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Application of Incomplete Self-Similarity Argument for Predicting Bed-Material Load Discharge

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Abstract: By applying the incomplete self-similarity argument, this study presents a structural analysis of models for predicting bed-material load discharge, which can be formulated consistently according to the number of independent variables considered. The coefficients involved in the proposed models are calibrated with published laboratory and field data (comprising almost 6,600 records). In comparison with the six bed-material formulas that are recommended in the recently updated ASCE manual on sedimentation engineering, the proposed models show significant improvements on the prediction of bed-material load discharge. This study also implies that the model developed, based on regular regression analysis, can be enhanced by considering interaction terms of independent variables.

CE Database subject headings: Sediment transport; Bed materials; Bed loads; Dimensional analysis; Predictions.

Author keywords: Sediment transport; Sediment discharge; Bed-material load; Total load; Dimensional analysis; Similarity.

Introduction

Several theories have been developed in the past decades to provide frameworks for the analysis of sediment transport rates. Some were developed on the basis of sound physical reasoning, but were largely subject to various assumptions and unknown parameters. For example, the resistance partition technique, which was employed in a few classical bedload functions, still remains questionable (Garcia 2008; Qian and Wan 1999). This partially explains the limited application of theoretical formulas in practice. Moreover, refinements of such theories, though

possible, may further complicate their application without adding much to accuracy of prediction (Ackers and White 1973).

To avoid difficulties inherent in theoretical attempts, an empirical correlation, proposed based on similarity principles or dimensional arguments, could be a good alternative to meeting various requirements in engineering practice. Excellent examples in this respect are attributable to Karim (1981) and Brownlie (1981b), whose empirical relations perform better than several theoretical functions in the prediction of total-load discharge. When deriving empirical relations from multiple regression analysis or other best-fit exercises, one has to consider the functional relation and variables that are to be selected for data analysis. Usually, the selection process is subjective and tedious. For example, in determining the form of the total-load discharge function, Karim (1981) first evaluated a series of combinations of various parameters and then ignored some that were considered less important.

The objective of this study is not to carry out regression analysis in the traditional sense. Rather, the study aims to make certain efforts to explore what form of functional relations should be formulated for computing bed-material load discharge. To this end, the so-called incomplete self-similarity argument is applied. This argument was developed by Barenblatt (1996), with successful applications in various research areas, including the derivation of the generalized power law for the description of velocity profiles in pipe flows and boundary layers. Ferro and Pecoraro (2000) and Carollo et al. (2005) extended the application of the argument to the evaluation of flow resistance in vegetated channels and gravel bed channels. The application presented in this study shows that working equations can be formulated in a systematic fashion according to the number of independent variables involved. This finding simplifies the procedure of data analysis, and the results obtained show significant improvements over the existing bed-material load predictions. However, the structural analysis of the models developed in this study has its inherent limitation in the investigation of the underlying physics.

Complete and Incomplete Self-Similarity

To illustrate the concept of incomplete self-similarity proposed by Barenblatt (1996), one may

consider the following function:

$$z_1 = f(x) \quad (1)$$

where x and z_1 = dimensionless parameters. Two kinds of similarities can be observed for certain cases when x is sufficiently small or large. The first is termed complete self-similarity, with which the function has a nonzero limit, and thus the effect of x disappears completely in the functional relation under the extreme condition. The concept of the complete self-similarity in the Reynolds number has been applied in the selection of scale ratios in physical model studies of river flows (Li and Jing 1981).

The second possible similarity appears when the function does not have a nonzero limit for sufficient small or large x -values. For this situation, the function can be effectively approximated by the power-law asymptotic:

$$z_1 = cx^\alpha \quad (2)$$

where c and α are constant. This case is referred to as the existence of an incomplete self-similarity in the variable x (Barenblatt 1996). It should be emphasized that (1) there may be other cases in which neither complete nor incomplete self-similarity is possible; and (2) Eq. (2) serves only as a good approximation for certain cases, and the power function, though facilitating the derivation of working equation as shown subsequently, may not be the sole option.

Structural Analysis of Working Equation

The following section attempts to extend the previous argument by considering two and more independent variables. First, consider the function given by

$$z_2 = f(x, y) \quad (3)$$

where z_2 , x , and y = dimensionless parameters. If an incomplete self-similarity exists in x , approximating the function in the power law yields

$$z_2 = Y_1(y)x^{Y_2(y)} \quad (4)$$

where both Y_1 and Y_2 depend on y only. On the other hand, if the incomplete self-similarity also

applies to y , one gets

$$z_2 = X_1(x)y^{X_2(x)} \quad (5)$$

where both X_1 and X_2 depend on x only. Both Eqs. (4) and (5), though somehow arbitrary, make easy a structural analysis of total-load formulas, as shown later in this paper. Next, differentiating Eq. (4) with respect to x and y , respectively, gives

$$\frac{\partial \ln z_2}{\partial x} = \frac{Y_2}{x} \quad (6)$$

$$\frac{\partial \ln z_2}{\partial y} = \frac{1}{Y_1} \frac{dY_1}{dy} + \frac{dY_2}{dy} \ln x \quad (7)$$

Similarly, from Eq. (5), one gets

$$\frac{\partial \ln z_2}{\partial x} = \frac{1}{X_1} \frac{dX_1}{dx} + \frac{dX_2}{dx} \ln y \quad (8)$$

$$\frac{\partial \ln z_2}{\partial y} = \frac{X_2}{y} \quad (9)$$

Comparing Eqs. (6) and (8) leads to

$$Y_2 = \frac{x}{X_1} \frac{dX_1}{dx} + x \frac{dX_2}{dx} \ln y \quad (10)$$

Because Y_2 is a function of y only, and both X_1 and X_2 depend on x only, $(x/X_1) dX_1/dx$ and xdX_2/dx included in Eq. (10) should be constant:

$$\frac{x}{X_1} \frac{dX_1}{dx} = \text{constant}; x \frac{dX_2}{dx} = \text{constant} \quad (11)$$

Integration of these two equations yields

$$X_1 = c_1 x^{c_2} \quad (12)$$

$$X_2 = c_3 \ln x + c_4 \quad (13)$$

where c_1 , c_2 , c_3 , and c_4 are constants.

Eqs. (12) and (13) provide a simple approach to evaluate the coefficient and exponent included in Eq. (5). Generally, the evaluation of X_1 and X_2 can be a challenging task in the application of the incomplete self-similarity. In his early work, Barenblatt (1999) demonstrated that significant efforts were needed in the evaluation of the coefficient and exponent of the power law for pipe flows. However, the method used by Barenblatt may not be applicable for determining coefficient and exponent in other cases. In comparison, the approach developed in this study, as shown in Eqs. (12) and (13), is much simpler. It is because of this improvement that the derivation of the general formula, as given later in Eq. (24), is made possible. Furthermore, both coefficient and exponent expressions obtained through series expansion by Barenblatt can also be represented approximately in the form of Eqs. (12) and (13), respectively.

With similar considerations, Eqs. (7) and (9) can be used to derive the following two functions:

$$Y_1 = c_5 y^{c_6} \quad (14)$$

$$Y_2 = c_7 \ln y + c_8 \quad (15)$$

Now substituting Eqs. (12) and (13) into Eq. (5), and Eqs. (14) and (15) into Eq. (4), respectively, one finally obtains

$$z_2 = c_1 x^{c_2} y^{c_3 \ln x + c_4} = c_5 y^{c_6} x^{c_7 \ln y + c_8} \quad (16)$$

With logarithmic transformation, Eq. (16) can be also revised as

$$\begin{aligned} \ln z_2 &= \ln c_1 + c_2 \ln x + c_4 \ln y + c_3 \ln x \ln y \\ &= \ln c_5 + c_8 \ln x + c_6 \ln y + c_7 \ln x \ln y \end{aligned} \quad (17)$$

Both expansions of $\ln(z_2)$ given in Eq. (17) are identical, and the constants are equivalent in pairs:

$$c_1 = c_5; \quad c_2 = c_8; \quad c_3 = c_7; \quad c_4 = c_6 \quad (18)$$

Eq. (17) also shows that with the incomplete self-similarity applied to the two independent variables, x and y , $\ln(z_2)$ can be simply expressed as a linear function of $\ln(x)$, $\ln(y)$, and the multiplicative or interaction term, $\ln(x) \ln(y)$:

$$\ln z_2 = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \alpha_{n_1 n_2} (\ln x)^{n_1} (\ln y)^{n_2} \quad (19)$$

where $\alpha_{n_1 n_2}$ represents the four coefficients. Comparing Eqs. (17) and (19) yields that $\alpha_{00} = \ln c_1$, $\alpha_{10} = c_2$, $\alpha_{01} = c_4$, and $\alpha_{11} = c_3$. The linear function in terms of logarithmically transformed variables, as given by Eq. (17), appears in the form similar to that described in regression analysis. In particular, the inclusion of the interaction term, which has been a much debated subject in multiple regression analysis (Cohen et al. 2003; Friedrich 1982), is shown to be necessary and important in this study. This also implies that the subjective process in the selection of term combinations (e.g., Karim 1981; Yang 1973) can be avoided.

Now, one may consider a new function, z_3 , which depends on z_2 and t :

$$z_3 = f(z_2, t) \quad (20)$$

where z_2 = a function of x and y . If z_3 does not have a nonzero limit when either z_2 or t is sufficiently small or large, one can then assume the incomplete self-similarity exists in the two variables, z_2 and t . By following the previous analysis, Eq. (20) can be further written in the form

$$z_3 = c_9 z_2^{c_{10}} t^{c_{11} \ln z_2 + c_{12}} \quad (21)$$

where c_9 , c_{10} , c_{11} , and c_{12} are constants. Substituting Eq. (16) into Eq. (21) and manipulating yields

$$\begin{aligned} \ln z_3 &= \ln a_1 + a_2 \ln x + a_3 \ln y + a_4 \ln t + a_5 \ln x \ln y + a_6 \ln x \ln t \\ &\quad + a_7 \ln y \ln t + a_8 \ln x \ln y \ln t \\ \text{or } z_3 &= a_1 x^{a_2} y^{a_3 + a_5 \ln x} t^{a_4 + a_6 \ln x + a_7 \ln y + a_8 \ln x \ln y} \end{aligned} \quad (22)$$

where a_1 to a_8 are constants. Eq. (22) shows that if the incomplete self-similarity applies to the three independent variables, x , y , and t , $\ln(z)$ can be expressed as a linear function of $\ln(x)$, $\ln(y)$,

$\ln(t)$ and their interaction terms:

$$\ln z_3 = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=1}^1 \alpha_{n_1 n_2 n_3} (\ln x)^{n_1} (\ln y)^{n_2} (\ln t)^{n_3} \quad (23)$$

where $\alpha_{n_1 n_2 n_3}$ denotes the 2^3 or eight coefficients as those given in Eq. (22).

Generally, the previous analysis can be applied to a function, z_N , with N independent variables, $x_1, x_2, \dots, x_N$. Applying of the incomplete self-similarity technique yields

$$\ln z_N = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \dots \sum_{n_N=0}^1 \alpha_{n_1 n_2 \dots n_N} (\ln x_1)^{n_1} (\ln x_2)^{n_2} \dots (\ln x_N)^{n_N} \quad (24)$$

The right-hand side of Eq. (24) is a sum of the 2^N terms, and therefore there are 2^N coefficients to be calibrated.

Dimensional Analysis for Bed-Material Load Discharge Prediction

For simplicity, it is assumed that (1) the flow under consideration is steady and uniform; (2) the channel is wide (or two-dimensional); (3) sediment transport occurs at the equilibrium stage; and (4) only the bed-material-load (or total load excluding wash load) is included. Without sediment, the average flow velocity, V , in an open channel varies depending on five variables:

$$V = f(S, h, \nu, k_s, g) \quad (25)$$

where S = energy slope; h = flow depth; ν = kinematic viscosity of fluid; k_s = length of bed roughness; and g = gravitational acceleration. This dependence can be mathematically validated by depth-averaging of the velocity profile (Cheng 2008):

$$(u^+)^m = (z^+)^m + \left[2.5 \ln \left(1 + \frac{9z^+}{1 + 0.3k_s^+ (1 - e^{-0.04k_s^+})} \right) \right]^m \quad (26)$$

where $u^+ = u/u_*$, $z^+ = u_* z/\nu$, $k_s^+ = u_* k_s/\nu$, $m = -3$, u = velocity at the distance z from the channel bed; and u_* = shear velocity ($= \sqrt{ghS}$). Eq. (26) serves as a generalized law-of-the-

wall function that predicts the velocity distribution for fully smooth, fully rough, and transitional boundary conditions. It reduces to the logarithmic law if the viscous sublayer is ignored.

In the presence of sediment, the flow velocity is modified by sediment properties, i.e., median grain size D_{50} , sediment density ρ_s or its difference from the fluid density ρ , sediment gradation or its geometric standard deviation σ_g , and sediment concentration C . Therefore, for an open channel flow over a mobile sediment boundary, if assuming that k_s is equivalent to D_{50} in the sense of roughness length scale, the dependence of the average flow velocity, V , can be generally expressed in the form

$$V = f(S, h, \nu, D_{50}, g, \sigma_g, \rho_s, \rho, C) \quad (27)$$

Eq. (27) suggests that the sediment concentration and thus the sediment transport rate per unit width, q_v , in volume, depend on the nine variables shown as follows:

$$q_v = f(S, V, h, \nu, D_{50}, \sigma_g, \rho_s, \rho, g) \quad (28)$$

q_v is related to C in the form

$$q_v = \frac{CQ}{10^6 - CB\rho_s} \quad (29)$$

where C (in ppm) is defined as the mass ratio of sediment discharge to water-sediment mixture discharge; Q = flow discharge; and B = flow width.

Using dimensional analysis by choosing ρ , g , and D_{50} as repeating variables, Eq. (28) can be transformed to

$$\frac{q_v}{\sqrt{gD_{50}^3}} = f\left[S, \frac{V}{\sqrt{gD_{50}}}, \frac{h}{D_{50}}, \left(\frac{g}{\nu^2}\right)^{\frac{1}{3}} D_{50}, \sigma_g, \frac{\rho_s}{\rho}\right] \quad (30)$$

By combining some dimensionless parameters, Eq. (30) can be further expressed as

$$\frac{q_v}{\sqrt{\Delta g D_{50}^3}} = f\left[\frac{hS}{\Delta D_{50}}, \frac{V}{\sqrt{\Delta g D_{50}}}, \frac{h}{D_{50}}, \left(\frac{\Delta g}{\nu^2}\right)^{\frac{1}{3}} D_{50}, \sigma_g, \Delta\right] \quad \text{or,}$$

$$\phi = f(\theta, F_g, h/D_{50}, D_*, \sigma_g, \Delta) \quad (31)$$

In Eq. (31), several dimensionless parameters are simply taken as those commonly employed in the literature. They are the Einstein number, $\phi = q_V / \sqrt{\Delta g D_{50}^3}$ where $\Delta = (\rho_s - \rho) / \rho$; the Shields number, $\theta = hS / (\Delta D_{50})$; the grain Froude number, $F_g = V / \sqrt{\Delta g D_{50}}$; and the dimensionless sediment diameter, $D_* = (\Delta g / \nu^2)^{1/3} D_{50}$. In these four parameters, Δ and g are combined in a product form to quantify the reduced effect of gravity for submerged sediment particles.

Theoretically, there should be a counterpart of Eq. (26), which is applicable for two-dimensional, open channel sediment-laden flows. However, such a general equation is not available in the literature owing to the limited understanding of two-phase flows. Therefore, it is impossible to theoretically validate the functional relation given by Eq. (27). On the other hand, one may reduce the list of independent variables in Eq. (27) on the grounds that for sediment transport at equilibrium, V , S , and h would be inter-related through the Colebrook-type resistance formula. As a result, one of the three variables, V , S , and h , should be excluded in the evaluation of the sediment concentration or discharge. Correspondingly, in terms of dimensionless parameters, either F_g or θ or h/D_{50} should be removed from the list of independent variables in Eq. (31).

Next, to apply the concept of incomplete self-similarity, one starts with the relationship of ϕ and θ . Because ϕ would approach infinity as θ is sufficiently large, the incomplete self-similarity argument is applicable for the formulation of the functional relation of $\phi - \theta$. As a result, ϕ can be simply related to θ in a power form similar to Eq. (2); one such example is attributable to Engelund and Hansen (1967).

Furthermore, it is reasonable to assume that ϕ would approach infinity for a very large V , or a very small D or Δ . Also, a very small h is equivalent to a very large V for a given flow rate, and a very small D would appear in the case of a very large σ_g . With the characteristics of ϕ for the various extreme situations, it is further assumed that the incomplete self-similarity argument also holds in relating ϕ to other variables, F_g , h/D_{50} , D_* , σ_g , and Δ . Therefore, Eq. (31) can be written in the same form as Eq. (24):

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \dots \sum_{n_6=0}^1 \alpha_{n_1 n_2 n_3 n_4 n_5 n_6} (\ln \theta)^{n_1} (\ln F_g)^{n_2} \left(\ln \frac{h}{D_{50}} \right)^{n_3} (\ln D_*)^{n_4} (\ln \sigma_g)^{n_5} (\ln \Delta)^{n_6} \quad (32)$$

Eq. (32) presents the general structure of working equations for the computation of bed-material load discharge. In comparison with the previous studies (Brownlie 1981b; Karim 1981; Yang 1973), the structure of the model given by Eq. (32) is well organized. This structure may be still considered arbitrary because of the power law approximation made earlier through Eq. (2). On the other hand, when ignoring the interaction terms, Eq. (32) resembles the mathematical formulation usually used in regular regression analysis. This resemblance facilitates the comparison of the present study with the previous formulas (Brownlie 1981b; Karim 1981; Yang 1973). However, as demonstrated later in this paper, it is essential to include all interaction terms in the regression. In other words, some of the interaction terms that may appear trivial for certain conditions should not be ignored.

The right-hand side of Eq. (32) is a sum of 2^6 or 64 terms if all parameters are considered. It reduces for simple conditions. For example, in the case of uniform sediment size (i.e., σ_g is close to 1.0) and constant sediment density (i.e., $\rho_s / \rho \approx 2.65$ or $\Delta \approx 1.65$), Eq. (32) reduces to

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \sum_{n_4=0}^1 \alpha_{n_1 n_2 n_3 n_4} (\ln \theta)^{n_1} (\ln F_g)^{n_2} \left(\ln \frac{h}{D_{50}} \right)^{n_3} (\ln D_*)^{n_4} \quad (33)$$

The right-hand side of Eq. (33) comprises 2^4 or 16 terms. For the coarse sediment (or large D_* or insignificant viscous effect), Eq. (33) can be further reduced to

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \theta)^{n_1} (\ln F_g)^{n_2} \left(\ln \frac{h}{D_{50}} \right)^{n_3} \quad (34)$$

The right-hand side of Eq. (34) comprises 2^3 or eight terms only. Moreover, if it is possible to relate θ to F_g and h/D_{50} , or otherwise through the resistance formula, then the terms included in the right-hand side of Eqs. (32)–(34) are reduced by half.

In the subsequent data-based analyses, the performance of various models proposed according to Eq. (32) are evaluated, and then differences in the prediction of total-load discharge are examined if one of θ , F_g , and h/D_{50} is excluded in the models. After that, comparisons are made between the proposed models and the published works.

Data Sources

The classical sediment transport data have been compiled by several investigators (e.g., Brownlie 1981a; Peterson and Howells 1973). The compilation of data presented by Brownlie is considered excellent, and was collected from over 70 sources comprising 5,263 laboratory records and 1,764 field records (i.e., 7,027 in total). Each record is described with 10 parameters including water discharge (Q), channel width (B), flow depth (h), water temperature (T), energy slope (S), median particle size (D_{50}), sediment gradation coefficient (σ_g), specific gravity (ρ_s/ρ), bed-material-load concentration (C), and type of bedform. In developing the total load discharge function, Brownlie (1981b) conducted his analysis based only on the data that were selected from 22 sources, subject to the conditions including (1) width-to-depth ratio ≥ 4 ; (2) $r/D_{50} > 100$, where r = hydraulic radius; (3) concentration ≥ 10 ppm; (4) $0.062 \text{ mm} \leq D_{50} \leq 2 \text{ mm}$ (sand only); and (5) $\sigma_g < 5$. In comparison, all records, except for those with missing information, are used for the analyses conducted in this study.

Preprocessing of Data

Before data analyses, all records included in the Brownlie compilation were sorted out first by removing those missing information about concentration, slope, particle size, or gradation coefficient, and also those with negative slope or zero concentration. However, the data with no records of water temperature were still taken into account by assuming it to be 20°C . This finally yielded a modified database with 6,590 records, of which the information is summarized in Table 1.

Sidewall Correction

For the laboratory data, the sidewall effect becomes considerable for small width-to-depth ratios. This can be removed by performing so-called sidewall correction following the Vanoni-Brooks procedure (Cheng and Chua 2005; Vanoni 1975), in which the wall-related friction factor f_w is evaluated using the formula proposed here:

$$f_w = 31 \left(\ln \frac{1.3R}{f} \right)^{-2.7} \quad (35)$$

where $f = 8grS/V^2$ and $R = 4rV/\nu$. Eq. (35) serves as one of alternatives to the graphical relation of f_w to R/f plotted by Vanoni and Brooks. The bed-related hydraulic radius is then computed as

$$r_b = h \left(1 - \frac{2h}{B + 2h} \frac{f_w}{f} \right) \quad (36)$$

In the subsequent analyses, h is replaced with r_b , and thus h/D_{50} and θ are changed to r_b/D_{50} and $r_b S/(\Delta D_{50})$, respectively.

Kinematic Viscosity of Water

The kinematic viscosity of water, ν , is related to the temperature, T , using the following empirical equation (Cheng et al. 2010):

$$\nu = \left(\frac{60}{T + 40} \right)^{1.45} 10^{-6} \quad (37)$$

where ν is in m^2/s , and T is in degrees Celsius. Eq. (37) provides a good representation of the standard database given by Linstrom and Mallard (2005), with errors less than 0.54% for $T = 0 - 100^\circ\text{C}$.

Settling Velocity and Critical Shear Velocity

The settling velocity, w , is computed by using

$$\frac{wD}{\nu} = \left(\sqrt{25 + 1.2D_*^2} - 5 \right)^{1.5} \quad (38)$$

Eq. (38) is applicable for the computation of the settling velocity of naturally worn sediment grains

(Cheng 1997). Eq. (38) performs better in representing experimental data than the Rubey equation, the latter being used in Molinas and Wu's total-load relation (2001). The critical shear velocity u_{*c} (or θ_c) can be evaluated using the formula given by Whitehouse et al. (2000):

$$\theta_c = \frac{0.3}{1 + 1.2D_*} + 0.055 \left[1 - \exp\left(\frac{-D_*}{50}\right) \right] \quad (39)$$

Eq. (39) is applicable for the condition of $D_* = 0.1 - 1000$. Eqs. (38) and (39) are used subsequently in applying some total-load formulas in which the estimates of w and u_{*c} are not specified.

Evaluation of Various Models

To perform data analyses by using the model function given by Eq. (32), different groupings of independent dimensionless parameters (IDP) are considered. Table 2 shows four types of models. The first model is simple in that effects of σ_g , D_* , and Δ are ignored by assuming that the sediment size is near uniform, the viscous effect is insignificant, and the specific gravity is almost constant. For this model, the coefficients are calibrated by conducting the regression analysis with the data in the range of (1) $\sigma_g < 1.3$, (2) $\rho_s/\rho = 2.6 - 2.7$, and (3) $D_{50} > 2$ mm. By noting that the three IDPs (i.e., θ , $\mathbf{F}_g$, r_b/D_{50}) are involved, the model is referred to as 3-IDP. Here, the viscous effect is considered insignificant for $D_{50} > 2$ mm, which can be explained as follows. If taking $\nu = 10^{-6} \text{ m}^2/\text{s}$ and $\rho_s/\rho = 2.65$, one gets $D_* > 50.6$ for $D_{50} > 2$ mm. The condition of $D_* > 50.6$ suggests that the critical Shields number is close to a constant for $D_{50} > 2$ mm with reference to the Shields diagram or Eq. (39), and therefore the sediment transport is considered not significantly affected by the viscous effect. The other three types of models described in Table 2 engage 4, 5, and 6 IDPs, respectively, together with different selections of the data by applying fewer restrictions.

To demonstrate what improvement can be made by applying the incomplete self-similarity argument, regular regression analysis was also conducted, i.e., the first order regression, by ignoring the interaction terms included in Eq. (32). The regression results obtained with and

without including the interaction terms are compared. For convenience, the formula including the interaction terms, exactly the same as given by Eq. (32), is referred to as the full model, while that without the interaction terms is referred to as the first-order model in terms of logarithmically transformed variables. For example, for model 3-IDP, the full model is formulated in the form similar to Eq. (22):

$$\phi_{\text{full}} = a_1 F_g^{a_2} \left(\frac{r_b}{D_{50}} \right)^{a_3 + a_4 \ln F_g} \theta^{a_5 + a_6 \ln F_g + a_7 \ln \left(\frac{r_b}{D_{50}} \right) + a_8 \ln F_g \ln \left(\frac{r_b}{D_{50}} \right)} \quad (40)$$

whereas the form of the first-order model reduces to

$$\phi_{st} = b_1 F_g^{b_2} \left(\frac{r_b}{D_{50}} \right)^{b_3} \theta^{b_4} \quad (41)$$

where a_1 to a_8 and b_1 to b_4 are constants. All these constants can be fixed by conducting regression analysis with the logarithmic transformation of Eqs. (40) and (41). The exponents appearing in Eqs. (40) and (41), respectively, are generally not comparable and should be interpreted differently. For the regular first-order model, all the exponents are constants, each describing the average dependence of ϕ on the particular variable. However, for the full model that includes the interaction terms, the exponent for each particular independent variable and thus the dependence of ϕ on that particular independent variable varies with the level of the other independent variables (Cohen et al. 2003).

Comparison of First-Order and Full Models

In Fig. 1, the predictions obtained using both the first-order and full model are compared with the data. This shows that (1) all the first-order models yield poor predictions, generally overestimating ϕ for the condition of low transport rates; and (2) all the full models perform much better than the corresponding first-order models, which implies the importance of the inclusion of the interaction terms. To quantify the goodness of prediction, two statistical parameters are used. They are (1) the average of the relative error

$$\text{Err}_1 = \frac{1}{n} \sum \frac{|\phi_{\text{cal}} - \phi_{\text{data}}|}{\phi_{\text{data}}} \quad (42)$$

where n = total number of data points used; and (2) the error sum of squares defined using $\log \phi$:

$$\text{Err}_2 = \sum (\log \phi_{\text{cal}} - \log \phi_{\text{data}})^2 \quad (43)$$

The value of Err_2 measures the overall deviation of the prediction from the perfect agreement, as visualized in the log-log plot.

The computed Err_1 and Err_2 are summarized in Table 2. For model 3-IDP, Err_1 reduces by 65 % from 5.4 for the first-order analysis to 1.9 for the full model, and Err_2 reduces by 46% from 331 to 180. More marked improvements are observed for the other models. For example, for model 4-IDP, Err_1 reduces by 97% and Err_2 reduces by 61%. Obviously, all these improvements are largely attributable to the inclusion of the interaction terms. Whether the interaction terms should be included or not in the regression analysis has been a much debated subject; the interpretation of interactions often appears important (Cohen et al. 2003; Friedrich 1982; Jaccard and Turrisi 2003). The consideration based on the incomplete self-similarity and the results obtained here offer a rational explanation of the necessity of the inclusion of the interaction terms. Given the significant difference between the predictions made using the first-order and full models, employed in the following analyses are the full models only.

Reduction of Full Models

In this section, the model performance is further examined by assuming that V , S , and r_b are interrelated through the resistance formula. For example, by ignoring $\mathbf{F}_g$, θ , or r_b/D_{50} , model 4-IDP, Eq. (33), can be reduced respectively to three 3-IDP models:

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \theta)^{n_1} \left(\ln \frac{r_b}{D_{50}} \right)^{n_2} (\ln D_*)^{n_3} \quad (44)$$

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \mathbf{F}_g)^{n_1} \left(\ln \frac{r_b}{D_{50}} \right)^{n_2} (\ln D_*)^{n_3} \quad (45)$$

$$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \theta)^{n_1} (\ln \mathbf{F}_g)^{n_2} (\ln D_*)^{n_3} \quad (46)$$

To compare model 4-IDP with its reduced versions defined previously, the computations were performed using 2,757 records that were selected under the condition of $\sigma_g < 1.3$ and $\rho_s/\rho = 2.6 - 2.7$, for which model 4-IDP applies (Table 2). The results are summarized in Table 3. Interestingly, Eq. (46), which includes S and V , but not r_b , performs much better than Eqs. (44) and (45), and the associated errors are even comparable to those related to model 4-IDP, i.e., Eq. (33). For example, Err_1 related to the reduced model decreases from 13.5 (if θ is ignored) and 2.9 (if $\mathbf{F}_g$ is ignored) to 1.6 (if r_b/D_{50} is ignored), while Err_2 decreases from 873 and 682 to 430. Better results are also observed for models 3-IDP, 5-IDP, and 6-IDP when r_b/D_{50} (rather than θ and $\mathbf{F}_g$) is ignored in the analyses (Table 3).

For the data used in the analysis, the variations in σ_g and Δ are relatively small. For example, the part of $\sigma_g > 3$ accounts only for 1.7% of the Brownlie compilation and that of $\Delta < 1.6$ and $\Delta > 1.7$ accounts only for 6.8%. Such limited variations would affect the quality of calibration of the coefficients used in models 5-IDP and 6-IDP. Considering this limitation, it is expected that the coefficients used in models 3-IDP and 4-IDP would be calibrated better than those in the other models, and thus models 3-IDP and 4-IDP are recommended for practical use. Tables 4 and 5 provide the calibrated coefficients used in models 3-IDP and 4-IDP, respectively, together with those used in their reduced versions, 3-IDP(r) and 4-IDP(r), in which r_b/D_{50} is ignored.

Comparison with Previous Formulas

Further comparisons are made between the proposed models and the six bed-material formulas recommended in the recently published ASCE manual on sedimentation engineering (Garcia 2008):

1. Engelund and Hansen's (1967) formula

$$\phi = 0.05 \frac{V^2}{ghS} \theta^{2.5} \quad (47)$$

2. Ackers and White's (1973) formula

$$C = 10^6 \beta \frac{\rho_s D}{\rho h} \left(\frac{V}{u_*} \right)^\eta \left(\frac{F_{gr}}{\alpha} - 1 \right)^\zeta \quad (48)$$

where C is in parts per million (ppm) by mass, $F_{rg} = u_*^\eta [V / (\sqrt{32} \log(10h/D_{50}))]^{1-\eta} / \sqrt{\Delta g D_{50}}$, $\alpha = 0.23 / \sqrt{D_*} + 0.14$, $\beta = 10^{2.86 \log D_* - (\log D_*)^2 - 3.53}$, $\zeta = 9.66/D_* + 1.34$, $\eta = 1 - 0.56 \log D_*$ for $1 < D_* < 60$; and $\alpha = 0.17, \beta = 0.025, \zeta = 1.5, \eta = 0$ for $D_* \geq 60$

3. Yang's (1973,1984) formula

$$\begin{aligned} \log C = & \left(a_1 + a_2 \log \frac{wD_{50}}{\nu} + a_3 \log \frac{u_*}{w} \right) \\ & + \left(a_4 + a_5 \log \frac{wD_{50}}{\nu} + a_6 \log \frac{u_*}{w} \right) \log \left(\frac{V - V_c}{w} S \right) \end{aligned} \quad (49)$$

where $V_c/w = 2.5/[\log(u_*D_{50}/\nu) - 0.06] + 0.66$ for $1.2 < u_*D_{50}/\nu \leq 70$, $V_c/w = 2.05$ for $u_*D_{50}/\nu > 70$, and a_1 to a_6 are equal to 5.435, -0.286 , -0.457 , 1.799, -0.409 , -0.314 , respectively, for sand, and 6.681, -0.633 , -4.816 , 2.784, -0.305 , -0.282 , respectively, for fine to medium gravel ($D_{50} = 2 - 10$ mm).

4. Karim's (1981) formula

$$\begin{aligned} \log \phi = & -2.2786 + 2.9719 \log F_g \\ & + 0.2989 \log(h/D_{50}) \log(\sqrt{\theta} - \sqrt{\theta_c}) \\ & + 1.06 \log F_g \log(\sqrt{\theta} - \sqrt{\theta_c}) \end{aligned} \quad (50)$$

where θ_c = critical Shields number used for the incipient sediment motion

5. Brownlie's (1981b) formula

$$C = 7115c_F(\mathbf{F}_g - \mathbf{F}_{gC})^{1.978} S^{0.6601} (R_h/D_{50})^{-0.3301} \quad (51)$$

where $c_F = 1$ (for laboratory data) and 1.268 (for field data), and $\mathbf{F}_{gC} = 4.596\tau_{*C}^{0.5293} S^{-0.1405} \sigma_g^{-0.1606}$ with $\tau_{*C} = 0.22Y + 0.06 (10)^{-7.7Y}$ and $Y = \left(\sqrt{\Delta g D_{50}^3}/\nu\right)^{-0.6} = D_*^{-0.9}$

6. Molinas and Wu's (2001) formula

$$C = \frac{1430(0.86 + \sqrt{\psi})\psi^{1.5}}{0.016 + \psi} \quad (52)$$

where $\psi = V^3/\{\Delta g h w [\log(h/D_{50})]^2\}$ and the settling velocity w is estimated using the Rubey equation.

These six bed-material formulas show that (1) the effect of fluid viscosity is ignored only by Engelund and Hansen (1967); (2) three of S , h , and V are present together in all the formulas except for that by Molinas and Wu (2001), who ignore S ; and (3) the effect of σ_g is considered only by Brownlie (1981b). Eqs. (48)–(51) contain difference terms such as $(V - V_C)$ and $(\sqrt{\theta} - \sqrt{\theta_C})$, and therefore are subject to the quantification of the critical flow condition for incipient sediment motion. The difference terms represent the extra driving force for sediment transport in terms of average or shear velocity. If their values are close to zero or negative, it does not necessarily mean that the sediment discharge is zero. However, Eqs. (48)–(51) fail to predict such weak transport conditions. For example, Karim's formula is not applicable for 436 records, which comprises 7% of the database used. Cheng (2002) also noted that almost all classical bedload functions cannot be used to predict low transport rates because of the involvement of the critical flow condition, which may be overestimated using the Shields diagram. Actually, by noting that θ_C is a function of D_* , as implied by Shields diagram, the effect of θ_C could be considered indirectly through D_* , which is actually demonstrated in this study by use of Eq. (32).

Therefore, to compare the present model with the six bed-material formulas, the writer only engaged records under the condition that allow possible predictions from all formulas. With this

limitation, 598 records that generally fall in the low transport regime were rejected, and the rest (i.e., 5,992 records or 91% of the data described in Table 1) remained for comparison analyses. Fig. 2 shows the predictions of the previous formulas and the present models, 4-IDP and 4IDP(r) (see Table 5), in comparison with the data used. The corresponding statistics, Err_1 and Err_2 , are summarized in Table 6.

From Fig. (2) and Table 6, the following remarks can be made. First, in comparison with the data excluding weak transport, the Engelund and Hansen formula, despite its simple form, almost performs best among the six bed-material formulas. Second, when compared with the six formulas excluding Molinas and Wu's formula that was proposed for large rivers only, the improvements made by using the present models, 4-IDP and 4IDP(r) (see Table 5), are very significant. On average, the errors, Err_1 and Err_2 , associated with the previous formulas are 5.1 and 1,281, respectively. They reduce to 1.6 (by 69%) and 840 (by 34%), respectively, if models 4-IDP and 4-IDP(r) are applied.

Effect of Sediment Gradation

The models proposed here can be conveniently used to check whether it is possible to use a representative sediment size in place of D_{50} to account for effect of sediment gradation. Two full models are chosen: model 4-IDP that includes D_{50} but without σ_g , and model 5-IDP that includes both D_{50} and σ_g . To conduct relevant analyses with particular sediment size, in model 4-IDP, D_{50} is replaced with D_5 , D_{10} , ..., D_{95} , respectively. Then the responding errors are evaluated and compared with those associated with model 5-IDP. In implementing both models, the data were sorted by applying the restriction of $\rho_s/\rho = 2.6 - 2.7$.

For model 4-IDP, a particular particle size can be evaluated with D_{50} and σ_g . Here, it is assumed that the size distribution can be approximated using the lognormal function. Therefore, any particular size D_p with percentile p can be related to D_{50} and σ_g by

$$D_p = D_{50} \sigma_g^m \quad (53)$$

where m depends on p . Empirically, the m - p relation can be estimated by the formula proposed here:

$$m = \frac{5p - 50}{4 \cdot 50} \left[1 - \left(\frac{p - 50}{50} \right)^2 \right]^{-1/4} \quad (54)$$

The prediction for $p = 10 - 90$ made using Eq. (54) has an accuracy of 0.7%. Fig. 3 shows the errors in the prediction of sediment discharge vary with p . It seems that for field data, a good representative sediment size has no much difference from D_{50} , but for laboratory data, the use of D_{80} could minimize prediction errors. However, the results also indicate that the inclusion of both D_{50} and σ_g always yields better predictions than the use of a single particle size without σ_g . In conducting computations associated with Fig. 3, the data points selected are those characterized by $\sigma_g > 2$, comprising 154 laboratory records ($\sigma_g = 2.1 - 3.8$) and 504 field records ($\sigma_g = 2.0 - 100.1$). Because of the small amount of the data used, the results presented in Fig. 3 should be considered preliminary.

Conclusions

The classical sediment discharge data were revisited by implementing the concept of incomplete self-similarity in the analysis. This results in a series of model equations, which are formulated in a systematic way according to the number of independent variables under consideration. When ignoring interaction terms, the models resemble the mathematical formulation usually employed in regular regression analysis.

In comparison with the commonly used bed-material load formulas, the models proposed in this study show significant improvements in the prediction of sediment discharge. In particular, these models do not involve the evaluation of the critical flow condition for incipient sediment motion, and thus are also applicable for weak transport conditions.

The additional analysis with the proposed models implies that it is impossible to use a single grain size, in place of the median diameter, to additionally account for effects of sediment gradation on sediment discharge prediction. This study also demonstrates that it is of great importance to

consider interaction terms of independent variables in performing regression analysis. The present models can be further calibrated using particular data with wide variations in the sediment gradation and density.

Notation

The following symbols are used in this paper:

- $a_1, a_2, \dots$ = coefficient;
- B = channel width;
- $b_1, b_2, \dots$ = coefficient;
- C = sediment concentration (in ppm);
- $c, c_1, c_2, \dots$ = coefficient;
- D_p = sediment size at percentile p ;
- D_{50} = median grain size;
- D_* = dimensionless sediment diameter
 $[(\Delta g / \nu^2)^{1/3} D_{50}]$;
- Err₁ = average of absolute relative error;
- Err₂ = sum of squared errors defined using
 $\log \phi$;
- F_g = grain Froude number ($= V / \sqrt{\Delta g D_{50}}$);
- f = friction factor ($= 8grS / V^2$);
- f_w = wall-related friction factor;
- g = gravitational acceleration;
- h = flow depth;
- k_s = length of bed roughness;
- m = constant or exponent;
- n = constant or exponent;
- $n_1, n_2, n_3, \dots$ = constant ($= 0, 1$);
- p = percentile used for sediment gradation;
- q_v = sediment transport rate per unit width;
- Q = flow discharge;
- R = Reynolds number ($= 4rV / \nu$);
- r = hydraulic radius;
- r_b = bed-related hydraulic radius;
- S = energy slope;
- t = independent variable;
- u = flow velocity at distance z from channel bed;
- $u^+ = u / u_*$;
- u_* = shear velocity;
- V = average flow velocity;
- V_C = critical average velocity;
- w = settling velocity of sediment particle;
- X_1, X_2 = function of x ;
- $x, x_1, x_2, \dots, x_N$ = independent variable;
- Y_1, Y_2 = function of y ;
- y = independent variable;
- z = vertical distance from the bed;
- $z_1, z_2, z_3, \dots, z_N$ = dependent variable;
- $z^+ = u_* z / \nu$;
- $\alpha_{n_1, n_2, n_3, \dots}$ = coefficient with $n_1 = 0, 1$; $n_2 = 0, 1$;
 $n_3 = 0, 1$; ...;
- β = coefficient;
- $\Delta = (\rho_s - \rho) / \rho$;
- θ = Shields number ($= hS / (\Delta D_{50})$);
- θ_c = critical Shields number;
- ν = kinematic viscosity of fluid;
- ρ = fluid density;
- ρ_s = sediment density;
- σ_g = geometric standard deviation; and
- ϕ = Einstein number ($= q_v / \sqrt{\Delta g D_{50}^3}$).

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Figure . 3. Effect of sediment gradation on sediment discharge prediction: (a) laboratory data; (b) field data

Measured variable	Flow rate Q (m ³ /s)	Width (m)	Depth (m)	D_{50} (mm)	Geometric standard deviation of sizes σ_g	Slope S	Sediment specific gravity ρ_s/ρ	Concentration (ppm)	Fluid viscosity ν (m ² /s)
Minimum	5.95×10^{-4}	0.07	0.0079	0.011	1	3×10^{-6}	1.03	0.001	4.57×10^{-7}
Maximum	2.88×10^4	1,109	17.282	76.11	100.1	0.037	4.22	1.11×10^5	1.74×10^{-6}
Dimensionless parameter	Einstein number ϕ	Shields number θ	Grain Froude number F_g	h/D_{50}	Dimensionless grain diameter D_*	Effective specific gravity Δ	Froude number $V/\sqrt{gh}$	Reynolds number VR_h/ν	Shear Reynolds number u_*D_{50}/ν
Minimum	2.05×10^{-9}	0.013	0.26	1.519	0.257	0.03	0.01	1,537	0.145
Maximum	4.19×10^4	11.73	259.5	9.25×10^4	1,860	3.22	3.64	7.16×10^7	2.02×10^4

Table 1

Model	Independent dimensionless parameter (IDP)	Restrictions	Data points used	Err ₁		Err ₂	
				First-order	Full	First-order	Full
3-IDP	$\theta, \mathbf{F}_g, r_b/D_{50}$	$\sigma_g < 1.3, D_{50} > 2 \text{ mm}, \rho_S/\rho = 2.6 - 2.7$	720	5.4	1.9	331	180
4-IDP	$\theta, \mathbf{F}_g, r_b/D_{50}, D_*$	$\sigma_g < 1.3, \rho_S/\rho = 2.6 - 2.7$	2,757	43.5	1.5	970	376
5-IDP	$\theta, \mathbf{F}_g, r_b/D_{50}, D_*, \sigma_g$	$\rho_S/\rho = 2.6 - 2.7$	6,129	38.1	1.4	1,862	892
6-IDP	$\theta, \mathbf{F}_g, r_b/D_{50}, D_*, \sigma_g, \Delta$		6,590	34.8	1.4	2,074	944

Table 2

Model	IDP	Prediction errors							
		If θ is ignored		If F_g is ignored		If r_b/D_{50} is ignored		If no IDP is ignored	
		Err ₁	Err ₂	Err ₁	Err ₂	Err ₁	Err ₂	Err ₁	Err ₂
3-IDP	$\theta, F_g, r_b/D_{50}$	12.6	393	3.9	349	2.0	201	1.9	180
4-IDP	$\theta, F_g, r_b/D_{50}, D_*$	13.5	873	2.9	682	1.6	430	1.5	376
5-IDP	$\theta, F_g, r_b/D_{50}, D_*, \sigma_g$	9.6	1,663	3.6	1,917	2.0	1,063	1.4	892
6-IDP	$\theta, F_g, r_b/D_{50}, D_*, \sigma_g, \Delta$	9.8	1,797	3.6	2,072	2.0	1,171	1.4	944

Table 3

Model 3-IDP			$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \theta)^{n_1} (\ln F_g)^{n_2} \left(\ln \frac{r_k}{D_{50}} \right)^{n_3}$ $\phi = e^{c_0} \left(\frac{r_k}{D_{50}} \right)^{c_1} F_g^{c_2 + c_3 \ln \frac{r_k}{D_{50}}} \theta^{c_4 + c_5 \ln \frac{r_k}{D_{50}} + c_6 \ln F_g + c_7 \ln \frac{r_k}{D_{50}}} \ln F_g$					
Coefficient	α_{000}	α_{001}	α_{010}	α_{011}	α_{100}	α_{101}	α_{110}	α_{111}
	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
	4.18	2.04	-0.52	-2.06	8.26	0.06	-4.62	-0.11
Reduced model 3-IDP(r)			$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \alpha_{n_1 n_2} (\ln \theta)^{n_1} (\ln F_g)^{n_2}$ $\phi = e^{c_0} F_g^{c_1} \theta^{c_2 + c_3 \ln F_g}$					
Coefficient	α_{00}	α_{01}	α_{10}	α_{11}				
	c_0	c_1	c_2	c_3				
	12.05	-9.04	9.29	-5.96				
Note: The models are applicable for the condition of $D_{50} > 2$ mm, $\sigma_g < 1.3$, and $\rho_S/\rho = 2.6 - 2.7$.								

Table 4

Model 4-IDP		$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \sum_{n_4=0}^1 \alpha_{n_1 n_2 n_3 n_4} (\ln \theta)^{n_1} (\ln \mathbf{F}_g)^{n_2} \left(\ln \frac{r_b}{D_{90}} \right)^{n_3} (\ln D_*)^{n_4}$ $\phi = e^{c_0} D_*^{c_1} \left(\frac{r_b}{D_{90}} \right)^{c_2 + c_3 \ln D_*} \mathbf{F}_g^{c_{F_g}} \theta^{c_\theta}$ <p>where</p> $c_{F_g} = c_4 + c_5 \ln D_* + c_6 \ln \frac{r_b}{D_{90}} + c_7 \ln D_* \ln \frac{r_b}{D_{90}}$ $c_\theta = c_8 + c_9 \ln D_* + c_{10} \ln \frac{r_b}{D_{90}} + c_{11} \ln D_* \ln \frac{r_b}{D_{90}} + \left(c_{12} + c_{13} \ln D_* + c_{14} \ln \frac{r_b}{D_{90}} + c_{15} \ln D_* \ln \frac{r_b}{D_{90}} \right) \ln \mathbf{F}_g$						
Coefficient	α_{0000}	α_{0001}	α_{0010}	α_{0011}	α_{0100}	α_{0101}	α_{0110}	α_{0111}
	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
	-8.49	4.40	0.46	-0.61	5.04	-2.09	-0.24	0.26
	α_{1000}	α_{1001}	α_{1010}	α_{1011}	α_{1100}	α_{1101}	α_{1110}	α_{1111}
	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
	-3.46	2.52	0.83	-0.33	2.12	-1.25	-0.28	0.12
Reduced model 4-IDP(r)		$\ln \phi = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \sum_{n_3=0}^1 \alpha_{n_1 n_2 n_3} (\ln \theta)^{n_1} (\ln \mathbf{F}_g)^{n_2} (\ln D_*)^{n_3}$ $\phi = e^{c_0} D_*^{c_1} \mathbf{F}_g^{c_2 + c_3 \ln D_*} \theta^{c_4 + c_5 \ln D_* + c_6 \ln \mathbf{F}_g + c_7 \ln D_* \ln \mathbf{F}_g}$						
Coefficient	α_{000}	α_{001}	α_{010}	α_{011}	α_{100}	α_{101}	α_{110}	α_{111}
	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
	-9.39	2.60	4.91	-1.23	-1.14	1.51	1.54	-0.95

^aNote: The models are applicable for the condition of $\sigma_g < 1.3$ and $\rho_S/\rho = 2.6 - 2.7$.

Table 5

Investigator		Err_1	Err_2
Engelund and Hansen (1967)		2.6	1,060
Ackers and White (1973)		3.3	1,231
Karim (1981)		9.9	1,174
Brownlie (1981b)		7.6	1,027
Yang (1973, 1984)		2.2	1,911
Molinas and Wu (2001)		14.3	2,679
Present models (see Table 5)	4-IDP (r) (θ , $\mathbf{F}_g$, D_*)	1.4	950
	4-IDP (θ , $\mathbf{F}_g$, r_b/D_{50} , D_*)	1.7	729

Table 6

Figure 1

Figure 2

Figure 3