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Power-law Index for Velocity Profiles in Open Channel Flows

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Abstract: Turbulence theory has demonstrated that the log law is one of the established theoretical results for describing velocity profiles, which is in principle applicable for the near-bed overlap region, being less than about 20% of the flow depth. In comparison, the power law that is often presented in an empirical fashion could apply to larger fraction of the flow domain. However, limited information is available for evaluating the power-law exponent or index. This paper attempts to show that the power law can be derived as a first-order approximation to the log law, and its power-law index is computed as a function of the Reynolds number as well as the relative roughness height. The result obtained also coincides with the fact that the one-sixth power included in the Manning equation is of prevalent acceptance, while higher indexes would be required for flows over very rough boundaries.

Introduction

Practical applications in hydraulic engineering have demonstrated that velocity profiles measured in wide open channels can be well represented by the power law [1, 2]. In comparison with the log law that can be derived based on dimensional reasoning or theoretical arguments [3], the power law is usually presented as the simplest data-driven formulation in spite of some recent theoretical efforts (e.g. [4]). However, the selection of the power law exponent or index appears quite empirical and even subjective.

Given the plentiful information available in the literature regarding the difference of the two laws, this paper is not to justify whether or not one law is better than the other; rather, it aims to provide an alternative theoretical consideration from which the power law could be derived analytically. The connection of the power law with the log law is then explored in light of improved understanding of wall turbulence in the past few years. This paper also details how the power-law index varies with flow and boundary conditions, and what possible approximations could be made for practical applications.

Background

Power law

For uniform equilibrium flows in a wide open channel, the power law can be expressed in the following form,

$$\frac{u}{u_{\max}} = \left(\frac{y}{h} \right)^{1/m} \quad (1)$$

where u is the streamwise, time-mean flow velocity, $u_{\max}$ is the maximum flow velocity taken at the free surface ($y = h$), y is the bed-normal distance measured upwards from the profile datum, h is the flow depth, and $1/m$ is referred to as the power-law exponent or index. In Eq. (1), the scaling parameters used are the maximum velocity and flow depth. If the power law is applied to the near-bed flow region only, $u_{\max}$ can be replaced by the shear velocity u_* and h replaced by either the wall unit ν/u_* (where ν is the kinematic viscosity of fluid) for a smooth bed or the roughness height k_s for a rough bed. However, it is always possible to transform the formulations that are derived with different scaling considerations into Eq. (1), which is used in the subsequent analysis.

In using the power law for the representation of velocity profile measurements, many m -values have been reported in the literature including studies concerning boundary layer, pipe and channel flows. One of well-cited examples is the classical case due to Prandtl, who suggested that the one-seventh power law ($m = 7$) can be used for turbulent flows over a smooth boundary to approximate the velocity profiles in boundary layer flows for low Reynolds numbers [5]. Use of this exponent is able to reproduce the Blasius' power-law resistance formulation for smooth pipe flows at low Reynolds numbers. Similar but more general information of the power-law index can be further inferred from the velocity profiles measured by Nikuradze for smooth-pipe flows [5, 6]. These profiles could be fitted well to the power law for the major fraction of the flow and the resulted m -value increases from 6 to 10 with increasing Reynolds numbers. Hinze [7] also demonstrated that the power law performs much better than the log law and could fit experimental data for $y/\delta = 0.01 \sim 1.0$, where δ is the boundary layer thickness and the lower limit roughly corresponds to 30 wall units. A recent example of such observations

was presented by Bergstrom et al [8], who employed open channel flow measurements to assess the ability of the power law.

Since the flow resistance equation can be derived from the average velocity, the same exponent can be also used for representing flow resistance in the form of power function. This can be easily verified using the power law based on the near-bed scales. The well-known Manning equation implies the average velocity as proportional to the flow depth to the one-sixth power (i.e. $m = 6$). However, the one-eighth power (i.e. $m = 8$) was also recommended [9]. Other indexes varying from $1/4$ to $1/12$ were also found in the literature, as reviewed by Chen [10]. By performing a nonlinear regression analysis based on the log law, Chen commented that each individual index is only applicable for a limited range of Reynolds numbers, and in particular, the exponent of $1/6$ could be more suitable at Reynolds numbers usually observed in practical applications. However, the exponent could require significant adjustment for some extreme cases, such as, in the presence of large-scale boundary roughness. Such a case was recently investigated by Smart et al. [11], who suggested that the m -value could reduce from 6 to 2 for channel flows subject to large-scale bed roughness.

Empirically, the power-law index could be simply related to the friction factor f for both smooth and rough pipe flows [7], yielding

$$m\sqrt{f} = C \quad (2)$$

where $C \approx 1.0$, and f is defined as $8(u_*/U)^2$ with U denoting the cross-section average velocity. Hinze [7] also reasoned that C could be taken as 1.2 based on some other approximations. If applying both power and log law to the entire flow domain from the channel bed to free surface, it can be shown that $C = \sqrt{8}\kappa$, which equals 1.13 for $\kappa = 0.4$.

This simple relation was previously used by Zimmermann and Kennedy [12] for approximating velocity distributions in meandering flows.

By extending his concept of incomplete similarity to wall-bounded turbulent flows, Barenblatt [13] argued that the power law was the sole correct scaling for mean flow velocity profiles. For smooth pipe flows, the power law derived by Barenblatt takes the form of

$$\frac{u}{u_*} = \beta \left(\frac{u_* y}{\nu} \right)^{1/m} \quad (3)$$

where both β and m depend on the Reynolds number, i.e.

$$\beta = 0.577 \ln \text{Re} + 2.5 \quad (4)$$

$$m = 0.667 \ln \text{Re} \quad (5)$$

where D is the pipe diameter and $\text{Re} = UD/\nu$. The three constants included in Eqs. (4) and (5) were estimated by Barenblatt and Prostokishin [4] with the pipe flow data collected by Nikuradze [14]. Other different formulations of β and m are also suggested by Zagarola and Smith [15] based on their own pipe flow measurements. It should be noted that the Reynolds number used here for pipe flows is based on the pipe diameter. This Reynolds number could be transformed to $4Uh/\nu$, if relevant relations are extended to wide open channel flows, based on the concept of the equivalent hydraulic diameter, which is equal to D for pipe flows and $4h$ for wide open channel flows. This transformation will be used and also discussed subsequently in this study.

Log law

The wide acceptance of the log law could be due to the fact that it can be justified with certain theoretical arguments, for example, Prandtl's mixing length assumption, von Karman's dimensional reasoning or Millikan's asymptotic analysis [3]. However, these arguments can be considered theoretically correct only in a limited region of flow (i.e. overlap region) although the log law may apply practically beyond the region. This can be explained briefly as follows. Generally, it is believed that wall-bounded turbulent flows are characterized by two kinds of length scales. In the inner region near a smooth boundary, fluid viscosity is important, and thus the acceptable length scale is the viscous length scale (or wall unit), ν/u_* . In the outer region that is sufficiently far from the boundary, flow inertia is significant and the corresponding length scale can be taken as the flow depth, h , for open channel flows. In addition, it is assumed that the shear velocity is a global velocity scale applicable both for the inner and outer region. With these considerations, the mean velocity profile can be scaled in the form

$$\frac{u}{u_*} = f\left(\frac{u_* y}{\nu}\right) \quad \text{for the inner region;} \quad (6)$$

and

$$\frac{u_{\max} - u}{u_*} = g\left(\frac{y}{h}\right) \quad \text{for the outer region.} \quad (7)$$

When the Reynolds number is sufficiently large, there is a region of overlap, $\nu/u_* \ll y \ll h$, where both scaling considerations and thus Eqs. (6) and (7) hold. Matching the two equations yields one of the classical results in turbulence theory, i.e., the logarithmic

velocity profiles or log law (see [3]). In terms of the inner variables, the log law is derived as

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \left(\frac{u_* y}{\nu} \right) + B \quad (8)$$

where κ and B are usually assumed to be constants. This log law also can be expressed in terms of the outer variables. Moreover, dimensional reasoning similar to the above can be made for the case of rough turbulent flows, for which the length scale associated with roughness size is used to replace the viscous length scale.

The overlap layer can be roughly defined as the region greater than 30 wall units and lower than 20% of the flow depth, i.e. $30\nu/u_* < y < 0.2h$ [16]. Such a range is quite narrow. In contrast, the power law could reproduce measured velocity profiles not only within the overlap region but also the majority of the outer region [7, 8]. This advantage of the power law actually serves as part of the motivation for this study.

Analytical Considerations

As stated in the foregoing section, turbulence theory has shown that the log law may apply only for the overlap region. Therefore, the analysis to be carried out in this study will be first limited to this region so that the log law can be engaged as an indirect theoretical basis. Then, the connection between the log law and power law is explored, which yields an analytical expression of the power-law index. On the other hand, as demonstrated by some previous studies [7, 8], the power law could perform much better than the log law when fitted to velocity measurements for the entire outer flow including

the overlap region. Therefore, it may be possible to extend the power law derived for the overlap region to the upper flow domain away from the channel bed.

In the following, we first derive possible power-form approximations to the log law within the overlap region. It is believed that the power function so obtained will share the same theoretical grounds as the log law. After that, the power-law index is evaluated based on flow resistance information available in the literature.

Consider wide open channel flows. For the overlap region, the log law is expressed in the following form

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y}{y_o} \right) \quad (9)$$

where y_o is the location of the zero-velocity predicted by the log law. Since it is related to the viscous length for a smooth bed or the roughness size for a rough bed, y_o is also called hydrodynamic roughness length.

The upper limit of the overlap layer is the same as that of the inner region. Here we denote it by $y = y_{ir}$ and the corresponding velocity by $u = u_{ir}$, with subscript ‘ir’ standing for the ‘inner region’. This is sketched in Fig. 1. Then with Eq. (9),

$$\frac{u_{ir}}{u_*} = \frac{1}{\kappa} \ln \left(\frac{y_{ir}}{y_o} \right) \quad (10)$$

Using Eqs. (9) and (10) to eliminate u_* , one gets

$$\frac{u}{u_{ir}} = \frac{1}{\alpha} \ln \left(\frac{y}{y_{ir}} \right) + 1 \quad (11)$$

where $\alpha = \ln(y_{ir}/y_o)$, or

$$\phi_L = \ln(\eta^{1/\alpha}) + 1 \quad (12)$$

where $\phi_L = u/u_{ir}$, $\eta = y/y_{ir}$, and subscript ‘L’ denotes the log law.

Similarly, the power law is expressed as

$$\frac{u}{u_{ir}} = \left(\frac{y}{y_{ir}} \right)^{1/m} \quad (13)$$

or

$$\phi_P = \eta^{1/m} \quad (14)$$

where subscript 'P' denotes the power law.

To compare the two functions given by Eqs. (12) and (14), the following two conditions are imposed:

$$\phi_L = \phi_P \quad (15)$$

$$\frac{d\phi_L}{d\eta} = \frac{d\phi_P}{d\eta} \quad (16)$$

With Eqs. (11) and (13), Eqs. (15) and (16) can be further written, respectively, as

$$\eta^{1/m} = \ln(\eta^{1/\alpha}) + 1 \quad (17)$$

$$\frac{1}{m} \eta^{(1/m)-1} = \frac{1}{\alpha \eta} \quad (18)$$

Combining Eqs. (17) and (18) by eliminating η yields the relationship between m and α ,

$$\ln \frac{\alpha}{m} = \frac{\alpha}{m} - 1 \quad (19)$$

The only solution to Eq. (19) is $m = \alpha$. This result suggests that the index used in the power law can be associated only with the ratio of the thickness of the inner region to the hydrodynamic roughness length, y_{ir}/y_o . A result similar to Eq. (19) was also presented by Chen [10], but by applying the log law to the entire flow region.

Using the result of $m = \alpha$, the power law is thus given by

$$\frac{u}{u_{ir}} = \left(\frac{y}{y_{ir}} \right)^{1/\ln(y_{ir}/y_o)} \quad (20)$$

Eq. (20) can be considered as a first approximation to the logarithmic law. This is explained in detail in the subsequent section.

Approximation based on series expansion

First consider the function of $\ln(1 - x)$ for $0 \leq x < 1$. It can be expanded in the form of Taylor series,

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad (21)$$

By taking $x = 1 - \eta^{1/\alpha}$ (where $0 < \eta \leq 1$, $\alpha > 0$) and substituting into Eq. (21), we have

$$\begin{aligned} \ln(\eta^{1/\alpha}) &= \ln[1 - (1 - \eta^{1/\alpha})] \\ &= -(1 - \eta^{1/\alpha}) - \frac{1}{2}(1 - \eta^{1/\alpha})^2 - \frac{1}{3}(1 - \eta^{1/\alpha})^3 - \frac{1}{4}(1 - \eta^{1/\alpha})^4 - \dots \end{aligned} \quad (22)$$

Substituting Eq. (22) into Eq. (11) yields

$$\phi = \eta^{1/\alpha} - \frac{1}{2}(1 - \eta^{1/\alpha})^2 - \frac{1}{3}(1 - \eta^{1/\alpha})^3 - \frac{1}{4}(1 - \eta^{1/\alpha})^4 - \dots \quad (23)$$

Obviously, Eq. (23) takes the power form, which is the same as Eq. (20), if the first term on the RHS of Eq. (23) is considered only. This means that the power function is the first-order approximation to the log law. For comparison, if the first two terms are included, Eq. (23) reduces to

$$\phi = \eta^{1/\alpha} - \frac{1}{2}(1 - \eta^{1/\alpha})^2 = \eta^{1/\alpha} \left(2 - \frac{1}{2}\eta^{1/\alpha} - \frac{1}{2}\eta^{-1/\alpha} \right) \quad (24)$$

Eq. (24) can be considered as the second-order approximation to the log law. From the series expansion, it follows that the approximations perform better for smaller values of $(1 - \eta^{1/\alpha})$, i.e. for the upper part of the inner flow region that is not so close to the channel bed. Fig. 2 shows the comparisons between the approximations of different degrees and the log law defined by Eq. (11).

Determination of power-law index

To evaluate the power-law index, the following analysis is performed based on the first-order approximation. This may cause certain deviations, which however will be corrected by validating the analysis with experimental results available in the literature. Since the power law could be extended up to the upper edge of the outer region, we may simply take the upper limit at $y = h$ where $u = u_{\max}$ when applying Eq. (20) for wide open channel flows without altering the exponent, which actually recovers Eq. (1). It is noted that this change in the power law formulation has no effects on the power-law index. Therefore, the power law defined by Eq. (1) can be used with the index given by

$$\frac{1}{m} = \ln \left(\frac{y_o}{y_{ir}} \right) \quad (25)$$

To evaluate this index, it is necessary to first understand how the hydrodynamic roughness length, y_o , and the thickness of the inner region, y_{ir} , are associated with flow and boundary conditions.

(a) Hydrodynamic roughness length, y_o

The hydrodynamic roughness length is generally dependent on two typical length scales, ν/u_* , the viscous length scale, and k_s , the height of the channel bed roughness that could be described by the size of bed sediment particles. To quantify the dependence, we here make use of the classical Nikuradze's pipe flow data. This is because similar comprehensive measurements are not yet available for open channel flows to the writer's knowledge. Nikuradze [6] conducted the seminal series of flow measurements with smooth and rough pipes over a wide range of Reynolds numbers. These measurements remain the benchmark in hydraulic engineering and have served as a crucial database for verifying any formulations proposed for velocity profiles and flow resistance. In Nikuradze's experiments, both smooth and rough pipes were tested. The rough pipes were prepared with pipe walls roughened by well-sorted sand particles, which yielded the relative roughness of r/k_s ranging from 15 to 507, where r is the pipe radius and k_s is the sand diameter.

From Nikuradze's data, empirical relations can be deduced to quantify the variation of y_o with both parameter, ν/u_* and k_s . If being composed of very fine roughness elements, the flow boundary may be considered to be hydraulically smooth. For turbulent flows over such a smooth boundary, if u_*k_s/ν is small, y_o is proportional to ν/u_* . Approximately, the relation takes the form $y_o \approx 0.11\nu/u_*$. For a boundary of coarse roughness elements which is hydraulically rough, y_o is proportional to the physical roughness height for large u_*k_s/ν . The experimental results show that $y_o \approx 0.033k_s$, where k_s = characteristic height of the rough boundary referred to as roughness height. An empirical interpolation between $y_o \approx 0.11\nu/u_*$ for the smooth boundary and $y_o \approx 0.033k_s$ for the rough boundary is proposed in this paper as

$$y_o = 0.11 \frac{v}{u_*} + 0.033k_s \left[1 - \exp\left(-0.04 \frac{u_* k_s}{v}\right) \right] \quad (26)$$

In Eq. (26), the varying viscous effect is characterized by the exponential damping function, $[1 - \exp(-0.04u_*k_s/v)]$, which is similar to that suggested by van Driest [5] for extending the mixing length to the flow boundary. Eq. (26) is plotted in Fig. 3, showing that it represents well the Nikuradze's experimental data for the flow boundaries roughened by well-sorted sediment particles. From the figure, it can be seen that the transitional bed situation between smooth and rough flows occurs roughly for $u_*k_s/v = 5 \sim 70$. It should be mentioned that the interpolation proposed previously by Colebrook [17] for commercial pipe flow resistance, if expressed in terms of y_o , is simply a linear combination of the scales, i.e. $y_o = 0.1v/u_* + 0.03k_s$, which fails to fit the Nikuradze's data for the transitional bed condition. Furthermore, it is believed that Eq. (26) may require some adjustments if boundary roughness elements are poor-sorted or mobile, as those observed in the process of sediment transport.

(b) Thickness of inner region, y_{ir}

Empirically, the thickness of the inner region is taken as 20% of the flow depth for wide open channels, i.e. $y_{ir}/h = 0.2$ [16]. This limit, although crude, is often taken as a guideline in open channel flow studies to identify the upper location where the log law should be applied so that the shear velocity can be evaluated from flow velocity measurements. Other similar limits can also be found in studies concerning boundary layer and pipe flows. For example, Zagarola and Smits [18] suggested that the fraction of the inner region was limited to 7% for smooth pipe flows, while Osterlund et al.'s [19]

analysis shows that the outer edge of the overlap region is located at $y/\delta = 0.2$ for turbulent boundary layers developed over a flat plane. Purtell et al. [20] reported that the fraction of the log-law region in the boundary layer thickness is a function of the Reynolds number. Tsuji and Nakamura [21] also argued that the fraction is not a constant but approaching the value of 20% as the Reynolds number increases.

In this study, we attempt to formulate this Reynolds number dependence based on turbulence statistics near the flow boundary. The thickness of the inner region can be considered as a measure of the extent to which the boundary effect interacts with the flow inertia. Such interactions can be also characterized by turbulence statistics, for example, the location of maximum velocity fluctuation or shear stress. For smooth pipe flows, Morrison et al [22] reported that the location of the maximum of the squared longitudinal velocity fluctuation ($\overline{u'^2}$), when scaled by the wall unit, increases with the shear Reynolds number,

$$\frac{u_* y_p}{\nu} = 1.3 \left(\frac{u_* D}{\nu} \right)^{0.52} \quad (27)$$

where y_p is the location of the maximum $\overline{u'^2}$. A similar relationship can be also proposed for the peak shear stress near the boundary [18, 23]. If using the Blasius friction factor equation, $f = 0.316 \text{ Re}^{-0.25}$, and noting the definition of $f = 8(u_*/U)^2$, Eq. (27) is re-written as

$$\frac{y_p}{D} = 2.8 \text{ Re}^{-0.42} \quad (28)$$

Eq. (28) shows that the peak location, when scaled with the pipe diameter, decreases with the Reynolds number. However, this relation could vary if other approaches are used for

quantifying the inner-outer interaction. For example, if the concept of ‘transitional critical layer’ is used for describing the interaction, the exponent given in Eq. (28) could change to $-0.15 \sim -0.29$ [22].

Given the above information, it is reasonable to assume here that for smooth open channel flows,

$$\frac{y_{ir}}{h} = \zeta \left(\frac{Uh}{\nu} \right)^{\lambda} \quad (29)$$

where ζ and λ are constants, and the exponent λ may be in the range of -0.1 to -0.5 based on the above discussion. Eq. (29) with the constants evaluated later is plotted in Fig. 4, showing that the relative thickness of the inner region, which is often taken as a constant (e.g., 0.1 - 0.2) in the previous studies, decreases gradually with increasing Reynolds number. This Reynolds number dependence may explain why different upper edges of the overlap region have been used by previous researchers for processing experimental data represented by the log law. For example, Zagorala and Smits’ [18] superpipe flow measurements were conducted at relatively high Reynolds numbers, so that the near-wall flow fraction suitable for evaluating the log law was found quite small, being less than 7% .

Substituting the relation $y_0 = 0.11\nu/u_*$ and Eq. (29) into Eq. (25), we get for the case of smooth boundary

$$m = \ln \left[\frac{\zeta}{0.11} \frac{u_*}{U} \left(\frac{Uh}{\nu} \right)^{\lambda+1} \right] \quad (30)$$

Noting that the Blasius equation, when applied for wide open channel flows, gives $u_*/U = 0.167(Uh/\nu)^{-0.125}$ for $Uh/\nu = 750$ - 25000 , Eq. (30) can be rewritten as

$$m = \ln \left[1.518 \zeta \left(\frac{Uh}{\nu} \right)^{\lambda+0.875} \right] \quad (31)$$

Comparing Eq. (31) to Eq. (5), which was derived from Nikuradze's data, and noting that $h = D/4$ yields $\lambda = -0.208$ and $\zeta = 1.661$. With these two constants and also the relation $h/y_o = 1.518 (Uh/\nu)^{0.875}$ derived from the Blasius equation, Eq. (29) can be rewritten as

$$\frac{y_{ir}}{h} = 1.834 \left(\frac{h}{y_o} \right)^{-0.238} \quad (32)$$

and thus

$$\frac{y_{ir}}{y_o} = 1.834 \left(\frac{h}{y_o} \right)^{0.762} \quad \text{or} \quad y_{ir} = 1.834 h^{0.762} y_o^{0.238} \quad (33)$$

Eqs. (32) and (33) are derived for smooth bed conditions at low Reynolds numbers, i.e. $Uh/\nu = 750-25000$.

Eq. (33) shows that the inner region grows with an increase in the flow depth h or the hydrodynamic roughness length y_o , which is physically understandable. Furthermore, since y_o is generally associated with ν/u_* and k_s , a similar relation to Eq. (33) is expected to also exist for rough boundary conditions. However, this cannot be verified at the current stage because no quantitative information is available for characterizing bed roughness effects on the logarithmic overlap region.

(b) Variation of m

From Eqs. (25), (26) and (33), it follows that m varies with the two dimensionless parameters, h/k_s and $u_* k_s/\nu$. Noting that $u_* k_s/\nu = (k_s/h)(Uh/\nu)(u_*/U) = (k_s/h)(Uh/\nu)\sqrt{f/8}$,

the variation of m can also be presented in terms of the flow-depth based Reynolds number Uh/ν and also the relative roughness height h/k_s . This is because the friction factor f is also a function of Uh/ν and h/k_s . In this study, f or u^*/U is computed based on the experimental relation provided by Nikuradze [6]. For convenience, the following formula is proposed here to represent Nikuradze's data for turbulent flows (as shown in Fig. 6),

$$f = (0.316 \text{Re}^{-0.25})^Z \left\{ \left[2 \log \left(\frac{r}{k_s} \right) + 1.74 \right]^{-2} \right\}^{1-Z} \quad (34)$$

where

$$Z = \frac{1}{1 + \left(\frac{\text{Re}}{320r/k_s} \right)^2} \quad (35)$$

In Eq. (34), the first part on the RHS is associated with flows over smooth boundary and the second part accounts for contributions of the boundary roughness. Note that the pipe diameter used for pipe flows can be modified $4h$ for wide open channel flows by assuming that the hydraulic diameter remains unchanged.

The computed variation of m with Re and r/k_s is plotted in Fig. 5. Also superimposed in the figure is the formula [Eq. (5)] proposed by Barenblatt and Prostokishin [4] for smooth pipe flows. For simplicity, relevant computations are performed for the Reynolds number and relative roughness height in the ranges to the same as those measured by Nikuradze [14]. Fig. 5 shows that for a given relative bed roughness, with increasing Reynolds number, the m -value first increases for the smooth bed condition, then maximizes in the transitional flow regime, and finally becomes constant for the rough bed

condition. For very large Reynolds numbers, m is only related to the relative roughness height, becoming relatively small for flows subject to large-scale bed roughness.

Fig. 5 also shows an interesting result that in the range of the Nikuradze's measurements ($15 < r/k_s < 507$), the power ranges approximately from $1/7.7$ to $1/4.7$ for $Re > 3000$, and in the middle of the range happens to be $1/6$, the index being included in the Manning equation. The value of $1/m$ varies, but within a small range. For example, if the relative roughness height increases by 100%, say from $r/k_s = 31$ to 60, the calculated m -value increases only by 9%, i.e. from 5.3 to 5.8, for large Reynolds numbers, say, $Re > 2 \times 10^5$. If considering the entire data range where r/k_s varies from 15 to 507, an increase by 3280%, the m -value increases from 4.7 to 7.7, i.e. only by 64%. This result indicates that the variation of m is not very sensitive to the variation of the relative roughness height. Its further implication is that that other power values might yield the best fit when applying the power law for particular flow measurements, but the power of $1/6$ would generally provide acceptable predictions.

(d) Approximate relation between m and f

Since both m and f depend on the Reynolds number and relative roughness height, the value of m could be computed based on f only. Such efforts were made previously, e.g. by Zimmermann and Kennedy [12] for investigating meandering flows. This possibility is further explored here based on the computations performed in this study. In Fig. 7, $m\sqrt{f}$ is plotted against Re with reference to Eq. (2). It shows that the calculated value of $m\sqrt{f}$ fluctuates but within a small range of 1.0-1.2. This means that the simple

analyses performed by Hinze [7] and Zimmermann and Kennedy [12], respectively, are acceptable, which suggest that $m\sqrt{f}$ be taken as a constant. Fig. 7 shows that if $m\sqrt{f}$ is approximated as 1.1, the error caused would be about $\pm 10\%$ for the flow conditions concerned.

In addition, Fig. 8 gives a plot of m against f using the same calculated results, showing that the best-fit curve is given by

$$m = 1.37 f^{-0.43} \quad (36)$$

which slightly deviates from the previous result of $m\sqrt{f}$ being taken as a constant. For comparisons, three curves computed by assuming that $m\sqrt{f} = 1.0, 1.1,$ and $1.2,$ respectively, are also superimposed on the figure.

In comparison to the procedure involved in applying the derived theoretical formula [Eq. (25)], the empirical relation [Eq. (36)] is simple and therefore could be used a good approximation for the selection of the power-law index.

Discussion

This paper attempts to analytically investigate the power-law index, but much information of flow structures used are originally derived from pipe flows. For example, the friction factor was computed with reference to the classical Nikuraze's pipe flow measurements based on the concept of the equivalent hydraulic diameter. Although the use of the hydraulic diameter concept is the common engineering practice of computing non-circular duct friction, it has been reported that the application of the approximation,

in particular, to open channel flows is subject to limitations [24, 25]. This is probably due to the fact that the presence of free surface in the wall-bounded turbulent flows would definitely differentiate open channel flows from pipe flows to some extent. To empirically offset the deviation induced by the use of the hydraulic diameter concept, some researchers suggested that the relation, $D = 4h$, be adjusted to, $D = (3-3.3)h$ for wide open channel flows [2, 24].

To investigate possible effects of the above adjustment on the results presented in this study, additional computations are performed. The results obtained show that if it is assumed that $D = 3h$, the m -value shown in Fig. 5 will be increased approximately by 0.22, and the best-fit curve shown in Fig. 8 will be changed to $m = 1.52f^{-0.41}$ in place of Eq. (36). Such changes seem insignificant. However, to further improve the current analysis, experimental works that are purposely designed for open channel flows are needed to be conducted in future.

The previous studies imply that the upper edge of the logarithmic overlap layer varies with the Reynolds number for flows over smooth boundary. However, it is not clear how the upper edge varies in the presence of the boundary roughness. In this paper, the scaling relation derived for the upper edge is in principle only applicable for smooth boundary conditions. This relation may not hold for other flow regimes; its verification and further improvement would depend on the availability of relevant turbulence statistics.

It has been clearly shown that the power-law index is closely related to the friction factor. This suggests that the flow resistance could be computed by first fitting measured velocity profile to the power law, and then estimating the friction factor from the power-

law index. This approach may serve as a good alternative to the use of the log law for evaluating the shear velocity or bed shear stress, in particular, by noting that the log law requires possible wake-function correction if data-fitting is performed far above the overlap region.

Conclusions

For studying wide open channel flows, the power law is conventionally considered as a simple but purely empirical formula for representing velocity profiles. This may be largely because complete information is not available for evaluating the power law exponent or index. In this paper, attempts are made to perform a relatively rigorous derivation of the power law based on the classical turbulence theory.

The result shows that the power law can be considered as a first-order approximation to the log law in the overlap region, but is applicable to the majority of the flow domain. The power-law index can be associated analytically with the Reynolds number and the relative roughness height, but also appears approximately related to the friction factor in a simple fashion. The finding of this study coincides with the fact that the one-sixth power as in the Manning equation is preferable for many engineering applications but higher power indexes are needed for flows over very rough boundaries.

Notation

The following symbols are used in this paper:

- $1/m$ = power-law exponent or index;
- C = constant;
- D = pipe diameter;
- f = friction factor defined as $8(u_*/U)^2$;
- h = flow depth;
- k_s = roughness height;
- r = pipe radius;
- Re = UD/ν = Reynolds number;
- u = streamwise, time-mean flow velocity;
- U = cross-sectional (or depth) average velocity;
- u_* = shear velocity;
- u_{ir} = streamwise velocity at $y = y_{ir}$;
- u_{max} = maximum flow velocity taken at the free surface ($y = h$);
- $\overline{u'^2}$ = squared longitudinal velocity fluctuation;
- y = bed-normal distance measured upwards from the profile datum;
- y_{ir} = bed-normal location of the upper edge of the overlap region or inner region;
- y_o = hydrodynamic roughness length or bed-normal location of zero-velocity predicted by the log law;
- Z = parameter used in Eq. (34);
- ν = kinematic viscosity of fluid;
- δ = boundary layer thickness;
- κ = von Karman constant;

$$\alpha = \ln(y_{ir}/y_o);$$

$$\beta = \text{coefficient};$$

$$\phi = u/u_{ir};$$

$$\eta = y/y_{ir};$$

$$\zeta = \text{constant; and}$$

$$\lambda = \text{constant.}$$

References

1. Chanson, H., *The hydraulics of open channel flow: an introduction*. 2nd ed. 2004, Oxford; Burlington, MA: Elsevier Butterworth-Heinemann. xlvii, 585 , [16] of plates.
2. Montes, S., *Hydraulics of open channel flow*. 1998, Reston, VA: ASCE Press. viii, 697.
3. Kundu, P.K., I.M. Cohen, and H.H. Hu, *Fluid mechanics*. 3rd ed. 2004, Amsterdam ; Boston: Elsevier Academic Press. xxiv, 759 p.
4. Barenblatt, G.I. and V.M. Prostokishin, *Scaling Laws for Fully-Developed Turbulent Shear Flows .2. Processing of Experimental-Data*. Journal of Fluid Mechanics, 1993. 248: p. 521-529.
5. Schlichting, H., *Boundary-layer theory*. 7th ed. McGraw-Hill series in mechanical engineering. 1979, New York: McGraw-Hill. xxii, 817 p.

6. Nikuradze, J., *Stromungsgesetze in rauhen Rohren*. Forschung auf dem Gebiete des Ingenieurwesens, 1933. Forschungsheft 361. VDI Verlag, Berlin, Germany (in German). (English translation: Laws of flow in rough pipes, NACA TM 1292, 1950).
7. Hinze, J.O., *Turbulence*. 2d ed. McGraw-Hill series in mechanical engineering. 1975, New York: McGraw-Hill. x, 790 p.
8. Bergstrom, D.J., M.F. Tachie, and R. Balachandar, *Application of power laws to low Reynolds number boundary layers on smooth and rough surfaces*. Physics of Fluids, 2001. 13(11): p. 3277-3284.
9. Qian, N. and Z. Wan, *Mechanics of sediment transport*. 1999, Reston, Va.: American Society of Civil Engineers. xix, 913 p.
10. Chen, C.I., *Unified Theory on Power Laws for Flow Resistance*. Journal of Hydraulic Engineering-ASCE, 1991. 117(3): p. 371-389.
11. Smart, G.M., M.J. Duncan, and J.M. Walsh, *Relatively rough flow resistance equations*. Journal of Hydraulic Engineering-ASCE, 2002. 128(6): p. 568-578.
12. Zimmermann, C. and J.F. Kennedy, *Transverse bed slopes in curved alluvial streams*. Journal of Engineering Mechanics-ASCE, 1978. 104(HY1): p. 33-48.
13. Barenblatt, G.I., *Scaling Laws for Fully-Developed Turbulent Shear Flows .1. Basic Hypotheses and Analysis*. Journal of Fluid Mechanics, 1993. 248: p. 513-520.
14. Nikuradze, J., *Stromungsgesetze in rauhen Rohren*. Forschungsheft, 1933. 361, Verein Deutscher Ingenieure, Berlin [English translation published as Technical Memorandum 1292, National Advisory Committee for Aeronautics (1950)].
15. Zagarola, M.V. and A.J. Smits, *Scaling of the mean velocity profile for turbulent pipe flow*. Physical Review Letters, 1997. 78(2): p. 239-242.

16. Nezu, I. and H. Nakagawa, *Turbulence in open-channel flows*. 1993, Rotterdam, Netherland: A.A. Balkema. xii, 281.
17. Colebrook, C.F., *Turbulent flow in pipes with particular reference to the transition region between the smooth and rough pipe laws*. Proceedings of the Institution of Civil Engineers, 1939. 12: p. 393-422.
18. Zagarola, M.V. and A.J. Smits, *Mean-flow scaling of turbulent pipe flow*. Journal of Fluid Mechanics, 1998. 373: p. 33-79.
19. Osterlund, J.M., et al., *A note on the overlap region in turbulent boundary layers*. Physics of Fluids, 2000. 12(1): p. 1-4.
20. Purtell, L.P., P.S. Klebanoff, and F.T. Buckley, *Turbulent Boundary-Layer at Low Reynolds-Number*. Physics of Fluids, 1981. 24(5): p. 802-811.
21. Tsuji, Y. and I. Nakamura, *Probability density function in the log-law region of low Reynolds number turbulent boundary layer*. Physics of Fluids, 1999. 11(3): p. 647-658.
22. Morrison, J.F., et al., *Scaling of the streamwise velocity component in turbulent pipe flow*. Journal of Fluid Mechanics, 2004. 508: p. 99-131.
23. Sreenivasan, K.R. and S. Anupam (1997) *The persistence of viscous effects in the overlap region, and the mean velocity in turbulent pipe and channel flows*. arXiv:physics/9708016 Volume, 1-15
24. White, F.M., *Viscous fluid flow*. 2nd ed. McGraw-Hill series in mechanical engineering. 1991, New York: McGraw-Hill. xxi, 614.
25. Yen, B.C., *Open channel flow resistance*. Journal of Hydraulic Engineering-ASCE, 2002. 128(1): p. 20-39.

Fig. 1 Sketch of the overlap region characterized by the log law.

Fig. 2 Power-law approximation to the log law in the inner region.

Fig. 3 Variations of the hydrodynamic roughness length for different flow regimes. Data points are reproduced from Schlichting [5].

Fig. 4 Calculated decrease in the thickness of the inner region, as given by Eq. (29).

Fig. 5 Calculated dependence of the power-law index on Reynolds number and relative roughness height.

Fig. 6 Friction factor evaluated based on Nikuradze's data for turbulent pipe flows. Data points are reproduced from Schlichting [5]. The solid lines are computed using Eq. (34).

Fig. 7 Calculated variations of $m\sqrt{f}$ with Reynolds number and relative roughness height.

Fig. 8 Calculated relation (denoted by symbols) between m and f , in comparison with empirical functions (denoted by lines).