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Variance Analysis of Robust State Estimation in Power System using Influence Function

Weng Khuen Ho^a, Tengpeng Chen^{b,*}, Keck Voon Ling^b, Lu Sun^a

^aDepartment of Electrical and Computer Engineering, National University of Singapore, Singapore

^bSchool of Electrical and Electronic Engineering, Nanyang Technological University, Singapore

Abstract

An analytical equation is derived using influence function approximation to calculate the variance of the state estimate for traditional robust state estimators such as the Quadratic-Constant, Quadratic-Linear, Square-Root, Schweppe-Huber Generalized-M and Multiple-Segment estimator. The equation gives insights into the precision of the estimation. Using the equation, the variance of a state estimate can be expressed as a function of measurement noise variances enabling the selection of sensors for a specified estimator precision. It can also be used to search for the optimum estimator parameters to give the minimum sum of variances. The well-known Weighted-Least-Squares variance formula is a special case of the equation and simulations on the IEEE 14-bus system are given to show the usefulness of the equation.

Keywords: Robust State Estimation, M-estimator, Influence Function, IEEE 14-Bus System.

Nomenclature

The notations used in the paper are summarized below for easy reference.

x	True state vector
$\hat{x}$	Estimated state vector
V_i	Magnitude of the voltage at bus i
δ_i	Phase angle at bus i
V_i^r	Real part of the voltage phasor at bus i
V_i^{im}	Imaginary part of the voltage phasor at bus i
z	Measurements from Phasor Measurement Units (PMUs)
y	Measurements from the SCADA system
$\epsilon(k)$	Vector of measurement noise at time index k
$\epsilon(k)^T$	Transpose of noise vector $\epsilon(k)$

*Corresponding author.

Email address: chen0887@e.ntu.edu.sg (Tengpeng Chen)

ε	Vector of measurement noise over a time interval, $[\epsilon(1)^T, \dots, \epsilon(N)^T]^T$
P_{ij}	Real power flow from bus i to j
15 Q_{ij}	Reactive power flow from bus i to j
e_i	Measurement residual i
$\rho(e_i)$	Cost function of e_i
J	Total cost function
$h_i(x)$	Nonlinear function relating the state vector x to measurement i
20 $f_i(\epsilon_i)$	Probability density function of ϵ_i
H	Jacobian matrix between measurements and states
H_i	The row i of matrix H
W_i	Weighting factor for measurement i
Ψ	Derivative of J wrt $\hat{x}$
25 $\text{IF}(\cdot)$	Influence function
$\text{erf}(\cdot)$	Gauss error function
$\beta(\cdot)$	Beta function
SV	Sum of variances
m	Number of measurements at time k
30 n	Number of states
N	Number of sets of measurements
i	Measurement index
k	Time index
a_i	Estimator first threshold parameter for e_i
35 b_i	Estimator second threshold parameter for e_i
r_i	Estimator third threshold parameter for e_i
σ_i	Standard deviation of ϵ_i
Ω, Λ	Diagonal matrix
v_i, α_i	t -distribution parameter
40 $\bar{x}$	Operating point

1. Introduction

The Gaussian noise assumption is commonly made in power system state estimation problems [1, 2]. However, this assumption is only an approximation to reality. For example, transient data in steady-state measurement, instrument failure, human error or model nonlinearity can generate non-Gaussian measurement errors [3, 4]. Outliers that are far away from the expected Gaussian distribution function can give rise to misleading estimation results [5]. Robust estimators with non-quadratic cost functions such as Quadratic-Constant (QC), Quadratic-Linear (QL), Square Root (SR), Schweppe-Huber Generalized-M (SHGM) and Multiple Segment (MS) have been introduced to solve the outliers problem in power system state estimation. Besides research papers, these robust estimators are also discussed in books [1, 6, 7] and surveys [8, 9, 10, 11, 12].

Recent technological advancement has enabled accurate monitoring of power system. For example, the introduction of Phasor Measurement Unit (PMU) [13, 14, 15, 16, 17] makes possible the measurements of voltages and currents in a synchronized manner with respect to the Global Positioning System clock. The PMUs can provide measurements with precise time synchronization, for example, 30 samples per second [18]. With PMUs, state estimation can now be formulated as a linear instead of a nonlinear problem as traditionally done. Robust estimation has also been used in conjunction with PMU recently [13, 19, 20].

The main contribution of this paper is the derivation of an analytical variance equation. The equation uses Influence Function (IF) to approximately calculate the variances of robust estimators such as QC, QL, SR, SHGM and MS. IF has previously been used as an analytical tool in robust statistics [21, 22] and filter design with non-Gaussian noise assumption [3].

The derived analytical variance equation is useful. Firstly, it can be used to express the variance of a state estimate as a function of measurement variances enabling the selection of sensors for specified estimation precision. Secondly, it can be used to design an optimal estimator. Finally, although numerical methods can also be used to find variance, the equation derived in this paper as a mathematical function is more insightful than just a numerical answer.

The paper is organized as follows. The robust state estimation problem is formulated in Section 2 and the IF analysis is discussed in Section 3. Simulation examples and conclusions are given in Sections 4 and 5 respectively.

2. Robust State Estimation

Robust state estimation for power systems has been discussed in the literature [1, 6, 8, 9, 10, 11, 12, 13, 19, 20, 23, 24, 25]. This section gives the equations necessary for the derivation of the results in this paper.

2.1. With Phasor Measurement Units (PMUs)

The rectangular coordinates are used in this paper when the measurements are collected from PMUs. With PMU, measurement $z(k)$ in voltages and currents can be expressed as a linear function of the state x [13, 26]:

$$z(k) = Hx(k) + \epsilon(k) \quad (1)$$

where

$$\begin{aligned} H &= [H_1^T, H_2^T, \dots, H_m^T]^T \\ x(k) &= [x_1(k), x_2(k), \dots, x_n(k)]^T \\ z(k) &= [z_1(k), z_2(k), \dots, z_m(k)]^T \\ \epsilon(k) &= [\epsilon_1(k), \epsilon_2(k), \dots, \epsilon_m(k)]^T \end{aligned}$$

The set of measurement $z_i(k)$, $i = 1, \dots, m$, is taken at each time instance $k = 1, \dots, N$. A traditional power system may be considered as a quasi-static system [18, 27] because load demands change slowly and hence the state changes slowly. The sampling time of PMU measurement is usually in the order of milliseconds while the estimates are usually updated once every few minutes if the measurements are collected from SCADA [28]. In this paper it is assumed that the PMUs collect the measurements every 30 milliseconds [18] and the states are estimated for every N sets of measurements. During this interval, it is assumed that the system state is constant, i.e. $\hat{x}(1) = \hat{x}(2) = \dots = \hat{x}(N) = \hat{x}$. $\epsilon_i(k)$ is the measurement noise.

Given N sets of m measurements, the estimated state vector $\hat{x}$ can be obtained by minimizing the following cost function:

$$J = \sum_{i=1}^m \sum_{k=1}^N \rho(e_i(k)) \quad (2)$$

where in general $\rho(e_i(k))$ is a nonlinear function. From (1) the measurement residual is given as

$$e_i(k) = z_i(k) - H_i \hat{x} \quad (3)$$

where $\hat{x}$ is the estimated state.

2.2. Without Phasor Measurement Units

In a traditional power system where PMU is not used, measurement $y(k)$ containing power injection and power flow is collected every five seconds or longer [18], and is a nonlinear function of the state $x(k)$:

$$y(k) = h(x(k)) + \epsilon(k) \quad (4)$$

where $x(k)$ and $\epsilon(k)$ are already defined in (1) and

$$\begin{aligned} h(x(k)) &= [h_1(x(k)), h_2(x(k)), \dots, h_m(x(k))]^T \\ y(k) &= [y_1(k), y_2(k), \dots, y_m(k)]^T \end{aligned}$$

The first-order Taylor series expansion about an operating point $\bar{x}$ is used to approximate the nonlinear function $h(x)$ in (4) to give

$$y(k) = h(\bar{x}) + H(x(k) - \bar{x}) + \epsilon(k) \quad (5)$$

where the Jacobian matrix

$$H = \left. \frac{\partial h(x)}{\partial x} \right|_{x=\bar{x}} \quad (6)$$

From (5) the measurement residual is given by

$$e_i(k) = y_i(k) - h_i(\bar{x}) + H_i\bar{x} - H_i\hat{x}$$

Let $z_i(k) = y_i(k) - h_i(\bar{x}) + H_i\bar{x}$ then (3) can be used to give the measurement residual. In this case, $\hat{x}$ can also be obtained by minimizing the cost function J in (2).

95 2.3. State Estimation

The minimization of the cost function (2) using the Newton's method of iteration or the iteratively re-weighted least squares method is well documented in the literature [1, 7]. The Newton's method approximates J at each iteration by a quadratic function about the previous estimate of x . The iteratively re-weighted least squares algorithm avoid the second derivative required in the Newton's method. This paper simply uses the iteratively re-weighted least squares algorithm given in [1, 7].

Differentiating the cost function (2) wrt $\hat{x}$ gives

$$\frac{\partial J}{\partial \hat{x}} = \frac{\partial J}{\partial e_i(k)} \frac{\partial e_i(k)}{\partial \hat{x}} = \sum_{i=1}^m \sum_{k=1}^N \frac{\partial \rho(e_i(k))}{\partial e_i(k)} \frac{1}{e_i(k)} e_i(k) \frac{\partial e_i(k)}{\partial \hat{x}} = - \sum_{i=1}^m \sum_{k=1}^N W_i(k) e_i(k) H_i^T \quad (7)$$

where

$$\begin{aligned} W_i(k) &= \frac{\partial \rho(e_i(k))}{\partial e_i(k)} \frac{1}{e_i(k)} \\ \frac{\partial e_i(k)}{\partial \hat{x}} &= -H_i^T \end{aligned} \quad (8)$$

Using (3), (7) can be written as

$$- \sum_{i=1}^m \sum_{k=1}^N W_i(k) (z_i(k) - H_i\hat{x}) H_i^T = -\bar{H}^T W (Z - \bar{H}\hat{x}) \quad (9)$$

$$= -\bar{H}^T W E \quad (10)$$

$$\triangleq \Psi(E)$$

where

$$\begin{aligned} \bar{H} &= \begin{bmatrix} H^T & \dots & H^T \end{bmatrix}^T \in \mathbb{R}^{Nm \times n} \\ Z &= \begin{bmatrix} z(1)^T & \dots & z(N)^T \end{bmatrix}^T \in \mathbb{R}^{Nm} \\ E &= \begin{bmatrix} e(1)^T & \dots & e(N)^T \end{bmatrix}^T \in \mathbb{R}^{Nm} \\ W &= \text{diag}(W_1(1), \dots, W_m(1), \dots, W_1(N), \dots, W_m(N)) \in \mathbb{R}^{Nm \times Nm} \end{aligned}$$

105 To minimize the cost function (2), set $\Psi(E) = 0$ in (9). The estimated results are given as

$$\hat{x} = (\bar{H}^T W \bar{H})^{-1} \bar{H}^T W Z \quad (11)$$

where the matrix $\bar{H}^T W \bar{H}$ in (11) is an invertible matrix since it is assumed that the system is observable. Using (6) and (8), the matrices H and W are updated and (11) is solved iteratively until the difference between the current and previous iteration for $\hat{x}$ is less than a specified tolerance [1, 7]. The diagonal matrix W for the MS, QC, QL, SR and SHGM estimator can be obtained as follows.

110 2.4. The Multiple-Segment (MS) Estimator

The MS estimator is sometimes also known as the Hampel estimator [22]. To obtain the matrix W for the MS estimator in (2), set

$$\rho(e_i(k)) = \begin{cases} \frac{(e_i(k))^2}{2\sigma_i^2} & |e_i(k)| \leq a_i\sigma_i \\ \frac{a_i|e_i(k)|}{\sigma_i} - \frac{a_i^2}{2} & a_i\sigma_i < |e_i(k)| \leq b_i\sigma_i \\ -\frac{a_i(r_i\sigma_i - |e_i(k)|)^2}{2(r_i - b_i)\sigma_i^2} + \frac{1}{2}a_i r_i + \frac{1}{2}a_i b_i - \frac{1}{2}a_i^2 & b_i\sigma_i < |e_i(k)| \leq r_i\sigma_i \\ \frac{1}{2}(a_i r_i + a_i b_i - a_i^2) & r_i\sigma_i < |e_i(k)| \end{cases} \quad (12)$$

where a_i, b_i and r_i are thresholds selected by the user and σ_i is the standard deviation of noise ϵ_i . Note that $a_i < b_i < r_i$.

115 Differentiating $\rho(e_i(k))$ wrt $e_i(k)$ and substituting into (8) gives

$$W_i(k) = \begin{cases} \frac{1}{\sigma_i^2} & |e_i(k)| \leq a_i\sigma_i \\ \frac{a_i}{\sigma_i|e_i(k)|} & a_i\sigma_i < |e_i(k)| \leq b_i\sigma_i \\ \frac{a_i(r_i\sigma_i - |e_i(k)|)}{(r_i - b_i)\sigma_i^2|e_i(k)|} & b_i\sigma_i < |e_i(k)| \leq r_i\sigma_i \\ 0 & r_i\sigma_i < |e_i(k)| \end{cases} \quad (13)$$

Note that $a_i < b_i < r_i$. The QC, QL and WLS estimators are special cases of MS estimator. The MS estimator reduces to the QC estimator when $b_i \rightarrow a_i, r_i \rightarrow a_i$, the QL estimator when $b_i \rightarrow \infty$ and the WLS estimator when $a_i \rightarrow \infty$ [1].

2.5. The Square-Root (SR) Estimator

120 According to [1], to obtain the matrix W for the SR estimator in (2), set

$$\rho(e_i(k)) = \begin{cases} \frac{(e_i(k))^2}{2\sigma_i^2} & |e_i(k)| \leq a_i\sigma_i \\ 2a_i^{3/2}\sqrt{\frac{|e_i(k)|}{\sigma_i}} - \frac{3}{2}a_i^2 & \text{otherwise} \end{cases} \quad (14)$$

Differentiating $\rho(e_i(k))$ wrt $e_i(k)$ and substituting into (8) gives

$$W_i(k) = \begin{cases} \frac{1}{\sigma_i^2} & |e_i(k)| \leq a_i\sigma_i \\ \sqrt{\frac{a_i^3}{\sigma_i|e_i(k)|^3}} & \text{otherwise} \end{cases} \quad (15)$$

2.6. The Schweppe-Huber Generalized-M (SHGM) Estimator

According to [29], the function $\rho(e_i(k))$ for the SHGM estimator is given as

$$\rho(e_i(k)) = \begin{cases} \frac{(e_i(k))^2}{2\sigma_i^2} & |e_i(k)| \leq a_i\kappa_i\sigma_i \\ \frac{a_i\kappa_i|e_i(k)|}{\sigma_i} - \frac{a_i^2\kappa_i^2}{2} & \text{otherwise} \end{cases} \quad (16)$$

125 where κ_i is the penalty factor chosen specifically to cancel the effect of any existing leverage points in the measurement set and defined as:

$$\kappa_i = \min\{1, [\frac{\chi_{\nu,p}^2}{PS_i}]\} \quad (17)$$

where $\chi_{\nu,p}^2$ is the Chi-square statistics, ν is the degrees of freedom, p is probability and is usually taken as 0.975. PS_i is the projection statistics for measurement i and its definition is given as [1]

$$PS_i = \max_{H_k} \frac{|H_i^T \cdot H_k|}{\lambda}, \text{ for } k = 1, 2, \dots, m$$

where $\lambda = \gamma \cdot \text{lomed}_i\{\text{lomed}_{j \neq i}\{|H_i^T H_k + H_j^T H_k|\}\}$, $1 \leq i, j, k \leq m$, the factor γ is usually chosen as 1.1926 and $\text{lomed}_i\{\cdot\}$ means the low median of the m numbers in the brace. If $PS_i > \chi_{\nu,p}^2$, then measurement i will be identified as a leverage point. More details about κ_i and PS_i can be found in [29]. When the penalty factor κ_i in (16) equals to 1, then the SHGM estimator reduces to a QL estimator.

Differentiating $\rho(e_i(k))$ wrt $e_i(k)$ and substituting into (8) gives

$$W_i(k) = \begin{cases} \frac{1}{\sigma_i^2} & |e_i(k)| \leq a_i \kappa_i \sigma_i \\ \frac{a_i \kappa_i}{\sigma_i |e_i(k)|} & \text{otherwise} \end{cases} \quad (18)$$

As a summary, $\rho(e_i)$ and their derivatives for the different estimators are shown in Figs. 1 and 2 where the thresholds are set to $a_i = 3, b_i = 4, r_i = 5$ for the MS estimator and $\kappa_i = 0.7$ for the SHGM estimator. It can be seen that the various robust estimators modified the cost function when measurement residuals are large to reduce the effect of outliers on the state estimates.

Figure 1: $\rho(e_i)$ of the Multiple-Segment (MS), Quadratic-Constant (QC), Quadratic-Linear (QL), Square-Root (SR) and Schweppe-Huber Generalized-M (SHGM) estimator

3. Influence Function Analysis

The IF has been derived in the literature [21, 22]. This section gives the equations necessary for the derivation of the state estimate variance.

The definition of the IF is given in [21, 22] as

$$\text{IF}(\varepsilon) = - \left[\int_{-\infty}^{\infty} \frac{\partial \Psi(\varepsilon)}{\partial \hat{x}} dF(\varepsilon) \right]^{-1} \Psi(\varepsilon) \approx \hat{x} - \bar{x} \quad (19)$$

Figure 2: The derivative of $\rho(e_i)$ for the Multiple-Segment (MS), Quadratic-Constant (QC), Quadratic-Linear (QL), Square-Root (SR) and Schweppe-Huber Generalized-M (SHGM) estimator.

where

$$\begin{aligned}\varepsilon &= [\epsilon(1)^T \cdots \epsilon(N)^T]^T \\ dF(\varepsilon) &= f(\varepsilon)d\varepsilon\end{aligned}$$

in which

$$\begin{aligned}f(\varepsilon) &= f_1(\epsilon_1(1)) \times f_2(\epsilon_2(1)) \times \cdots \times f_m(\epsilon_m(1)) \times \cdots \times f_1(\epsilon_1(N)) \times f_2(\epsilon_2(N)) \times \cdots \times f_m(\epsilon_m(N)) \\ d\varepsilon &= d\epsilon_1(1) \times d\epsilon_2(1) \times \cdots \times d\epsilon_m(1) \times \cdots \times d\epsilon_1(N) \times d\epsilon_2(N) \times \cdots \times d\epsilon_m(N)\end{aligned}$$

Equation (19) can be understood intuitively as follows.

The first-order Taylor series expansion of $g(x)$ about the operating point $(\bar{x}, g(\bar{x}))$ is given by

$$g(x) \approx g(\bar{x}) + \left. \frac{dg(x)}{dx} \right|_{x=\bar{x}} (x - \bar{x})$$

Let $x = \hat{x}$, $g(\bar{x}) = \Psi(\varepsilon)$, $g(x) = 0$ and taking expectation of $\frac{d\Psi(\varepsilon)}{dx}$ give (19). A formal derivation of the influence function is given in [22].

Using (19), as given in [21, 22], the variance of the state estimate can be approximated as

$$\begin{aligned}\text{Var}(\hat{x}) &= \text{Var}(\hat{x} - \bar{x}) = \int_{-\infty}^{\infty} \text{IF}(\varepsilon) \text{IF}(\varepsilon)^T dF(\varepsilon) \\ &= \left[\int_{-\infty}^{\infty} \frac{\partial \Psi(\varepsilon)}{\partial \hat{x}} dF(\varepsilon) \right]^{-1} \int_{-\infty}^{\infty} \Psi(\varepsilon) (\Psi(\varepsilon))^T dF(\varepsilon) \left[\int_{-\infty}^{\infty} \frac{\partial \Psi(\varepsilon)}{\partial \hat{x}} dF(\varepsilon) \right]^{-T}\end{aligned}\quad (20)$$

Using (10) and let the residue E be given by the noise ε , equation (20) gives

$$\begin{aligned}\text{Var}(\hat{x}) &= \left[\bar{H}^T \int_{-\infty}^{\infty} \frac{\partial W\varepsilon}{\partial \varepsilon} dF(\varepsilon) \bar{H} \right]^{-1} \bar{H}^T \int_{-\infty}^{\infty} W\varepsilon \varepsilon^T W^T dF(\varepsilon) \bar{H} \left[\bar{H}^T \int_{-\infty}^{\infty} \frac{\partial W\varepsilon}{\partial \varepsilon} dF(\varepsilon) \bar{H} \right]^{-T} \\ &= [\bar{H}^T \Omega \bar{H}]^{-1} [\bar{H}^T \Lambda \bar{H}] [\bar{H}^T \Omega \bar{H}]^{-T}\end{aligned}\quad (21)$$

where

$$\Omega = \int_{-\infty}^{\infty} \frac{\partial W_{\varepsilon}}{\partial \varepsilon} dF(\varepsilon) = \text{diag}(\Omega_1(1), \dots, \Omega_m(1), \dots, \Omega_1(N), \dots, \Omega_m(N)) \quad (22)$$

$$\Lambda = \int_{-\infty}^{\infty} W_{\varepsilon} \varepsilon^T W^T dF(\varepsilon) = \text{diag}(\Lambda_1(1), \dots, \Lambda_m(1), \dots, \Lambda_1(N), \dots, \Lambda_m(N)) \quad (23)$$

150 Equation (21) is useful. It can be used to calculate the variance of the estimate $\hat{x}$ approximately. For the robust state estimators, the elements of the diagonal matrices Ω and Λ in (21) can be obtained as shown in the next three sections.

3.1. Variance of MS Estimator

Substituting (13) into (22) and (23) gives

$$\Omega_i(k) = \frac{2}{\sigma_i^2} \int_0^{a_i \sigma_i} f_i(\epsilon_i) d\epsilon_i - \frac{2}{\sigma_i^2} \int_{b_i \sigma_i}^{r_i \sigma_i} \frac{a_i f_i(\epsilon_i)}{r_i - b_i} d\epsilon_i \quad (24)$$

$$\Lambda_i(k) = 2 \left(\int_0^{a_i \sigma_i} \frac{\epsilon_i^2}{\sigma_i^4} f_i(\epsilon_i) d\epsilon_i + \int_{a_i \sigma_i}^{b_i \sigma_i} \frac{a_i^2}{\sigma_i^2} f_i(\epsilon_i) d\epsilon_i + \int_{b_i \sigma_i}^{r_i \sigma_i} \frac{a_i^2 (r_i \sigma_i - \epsilon_i)^2}{\sigma_i^4 (r_i - b_i)^2} f_i(\epsilon_i) d\epsilon_i \right) \quad (25)$$

155 3.2. Variance of SR Estimator

Substituting (15) into (22) and (23) gives

$$\Omega_i(k) = \frac{2}{\sigma_i^2} \int_0^{a_i \sigma_i} f_i(\epsilon_i) d\epsilon_i - \int_{a_i \sigma_i}^{\infty} \sqrt{\frac{a_i^3}{\sigma_i |\epsilon_i|^3}} f_i(\epsilon_i) d\epsilon_i \quad (26)$$

$$\Lambda_i(k) = \int_0^{a_i \sigma_i} \frac{2\epsilon_i^2}{\sigma_i^4} f_i(\epsilon_i) d\epsilon_i + \int_{a_i \sigma_i}^{\infty} \frac{2a_i^3}{\sigma_i |\epsilon_i|} f_i(\epsilon_i) d\epsilon_i \quad (27)$$

3.3. Variance of SHGM Estimator

Substituting (18) into (22) and (23) gives

$$\Omega_i(k) = \frac{2}{\sigma_i^2} \int_0^{a_i \kappa_i \sigma_i} f_i(\epsilon_i) d\epsilon_i \quad (28)$$

$$\Lambda_i(k) = 2 \left(\int_0^{a_i \kappa_i \sigma_i} \frac{\epsilon_i^2}{\sigma_i^4} f_i(\epsilon_i) d\epsilon_i + \int_{a_i \kappa_i \sigma_i}^{\infty} \frac{a_i^2 \kappa_i^2}{\sigma_i^2} f_i(\epsilon_i) d\epsilon_i \right) \quad (29)$$

4. Simulation Examples

160 In Example 1, a simple power system example taken from [7] is used to illustrate the detail calculations for variance equation (21). State estimation with PMU measurements on the IEEE 14-bus system is given in Examples 2 and 3. Equation (21) is used to approximate state estimation variance in the IEEE 14-bus system.

For easy reference, the parameters of the robust state estimators and probability density functions of the “normal” noise and outliers are summarized in Table 1.

Table 1: Parameters of estimators and noises used in the simulation examples

Example	1	2		3	
Model	4 Measurements, 3 States	IEEE 14-bus System with PMU Measurements		IEEE 14-bus System with PMU Measurements	
Measure- ment i	$1, \dots, 4$	$1, \dots, 12$	$13, \dots, 58$	$1, \dots, 12$	$13, \dots, 58$
Normal Noise	$*N(0.02)$ $N(0.03)$ $N(0.03)$ $N(0.04)$	$0.99 \times$ $N(0.006)$	$0.97 \times$ $N(0.003)$	t_3	t_5
Outliers	—	$0.01 \times$ $N(0.06)$	$^\dagger 0.03 \times$ $U(0.03)$		
‡ Esti- mator	WLS or MS(5,6 ,7)	QC(2.5) QL(2.5) SR(2.5) SHGM(2.5) MS(2.5,3.5, 4.5)	QC(3) QL(3) SR(3) SHGM(3) MS(3,4 ,5)	MS(2.5,3.5, 4.5) MS(0.6,1.5, 6)	MS(3,4,5) MS(0.8,2.5, 6)
Figure	3	4, 5		6	
Table	—	2		3	

$$*N(\sigma_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{\epsilon_i^2}{2\sigma_i^2}}.$$

$$^\dagger U(\sigma_i) = \frac{1}{2\sigma_i} \text{ for } -\sigma_i \leq \epsilon_i \leq \sigma_i.$$

‡ Parameter a_i is given in the bracket of QC(a_i), QL(a_i), SR(a_i) and SHGM(a_i) while parameters a_i , b_i , r_i are given in the bracket of MS(a_i, b_i, r_i).

165 Example 1: Four Measurements, Three States

Consider Fig. 3, where voltage measurements V_1 , V_2 , real and reactive power flow measurements P_{21} , Q_{21} are given. Gaussian noise is assumed, Bus 1 is taken as the slack bus and δ_2 is the phase angle of Bus 2.

The measurement model is given as

$$y(k) = h(x(k)) + \epsilon(k)$$

where

$$\begin{aligned} x(k) &= \begin{bmatrix} V_1(k) & V_2(k) & \delta_2(k) \end{bmatrix}^T \\ y(k) &= \begin{bmatrix} V_1(k) & V_2(k) & P_{21}(k) & Q_{21}(k) \end{bmatrix}^T \\ \epsilon(k) &= \begin{bmatrix} \epsilon_1(k) & \epsilon_2(k) & \epsilon_3(k) & \epsilon_4(k) \end{bmatrix}^T \end{aligned}$$

170 and the Jacobian matrix in [7] is given as

$$H = \begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 1.00 & 0 \\ -1.57 & -1.57 & 14.83 \\ -14.88 & 15.05 & -1.56 \end{bmatrix} \quad (30)$$

The variance formula in [7] for the WLS estimator gives

$$\text{Var}(\hat{x}) = (H^T W H)^{-1} = \begin{bmatrix} 2.76 & 2.76 & 0.58 \\ 2.76 & 2.84 & 0.60 \\ 0.58 & 0.60 & 0.17 \end{bmatrix} \times 10^{-4} \quad (31)$$

where

$$W = \text{diag} \left(\frac{1}{\sigma_1^2}, \frac{1}{\sigma_2^2}, \frac{1}{\sigma_3^2}, \frac{1}{\sigma_4^2} \right)$$

and the standard deviation of the Gaussian noise $\sigma_1 = 0.02$, $\sigma_2 = \sigma_3 = 0.03$, $\sigma_4 = 0.04$.

Figure 3: A simple example for power system state estimation with voltage measurements V_1 , V_2 , real and reactive power flow measurements P_{21} , Q_{21} [7]

175 The results in (31) given in [7] can also be obtained from variance equation (21) in the paper as follows. The variance of the WLS estimator can be obtained by choosing threshold parameter “ a_i ” large [1] such that the probability density function $f_i(\epsilon_i)$ is zero outside the threshold “ $a_i\sigma_i$ ”. Threshold parameters “ b_i ” and “ r_i ” can be chosen arbitrarily as $r_i > b_i > a_i$ and are outside the noise distribution. For the Gaussian distribution, $a_i = 5$ is good enough.

Using $a_1 = 5$, $b_1 = 6$, $r_1 = 7$ in (22) and (23) give

$$\begin{aligned} \Omega_1(1) &= \frac{2}{\sigma_1^2} \int_0^{5\sigma_1} f_1(\epsilon_1) d\epsilon_1 - \frac{2}{\sigma_1^2} \int_{6\sigma_1}^{7\sigma_1} \frac{5f_1(\epsilon_1)}{7-6} d\epsilon_1 = 2500, \\ \Lambda_1(1) &= 2 \left(\int_0^{5\sigma_1} \frac{\epsilon_1^2}{\sigma_1^4} f_1(\epsilon_1) d\epsilon_1 + \int_{5\sigma_1}^{6\sigma_1} \frac{5^2}{\sigma_1^2} f_1(\epsilon_1) d\epsilon_1 + \int_{6\sigma_1}^{7\sigma_1} \frac{5^2(7\sigma_1 - \epsilon_1)^2}{\sigma_1^4(7-6)^2} f_1(\epsilon_1) d\epsilon_1 \right) = 2500 \end{aligned}$$

where $f_1(\epsilon_1)$ is the probability density function of the zero-mean Gaussian distribution with standard deviation σ_1 .

Note that the definite integral in Ω_1 and Λ_1 can be calculated easily with the help of the Gauss error function [30]

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Similar calculations give $\Omega_2(1) = \Omega_3(1) = 1111$, $\Omega_4(1) = 625$, $\Lambda_2(1) = \Lambda_3(1) = 1111$, $\Lambda_4(1) = 625$. Variance Equation (21) then gives the same results in (31). Other than the WLS estimator with the usual Gaussian noise assumption, formulas for the variance of the estimates for QC, QL, MS, SR and SHGM estimators with non-Gaussian noise are not available in the literature and this is where variance equation (21) derived in this paper becomes useful. This will be shown in Example 2.

Example 2: IEEE 14-bus System with PMU measurements

The PMUs are placed according to [14]. With PMU, the measurement model in (1) is applicable. The matrix H is chosen according to [31]. There are $n = 28$ states in the vector $x = [V_1^r \ V_2^r \ \dots \ V_{14}^r \ V_1^{im} \ \dots \ V_{14}^{im}]^T$ where the real and imaginary part of the voltage phasor are given as V_i^r and V_i^{im} respectively. Fifty-eight measurements, z_i , $i = 1, \dots, 58$, comprising of 12 voltages ($i = 1, \dots, 12$) and 46 currents ($i = 13, \dots, 58$) are taken at each time instance k and $N = 3$ sets of measurements, i.e. $k = 1, 2, 3$, are used to give one set of estimates. Measurement outliers are removed by the robust estimator when they are outside the estimator cut-off “ r ”. As such, there could be insufficient data for the equations to solve for the estimates if only $N = 1$ set of measurements is used.

For the voltage measurements z_i , $i = 1, \dots, 12$, the parameters for the QC, QL, SR and SHGM estimators are chosen according to [24, 25] as $a_i = 2.5$ while the parameters for the MS estimator are $a_i = 2.5$, $b_i = 3.5$, $r_i = 4.5$. The penalty κ_i , $i = 1, \dots, 58$ of SHGM estimator is calculated by (17). The standard deviation for measurement from PMU is between 0.0005 and 0.01 [32]. Here, the noise ϵ_i is associated with the probability density function

$$f_i(\epsilon_i) = \frac{0.99}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) + \frac{0.01}{\sqrt{2\pi(10\sigma_i)^2}} \exp\left(-\frac{\epsilon_i^2}{2(10\sigma_i)^2}\right) \quad (32)$$

where $\sigma_i = 0.006$, $i = 1, \dots, 12$. The first term in the above probability density function represents the 99% of “normal” noise while the second term represents outlier by the 1% Gaussian noise with standard deviation $10\sigma_i$.

For the current measurements z_i , $i = 13, \dots, 58$, the parameters for the QC, QL, SR and SHGM estimators are chosen as $a_i = 3$ while the parameters for the MS estimator are $a_i = 3$, $b_i = 4$, $r_i = 5$. The noise ϵ_i is associated with the probability density function

$$f_i(\epsilon_i) = \begin{cases} \frac{0.97}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) + \frac{0.03}{2 \times 10\sigma_i} & |\epsilon_i| \leq 10\sigma_i \\ \frac{0.97}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) & \text{otherwise} \end{cases} \quad (33)$$

where $\sigma_i = 0.003$, $i = 13, \dots, 58$. The probability density function in the form of a mixture distribution in (33) is also given in [25]. The 3% of uniform distribution $\frac{0.03}{2 \times 10\sigma_i}$ in (33) is useful for modeling initial conditions, disturbances and measurement errors that are equally likely to occur anywhere within a given interval. In the study of outliers, the “normal” noise is typically represented by the Gaussian distribution, while the outliers by Gaussian with a larger standard deviation or non-Gaussian distributions [22, 25].

In this example, the QC, QL, SR, MS and SHGM robust state estimators are used to estimate the states of the IEEE 14-bus system. The non robust estimator (WLS) is a special case of MS (when parameters a_i , b_i and r_i tend to ∞), and its results are also given in Table 2. The detail calculation steps follow those already illustrated in Example 1.

The variances calculated from variance equation (21) are given in Table 2 under column “Eq. (21)”. To give an idea of the accuracy, the variance calculated from (21) is compared with the variance obtained from 10,000 simulation runs and the results are also given in Table 2 under column “Sim.”. The percentage errors between the variances from simulation and from (21) are given under column “% Er.”. The maximum percentage error is less than 5%, which in practice should be good enough. Furthermore, when the outliers increase to 5%,

$$f_i(\epsilon_i) = \frac{0.95}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) + \frac{0.05}{\sqrt{2\pi(10\sigma_i)^2}} \exp\left(-\frac{\epsilon_i^2}{2(10\sigma_i)^2}\right)$$

where $\sigma_i = 0.006$, $i = 1, \dots, 12$, and

$$f_i(\epsilon_i) = \begin{cases} \frac{0.95}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) + \frac{0.05}{2 \times 10\sigma_i} & |\epsilon_i| \leq 10\sigma_i \\ \frac{0.95}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) & \text{otherwise} \end{cases}$$

where $\sigma_i = 0.003$, $i = 13, \dots, 58$, the maximum percentage error increases to 7%.

Equation (21) can also be used to obtain the variance of a state estimate as a function of measurement variance. Consider the MS estimation. When $\sigma_i = 0.006$, $i = 1, \dots, 12$, and $\sigma_i = 0.003$, $i = 13, \dots, 58$, equation (21) gives $\text{Var}(\hat{V}_{12}^r) = 2.26 \times 10^{-6}$ in Table 2 (see Row “ $\hat{V}_{12}^r$ ”, Column “MS-Eq. (21)”). If it is not given that $\sigma_3 = 0.006$, then σ_3 can be left as a variable and equation (21) gives

$$\text{Var}(\hat{V}_{12}^r) = \frac{6.27 \times 10^5 \sigma_3^4 + 4.93 \sigma_3^2 + 1.90 \times 10^{-6}}{(4.82 \times 10^5 \sigma_3^2 + 3.61)^2} \quad (34)$$

Equation (34) is plotted as the solid-line in Fig. 4. The variances of the other 27 state estimates as a function of σ_3 were likewise obtained and plotted in the figure as dashed-lines. Fig. 4 now summarizes, for the MS estimation, how the variances of all 28 state estimates are affected by σ_3 . Obviously, instead of σ_3 , (21) can also give $\text{Var}(\hat{V}_{12}^r)$ as a function of the other σ_i as shown in Fig. 5 where (34) is again plotted as the solid-line. These figures can help us in selecting sensors to achieve a specified estimation precision. For example, if the variance of the state V_{12}^r , $\text{Var}(\hat{V}_{12}^r) \leq 1.6 \times 10^{-6}$ is specified then the solid-line in Fig. 4 shows that Sensor 3 with standard deviation $\sigma_3 \leq 3 \times 10^{-3}$ should be selected. The crosses in both figures are variances obtained from simulations and they are well approximated by the solid-line obtained with (21).

Example 3: Optimal Estimator Design

In this example equation (21) is used to find the optimum parameters a_i , b_i and r_i of the MS estimator to give the minimum sum of the state estimate variances defined as

$$\text{SV} = \sum_{i=1}^{28} \text{Var}(\hat{x}_i) \quad (35)$$

Consider the same IEEE 14-bus system with PMU measurements in Example 2 but let the noise ϵ_i be associated

Table 2: Variances of the state estimates in the IEEE 14-bus system with PMU measurements.

	WLS			QC			QL			SR			MS			SHGM		
	Sim.	Eq. (21)	% Er.	Sim.	Eq. (21)	% Er.	Sim.	Eq. (21)	% Er.	Sim.	Eq. (21)	% Er.	Sim.	Eq. (21)	% Er.	Sim.	Eq. (21)	% Er.
	$a_i \rightarrow \infty,$ $b_i \rightarrow \infty,$ $r_i \rightarrow \infty,$ $i=1, \dots, 58.$			$a_i = 2.5, b_i \rightarrow a_i,$ $r_i \rightarrow a_i,$ $i = 1, \dots, 12;$ $a_i = 3, b_i \rightarrow a_i,$ $r_i \rightarrow a_i,$ $i = 13, \dots, 58.$			$a_i = 2.5, b_i \rightarrow \infty,$ $r_i \rightarrow \infty,$ $i = 1, \dots, 12;$ $a_i = 3, b_i \rightarrow \infty,$ $r_i \rightarrow \infty,$ $i = 13, \dots, 58.$			$a_i = 2.5,$ $i = 1, \dots, 12;$ $a_i = 3,$ $i = 13, \dots, 58.$			$a_i = 2.5, b_i = 3.5,$ $r_i = 4.5,$ $i = 1, \dots, 12;$ $a_i = 3, b_i = 4,$ $r_i = 5,$ $i=13, \dots, 58.$			$a_i = 2.5,$ $i = 1, \dots, 12;$ $a_i = 3,$ $i=13, \dots, 58.$		
$\hat{V}_1^r$	4.14	4.08	-1.4	2.22	2.31	4.1	2.19	2.20	0.5	2.16	2.13	-1.4	2.13	2.11	-0.9	2.22	2.22	0.0
$\hat{V}_2^r$	4.14	4.07	-1.7	2.22	2.31	4.1	2.17	2.19	0.9	2.15	2.13	-0.9	2.12	2.10	-0.9	2.21	2.21	0.0
$\hat{V}_3^r$	4.17	4.11	-1.4	2.24	2.33	4.0	2.22	2.22	0.0	2.18	2.15	-1.4	2.16	2.12	-1.9	2.24	2.26	0.9
$\hat{V}_4^r$	4.08	4.03	-1.2	2.19	2.28	4.1	2.15	2.16	0.5	2.12	2.10	-0.9	2.10	2.07	-1.4	2.18	2.17	-0.4
$\hat{V}_5^r$	4.09	4.03	-1.5	2.19	2.28	4.1	2.15	2.16	0.5	2.12	2.10	-0.9	2.10	2.08	-1.0	2.18	2.18	0.0
$\hat{V}_6^r$	4.13	4.10	-0.7	2.23	2.32	4.0	2.21	2.21	0.0	2.16	2.15	-0.5	2.14	2.12	-0.9	2.23	2.25	0.9
$\hat{V}_7^r$	4.09	4.05	-1.0	2.20	2.29	4.1	2.16	2.17	0.5	2.13	2.11	-0.9	2.10	2.09	-0.5	2.20	2.19	-0.5
$\hat{V}_8^r$	4.27	4.23	-0.9	2.30	2.39	3.9	2.29	2.29	0.0	2.25	2.22	-1.3	2.22	2.19	-1.4	2.36	2.36	0.0
$\hat{V}_9^r$	4.10	4.06	-1.0	2.21	2.30	4.1	2.15	2.18	1.4	2.15	2.12	-1.4	2.11	2.09	-0.9	2.21	2.20	-0.5
$\hat{V}_{10}^r$	4.14	4.11	-0.7	2.23	2.33	4.5	2.18	2.21	1.4	2.18	2.15	-1.4	2.14	2.12	-0.9	2.24	2.25	0.3
$\hat{V}_{11}^r$	4.39	4.39	0.0	2.39	2.48	3.8	2.41	2.39	-0.8	2.33	2.31	-0.9	2.31	2.28	-1.3	2.51	2.51	0.0
$\hat{V}_{12}^r$	4.38	4.35	-0.7	2.37	2.46	3.8	2.36	2.37	0.4	2.33	2.29	-1.7	2.28	2.26	-0.9	2.48	2.47	-0.4
$\hat{V}_{13}^r$	4.16	4.13	-0.7	2.25	2.34	4.0	2.21	2.22	0.5	2.17	2.16	-0.5	2.15	2.13	-0.9	2.23	2.27	1.8
$\hat{V}_{14}^r$	4.36	4.33	-0.7	2.38	2.45	2.9	2.36	2.35	-0.4	2.29	2.28	-0.4	2.25	2.24	-0.4	2.41	2.45	1.7
$\hat{V}_1^{im}$	3.95	4.08	3.3	2.26	2.31	2.2	2.23	2.20	-1.3	2.16	2.13	-1.4	2.11	2.11	0.0	2.27	2.22	-2.2
$\hat{V}_2^{im}$	3.95	4.07	3.0	2.25	2.31	2.7	2.22	2.19	-1.4	2.15	2.13	-0.9	2.11	2.10	-0.5	2.27	2.21	-2.6
$\hat{V}_3^{im}$	3.97	4.11	3.5	2.24	2.33	4.0	2.26	2.22	-1.8	2.18	2.15	-1.4	2.13	2.12	-0.5	2.28	2.26	-0.9
$\hat{V}_4^{im}$	3.90	4.03	3.3	2.21	2.28	3.2	2.18	2.16	-0.9	2.12	2.10	-0.9	2.08	2.07	-0.5	2.23	2.17	-2.7
$\hat{V}_5^{im}$	3.90	4.03	3.3	2.21	2.28	3.2	2.18	2.16	-0.9	2.12	2.10	-0.9	2.08	2.08	0.0	2.22	2.18	-1.8
$\hat{V}_6^{im}$	3.98	4.10	3.0	2.26	2.32	2.7	2.23	2.21	-0.9	2.17	2.15	-0.9	2.13	2.12	-0.5	2.23	2.25	0.9
$\hat{V}_7^{im}$	3.92	4.05	3.3	2.23	2.29	2.7	2.21	2.17	-1.8	2.14	2.11	-1.4	2.09	2.09	0.0	2.23	2.19	-1.8
$\hat{V}_8^{im}$	4.10	4.23	3.2	2.33	2.39	2.6	2.33	2.29	-1.7	2.28	2.22	-2.6	2.20	2.19	-0.5	2.36	2.36	0.0
$\hat{V}_9^{im}$	3.94	4.06	3.0	2.23	2.30	3.1	2.21	2.18	-1.4	2.15	2.12	-1.4	2.09	2.09	0.0	2.24	2.20	-1.8
$\hat{V}_{10}^{im}$	4.00	4.11	2.7	2.26	2.33	3.1	2.25	2.21	-1.8	2.19	2.15	-1.8	2.12	2.12	0.0	2.28	2.25	-1.3
$\hat{V}_{11}^{im}$	4.26	4.39	3.1	2.42	2.48	2.5	2.47	2.39	-3.2	2.38	2.31	-2.9	2.30	2.28	-0.9	2.53	2.51	-0.8
$\hat{V}_{12}^{im}$	4.25	4.35	2.4	2.42	2.46	1.7	2.40	2.37	-1.2	2.32	2.29	-1.3	2.27	2.26	-0.4	2.47	2.47	0.0
$\hat{V}_{13}^{im}$	4.00	4.13	3.3	2.28	2.34	2.6	2.23	2.22	-0.4	2.19	2.16	-1.4	2.14	2.13	-0.5	2.24	2.27	1.3
$\hat{V}_{14}^{im}$	4.20	4.33	3.1	2.38	2.45	2.9	2.36	2.35	-0.4	2.31	2.28	-1.3	2.24	2.24	0.0	2.42	2.45	1.2

Variances unit: $\times 10^{-6}$;% Er.=(Eq. (21) - Sim.)/Sim. $\times 100\%$.

Figure 4: Variances of all the 28 state estimates versus σ_3 . The solid-line is the variance of state estimate $\hat{V}_{12}^r$. The crosses are the variances of state estimate $\hat{V}_{12}^r$ obtained from simulation.

Figure 5: Variance of state estimate $\hat{V}_{12,r}$ versus $\sigma_i, i = 1, \dots, 58$. The solid-line shows the variance of state estimate $\hat{V}_{12}^r$ versus σ_3 . The crosses are the variances of state estimate $\hat{V}_{12}^r$ versus σ_3 obtained from simulation.

with the following t -distribution which is commonly used to model noise with outliers [22]

$$f_i(\epsilon_i) = \frac{1}{\sqrt{v_i} \alpha_i \beta \left(\frac{1}{2}, \frac{v_i}{2}\right) \left(1 + \frac{1}{v_i} \left(\frac{\epsilon_i}{\alpha_i}\right)^2\right)^{\frac{v_i+1}{2}}},$$

where $v_i = 3$ (t_3 distribution), $\alpha_i = 0.006$ for voltage measurements i.e. $i = 1, \dots, 12$ and $v_i = 5$ (t_5 distribution), $\alpha_i = 0.003$ for current measurements i.e. $i = 13, \dots, 58$.

A search was conducted in the ranges $a_i = 0.5, 0.6, \dots, 1.4$, $b_i = 0.5, 0.6, \dots, 1.4$, $r_i = 6, 7, \dots, 15$ to find the minimum SV in (35). Since each range of a_i , b_i and r_i consists of 10 points and 2 independent sets of a_i , b_i and r_i for the voltage measurements and current measurements need to be chosen, a total of $(10 \times 10 \times 10 - 450)^2 = 302,500$ combinations of points were searched. The subtraction of 450 is to account for the constraint $a_i \leq b_i \leq r_i$. For each of the 302,500 combinations, equation (21) was used to calculate a set of 28 variances which were then added together according to (35) to give SV. The search then gave the minimum SV = 87.51×10^{-6} with $a_i = 0.6$, $b_i = 1$, $r_i = 7$ for $i = 1, \dots, 12$ and $a_i = 1$, $b_i = 1.1$, $r_i = 13$ for $i = 13, \dots, 58$ as shown in Table 3 (see Row “SV”, Column “MS (optimal)-Eq. (21)”). In contrast, if typical values of $a_i = 2.5$, $b_i = 3.5$, $r_i = 4.5$ for $i = 1, \dots, 12$ and $a_i = 3$, $b_i = 4$, $r_i = 5$ for $i = 13, \dots, 58$ were chosen then SV = 117.64×10^{-6} as shown in Table 3 (see Row “SV”, Column “MS (typical)-Eq. (21)”). The optimum parameters reduced the sum of variances by about $(117.64 - 87.51)/(117.64) \times 100\% = 26\%$. Hence the optimal estimator gives more precise state estimates.

Monte Carlo simulation method can also be used to determine the variance of $\hat{x}$, but many data points are needed and hence many simulation runs are required to achieve convergence. Ten thousand simulation runs using Matlab version R2015a on a *i7* Windows 10 computer with 8 GB RAM took 145s to give a set of 28 variances. On the same equipment, equation (21) took less than 0.001 s to perform the same task. Even if a lower level of precision is accepted and the simulation runs are reduced to 1000, it will still take 14.5 s. At this lower level of precision, going through 302,500 combinations requires $14.5s \times 302,500 \approx 50$ days. In contrast, equation (21) takes $0.001s \times 302,500 \approx 5$ minutes, making the optimal estimator design using (21) a more tractable problem. In this paper, for simplicity exhaustive search is used. Independent of the search method, Equation (21) is 14500 ($= 14.5/0.001$) times faster than Monte Carlo simulation.

To put things in perspective, the maximum likelihood estimator was also designed for the given IEEE 14-bus system and its IF plotted in Fig. 6. The IF of the MS (optimal) and MS (typical) estimators given in Table 3 are superimposed onto the figure. It is clear from the figure that the IF of the MS (optimal) estimator is close to optimum i.e. the maximum likelihood while the MS (typical) estimator is not. Note that the IF is a function of ϵ_i $i = 1, \dots, 58$ but for simplicity, the IF curve assuming $\epsilon_1 = \epsilon_2 = \dots = \epsilon_{58}$ has been plotted.

5. Conclusions

In this paper, an analytical equation is derived using influence function approximation to calculate the variance of the state estimate approximately for common robust state estimators such as the Quadratic-Constant, Quadratic-Linear, Square-Root, Multiple-Segment and Schweppe-Huber Generalized-M estimator. The well-known Weighted-Least-Squares variance formula is a special case of the equation. The derived equation can be used to: (1) design

Table 3: Variances of the state estimates of the MS (optimal) and MS (typical) estimators in the IEEE 14-bus system.

	MS (typical) $a_i = 2.5, b_i = 3.5, r_i = 4.5,$ for $i = 1, \dots, 12;$ $a_i = 3, b_i = 4, r_i = 5,$ for $i = 13, \dots, 58.$			MS (optimal) $a_i = 0.6, b_i = 1, r_i = 7,$ for $i = 1, \dots, 12;$ $a_i = 1, b_i = 1.1, r_i = 13,$ for $i = 13, \dots, 58.$		
	Sim.	Eq.(21)	% Er.	Sim.	Eq. (21)	% Er.
$\hat{V}_1^r$	4.23	4.15	-1.8	3.15	3.08	-2.4
$\hat{V}_2^r$	4.23	4.14	-1.9	3.14	3.07	-2.2
$\hat{V}_3^r$	4.28	4.17	-2.6	3.17	3.10	-2.2
$\hat{V}_4^r$	4.20	4.11	-2.2	3.11	3.04	-2.2
$\hat{V}_5^r$	4.19	4.11	-2.0	3.11	3.04	-2.1
$\hat{V}_6^r$	4.27	4.17	-2.3	3.18	3.10	-2.7
$\hat{V}_7^r$	4.21	4.13	-2.0	3.13	3.06	-2.4
$\hat{V}_8^r$	4.35	4.27	-2.0	3.29	3.18	-3.1
$\hat{V}_9^r$	4.22	4.13	-2.0	3.15	3.07	-2.6
$\hat{V}_{10}^r$	4.27	4.17	-2.3	3.18	3.10	-2.4
$\hat{V}_{11}^r$	4.52	4.39	-2.8	3.42	3.29	-3.7
$\hat{V}_{12}^r$	4.47	4.36	-2.5	3.34	3.26	-2.4
$\hat{V}_{13}^r$	4.28	4.19	-2.3	3.20	3.11	-2.9
$\hat{V}_{14}^r$	4.41	4.34	-1.6	3.35	3.25	-3.1
$\hat{V}_1^{im}$	4.20	4.15	-1.3	3.20	3.08	-3.9
$\hat{V}_2^{im}$	4.19	4.14	-1.1	3.20	3.07	-3.8
$\hat{V}_3^{im}$	4.20	4.17	-0.7	3.22	3.10	-3.7
$\hat{V}_4^{im}$	4.15	4.11	-0.9	3.15	3.04	-3.5
$\hat{V}_5^{im}$	4.15	4.11	-0.9	3.15	3.04	-3.5
$\hat{V}_6^{im}$	4.19	4.17	-0.5	3.22	3.10	-3.9
$\hat{V}_7^{im}$	4.15	4.13	-0.6	3.18	3.06	-3.9
$\hat{V}_8^{im}$	4.30	4.27	-0.6	3.33	3.18	-4.5
$\hat{V}_9^{im}$	4.15	4.13	-0.5	3.18	3.07	-3.7
$\hat{V}_{10}^{im}$	4.19	4.17	-0.6	3.22	3.10	-3.7
$\hat{V}_{11}^{im}$	4.43	4.39	-1.0	3.44	3.29	-4.3
$\hat{V}_{12}^{im}$	4.40	4.36	-1.0	3.42	3.26	-4.5
$\hat{V}_{13}^{im}$	4.19	4.19	-0.2	3.23	3.11	-3.8
$\hat{V}_{14}^{im}$	4.37	4.34	-0.6	3.38	3.25	-4.0
SV	119.39	117.64	-1.5	90.45	87.51	-3.3

Variances unit: $\times 10^{-6}$;% Er.=(Eq. (21) -Sim.)/Sim. $\times 100\%$.

Figure 6: IF of estimators: maximum likelihood (dash-dotted-line), optimal MS (solid-line), typical MS (dash-line)

an optimal state estimator; (2) express the variance of a state estimate as a function of measurement variances enabling the selection of sensors for specified estimator precision.

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