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Completeness of Randomized Kinodynamic Planners with State-based Steering

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Abstract

Probabilistic completeness is an important property in motion planning. Although it has been established with clear assumptions for geometric planners, the panorama of completeness results for *kinodynamic* planners is still incomplete, as most existing proofs rely on strong assumptions that are difficult, if not impossible, to verify on practical systems. In this paper, we focus on an important class of kinodynamic planners, namely those that interpolate trajectories in the state space. We provide a proof of probabilistic completeness for such planners under assumptions that can be readily verified from the system's equations of motion and the user-defined interpolation function. Our proof relies crucially on a property of interpolated trajectories, termed *second-order continuity* (SOC), which we show is tightly related to the ability of a planner to benefit from denser sampling. We analyze the impact of this property in simulations on a low-torque pendulum. Our results show that a simple RRT using a second-order continuous interpolation swiftly finds solution, while it is impossible for the same planner using standard Bezier curves (which are not SOC) to find any solution.¹

Keywords: kinodynamic planning, probabilistic completeness

1. Introduction

A deterministic motion planner is said to be *complete* if it returns a solution whenever one exists [2]. A *randomized* planner is said to be *probabilistically complete* if the probability of returning a solution, when there is one, tends to one as execution time goes to infinity [3]. Although these two notions might seem theoretical, they are of notable practical interest, as proving completeness requires one to formalize the problem by hypotheses on the robot, the environment, etc. While

¹ This paper is a revised and expanded version of [1], which was presented at the *International Conference on Robotics and Automation*, 2014. The proof (Section 3) has been rewritten using Landau notation for easier reading, and a new evaluation on the low-torque pendulum has been appended, including a proof of incompleteness for fixed-time Bezier interpolation and empirical evaluation in simulations (Section 4).

17 experiments can show that a planner works for a given robot, in a given environ-
 18 ment, for a given query, etc., a proof of completeness is a certificate that the planner
 19 works for a precise *set* of problems. The size of this set depends on how strong the
 20 assumptions required to make the proof are: the weaker the assumptions, the larger
 21 the set of solvable problems.

22 Probabilistic completeness has been established for systems with *geometric*
 23 constraints [4, 3] such as *e.g.* obstacle avoidance [5]. However, proofs for sys-
 24 tems with *kinodynamic* constraints [6, 7, 8, 9] have yet to reach the same level
 25 of generality. Proofs available in the literature often rely on strong assumptions
 26 that are difficult to verify on practical systems (as a matter of fact, none of the
 27 previously mentioned works verified their hypotheses on non-trivial systems). In
 28 this paper, we establish probabilistic completeness (Section 3) for a large class of
 29 kinodynamic planners, namely those that interpolate trajectories in the state space.
 30 Unlike previous works, our assumptions can be readily verified from the system’s
 31 equations of motion and the user-defined interpolation function.

32 The most important of these properties is *second-order continuity* (SOC), which
 33 states that the interpolation function varies smoothly and locally between states
 34 that are close. We evaluate the impact of this property in simulations (Section 4)
 35 on a low-torque pendulum. Experiments validate our completeness theorem, and
 36 suggest that SOC is an important design guideline for kinodynamic planners that
 37 interpolate in the state space.

38 2. Background

39 2.1. Kinodynamic Constraints

40 Motion planning was first concerned only with *geometric* constraints such as
 41 obstacle avoidance or those imposed by the kinematic structures of manipula-
 42 tors [10, 4, 8, 6]. More recently, *kinodynamic* constraints, which stem from differ-
 43 ential equations of dynamic systems, have also been taken into account [11, 6, 5, 9].

44 Kinodynamic constraints are more difficult to deal with than geometric con-
 45 straints because they cannot in general be expressed using only *configuration-space*
 46 *variables* – such as the joint angles of a manipulator, the position and the orienta-
 47 tion of a mobile robot, etc. Rather, they involve higher-order derivatives such as
 48 velocities and accelerations. There are two types of kinodynamic constraints:

49 **Non-holonomic constraints:** non-integrable *equality* constraints on higher-order
 50 derivatives, such as found in wheeled vehicles [12], under-actuated manipu-
 51 lators [13] or space robots.

52 **Hard bounds:** *inequality* constraints on higher-order derivatives such as torque
 53 bounds for manipulators [14], support areas [15] or wrench cones for hu-
 54 manoid stability [16], etc.

55 Some authors have considered systems that are subject to both types of constraints,
 56 such as under-actuated manipulators with torque bounds [13].

57 2.2. Randomized Planners

58 Randomized planners such as Probabilistic Roadmaps (PRM) [4] or
 59 Rapidly-exploring Random Trees (RRT) [6] build a roadmap on the state space.
 60 Both rely on repeated random sampling of the free state space, *i.e.* states with non-
 61 colliding configurations and velocities within the system bounds. New states are
 62 connected to the roadmap using a *steering* function, which is a method used to
 63 drive the system from an initial to a goal configuration. The steering method may
 64 be imperfect, *e.g.* it may not reach the goal exactly, not take environment collisions
 65 into account, only apply to states that are sufficiently close, etc. The objective of the
 66 motion planner is to overcome these limitations, turning a local steering function
 67 into a global planning method.

68 PRM builds a roadmap that is later used to generate motions between many
 69 initial and final states (multiple queries). When new samples are drawn, they are
 70 connected to *all* neighboring states in the roadmap using the steering function,
 71 resulting in a connected graph. Meanwhile, RRT focuses on driving the system
 72 from *one* initial state x_{init} towards a goal area (single query). It grows a tree by
 73 connecting new samples to *one* neighboring state, usually their closest neighbor.

74 Both PRM's and RRT's *extension* step are represented by Algorithm 1, which
 75 relies on the following sub-routines (see Fig. 1 for an illustration):

- 76 • SAMPLE(S): randomly sample an element from a set S ;
- 77 • PARENTS(x, V): return a set of states in the roadmap V from which steer-
 78 ing towards x will be attempted;
- 79 • STEER(x, x'): generate a system trajectory from x towards x' . If success-
 80 ful, return a new node x_{steer} ready to be added to the roadmap. Depending
 81 on the planner, the success criterion may be “reach x' exactly” or “reach a
 82 vicinity of x' ”.

The design of each sub-routine greatly impacts the quality and even the com-
 pleteness of the resulting planner. In the literature, SAMPLE(S) is usually im-
 plemented as uniform random sampling over S , but some authors have suggested
 adaptive sampling as a way to improve planner performance [17]. In geometric

Algorithm 1 Extension step of randomized planners (PRM or RRT)

Require: initial node x_{init} , number of iterations N

```
1:  $(V, E) \leftarrow (\{x_{init}\}, \emptyset)$ 
2: for  $N$  steps do
3:    $x_{rand} \leftarrow \text{SAMPLE}(\mathcal{X}_{free})$ 
4:    $X_{parents} \leftarrow \text{PARENTS}(x_{rand}, V)$ 
5:   for  $x_{parent}$  in  $X_{parents}$  do
6:      $x_{steer} \leftarrow \text{STEER}(x_{parent}, x_{rand})$ 
7:     if  $x_{steer}$  is a valid state then
8:        $V \leftarrow V \cup \{x_{steer}\}$ 
9:        $E \leftarrow E \cup \{(x_{parent}, x_{steer})\}$ 
10:    end if
11:  end for
12: end for
13: return  $(V, E)$ 
```

planners, $\text{PARENTS}(q, V)$ is usually implemented from the Euclidean norm over $\mathcal{C}$ as

$$\text{PARENTS}(q, V) := \arg \min_{q' \in V} \|q' - q\|.$$

83 This choice results in the so-called Voronoi bias of RRTs [6]. Both experiments and
84 theoretical analysis support this choice for geometric planning, however it becomes
85 inefficient for kinodynamic planning, as was showed by Shkolnik et al. [18] on
86 systems as simple as the torque-limited pendulum.

87 2.3. Steering Methods

88 This paper focuses on steering functions. These can be classified into three
89 categories: analytical, state-based and control-based steering.

90 *Analytical steering.* This category corresponds to the ideal case when one can
91 compute analytical trajectories respecting the system's differential constraints, which
92 are usually called (perfect) *steering functions* in the literature [6, 19]. Unfortu-
93 nately, it only applies to a handful of systems. Reeds and Shepp curves for cars are
94 a notorious example of this [12].

95 *Control-based steering.* Generate a control $u : [0, \Delta t] \rightarrow \mathcal{U}_{adm}$, where $\mathcal{U}_{adm}$ de-
96 notes the set of *admissible* controls, and compute the corresponding trajectory by
97 *forward dynamics*. This approach has been called *incremental simulation* [20],
98 *control application* [6] or *control-space sampling* [19, 21] in the literature. It is

Figure 1: Illustration of the extension routine of randomized planners. To grow the roadmap toward the sample x' , the planner selects a number of parents $\text{PARENTS}(x') = \{P_1, P_2, P_3\}$ from which it applies the $\text{STEER}(P_i, x')$ method.

99 widely applicable, as it only requires forward-dynamic calculations, but usually
 100 results in weak steering functions as the user has no or limited control over the
 101 destination state. In works such as [6, 5], random functions u are sampled from
 102 a family of primitives (*e.g.* piecewise-constant functions), a number of them are
 103 tried and only the one bringing the system closest to the target is retained. Linear-
 104 Quadratic Regulation (LQR) [22, 23] also qualifies as control-based steering: in
 105 this case, u is computed as the optimal policy for a linear approximation of the
 106 system given a quadratic cost function.

107 *State-based steering.* Interpolate a trajectory $\gamma_{int} : [0, \Delta t] \rightarrow \mathcal{C}$, for instance a
 108 Bezier curve matching the initial and target configurations and velocities, and com-
 109 pute a control that makes the system track that trajectory. For fully-actuated sys-
 110 tem, this is typically done using *inverse dynamics*. An interpolated trajectory is
 111 rejected if no suitable control can be found. Algorithm 2 gives the prototype of
 112 state-based steering functions.

Algorithm 2 Prototype of state-based steering functions $\text{STEER}(x, x')$

- 1: $\gamma_{int} \leftarrow \text{INTERPOLATE}(x, x')$
 - 2: $u_{int} := \text{INVERSE_DYNAMICS}(\gamma_{int}(t), \dot{\gamma}_{int}(t), \ddot{\gamma}_{int}(t))$
 - 3: **if** $\forall t \in [0, \Delta t], u_{int}(t) \subset \mathcal{U}_{adm}$ **then**
 - 4: **return** the last state of γ_{int}
 - 5: **end if**
 - 6: **return** failure
-

113 Compared to control-based steering, this approach seems to apply to a more

114 limited range of systems (those where inverse dynamic solvers are readily avail-
 115 able). However, it delivers control over the destination state, which is crucial, for
 116 instance, in any robotic application where contacts need to be actively maintained.
 117 Humanoids provide a canonical example for this: their high degree of freedom
 118 and their need to maintain geometric constraints while moving (*e.g.* feet on the
 119 floor) make them particularly unsuitable to control sampling. As a matter of fact,
 120 to the best of our knowledge, all humanoid motion planners reported so far in the
 121 literature (see *e.g.* [24, 25, 26]) are based on trajectory interpolation.

122 2.4. Previous works

123 Randomized planners such as RRT and PRM are both simple to implement² yet
 124 efficient for geometric planning. The completeness of these planners has been es-
 125 tablished for geometric planning in [6, 7, 8]. In their proof, Hsu et al. [8] quantified
 126 the problem of narrow passages in configuration space with the notion of (α, β) -
 127 expansiveness. The two constants α and β express a geometric lower bound on the
 128 rate of expansion of reachability areas.

129 There is, however, a gap between geometric and *kinodynamic* planning [11]
 130 in terms of proving probabilistic completeness. When Hsu et al. extended their
 131 solution to kinodynamic planning [5], they applied the same notion of expansive-
 132 ness, but this time in the $\mathcal{X} \times \mathcal{T}$ (state and time) space with control-based steering.
 133 Their proof states that, when $\alpha > 0$ and $\beta > 0$, their planner is probabilistically
 134 complete. However, whether $\alpha > 0$ or $\alpha = 0$ in the non-geometric space $\mathcal{X} \times \mathcal{T}$
 135 remains an open question. As a matter of fact, the problem of evaluating (α, β)
 136 has been deemed as difficult as the initial planning problem [8]. In a parallel line
 137 of work, LaValle et al. [6] provided a completeness argument for kinodynamic
 138 planning, based on the hypothesis of an *attraction sequence*, *i.e.* a covering of the
 139 state space where two major problems of kinodynamic planning are already solved:
 140 steering and antecedent selection. Unfortunately, the existence of such a sequence
 141 was not established.

142 In the two previous examples, completeness is established under assumptions
 143 whose verification is at least as difficult as the motion planning problem itself. Ar-
 144 guably, too much of the complexity of kinodynamic planning has been abstracted
 145 into hypotheses, and these results are not strong enough to hold the claim that
 146 their planners are probabilistically complete in general. This was exemplified re-
 147 cently when Kunz and Stilman [27] showed that RRTs with control-based steering
 148 were *not* probabilistically complete for a family of control inputs (namely, those

² For instance, the RRT used in the simulations of this paper was implemented in less than a hundred lines of Python code.

149 with fixed time step and best-input extension). At the same time, Papadopoulos et
 150 al. [19] established probabilistic completeness for the same planner using a differ-
 151 ent family of control inputs (randomly sampled piecewise-constant functions). The
 152 picture of completeness for kinodynamic planners therefore seems to be a nuanced
 153 one.

154 Karaman et al. [7] introduced the RRT* path planner and extended it to kinody-
 155 namic planning with differential constraints in [28], providing a sketch of proof for
 156 the completeness of their solution. However, they assumed that their planner had
 157 access to the optimal cost metric and optimal local steering, which restricts their
 158 analysis to systems for which these ideal solutions are known. The same authors
 159 tackled the problem from a slightly different perspective in [29] where they sup-
 160 posed that the PARENTS function had access to w -weighted boxes, an abstraction
 161 of the system’s local controllability. However, they did not show how these boxes
 162 can be computed in practice³ and did not prove their theorem, arguing that the
 163 reasoning was similar to the one in [7] for kinematic systems.

164 While proof of completeness have been given for kinodynamic planners using
 165 control-based steering (e.g. [19, 21]), to the best of our knowledge, the present
 166 paper is the first to provide a proof of probabilistic completeness for kinodynamic
 167 planners using state-based steering.

168 2.5. Terminology

169 A function is *smooth* when all its derivatives exist and are continuous. Let $\|\cdot\|$
 170 denote the Euclidean norm. A function $f : A \rightarrow B$ between metric spaces is
 171 Lipschitz when there exists a constant K_f such that

$$\forall (x, y) \in A, \|f(x) - f(y)\| \leq K_f \|x - y\|.$$

172 The (smallest) constant K_f is called the Lipschitz constant of the function f .

173 Let $\mathcal{C}$ denote n -dimensional configuration space, where n is the number of
 174 degrees of freedom of the robot. The *state space* $\mathcal{X}$ is the $2n$ -dimensional manifold
 175 of configuration and velocity coordinates $x = (q, \dot{q})$. A trajectory is a continuous
 176 function $\gamma : [0, \Delta t] \rightarrow \mathcal{C}$, and the distance of a state $x \in \mathcal{X}$ to a trajectory γ is

$$dist_\gamma(x) := \min_{t \in [0, \Delta t]} \|(\gamma, \dot{\gamma})(t) - x\|. \quad (1)$$

177 A kinodynamic system can be written as a time-invariant differential system:

$$\dot{x}(t) = f(x(t), u(t)), \quad (2)$$

³ The definition of w -weighted boxes is quite involved: it depends on the joint flow of vector fields spanning the tangent space of the system’s manifold.

178 where $u \in \mathcal{U}$ denotes the control input and $x(t) \in \mathcal{X}$. Let $\mathcal{U}_{adm} \subset \mathcal{U}$ denote
 179 the subset of admissible controls. (For instance, $\mathcal{U}_{adm} = [\tau_{\min}, \tau_{\max}] \subset \mathcal{U} = \mathbb{R}$
 180 represents bounded torques for a single joint.) A trajectory γ that is solution to
 181 the differential system (2) using only controls $u(t) \in \mathcal{U}_{adm}$ is called an *admissible*
 182 trajectory. The kinodynamic motion planning problem is to find an admissible
 183 trajectory from q_{init} to q_{goal} .

184 A control function $u : [0, \Delta t] \rightarrow \mathcal{U}$ has δ -clearance when its image is in the
 185 δ -interior of $\mathcal{U}_{adm}$, i.e. for any time t , $\mathcal{B}(u(t), \delta) \subset \mathcal{U}_{adm}$. By extension, we will say
 186 that a trajectory γ has δ -clearance in control space when the control function $u(t)$
 187 computed by inverse dynamics has such clearance.

188 3. Completeness Theorem

189 3.1. System assumptions

190 Our model for an $\mathcal{X}$ -state randomized planner is given by Algorithm 1 using
 191 state-based steering. We first assume that:

192 **Assumption 1.** *The system is fully actuated.*

193 Full actuation restricts our analysis to a range of systems that includes *e.g.* in-
 194 dustrial manipulators (with or without kinematic redundancy) or space robots with
 195 thrusters. It allows us to write the equations of motion of the system in generalized
 196 coordinates as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u, \quad (3)$$

197 where $u \in \mathcal{U}_{adm}$ and we assume that the set of admissible controls $\mathcal{U}_{adm}$ is compact
 198 with non-empty interior. Since torque constraints are our main concern, we will
 199 focus on

$$\mathcal{U}_{adm} := \{u \in \mathcal{U}, |u| \leq \tau_{\max}\}, \quad (4)$$

200 which is indeed compact.⁴ (Vector comparisons are component-wise.) Finally,
 201 we suppose similarly to [21] that forward and inverse dynamics mappings have
 202 Lipschitz smoothness:

203 **Assumption 2.** *The forward dynamics function f is Lipschitz continuous in both of*
 204 *its arguments, and its inverse f^{-1} (the inverse dynamics function $u = f^{-1}(x, \dot{x})$)*
 205 *is Lipschitz in both of its arguments.*

⁴ The application of our proof of completeness to an arbitrary compact set presents no technical difficulty: one can just replace $|u| \leq \tau_{\max}$ with $d(u, \partial\mathcal{U}_{adm})$, with $\partial\mathcal{U}_{adm}$ the boundary of $\mathcal{U}_{adm}$. Using Equation (4) avoids this level of verbosity.

Figure 2: Single (A) and double (B) pendulums. Under torque bounds, these systems must swing back and forth several times before they can reach for the upright position, as depicted in (B) (lighter images represent earlier times).

206 These two assumptions are satisfied when f is given by (3) as long as the
 207 matrices $M(q)$ and $C(q, \dot{q})$ are bounded and the gravity term $g(q)$ is Lipschitz.
 208 Indeed, for a small displacement between x and x' ,

$$\|u' - u\| \leq \|M\| \|\ddot{q}' - \ddot{q}\| + \|C(q, \dot{q})\| \|\dot{q}' - \dot{q}\| + K_g \|q' - q\| \quad (5)$$

209 Let us illustrate this on the double pendulum depicted in Figure 2. When both links
 210 have mass m and length l , the gravity term

$$g(\theta_1, \theta_2) = \frac{mgl}{2} [\sin \theta_1 + \sin(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2)]$$

211 is Lipschitz with constant $K_g = 2mgl$, while the inertial term is bounded by
 212 $\|M\| \leq 3ml^2$. When joint angular velocities are bounded by ω , the norm of
 213 the Coriolis tensor is bounded by $2\omega ml^2$. Using (5), one can therefore derive the
 214 Lipschitz constant $K_{f^{-1}}$ of the inverse dynamics function.

215 3.2. Interpolation assumptions

216 We also require smoothness for the interpolated trajectories:

217 **Assumption 3.** The interpolation $\gamma_{int} : [0, \Delta t] \rightarrow \mathcal{C}$ returned by INTERPOLATE(x, x')
 218 is a smooth Lipschitz functions, and its time-derivative $\dot{\gamma}_{int}$ is also Lipschitz. It sat-
 219 isfies the boundary conditions $(\gamma(0), \dot{\gamma}(0)) = x$ and $(\gamma(\Delta t), \dot{\gamma}(\Delta t)) = x'$.

220 The following two assumptions ensure a continuous behavior of the interpola-
 221 tion procedure:

222 **Assumption 4** (Local boundedness). *Interpolated trajectories stay within a neigh-*
 223 *borhood of their start and end states, i.e. there exists a constant η such that, for*
 224 *any $(x, x') \in \mathcal{X}^2$, the interpolated trajectory $\gamma_{\text{int}} : [0, \Delta t] \rightarrow \mathcal{C}$ resulting from*
 225 *INTERPOLATE(x, x') is included in a ball of center x and radius $\eta \|x' - x\|$.*

226 Let us denote by $\Delta x = (\Delta q, \Delta \dot{q}) = x' - x$ the displacement between two
 227 states $x = (q, \dot{q})$ and $x' = (q', \dot{q}')$. We define the *average velocity* and *discrete*
 228 *time step* between the two states as:

$$\dot{q}_{\text{avg}} := \frac{\dot{q} + \dot{q}'}{2}, \quad \Delta t_{\text{disc}} := \frac{\Delta q \cdot \dot{q}_{\text{avg}}}{\|\dot{q}_{\text{avg}}\|^2}. \quad (6)$$

229 In order for Δt_{disc} to be well defined, one needs to ensure that:

- 230 **P1.** the scalar product $\Delta q \cdot \dot{q}_{\text{avg}}$ is positive: this reflects the fact that displacements
 231 and velocities should be aligned as the former gets small;
- 232 **P2.** the velocity $\dot{q}_{\text{avg}}$ is non-zero: in what follows, we will further assume that
 233 $\|\dot{q}_{\text{avg}}\| \geq \epsilon$, where ϵ is a positive constant. The value of this parameter will
 234 be derived thereafter.

235 We say that a pair of state (x, x') is *locally consistent* when it satisfies the two
 236 prerequisites above, as well as the following:

- 237 **P3.** the velocity $\dot{q}_{\text{avg}}$ aligns with the displacement Δq as x' and x are close from
 238 each other: formally, $\sin(\Delta q, \dot{q}_{\text{avg}}) = O(\Delta x)$.

239 The *discrete acceleration* between two locally-consistent states x and x' is then
 240 defined by $\ddot{q}_{\text{disc}}(x, x') := \frac{\Delta \dot{q}}{\Delta t_{\text{disc}}}$, which can also be written:

$$\ddot{q}_{\text{disc}}(x, x') = \frac{\Delta \dot{q} \|\dot{q}_{\text{avg}}\|^2}{\|\Delta q\| \|\dot{q}_{\text{avg}}\| \sqrt{1 - \sin(\Delta q, \dot{q}_{\text{avg}})^2}} = \frac{\Delta \dot{q} \|\dot{q}_{\text{avg}}\|}{\|\Delta q\|} + O(\Delta x) \quad (7)$$

241 The rationale behind these expressions is that the continuous analog of $\ddot{q}_{\text{disc}}$ when
 242 $x' \rightarrow x$ is $\frac{\|\dot{q}\| d\dot{q}}{\|d\dot{q}\|} = \frac{\|\dot{q}\| d\dot{q}}{\|\dot{q}\| dt} = \frac{d\dot{q}}{dt}$, i.e. the continuous acceleration.

243 Prerequisite P2 is a loose restriction: The probability that uniform random sam-
 244 pling returns a pair (x, x') violating it is on the scale of ϵ (divided by the maximum
 245 velocity), which can be chosen arbitrarily small. Meanwhile, P1 delimits a half-
 246 space where the angle between Δq and $\dot{q}_{\text{avg}}$ is acute, thus non-empty and sampled
 247 with 1/2 probability.⁵ Also, note that local consistency does not preclude $\dot{q}$ or $\dot{q}'$

⁵ The sampling routine could also be adapted to always return new samples x' locally consistent with a given x in the roadmap.

248 from being zero, as usually happens at x_{init} and x_{goal} . The only case excluded is
 249 when *both* are zero, or $\dot{q}' = -\dot{q}$, which would imply that the trajectory halts at
 250 some intermediate state. As we will see, such halts can always be excluded by
 251 leveraging the fact that $\mathcal{U}_{adm}$ has non-empty interior.

Assumption 5 (Discrete-acceleration convergence). *When start and end states become close, accelerations of interpolated trajectories uniformly converge to the discrete acceleration between them, i.e. there exists some $\nu > 0$ such that, if $\gamma_{int} : [0, \Delta t] \rightarrow \mathcal{C}$ results from $\text{INTERPOLATE}(x, x')$ for a locally consistent pair (x, x') , then*

$$\forall \tau \in [0, \Delta t], \|\ddot{\gamma}_{int}(\tau) - \ddot{q}_{disc}(x, x')\| \leq \nu \|\Delta x\|.$$

252 These three assumptions ensure that the planner interpolates trajectories locally
 253 and “continuously” when x and x' are close and locally consistent (we will see that
 254 the subset of pairwise locally-consistent states is sufficient to establish completeness). We will call them altogether *second-order continuity*, where “second-order”
 255 refers to the discrete acceleration (7) used in Assumption 5. This continuous behavior plays a key role in our proof of completeness, as it ensures that denser sampling
 256 will allow finding arbitrarily narrow state-space passages.
 257

258 Let us consider the example of a single pendulum with joint angle q and state
 259 $x = (q, \dot{q}) \in \mathbb{R}^2$, for the interpolation function $\gamma = \text{INTERPOLATE}(x, x')$ given
 260 by
 261

$$\begin{aligned} \gamma_s : [0, \Delta t] &\rightarrow]-\pi, \pi] \\ t &\mapsto \gamma_s(t) = \frac{\Delta \dot{q}}{2\Delta t} t^2 + \dot{q}t + q. \end{aligned} \quad (8)$$

for $\Delta t = \Delta t_{disc} = (\Delta q \dot{q}_{avg})/(\dot{q}_{avg}^2) = \Delta q/\dot{q}_{avg}$. One can then check that $\dot{\gamma}_s(0) = \dot{q}$, $\dot{\gamma}_s(\Delta t) = \dot{q}'$, $\gamma_s(0) = q$ and

$$\gamma_s(\Delta t) = \frac{1}{2}(\Delta \dot{q} + 2\dot{q})\Delta t + q = \dot{q}_{avg}\Delta t + q = \Delta q + q = q'.$$

The function being a second-order polynomial, itself and all of its derivatives are smooth and Assumption 3 is verified. One can check Assumption 4 by:

$$\|\gamma_s(t) - q\| \leq \frac{|\Delta \dot{q}| + 2\dot{q}}{2} \Delta t \leq \frac{|\Delta \dot{q}| + 2\dot{q}}{2\epsilon} |\Delta q| \leq \frac{1 + 2\omega}{2\epsilon} \|x' - x\|$$

262 where ω is again the maximum angular velocity, while $\|\dot{\gamma}_s(t) - \dot{q}\| \leq |\Delta \dot{q}| \leq$
 263 $\|x' - x\|$; the assumption follows by taking e.g. $\eta = 1 + (1 + 2\omega)/2\epsilon$. Finally,
 264 $\ddot{\gamma}_s(t) = \frac{\Delta \dot{q}}{\Delta t} = \ddot{q}_{disc}$ by construction, so that Assumption 5 is also verified. The
 265 function γ_s given by Equation (8) provides a valid interpolator satisfying all our
 266 assumptions for the simple pendulum.

267 3.3. Completeness theorem

268 In order to prove the theorem, we will use the following two lemmas, which
269 are proved in Appendix A.

Lemma 1. *Let $g : [0, \Delta t] \rightarrow \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, \Delta t]^2$,*

$$\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \leq \frac{K_g}{2} |t' - t|.$$

270 **Lemma 2.** *If there exists an admissible trajectory γ with δ -clearance in control
271 space, then there exists $\delta' < \delta$ and a neighboring admissible trajectory γ' with δ' -
272 clearance in control space whose acceleration and velocity never vanish, i.e. such
273 that $\|\ddot{\gamma}'\|$ (resp. $\|\dot{\gamma}'\|$) is always greater than some constant $\ddot{m} > 0$ (resp. $\dot{m} > 0$).*

274 We can now state our main theorem:

275 **Theorem 1.** *Consider a time-invariant differential system (2) with Lipschitz-continuous
276 f and full actuation over a compact set of admissible controls $\mathcal{U}_{adm}$. Suppose
277 that the kinodynamic planning problem between two states x_{init} and x_{goal} admits
278 a smooth Lipschitz solution $\gamma : [0, T] \rightarrow \mathcal{C}$ with δ -clearance in control space. A
279 randomized motion planner (Algorithm 1) using a second-order continuous inter-
280 polation is probabilistically complete.*

281 *Proof.* Let $\gamma : [0, T] \rightarrow \mathcal{C}, t \mapsto \gamma(t)$ denote a smooth Lipschitz admissible
282 trajectory from x_{init} to x_{goal} , and $u : [0, T] \rightarrow \mathcal{U}_{adm}$ its associated control tra-
283 jectory with δ -clearance in control space. Consider two states $x = (q, \dot{q})$ and
284 $x' = (q', \dot{q}')$. Given a sufficiently dense sampling of the state space, we suppose
285 that $dist_\gamma(x) \leq \rho$ and $dist_\gamma(x') \leq \rho$, where ρ is the radius of a tube around the
286 trajectory γ . Given the minimum velocity $\dot{m} > 0$ of the trajectory γ , we take ϵ as
287 $\dot{m}/2$ and enforce that $\rho < \dot{m}/3$. Assuming again a sufficiently dense sampling,
288 the two states can be close enough so that (1) $\|\Delta \dot{q}\| < \dot{m}/3$ and (2) $\|\Delta \dot{q}\| < 2\|\dot{q}\|$.
289 Then:

$$\begin{aligned} \|\dot{q}\| &\geq \|\dot{q} - \dot{\gamma}\| - \|\dot{\gamma}\| \geq \|\dot{\gamma}\| - \rho \geq \dot{m} - \rho \geq 2\dot{m}/3 \\ \|\dot{q}_{avg}\| &\geq \|\dot{q}_{avg} - \dot{q}\| - \|\dot{q}\| \geq \|\dot{q}\| - \|\Delta \dot{q}\|/2 \geq 2\dot{m}/3 - \dot{m}/6 = \epsilon. \end{aligned}$$

290 All states sampled in the tube of radius ρ therefore satisfy property P2. Consider
291 their corresponding time instants on the trajectory:

$$\begin{aligned} t &:= \arg \min_t \|(\gamma(t), \dot{\gamma}(t)) - x\|, \\ t' &:= \arg \min_t \|(\gamma(t), \dot{\gamma}(t)) - x'\|. \end{aligned}$$

Supposing without loss of generality that $t' > t$, we denote by $\Delta t = t' - t > 0$. Next, we enforce that $\rho/\Delta t = O(\Delta t)$, *i.e.* the radius ρ is at most quadratic in the time difference. Then,

$$\Delta q \cdot \dot{q}_{avg} = (\Delta q - \Delta \gamma) \cdot \dot{q}_{avg} + \Delta \gamma \cdot (\dot{q}_{avg} - \dot{\gamma}_{avg}) + \Delta \gamma \cdot \dot{\gamma}_{avg}.$$

The first two terms are $O(\rho) = O(\Delta t^2)$, while the third term $\Delta \gamma \cdot \dot{\gamma}_{avg} = \|\dot{\gamma}\|^2 \Delta t + O(\Delta t^2)$. Therefore,

$$\Delta q \cdot \dot{q}_{avg} > \dot{m}^2 \Delta t + r(\Delta t)$$

292 where the residual $r(\Delta t) = O(\Delta t^2)$. Relying again on dense sampling, we assume
 293 that Δt is small enough so that $r(\Delta t) < \dot{m}^2 \Delta t/2$. It follows that $\Delta q \cdot \dot{q}_{avg} > 0$,
 294 thus the pair (x, x') satisfies P1. Next,

$$|\sin(\Delta q, \dot{q}_{avg})| \leq |\sin(\Delta q, \Delta \gamma)| + |\sin(\Delta \gamma, \dot{\gamma})| + |\sin(\dot{\gamma}, \dot{q}_{avg})|.$$

295 Using the inequality $|\sin(\lambda v + u, v)| \leq \|u\|/\lambda \|v\|$ for any $\lambda > 0$, we can upper-
 296 bound the last two terms by

$$\begin{aligned} |\sin(\Delta \gamma, \dot{\gamma})| &= |\sin(\dot{\gamma} \Delta t + O(\Delta t^2), \dot{\gamma})| = O(\Delta t^2)/\|\dot{\gamma}\| \Delta t = O(\Delta t)/\dot{m}, \\ |\sin(\dot{\gamma}, \dot{q}_{avg})| &= |\sin(\dot{q}_{avg} + O(\rho), \dot{q}_{avg})| = O(\rho)/\|\dot{q}_{avg}\| = O(\rho)/\dot{m} \Delta t. \end{aligned}$$

Both right-hand side expressions are $O(\Delta t)$. Meanwhile, simple vector geometry shows that

$$\sin(\Delta q, \Delta \gamma) \leq \frac{\text{dist}_\gamma(x) + \text{dist}_\gamma(x')}{\|\Delta \gamma\|} \leq \frac{\rho}{\dot{m} \Delta t},$$

297 so that the first term of the inequality is $O(\Delta t)$ as well. Finally, note that $\|\Delta q\| =$
 298 $\|\Delta \gamma\| + O(\rho) = \|\dot{\gamma}\| \Delta t + O(\Delta t^2) \geq \dot{m} \Delta t + O(\Delta t^2)$. Since $\dot{m} > 0$, this implies
 299 that Δt is $O(\Delta q)$, from which we conclude that the sine above is $O(\Delta q)$ and the
 300 couple (x, x') satisfies P3. To summarize, when sampling is dense enough with the
 301 conditions that we have defined so far, we have seen how the states x and x' are
 302 locally consistent.

303 We can now define $\Delta t_{disc} = (\Delta q \cdot \dot{q}_{avg})/\|\dot{q}_{avg}\|^2 > 0$ the discrete time step
 304 between two locally consistent states x and x' . We further assume that $\rho/\Delta t_{disc} =$
 305 $O(\Delta t)$, a quadratic upper bound similar to that already enforced on ρ , which will
 306 prove useful thereafter.

307 Let $\gamma_{int} : [0, \Delta t] \rightarrow \mathcal{C}$ denote the result of the interpolation between x and
 308 x' . For $\tau \in [0, \Delta t]$, the torque required to follow the trajectory γ_{int} is $u_{int}(\tau) :=$
 309 $f(\gamma_{int}(\tau), \dot{\gamma}_{int}(\tau), \ddot{\gamma}_{int}(\tau))$. Since u has δ -clearance in control space,

$$\begin{aligned} |u_{int}(\tau)| &\leq |u_{int}(\tau) - u(t)| + |u(t)| \\ &\leq |f(\gamma_{int}(\tau), \dot{\gamma}_{int}(\tau), \ddot{\gamma}_{int}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))| + (1 - \delta) \tau_{max}, \end{aligned}$$

310 (As previously, vector inequalities are component-wise.) Let us denote by $|\widetilde{u}_{int}|$
 311 the first term of this inequality. We will now show that $|\widetilde{u}_{int}| = O(\Delta t) \rightarrow 0$ when
 312 $\Delta t \rightarrow 0$, and therefore that $|u_{int}(\tau)| \leq \tau_{max}$ for a small enough Δt (*i.e.* when
 313 sampling density is high enough). Let us first rewrite it as follows:

$$\begin{aligned}
 |\widetilde{u}_{int}| &= |f(\gamma_{int}(\tau), \dot{\gamma}_{int}(\tau), \ddot{\gamma}_{int}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))| \\
 &\leq \|f(\gamma_{int}(\tau), \dot{\gamma}_{int}(\tau), \ddot{\gamma}_{int}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))\|_{\infty} \\
 &\leq K_f \|(\gamma_{int}(\tau), \dot{\gamma}_{int}(\tau)) - (\gamma(t), \dot{\gamma}(t))\| + K_f \|\ddot{\gamma}_{int}(\tau) - \ddot{\gamma}(t)\| \\
 &\leq \underbrace{K_f [(\eta + \nu) \|\Delta x\| + \text{dist}_{\gamma}(x)]}_{\text{position-velocity term (PV)}} + \underbrace{K_f \left\| \frac{\|\dot{q}\|}{\|\Delta q\|} \Delta \dot{q} - \ddot{\gamma}(t) \right\|}_{\text{acceleration term (A)}}.
 \end{aligned}$$

314 The replacement of the norm $\|\cdot\|$ by $\|\cdot\|_{\infty}$ is possible because all norms of $\mathbb{R}^n$ are
 315 equivalent (a change in norm will be reflected by a different constant K_f). The
 316 transition from the second to the third row uses Lipschitz smoothness of f , as well
 317 as the triangular inequality to separate position-velocity and acceleration coordi-
 318 nates. The transition from the third to the fourth row relies on the two interpolation
 319 assumptions: local boundedness (yields the η factor in the distance term) and, since
 320 x and x' are locally consistent, convergence to the discrete-acceleration (yields the
 321 ν factor in the distance term, as well as the acceleration term—note how, for the
 322 latter, we used the expression from Equation (7), which is possible since x and x'
 323 are locally consistent).

The position-velocity term (PV) satisfies:

$$(\text{PV}) \leq (2\rho + \|\Delta\gamma\|)(\eta + \nu) + \rho \leq \frac{1}{2}K_{\gamma}(\eta + \nu)\Delta t + (1 + 2(\eta + \nu))\rho.$$

324 Since $\rho = O(\Delta t)$, we have $(\text{PV}) = O(\Delta t)$ and thus $|\widetilde{u}| \leq (\text{A}) + O(\Delta t)$. Next,
 325 the difference (A) can be bounded as:

$$\begin{aligned}
 (\text{A}) &\leq \underbrace{\left\| \Delta \dot{q} \frac{\|\dot{q}\|}{\|\Delta q\|} - \Delta \dot{\gamma} \frac{\|\dot{\gamma}(t)\|}{\|\Delta \gamma\|} \right\|}_{(\Delta)} + \underbrace{\left\| \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} \left\| \dot{\gamma}(t) \right\| - \frac{\|\Delta \gamma\|}{\Delta t} \right\|}_{(\text{A}')} \\
 &\quad + \underbrace{\left\| \frac{\Delta \dot{\gamma}}{\Delta t} - \ddot{\gamma}(t) \right\|}_{(\text{A}'')}.
 \end{aligned}$$

326 From Lemma 1, the two terms (A') and (A'') satisfy:

$$\begin{aligned}
 (\text{A}') &\leq \frac{K_{\dot{\gamma}}}{2} \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} \Delta t = O(\Delta t), \\
 (\text{A}'') &\leq \frac{K_{\dot{\gamma}}}{2} \Delta t = O(\Delta t),
 \end{aligned}$$

327 where the first upper bound $O(\Delta t)$ comes from the fact that $\frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} \underset{\Delta t \rightarrow 0}{\sim} \Delta t$. We
 328 now have $|\tilde{u}| \leq (\Delta) + O(\Delta t)$. The term (Δ) can be seen as the deviation between
 329 the discrete accelerations of γ_{int} and γ . Let us decompose it in terms of norm and
 330 angular deviations:

$$\begin{aligned}
 (\Delta) &\leq \left\| \left(\frac{\Delta \dot{\gamma}}{\|\Delta \dot{\gamma}\|} - \frac{\Delta \dot{q}}{\|\Delta \dot{q}\|} \right) \frac{\|\dot{\gamma}\| \|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} + \frac{\Delta \dot{q}}{\|\Delta \dot{q}\|} \left(\frac{\|\Delta \dot{\gamma}\| \|\dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\| \|\dot{q}\|}{\|\Delta q\|} \right) \right\| \\
 &\leq \underbrace{2 \frac{\|\dot{\gamma}\| \|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} (1 - \cos(\Delta \dot{q}, \Delta \dot{\gamma}))}_{\text{angular deviation term } (\theta)} + \underbrace{\left| \frac{\|\dot{\gamma}\| \|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\| \|\dot{q}\|}{\|\Delta q\|} \right|}_{\text{norm deviation term (N)}}
 \end{aligned}$$

The factor $\frac{2\|\dot{\gamma}\|\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|}$ before (θ) is $O(1)$ when $\Delta t \rightarrow 0$, while simple vector geometry then shows that

$$\sin(\Delta \dot{q}, \Delta \dot{\gamma}) \leq \frac{\text{dist}_\gamma(x) + \text{dist}_\gamma(x')}{\|\Delta \dot{\gamma}\|} \leq \frac{\rho}{\ddot{m} \Delta t},$$

331 where $\ddot{m} := \min_t \|\ddot{\gamma}(t)\|$. From Lemma 2, we can assume this minimum acceleration
 332 to be strictly positive. Then, it follows from $\rho = O(\Delta t^2)$ that the sine above
 333 is $O(\Delta t)$. Recalling the fact that $1 - \cos \theta < \sin \theta$ for any $\theta \in [0, \pi/2]$, we have
 334 $(\theta) = O(\Delta t)$.

335 Finally,

$$\begin{aligned}
 (\text{N}) &\leq \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} \left| \|\dot{\gamma}\| - \|\dot{q}\| + \|\dot{q}\| \left| \frac{\|\Delta \dot{\gamma}\|}{\|\Delta \gamma\|} - \frac{\|\Delta \dot{q}\|}{\|\Delta q\|} \right| \right| \\
 &\leq O(\Delta t \cdot \rho) + \|\dot{q}\| \frac{(\|\Delta q\| + \|\Delta \dot{q}\|) O(\rho)}{\|\Delta q\| (\|\Delta q\| + O(\rho))} \\
 &\leq O(\Delta t \cdot \rho) + \frac{\|\dot{q}\| \rho}{\|\Delta q\| + O(\rho)} + \frac{\|\dot{q}\| \|\Delta \dot{q}\|}{\|\Delta q\|} \frac{O(\rho)}{\|\Delta q\| + O(\rho)}
 \end{aligned}$$

336 Where we used the fact that $\|\Delta \gamma\| \leq \text{dist}_\gamma(x) + \|\Delta q\| + \text{dist}_\gamma(x') = \|\Delta q\| + O(\rho)$,
 337 and similarly for $\|\Delta \dot{\gamma}\|$. Because $\|\Delta q\| = \|\dot{q}\| \Delta t_{disc} + O(\Delta t_{disc}^2)$ and $\rho / \Delta t_{disc} =$
 338 $O(\Delta t)$, the last two fractions are $O(\Delta t)$, so our last term $(\text{N}) = O(\Delta t)$.

339 Overall, we have derived an upper bound $|u(\tau)| \leq (1 - \delta) \tau_{max} + O(\Delta t)$.
 340 As a consequence, there exists a constant $\delta t > 0$ such that, whenever $\Delta t \leq$
 341 δt , interpolated torques satisfy $|u| \leq \tau_{max}$ and the interpolated trajectory $\gamma_{int} =$
 342 $\text{INTERPOLATE}(x, x')$ is admissible. Note that the constant δt is uniform, in the
 343 sense that it does not depend on the index t on the trajectory.

344 *Conclusion of the Proof.* We have effectively constructed the attraction sequence
 345 conjectured in [6]. We can now conclude the proof similarly to the strategy sketched
 346 in that paper. Let us denote by $\mathcal{B}_t := \mathcal{B}((\gamma, \dot{\gamma})(t), \delta\rho)$, the ball of radius $\delta\rho$ cen-
 347 tered on $(\gamma, \dot{\gamma})(t) \in \mathcal{X}$, where $\delta\rho = O(\delta t^2)$ as before. Suppose that the roadmap
 348 contains a state $x \in \mathcal{B}_t$, and let $t' := t + \delta t$. If the planner samples a state $x' \in \mathcal{B}_{t'}$,
 349 the interpolation between x and x' will be successful and x' will be added to the
 350 roadmap. Since the volume of $\mathcal{B}_{t'}$ is non-zero, the event $\{\text{SAMPLE}(\mathcal{X}_{\text{free}}) \in \mathcal{B}_{t'}\}$
 351 will happen with probability one as the number of extensions goes to infinity. At
 352 the initialization of the planner, the roadmap is reduced to $x_{\text{init}} = (\gamma(0), \dot{\gamma}(0))$.
 353 Therefore, using the property above, by induction on the number of time steps δt ,
 354 the last state $x_{\text{goal}} = (\gamma(T), \dot{\gamma}(T))$ will be eventually added to the roadmap with
 355 probability one, and the planner will find an admissible trajectory connecting x_{init}
 356 to x_{goal} . ■

357 4. Completeness and state-based steering in practice

358 Shkolnik et al. [18] showed how RRTs could not be directly applied to kinody-
 359 namic planning due to their poor expansion rate at the boundaries of the roadmap.
 360 They illustrated this phenomenon on the planning problem of swinging up a (sin-
 361 gle) pendulum vertically against gravity. Let us consider the same system, *i.e.* the
 362 1-DOF single pendulum depicted in Figure 2 (A), with length $l = 20$ cm and mass
 363 $m = 8$ kg. It satisfies the system assumptions of Theorem 1 *a fortiori*, as we saw
 364 that they apply to the double pendulum.

365 We assume that the single actuator of the pendulum, corresponding to the joint
 366 angle θ in Figure 2, has limited actuation power: $|\tau| \leq \tau_{\text{max}}$. The static equilibrium
 367 of the system requiring the most torque is given at $\theta = \pm\pi/2$ with $\tau = \frac{1}{2}lmg \approx$
 368 7.84 Nm. Therefore, when $\tau_{\text{max}} < 7.84$ Nm, it is impossible for the system to
 369 raise upright directly, and the pendulum rather needs to swing back and forth to
 370 accumulate kinetic energy before it can swing up. For any $\tau_{\text{max}} > 0$, the pendulum
 371 can achieve the swingup in a finite number of swings N , with $N \rightarrow \infty$ as $\tau_{\text{max}} \rightarrow$
 372 0.

373 4.1. Bezier interpolation

A common solution [30, 31, 32] to connect two states $(q, \dot{q})$ and $(q', \dot{q}')$ is the
 cubic Bezier curve (also called “Hermit curve”) which is the quadratic function
 $B(t)$ such that $B(0) = q$, $\dot{B}(0) = \dot{q}$, $B(T) = q'$ and $\dot{B}(T) = \dot{q}'$, where T is the
 fixed duration of the interpolated trajectory. Its expression is given by:

$$B(t) = \frac{-2\Delta q + T(\dot{q} + \dot{q}')}{T^3} t^3 + \frac{3\Delta q - 2\dot{q} - \dot{q}'}{T^2} t^2 + \dot{q}t + q$$

374 This interpolation is straightforward to implement, however it does not verify our
 375 Assumption 5, as for instance

$$\ddot{B}(0) = \frac{6\Delta q - 4\dot{q} - 2\dot{q}'}{T^2} \xrightarrow{\Delta x \rightarrow 0} \frac{-6\dot{q}}{T^2} \neq 0. \quad (9)$$

376 Our proof of completeness does not apply to such interpolators: even though a
 377 feasible trajectory is sampled as closely as possible ($\Delta x \rightarrow 0$), the interpolated
 378 acceleration will *not* approximate the smooth acceleration underlying the feasible
 379 trajectory.

380 **Proposition 1.** *A randomized motion planner interpolating pendulum trajectories*
 381 *by Bezier curves with a fixed duration T cannot find non-quasi static solutions by*
 382 *increasing sampling density.*

Proof. When actuation power decreases, the pendulum needs to store kinetic energy in order to swing up, which implies that all swingup trajectories go through velocities $|\dot{\theta}| > \dot{\theta}_{\text{swingup}}(\tau_{\text{max}})$. The function $\dot{\theta}_{\text{swingup}}$ increases to a positive limit $\dot{\theta}_{\text{swingup}}^{\text{lim}}$ as $\tau_{\text{max}} \rightarrow 0$, where $\dot{\theta}_{\text{swingup}}^{\text{lim}} > \sqrt{8g/l}$ from energetic considerations.⁶ Yet, feasible accelerations are also bound by $|\ddot{\theta}| \leq K\tau_{\text{max}}$ for some constant $K > 0$. Combining both observations in (9) yields:

$$K\tau_{\text{max}} \geq 6\frac{|\dot{\theta}|}{T^2} > 6\frac{\dot{\theta}_{\text{swingup}}(\tau_{\text{max}})}{T^2} \Rightarrow \dot{\theta}_{\text{swingup}}(\tau_{\text{max}}) \leq \frac{KT^2}{6}\tau_{\text{max}}.$$

383 Since the planner uses a constant T and $\dot{\theta}_{\text{swingup}}$ increases to $\dot{\theta}_{\text{swingup}}^{\text{lim}} > \sqrt{8g/l}$
 384 when τ_{max} decreases to 0, this inequality cannot be satisfied for arbitrary small ac-
 385 tuation power τ_{max} . Hence, even with an arbitrarily high sampling density around
 386 a feasible trajectory $\gamma(t)$, the planner will not be able to reconstruct a feasible ap-
 387 proximation $\gamma_{\text{int}}(t)$. $\square$

388 4.2. Second-order continuous interpolation

389 Since the system has only one degree of freedom, one can interpolate trajec-
 390 tories that comply with our Assumption 5 using constant accelerations by selecting

⁶ The expression $\dot{\theta} = \sqrt{8g/l}$ corresponds to the kinetic energy $\frac{1}{4}ml\dot{\theta}^2 = mgl$, the latter being the (potential) energy of the system at rest in the upward equilibrium. During a successful last swing, the kinetic energy at $\theta = 0$ is $\frac{1}{4}ml\dot{\theta}_{\text{swingup}}^2 + \mathcal{W}_g + \mathcal{W}_\tau = mgl$, with $\mathcal{W}_g < 0$ the work of gravity and $\mathcal{W}_\tau$ the work of actuation forces between $\theta = 0$ and $\theta = \pi$. The work $\mathcal{W}_\tau$ vanishes when $\tau_{\text{max}} \rightarrow 0$.

a suitable trajectory duration. We propose here the example previously given in Equation (8):

$$\begin{aligned} \gamma_s : [0, \Delta t] &\rightarrow] - \pi, \pi] \\ t &\mapsto \gamma_s(t) = \frac{\Delta \dot{q}}{2\Delta t} t^2 + t\dot{q} + q. \end{aligned} \quad (10)$$

We showed in Section 3.2 how this interpolation satisfies all Assumptions 3, 4 and 5. Note that it only applies to single-DOF systems. For multi-DOF systems, the correct duration Δt_C used to transfer from one state to another is different for each DOF, hence constant accelerations cannot be used. One can then apply optimization techniques [22, 9] or use a richer family of curves such as piecewise linear-quadratic segments [33].

4.3. Comparison in simulations

According to Theorem 1 and our previous discussion, a randomized planner based on Bezier interpolation is not expected to be probabilistically complete as $\tau_{\max} \rightarrow 0$, while the same planner using the SOC interpolation will be complete at any rate. We asserted this statement in simulations of the pendulum with RRT [34].

Our implementation of RRT is that described in Algorithm 1, with the addition of the *steer-to-goal* heuristic: every $m = 100$ steps, the planner tries to steer to x_{goal} rather than x_{rand} . This extra step speeds up convergence when the system reaches the vicinity of the goal area. We use uniform random sampling for $\text{SAMPLE}(S)$, while for $\text{PARENTS}(x', V)$ returns the $k = 10$ nearest neighbors of x' in the roadmap V . All the source code used in these experiments can be accessed at [35].

We compared the performance of RRT with the Bezier and SOC interpolations, all other parameters being the same, on a single pendulum for a range of maximum torque values. For a meaningful comparison, we set up a benchmark where both RRTs are run on the same random samples x_{rand} . The benchmark consists of four settings $\tau_{\max} \in \{5, 6, 7, 8\}$ Nm, each setting comprising 20 runs in total, each run starting from a random state $x = (\theta, \dot{\theta})$ with $\theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$ (we are interested in swingup trajectories). We measure the performance of a planner over time by its distance to goal, *i.e.* the distance from the closest roadmap node to the goal, where our distance function is:

$$d(x_0, x_1) = \frac{\sqrt{1 - \cos(\theta_0 - \theta_1)}}{2\sqrt{2}} + \frac{|\dot{\theta}_0 - \dot{\theta}_1|}{4\dot{\theta}_{\max}}, \quad (11)$$

Figure 3 shows the results of the benchmark, with green and red trajectories corresponding respectively to the Bezier and SOC RRTs. Vertical lines mark times at which a planner finds a solution. There are no green vertical line for $\tau_{\max} = 5$ Nm as no RRT using Bezier interpolation succeeded in reaching the goal, which concurs with Proposition 1. The data generated in this benchmark is released at:

Figure 3: Results of the benchmark comparing the performance of RRT on a kinodynamic planning task (swinging up a pendulum) when using either Bezier (in green) or Second-Order Continuous (in red) interpolation functions. The benchmark consists of 20 runs for each setting $\tau_{\max} \in \{5, 6, 7, 8\}$ Nm. Each run extending an RRT-Bezier and an RRT-SOC on the same random samples. The curves above show the minimum distance-to-goal of roadmaps over time. Average values over all runs of the benchmark are drawn in bold. The area between worst and best cases are filled in light green and red for Bezier and SOC, respectively. Red (resp. green) vertical lines indicate times when an RRT-SOC (resp. RRT-Bezier) successfully connected to the goal state. When it is possible to reach the goal in a single swing ($\tau_{\max} \geq 7$ Nm), RRT-Bezier finds solutions faster. However, for $\tau_{\max} = 6$ Nm, more than half RRT-Bezier instances fail to find a solution after 10^5 iterations, and all of them fail for $\tau_{\max} = 5$ Nm. Meanwhile, the RRT-SOC variant with suitable interpolations always finds solutions (Theorem 1).

425 `https://scaron.info/files/pendulum-benchmark.tar.bz2`
426 (File size: 229 MB, SHA1 sum: 71d4de12980a103c2496a25e93c703321d9069bf).
427 It includes all generated random samples, so that results can be reproduced using
428 the source code [35].

429 To understand where RRT-Bezier failures stem from, let us compare the im-
430 pact of the two interpolation functions on a particular run. In the roadmap depicted
431 in Figure 4, the RRT-SOC combo found a four-swing solution after 8,800 RRT
432 extensions. Meanwhile, even after more than 100,000 RRT extensions, the RRT-
433 Bezier combo does not find any solution. Its roadmap, depicted in Figure 5, counts
434 26,663 roughly distributed between two zones. The first one is a dense, diamond-
435 shape area near the downward equilibrium $\theta = 0$. It corresponds to states that are
436 straightforward to connect by Bezier interpolation, and as expected from Proposi-
437 tion 1, velocities $\dot{\theta}$ in this area decrease sharply with θ . The second one consists of
438 two cones directed towards the goal. Both areas exhibit a higher density near the
439 axis $\dot{\theta} = 0$, which is also consistent with Proposition 1.

440 The comparison of the two roadmaps is clear: with a second-order continuous
441 interpolation, the RRT-SOC planner leverages additional sampling into exploration
442 of the state space. Conversely, RRT-Bezier lacks this property (Proposition 1), and
443 its roadmap stays confined to a subset of the pendulum’s reachable space.

444 5. Conclusion

445 In this paper, we provided the first “operational” proof of probabilistic com-
446 pleteness for a large class of randomized kinodynamic planners, namely those that
447 interpolate state-space trajectories. We observed that an important ingredient for
448 completeness is the “continuity” of the interpolation procedure, which we char-
449 acterized by the *second-order continuity* (SOC) property. In particular, we found
450 in simulation experiments that this property is critical to planner performances: a
451 standard RRT with second-order continuous interpolation has no difficulty finding
452 swingup trajectories for a low-torque pendulum, while the same RRT with Bezier
453 interpolation (which are not SOC) could not find any solution. This experimentally
454 confirms our completeness theorem and suggests that second-order continuity is an
455 important design guideline for kinodynamic planners with state-based steering.

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Figure 4: Phase-space portrait of a roadmap constructed by RRT using the second-order continuous (SOC) interpolation. In this run, the planner found a successful trajectory (thick red curve) after 8,800 extensions. This planner is probabilistically complete (Theorem 1) thanks to the fact that SOC curves satisfy Assumption 5.

Figure 5: Roadmap constructed by RRT after 100,000 extensions using the Bezier interpolation. Reachable states are distributed in two major areas: a central, diamond shape corresponding to the states that the planner can connect at any rate, and two cones directed towards the goal ($\theta = \pi$ or $\theta = -\pi$). Even after several days of computations, this planner could not find a successful motion plan in this run. Our completeness theorem does not apply to this planner because Bezier curves do not comply with discrete accelerations (Assumption 5).

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552 Appendix A. Proofs of the lemmas

Lemma 1. *Let $g : [0, \Delta t] \rightarrow \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, \Delta t]^2$,*

$$\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \leq \frac{K_g}{2} |t' - t|.$$

553 *Proof.* For $t' > t$,

$$\begin{aligned}
\left\| \dot{g}(t) - \frac{g(t') - g(t)}{t' - t} \right\| &\leq \frac{1}{t' - t} \left\| \int_t^{t'} (\dot{g}(t) - \dot{g}(w)) \, dw \right\| \\
&\leq \frac{1}{t' - t} \int_t^{t'} \|\dot{g}(t) - \dot{g}(w)\| \, dw \\
&\leq \frac{K_g}{t' - t} \int_t^{t'} |t - w| \, dw \\
&\leq \frac{K_g}{2} (t' - t). \quad \square
\end{aligned}$$

554 **Lemma 2.** *If there exists an admissible trajectory γ with δ -clearance in control*
555 *space, then there exists $\delta' < \delta$ and a neighboring admissible trajectory γ' with δ' -*
556 *clearance in control space whose acceleration and velocity never vanish, i.e. such*
557 *that $\|\ddot{\gamma}'\|$ (resp. $\|\dot{\gamma}'\|$) is always greater than some constant $\ddot{m} > 0$ (resp. $\dot{m} > 0$).*

558 *Proof.* The main idea behind the proof is to leverage the non-empty interior $\mathcal{U}_{adm}$
559 to add arbitrarily small controls around all zeros of the trajectory γ . First, imagine
560 that there is a time interval $[t, t']$ on which $\ddot{\gamma} \equiv 0$. Then, suffices to add a wavelet
561 function $\delta\ddot{\gamma}_i$ of arbitrary small amplitude $\delta\ddot{q}_i$ and zero integral over $[t, t']$ to gener-
562 ate a new trajectory $\ddot{\gamma} + \delta\ddot{\gamma}$ where the acceleration cancels on at most a discrete
563 number of time instants. Adding accelerations $\delta\ddot{\gamma}_i$ directly is possible thanks to
564 full actuation. The amplitude of the wavelet $\delta\ddot{\gamma}_i$ needs to be sufficiently small to
565 achieve δ' -clearance in $\mathcal{U}_{adm}$ for some $\delta' \leq \delta$ (a solution always exists thanks to
566 the openness and non-emptiness of its interior).

567 Suppose now that the roots of $\ddot{\gamma}$ form a discrete set $\{t_0, t_1, \dots, t_m\}$. Let t_0
568 be one of the first of these roots, and let $[t, t']$ denote a neighborhood of t_0 . We
569 now add two wavelet functions $\delta\ddot{\gamma}_i$ and $\delta\ddot{\gamma}_j$ of zero integral over $[t, t']$ and arbitrary
570 small amplitude to two coordinates i and j , but this time enforcing that the sum
571 of the two wavelets satisfies $|\delta\ddot{\gamma}_i + \delta\ddot{\gamma}_j| \geq \epsilon_{ij} > 0$. This method ensures that
572 the root t_0 is eliminated (either $\ddot{\gamma}_i(t_0) \neq 0$ or $\ddot{\gamma}_j(t_0) \neq 0$) while the positivity of
573 $|\delta\ddot{\gamma}_i + \delta\ddot{\gamma}_j|$ ensures that no new root is created at the zero-crossing points of $\ddot{\gamma}_i$ or
574 $\ddot{\gamma}_j$. We conclude by iterating the process on the finite set of roots.

575 Once $\|\ddot{\gamma}\| \geq \ddot{m}$ for some $\ddot{m} > 0$, the velocity $\dot{\gamma}$ cannot be uniformly zero over
576 any interval. Let us suppose that $\dot{\gamma}(t_0) = 0$ for some time t_0 , and take an interval
577 $[t, t']$ around it. Again, we add two wavelet functions $\delta\dot{\gamma}_i$ and $\delta\dot{\gamma}_j$ of zero integral
578 over $[t, t']$, arbitrary small amplitude $\delta a_i, \delta a_j$ and such that $|\delta\dot{\gamma}_i + \delta\dot{\gamma}_j| \geq \epsilon_{ij} > 0$.
579 (An example of such function is $\dot{\gamma}_i(w) = \delta a_i(w) \sin\left(\pi \frac{w-t}{t'-t}\right)$.) For $\|\delta\dot{a}_i\|$ and
580 $\|\delta\dot{a}_j\|$ small enough, the wavelets can be added to the trajectory γ while enforcing
581 δ' -clearance for some $\delta' < \delta$, which concludes the argument. $\square$