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Tracking of a Moving Ground Target by a Quadrotor using a Backstepping Approach based on a Full State Cascaded Dynamics

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Abstract

In this paper, a tracking controller is formulated for a quadrotor to track a moving ground target. The quadrotor exhibits distinct hierarchical dynamics that allows its position to be controlled by its attitude. This motivates the use of backstepping control on the underactuated quadrotor. Most backstepping architecture controls the quadrotor position and attitude independently, and couples them with inverse kinematics. Inverse kinematics computes the attitude angles required to achieve a desired acceleration. However unmodelled effects are shown to cause inexact inversion resulting in tracking error. The approach proposed in this paper uses a re-formulated full state cascaded dynamics to eliminate the need for inverse kinematics in a full state backstepping control architecture. It is shown that zero steady state error is achieved in the presence of unmodelled aerodynamics effect and wind disturbance despite no integral action. In addition, a backstepping formulation is derived using contraction theory that guarantees the boundedness of state response under bounded disturbances such as wind. This improves the system performance. Numerical simulations are performed using the proposed controller to track a target moving along predefined paths and the results are compared with a benchmark controller derived using inverse kinematics. The results show that the proposed controller is able to achieve better tracking performance under unmodelled aerodynamic effects and wind disturbance as compared with the benchmark controller.

Keywords: quadrotor, UAV, backstepping, contraction, tracking

1. Introduction

In recent years, the use of multi-rotor unmanned aerial vehicle (UAV) has gained popularity, especially the four-rotor UAV commonly known as the quadrotor. This is because the multi-rotor UAV is able to perform vertical takeoff and landing (VTOL), hover and possesses exceptional agility. These abilities allow the multi-rotor UAV to perform a diverse range of tasks such as rescue, surveillance, inspection and military operations [1, 2].

Tracking of a moving target is critical in missions such as surveillance, reconnaissance and operations that require the UAV to track a moving platform [3]. The target tracking of an UAV requires the ground path of an UAV to follow a randomly moving ground target. Its challenge primarily arises from the unknown nature of the path traced by the target thus only the position information of the target is available for feedback.

Research has been conducted on tracking of a moving target using multi-rotor UAVs, especially using the quadrotor. Different types of control methodologies are applied that include PD control of the inner and outer-loop dynamics [4, 5], nested saturation control [6, 7, 8], model predictive control [9, 10], dynamic inversion control [11], sliding mode control [12], adaptive control [13], vision-based control [3] and backstepping control [1, 14, 15, 16, 17, 18, 19].

This paper focuses on the backstepping control methodology as it is the most commonly used control method. The dynamics of the multi-rotor UAV is distinctly divided into rotational and translational dynamics, with the translational dynamics affected by the rotational motion but not vice versa. This underactuated structure [6, 18, 20, 21] is well-suited for the backstepping technique as the multi-rotor UAV dynamics has a cascaded structure. Figure 1 shows the structure of a typical multi-rotor UAV highlighting the suitability of using backstepping control. Backstepping control is also widely used in other control systems such as [22, 23, 24].

The separation of the backstepping control architecture into translational and rotational control often requires the use of inverse kinematics to couple the output of the rotational controller and the input of the translational controller [1, 4, 5, 12, 18]. Inverse kinematics is an algebraic inversion that computes the attitude angles that are required to achieve a certain acceleration based on a quadrotor dynamic model. Unmodelled dynamics often causes the computed attitude angles to be erroneous leading to an offset or inaccuracies in the position tracking. The following example illustrates this

Figure 1: Structure of the multi-rotor UAV. The rotational dynamics is controlled by the moment input while the translational dynamics is affected by both the attitude angles and the thrust input. Backstepping control is able to utilize this structure to control the translational motion of the UAV.

effect.

Example 1. Consider a system

$$\dot{x}_1 = x_2, \quad (1a)$$

$$x_2 = h(x_3), \quad (1b)$$

$$\dot{x}_3 = u, \quad (1c)$$

where x_1 and x_3 are the system states, u is the control input and x_2 is related to x_3 by a nonlinear function $h(\cdot)$.

Assume that there exists a backstepping control law $u = g(x_1, x_3)$ that makes x_1 tracks $x_{1,d}$ by computing the desired values of x_2 and x_3 which are $x_{2,d}$ and $x_{3,d}$ respectively.

Inverse kinematics is the relationship $x_{3,d} = h^{-1}(x_{2,d})$ that relates the desired value $x_{2,d}$ to $x_{3,d}$ so that $x_3 \rightarrow x_{3,d} \Rightarrow x_2 \rightarrow x_{2,d}$. This works well if the inversion matches the actual physical system.

However suppose that there exists unmodelled effects such that the actual relationship between x_2 and x_3 is $x_2 = h(x_3) + d$, then x_2 will track a value $x_{2,d} + d$ instead of the intended $x_{2,d}$.

This will result in a steady state error $(x_1 - x_{1,d})$ for any nonzero function d .

This is a problem commonly associated with the use of inverse kinematics. The impact of unmodelled effects such as wind becomes more pronounced under windy environment. This problem can be solved by using integral action such as in the work done by Raffo *et. al.* [25]. Nevertheless, the present paper proposes a novel re-formulation of the quadrotor dynamics into a full state cascaded form and eliminates the use of inverse kinematics.

Thereafter a full state backstepping controller is proposed that achieves the effect of integral action without increasing the order of the controller (see details in Remark 6 and illustration in Example 2 in Section 5). Without using integral action, the order of the proposed controller is not increased thus is advantageous in terms of implementation complexity.

The effect of improved robustness using the proposed controller is similar to the introduction of integral action. This is because of the use of a re-formulated full state cascaded dynamics that eliminates the use of integral action from the rotational to the translational dynamics. Nevertheless, there are other forms of robust control such as sliding mode control [26], adaptive control [27] and other robust control methods [28]. These methods apply compensation techniques for external disturbances while our proposed controller uses the inherent structure of the dynamics for an integral action effect. This allows for a lower order controller design using the proposed method.

In this paper, backstepping control of a quadrotor is proposed to achieve precise tracking of a moving ground target using a re-formulated full state cascaded dynamics to eliminate the use of inverse kinematics and its associated disadvantage. Full state cascaded dynamics is formulated from a six degrees of freedom quadrotor dynamics to achieve acceleration control in a nonlinear affine form. This also transforms the dynamics of the system into a cascaded form required for applying backstepping control. This formulation is the main contribution of this paper.

Furthermore, this paper also presents a backstepping control synthesis methodology using contraction theory [29]. A contracting system is a strong form of stability that is defined by decreasing virtual displacement between neighboring state trajectories. In addition, a contracting system is also bounded in the presence of bounded disturbance. Therefore the proposed controller derived using contraction theory-based backstepping control exhibits good robust performance that is demonstrated in this paper. This is the secondary contribution of this paper.

The paper is organized as follows: Section 2 introduces contraction theory and Section 3 combines the backstepping control formulation with contraction theory. Section 4 describes the quadrotor dynamics. Section 5 formulates the tracking controller based on the derived full state cascaded dynamics and contraction-based backstepping control. Section 6 provides the results of the numerical simulation. Concluding remarks are given in Section 7.

2. Contraction Theory

The contraction theory, developed by Slotine [29, 30, 31, 32], is a stability analysis method that considers infinitesimal changes between state trajectories using the concept of virtual displacements described in a differential framework. Stability in the sense of contraction implies that any two trajectories will converge onto each other over time.

In backstepping control, a state variable is used as a virtual control and the control objective is achieved when the state variable is able to converge onto the virtual control. Contraction analysis is thus well-suited for backstepping control as stability in the sense of contraction implies the convergence of the state variable to the virtual control.

In this section, the fundamentals of contraction theory that is relevant to the formulation of backstepping control is provided. Interested readers can refer to [29] for more details on contraction theory.

2.1. Fundamentals of Contraction Theory

Consider a nonlinear system given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system and t is the time. Using the concept of virtual displacement, the virtual dynamics is written as

$$\delta \dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \delta \mathbf{x}, \quad (2)$$

where $\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}$ is the Jacobian of the system.

The time variation of the term $\delta \mathbf{x}^T \delta \mathbf{x}$ is computed as

$$\frac{d}{dt} (\delta \mathbf{x}^T \delta \mathbf{x}) = \delta \mathbf{x}^T \left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right) \delta \mathbf{x}, \quad (3)$$

where $\delta \mathbf{x}^T \delta \mathbf{x}$ physically indicates the squared distance between infinitesimally close trajectories. It is noted that the virtual displacement is a well-defined mathematical object that describes the infinitesimal change in the state trajectories at a fixed time.

If the real part of the Jacobian $\left[\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right]$ is uniformly negative definite, then the distance between state trajectories decreases exponentially to zero. Hence the dynamics is stable since the state trajectories converge into each other. A region in the state space where the above condition is satisfied is called a *contraction region*. The fundamental theorem of contraction theory is summarized below.

Theorem 1. [29, p.10] *Given the system equations, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory in the state space which starts in a ball of constant radius centered about a given trajectory and contained at all times in a region satisfying $\left(\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}\right)^T + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} < 0$ (i.e. the contraction region), remains in the ball and converges exponentially to that given trajectory.*

Note that $\mathbf{A} < 0$ for any symmetric matrix (i.e. $\mathbf{A} = \mathbf{A}^T$) refers to matrix negative definiteness. Furthermore if $\mathbf{A}$ is not symmetric, $\mathbf{A} < 0$ refers to $\mathbf{A} + \mathbf{A}^T < 0$. This notation is used for the rest of this paper.

Stability in terms of a contracting system is a stronger notion for stability than Lyapunov stability. Therefore state trajectories in the contraction region exhibits various useful properties. In particular, all state trajectories are contained within a ball of a bounded radius centered about a given state trajectory under the influence of bounded disturbances. The proof can be found in [29]. This property is critical to the application of contraction theory to backstepping control. This is because instances when a state variable has not converged onto the commanded virtual control input can be regarded as a presence of a bounded disturbance. This results in a bounded state response, thus ensuring the eventual convergence of the system.

Theorem 2. [29, p.22] *Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ in a contraction region, and a perturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{d}(\mathbf{x}, t)$. Let the largest eigenvalue of the symmetric part of the Jacobian of $\mathbf{f}(\mathbf{x}, t)$ be $\lambda_{\max} < 0$. Then the relative velocity between trajectories follows $\dot{R} + |\lambda_{\max}|R \leq \|\mathbf{d}\|$. Furthermore if the perturbation is bounded by $\|\mathbf{d}\| < d$ for some $d(\mathbf{x}, t) > 0$. Then the response of the perturbed system will converge to that of the original system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ within a boundary ball of radius R governed by $\dot{R} + |\lambda_{\max}|R \leq d$.*

Theorem 2 provides an upper bound to the ball for which trajectories converge into given the size of the bounded disturbance. It also suggests that in the case of exponentially decreasing disturbance, the ball decreases exponentially to zero.

2.2. Partial Contraction

Partial contraction stems from a more general contraction theory. It is commonly used in the synchronization of coupled systems [33, 34, 35]. It can also be used in the formulation of a backstepping control law where the state variable is synchronized with the commanded virtual control input.

Consider an auxiliary system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$, where $\mathbf{y} \in \mathbb{R}^n$, which is contracting with respect to $\mathbf{y}$. This auxiliary system is chosen such that a given nonlinear system can be expressed as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$, where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system and t is the time. It is clear that $\mathbf{x}$ is a particular solution of the auxiliary system when $\mathbf{y} = \mathbf{x}$. If the auxiliary system also has a solution with a specific property, then the contraction nature of the auxiliary system guarantees that the solution $\mathbf{x}(t)$ converges exponentially to the solution with the specific property. Therefore $\mathbf{x}(t)$ verifies the specific property exponentially. The specific property in this case is a particular state trajectory. The nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$ is said to be partially contracting. Partial contraction is stated below.

Theorem 3. [33] *Given the system equations, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$, and an auxiliary system, $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t)$. If the auxiliary system is contracting with respect to $\mathbf{y}$ and a particular solution of the auxiliary system verifies a specific property, then all the trajectories of the original system verify the specific property exponentially.*

Remark 1. *In particular, if the commanded virtual control is a solution to the auxiliary system, then the state variable converges onto the commanded virtual control exponentially. This property is used in the next section.*

3. Backstepping

Backstepping control law can be formulated by applying contraction analysis. Contraction analysis has been used in backstepping control in [36, 37, 38, 39, 40]. For the use of backstepping control, a system has to be in a cascaded form where a state variable of a subsystem can be used as a virtual control of the state variable of a subsequent subsystem. The controlled state variable is then the virtual control for the next state variable. The system equations for a system in a *cascaded affine form* is written as

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{x}_2, \quad (4a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t) + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{x}_3, \quad (4b)$$

$$\vdots$$

$$\dot{\mathbf{x}}_{n-1} = \mathbf{f}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t) + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{x}_n, \quad (4c)$$

$$\dot{\mathbf{x}}_n = \mathbf{f}_n(\mathbf{x}_n, t) + \mathbf{g}_n(\mathbf{x}_n, t)\mathbf{u}, \quad (4d)$$

where $\mathbf{x}_i$ for $i = 1, 2, \dots, n$ are the state variables, $\mathbf{u}$ is the control input and t is the time. Transforming dynamics of more complex systems such as aircraft dynamics [41] into this form often requires Lie derivatives and results in complicated expressions [1].

In backstepping control, $\mathbf{x}_j$ for $j = 2, 3, \dots, n$ are used as the virtual control for $\mathbf{x}_{j-1}$ and the control input $\mathbf{u}$ is used to control $\mathbf{x}_n$. In this way, the control law for each virtual control and the control input is formulated in a systematic way and the control of $\mathbf{x}_1$, which is often the output, is achieved.

Consider the two-subsystem case,

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, t)\mathbf{x}_2, \quad (5a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t) + \mathbf{g}_2(\mathbf{x}_2, t)\mathbf{u}. \quad (5b)$$

Assume that $\mathbf{g}_i$ for $i = 1, 2$ are invertible, hence dynamic inversion can be used to obtain the control law.

Contraction theory is used to ensure the tracking of the commanded virtual control by the state variable. In this case, $\mathbf{x}_2$ is used as a virtual control to ensure that $\mathbf{x}_1$ tracks $\mathbf{x}_1^*$. The virtual control is chosen as,

$$\mathbf{x}_2^* = \mathbf{g}_1^{-1}(\mathbf{x}_1, \mathbf{x}_2, t)[-\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t) + \dot{\mathbf{x}}_1^* - \mathbf{K}_1(\mathbf{x}_1 - \mathbf{x}_1^*)], \quad (6)$$

where $\mathbf{x}_1^*$ is the desired trajectory of $\mathbf{x}_1$, $\mathbf{x}_2^*$ is the desired trajectory of $\mathbf{x}_2$ and $\mathbf{K}_1 > 0$ is the control gain.

The closed-loop system when $\mathbf{x}_2 = \mathbf{x}_2^*$ is

$$\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_1^* - \mathbf{K}_1(\mathbf{x}_1 - \mathbf{x}_1^*). \quad (7)$$

To proof the stability of the closed-loop system (7), consider the following auxiliary system,

$$\dot{\mathbf{y}}_1 = \dot{\mathbf{x}}_1^* - \mathbf{K}_1(\mathbf{y}_1 - \mathbf{x}_1^*), \quad (8)$$

where $\mathbf{y}_1$ is the auxiliary state variable. It is noted that the auxiliary system is contracting with respect to $\mathbf{y}_1$ as $\mathbf{K}_1 > 0$. Furthermore, it is obvious that $\mathbf{y}_1 = \mathbf{x}_1^*$ is a particular solution of the auxiliary system. Hence all state trajectories of the closed-loop system (7) approach the desired trajectory $\mathbf{x}_1^*$ exponentially. Hence tracking is achieved exponentially.

If $|\mathbf{x}_2 - \mathbf{x}_2^*|$ converges to zero exponentially, then tracking of the state $\mathbf{x}_1$ is exponentially achieved. Furthermore, the contracting nature of the system ensure that $|\mathbf{x}_2 - \mathbf{x}_2^*|$ is bounded.

Next, exponential convergence of $\mathbf{x}_2$ to $\mathbf{x}_2^*$ needs to be ensured. A similar procedure and analysis is used to formulate the control input $\mathbf{u}$.

The control input $\mathbf{u}$ is proposed as

$$\mathbf{u} = \mathbf{g}_2^{-1}(\mathbf{x}_2, t)[- \mathbf{f}_2(\mathbf{x}_2, t) + \dot{\mathbf{x}}_2^* - \mathbf{K}_2(\mathbf{x}_2 - \mathbf{x}_2^*)], \quad (9)$$

where $\mathbf{x}_2^*$ is the desired trajectory of $\mathbf{x}_2$ and $\mathbf{K}_2 > 0$ is the control gain

The closed-loop system becomes

$$\dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_2^* - \mathbf{K}_2(\mathbf{x}_2 - \mathbf{x}_2^*). \quad (10)$$

To proof the stability of the closed-loop system (10), consider the following auxiliary system,

$$\dot{\mathbf{y}}_2 = \dot{\mathbf{x}}_2^* - \mathbf{K}_2(\mathbf{y}_2 - \mathbf{x}_2^*), \quad (11)$$

where $\mathbf{y}_2$ is the auxiliary state variable. The auxiliary system is contracting with respect to $\mathbf{y}_2$ as $\mathbf{K}_2 > 0$. Furthermore, it is obvious that $\mathbf{y}_2 = \mathbf{x}_2^*$ is a particular solution of the auxiliary system. Hence all state trajectories of the closed-loop system (10) approach the desired trajectory $\mathbf{x}_2^*$ exponentially. Convergence of $\mathbf{x}_2$ to $\mathbf{x}_2^*$, hence tracking of $\mathbf{x}_1$ to $\mathbf{x}_1^*$ is ensured.

For the system given by (5), the control law given by (6) and (9) is able to ensure tracking of the state variable $\mathbf{x}_1$.

The backstepping control of the two-subsystem case above can be generalized into a n -subsystem case. Theorem 4 states the formulation of the control law.

Theorem 4. *For a system given by (4), the following backstepping control law results in the exponential convergence of the state variables $\mathbf{x}_j$ to their desired trajectory $\mathbf{x}_j^*$ for $j = 1, 2, \dots, n$.*

$$\mathbf{x}_2^* = \mathbf{g}_1^{-1}[-\mathbf{f}_1 + \dot{\mathbf{x}}_1^* - \mathbf{K}_1(\mathbf{x}_1 - \mathbf{x}_1^*)], \quad (12a)$$

$$\mathbf{x}_3^* = \mathbf{g}_2^{-1}[-\mathbf{f}_2 + \dot{\mathbf{x}}_2^* - \mathbf{K}_2(\mathbf{x}_2 - \mathbf{x}_2^*)], \quad (12b)$$

$\vdots$

$$\mathbf{x}_n^* = \mathbf{g}_{n-1}^{-1}[-\mathbf{f}_{n-1} + \dot{\mathbf{x}}_{n-1}^* - \mathbf{K}_{n-1}(\mathbf{x}_{n-1} - \mathbf{x}_{n-1}^*)], \quad (12c)$$

$$\mathbf{u} = \mathbf{g}_n^{-1}[-\mathbf{f}_n + \dot{\mathbf{x}}_n^* - \mathbf{K}_n(\mathbf{x}_n - \mathbf{x}_n^*)], \quad (12d)$$

where $\mathbf{K}_i > 0$ for $i = 1, 2, \dots, n$.

Proof. For the virtual control given by Theorem 4 for the j^{th} subsystem, the closed-loop system when $\mathbf{x}_{j+1} = \mathbf{x}_{j+1}^*$ becomes

$$\dot{\mathbf{x}}_j = \dot{\mathbf{x}}_j^* - \mathbf{K}_j(\mathbf{x}_j - \mathbf{x}_j^*), \quad (13)$$

for $j = 1, 2, \dots, n-1$. For $j = n$, the closed-loop system becomes

$$\dot{\mathbf{x}}_n = \dot{\mathbf{x}}_n^* - \mathbf{K}_n(\mathbf{x}_n - \mathbf{x}_n^*). \quad (14)$$

The stability of the j^{th} subsystem is proven by considering the auxiliary system,

$$\dot{\mathbf{y}}_j = \dot{\mathbf{x}}_j^* - \mathbf{K}_j(\mathbf{y}_j - \mathbf{x}_j^*), \quad (15)$$

where $j = 1, 2, \dots, n$, $\mathbf{y}_j$ is the auxiliary state variable, $\mathbf{x}_j^*$ is the desired trajectory for the state variable $\mathbf{x}_j$ and $\mathbf{K}_j > 0$ is the control gain. The auxiliary system is contracting with respect to $\mathbf{y}_j$ as $\mathbf{K}_j > 0$. Furthermore, it is obvious that $\mathbf{y}_j = \mathbf{x}_j^*$ is a particular solution of the auxiliary system. Hence all possible state trajectories of $\mathbf{x}_j$ converges exponentially to $\mathbf{x}_j^*$ for $j = 1, 2, \dots, n$.

Through backstepping reasoning and mathematical induction, if $|\mathbf{x}_j - \mathbf{x}_j^*|$ converges to zero exponentially, then $|\mathbf{x}_{j-1} - \mathbf{x}_{j-1}^*|$ also converges to zero exponentially for $j = 2, 3, \dots, n$. Therefore the state variables of the system given by (4) converge to their desired trajectories exponentially. $\square$

Theorem 4 provides a systematic way to formulate a control law for a system in a cascaded form and yields a simple expression for the control law. This is used to formulate the tracking controller in Section 5.

Remark 2. *It is noted that the system considered in this section is an upper-triangular form, as opposed to the lower-triangular form that was used in the original backstepping approach [42]. So Theorem 4 extends the backstepping approach of [42] to the upper-triangular form systems. This type of upper-triangular system is also referred to as cascaded system.*

3.1. Bounded Disturbance Response

An advantage of this formulation due to the contraction property of the closed-loop system is the boundedness of the system state responses in the presence of external disturbances. This property also allows a quantitative estimation of the system state response bounds in relation to an external disturbance bound. It is stated in the following theorem.

Theorem 5. *For a system given by*

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t) + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{x}_2 + \mathbf{d}_1, \quad (16a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t) + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{x}_3 + \mathbf{d}_2, \quad (16b)$$

$\vdots$

$$\dot{\mathbf{x}}_{n-1} = \mathbf{f}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t) + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{x}_n + \mathbf{d}_{n-1}, \quad (16c)$$

$$\dot{\mathbf{x}}_n = \mathbf{f}_n(\mathbf{x}_n, t) + \mathbf{g}_n(\mathbf{x}_n, t)\mathbf{u} + \mathbf{d}_n, \quad (16d)$$

and a control law given by (12), the state $\mathbf{x}_i$ and the desired state $\mathbf{x}_i^*$ remain within a ball of radius R_i governed by $\dot{R}_i + |\lambda_{\max, i}|R_i \leq L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$ for $i = 1, 2, \dots, n-1$ and $\dot{R}_n + |\lambda_{\max, n}|R_n \leq d_n$ where $\|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\| < L_i(R_i, R_{i+1}, \dots, R_n, t)$, $\|\mathbf{d}_i\| < d_i$ and $\lambda_{\max, i}$ is the largest eigenvalue of $\mathbf{K}_i + \mathbf{K}_i^T$.

Proof. Observe the closed-loop system,

$$\dot{\mathbf{z}}_1 = -\mathbf{K}_1\mathbf{z}_1 + \mathbf{g}_1(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, t)\mathbf{z}_2 + \mathbf{d}_1, \quad (17a)$$

$$\dot{\mathbf{z}}_2 = -\mathbf{K}_2\mathbf{z}_2 + \mathbf{g}_2(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n, t)\mathbf{z}_3 + \mathbf{d}_2, \quad (17b)$$

$\vdots$

$$\dot{\mathbf{z}}_{n-1} = -\mathbf{K}_{n-1}\mathbf{z}_{n-1} + \mathbf{g}_{n-1}(\mathbf{x}_{n-1}, \mathbf{x}_n, t)\mathbf{z}_n + \mathbf{d}_{n-1}, \quad (17c)$$

$$\dot{\mathbf{z}}_n = -\mathbf{K}_n\mathbf{z}_n + \mathbf{d}_n, \quad (17d)$$

where $\mathbf{z}_i = \mathbf{x}_i - \mathbf{x}_i^*$ for $i = 2, 3, \dots, n$.

The “disturbance” term for the i^{th} subsystem is $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{z}_{i+1} + \mathbf{d}_i$. An estimate of the bound for the first disturbance term depends on the nonlinear function $\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{z}_{i+1}$. This term is bounded by $\|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\mathbf{z}_{i+1}\| < L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1}$, for some function $L_i(R_i, R_{i+1}, \dots, R_n, t) \geq \|\mathbf{g}_i(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n, t)\|$ and remains bounded if R_j for $j \geq i$ are bounded. This term is caused by the difference between the state $\mathbf{x}_{i+1}$ and the desired state $\mathbf{x}_{i+1}^*$, and is expected to converge to zero if there is no external disturbance $\mathbf{d}_i = \mathbf{0}$.

The external disturbance $\mathbf{d}_i$, on the other hand, is bounded by $\|\mathbf{d}_i\| < d_i$ therefore the bound for the two disturbances terms is conservatively estimated as $L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$.

According to Theorem 2, the state $\mathbf{x}_i$ is bounded by a ball of radius R_i centered on the desired state $\mathbf{x}_i^*$ that is governed by $\dot{R}_i + |\lambda_{\max, i}|R_i \leq L_i(R_i, R_{i+1}, \dots, R_n, t)R_{i+1} + d_i$ for $i = 1, 2, \dots, n-1$. For the n^{th} subsystem, this ball is governed by $\dot{R}_n + |\lambda_{\max, n}|R_n \leq d_n$. $\square$

This theorem provides a proof on the closed-loop stability of the system when it is subjected to external disturbances. Furthermore, Theorem 5 also shows a relationship between the performance bound R_i and the size of the disturbance d_i . In addition, increasing $|\lambda_{\max,n}|$ reduces the effect of disturbance by reducing the size of the performance bound R_i . Since $\lambda_{\max,n} < 0$ is the maximum eigenvalue of $\mathbf{K}_i + \mathbf{K}_i^T$, the proper design of control gain allows the reduction of the sensitivity to external disturbance. Note that a larger $\mathbf{K}_i$ also results in larger control action, therefore a tradeoff needs to be considered.

4. Quadrotor Dynamics

This section describes the dynamics of a quadrotor. The dynamics of quadrotor can be found in many references and appears in many forms including the Newton-Euler form [7, 18] and the Euler-Lagrange form [1]. In the present paper, the Newton-Euler form is used.

The dynamics of the quadrotor is separated into translational and rotational motions. In the translational motion, the kinematics describes the translational movement of the quadrotor while the dynamics relates the forces acting on the quadrotor to its linear acceleration. Similarly for rotational motion, the kinematics describes the rotational movement of the quadrotor while the dynamics relates the moments acting on the quadrotor to its rotational acceleration.

Figure 2 shows the quadrotor in an inertial frame and the Euler angles associated with the attitude of the quadrotor.

4.1. Control Forces and Moments

The quadrotor is controlled by thrusts provided by its rotors. By altering the thrust produced by each rotor, the total thrust and the moments acting on the quadrotor are controlled.

The force f_i acting on a single rotor is approximated as

$$f_i = k_t \Omega_i^2, \quad i = 1, 2, 3, 4 \quad (18)$$

where k_t is the lift factor and Ω_i is the rotation rate of the i^{th} rotor.

The total thrust acting on the quadrotor is

$$T = \sum_{i=1}^4 f_i. \quad (19)$$

Figure 2: A diagram showing the quadrotor and the Euler angles associated with the attitude of the quadrotor. The force on the quadrotor is acting on each of the rotors and pointing in the negative body z -axis. The reference frames, Earth-fixed inertial frame (X - Y - Z) and body frame (x - y - z), are also shown in the diagram.

It acts in the negative body z -axis direction and is written as

$$\mathbf{T} = -T\hat{\mathbf{e}}_3, \quad (20)$$

where $\hat{\mathbf{e}}_3$ is the unit vector along the z -axis.

The torque τ_i acting on a single rotor is approximated as

$$\tau_i = k_q \Omega_i^2, \quad i = 1, 2, 3, 4 \quad (21)$$

where k_q is the drag factor respectively.

The total control moment is obtained by summing up the moments acting at the center of the quadrotor by each of the thrust and torque acting on the rotors. The total control moments acting on the quadrotor $\mathbf{M}$ is

$$\mathbf{M} = \sum_{i=1}^4 [\mathbf{r}_i \times (-f_i \hat{\mathbf{e}}_3) + (-1)^i \tau_i \hat{\mathbf{e}}_3], \quad (22)$$

where $\mathbf{r}_i$ for $i = 1, 2, 3, 4$ is the position vector of the center of the i^{th} rotor from the center of the quadrotor¹.

If $\mathbf{r}_i$ are $-l\hat{\mathbf{e}}_1$, $-l\hat{\mathbf{e}}_2$, $l\hat{\mathbf{e}}_1$ and $l\hat{\mathbf{e}}_2$ for $i = 1, 2, 3, 4$ respectively with l being the distance of the rotor to the quadrotor center of mass, the total control

¹Rotors 1 and 3 are rotating in the positive z direction, while rotors 2 and 4 are rotating in the negative z direction.

moment is written as

$$\mathbf{M} = \begin{pmatrix} lk_t (\Omega_2^2 - \Omega_4^2) \\ lk_t (\Omega_1^2 - \Omega_3^2) \\ k_q (\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \end{pmatrix}. \quad (23)$$

Given the required total thrust and control moment, the desired rotor speeds are calculated by

$$\begin{pmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{pmatrix} = \begin{pmatrix} -k_t & -k_t & -k_t & -k_t \\ 0 & lk_t & 0 & -lk_t \\ lk_t & 0 & -lk_t & 0 \\ -k_q & k_q & -k_q & k_q \end{pmatrix}^{-1} \begin{pmatrix} T \\ M_x \\ M_y \\ M_z \end{pmatrix}, \quad (24)$$

where M_x, M_y, M_z are the torques along the x, y, z -axes respectively and the matrix is invertible by the construction of the quadrotor. This process is called control allocation where the rotation rates of the rotors are calculated from the thrust and moments required.

4.2. Translational Kinematics

The translational kinematics describes the motion of the quadrotor in the Earth-fixed inertial frame. It is written as

$$\dot{\mathbf{p}} = \mathbf{v}, \quad (25)$$

where $\mathbf{p}$ and $\mathbf{v}$ are the position and velocity of the quadrotor center of mass in the Earth-fixed inertial frame.

4.3. Translational Dynamics

The forces acting on the quadrotor influence the quadrotor according to the Newton's second law. The translational dynamics is formulated in the Earth-fixed inertial frame as

$$\dot{\mathbf{v}} = -\frac{1}{m} \mathbf{R}^T U_1 \hat{\mathbf{e}}_3 + g \hat{\mathbf{e}}_3, \quad (26)$$

where $U_1 = T$, $\mathbf{R}$ is the direction cosine matrix from the Earth-fixed inertial frame to the body frame, m is the mass of the quadrotor and g is the gravitational acceleration.

The direction cosine matrix is

$$\mathbf{R} = \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{pmatrix}, \quad (27)$$

where $s(\cdot)$ and $c(\cdot)$ are $\sin(\cdot)$ and $\cos(\cdot)$ respectively and ψ , θ and ϕ are the yaw, pitch and roll angles associated with the rotation about the z , y and x axes from the Earth-fixed inertial frame to the body frame in that sequence respectively.

In this formulation, small forces such as aerodynamic forces are neglected.

Remark 3. *The translational dynamics exhibits strong coupling between the attitude and the thrust control input. Therefore changes in either control U_1 or directional cosine matrix $\mathbf{R}$ causes a reaction in the translational motion of the quadrotor. This is because U_1 and $\mathbf{R}$ appear as a product in (26). Furthermore, this also increases the difficulty in expressing the dynamics in a full state cascaded form.*

4.4. Rotational Kinematics

The rotational kinematics describes how the attitude angles of the quadrotor change and is given by

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & t\theta s\phi & t\theta c\phi \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad (28)$$

where $t(\cdot)$ is $\tan(\cdot)$, p , q and r are the rotation rates of the quadrotor about the body x , y and z -axis respectively.

4.5. Rotational Dynamics

The rotational dynamics is derived from the Newton's second law in the body-fixed frame. It is written as

$$\dot{\omega} = \mathbf{J}^{-1} (\mathbf{M} - \omega \times \mathbf{J}\omega), \quad (29)$$

where $\mathbf{J}$ is the moment of inertia matrix and $\omega = (p \ q \ r)^T$.

In this formulation, small moments and torques acting on the quadrotor such as aerodynamic moments are neglected.

By control input transformation $\mathbf{U}_2 = \mathbf{J}^{-1}(\mathbf{M} - \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega})$, (29) is written as

$$\dot{\boldsymbol{\omega}} = \mathbf{U}_2. \quad (30)$$

The equations of motion of the quadrotor system are hence given by (25), (26), (28) and (30). The control inputs are given by U_1 and $\mathbf{U}_2$. These equations are transformed into a full state cascaded form in the following section.

5. Tracking Controller

In this section, the tracking controller is formulated. The quadrotor is required to track a moving ground target at a given altitude with feedback of the position or relative position of the ground target.

The altitude control is decoupled from the planar position and yaw control. The planar position refers to the X, Y position of the quadrotor which is the ground track of the quadrotor. This resolves the bilinear control issue as the thrust is used to control the altitude and allows backstepping control to be applied to the planar position and yaw control separately.

5.1. Altitude Control

The altitude dynamics is decoupled from the translational dynamics given in (26). The z -component of (26) is

$$\ddot{h} = U_1 \cos \theta \cos \phi - mg, \quad (31)$$

where $h = -Z$ is the altitude of the quadrotor.

Letting $U_1 \cos \theta \cos \phi - mg = U'_1$, (31) becomes

$$\ddot{h} = U'_1. \quad (32)$$

As altitude control is not the focus of this paper, a simple pole placement technique is used for the altitude control. The control is

$$U'_1 = \ddot{h}_{\text{cmd}} - k_1^{\text{alt}}(\dot{h} - \dot{h}_{\text{cmd}}) - k_2^{\text{alt}}(h - h_{\text{cmd}}), \quad (33)$$

where h_{cmd} is the commanded altitude, k_1^{alt} and k_2^{alt} are the control gains to be chosen by pole placement technique. The altitude dynamics given by (32) becomes

$$\ddot{e}_h + k_1^{\text{alt}}\dot{e}_h + k_2^{\text{alt}}e_h = 0, \quad (34)$$

where $e_h = h - h_{\text{cmd}}$ is the altitude error. The control gains are chosen by making the real parts of the poles negative.

Therefore the resultant altitude controller is given by

$$U_1 = \frac{U'_1 + mg}{\cos \theta \cos \phi} = \frac{\ddot{h}_{\text{cmd}} - k_1^{\text{alt}} \dot{e}_h - k_2^{\text{alt}} e_h + mg}{\cos \theta \cos \phi}. \quad (35)$$

This thrust input is used by the planar position and yaw control in the backstepping formulation.

Remark 4. *There is a singularity in (35) when $\phi = \frac{\pi}{2}$ or $\theta = \frac{\pi}{2}$. However this is not a problem as the attitude of the quadrotor is assumed to satisfy $|\phi| < \frac{\pi}{2}$ and $|\theta| < \frac{\pi}{2}$.*

5.2. Planar Position and Yaw Control

Given the position feedback of the target, the velocity and acceleration of the quadrotor are commanded to track the target. The rotation rate ω of the quadrotor is controlled to achieve the desired acceleration. Lastly the control moments are commanded to control the quadrotor rotation rate ω . In this way, backstepping is used to achieve tracking of a moving target. Figure 3 illustrates this backstepping procedure.

Figure 3: The position of the quadrotor is related through kinematics relation to its velocity and acceleration. The desired acceleration and yaw angle are achieved by defining a desired rotation rate. The desired rotation rate is achieved by the control moments.

For the rest of this subsection, position refers to the planar position of the quadrotor.

The equations of motion is transformed into a full state cascaded form before applying backstepping control.

The position and velocity kinematics of the quadrotor is

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (36)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}, \quad (37)$$

where $\mathbf{p}_{\text{planar}} = (X \ Y)^T$, $\mathbf{v}_{\text{planar}}$ and $\mathbf{a}_{\text{planar}}$ are the the planar position, velocity and acceleration of the center of mass of the quadrotor in the Earth-fixed inertial frame respectively.

To control the acceleration of the quadrotor, the time derivative of acceleration or jerk is derived. By differentiating (26) with respect to time and using Coriolis' theorem, the acceleration dynamics is

$$\ddot{\mathbf{v}} = -\frac{1}{m}\dot{U}_1\mathbf{R}^T\hat{\mathbf{e}}_3 + \frac{1}{m}U_1(\mathbf{R}^T\hat{\mathbf{e}}_3) \times \omega. \quad (38)$$

The cross product term is transformed into a matrix product by the following

relationship: $\mathbf{a} \times \mathbf{b} = \mathbf{a}_{\times} \mathbf{b}$ where $\mathbf{a}_{\times} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$. Then (38)

becomes

$$\ddot{\mathbf{v}} = -\frac{1}{m}\dot{U}_1\mathbf{R}^T\hat{\mathbf{e}}_3 + \frac{1}{m}U_1(\mathbf{R}^T\hat{\mathbf{e}}_3)_{\times} \omega, \quad (39)$$

where

$$(\mathbf{R}^T\hat{\mathbf{e}}_3)_{\times} = \begin{pmatrix} 0 & -c\phi c\theta & -s\phi c\psi + c\phi s\theta s\psi \\ c\phi c\theta & 0 & -s\phi s\psi - c\phi s\theta c\psi \\ s\phi c\psi - c\phi s\theta s\psi & s\phi s\psi + c\phi s\theta c\psi & 0 \end{pmatrix}. \quad (40)$$

The planar jerk is given by

$$\dot{\mathbf{a}}_{\text{planar}} = -\frac{1}{m}\mathbf{E}\dot{U}_1\mathbf{R}^T\hat{\mathbf{e}}_3 + \frac{1}{m}\mathbf{E}U_1(\mathbf{R}^T\hat{\mathbf{e}}_3)_{\times} \omega, \quad (41)$$

where $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ extracts the dynamics in the X, Y -axes.

The planar jerk in (41) is simplified as

$$\dot{\mathbf{a}}_{\text{planar}} = \mathbf{F}_1 + \mathbf{G}_1\omega, \quad (42)$$

where $\mathbf{F}_1 = -\frac{1}{m}\mathbf{E}\dot{U}_1\mathbf{R}^T\hat{\mathbf{e}}_3$ and $\mathbf{G}_1 = \frac{1}{m}\mathbf{E}U_1(\mathbf{R}^T\hat{\mathbf{e}}_3)_{\times}$.

The dynamics in (42) shows that the rotation rates can be used to control the planar acceleration of the quadrotor. Furthermore, the rotation rates are also used to control the yaw angle. From (28), the yaw kinematics is

$$\dot{\psi} = \frac{\sin \phi}{\cos \theta} q + \frac{\cos \phi}{\cos \theta} r, \quad (43a)$$

$$= \mathbf{G}_{\text{yaw}}\omega, \quad (43b)$$

where $\mathbf{G}_{\text{yaw}} = \begin{pmatrix} 0 & \sin \phi / \cos \theta & \cos \phi / \cos \theta \end{pmatrix}$.

The planar acceleration dynamics (42) and the yaw kinematics (43b) are concatenated as both subsystems use the same virtual control, ω .

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix} \omega. \quad (44)$$

It is observed that $\text{rank} \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix} = 3$ when $|\phi| < \frac{\pi}{2}$, $|\theta| < \frac{\pi}{2}$ and $U_1 > 0$, therefore $\begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix}$ is always invertible within these bounds.

Lastly, (30) completes the equations of motion that is used for backstepping formulation.

The equations of motion of the quadrotor can now be written in the following 10th order form (less 2 order from the separation of altitude control)

$$\dot{\mathbf{p}}_{\text{planar}} = \mathbf{v}_{\text{planar}}, \quad (45a)$$

$$\dot{\mathbf{v}}_{\text{planar}} = \mathbf{a}_{\text{planar}}, \quad (45b)$$

$$\frac{d}{dt} \begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix} \omega, \quad (45c)$$

$$\dot{\omega} = \mathbf{U}_2, \quad (45d)$$

which is in a full state cascaded form and is ready for the application of the backstepping method.

Using Theorem 4, backstepping control is formulated as follows. Given the target position $\mathbf{p}_{\text{planar},d}(t)$ (desired quadrotor planar position), the desired velocity of the quadrotor is

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},d} - \mathbf{K}_1(\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},d}), \quad (46)$$

where $\mathbf{v}_{\text{planar},d}$ is the desired velocity of the quadrotor, $\mathbf{p}_{\text{planar},d}$ is the target position which is also the desired quadrotor position and $\mathbf{K}_1 > 0$ is the control gain.

The desired acceleration becomes

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - \mathbf{K}_2(\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (47)$$

where $\mathbf{K}_2 > 0$ is the control gain.

The desired acceleration is concatenated with the desired yaw angle to form the desired planar acceleration $\mathbf{a}_{\text{planar},d}$ and yaw angle ψ_d . It is noted that the desired yaw angle can be any arbitrary user-defined value. The desired rotation rate becomes

$$\omega_d = \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix}^{-1} \left(- \begin{pmatrix} \mathbf{F}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\mathbf{a}}_{\text{planar},d} \\ \dot{\psi}_d \end{pmatrix} - \mathbf{K}_3 \left(\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} \right) \right), \quad (48)$$

where $\mathbf{K}_3 > 0$ is the control gain.

The commanded control input, $\mathbf{U}_2$, becomes

$$\mathbf{U}_2 = \dot{\omega}_d - \mathbf{K}_4(\omega - \omega_d), \quad (49)$$

where $\mathbf{K}_4 > 0$ is the control gain.

The equations given in (46) - (49) form the control law for the tracking controller to track a moving ground target. Each backstepping step calculates the desired state variable trajectory that will result in the precise tracking of the target.

Remark 5. *The control gains $\mathbf{K}_i$ for $i = 1, 2, 3, 4$ can be any positive definite matrix. There is no requirement for the inner-loop to have a higher gain than the outer-loop. Therefore there is not time-scale restriction on the inner and outer-loop unlike in the time-scale separation approaches.*

In addition, the resultant control structure also results in an integral action-like behavior for position control. This effect is highlighted in the following example in contrast with Example 1 in Section 1.

Example 2. *With the re-formulated kinematics and dynamics approach, the system model in Example 1 becomes*

$$\ddot{x}_1 = \frac{dh}{dx_3} u, \quad (50)$$

while the actual dynamics with unmodelled effect is

$$\ddot{x}_1 = \frac{dh}{dx_3} u + \dot{d}. \quad (51)$$

The effect of d is captured by the measurement $\dot{x}_1$ in our re-formulated dynamics. Furthermore if the unmodelled effect is a constant bias ($\dot{d} = 0$), then the control law derived is able to achieve exact tracking of x_1 .

Furthermore, this integral effect is achieved without explicitly using any integrators resulting in a reduced control law order. The following remark highlight this point.

Remark 6. *The re-formulated equations of motion (45) has a degree of 10 (less 2 degrees of freedom from separate altitude control) which is identical to the degree of freedom in the conventional equations of motion in (25), (26), (28) and (30) that are typically used in quadrotor controller designs. Furthermore the order of control law proposed in (46) - (49) is 10. In view of this, no integral action is included explicitly in the proposed design, yet it is shown that it is able to achieve zero steady state error in the presence of constant bias similar to integral control.*

5.3. Linear Tracking Differentiator

It is evident that the above formulation requires a significant amount of derivatives to be evaluated. Obtaining the exact expressions for the derivatives results in a complex controller, therefore a linear tracking differentiator is employed to extract the derivative from a filtered signal.

For instance, the signal $\mathbf{p}_{\text{planar},d}$ is passed through the following command filter to evaluate its derivative, it is given by

$$\dot{\mathbf{p}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{p}_{\text{planar},\text{cmd}} - \mathbf{p}_{\text{planar},d}), \quad (52)$$

where $\mathbf{p}_{\text{planar},\text{cmd}}$ is the filtered signal and $\tau > 0$ is the command filter time constant.

The filtered signal $\mathbf{p}_{\text{planar},\text{cmd}}$ replaces the input signal $\mathbf{p}_{\text{planar},d}$ in the control law and is therefore named the command signal. Therefore the derivative required is given by (52). This eliminates the need to obtain an analytical expression of the derivative, thus simplifying the control law.

The command filter gain τ needs to be sufficiently large such that singular perturbation theory can be applied to show that $\mathbf{p}_{\text{planar},\text{cmd}} \simeq \mathbf{p}_{\text{planar},d}$.

The control law (46) - (49) is therefore rewritten as

$$\mathbf{v}_{\text{planar},d} = \dot{\mathbf{p}}_{\text{planar},\text{cmd}} - \mathbf{K}_1(\mathbf{p}_{\text{planar}} - \mathbf{p}_{\text{planar},\text{cmd}}), \quad (53a)$$

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},\text{cmd}} - \mathbf{K}_2(\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},\text{cmd}}), \quad (53b)$$

$$\omega_d = \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_{\text{yaw}} \end{pmatrix}^{-1} \left(- \begin{pmatrix} \mathbf{F}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \dot{\mathbf{a}}_{\text{planar},\text{cmd}} \\ \dot{\psi}_{\text{cmd}} \end{pmatrix} \right) \quad (53c)$$

$$- \mathbf{K}_3 \left(\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix} - \begin{pmatrix} \mathbf{a}_{\text{planar},\text{cmd}} \\ \psi_{\text{cmd}} \end{pmatrix} \right),$$

$$\mathbf{U}_2 = \dot{\omega}_{\text{cmd}} - \mathbf{K}_4(\omega - \omega_{\text{cmd}}), \quad (53d)$$

and the command filters that the desired variables are passed through are

$$\dot{\mathbf{p}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{p}_{\text{planar},\text{cmd}} - \mathbf{p}_{\text{planar},d}), \quad (54a)$$

$$\dot{\mathbf{v}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{v}_{\text{planar},\text{cmd}} - \mathbf{v}_{\text{planar},d}), \quad (54b)$$

$$\dot{\mathbf{a}}_{\text{planar},\text{cmd}} = -\tau(\mathbf{a}_{\text{planar},\text{cmd}} - \mathbf{a}_{\text{planar},d}), \quad (54c)$$

$$\dot{\psi}_{\text{cmd}} = -\tau(\psi_{\text{cmd}} - \psi_d), \quad (54d)$$

$$\dot{\omega}_{\text{cmd}} = -\tau(\omega_{\text{cmd}} - \omega_d), \quad (54e)$$

where $(\cdot)_{\text{cmd}}$ are the commanded signals and $\tau > 0$ is sufficiently large.

In this case, the command filter dynamics (54) acts with a faster dynamics compared to the control law (53) in a two time-scale system. Mathematically it means that $\tau \gg \max(\text{eig}(\mathbf{K}_i))$ for all $i = 1, 2, 3, 4$.

Remark 7. *The command filter gains τ in (54) are set to be the same for simplicity. This is not a strict requirement and the gains can be made different as long as they are sufficiently large (i.e. $\tau \gg \max(\text{eig}(\mathbf{K}_i))$ for all $i = 1, 2, 3, 4$).*

6. Numerical Simulation and Discussions

The proposed tracking controller is simulated on Matlab/Simulink. In the simulation, the quadrotor is commanded to track a ground target moving along predefined paths. The performance of the proposed controller is compared with a backstepping controller using inverse kinematics. In addition, aerodynamic effects are simulated to compare the robustness of each controller with respect to neglected small forces and moments. Lastly, the responses of both controllers to wind disturbances are also evaluated and compared.

6.1. Benchmark Controller

In Section 1, it has been mentioned that the use of full state cascaded dynamics eliminates the need of inverse kinematics. Inverse kinematics is used in some research works like those presented in [11, 17, 18, 19]. It is an algebraic inversion step used to obtain the attitude angles required to achieve the desired acceleration. As this is used as the *benchmark controller* in this paper, its formulation is described here. Figure 4 shows the controller architecture.

Figure 4: Benchmark controller. The architecture is similar to that of the proposed tracking controller. The only difference lies in the application of the of the inverse kinematics to obtain the roll and pitch angles required to achieve the desired acceleration.

The desired velocity and acceleration for the benchmark controller are obtained through an identical process as the proposed controller as shown in (46) and (47).

To obtain the desired acceleration, the following equations are used to calculate the roll and pitch angles required.

$$\theta_d = -\arctan\left(\frac{a_x \cos \psi_d + a_y \sin \psi_d}{a_z - g}\right), \quad (55a)$$

$$\phi_d = \arcsin\left(\frac{a_y \cos \psi_d - a_x \sin \psi_d}{\sqrt{a_x^2 + a_y^2 + (g - a_z)^2}}\right), \quad (55b)$$

where a_x , a_y and a_z are the desired acceleration in the Earth-fixed x , y and z -axes respectively. It is noted that $a_z = -\ddot{h}$ can be obtained via (32).

The desired attitude angles are then controlled by the angular rates given by

$$\omega_d = \mathbf{W}^{-1} \left(\dot{\Theta} - \mathbf{K}_5 (\Theta - \Theta_d) \right), \quad (56)$$

where $\Theta = (\phi \ \theta \ \psi)^T$, $\Theta_d = (\phi_d \ \theta_d \ \psi_d)^T$, $\mathbf{K}_5 > 0$ is the control gain and $\mathbf{W} = \begin{pmatrix} 1 & t\theta s\phi & t\theta c\phi \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{pmatrix}$. Furthermore, $\mathbf{W}^{-1} = \begin{pmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & c\theta s\phi \\ 0 & -s\phi & c\phi c\theta \end{pmatrix}$ and

always defined.

Lastly, the desired angular rates ω_d are controlled by the moment input in the same way as (48) and the control law in (49).

Remark 8. *The command filters in (54) are also used in the benchmark controller so that a simplified expression can be obtained.*

Remark 9. *The application of the inverse kinematics separates the controller into the translational and rotational part. However, the time-scale separation method is not required to be applied as mentioned in [18].*

It is noted that the performance of the benchmark controller is similar to those presented in [4, 12, 16, 19] under both windy and enclosed environment. This provides certain validation for the simulation results against experimental platforms.

6.2. Simulation Parameters

Numerical simulation is performed on a generic quadrotor using parameters obtained from [19]. The parameters are listed in Table 1.

Table 1: A list of quadrotor parameters used in the simulation [19].

Parameters	Values
Mass (m)	1.25kg
Distance between the rotor and center (l)	0.5m
Moment of inertia to x -axis (J_{xx})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to y -axis (J_{yy})	$8.1 \times 10^{-3} \text{Nms}^{-2}$
Moment of inertia to z -axis (J_{zz})	$14.2 \times 10^{-3} \text{Nms}^{-2}$
Lift factor k_t	$54.2 \times 10^{-6} \text{Ns}^2$
Drag factor k_q	$1.1 \times 10^{-6} \text{Ns}^2$

Two target trajectories are chosen: A straight line path and a circular path. The quadrotor is commanded to track a ground target moving along these trajectories. For the linear trajectory, the target is initially placed at a location $(X_{d0} \ Y_{d0})^T = (10\text{m} \ 10\text{m})^T$, moving at a speed of $V_{\text{target}} = 10\text{m/s}$

and with a heading angle of $\chi_{\text{target}} = 30^\circ$. Mathematically, it is

$$X_d = X_{d0} + V_{\text{target}} \cos \chi_{\text{target}} t, \quad (57a)$$

$$Y_d = Y_{d0} + V_{\text{target}} \sin \chi_{\text{target}} t. \quad (57b)$$

In the other trajectory, the target is moving along a circular path of radius $R_{\text{target}} = 10\text{m}$ with a period of $T_{\text{target}} = 30\text{s}$. Mathematically, it is

$$X_d = R_{\text{target}} \cos \left(\frac{2\pi}{T_{\text{target}}} t \right), \quad (58a)$$

$$Y_d = R_{\text{target}} \sin \left(\frac{2\pi}{T_{\text{target}}} t \right). \quad (58b)$$

6.3. Tracking a Moving Target

Numerical simulations are performed using the proposed and the benchmark controllers on the quadrotor with parameters defined in Table 1. The quadrotor is initially hovering at 10m above ground and $(X \ Y)^T = (0 \ 0)^T$. The quadrotor is then commanded to track a ground target moving along a prescribed trajectory. The results using both controllers are plotted on the same graph for easier comparison and are shown in Figures 5 and 6. The X and Y position of the quadrotor against time is shown to highlight the tracking of the target. The roll and pitch angles show the attitude response of the quadrotor. Finally, the tracking error presents the absolute distance between the quadrotor and the target in the x - y plane.

From the results, both controllers are able to track the ground target for both target trajectories. The tracking errors in Figures 5(e) and 6(e) show exponential convergence for both controllers. This is a consequence of the quadrotor behaving like a contracting system and ensuring asymptotic convergence of the quadrotor position to the target position. For the case of tracking a target that is moving linearly (i.e. Figure 5), the quadrotor starts from hovering whereas the target starts with an initial speed, therefore the tracking error increases slightly before decreases as the quadrotor catches up. Whereas for a target moving in a circle (i.e. Figure 6), the quadrotor starts at $(X \ Y)^T = (0 \ 0)^T$ and moves towards the target while adjusting its direction as the target moves. Despite this, exponential convergence is observed for both trajectories. Under ideal situation, both controllers are able to achieve tracking of the ground target.

6.4. Aerodynamic Effects

In the quadrotor model, small forces and moments such as aerodynamic effects are neglected. In situations when the quadrotor is flying at a sufficiently high speed, drag acting on the quadrotor becomes significant. Backstepping control is a model-based control, therefore it is important to assess the performance of the proposed control law in the presence of aerodynamic effects. For comparison, the benchmark controller is also simulated in the presence of aerodynamic effects.

The aerodynamic effects are modelled in a similar way as [18, 17], and is given by

$$\mathbf{F}_{\text{aero}} = -\mathbf{K}_d \mathbf{v}, \quad (59a)$$

$$\mathbf{M}_{\text{aero}} = -\mathbf{K}_m \boldsymbol{\omega}, \quad (59b)$$

where $\mathbf{F}_{\text{aero}}$, $\mathbf{M}_{\text{aero}}$ are the aerodynamic force and moment respectively and $\mathbf{K}_d$ and $\mathbf{K}_m$ are the coefficient matrices of the aerodynamic force and moment respectively.

In this simulation, $\mathbf{K}_d$ and $\mathbf{K}_m$ are chosen to be diagonal matrices with same diagonal terms having a value of 10^{-1}Ns/m and 10^{-2}Nms/rad respectively for simplicity.

The results for the numerical simulations of the proposed and benchmark controllers in tracking the target in the linear and circular paths are shown in Figures 7 to 10.

Comparing both the proposed and the benchmark controllers, it is evident that the proposed control performs better in terms of rejection of unmodelled aerodynamic effects. Figure 7(e) indicates a nonzero steady state tracking error with the benchmark controller but a zero steady state tracking error with the proposed controller. Similarly in 8(e) (though not as significant as in Figure 7(e)), there is a small but nonzero tracking error with the benchmark controller, while such error is totally absent with the proposed controller.

The nonzero steady state error that occurs with the benchmark controller is due to the use of inverse kinematics (an algebraic inversion method) with the lack of integral action in the presence of external disturbances. The aerodynamic effects are not accounted for during the inversion. Furthermore, there is no feedback information on the acceleration. These result in a nonzero steady state tracking error which is also observed by Zuo [18].

On the other hand, the proposed controller handles aerodynamic effects effectively. Rotation rate $\boldsymbol{\omega}$ is used to control the planar acceleration $\mathbf{a}_{\text{planar}}$,

which is affected by the aerodynamic effects, according to the control law given by (53c). The availability of acceleration feedback results in a near zero steady state tracking error.

In particular, for the linear path where the desired acceleration is zero, convergence to zero tracking error is achieved. Considering the aerodynamic force acting on the quadrotor, the translational dynamics in (26) becomes

$$\dot{\mathbf{v}} = -\frac{1}{m}\mathbf{R}^T U_1 \hat{\mathbf{e}}_3 + g\hat{\mathbf{e}}_3 - \frac{1}{m}\mathbf{K}_d \mathbf{v}. \quad (60)$$

The planar jerk of the quadrotor then becomes

$$\dot{\mathbf{a}}_{\text{planar}} = -\frac{1}{m}\mathbf{E}\dot{U}_1\mathbf{R}^T\hat{\mathbf{e}}_3 + \frac{1}{m}\mathbf{E}U_1(\mathbf{R}^T\hat{\mathbf{e}}_3)_{\times}\omega - \frac{1}{m}\mathbf{E}\mathbf{K}_d\mathbf{E}^T\mathbf{a}_{\text{planar}}, \quad (61)$$

for a diagonal $\mathbf{K}_d$. Therefore in the presence of aerodynamic effects, the aerodynamic force is a stabilizing term which tends to bring the planar acceleration to zero. In particular when $\mathbf{a}_{\text{planar},d} = 0, \psi_d = 0 \Rightarrow \begin{pmatrix} \mathbf{a}_{\text{planar},d} \\ \psi_d \end{pmatrix} = \mathbf{0}$ (linear path), $\begin{pmatrix} \mathbf{a}_{\text{planar}} \\ \psi \end{pmatrix}$ converges asymptotically to zero.

On the other hand for inverse kinematics-based backstepping controller, the inverse kinematics that is supposed to convert the desired planar acceleration into desired roll and pitch angles now lack an aerodynamic term. Therefore the roll and pitch angles calculated that supposedly results in a certain planar acceleration value now contains an offset. Suppose the attitude angles track desired values that are given by inverse kinematics (55a), the acceleration (obtained from (26)) becomes

$$\mathbf{a}_{\text{planar}} = \dot{\mathbf{v}}_{\text{planar},d} - \mathbf{K}_2(\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}) - \frac{1}{m}\mathbf{E}\mathbf{K}_d\mathbf{v}, \quad (62)$$

and this differs from the desired acceleration

$$\mathbf{a}_{\text{planar},d} = \dot{\mathbf{v}}_{\text{planar},d} - \mathbf{K}_2(\mathbf{v}_{\text{planar}} - \mathbf{v}_{\text{planar},d}), \quad (63)$$

by a nonzero term $\frac{1}{m}\mathbf{E}\mathbf{K}_d\mathbf{v}$ resulting in inexact tracking of the planar acceleration to the true desired planar acceleration, hence a steady state error in tracked position.

It is also noted that integral action is not added explicitly to the proposed controller yet it is able to achieve a result similar to the addition of an integral

action. The order of the proposed controller in (53) is 10 which is the same as the benchmark controller. For the benchmark controller to achieve the same result as the proposed controller, integral action needs to be added that will increase its order to 12.

Therefore it is shown conclusively that the proposed controller is more superior to the inverse kinematics-based backstepping controller in terms of unmodelled effects, especially those that appears in the inverse kinematics step.

6.5. Wind Disturbances

The performance of the proposed controller is also evaluated and compared with the benchmark controller in the presence of wind disturbances. Wind disturbances act on the quadrotor by changing the airspeed and thus the aerodynamic force and moment acting on the quadrotor.

Given a wind velocity $\mathbf{v}_{\text{wind}}$ and wind angular velocity ω_{wind} , the aerodynamic force and moment generated are

$$\mathbf{F}_{\text{aero}} = -\mathbf{K}_d(\mathbf{v} - \mathbf{v}_{\text{wind}}), \quad (64a)$$

$$\mathbf{M}_{\text{aero}} = -\mathbf{K}_m(\omega - \omega_{\text{wind}}). \quad (64b)$$

The Dryden wind turbulence model is used to generate random wind disturbances. In addition, a steady wind bias of $\mathbf{v}_{\text{wind}} = (10 \ 0 \ 0)^T \text{ m/s}$ is also added. Examples of the wind disturbances generated are shown in Figures 11 and 12.

The results of the numerical simulation of the proposed and benchmark controllers under wind disturbances are shown in Figures 13 to 16.

The proposed controller is able to handle wind disturbances very well as compared to the benchmark controller. Due to the use of inverse kinematics and lack of integral action, the benchmark controller is unable to compensate for the wind disturbance leading to a nonzero steady state error. Furthermore, it is also noted that there is a constant offset or steady state error when using the benchmark controller due to the biased component of the wind disturbance. The fluctuation due to random wind is also greater for the benchmark controller.

The proposed controller, on the other hand, is able to achieve near zero steady state error. The contracting nature of the closed-loop system guarantees a bounded response to a bounded disturbance, which is a bounded wind disturbance in this case. Furthermore, the proposed controller is a proper

backstepping controller without the use of inverse kinematics separating the translational and rotational part, therefore the steady state tracking error is bounded more tightly to zero.

Therefore it is shown in this simulation that in applications that require the quadrotor to hover or track above a spot in adverse and windy environment, the proposed controller is expected to provide better accuracy and precision.

7. Concluding Remarks

In this paper, a contraction-based backstepping controller that is formulated using full state cascaded dynamics is proposed to control a quadrotor to track a moving ground target. Contraction theory is used as to derive a backstepping controller while ensuring the state variables converge to their respective virtual control at each step. A contracting closed-loop system also ensures that the system response remains bounded under external disturbances. The use of full state cascaded dynamics eliminates the need for inverse kinematics and enables the use of acceleration information in feedback with rotation rate as the virtual control. This achieves integral action effect without increasing the order of the controller like an integral controller. A linear tracking differentiator is also employed to eliminate the need of deriving complicated derivative expressions. Numerical simulations are performed to evaluate the performance of the proposed controller as compared to a benchmark backstepping controller using inverse kinematics. The performance of both controllers are comparable for the ideal case with accurate tracking achieved. However in the presence of aerodynamic effects and wind disturbance, the proposed controller performs better by tracking the target more closely. The proposed controller has demonstrated precise tracking of a moving target with improved rejection of unmodelled aerodynamic effects and wind disturbance. Future directions include the actual implementation of this control scheme and potential applications of the contraction theory-based backstepping controller to other systems.

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(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 5: Results from the numerical simulation of a quadrotor tracking a target moving along a linear path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 5(a) and 5(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 5(c) and 5(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 5(e).

(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 6: Results from the numerical simulation of a quadrotor tracking a target moving along a circular path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 6(a) and 6(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 6(c) and 6(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 6(e).

(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 7: Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a linear path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 7(a) and 7(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 7(c) and 7(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 7(e).

(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 8: Results from the numerical simulation with aerodynamic effects of a quadrotor tracking a target moving along a circular path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 8(a) and 8(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 8(c) and 8(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 8(e).

Figure 9: The ground track of the quadrotor tracking a target moving along a circular path on the ground with aerodynamic effects.

Figure 10: The ground track of the quadrotor equipped with the benchmark controller tracking a target moving along a circular path on the ground with aerodynamic effects.

Figure 11: The random wind speed profile generated by the Dryden wind turbulence model.

Figure 12: The random wind rotation speed profile generated by the Dryden wind turbulence model.

(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 13: Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a linear path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 13(a) and 13(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 13(c) and 13(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 13(e).

(a) X Position

(b) Y Position

(c) Roll Angle

(d) Pitch Angle

(e) Tracking Error

Figure 14: Results from the numerical simulation with wind disturbances of a quadrotor tracking a target moving along a circular path on the ground with the proposed and benchmark controller. The quadrotor is maintained at a height of 10m and a yaw angle of 0^0 . The X and Y position of the quadrotor and the target against time is shown in Figure 14(a) and 14(b) (a zoomed view is shown). The roll and pitch angles against time are also shown in Figure 14(c) and 14(d) respectively. Lastly, the tracking error between the quadrotor and the target is plotted in Figure 14(e).

Figure 15: The ground track of the quadrotor tracking a target moving along a circular path on the ground with wind disturbance.

Figure 16: The ground track of the quadrotor equipped with the benchmark controller tracking a target moving along a circular path on the ground with wind disturbance.