state the following lemma:

Lemma 2. Consider the following structured network system:

$$\begin{cases} \dot{x} = -\mathcal{L}_s x \\ y = C_i x \end{cases}, \quad (4)$$

where $\mathcal{L}_s$ is the structured Laplacian matrix (i.e. the weighted Laplacian matrix in which weights can be freely chosen). If there is no path from node j to the output node i , then

$x^* = \begin{bmatrix} 0, 0, \dots, 0, \underbrace{\alpha}_{j^{\text{th}} \text{ element}}, 0, \dots, 0 \end{bmatrix}^T$ is an unobservable state of the system, and hence the network system is structurally unobservable.

We have shown that the existence of a path from all nodes to the output node is a necessary condition for the *structural observability*. In the following, we define the *structurally unobservable node* based on the above results on the existence of the unobservable state in network systems.

Definition 3. Consider the network system (1) with any arbitrary output node $i = 1, 2, \dots, N$. Node j is said to be a *structurally unobservable node* of the network, if there is no path from node j to at least one arbitrary output node.

Hence, one can deduce that there is no arborescence sub-digraph diverging from a *structurally unobservable node*, and consequently it is not a globally reachable node. In network systems, identifying *structurally unobservable nodes* are useful, since it provides the knowledge about the information flow in the network. Moreover, it shows which nodes cannot be assigned as the leader node. Therefore, it is desirable to find a simple and easy way to identify *structurally unobservable nodes*. In the following, we show how principal minors of the Laplacian matrix can be used to identify the *structurally unobservable nodes*.

Lemma 3. Consider a network system with the associate digraph $\mathcal{G}$ and the corresponding Laplacian matrix $\mathcal{L}$. The principal minor of $\mathcal{L}$ corresponding to an arbitrary node i is zero if and only if there is no arborescence sub-digraph diverging from node i .

Proof: Necessary condition: we assume that there is a sub-digraph which is an arborescence diverging from node i (without loss of generality we assume $i = 1$), then we show that the principal minor corresponding to node i is not zero. Consider the sub-digraph $\mathcal{G}'$ which is

Fig. 1: An example of the partitioning $\mathcal{G}$ into two subdigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$

obtained by removing the incoming links to node i . Since removing the incoming links to node i has no effect on the arborescence sub-digraph diverging from node i , hence $\mathcal{G}'$ contains an arborescence. Using lemma 1, the associate Laplacian matrix $\mathcal{L}'$ of $\mathcal{G}'$ has rank $N - 1$. Since elementary row/column operations do not change the rank of a matrix, we add columns 2,...,N to the column 1, therefore we obtain

$$\text{rank}(\mathcal{L}'(\mathcal{G})) = \text{rank} \left(\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{L}(\mathcal{G}/1) & \\ 0 & & & \end{bmatrix} \right) = N - 1. \quad (5)$$

The matrix $\mathcal{L}'(\mathcal{G})$ has rank $N - 1$ while first row and column are dependent, hence we can conclude that the matrix $\mathcal{L}(\mathcal{G}/1)$ is full rank. Moreover, since $\mathcal{L}(\mathcal{G}/1)$ is a diagonally dominant matrix with positive diagonal entries, therefore we obtain $\det(\mathcal{L}(\mathcal{G}/1)) > 0$.

Sufficient condition: Let $\mathcal{G}$ be a digraph which has no arborescence sub-digraph diverging from node i . Therefore $\mathcal{G}$ can be decomposed into two subdigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ as illustrated in figure 1, where $\mathcal{G}_1$ contains r nodes which includes node i plus those nodes that have access to information from node i , and $\mathcal{G}_2$ contains $N - r$ nodes which have no access to the information from node i . By renumbering of the nodes of $\mathcal{G}$, the Laplacian matrix of the directed graph $\mathcal{G}$ can be expressed in the following decomposed form:

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_{12} \\ \mathbf{0} & \mathcal{L}_2 \end{bmatrix}, \quad (6)$$

where $\mathcal{L}_1$ is the $r \times r$ matrix associated to the sub-digraph $\mathcal{G}_1$, $\mathcal{L}_2$ is the $(N - r) \times (N - r)$ matrix associated to the sub-digraph $\mathcal{G}_2$, $\mathcal{L}_{12}$ is the $r \times (N - r)$ matrix associated to the information

transfer from $\mathcal{G}_2$ to $\mathcal{G}_1$, and $\mathbf{0}$ is $(N-r) \times r$ zero matrix. Since information is only transferred from $\mathcal{G}_2$ to $\mathcal{G}_1$, and $\mathcal{L}$ has zero row sums, $\mathcal{L}_2$ has also zero row sums. The principal minor corresponding to node i can be expressed as follows:

$$\det(\mathcal{L}(\mathcal{G}/1)) = \begin{vmatrix} \mathcal{L}'_1 & \mathcal{L}'_{12} \\ \mathbf{0}' & \mathcal{L}_2 \end{vmatrix}, \quad (7)$$

where $\mathcal{L}'_1$, $\mathcal{L}'_2$, $\mathcal{L}'_{12}$ and $\mathbf{0}'$ are obtained by removing the row and column corresponding to node i from $\mathcal{L}$. Using the determinant of the partitioned matrices we obtain

$$\det(\mathcal{L}(\mathcal{G}/1)) = \det(\mathcal{L}'_1) \cdot \det(\mathcal{L}_2). \quad (8)$$

Since matrix with zero row sums has zero determinant, therefore, $\det(\mathcal{L}_2) = 0$ which lead to $\det(\mathcal{L}(\mathcal{G}/1)) = 0$. ■

A direct conclusion of the lemma 3 is expressed in the following corollary:

Corollary 1. *A node is globally reachable node, if and only if the corresponding principal minor of the Laplacian matrix is greater than zero.*

Theorem 1. *Consider a network with corresponding directed graph $\mathcal{G}$. Node i is a structurally unobservable node if and only if, the i^{th} principal minor of the Laplacian matrix is zero.*

Proof: The proof can be obtained by using definition 3 and lemma 3. ■

After presenting the method for identifying *structurally unobservable nodes* of a network system, we show the relationship between the average consensus problem and the existence of *structurally unobservable nodes* in the next section.

IV. DISTRIBUTED AVERAGE CONSENSUS AND STRUCTURAL WEIGHT-BALANCEABILITY

In many applications such as distributed computation [39], distributed filtering [40],[41], and distributed sensor fusion [42], it is desirable to reach the average consensus. The distributed average consensus is achieved in a network, if $\lim_{t \rightarrow \infty} (x_i - \frac{1}{N} \sum_{j=1}^N x_j(0)) = 0$, for every $i \in \{1, \dots, N\}$, where $x_j(0)$ refer to initial value of state variables. For undirected graphs, existence of spanning tree guarantees that the network can reach average consensus; however, for directed graph existence of directed spanning tree is a necessary but not sufficient condition. It has been shown [43], that the weight-balanced and strong connectivity are necessary and sufficient conditions

for reaching the average consensus in networks with directed topology. Now, we present the following definition:

Definition 4. A Laplacian matrix, $\mathcal{L}$, is said to be *structurally weight-balanceable* if there exists a weight-balanced Laplacian matrix $\mathcal{L}^*$ with the same structure as $\mathcal{L}$. Consequently, a network with structurally weight-balanceable Laplacian matrix is called *structurally weight-balanceable network*.

Remark 1. In sociology, the notion of structural balance has been introduced for social signed networks as the absence of cycles with an odd number of negative links [44],[45]. Here, we consider the weight-balancing problem in unsigned networks, and the notion of structural weight-balance is introduced for unsigned networks as the existence of a set of weights that make a network weight-balanced.

It can be said that any arbitrary network is able to reach average consensus by modifying links' weights if and only if it is structurally weight-balanceable. Though it has been shown that the distributed average consensus is achievable in weight-balanced networks, a simple technique to identify a structurally weight-balanceable network and make it weight-balance, has not been presented. In this section, we show how the existence of *structurally unobservable nodes* leads to loss of structural weight-balanceability. In addition, we provide a rigorous mathematical formulation to check the structural weight-balanceability of large scale networks and modify a structurally weight-balanceable network into a weight-balanced one. The following theorem presents the relationship between structural weight-balanceability and the results on the *structural observability* of network systems.

Theorem 2. Consider a network with corresponding directed graph $\mathcal{G}$. The network is *structurally weight-balanceable* or equivalently *average consensusable*, if and only if it has no *structurally unobservable nodes*. Consequently, the network can be weight-balanced by multiplying each row of the Laplacian matrix of $\mathcal{G}$ by the corresponding principal minor.

Proof: To prove the necessary condition, it must be shown that if the network has no *structurally unobservable nodes*, then it is average consensusable. Using theorem 1, since the network has no *structurally unobservable nodes*, therefore all the principal minors are greater

than zero. To show the average consensusability of the network, we first show that the network is structurally weight-balanceable. To do so, we start to calculate the determinant of the Laplacian matrix by expanding along the first column as follows:

$$\det(\mathcal{L}) = a_{11} \begin{vmatrix} a_{22} & \cdots & -a_{2N} \\ \vdots & \ddots & \vdots \\ -a_{N2} & \cdots & a_{NN} \end{vmatrix} + \cdots + (-1)^{N+1} (-a_{N1}) \begin{vmatrix} -a_{12} & \cdots & -a_{1N} \\ \vdots & \ddots & \vdots \\ -a_{(N-1)2} & \cdots & -a_{(N-1)N} \end{vmatrix}. \quad (9)$$

Since, the sum over each row is zero, therefore by elementary operations, the second cofactor can be rewritten as follows:

$$\begin{aligned} \begin{vmatrix} -a_{12} & -a_{13} & \cdots & -a_{1N} \\ -a_{32} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N2} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} &= \begin{vmatrix} -\sum_{j=2}^N a_{1j} & -a_{13} & \cdots & -a_{1N} \\ a_{33} - \sum_{(j=2) \& (j \neq 3)}^N a_{3j} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{NN} - \sum_{(j=2) \& (j \neq N)}^N a_{Nj} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} \\ &= - \begin{vmatrix} a_{11} & -a_{13} & \cdots & -a_{1N} \\ -a_{31} & a_{33} & \cdots & -a_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N3} & \cdots & a_{NN} \end{vmatrix} = -\mathcal{M}_{22}, \end{aligned} \quad (10)$$

where $\mathcal{M}_{22}$ represents the second principal minor. Similarly, all other cofactors in (9) can be expressed as the corresponding principal minors. Hence, the determinant of the Laplacian matrix can be represented as the following linear combination of the principal minors:

$$\det(\mathcal{L}) = a_{11}\mathcal{M}_{11} - a_{21}\mathcal{M}_{22} - \cdots - a_{N1}\mathcal{M}_{NN}, \quad (11)$$

where $\mathcal{M}_{ii}$ represents the i^{th} principal minor of $\mathcal{L}$. Similarly, the determinant of the Laplacian matrix can be expanded over other columns. Since the determinant of the Laplacian matrix is zero, the following set of linear combinations of the principal minors can be obtained:

$$\begin{bmatrix} a_{11} & -a_{21} & \cdots & -a_{N1} \\ -a_{12} & a_{22} & \cdots & -a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1N} & -a_{2N} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \vdots \\ \mathcal{M}_{NN} \end{bmatrix} = 0. \quad (12)$$

Since all the principal minors are greater than zero, equation (12) can be expressed as follows:

$$\begin{bmatrix} \mathcal{M}_{11}a_{11} & -\mathcal{M}_{22}a_{21} & \cdots & -\mathcal{M}_{NN}a_{N1} \\ -\mathcal{M}_{11}a_{12} & \mathcal{M}_{22}a_{22} & \cdots & -\mathcal{M}_{NN}a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathcal{M}_{11}a_{1N} & -\mathcal{M}_{22}a_{2N} & \cdots & \mathcal{M}_{NN}a_{NN} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0, \quad (13)$$

which implies that, if all the principal minors are greater than zero, then by multiplying each row of the Laplacian matrix by the corresponding principal minor, a weight-balanced Laplacian matrix can be achieved, i.e. the following Laplacian matrix satisfies the weight-balanced condition:

$$\mathcal{L}' = \begin{bmatrix} a'_{11} & -a'_{12} & \cdots & -a'_{1N} \\ -a'_{21} & a'_{22} & \cdots & -a'_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a'_{N1} & -a'_{N2} & \cdots & a'_{NN} \end{bmatrix}, \quad (14)$$

where $a'_{ij} = \mathcal{M}_{ii}a_{ij}$, and $\sum_{(j=1) \& (j \neq i)}^N a'_{ij} = \sum_{(j=1) \& (j \neq i)}^N a'_{ji}$. Since the network is both consensusable and weight-balanceable, then it is average consensusable.

To prove the sufficient condition, we must show an average consensusable network has no *structurally unobservable nodes*. From theorem 1, we know that a node is structurally unobservable if and only if the corresponding principal minor of the Laplacian matrix is equal to zero. We proceed to show that in network systems all the principal minors of the Laplacian matrix associated with an average consensusable network are nonzero. Since the network is average consensusable, without loss of generality we assume that the Laplacian matrix $\mathcal{L}$ associated with the network is weight-balanced. Multiplying the Laplacian matrix with nonsingular matrices, we obtain

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & \mathbf{I}_{N-1} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & -a_{1N} \\ \vdots & \ddots & \vdots \\ -a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & \mathbf{I}_{N-1} & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{N2} & \cdots & a_{NN} \end{bmatrix}, \quad (15)$$

where $\mathbf{I}_{N-1}$ is an identity matrix with dimension $N-1$. Since the network is average consensusable, the rank of the Laplacian matrix is $N-1$ (see lemma 1). Since the rank of a matrix is invariant under multiplication by a non-singular matrix, the resultant matrix of (15) has rank $N-1$. The first row and column are dependent, hence rows 2 to N and columns 2 to N are

independent. Therefore, $\mathcal{M}_{11}$ which is the first principal minor, is nonzero. Similarly, we can show that the other principal minors are nonzero. ■

The following result follows directly from equations (12) and (13):

Corollary 2. *Distributed average consensus is achievable for a network of nodes, if and only if, the corresponding Laplacian matrix has no zero principal minors.*

In theorem 2, it is shown that the principal minors of the Laplacian matrix can be used to examine the structural weight-balanceability of a network and making a structurally weight-balanceable network into a weight-balanced one. However, the computation of principal minors requires the knowledge of the entire Laplacian matrix. In the following, we present a method to compute the normalized principal minors based on the distributed consensus-like estimators.

A. Distributed computation of the normalized principal minors

Let $\bar{m} = \frac{m}{\|m\|_1}$ be the vector of the normalized principal minors where $m = [\mathcal{M}_1, \dots, \mathcal{M}_N]^T \in R^N$ is the vector of principal minors, and let the estimation of $\bar{m}$ by node i be represented by $\hat{\hat{m}}_i \in R^N$. To compute $\hat{\hat{m}}_i$, without requiring knowledge of the whole Laplacian matrix, we present the following theorem:

Theorem 3. *Consider a network system with the associate digraph $\mathcal{G}$ and the corresponding Laplacian matrix $\mathcal{L}$. The following consensus-like estimator is able to compute the normalized principal minors of the Laplacian matrix:*

$$\dot{\hat{\hat{m}}}_i = - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij} (\hat{\hat{m}}_i - \hat{\hat{m}}_j), \quad \hat{\hat{m}}_i(0) = \underbrace{[0, \dots, 0, 1, 0, \dots, 0]^T}_{\text{the } i^{\text{th}} \text{ element is } 1} \quad (16)$$

where a_{ij} is positive weight if the agent i can obtain information from the agent j , zero otherwise.

Proof: Equation (16) can be written as the following closed form:

$$\dot{\hat{\hat{M}}} = -\mathcal{L}\hat{\hat{M}}, \quad \hat{\hat{M}}(0) = I_N \quad (17)$$

where $\hat{\hat{M}} = [\hat{\hat{m}}_1, \dots, \hat{\hat{m}}_N]^T \in R^{N \times N}$ and I_N is the identity matrix. Multiply both sides of equation (17) by m^T , yields

$$m^T \dot{\hat{\hat{M}}} = -m^T \mathcal{L} \hat{\hat{M}} = 0 \quad (18)$$

Hence, $m^T \hat{M}(t) = m^T \hat{M}(0) = m^T$. Let $\tilde{m} = [\tilde{m}_1, \dots, \tilde{m}_N]^T$ be the consensus value of (16), i.e. $\hat{M}(\infty) = [\tilde{m}, \dots, \tilde{m}]^T$, therefore, we have

$$\begin{aligned}
 m^T \hat{M}(\infty) &= [\mathcal{M}_1, \dots, \mathcal{M}_N] \begin{bmatrix} \tilde{m}^T \\ \vdots \\ \tilde{m}^T \end{bmatrix} \\
 &= \begin{bmatrix} \tilde{m}_1 \sum_{i=1}^N \mathcal{M}_i & \tilde{m}_2 \sum_{i=1}^N \mathcal{M}_i & \dots & \tilde{m}_N \sum_{i=1}^N \mathcal{M}_i \end{bmatrix} \\
 &= \tilde{m}^T \sum_{i=1}^N \mathcal{M}_i
 \end{aligned} \tag{19}$$

Hence,

$$\begin{cases} m^T \hat{M}(\infty) = m^T \\ m^T \hat{M}(\infty) = \tilde{m}^T \sum_{i=1}^N \mathcal{M}_i \end{cases} \Rightarrow \tilde{m} = \frac{m}{\sum_{i=1}^N \mathcal{M}_i} = \bar{m} \tag{20}$$

■

Remark 2. The weight-balancing technique can be performed in the following ways:

- i) Adjusting the weights before applying the consensus protocol.
 - (a) Compute the normalized principal minors using the proposed estimator in theorem 3.
 - (b) Check whether the estimated principal minors are all nonzero, and if not the network is not average consensusable or structurally weight-balanceable.
 - (c) If the network is structurally weight-balanceable, obtain the weight-balanced network by using the proposed wight-balancing method in theorem 2, as follows:

$$\dot{x}_i = - \sum_{j=1}^N \tilde{m}_i a_{ij} (x_i - x_j) \tag{21}$$

where $\tilde{m}_i$ is the i^{th} value of the consensus value of (16).

- ii) During consensus process agents modify their weights as follows:

$$\begin{cases} \dot{\hat{m}}_i = -k_e \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij} (\hat{m}_i - \hat{m}_j), & \hat{m}_i(0) = \begin{bmatrix} \underbrace{0, \dots, 0, 1, 0, \dots, 0}_{\text{the } i^{th} \text{ element is } 1} \end{bmatrix}^T \\ \dot{x}_i = - \sum_{j=1}^N \hat{m}_i^i a_{ij} (x_i - x_j) \end{cases} \tag{22}$$

where k_e is a positive constant. The gain k_e can be chosen large enough to achieve fast convergence of $\hat{m}_i$.

V. SIMULATIONS

In this section, we present simulation results to illustrate the performance of the proposed methods for networks with large number of nodes. For the numerical calculations and simulations we used MATLAB software. We consider a network of 1000 nodes which are distributed randomly within a square region as shown in figure 2. The communication link is generated between neighboring nodes with the probability of 0.65. The algebraic connectivity of the graph $\lambda_2(\mathcal{L}) = 0.36$. The simulation is done for two cases: I. In the first case, the consensus protocol is applied for unbalance network, depicted in figure 3(a). The initial values are randomly selected from $[0,100]$. It is shown that since the network is not weight-balanced, the consensus value, which is 41.67, has considerable difference with average of initial values which is 51.10. II. In the second case, we implement the proposed estimator expressed by equation (16) by using the Runge-Kutta method to obtain the normalized principal minors. Since all the principal minors are positive, we conclude the network is structurally weight-balanceable. Then by applying the proposed method expressed in theorem 2, the weight-balanced network is obtained. The result of applied consensus protocol for the weight-balanced network is illustrated in figure 3(b). It is shown that since the network is structurally weight-balanceable the distributed average consensus is obtained by modifying the communication weights between nodes.

For comparison purpose, we considered the imbalance-correcting algorithm proposed in [22]. The imbalance-correcting algorithm was presented for strongly connected networks, however a mathematical representation of strong connectivity has not been investigated in [22]. Moreover, the imbalance-correcting algorithm is based on the assumption that each node is aware of its outgoing edges, which requires a feasible method for identification of the outgoing neighbors and the computation of the out-degree. We applied the imbalance-correcting algorithm for the network represented in figure 2. The algorithm required a very long convergence time, which shows the inability of the imbalance-correcting algorithm in dealing with networks with large number of nodes. The result is depicted in figure 3(c).

VI. CONCLUSION

In this paper, we have presented the concept of structural weight-balanceability to check the average consensus in network systems. The notion of *structurally unobservable node* has been introduced based on the structural observability concept. It has been shown that the existence of

Fig. 2: Network consists of 1000 nodes

Fig. 3: Applying consensus protocol on (a) a structurally weight-balanceable network (b) a weight-balanced network obtained by the proposed method (c) a weight-balanced network obtained by the imbalance-correcting algorithm purposed in [22].

structurally unobservable node leads to loss of structural weight-balanceability. A mathematical method has been presented based on principal minors of the in-degree Laplacian matrix to determine the structural weight-balanceability of a network and make the structurally weight-balanceable network, weight-balanced. In addition, a distributed method has been proposed to compute the normalized principal minors in a network with large number of nodes. Simulation results have been presented to illustrate the performance of the proposed methods in dealing with large-scale networks.

In the proposed method in theorem 3 for computation of the principal minors, each node computes

all the principal minors of other nodes. Future research could be devoted to explore the possibility of further improving the proposed method to compute only the principal minor associated with each node. It is also interesting to investigate the effects of uncontrollable nodes on the structural weight-balanceability of the network systems.

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APPENDIX

Lemma A1. *Let $\mathcal{L}$ be the (in-degree) Laplacian matrix of a digraph $\mathcal{G}$. Consider the following matrix:*

$$\mathcal{P} = \sum_{i=1}^{N-1} \beta_i \mathcal{L}^i \quad (23)$$

where β_i are scalars. $(\mathcal{P})_{ij}$ is zero for any arbitrary values of β_i , if there is no path of any length from node j to node i .

Proof: To proof the lemma, we show that the $(i, j)^{th}$ element of all matrices $\mathcal{L}^i$ where $i = 1, 2, \dots, N$ are zero, if there is no path of any length from node j to node i . Since there is no adjacent path from node j to i , then $a_{ij} = 0$. Therefore, the $(i, j)^{th}$ element of the $\mathcal{L}^2$ can be expressed as follows:

$$(\mathcal{L}^2)_{ij} = \sum_{\substack{k_1=1 \\ k_1 \neq j}}^N a_{ik_1} a_{k_1j} = (\mathcal{A}^2)_{ij} \quad (24)$$

Using lemma A2, we obtain $(\mathcal{L}^2)_{ij} = 0$. Therefore, the $(i, j)^{th}$ element of the $\mathcal{L}^3$ can be expressed as follows:

$$(\mathcal{L}^3)_{ij} = \sum_{\substack{k_1=1 \\ k_1 \neq i, j}}^N \sum_{\substack{k_2=1 \\ k_2 \neq i, j}}^N a_{ik_1} a_{k_1k_2} a_{k_2j} = (\mathcal{A}^3)_{ij} \quad (25)$$

Using lemma A2, we obtain $(\mathcal{L}^3)_{ij} = 0$. Similarly, we can proceed for $\mathcal{L}^4, \mathcal{L}^5, \dots, \mathcal{L}^{N-1}$, and show that $(\mathcal{L}^k)_{ij} = 0$ for $k = 1, 2, \dots, N-1$. ■

Lemma A2. [46] *Let $\mathcal{A}$ be the adjacency matrix of a digraph $\mathcal{G}$, then $(\mathcal{A}^k)_{ij}$ is greater than zero if and only if there is a path of length k from node j to node i .*