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Comparison of settling-velocity based formulas for threshold of sediment motion

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ABSTRACT: The critical condition for incipient sediment motion is formulated in this note based on the settling velocity. The formula obtained is simple, relating the ratio of critical shear velocity to settling velocity to the dimensionless sediment diameter. Comparisons are then made with other settling-velocity based formulas available in the literature. To facilitate the computation of the effective near-bed velocity at the threshold condition, a generalized law-of-the-wall function is proposed for predicting the velocity distribution under various boundary conditions. This study demonstrates that the settling velocity is equivalent to the critical near-bed velocity, which is experienced by a typical bed sediment particle under the threshold condition, but only for large sediment sizes such as sand and gravel. Comparison results show that Yang's formula is suitable for flows with small flow depth relative to sediment size while Le Roux's formula may overestimate the threshold condition for fine particles by up to 30%.

Introduction

As an important parameter for characterizing sediment grains, the settling velocity, w , is often included as an independent variable in various considerations related to sediment transport. For example, it can be used for the formulations of the critical condition for incipient sediment motion (Cheng and Chiew 1999; Egiazaroff 1965; Komar and Clemens 1986; Le Roux 1998; Yang 1973), sediment transport rate (Bagnold 1973) and sediment transport capacity (see Qian and Wan 1999), and also for the clarification of bedforms (Liu 1957).

Rubey (1933), who seems to be the first to involve w in the description of the threshold of sediment motion, noticed that the flow velocity required to initiate the motion of a sediment grain at the bed is approximately of the same order as w . This approximation was later adopted by Egiazaroff (1965) in an investigation of the critical condition for incipient motion of non-uniform sediment grains. Yang (1973) formulated the critical condition using the ratio of depth-averaged velocity, U , to w in relation to the shear Reynolds number.

In developing his bed load function, Einstein (1950) used the ratio of particle diameter, D , to w for characterizing the time scale of sediment entrainment. Liu (1957) preferred to engage w rather than the Shields number in identifying the formation of ripples. The ratio, U/w , can be considered an essential parameter for investigating sediment transport capacity, as demonstrated in many dimensional analyses (see Qian and Wan 1999).

As mentioned by Komar and Clemens (1986), an intuitive view exists that a connection between threshold condition and w is unlikely or at most spurious because the threshold occurs when the grain is essentially at rest, whereas sediment settling becomes important only when the particle is transported as part of the suspended load. However, a direct connection that has been explored in many relevant studies is that fluid flowing around a bed particle induces a drag which is comparable to the resistance exerted by fluid as the particle settles. In addition, the use of w in an analysis could provide an implicit consideration of the shape effect of sediment grains. The settling velocity may also serve as a useful velocity scale to simplify some physical reasoning, as shown later in this note.

With the theoretical result from the probabilistic considerations by Cheng and Chiew (1998; 1999), a simple formula is proposed in this study to relate critical shear velocity, u_{*c} , to w , which could serve as an alternative interpretation of the Shields diagram. Comparisons are then made between the present result and other relevant formulas available in the literature.

Formulation of Critical Condition Using Settling Velocity

By applying the concept of probability, Cheng and Chiew (1998; 1999) demonstrated how near-bed flow conditions affect sediment entrainment and initiation of suspension. The probability of a sediment particle being transported by turbulent flow is generally dependent both on the dimensionless bed shear stress (or Shields number), τ_* , and the shear velocity based Reynolds number, Re_* , where $\tau_* = u_*^2/(\Delta g D)$, $Re_* = u_* D/\nu$, u_* is the

shear velocity, $\Delta = (\rho_s/\rho - 1)$, ρ_s is the sediment density, ρ is the fluid density, g is the gravitational acceleration and ν is the kinematic viscosity of fluid.

A very small probability can be used for quantifying the threshold condition for sediment entrainment, so a probabilistic approach can simulate well the Shields diagram (Cheng 2006; Cheng and Chiew 1999). Further manipulation of the theoretical result of Cheng and Chiew (1999) yields

$$\tau_{*c} = \frac{1.32 Re_{*c}^2}{\left[\left(\sqrt[3]{\alpha [Re_{*c} - Re_{*c} \exp(-0.093 Re_{*c}^{1.3})]^2 + 5} \right)^2 - 25 \right]^{3/2}} \quad (1)$$

where the subscript ‘c’ denotes the value taken for the critical condition, α is a constant associated with the critical probability p_c in the form of $\alpha = -0.5\pi \ln(4p_c - 4p_c^2)$, and $\alpha = 23.14$ for the incipient sediment motion at $p_c = 10^{-7}$ and 5.07 for the threshold of suspension at $p_c = 0.01$. Because both τ_{*c} and Re_{*c} include u_{*c} , a trial procedure is needed when applying Eq. (1) for the computation of the critical shear velocity. A simplified form of Eq. (1) is derived in this note based on w .

The relative critical shear velocity, u_{*c}/w , and the dimensionless grain diameter, $D_* = (\Delta g/\nu^2)^{1/3} D$, are used in place of Re_{*c} and τ_{*c} , the latter being included in the Shields diagram. Noting that $u_{*c}/w = Re_{*c}/Re_w$ where $Re_w = wD/\nu$ and $D_* = Re_{*c}^{2/3} \tau_{*c}^{-1/3}$, the relation given by Eq. (1) can be also presented in terms of u_{*c}/w and D_* , as plotted in Fig. 1.

For a fully-rough boundary, Re_{*c} is very large and Eq. (1) reduces to $\tau_{*c} = 0.057$, while for a fully-smooth boundary, Re_{*c} is very small and Eq. (1) is simplified to $\tau_{*c} = 0.094 Re_{*c}^{-0.3}$. The two reduced relations can be re-written, respectively, as

$$\frac{u_{*c}}{w} = \frac{8.6}{D_*^{1.7}} \quad \text{for } D_* < 2 \quad (2)$$

$$\frac{u_{*c}}{w} = 0.21 \quad \text{for } D_* > 20 \quad (3)$$

Eqs. (2) and (3) are also plotted in Fig. 1. Here, w is computed using the following equation (Cheng 1997),

$$Re_w = (\sqrt{25 + 1.2D_*^2} - 5)^{1.5} \quad (4)$$

Eq. (4) applies to naturally worn sediment grains.

To describe the transition region as shown in Fig. 1, we apply the interpolation function in the following power-sum form,

$$\left(\frac{u_{*c}}{w}\right)^{1/n} = \left(\frac{8.6}{D_*^{1.7}} + 0.16\right)^{1/n} + 0.21^{1/n} \quad \text{or} \quad \frac{u_{*c}}{w} = 0.21 \left[\left(\frac{41}{D_*^{1.7}} + 0.76\right)^{1/n} + 1 \right]^n \quad (5)$$

where n is determined by fitting to be 0.05. A similar interpolation approach was also used by Cheng (1997) for computing the settling drag coefficient. Eq. (5) is explicit and easy to use, and agrees with Eq. (1) within $\pm 3.3\%$ for Re_{*c} ranging from 0.01 to 10^4 .

Comparisons with Other w -Based Formulas

In addition to u_{*c} , the critical condition for initial sediment motion can be also formulated in terms of the critical near-bed flow velocity, u_{bc} , or the critical depth-averaged flow velocity, U_c .

Threshold criteria in terms of u_{bc}/w

Egiazaroff (1965) assumed that $u_{bc} = w$. This simple assumption appears very crude and has been considered not quite correct by other researchers (e.g. Bagnold 1973, Komar and Clemens 1986). In spite of its simplicity, the theoretical analysis of the threshold condition for the entrainment of non-uniform sediment based on this assumption fits well some experimental observations.

A theoretical basis for using the Egiazaroff's assumption can be developed by relating u_{bc} to u_{*c} . A typical sediment grain at the bed surface may experience different flow situations depending on its size relative to the viscous length scale. As a result, u_{bc} should be evaluated using the appropriate velocity profile that may be associated with the viscous sublayer, buffer layer or logarithmic layer. The following power-sum equation for computing u_{bc} is proposed

$$(u^+)^{-3} = (y^+)^{-3} + \left[2.5 \ln \left(1 + \frac{9y^+}{1 + 0.3k_s^+(1 - e^{-0.04k_s^+})} \right) \right]^{-3} \quad (6)$$

where $u^+ = u/u_*$, $y^+ = u_*y/\nu$, $k_s^+ = u_*k_s/\nu$, u is the time-mean longitudinal velocity, y is the distance measured from the theoretical bed surface, and k_s is the boundary roughness height related to sediment size. Eq. (6) is derived in Appendix. It is a generalized law-of-the-wall function predicting the velocity distribution throughout the viscous sublayer, buffer layer and logarithmic layer for fully-smooth, fully-rough and transitional boundary conditions. For rough boundaries, the time-mean velocity represents the flow information averaged in the domain of the two-dimensional plane parallel to the bed surface; local variations induced by the irregularities inherent in the bed configuration and turbulent flow are not considered.

To estimate u_{bc} , it is assumed that u_{bc} is measured at the location of $y_b = (0.25-0.75)D$, and the roughness size $k_s = (1-4)D$, which therefore yields that $y_b^+ = (0.25-0.75)Re_{*c}$ and $k_s^+ = (1-4)Re_{*c}$ where $Re_{*c} = u_{*c}D/\nu$. The variations in the two coefficients of proportionality are selected from previous studies (see Bridge and Bennett 1992; Qian and Wan 1999; Raudkivi 1998). To relate u_{bc} to w , Eqs. (4), (5) and (6) are combined. With a given D_* , first compute u_{*c}/w and Re_{*c} using Eqs. (4) and (5). Then, find out u_{bc}/u_{*c} using Eq. (6). This finally provides the variation of u_{bc}/w with D_* . The results obtained for over 100 cases were then averaged to yield an average relation between u_{bc}/w and D_* . Fig. 2 shows the average curve, together with some extreme cases computed. It can be observed that the averaged u_{bc}/w varies slightly for $D_* > 2$, but increases significantly for $D_* < 2$. Additional computations indicate that the average curve is very close to that computed with $y_b/D = 0.5$ and $k_s/D = 2.5$.

The average relation implies that the assumption of $u_{bc}/w = 1$ used by Egiazaroff (1965) is very reasonable for natural sediment in the size of sand or larger (D_* ranging from 2 – 1600) but not applicable for finer particles. This result could be helpful in the formulation of hydrodynamic forces, lift and drag, for bed particles at the threshold of motion. By engaging w rather than u_{bc} , some simplifications could be also done for relevant analytical considerations and numerical simulations because u_{bc} depends on the near-bed flow structure that is generally complex.

The Egiazaroff's approximation also illuminates the bedload transport model of Bagnold (1973). In studying bedload transport, Bagnold used w for the description of the velocity of a bedload particle and concluded that the transport rate is proportional to the difference, $u_b - w$. Because u_b and w have the different directions, the use of the velocity

difference in such a way is not easy to understand. However, with $w \approx u_{bc}$, the Bagnold's argument can be appreciated more easily with the difference, $(u_b - u_{bc})$ or equivalently $(u_* - u_{*c})$, the latter being used commonly in the literature.

Threshold criteria in terms of U_c/w

Yang (1973) developed a criteria for incipient sediment motion based on U_c/w

$$\frac{U_c}{w} = \begin{cases} \frac{2.5}{\log(u_{*c}D/\nu) - 0.06} + 0.66 & \text{for } 1.2 < \frac{u_{*c}D}{\nu} < 70 \\ 2.05 & \text{for } \frac{u_{*c}D}{\nu} \geq 70 \end{cases} \quad (7)$$

To obtain U_c/w using the current approach, we first compute u_{*c}/w using Eq. (5), and then find out U_c/u_{*c} by integrating Eq. (6). The resulting relation is flow-depth dependent (Fig. 3). Yang's formula is suitable only for small relative flow depth, h/D . This is understandable because the constants included in Eq. (7) were determined from laboratory data, for which the range of h/D is small and limited. If Eq. (7) is extended to the case of large h/D as in river flows, U_c would be underestimated (Fig. 3). In other words, u_{*c} may be considered more suitable than the bulk-flow parameter such as U_c for incorporating the effect of flow depth. As only two calibration data points are available for $Re_{*c} < 2$, application of Yang's formula to flows with small Re_{*c} (i.e. fully-smooth boundary) would overestimate U_c .

Formulas by Komar and Clemens (1986) and Le Roux (1998)

Komar and Clemens (1986) reported that it is not necessary to include the grain size for the formulation of the critical shear velocity in the Stokes range. Their formula is given by

$$u_{*c} = 0.482(\Delta g v)^{0.282} w^{0.154} \quad (8)$$

By use of the relation, $Re_w = 0.042D_*$, which is derived from Eq. (4) for Stokes flow, Eq. (8) can be re-written as $u_{*c} / w = 7.1D_*^{-1.7}$, which is comparable to Eq.(2).

Le Roux (1998) expressed τ_{*c} as an empirical function of the dimensionless settling velocity, w_* ($= w / \sqrt[3]{\Delta g v} = Re_w / D_*$). The comparison of Le Roux's formula with the present study and other two empirical formulas by Yalin and Da Silva (2001) and Whitehouse et al. (2000) indicates that Le Roux's formula over-predicts the critical shear stress by up to 30% for $w_* = 0.001-1$ or $D_* = 0.15-6$.

Conclusions

This study presents a simple formula for the evaluation of the critical shear stress for incipient sediment motion. The formula relates the critical shear velocity, being scaled with the settling velocity, to the dimensionless sediment diameter. It agrees well with the published empirical relations that represent experimental data.

With the proposed formula, this study also demonstrates that the settling velocity can be used to represent the effective near-bed velocity, which is experienced by a typical bed sediment particle under the threshold condition, but only for large sediment sizes such as sand and gravel. Comparison results show that Yang's (1973) formula is suitable for flows with small relative flow depth while Le Roux's (1998) formula may over-

predict the critical condition for fine particles. In addition, this study also develops a single but generalized law-of-the-wall function that predicts the near-bed velocity distribution throughout the viscous sublayer, buffer layer and logarithmic layer for fully-smooth, fully-rough and transitional boundary conditions.

Appendix: Generalized law-of-the-wall for velocity distribution

The flow near a smooth bed is generally composed of viscous sublayer, buffer layer and logarithmic layer. The velocity in the viscous sublayer ($y^+ = u_* y / \nu < 5$) follows a linear distribution. The logarithmic layer ($y^+ > 30$) is characterized by the log-law, which is affected by the bed roughness related to sediment size. The buffer layer ($5 < y^+ < 30$) serves as a transition between the other two. A velocity profile throughout the three layers is formulated here in the power-sum fashion,

$$(u^+)^m = (u_1^+)^m + (u_2^+)^m \quad (9)$$

where $u^+ = u/u_*$, $u_1^+ = y^+$ (for the viscous sublayer), $u_2^+ = 2.5 \ln(y^+/y_o^+)$ (for the logarithmic layer), $y_o^+ = u_* y_o / \nu$, y_o is the hydrodynamic roughness length and m = an exponent.

Eq. (9) is subject to a singularity at $y^+ = y_o^+$ because of the logarithmic term. This shortcoming can be avoided by making a slight modification by considering that the log law is physically correct only for $y^+ > 30$. For fully-smooth boundaries, $y_o^+ \approx 0.11$, and thus the lower boundary for the log law is taken as $y^+/y_o^+ > 270$. Therefore, y^+/y_o^+ included in Eq. (9) can be replaced with $(1 + y^+/y_o^+)$ without causing significant errors. As a result, with $y_o \approx 0.11 \nu / u_*$, Eq. (9) is rewritten for fully-smooth turbulent flows as

$$(u^+)^m = (y^+)^m + [2.5 \ln(1 + 9y^+)]^m \quad (10)$$

To estimate m , Eq. (10) is compared with two law-of-the-wall formulas for turbulent flows over fully-smooth boundaries (White 1991), one given by van Driest in 1956 and the other by Spalding in 1961. Good agreement is achieved with $m = -3$.

Now, consider how y_o is modified in the presence of bed roughness. Nikuradze's pipe flow data (see Schlichting 1979) shows that for a fully-rough boundary, $y_o \approx 0.033k_s$. The transition between $y_o \approx 0.11\nu/u_*$ for a fully-smooth boundary and $y_o \approx 0.033k_s$ for a fully-rough boundary is described by (Cheng 2007)

$$y_o^+ = 0.11[1 + 0.3k_s^+(1 - e^{-0.04k_s^+})] \quad (11)$$

where $k_s^+ = u_*k_s/\nu$. Substituting Eq. (11) into Eq. (10) yields Eq. (6), the latter being plotted in Fig. 4. It shows that the log-law profile systematically shifts downwards with increasing bed roughness heights, which then extends to the boundary through a smooth transition. Further studies are needed to verify Eq. (6) or Fig. 4.

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Notation

The following symbols are used in this paper:

- D = sediment grain diameter;
- D^* = dimensionless grain diameter $[(\Delta g/v^2)^{1/3}D]$;
- g = gravitational acceleration;
- h = flow depth;
- k_s = boundary roughness size;
- k_s^+ = $u_* k_s / \nu$;
- m = constant;
- n = constant;

p_c = critical probability;
 Re^* = shear velocity based Reynolds number ($= u_* D / \nu$);
 Re_{*c} = $u_{*c} D / \nu$;
 Re_w = $w D / \nu$;
 u = time-mean longitudinal velocity;
 u_* = shear velocity;
 u_{*c} = critical shear velocity;
 u^+ = u / u_* ;
 u_{bc} = effective near-bed velocity for incipient sediment motion;
 U_c = critical depth-averaged velocity;
 w = settling velocity;
 w^* = dimensionless settling velocity ($= w / \sqrt[3]{\Delta g \nu}$);
 y = distance measured from the channel bed;
 y^+ = $u_* y / \nu$;
 y_o = hydrodynamic roughness length used in log-law;
 y_o^+ = $u_* y_o / \nu$;
 y_b = location at which u_{bc} is measured;
 α = a constant associated with the critical probability;
 ρ = fluid density;
 ρ_s = sediment density;
 Δ = $(\rho_s / \rho - 1)$;
 ν = kinematic viscosity of fluid;
 τ^* = dimensionless bed shear stress or Shields number [$= u_*^2 / (\Delta g D)$]; and

τ_{*c} = dimensionless critical shear stress $[= u_{*c}^2/(\Delta g D)]$.

List of figure captions

Fig. 1. Simplification of Eq. (1) based on the settling velocity.

Fig. 2. Variation of u_{bc}/w with D^* . The average curve is obtained based on the computations made for $a \equiv k_s/D = 1-4$ and $b \equiv y_b/D = 0.25-0.75$.

Fig. 3. Comparison of the present study with Yang's (1973) formula.

Fig. 4. A generalized wall-layer function for the prediction of velocity distribution.

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Figure2
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Figure3
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