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Device-independent characterizations of a shared quantum state independent of any Bell inequalities

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In a Bell experiment two parties share a quantum state and perform local measurements on their subsystems separately, and the statistics of the measurement outcomes are recorded as a Bell correlation. For any Bell correlation, it turns out that a quantum state with minimal size that is able to produce this correlation can always be pure. In this work, we first exhibit two device-independent characterizations for the pure state that Alice and Bob share using only the correlation data. Specifically, we give two conditions that the Schmidt coefficients must satisfy, which can be tight, and have various applications in quantum tasks. First, one of the characterizations allows us to bound the entanglement between Alice and Bob using Renyi entropies and also to bound the underlying Hilbert space dimension. Second, when the Hilbert space dimension bound is tight, the shared pure quantum state has to be maximally entangled. Third, the second characterization gives a sufficient condition that a Bell correlation cannot be generated by particular quantum states. We also show that our results can be generalized to the case of shared mixed states.

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I. INTRODUCTION

In the study of quantum physics, frequently the internal workings of a quantum device are not exactly known. For example, it is often the case that we do not have sufficient knowledge of the internal physical structure, or the precision of the quantum controls is very limited, or even the devices we are using cannot be trusted. In these cases, it could be that the only reliable information available is the measurement statistics from observing the quantum system. However, sometimes we still want to draw nontrivial conclusions on the quantum properties of the involved system. This sounds like a challenging, or even impossible task, but it has been shown to be possible in many cases [1–14]. These kinds of tasks are called *device independent*, as their application assumes only the correctness of quantum mechanics as a valid description of nature and is independent of the internal workings of the devices used. Device independence is a very valuable property in physical implementations of various quantum schemes. Typical examples of its usefulness include the transmission of information safely using untrusted devices and easy monitoring of the overall performance of vulnerable quantum devices [11–14].

We consider in this paper the setting of a Bell experiment, i.e., two spatially separated parties sharing a quantum state and performing local measurements on their subsystems. The corresponding statistics of the measurement outcomes is called a Bell correlation. It has been shown that the dimension and the entanglement of the underlying quantum state can be quantified in a device-independent way using only the Bell correlation data [5,8–10]. In fact, some quantum states can even be pinned down completely by their violations of particular Bell inequalities, but this is known to be possible only for some special cases [1–4,15–17].

In a Bell experiment, suppose a correlation is generated by measuring the shared quantum state ρ . We often hope the dimension of ρ is as small as possible due to the fact that quantum dimensionality is a precious resource. Interestingly, for an arbitrary Bell correlation, it is known that this quantum state with minimal dimension can always be pure [9]. Conveniently, a pure state can be described using its Schmidt decomposition, where the Schmidt coefficients completely capture its quantum properties.

In this paper, we give two device-independent characterizations of the Schmidt coefficients of the state used in a general Bell experiment. In particular, these characterizations are independent of any Bell inequalities and are very easy to calculate using only the correlation data. We show that these characterizations enjoy various applications in many device-independent tasks. Concerning the first characterization, we provide examples for which it is tight and where the shared pure quantum states are actually pinned down completely. Second, we show that it implies lower bounds on both the dimension and the amount of entanglement of the underlying quantum state, which are device-independent tasks that have drawn much attention recently [5,8,9]. We then show that the second characterization allows us to exclude the pure state that can produce a given Bell correlation from being particular states. We also show that both of the characterizations can be generalized to the case of shared mixed quantum states, where Schmidt coefficients are replaced by eigenvalues of the reduced density matrices of the two parties.

II. SCENARIO

In a Bell scenario, the two separated parties, Alice and Bob, share a pure quantum state $|\psi\rangle$ acting on $\mathbb{C}^d \otimes \mathbb{C}^d$. They each have a local measurement apparatus and can choose different settings to measure their respective subsystems. We denote the sets of the measurement settings of Alice and Bob

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FIG. 1. Characterizing the quantum state in a Bell experiment with unknown internal workings.

by X and Y , respectively. For any $x \in X$, the measurement x is described as a local positive-operator valued measure (POVM) $\{M_{xa} : a \in A\}$, and similarly, any measurement $y \in Y$ is described as a POVM $\{N_{yb} : b \in B\}$, where A and B are the sets of the measurement outcomes of Alice and Bob, respectively, as illustrated in Fig. 1. A Bell correlation p is the collection of the joint conditional probabilities $p(ab|xy)$ Alice and Bob observe, i.e.,

$$p(ab|xy) = \langle \psi | M_{xa} \otimes N_{yb} | \psi \rangle. \quad (1)$$

Up to local change of bases, $|\psi\rangle$ can be Schmidt decomposed into the computational basis as

$$|\psi\rangle = \sum_{k=1}^d \sqrt{\lambda_k} |k\rangle |k\rangle, \quad (2)$$

where the Schmidt coefficients $(\lambda_1, \dots, \lambda_d)$ are non-negative. Define $D \equiv \text{diag}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_d}\}$. It can be shown that

$$p(ab|xy) = \text{Tr}(M_{xa} \cdot D N_{yb}^* D) = \text{Tr}(D M_{xa} D \cdot N_{yb}^*), \quad (3)$$

$$p(a|x) = \sum_b p(ab|xy) = \text{Tr}(D M_{xa} D), \quad (4)$$

$$p(b|y) = \sum_a p(ab|xy) = \text{Tr}(D N_{yb}^* D), \quad (5)$$

where S^* denotes the complex conjugate of the matrix S .

III. CHARACTERIZATION OF THE SCHMIDT COEFFICIENTS

For fixed y and b , if $p(b|y) \neq 0$, define the quantum state

$$\rho_{yb} \equiv \frac{1}{p(b|y)} D N_{yb}^* D \quad (6)$$

and notice that the probability that measurement x outputs a , when applied to ρ_{yb} , is given by $\frac{p(ab|xy)}{p(b|y)}$.

Now we want to estimate the distances between these quantum states. For this, we utilize the fact that when two quantum states are measured by the same measurement, the fidelity between two quantum states ρ and σ , defined by $\mathbf{F}(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$, is upper bounded by that between the probability distributions of measurement outcomes [18]. Thus, for any $x \in X$, $y_1, y_2 \in Y$, and $b_1, b_2 \in B$, if $p(b_1|y_1) > 0$ and

$$p(b_2|y_2) > 0,$$

$$\mathbf{F}(\rho_{y_1 b_1}, \rho_{y_2 b_2}) \leq \sum_a \sqrt{\frac{p(ab_1|x y_1)}{p(b_1|y_1)} \frac{p(ab_2|x y_2)}{p(b_2|y_2)}},$$

which means that

$$\mathbf{F}(\rho_{y_1 b_1}, \rho_{y_2 b_2}) \leq \min_x \sum_a \sqrt{\frac{p(ab_1|x y_1)}{p(b_1|y_1)} \frac{p(ab_2|x y_2)}{p(b_2|y_2)}}. \quad (7)$$

Combining this with the fact that $\text{Tr}(\rho\sigma) \leq \mathbf{F}(\rho, \sigma)^2$ for any quantum states ρ and σ , we obtain that

$$\text{Tr}(\rho_{y_1 b_1} \rho_{y_2 b_2}) \leq \min_x \left(\sum_a \sqrt{\frac{p(ab_1|x y_1)}{p(b_1|y_1)} \frac{p(ab_2|x y_2)}{p(b_2|y_2)}} \right)^2.$$

Recalling the definition of ρ_{yb} , we have

$$\text{Tr}(N_{y_1 b_1}^* D^2 N_{y_2 b_2}^* D^2) \leq \min_x \left(\sum_a \sqrt{p(ab_1|x y_1) p(ab_2|x y_2)} \right)^2, \quad (8)$$

which is also true when $p(b_1|y_1) = 0$ or $p(b_2|y_2) = 0$.

On the other hand, for any $y \in Y$ it holds that $\sum_b D N_{yb}^* D = D^2$, as $\{N_{yb}^* : b \in B\}$ is a POVM. Thus, for any $y_1, y_2 \in Y$, we have that

$$\sum_{i=1}^d \lambda_i^2 = \text{Tr}(D^4) = \sum_{b_1, b_2} \text{Tr}(N_{y_1 b_1}^* D^2 N_{y_2 b_2}^* D^2). \quad (9)$$

Combining (8) and (9), we obtain that $\sum_i \lambda_i^2$ is upper bounded by

$$\min_{y_1, y_2} \sum_{b_1, b_2} \min_x \left(\sum_a \sqrt{p(ab_1|x y_1) p(ab_2|x y_2)} \right)^2. \quad (10)$$

Note that we could have regarded $\{N_{yb}\}$ as a measurement and $D M_{xa}^* D / p(a|x)$ as a quantum state to view the correlation data. In this case, by repeating the discussion above we conclude that $\sum_i \lambda_i^2$ is also upper bounded by

$$\min_{x_1, x_2} \sum_{a_1, a_2} \min_y \left(\sum_b \sqrt{p(a_1 b|x_1 y) p(a_2 b|x_2 y)} \right)^2. \quad (11)$$

Therefore, we have the following characterization for the Schmidt coefficients.

Theorem 1. If a Bell correlation p can be generated by the state $|\psi\rangle$ with Schmidt coefficients $(\lambda_1, \dots, \lambda_d)$, then

$$\sum_{i=1}^d \lambda_i^2 \leq \min\{f_1(p), f_2(p)\}, \quad (12)$$

where $f_1(p)$ and $f_2(p)$ denote the values given in (10) and (11), respectively.

We now remark on Theorem 1. First, note that in the discussion above, the dimension of the pure state can be arbitrary; thus (12) is valid for any pure state that generates p , not just one of a particular dimension. For example, suppose $|\psi\rangle$ is a quantum state generating some Bell correlation. We can replace it with $|\psi\rangle \otimes |\Phi\rangle$ to produce the same correlation, where $|\Phi\rangle$ is a redundant Einstein-Podolsky-Rosen (EPR) pair

shared by Alice and Bob. It is easy to verify that for this new quantum state, the sum of squares of the Schmidt coefficients has decreased, which makes the bound (12) looser. Therefore, Theorem 1 tends to provide a more meaningful result when the dimension of the underlying system is close to minimal. We illustrate this in a later example. This also proves that one cannot hope to find a lower bound on $\sum_i \lambda_i^2$ as a function of only the correlation data.

Second, we now consider the case when Alice and Bob share a mixed state ρ . In this case, we can bound the *purity* of ρ_A or ρ_B , where $\rho_A \equiv \text{Tr}_B(\rho)$ and $\rho_B \equiv \text{Tr}_A(\rho)$. The purity of a quantum state ρ is defined as $\text{Tr}(\rho^2)$ (see [18]), and $\text{Tr}(\rho_A^2)$ is precisely $\sum_{i=1}^d \lambda_i^2$ in the case of the pure state (2). To see how to bound the purity of ρ_A , suppose Bob introduces a third subsystem C on his side to purify ρ to be $|\psi\rangle_{ABC}$. Then by performing an isometry, he maps his subsystem to a smaller one with the same dimension as that of Alice (seen to be possible by viewing its Schmidt decomposition). Next, he adjusts the measurements he uses by the same isometry. Then it can be verified that Alice and Bob now have a Bell experiment that generates the same correlation as before, where they share a pure quantum state on $\mathbb{C}^d \otimes \mathbb{C}^d$ for some d . Note that Alice's reduced density matrix remains unchanged in the whole process, and its eigenvalues are exactly the Schmidt coefficients of the new pure state. Therefore Theorem 1 gives an upper bound for the purity of ρ_A . Later we discuss how this allows us to estimate the *entanglement of formation* of ρ .

IV. SEVERAL TIGHT EXAMPLES

To show that the bound (12) can be tight, we first consider an example in which $A = B = X = Y = \{0, 1\}$ and the correlation p is given by

$$p(ab|xy) = \begin{cases} (2 + \sqrt{2})/8, & \text{if } a \oplus b = xy, \\ (2 - \sqrt{2})/8, & \text{if } a \oplus b \neq xy, \end{cases} \quad (13)$$

where $\oplus$ denotes the logical exclusive OR of two bits. This correlation corresponds to the optimal strategy for the Clauser-Horne-Shimony-Holt game [19] and can be generated by the maximally entangled state in $\mathbb{C}^2 \otimes \mathbb{C}^2$. From (12) we can see that $\sum_{i=1}^d \lambda_i^2 \leq 1/2$, which is tight.

For a second example, we now apply our bound to an extreme point of the no-signaling polytope in the setting $|X| = |Y| = |A| = |B| = 3$ (see Table III of [20]). We find that $f_1(p) = 0$ (seen by choosing $y_1 = 0, y_2 = 2$). Thus, $\lambda_i = 0$ for every i , implying no finite-dimensional quantum state exists which generates this correlation. Thus, we can certify the nonquantumness of particular correlations.

As the last example, we set $X = Y = \{1, 2, 3\}$ and $A = B = \{0, 1\}^3$ and consider the correlation

$$p(ab|xy) = \begin{cases} 1/8, & \text{if } a_y = b_x, a \text{ has even parity,} \\ & \text{and } b \text{ has odd parity,} \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

This correlation is optimal for the magic square game [21–23] and can be generated if Alice and Bob share the maximally entangled state in $\mathbb{C}^4 \otimes \mathbb{C}^4$.

We now ask the question of whether it is possible to generate this correlation with any other pure state of the

same dimension. The answer is no, and we can prove this using Theorem 1. By straightforward calculation, it can be shown that the right side of (12) for this case is $1/4$, which again is tight. Moreover, for a pure state on $\mathbb{C}^4 \otimes \mathbb{C}^4$, the minimum value of $\sum_i \lambda_i^2$ is $1/4$, and it can be achieved only by a maximally entangled state. Therefore, in this case, Theorem 1 certifies that the pure quantum state on $\mathbb{C}^4 \otimes \mathbb{C}^4$ that can generate (14) is unique up to local unitary transformations. Actually, even if we allow the shared state to be mixed, it has been shown in a recent work [17] that the state must still be maximally entangled on $\mathbb{C}^4 \otimes \mathbb{C}^4$. These results are useful in the line of research known as *self-testing* [1–4]. Note that a similar analysis can be applied to the correlation (13).

V. RELATION TO DEVICE-INDEPENDENT DIMENSION TEST

Device-independent lower bounds on the dimension of a quantum state used in a Bell setting is a very interesting problem that has attracted much attention recently [5, 9]. Recall that for any Bell correlation, a quantum state with the minimal size that produces this correlation can always be pure [9]. In our notation, if $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ generates the correlation p , we would like to lower bound d using only the correlation data. Noting that $d \geq 1/(\sum_{i=1}^d \lambda_i^2)$ is valid for any pure state, (12) immediately implies the two *lower bounds* for the underlying Hilbert space dimension,

$$\left[\min_{y_1, y_2} \left(\sum_{b_1, b_2} \min_x \sum_a \sqrt{p(ab_1|xy_1)p(ab_2|xy_2)} \right)^2 \right]^{-1}, \quad (15)$$

$$\left[\min_{x_1, x_2} \left(\sum_{a_1, a_2} \min_y \sum_b \sqrt{p(a_1b|x_1y)p(a_2b|x_2y)} \right)^2 \right]^{-1}, \quad (16)$$

recovering the main result in Ref. [9]. However, these lower bounds on the dimension do not imply our result, Theorem 1.

VI. QUANTIFICATION OF ENTANGLEMENT

Since quantum properties of a bipartite pure quantum state are captured completely by the Schmidt coefficients, our bound (12) can be used to characterize other properties as well in a device-independent manner. As a natural application, we now consider quantifying the amount of entanglement shared by Alice and Bob.

For this, we first recall that the generalized Renyi entanglement entropies of a mixed state ρ are defined as

$$S_n(\rho) \equiv \frac{1}{1-n} \log[\text{Tr}(\rho^n)], \quad (17)$$

where $n > 0$ is a real number. It can be shown that S_n is a nonincreasing function in n , which, as n approaches 1, converges onto the well-known von Neumann entropy

$$S(\rho) \equiv -\text{Tr}[\rho \log(\rho)]. \quad (18)$$

As a result, $S_2(\rho)$ is a natural lower bound for the von Neumann entropy $S(\rho)$. Revisiting Theorem 1, if a pure state generates a correlation p , it is clear that we can bound $S_2(\rho_A)$, where ρ_A is Alice's reduced density matrix, as

$$S(\rho_A) \geq S_2(\rho_A) \geq -\log(\min\{f_1(p), f_2(p)\}). \quad (19)$$

Note again that a lower bound on the dimension does not directly imply any lower bounds on the entropy. On the other hand, for a fixed Bell correlation p , there does not exist a general upper bound on $S(\rho_A)$, since Alice and Bob can always carry redundant EPR pairs and still generate the same correlation.

As an example, we now use (19) to consider the $I3322$ Bell inequality [24], which is quite interesting, as numerical evidence suggests that to violate this Bell inequality maximally, infinite-dimensional Hilbert spaces are required [25]. By applying (19) to a Bell correlation produced by a quantum state in $\mathbb{C}^{49} \otimes \mathbb{C}^{49}$ that approximates the maximal violation given in Ref. [25], we obtain that the von Neumann entanglement entropy needed to produce this correlation from a shared pure state is at least 0.67.

Since in practical experiments quantum states are often mixed, we next briefly discuss the case when the shared state ρ is unknown but assumed to be close to pure, i.e., that $\text{Tr}(\rho^2) > 1 - \eta$, where η is a small positive number. Note that with this assumption, it is not completely device independent any longer. However, this is still a realistic setting due to the remarkable improvements in quantum experimentation in recent years. We now show that our results allow us to estimate the *entanglement of formation* of ρ , denoted by $E_f(\rho)$ and defined to be

$$E_f(\rho) \equiv \min \sum_i p_i S(\rho_i), \quad (20)$$

where the minimum is taken over all ensembles $\{p_i, |\alpha_i\rangle\}$ generating ρ , and $\rho_i = \text{Tr}_B(|\alpha_i\rangle\langle\alpha_i|)$. Suppose an orthogonal decomposition of ρ is $\rho = \sum_{i=1}^k a_i |\psi_i\rangle\langle\psi_i|$, where $a_i \geq a_j$ for $i < j$. Then it can be shown that

$$a_1 \geq \frac{1}{2} + \sqrt{\frac{1}{2} \left(\frac{1}{2} - \eta \right)} \approx 1 - \frac{1}{2}\eta. \quad (21)$$

Thus the distance between ρ and $|\psi_1\rangle\langle\psi_1|$ is small. Also, we have $\text{Tr}(\rho_{A1}^2) \leq \frac{1}{a_1^2} \text{Tr}(\rho_A^2)$, where $\rho_A = \text{Tr}_B(\rho)$ and $\rho_{A1} = \text{Tr}_B(|\psi_1\rangle\langle\psi_1|)$. Combining this fact with the upper bound for $\text{Tr}(\rho_A^2)$ mentioned above, one can lower bound the entanglement entropy of $|\psi_1\rangle\langle\psi_1|$, which is also its entanglement of formation $E_f(|\psi_1\rangle\langle\psi_1|)$. Last, according to the continuous property of the entanglement of formation [26], it holds that

$$|E_f(\rho) - E_f(|\psi_1\rangle\langle\psi_1|)| \leq \sqrt{2\eta} [9 \log(d) - \log(2\eta)]. \quad (22)$$

This way one can obtain a lower bound for $E_f(\rho)$.

VII. THE SMALLEST SCHMIDT COEFFICIENT

In this section, we give another necessary condition that the set of Schmidt coefficients must satisfy. Suppose we define $\lambda_{\min}$ as the least nonzero Schmidt coefficient of the pure state that generates a correlation p . We now show that it can be upper bounded in a device-independent manner by a function of the correlation data.

Using the isometry argument mentioned before, we can assume without loss of generality that the number of nonzero Schmidt coefficients is d , i.e., the shared pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ has full Schmidt rank. Note that for any positive

semidefinite matrices A and B , we have that $\text{Tr}(AB) \leq \text{Tr}(A)\text{Tr}(B)$. Then using (3), we have

$$p(ab|xy) \leq \text{Tr}(M_{xa})p(b|y), \quad (23)$$

$$p(ab|xy) \leq \text{Tr}(N_{yb})p(a|x). \quad (24)$$

By (4) and (5) we have

$$\text{Tr}(M_{xa}) \leq \frac{p(a|x)}{\lambda_{\min}} \quad \text{and} \quad \text{Tr}(N_{yb}) \leq \frac{p(b|y)}{\lambda_{\min}}. \quad (25)$$

Considering that these inequalities are valid for any choice of parameters, we obtain the following theorem.

Theorem 2. If a Bell correlation p can be generated by the state $|\psi\rangle$ with least nonzero Schmidt coefficient $\lambda_{\min}$, it holds that

$$\lambda_{\min} \leq \min_{x,y,a,b} \frac{p(a|x)p(b|y)}{p(ab|xy)}. \quad (26)$$

We now comment on how Theorem 2 can be tight. As an example, consider the BB84 correlation defined as $p(ab|xy) = \frac{1+ab\delta_{xy}}{4}$ [27], where $a,b,x,y \in \{-1,1\}$. This Bell correlation can be generated by the maximally entangled state in $\mathbb{C}^2 \otimes \mathbb{C}^2$. A quick calculation of (26) shows that $\lambda_{\min} \leq 1/2$, which is tight. Again, if Alice and Bob share a mixed state ρ , Theorem 2 can be used to upper bound the minimum nonzero eigenvalues of ρ_A and ρ_B .

One may ask whether we can lower bound the greatest Schmidt coefficient based only on the correlation data. It turns out that it is not possible. Again, for any Bell experiment, if Alice and Bob introduce a redundant pure state, the greatest Schmidt coefficient can become arbitrarily small while still generating the same correlation.

Above we have seen examples where pure states of certain dimensions which generate particular Bell correlations have to be maximally entangled. We now use Theorem 2, in the opposite manner, to show that a correlation *cannot* be generated using a particular state, again under dimension assumptions. For this, suppose that p is generated by $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Then if (26) certifies that $\lambda_{\min} < 1/d$, we can conclude that, independent of the local measurements Alice and Bob may apply, $|\psi\rangle$ cannot be maximally entangled. In other words, p cannot be reproduced by any maximally entangled state of local dimension up to d . Of course, this can be used to rule out other states as well, depending on the dimension and bound on $\lambda_{\min}$.

We now illustrate this with a concrete example. Suppose Alice and Bob fix some choice of measurements x and y , and each measurement has three outcomes $\{1,2,3\}$. We specify some of the probabilities in a possible correlation p below:

$$\begin{bmatrix} 1/10 & 1/100 & 1/100 \\ 1/100 & * & * \\ 1/100 & * & * \end{bmatrix},$$

where the (a,b) entry is $p(ab|xy)$, and the asterisks represent unspecified probabilities. According to Ref. [28], the minimum size of quantum state that can generate such a partial correlation p has local dimension of at most 3. Meanwhile, it can be verified using (26) that $\lambda_{\min} \leq 18/125$, which is strictly less than $1/3$. Therefore, it is clear that any pure state

in $\mathbb{C}^3 \otimes \mathbb{C}^3$ which generates p cannot be maximally entangled. In fact, we would require a maximally entangled state to have local dimension of at least 7 to generate p . Furthermore, if we restrict to a state in $\mathbb{C}^2 \otimes \mathbb{C}^2$, then such a correlation cannot be generated by any state of the form $\sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle$ where $a \in (18/125, 107/125)$.

VIII. CONCLUSIONS

For an arbitrary Bell correlation produced by locally measuring a bipartite pure quantum state, we have given two characterizations for its Schmidt coefficients, which can be generalized to the case of shared mixed states. Also, we showed that they have various applications in many device-independent quantum processing tasks. Since our bounds only involve simple functions of the Bell correlation data, they are

quite robust against errors in statistical data, making them usable in practical quantum tasks. We hope these results will lead to more nontrivial applications in quantum physics and quantum information theory, and particularly, we hope the entanglement quantification application can be helpful in future quantum experiments.

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