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2016

Han, L., Sun, H., Wang, D. Z. W., & Zhu, C. (2016). A stochastic process traffic assignment model considering stochastic traffic demand. *Transportmetrica B: Transport Dynamics*, in press.

<https://hdl.handle.net/10356/83022>

<https://doi.org/10.1080/21680566.2016.1240051>

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A stochastic process traffic assignment model considering stochastic traffic demand

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Abstract

In real traffic network, both link capacity and traffic demand are subject to stochastic fluctuations. These random fluctuations are major sources of travel time uncertainty. All existing stochastic process traffic assignment model models considering the uncertainty of travel time are presented with fixed traffic demand. In this study, a stochastic process traffic assignment model is presented to consider stochastic traffic demand. The traffic demand is assumed to be comprised of two groups of travelers: commuters with fixed traffic demand and irregular travelers with discrete random demand. With mild conditions, it is proved that our stochastic process traffic assignment model is ergodic and has a unique stable distribution. An algorithm is given to describe the stochastic process model. By conducting numerical test, we analyze the effect of commuters' memory length, irregular travelers' demand and commuters' perception error on the stable distribution of our model.

Keywords: stochastic process, day-to-day dynamical model, traffic assignment model, stochastic traffic demand

1. Introduction

Traditionally, a deterministic approach is used to describe the performance of traffic network, with the assumption that both the link capacity (traffic supply) and the traffic demand between origin-destination (OD) are

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perfectly known. In real traffic network, however, both link capacity and traffic demand are subject to stochastic fluctuations. The traffic supply uncertainty is caused by different disturbances on the road (e.g. accident, road work, illegal parking, or weather) affecting the driving condition and hence the road capacity (Chen and Zhou, 2010; Lam et al., 2008). Traffic control devices, such as signal timing and ramp metering, also contribute to the variability of link capacity. The uncertainty of supply can be referred as stochastic link capacity variations, and typically result in non-recurrent congestion (Al-Deek and Emam, 2006; Chen et al., 2002; Lo et al., 2006). On the other hand, there are also several sources of uncertainty in the traffic demand. Travel demand fluctuations can be caused by temporal factors (time of day, day of week or seasonal effects), special events, population characteristics (age, car ownership, and household income). Traffic information provided by Advanced Traveler Information Systems (ATIS) can also influence the travelers' trip decision, including their departure time, destination, mode, and route choice, which consequently affect the traffic demand. These demand variations usually induce a recurrent congestion (Asakura and Kashiwadani, 1991; Clark and Watling, 2005). The coupling of the demand and supply uncertainties result in the recurrent variability and unreliability of the travel time and traffic condition (Asakura and Kashiwadani, 1991). Jackson and Jucker (1982) and Abdel-Aty et al. (1995) indicated that the perceived reliability of travel time plays an important role on travelers' route choice decisions. So, in traffic assignment, it is imperative to consider the effect of uncertain travel time. In this study, we will study the travelers' long-term route choice behavior in presence of uncertain travel time.

From the literature, one can find that there exists two main methods for studying the issue. One is the method of equilibrium model. Nicholson and Du (1997) proposed a theoretical framework for analyzing degradable traffic systems. They used a conventional integrated network equilibrium model with variable demand to describe flows in a network with degradable link capacities. Bell and Cassir (2002) applied the game theory to build a risk-averse user equilibrium model. They assumed that the network user "plays through" all the possible eventualities before selecting his best route. Kurauchi et al. (2003) made use of the absorbing Markov chains to solve the capacity constrained transit network loading problem taking common line into account. In their model, passengers will not be able to board because of the absence of sufficient space, and the choice of route depends on frequency of arrivals. Lo and Tung (2003) characterized the route choice behavior in

the face of uncertain travel times with the notion of probabilistic user equilibrium (PUE), which was developed with a nonlinear mathematical program to study the tradeoff between the maximum flow that a network can carry and the extent of satisfying the PUE reliability constraints. [Clark and Watling \(2005\)](#) proposed a technique for estimating the probability distribution of total network travel time, in the light of normal day-to-day variations in the travel demand matrix over a road traffic network. [Sumalee et al. \(2006\)](#) applied the model of [Clark and Watling \(2005\)](#) to develop a Network Design Problem model considering the effect of stochastic demand on network reliability. [Lo et al. \(2006\)](#) extended the method of [Nakayama and Takayama \(2003\)](#) and proposed a model which considers the road capacity as a uniform distributed random variable subject to traffic incidents. [Shao et al. \(2006\)](#) extended the study of [Lo et al. \(2006\)](#) and presented a demand driven travel time reliability-based user equilibrium model to consider the effects of daily demand fluctuations. [Chen and Zhou \(2010\)](#) proposed a model called the a-reliable mean-excess traffic equilibrium (METE) model that explicitly considers both reliability and unreliability aspects of travel time variability in the route choice decision process. [Zhang et al. \(2011\)](#) applied the expected residual minimization (ERM) model to provide a robust traffic assignment with an emphasis on the planner’s perspective under stochastic demand and supply. [Wang et al. \(2014\)](#) presented a bi-objective user equilibrium model of travel time reliability in road network. They proved that their model is a general framework for the study of reliability related user equilibrium. [Nakayama and Watling \(2014\)](#) presented an internally-consistent network equilibrium approach. Their model considers two potential sources of flow variability: (i) daily variation in route choice and (ii) daily variation in origin-destination demand. [Nikolova and Stier-Moses \(2014\)](#) developed a mean-risk model of the traffic assignment problem with stochastic travel times. In their model, travelers are risk-averse who capture the trade-off between travel times and their variability in a mean-standard deviation objective, defined as the mean travel time plus a risk-aversion factor times the standard deviation of travel time along a path. Other than above mentioned papers for general traffic network, some researchers also considered the uncertain travel times in the special network-commuter corridor case. These researches can be referred to [Wang and Du \(2013\)](#), [Du and Wang \(2014\)](#) and [Wang and Du \(2016\)](#).

All of the studies mentioned above belong to the class of static equilibrium model, in which equilibrium modeling approaches have been adopted to consider the effects of network uncertainties on long-run behavior of travelers.

The other method is dynamical traffic model, which assumes that the state of traffic system at any given time is not predictable a priori due to the intrinsic randomness of the travel time and it depends on previous states determined by travelers' choices in the past. The implication of these assumptions is that the traffic system evolves over successive time periods as a stochastic process, the type of which depends on the particular choice mechanism followed by travelers (Cascetta, 1989). In this class of model, each traveler independently makes route choices every "day" based on his/her past experiences of travel costs, and this type of model can estimate the day-to-day variation of traffic state more directly. A previous study on this class of model was represented by Daganzo (1977), who modeled the system evolution using a Markovian chain and also established the coincidence of time-average flows and stochastic user equilibrium flows for uncongested networks and large OD demand. Cascetta (1989) presented a stochastic process approach to the analysis of the day-to-day dynamics in transportation network. In the paper, they gave the sufficient conditions for the stochastic process stationarity and compared expected path and link flows of the final steady distribution of traffic flows with those of stochastic user equilibrium (SUE). Cascetta and Cantarella (1991) extended the model of Cascetta (1989). They develop a stochastic process model to model day-to-day and the within-day dynamic flow fluctuations. To make the type of Markovian traffic assignment model more practical, Hazelton (2002) proposed a method to solve the Day-to-day variation in Markovian assignment processes. Hazelton and Watling (2004) gave an alternative method of computing the equilibrium distribution, which is applicable to the class of Markovian models with linear exponential learning filter. Recently, there has been a growing interest in models of transportation networks which explicitly represent epoch-to-epoch adaptive behavior of travelers, such as the day-to-day dynamics of drivers' route choices. Watling and Cantarella (2013) made several new advances to the existing body of theory including deterministic and stochastic traffic dynamical system to model sources of variation in transportation systems. Smith et al. (2014) developed a long term behavior of day-to-day traffic assignment models which can exhibit characteristics of both deterministic models and stochastic models.

It should be noted that all above studies on stochastic process traffic assignment model are presented with fixed traffic demand. Although, Hazelton and Watling (2004) contended that their model could consider the uncertainty of traffic demand simply by introducing a single "dummy route" to

denote the option of not travelling for each interzonal movement, the method may have the following issues that are difficult to be handled:

- (I) By the method, the travelers, who are assigned to the “dummy routes” on day m , do not travel on day m in practice. However, they can still gain the travel experience of m th day.
- (II) If the distribution of traffic demand is known, the travel cost function of the “dummy route” should be developed to ensure that the specific demand distribution is achieved in traffic assignment.

The issue (I) can be solved if certain assumptions are made to make sure every traveler who no matter travels or not on day m can obtain the traffic information of that day. However, it is very difficult to develop a travel cost function on the “dummy route” to ensure that the specific distribution of traffic demand is exactly achieved, as too many factors (such as, the travelers’ perception, route choice behavior) should be considered in traffic assignment. Meanwhile, we also think that the “dummy route” method may not be able to deal with the travel demand comprising of both commuters and irregular travelers. In urban traffic network, the majority of travelers are commuters during morning peak or evening peak. Commuters regularly travel from home in a suburb to work in a city, and their traffic demand is quite stable. But, there are also many irregular travelers for tour or special errands. They do not make the same trip frequently and their travel may make the congested traffic system worse. The importance of the analysis of the effects from irregular travelers could be justified by the practical traffic control measures on nonlocal vehicles implemented in Beijing and Shanghai, China. In these two cities, the nonlocal vehicles are restricted from entering the center of city during the peak periods. The effectiveness of implementing this traffic control measures strongly reflects the significant effects of irregular travelers towards the performance of the urban traffic system. The lack of analysis of this effect in the literature motivates us to fill in this research gap by explicitly analyzing how the irregular travelers would affect the traffic performance.

In this paper, it is assumed that the travelers in urban traffic network can be segregated into two groups: commuters/regular travelers and irregular travelers. The demand of commuters is assumed to be fixed, as the number of commuters who travel regularly from home to workplace is very stable and can be considered as fixed. However, the number of irregular travelers has higher level of stochastic variations. The commuters between an OD pair make daily trips and are very familiar with the traffic conditions. Therefore,

they have the comprehensive travel experiences and make route choice to minimize their perceived travel cost. On the other hand, irregular travelers do not have the complete information of the traffic conditions in the past. They can rely on some supplementary tools (e.g., map, route guidance system) to choose their preferred routes. As was done in [Siu and Lo \(2008\)](#), it is assumed that irregular travelers would select routes based on shortest distance (or free travel time) to finish their trips. Based on these assumptions, we would present a stochastic process traffic assignment model to study the day-to-day evolution of traffic system in peak periods.

This paper is organized as follows. In Section 2, the notation and statement used in this paper are given. The sufficient conditions for the stationarity of the stochastic process traffic assignment model are presented in Section 3. And the specific formulation of our model is shown in Section 4. Section 5 shows discussions and the results of simulation on our model. The conclusions and the future work would be given in Section 6.

2. Notation and Statement of the Problem

Let us consider a traffic network $G(N, L)$, consisting of a node set N , a set L of directed links and a Origin-Destination (OD) set W . Let $\mathbf{d}^1$ be the demand vector of commuters, and its component d_w^1 is the number of commuting travelers between OD w and it is a constant for every day. $\mathbf{d}^{2,t}$ is the vector of irregular travelers' demand on day t , whose w th component $d_w^{2,t}$ is the number of irregular travelers between the OD w . $d_w^{2,t}$ is a bounded discrete random variable and $0 \leq d_w^{2,t} \leq d_w^{2,up}$, where $d_w^{2,up}$ is its supremum. It is assumed that each of OD i ($i = 1, 2, \dots, |W|$) has a set of routes, RR_i , with $m_i = |RR_i|$ and $M = \sum_i^N m_i$. Routes in RR_1 are numerated as $1, 2, \dots, m_1$; $m_1 + 1, m_2 + 2, \dots, m_1 + m_2$ for routes in RR_2 , and $M - m_N + 1, M - m_N + 2, \dots, M$ for routes in RR_N . The route set in $G(N, L)$ is denoted as $RR = \bigcup_{i \in W} RR_i$. In this study, the routes referred in traffic network are feasible acyclic routes.

Because it is assumed that irregular travelers consistently choose route with shortest of free travel time, the route choice of irregular travelers between the same OD pair can be regarded as a single traveler. In this study, the irregular travelers making their inter-zonal movements is aggregately dealt as one traveler. The number of travelers making route choice between OD w is $d_w^1 + 1$. The total number of travelers making route choice in transport network is $\sum_{w \in W} d_w^1 + |W|$.

To study the evolution of the traffic system over a sequence of times, $\dots, t-1, t, t+1, \dots$, we firstly give the most basic representation of system state occupied by system at a given referred period. The system state represents the route choice of every traveler; it is denoted by a vector $\mathbf{R}$ and its dimension is $\sum_{w \in W} d_w^1 + |W|$. Since the traffic demand is grouped into two parts, $\mathbf{R}$ can be looked upon as $\mathbf{R} = (R_1, R_2)$, where $\mathbf{R}_1$ is the route choice vector of commuters and its dimension is $\sum_w d_w^1$, $\mathbf{R}_2$ is the route choice vector of irregular travelers. The dimension of $\mathbf{R}_2$ is $|W|$. The u th component of $\mathbf{R}_1$ being:

$$R(u) = [\text{index of the route chosen by the } u\text{th commuter}], R(u) \in I(u) \quad (1)$$

where $I(u)$ is the set of feasible routes considered by the u th commuter. The w th component of $\mathbf{R}_2$ is the index of the route chosen by irregular commuters between OD w . For $0 \leq d_w^{2,t} \leq d_w^{2,up}$, the irregular travelers' demand has $2^{d_w^{2,up}}$ possible states. Then the number of feasible $\mathbf{R}_2$ is $\prod_w 2^{d_w^{2,up}}$. Meanwhile, the number of feasible acyclic routes is finite (Cascetta, 1989; Cascetta et al., 1997). So the number of feasible path choice vectors by commuters n_{R_1} is finite and $n_{R_1} = \prod_{w \in W} (m_w)^{d_w^1}$. The number of feasible $\mathbf{R}$ is finite and $n_R = n_{R_1} n_{R_2}$. The set of feasible route vectors is S_R :

$$S_R = (\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_{n_R}). \quad (2)$$

When travelers complete route choice, the path flow vector would be determined. Here, the path flow vector is denoted as $\mathbf{F}$, its k th component is defined as follows:

$$F(k) = [\text{the number of travelers on the } k\text{th route}] \quad (3)$$

It is assumed that the route k is between OD w . Because the traffic demand is comprised of commuters and irregular travelers, the specific number of travelers on route k is defined as follows:

$$F(k) = \begin{cases} \text{Num}_{k,com} + d_w^2, & \text{if } k \text{ has minimum free travel time between OD } w \\ \text{Num}_{k,com}, & \text{otherwise} \end{cases} \quad (4)$$

where $\text{Num}_{k,com}$ is the number of commuters on route k . Every $\mathbf{F}$, can also be decomposed of two parts: the path flow vector $\mathbf{F}^1$ generated by commuters

and the path flow vector $\mathbf{F}^2$ produced by irregular travelers, and $\mathbf{F} = \mathbf{F}^1 + \mathbf{F}^2$. For a route choice vector can only produce a path flow vector. So the number of feasible path flow vector n_F is finite and $n_F \leq n_R$.

let S_F be the set of all feasible path flow vector, then:

$$S_F = (\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_{n_F}).$$

In general, some route choice vectors may correspond to the same path flow vector $\mathbf{F}_k$; Let IF_k denote the subset of feasible route choice vectors giving rise to the vector $\mathbf{F}_k$.

Using the link-path incidence matrix, the third system state representation considered in this paper-link flow is calculated following:

$$\mathbf{v} = \mathbf{A}\mathbf{F}. \quad (5)$$

where $\mathbf{A}$ is the link-path incidence matrix. The components of link flow vector $\mathbf{v}$ is given by:

$$v(i) = [\text{no.of travelers using link } i \text{ in the reference period}]. \quad (6)$$

And the number of feasible link flow vector is finite n_v as the n_F is finite. Let Iv_i denote the subset of feasible path flow vectors giving rise to the link flow vector $\mathbf{v}_i$.

Let $p_u^t(k)$ denote the probability that the u th commuter follows route k at time t . And travelers are assumed to choose routes independently. The probability that the system is in the feasible state $\mathbf{R}_h$ at time t can be expressed as:

$$\mathbf{Prob}[\mathbf{R}^t = \mathbf{R}_h] = \prod_u p_u^t[R_h(u)] \prod_{w \in W} p_w^t[d_w^{2,t} = d_w^2]. \quad (7)$$

where $d_w^{2,t}$ is the number of irregular travelers in reference period t , $p_w^t[d_w^{2,t} = d_w^2]$ is the probability that irregular travelers between OD w is d_w^2 at time t .

Above statements on the state occupied by the traffic system at each reference time implies that the evolution of the traffic system over different times can be realized by a stochastic process with discrete time and state spaces. In the following, we would present the sufficient conditions for stationarity and ergodicity of the stochastic process.

3. Sufficient Conditions for the Stationarity of the Stochastic Process

It is known that the type of stochastic process describing traffic system evolution over time depends on the $p_u^t(k)$, or on how travelers make their route choices at each referred time t . Here we introduce the similar assumptions of [Cascetta \(1989\)](#):

- The number of travelers making travel decisions in each epoch is uniformly distributed over the reference period so that usual definitions of path and link flows maintain their significance.
- All commuters moving at each time t do not know in advance the travel costs at $t + 1$ but they can make their choices depending on available information relative to costs incurred in previous epochs. This can be expressed by supposing that the path choice probability p_u^t of u th commuter route choice depend on the states occupied by the system in previous $t - 1, t - 2, \dots$

$$p_u^t(k) = p_u^t(k)[\mathbf{R}^{t-1}, \mathbf{R}^{t-2}, \dots] \quad (8)$$

Next we give the sufficient conditions for the stability and ergodicity of the stochastic process describing the time evolution of traffic system in route choice vector space with stochastic traffic demand.

Proposition 1. *The stochastic process representing evolution of the traffic system in the route choice space has a unique steady-state probability distribution $\pi = (\pi_1, \pi_2, \dots, \pi_{n_R})$ and it is ergodic under the following conditions:*

- Commuters' path choice probabilities are time homogeneous, i.e., they are invariant under a temporal translation given the set of previous states.*

$$p_u^t(k)[R^{t-1} = R_h, R^{t-2} = R_k, \dots] = p_u^{t'-1}(k)[R^{t'-1} = R_h, R^{t'-2} = R_k, \dots] \quad (9)$$

- Commuters' path choice probabilities are positive for all feasible paths.*

$$p_u^t(k) > 0 \quad \forall k \in I(u) \quad (10)$$

- (iii) *The probability of irregular travelers' demand between each OD pair are positive for all possible values.*

$$p_w^t[d_w^{2,t}] > 0 \quad \forall t \in T, w \in W \quad (11)$$

- (iv) *Commuters' path choice probabilities depend on not more than a finite number (m) of previous states.*

$$p_u^t(k)[R^{t-1}, R^{t-2}, \dots, R^{t-m}] = p^{t'-1}(k)[R^{t'-1}, R^{t'-2}, \dots, R^{t'-m}] \quad (12)$$

Conditions (i) and (iv) ensure that the stochastic process of traffic system's evolution with stochastic traffic demand is a m -dependent Markovian chain. Conditions (ii) ensure that all feasible paths have a positive probability of being chosen by commuters at each time. Condition (iii) entails all "feasible" irregular traveler demand have a positive probability.

The proof of the proposition is as follows: By transformation of the state space, a homogeneous m -dependent Markovian chain can be reduced to an homogeneous (1-dependent) Markovian chain. The transformed state $\mathbf{R}^*$ is a collection of m feasible route choice states:

$$\mathbf{R}_{h,s,\dots,j}^* = (\mathbf{R}_h, \mathbf{R}_s, \dots, \mathbf{R}_j) \quad (13)$$

and the number of such states is n_R^m .

The matrix of transition probabilities between transformed states can be determined as follows:

$$\begin{aligned} \pi_{h,\dots,l,j,qh,\dots,l} &= \mathbf{Prob}[\mathbf{R}^{*t} = \mathbf{R}_{qh,\dots,l}^* / \mathbf{R}^{*t-1} = \mathbf{R}_{h\dots lj}^*] \\ &= \prod_u p_{Rq(u)}[\mathbf{R}_h, \dots, \mathbf{R}_l, \mathbf{R}_j] \end{aligned} \quad (14)$$

Conditions (iii) and (iv) ensures that transition probabilities between transformed states is a positive probability:

$$\pi_{h,\dots,l,j,qh,\dots,l} > 0 \quad \forall q \in S_R \quad (15)$$

Using the same method of [Cascetta \(1989\)](#), we can easily prove Prop.1.

The stochastic process of evolution on traffic system in route or link flow space also has the similar conclusion ([Cascetta, 1989](#)):

Proposition 2. *If the process in route choice space is stationary and ergodic, the same properties hold for the route and link flow spaces.*

The above proposition can be proved by specifying the relations among route choice, path flow and link flow vectors. Let the sets of route choice vector which respectively produce route flow vectors $\mathbf{F}_{k_1}, \mathbf{F}_{k_2}, \dots, \mathbf{F}_{k_n}$ be $IF_{k_1}, IF_{k_2}, \dots, IF_{k_n}$. For all Δt , n , and $t_1, t_2, \dots, t_n$, the probability:

$$\begin{aligned}
& P(\mathbf{F}^{t_1} = \mathbf{F}_{k_1}, \mathbf{F}^{t_2} = \mathbf{F}_{k_2}, \dots, \mathbf{F}^{t_n} = \mathbf{F}_{k_n}) \\
&= \sum_{\mathbf{R}_{k_1} \in IF_{k_1}, \dots, \mathbf{R}_{k_n} \in IF_{k_n}} P(\mathbf{R}^{t_1} = \mathbf{R}_{k_1}, \dots, \mathbf{R}^{t_n} = \mathbf{R}_{k_n}) \\
&= \sum_{\mathbf{R}_{k_1} \in IF_{k_1}, \dots, \mathbf{R}_{k_n} \in IF_{k_n}} P(\mathbf{R}^{t_1 + \Delta t} = \mathbf{R}_{k_1}, \dots, \mathbf{R}^{t_n + \Delta t} = \mathbf{R}_{k_n}) \\
&= P(\mathbf{F}^{t_1 + \Delta t} = \mathbf{F}_{k_1}, \mathbf{F}^{t_2 + \Delta t} = \mathbf{F}_{k_2}, \dots, \mathbf{F}^{t_n + \Delta t} = \mathbf{F}_{k_n})
\end{aligned}$$

Therefore, the the process in path flow space is stationary. Its ergodicity can be also shown by using the same method. Further, based on the relationship between path flow and link flow vectors, the stationary and ergodicity of the process in link flow space can be verified.

4. The stochastic process traffic assignment model with stochastic traffic demand

Although we introduce the random irregular travelers into the traffic system, it can be found that our stochastic process traffic assignment model describing the evolution of traffic system still have the nice property of the model presented by [Cascetta \(1989\)](#):

- Various stochastic processes with their moments can be obtained by assuming different choice and information acquisition models with various degrees of realism.

Here we also use the assumptions in [Cascetta \(1989\)](#) to describe the commuters' route choice and the traffic information acquisition:

- (i) All commuters moving between the same OD w have the same set of "feasible" paths I_w .

- (ii) All commuters are rational decision makers. At each time t they associate a perceived (generalized) travel cost $C'_k(t)$ to each alternative path and choose so as to minimize this cost.
- (iii) All commuters have the same information acquisition (learning) mechanism.

Because of random fluctuations in link capacity and traffic demand, it is assumed that perceived path costs are random variable with mean value $C_k^+(t)$ and the perceived cost can be described in the following:

$$C'_k(t) = C_k^+(t) + \epsilon_k(t) \quad (16)$$

where the parameter $\epsilon_k(t)$ is a random variable taking into account commuter's perception errors around the mean value $C_k^+(t)$. Under this assumption the u th commuters' path choice probabilities $p_u^t(k)$ can be computed as:

$$p_u^t(k) = \text{Prob}[C'_k(t) < C'_h(t) \quad \forall h \neq k \quad h \in I_w] \quad (17)$$

The functional form of path choice probabilities $p_u^t(k)$ depends on the joint probability density function for the term of perception error ϵ_k .

If the ϵ_k independently and identically follows Gumbel $G(0, \theta)$ distribution, the probability of route choice is calculated by Logit model. It is well known that logit model applied to the route choice problem has a drawback: inability to account for overlapping (or correlation) among routes. It stems from the underlying assumptions of logit model that the random error terms are independently and identically distributed (IID) with the same, fixed variances (Sheffi, 1985). To avoid the problem, several extended logit models are available including C-logit (Cascetta et al., 1996; Zhou et al., 2012), path-size logit (Ben-Akiva and Bierlaire, 1999), cross-nested logit (CNL) (Prashker and Bekhor, 1998; Vovsha and Bekhor, 1998), paired combinatorial logit (PCL) (Pravinongvuth and Chen, 2005), generalized nested logit (GNL) (Bekhor and Prashker, 2001), and logit kernel (Bekhor et al., 2002). Except logit and its extended models, probit model can also be used to obtain the route choice probability if the residuals are assumed to be jointly distributed as a Multivariate Normal of zeros mean.

According to the condition (iv) of Prop.1, commuters only use the information of previous m reference times. This study presents two methods to

calculate the mean perceived travel cost. One is that the perceived travel cost on path k at time t is a weighted average of costs actually incurred in a finite number m of previous times $C_k(t)$:

$$C_k^+(t) = \sum_{i=1}^m w_i C_k(t-i) \quad (18)$$

This type of formulation implies that the commuters are risk neutral. It is to say that commuters only concern the mean travel time and do not consider its error.

Another expression of the mean perceived travel cost is calculated as follows:

$$C_k^+(t) = \sum_{i=1}^m w_i C_k(t-i) + \eta \mathbf{var}[C_k(t-1), C_k(t-2), \dots, C_k(t-m)] \quad (19)$$

where $\mathbf{var}[C_k(t-1), C_k(t-2), \dots, C_k(t-m)]$ is mean deviation of travelers' actual experienced travel time in previous referred m periods:

$$\mathbf{var}[C_k(t-1), C_k(t-2), \dots, C_k(t-m)] = \left\{ \sum_{i=1}^m w_i [C_k(t-i) - \sum_{n=1}^m w_n C_k(t-n)]^2 \right\}^{1/2} \quad (20)$$

And parameter η is defined as the requirement of punctual arrivals or the probability of arriving at the destination within the travel time budget (Siu and Lo, 2008). The Eq.19 adds a travel time margin to the mean trip time to avoid late arrivals and schedule delay penalty. In other words, commuters consider a longer travel time budget to hedge against travel time uncertainty. This type of behavior can be recognized as risk averse.

The assumptions on commuters route choice and information acquisition (i)-(iii) in this section assure that commuters are homogenous. So commuters between the same OD pair have the same probability of route choice, i.e. $p_{k,w}^t = p_u^t(k)$. With these homogeneous assumptions, the stochastic process is also Markovian in path and link flow spaces (Bhat, 1972; Kemeny and Snell, 1960). For commuters independently make route choice, the path flow $\mathbf{F}_1$ generated by commuters follows a multinomial distribution as follows:

$$\mathbf{Prob}[\mathbf{F}^{1,t} = \mathbf{F}_h^1] = \prod_w d_w^1! \prod_{k \in I_w} (p_{kw}^t)^{f_h^u(k)} / F_h^u(k)! \quad (21)$$

And the probability distribution of $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ is following:

$$\text{Prob}[\mathbf{F}^t = \mathbf{F}_h] = \prod_w d_w^1! \prod_{k \in I_w} (p_{kw}^t)^{f_h^u(k)} / F_h^u(k)! \prod_w [d_{t,w}^2 = d_w^2] \quad (22)$$

Where $d^{2,w}$ is the w th component of $\mathbf{F}_2$.

Synthesize the Eq.16-17 and Eq.22, the traffic system evolution for a sequence referred periods in the route choice space can be described by the stochastic process process traffic assignment model of following formulation:

$$\begin{cases} C'_k(t) = C_k^+(t) + \epsilon_k(t) \\ p_u^t(k) = \text{Prob}[C'_k(t) < C'_h(t) \quad \forall h \neq k \quad h \in I_w] \\ \text{Prob}[\mathbf{F}_h] = \prod_w d_w^1! \prod_{k \in I_w} (p_{kw}^t)^{f_h^u(k)} / F_h^u(k)! \prod_w [d_{t,w}^2 = d_w^2] \end{cases} \quad (23)$$

5. Algorithm

The stochastic process traffic assignment model given by Eq.23 has an unique steady-state distribution and is ergodic. So, it is easy to develop algorithm to realize the stochastic process process. An algorithm is proposed in this section and have the following general structure:

- step 0: Give the travel costs in the initial m periods, and set $t = 1$;
- step 1: Compute mean perceived costs $C_k^+(t)$ by using last m iteration flows via Eq.18 or Eq.19;
- step 2: Calculate the path choice probability $p_{k,w}^t(k) (\forall k \in R_w, \forall w \in W)$ that the route k is chosen by commuters using Eq.17;
- step 3: Sample path flow $\mathbf{F}^{1,t}$ caused by commuters via the Eq.21 and the irregular traffic demand $d_{t,w}^2$ for $\forall w \in W$ using its random distribution to generate the path flow $\mathbf{F}^{2,t}$. Then the path flows $\mathbf{F}^t$ for time t is $\mathbf{F}^t = \mathbf{F}^{1,t} + \mathbf{F}^{2,t}$ and the link flow $\mathbf{v}^t = A\mathbf{F}^t$;
- step 4: Mean link flow and average deviation of link volumes over all iterations.

$$v_l^+(t) = 1/t \sum_{i=1}^t v_l^i \quad (24a)$$

$$\text{Var}(v_l(t)) = 1/t \sum_{i=1}^t (v_l^i - v_l^+(t))^2 \quad (24b)$$

If the stop criterion is not met, $t = t + 1$ and go to step 1.

In the following section, we would use the algorithm to study the effect of parameters in the stochastic process traffic assignment model based on two testing examples.

6. Numerical Study

We have proved the stability and ergodicity of the stochastic process traffic assignment model. In this section, we would use two examples to analyze the effect of parameters in the model. The first example is an analog traffic network shown in Fig. 1. The traffic network has 5 nodes, 6 links and one OD pair (1-5). The link free travel time t_a^0 respectively is 2,1,5,1,1; the link capacity c_a is 500,800,800,800,500,500. The network has three paths which are denoted by node sequence, path 1: 1-2-5; path 2: 1-5; path 3: 1-3-4-5. Therefore, irregular travelers choose path 3 to travel. Since these three paths have no common link, Logit model is used to calculate the commuters' route choice probability in this small test example.

Figure 1: The first traffic network example

The second example is the aggregated network of the city Sioux Falls, South Dakota. It has 24 nodes and 76 links shown in Fig.2. And it is assumed that there are only two OD pairs: (4-20) and (6-24). Their feasible paths are shown in Tab.1. The shortest paths of free travel time are respectively path 1-1 and path 2-1. In Sioux Falls network, paths between the same OD pair usually share a number of links. Therefore, the standard logit model may produce unreasonable route choice probabilities due to its overlapping problem. To overcome this problem, this study applies the C-logit model developed by [Cascetta et al. \(1996\)](#) to gain route choice probability. Its functional formulation is as following:

$$p_k = \frac{\exp \theta(V_k - CF_k)}{\sum_{l \in R_w} \exp \theta(V_l - CF_l)} \quad (25)$$

Table 1: The feasible paths in Sioux falls

feasible paths of OD (4,20)	feasible paths of OD (6,24)
path 1-1: 4→5→6→8→7→18→20	path 2-1: 6→8→7→18→20→21→24
path 1-2: 4→11→14→15→19→20	path 2-2: 6→5→4→3→12→13→24
path 1-3: 4→5→9→10→16→18→20	path 2-3: 6→5→4→11→14→23→24
path 1-4: 4→5→9→10→15→19→20	path 2-4: 6→2→1→3→12→13→24
	path 2-5: 6→8→16→17→19→15→22→21→24

where V_k is the commuters' travel utility of path k and θ is the parameter of commuters' perception error. CF_k is the commonality factor of path k , which is a measure of degree of similarity of path k with other paths between the same OD pair. The role of CF_k is that heavily overlapping paths own larger commonality factors and thus a larger generalized travel cost with respect to similar, but independent paths. CF_k can be specified in different ways, this study adopt the formula of [Cascetta et al. \(1996\)](#) shown in Eq.26:

$$CF_k = \beta \ln \sum_l \left(\frac{L_{kl}}{L_k^{1/2} L_l^{1/2}} \right)^\gamma \quad (26)$$

where L_{kl} is the free travel time of links common to path k and l . L_k , L_l are the overall free travel time (the sum of free travel time of related links) of paths k and l , respectively. According to [Cascetta et al. \(1996\)](#), parameters $\beta > 0$ and $\gamma > 0$ are respectively valued $\beta = 1$ and $\gamma = 2$ in this study.

In this study, we only consider that the actual travel cost of link a only depend on its flow v_a , with special formulation of BPR function (Bureau of Public Roads, BPR) given in Eq.27. The value of parameters of Sioux Falls can be referred to the work of [Suwansirikul et al. \(1987\)](#)

$$t_a = t_a^0 \left[1 + \left(\frac{v_a}{c_a} \right)^4 \right] \quad (27)$$

The irregular travelers' traffic demand $d_w^{2,t}$ of OD w follows Poisson distribution and the corresponding parameter is λ_w . The supremum of $d_w^{2,up}$ is set as $10\lambda_w$.

6.1. Effect of commuters' memory length

In this study, commuters estimating the travel cost on time t only use the travel costs of previous m reference times. So m is the measure of commuters'

Figure 2: The second traffic network example

memory length. If commuters are risk neutral, their mean travel costs are calculated by Eq.18 and parameter $w_i = 1/m$. In the first example, the demand of commuters is 2000, and the mean demand of irregular travelers λ is 500. The commuters' demand of two OD pairs in Sioux Falls are both 10000 and the mean irregular travelers' demand are both 2000.

Fig.3 shows the mean flows of path 1 and path 3 with $m = 10, 20, 40$ when commuters are risk neutral in the first example. One can observe that the differences among these mean path flows on the same path for those values of m are very small. The average flow deviations of path 1 and path 3 for $m = 10, 20, 40$ are shown in Fig.4. There are still small differences among the average deviations of path flow. We also study the effect of m on the mean and average deviation of path flow in the second example when commuters are risk neutral. Fig.5 shows the mean flows of path 1-1 and path 1-3 between OD (4,20), path 2-1 and 2-4 between OD (6,24) of the second example when $m = 10, 20, 30, 40$. Their average deviations of flow are shown in Fig.6. It can be found that the differences among the mean or average deviations of flow on a path with $m = 10, 20, 30, 40$ are all very small.

When commuters take the uncertainty of travel time into their travel cost, they are risk averse. Then their travel costs are calculated by Eq.19

Figure 3: The mean path flow for different m when commuters are risk neutral in the first example

and Eq.20. The parameter $\eta = 1.6$ in this study. Fig.7 and Fig.8 respectively show the mean and average flow deviations of path 1 and path 3 in the first example. Those values of path 1-1 and path 1-3, path 2-1 and path 2-4 in the second example are presented in Fig.9 and Fig.10 when $m = 20, 30, 40$. The conclusions are very similar with those of risk neutral case. However, those similarities do not retain when the memory length of commuter is small. The conclusion can be obtained from Fig.11. Fig.11(a) shows the average deviations of flow on path 3 of the first example with $m = 5, 10$. Fig.11(b) presents the mean flows on path 1-1 of the second example when $m = 10, 15, 20$. It can be found that there are obvious differences among the mean path flow or the average deviation of path flow when m is small.

6.2. Effect of irregular travelers' demand

In this section, we will study the effect of irregular travelers' demand on the stable distribution of stochastic process model. Based on the discussion in Section 6.1, m is set to be 20 in the following sections. The demand of commuters d^1 is 2000, and let the mean irregular traveler demand $\lambda = 500, 1000, 1500$ in the first example. Fig.12 shows the mean path flow, the corresponding average deviation of path flow is shown in Fig.13. In both figures, the solid lines denote the results when commuters are risk neutral and the dash lines show the results for the risk averse case.

From Fig.12, it can be found that the mean flow of every path increases as λ goes up when commuters are risk neutral or averse. The increase of mean flow on path 3 is very natural, because the irregular travelers all choose path

Figure 4: The variation of path flow for different m when commuters are risk neutral in the first example.

Figure 5: The mean path flow for different m when commuters are risk neutral in Sioux Falls.

3 to travel. When mean irregular traveler increases on path 3, the mean travel cost jacks up. Then some commuters originally on path 3 have to choose path 1 or path 2 to complete trip. So, the mean path flow of path 1 and path 2 increase with λ .

However, the average deviation of path flow does not always increase with λ (Fig.13). It can be found that the average deviations of flow on path 1 and path 2 attain at maximum when $\lambda = 1000$. This is due to the path flow produced by commuters' route choice following a multinomial distribution described by Eq.21. Although the multinomial distribution just describes the path flow distribution on a reference time t , it can also describe the stable

Figure 6: The variation of path flow for different m when commuters are risk neutral in Sioux Falls.

Figure 7: The mean path flow for different $m = 10, 20, 40$ when commuters are risk averse in the first example.

distribution of path flow the stochastic process traffic assignment model if the probability of commuters' route choice is known. Let p_1 and p_2 respectively denote the probability of path 1 and path 2 chosen by commuters. The variation of flow on path 1 is $d^1 p_1 (1 - p_1)$ and that of path 2 is $d^1 p_2 (1 - p_2)$. So the average deviations of flow on path 1 and path 2 do not always increase with λ . For the flow on path 3 is comprised of irregular travelers and commuters. Let var_{com} denote the variation of path flow on path 3 induced by commuters. The variation generated by commuters on path 3 is λ because the irregular travelers' demand follows Poisson distribution. The covariance between commuters and irregular travelers is $cov_{com,nocom}$. Then the variation

Figure 8: The variation of path flow for different m when commuters are risk averse in the first example.

Figure 9: The mean path flow for different m when commuters are risk averse in Sioux Falls.

of path flow on path 3 can be express by: $var_{com} + 2cov_{com,nocom} + \lambda$. So the average deviation of flow on path 3 also does not always increase with λ .

Compared with the mean path flow in risk neutral case, the mean flows on path 1 and path 2 are not less than those in risk averse case. Especially, the mean flow of path 2 obviously become larger. This can be interpreted by the fact that less commuters take path 3 to travel when commuters factor the fluctuation of travel time into their travel cost. From Fig.13, one can find that the average deviations of flow in both cases do not have larger difference.

In the numerical test of the second example, the commuters' demand between OD (4,20) and (6,24) are both 10000 and $\lambda = 1000, 2000, 4000$. Since

Figure 10: The variation of path flow for different m when commuters are risk averse in Sioux Falls.

Figure 11: The mean or average deviation of path flow when the value of m is less.

there are correlations of paths between the same OD pair in the example. It is found that the mean path flow does not follow the rules in the first example in which the paths have no correlations. For example, the mean flows on path 1-2 between the OD (4,20) and path 2-5 between OD (6,24) are respectively shown in Fig.14. From Fig.14(a), it can be found that the mean path flow on path 1-2 is minimum when the mean irregular demand is 4000, and the result on path 2-5 between OD pair (6,24) decreases with λ presented in Fig.14(b). But the average deviation of path flow of the second example has similar rules with that presented in the first example. Fig.15 shows the average deviations of flow on path 1-1 and path 2-1. It can be found that

Figure 12: For fixed commuters' traffic demand and different irregular travelers' mean demand, the mean path flow when commuters are risk neutral and averse in the first example.

the average deviations of flow path 1-1 and path 2-1 do not increase with the mean demand of irregular traveler.

6.3. The effect of commuters' perception error

In this study, the route choice probability is calculated by Logit or C-logit model. The parameter θ in the model is related with commuters' perception error. In this section, to evaluate the effect of this parameter, the values of parameters of our model except θ are assumed to be the same as those in Section 6.2.

Fig.16 and Fig.17 respectively show the mean path flows and average deviations of path flow for different values of θ in the first example. In the both figures, the solid lines describe the results when commuters are risk

Figure 13: For fixed commuters' traffic demand and different irregular travelers' mean demand, the deviation of path flow when commuters are risk neutral and averse in the first example.

neutral and the dash lines show the results of the risk averse case. It can be found that the mean path flow on path 3 decreases with increasing θ . This is because larger θ means less perception error. With reduced commuters' perception error, less commuters originally choosing path 3 would bear more travel cost induced by irregular travelers. Further, mean flows on path 1 and path 2 all increase with θ . Fig.17 shows the average deviation of path flow. It can be seen that the fluctuations of path 1 and path 2 become larger with higher value of θ . Because the mean irregular travelers' demand is fixed in this section, the fluctuation of path flow is caused by the route choice of commuters. That is, more and more commuters want to minimize their perceived travel cost by changing route choice.

It can also be found that the commuters on path 3 become less when the

Figure 14: For fixed commuters' traffic demand and different value of irregular travelers' mean demand, the mean path flow when commuters are risk neutral and averse in Sioux Falls.

uncertainty of travel time is factored into commuters' travel cost for a given θ . Therefore, the mean flows on path 1 and path 2 are not less than those of risk neutral case. After considering uncertainty of travel time in travel cost, the fluctuations of path flows are larger than those without considering uncertainty of travel time with the same value of θ . That is, the commuters are more likely to change route choice as compared to the risk neutral case.

Although there are correlations among paths in the second example, its mean and average deviation of path flow also have the same rules of the first example. Fig.18 shows the mean flows of path 1-1 and path 2-1 increase with the commuters' perception error. Meanwhile, the mean deviations of corresponding path flow (Fig.19) also increase since the increase of mean path flow.

7. Conclusions

In this study, we assume that the traffic demand is grouped into two parts: commuters with fixed demand and irregular travelers with discrete random demand. Commuters have the complete travel experience and make route choice to minimize their perceived travel cost. Infrequent travelers, on the other hand, are not familiar with the traffic conditions and they are assumed to select routes based on shortest distance to complete their trips. With very mild conditions, it is proved that the stochastic process describing the evolution of traffic system with random demand has ergodicity

Figure 15: For fixed commuters' traffic demand and different value of irregular travelers' mean demand, the deviation of path flow when commuters are risk neutral and averse in Sioux Falls.

and a unique stable probability distribution. Based on the assumptions about commuters' route choice and information acquisition mechanism, we give a specific stochastic process traffic assignment model. In the model, we consider two attitudes toward the uncertainty of travel time: risk neutral and risk averse. We also propose an algorithm to describe the stochastic process model. In the numerical tests, it is found the commuters' memory length has less effect on the mean and average deviation of the stable distribution of the stochastic process in both risk neutral and averse cases when it is enough sufficiently. But, it has substantial impact on the stable distribution when it is small. It is also found that the mean irregular travelers' demand and commuters' perception error also have important effect on the stable distribution of our stochastic process model.

Acknowledgement

Dr. Sun was supported by the National Natural Science Foundation of China(71322102, 71271023) and Research Foundation of State Key Laboratory of Rail Traffic Control and Safety (RCS2016ZT002). Dr. Han was supported by the China National Funds for Distinguished Young Scientists(71525002) and Singapore Ministry of Education (MOE) AcRF Tier 2 Grant ARC21/14 (MOE2013-T2-2-088).

Figure 16: The mean path flow for different θ when commuters are risk neutral or averse in the first example

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Figure 17: The deviation of path flow for different θ when commuters are risk neutral or averse in the first example

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Figure 18: The mean path flow for different θ when commuters are risk neutral or averse in Sioux Falls

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Figure 19: The deviation of path flow for different θ when commuters are risk neutral or averse in Sioux Falls

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