Consensusability of Discrete-time Multi-agent Systems via Relative Output Feedback

Keyou You and Lihua Xie

School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.
E-mail: {youk0001,elhxie}@ntu.edu.sg.

Abstract

This paper investigates the joint effects of agent dynamic and network topology on the consensusability of linear discrete-time multi-agent systems via relative output feedback. An observer-based distributed control protocol is proposed. A necessary and sufficient condition for consensusability under this control protocol is given, which explicitly reveals how the intrinsic entropy rate of the agent dynamic and the *eigenratio* of the *undirected* communication graph affect consensusability. As a special case, the discrete-time double integrator system is discussed where a simple control protocol directly using the two-step relative position feedback is provided to reach a consensus. The theoretic results are illustrated by a simulation example.

Index Terms—Multi-agent systems, communication graphs, consensusability and eigenratio.

I. INTRODUCTION

In recent years there has been an increasing interest in the study of the interplay between communications and control. It has been realized that information flow and communication constraints may significantly affect the performance of a control system. The research in this field has ushered new control paradigms such as quantized feedback control, networked control and cooperative control, see the special issue [1] and the references therein.

Distributed coordination of multiple agents has attracted a considerable interest in various scientific communities due to broad applications in many areas including formation control [6], distributed sensor networks [4], flocking [17], distributed computation [13], and synchronization of coupled chaotic oscillators [3], [5]. The common property of those applications is that each individual agent lacks global knowledge of the whole system and can only interact with its neighbors to achieve certain global behaviors. Within this framework, communication graph (topology), which determines what information is available for each agent at a given time instant, is an important aspect of information flow in distributed coordination. For example, to achieve an average consensus which requires the states of all agents to asymptotically converge to the average of their initial values, the communication graph must be connected for a fixed topology [18] while for a switching topology, the union of the communication graphs should contain a spanning tree frequently enough as the system evolves [10], [12], [19]. Also, it has been known that the convergence rate to consensus directly relies on the second smallest eigenvalue of the graph Laplacian matrix [16], [18].

A fundamental problem dealing with the existence of consensus protocols has been studied in the recent works [14], [26], which respectively give a necessary and sufficient condition for the consensusability of continuous-time and discrete-time multi-agent systems with respect to a common consensus protocol. By merging ideas from algebraic graph theory and control theory, they are able to characterize the interplay between communication topology and agent dynamic. In particular, it is shown that the minimum requirement for consensusability is that the dynamic of each identical agent has to be stabilizable and the *eigenratio* of the *undirected* communication graph must be greater than a threshold, which is determined by the intrinsic entropy rate [15] of the agent dynamic. However, the results were established under a critical assumption that each agent can precisely measure the state feedback relative to its neighboring agents. Extending our previous work of [26], we restrict ourselves to the case that each agent can only access the output relative to that of its neighbors. The problem arises in the situation that the agent does not know its position in a global coordinate system but can measure its position relative to that of its neighbors.

In this paper, an observer-based distributed control protocol is proposed to study the consensusability problem. The idea of designing a dynamic control protocol for synchronization of networked multi-agent systems was recently adopted in [20], [21]. The main shortcoming of those works lies in the assumption that the agent dynamic is neutrally stable [2]. This assumption significantly simplifies the problem under investigation and conceals the fact that the strictly unstable modes of the agent dynamic pose a fundamental limitation on the *eigenratio* of the *undirected* communication graph. Here the *eigenratio* is referred to the ratio of the second smallest eigenvalue and the largest one of the *undirected* graph Laplacian matrix. We shall explore their relationship in the current work. As a special case, the consensusability of the discrete-time double integrator multi-agent systems is discussed. By exploiting the property of double integrator systems, a simple control protocol directly using the two-step relative position is provided to reach a consensus of the multi-agent system.

The rest of the paper is organized as follows. Section II introduces the concepts of communication graphs and the consensusability on graphs. Two admissible control protocols are proposed as well. The consensusability analysis is proceeded in Section III, where some necessary and sufficient conditions are given for the consensusability under the corresponding control protocols. In Section IV, the consensusability of the discrete-time double integrator systems is discussed, whose results are demonstrated by a simulation example in Section V. The conclusion remarks are drawn in Section VI.

Notations: $\rho(A)$ denotes the spectral radius of matrix A. We use conventions that $\frac{a}{0}=\infty$, $\forall_a > 0$ and $\frac{0}{0}=0$. For any positive integer N, let $\mathcal{N}=\{1,\ldots,N\}$. I_n is the identity matrix with dimension $n\times n$. $\|\cdot\|$ represents the standards ℓ^2 norm on vectors or their induced norms on matrices. The transpose of matrix A is denoted by A^T. diag $(A_1,\ldots,A_n)$ is a block diagonal matrix with main diagonal blocks matrices $A_j, j=1,\ldots,n$ and zero off-diagonal block matrices. The

Kronecker product [9], denoted by $\otimes$, facilitates the manipulation of matrices by the following properties: (1) $(A \otimes B)(C \otimes D) = AC \otimes BD$; (2) $(A \otimes B)^T = A^T \otimes B^T$.

II. PROBLEM FORMULATION

A. Communication graph

Let $\mathcal{V} = [1, \dots, N\}$ be an index set of N agents with v_i representing the i-th agent. An *undirected* graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ will be utilized to model the interactions among these agents, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of paired agents and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative elements is the weighted adjacency matrix of the graph $\mathcal{G}$. Self-edges (i, i) are not allowed, i.e., $(i, i) \notin \mathcal{E}, \forall i \in \mathcal{V}$. $(i, j) \in \mathcal{E}$ if and only if $a_{ij} > 0$. Note that for an undirected graph $\mathcal{G}$, $\mathcal{A}$ is a symmetric matrix. The neighborhood of the i-th agent is denoted by $\mathcal{N}_i \triangleq \{j|(i, j) \in \mathcal{E}\}$. The degree of the agent i is represented by $deg_i = \sum_{j=1}^{N} a_{ij}$. Denote the diagonal matrix $\mathcal{D} \triangleq \text{diag}\,(deg_1, \dots, deg_N)$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}_{\mathcal{G}} = \mathcal{D} - \mathcal{A}$, which is a symmetric positive semi-definite matrix and its eigenvalues in an ascending order are wriiten as $0 = \lambda_1 (\mathcal{L}_{\mathcal{G}}) \leq \lambda_2 (\mathcal{L}_{\mathcal{G}}) \leq \cdots \leq \lambda_N (\mathcal{L}_{\mathcal{G}})$.a sequence of edges $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)$ with $(i_{j-i}, i_j) \in \mathcal{E}, \forall j \in \{2, \dots, k\}$ is called a path from agent i_1 to agent i_k. A communication graph $\mathcal{G}$ is said to be *connected* if for any two agents $i, j \in \mathcal{V}$, there is a path from agent i to agent j. A graph is called *complete* if each pair of agents can directly communicate with each other, i.e., $(i, j) \in \mathcal{E}, \forall i \neq j$.

B. Consensusability on graphs

The dynamic of agent i takes the following form:

$$\begin{cases} X_i(k+1) &= AX_i(k) + Bu_i(k), \\ Y_i(k) &= CX_i(k), \end{cases} \quad \forall i \in \mathcal{N}, k \in \mathbb{N}, \quad (1)$$

where $X_i(k) \in \mathbb{R}^n, u_i(k) \in \mathbb{R}$ and $Y_i(k) \in \mathbb{R}^m$ respectively represent the state, control input and output of agent i. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{m\times n}$ are respectively the state, input and observation matrices.

Consider the situation where each agent does not know its exact output but can measure the output relative to those of its neighboring agents. For instance, in vehicle coordination, the vision-based sensor on a vehicle can not directly locate the position of the vehicle in a global coordinate system but can measure the relative position to its neighbors. While in networked clock synchronization, we are more concerned with the time difference between each pair of clocks. In addition, the communication link of $\mathcal{E}$ is assumed to be perfect in the sense that we ignore effects due to quantization, packet dropout, transmission errors and delays.

By adapting to the available information for each agent, we say a control protocol *admissible* if each agent generates its control input signal by relying on relative outputs. Generally, *admissible* control protocols can be categorized depending on whether they are *static* or *dynamic*. A *dynamic* protocol uses memory and can be potentially more powerful. In the sequel, two *admissible* control protocols will be proposed. Precisely, we first adopt a *static* control protocol:

$$u_i(k) = F \sum_{j \in \mathcal{N}_i} a_{ij}(Y_j(k) - Y_i(k)) \triangleq F\zeta_i(k), F \in \mathbb{R}^{1\times m}. \tag{2}$$

Definition 2.1: Given an *undirected* communication graph $\mathcal{G}$, the discrete-time multi-agent systems of (1) are said to be *consensusable* under the *static* protocol (2) if for any finite $X_i(0), \forall i \in \mathcal{N}$, the control protocol can asymptotically drive all agents close to each other, i.e.,

$$\lim_{k\to\infty} \|X_i(k) - X_j(k)\| = 0, \forall i, j \in \mathcal{N}. \tag{3}$$

The second *admissible* control protocol is an observer-based *dynamic* protocol that depends on an internal controller state. In our previous work [26], the following consensus protocol using the relative state feedback is exploited:

$$u_i(k) = K \sum_{j \in \mathcal{N}_i} a_{ij}(X_j(k) - X_i(k)) \triangleq K\xi_i(k), K \in \mathbb{R}^{1\times n}. \tag{4}$$

Since $\xi_i(k)$ is no longer available in the current framework, a very natural thing is to design an observer to estimate $\xi_i(k)$ for the control design. In consideration of the agent dynamic, the following observer-based control protocol for agent i will be studied.

$$\begin{cases} \hat{\xi}_i(k+1) &= A\hat{\xi}_i(k) + B\sum_{j\in\mathcal{N}_i} a_{ij}(u_j(k) - u_i(k)) \\ & \quad +L(\zeta_i(k) - C\hat{\xi}_i(k)), \\ u_i(k) &= K\hat{\xi}_i(k), L \in \mathbb{R}^{n\times m}, K \in \mathbb{R}^{1\times n}. \end{cases} \tag{5}$$

At time k, agent i computes the aggregate relative measurements to those of its neighbors, $\zeta_i(k)$. Together with control inputs from its neighbors, $u_j(k), j \in \mathcal{N}_i$, which will be received before time $k+1$, the agent updates its internal controller state to obtain $\hat{\xi}_i(k+1)$ and produces the control input $u_i(k+1)$. It is clear that the *dynamic* control protocol in (5) is *admissible*. Compared to the *static* protocol in (2), this *dynamic* protocol requires each agent to broadcast its control input to its neighboring agents.

Observe the special case that the initial estimate is perfect, i.e., $\xi_i(0) = \hat{\xi}_i(0)$, it can be easily shown that $\xi_i(k) = \hat{\xi}_i(k)$, $\forall k \in \mathbb{N}$. When the consensus is reached, the internal controller state $\hat{\xi}_i(k)$ of this case becomes zero. By taking this into consideration, it is reasonable to impose an additional condition on the definition of consensus that all controller internal states $\hat{\xi}_i(k)$, $\forall i \in \mathcal{N}$ should asymptotically converge to zero.

Definition 2.2: Given an *undirected* communication graph $\mathcal{G}$, the discrete-time multi-agent systems of (1) are said to be *consensusable* under the *dynamic* protocol (5) if for any finite $X_i(0)$, $\forall i \in \mathcal{N}$, the control protocol can asymptotically drive the states of all agents close to each other and all the controller internal states to zero, i.e.

$$\lim_{k\to\infty} \|X_i(k) - X_j(k)\| = 0 \;\&\; \lim_{k\to\infty} \|\widehat{\xi}_i(k)\| = 0, \forall i, j \in \mathcal{N}. \tag{6}$$

One of the main objectives of this paper is to derive a necessary and sufficient condition to ensure the consensusability of the multi-agent systems under the corresponding *admissible* control protocols.

III. CONSENSUSABILITY ANALYSIS

Since our focus is on *undirected* graphs, all the eigenvalues of $\mathcal{L}_{\mathcal{G}}$ are nonnegative and real. For notational simplicity, we rewrite $\lambda_j(\mathcal{L}_{\mathcal{G}})$ as $\lambda_j, j \in \mathcal{N}$ in the rest of the paper.

Theorem 3.1: Given an *undirected* communication graph $\mathcal{G}$, the discrete-time multi-agent systems in (1) are consensusable under the *static* control protocol in (2) *if and only if* there exists a common gain $F \in \mathbb{R}^{1\times m}$ such that $\rho(A - \lambda_j BFC) < 1, \forall j \in \{2, \dots, N\}$.

Proof: Denote the average state of all agents by $\bar{X}(k) \triangleq \frac{1}{N}\sum_{i=1}^{N} X_i(k) = \frac{1}{N}(\mathbf{1}^T \otimes I_n)X(k)$, where $X(k) \triangleq [X_1^T(k), \dots, X_N^T(k)]^T$, and the deviation of each state from the average state by $\delta_i(k) \triangleq X_i(k) - \bar{X}(k)$, where $\mathbf{1}$ is a compatible dimension vector with each element of one and similar for $\mathbf{0}$ in the sequel. By Definition 2.1, it yields that

$$\lim_{k\to\infty} \|\delta_i(k)\| \le \frac{1}{N}\sum_{j=1}^{N} \lim_{k\to\infty} \|X_i(k) - X_j(k)\| = 0.$$

Conversely, $\lim_{k\to\infty}\|\delta_i(k)\| = 0, \ \forall i \in \mathcal{N}$ immediately implies the consensusability of the multi-agent systems in (1). Thus, it is equivalent to find a necessary and sufficient condition such that $\lim_{k\to\infty}\|\delta_i(k)\| = 0, \ \forall i \in \mathcal{N}$. Inserting the control protocol in (2) into each agent dynamic, the dynamical equation of $X(k)$ can be written as

$$X(k+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}} \otimes BFC)X(k). \tag{7}$$

Considering that $\mathbf{1}^T\mathcal{L}_{\mathcal{G}} = \mathbf{0}^T$, we obtain

$$\begin{aligned}\bar{X}(k+1) &= \frac{1}{N}(\mathbf{1}^T \otimes A)X(k) - \frac{1}{N}(\mathbf{1}^T\mathcal{L}_{\mathcal{G}} \otimes BFC)X(k)\\ &= A\bar{X}(k).\end{aligned} \tag{8}$$

Let $\delta(k) = [\delta_1^T(k), \dots, \delta_N^T(k)]^T$, subtracting (7) from (8) leads to that

$$\delta(k+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}} \otimes BFC)\delta(k). \tag{9}$$

Select $\phi_i \in \mathbb{R}^N$ such that $\phi_i^T\mathcal{L}_{\mathcal{G}} = \lambda_i \phi_i^T$ and form the unitary matrix $\Phi = \left[\frac{1}{\sqrt{N}}, \phi_2, \dots, \phi_N\right]$ to transform $\mathcal{L}_{\mathcal{G}}$into a diagonal form: diag $(0, \lambda_2, \dots, \lambda_N) = \Phi^T\mathcal{L}_{\mathcal{G}}\Phi$. Further, using the property of Kronecker product gives that

$$(\Phi \otimes I_n)^T (I_N \otimes A - \mathcal{L}_{\mathcal{G}} \otimes BFC)(\Phi \otimes I_n)$$
$$= I_N \otimes A - \Phi^T \mathcal{L}_{\mathcal{G}} \Phi \otimes BFC$$
$$= \mathrm{diag}(A, A - \lambda_2 BFC, \ldots, A - \lambda_N BFC). \quad (10)$$

Denote $\tilde{\delta}(k) = (\Phi \otimes I_n)^T \delta(k)$ and partition $\tilde{\delta}(k) \in \mathbb{R}^{nN}$ into two parts, i.e., $\tilde{\delta}(k) = [\tilde{\delta}_1^T(k), \tilde{\delta}_2^T(k)]^T$ where $\tilde{\delta}_1(k) \in \mathbb{R}^n$ is a vector consisting of the first n elements of $\tilde{\delta}(k)$. Then, $\tilde{\delta}_1(k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \delta_i(k) = \mathbf{0}$. In view of (9) and (10), it yields that

$$\tilde{\delta}_2(k+1) = \mathrm{diag}(A - \lambda_2 BFC, \ldots, A - \lambda_N BFC)\tilde{\delta}_2(k). \quad (11)$$

The rest of the proof is straightforward.

Theorem 3.2: Given an *undirected* communication graph $\mathcal{G}$, the discrete-time multi-agent systems in (1) are consensusable under the *dynamic* control protocol in (5) *if and only if* the following conditions hold.

(a) (A, B, C) is stabilizable and detectable;
(b) Each agent cannot change too fast. Precisely, the product of the unstable eigenvalues of A is upper bounded by the following strict inequality:

$$\prod_j |\lambda_j^u(A)| < \frac{1 + \lambda_2/\lambda_N}{1 - \lambda_2/\lambda_N}, \quad (12)$$

where $\lambda_j^u(A)$ represent an unstable eigenvalue of A. λ_2 and λ_N are respectively the second smallest and largest eigenvalues of an *undirected* graph.

Moreover, if the above conditions hold, a control gain K that solves the consensus problem can be selected as $K = \frac{2}{\lambda_2 + \lambda_N} \frac{B^T PA}{B^T PB}$, where P is a positive definite solution to the following discrete-time algebraic Riccati inequality:

$$P - A^T PA + \frac{A^T PBB^T PA}{B^T PB} > 0. \quad (13)$$

The observed gain L is chose to make $\rho(A - LC) < 1$.

In the sequel, λ_2/λ_N is termed as the *eigenratio* of an *undirected* graph. By Lemmas A.1-A.2 [11], an upper bound of the *eigenratio* is given by $\frac{\lambda_2}{\lambda_N} \leq \frac{min_i deg_i}{max_i deg_i}$.

Remark 3.1:

1) For the average consensus problem in [18], the state of each agent is scalar and $A = B = C = 1$. The condition in item (a) is automatically satisfied while the inequality of (12) implies that $\lambda_2 > 0$. That is, the communication graph has to be connected which is consistent with the result in [18].
2) If the adjacent matrix $\mathcal{A}$ of the graph $\mathcal{G}$ is selected as a symmetric (0,1)-matrix, the eigenratio $\lambda_2/\lambda_N \to 1$ means that the communication graph is almost complete [7]. In

this case, the controller can be designed almost in a centralized fashion. Then, it is clear that if (A, B, C) are stabilizable and detectable, consensusability can be achieved.

3) It is well known that the convergence rate of the average consensus over an undirected graph is determined by λ_2 [16],[18]. By the Courant-Wely interlacing inequalities [9], adding an edge to the graph $\mathcal{G}$ will never decrease λ_2, suggesting that the consensus performance will not deteriorate. Note that adding an edge to a graph may lead to a smaller *eigenratio*. Theorem 3.2 implies that it is possible to lose the consensusability of the multi-agent systems in (1) under the protocol (5) by adding an edge. It appears to be counter-intuitive since the communication graph with a "better" connectivity may correspond to a worse consensusability. Whether the *eigenratio* will increase or decrease by adding an edge is not conclusive, see [5] for a more detailed discussion.
4) The importance of the intrinsic entropy rate of a system, quantified by $\sum_j \log_2 | \lambda_j^u (A)|$, has been widely recognized in networked control systems, e.g., [8], [23]–[25] as it determines the minimum data rate for stabilization of an unstable system. From this perspective, our result provides a bridge between information flow and communication data rate constraint.

The proof depends on the following lemmas.

Lemma 3.1: [25] Suppose that the sequence $\{z_k\} \subset \mathbb{R}$ is recursively computed by the formula $z_{k+1} = (1 - a_k)z_k + b_k$, $\forall k \in \mathbb{N}$ and $a_k \in [0,1), \sum_{k=0}^{\infty} a_k = \infty$, $|z_0| < \infty$. Then if $\lim_{k\to\infty} \frac{b_k}{a_k}$ exists, we have $\lim_{k\to\infty} z_k = \lim_{k\to\infty} \frac{b_k}{a_k}$.

Lemma 3.2: [22] For any $A \in \mathbb{R}^{n\times n}$ and $\epsilon > 0$, it holds that

$$\|A^k\| \leq M\eta^k, \forall k \geq 0, \tag{14}$$

where $M = \sqrt{n}(1 + \frac{2}{\epsilon})^{n-1}, \eta = \rho(A) + \epsilon\|A\|$.

Proof of Theorem 3.2: Define $\tilde{\xi}_i(k)$ as the estimation error of $\xi_i(k)$, i.e., $\tilde{\xi}_i(k) = \hat{\xi}_i(k) - \xi_i(k)$. Inserting the control protocol in (5) into each agent dynamic, the dynamical equation of $X(k)$ can be written as

$$X(k+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}} \otimes BK)X(k) + (I_N \otimes BK)\tilde{\xi}(k). \tag{15}$$

Similarly, it is easy to show that

$$\delta(k+1) = (I_N \otimes A - \mathcal{L}_{\mathcal{G}} \otimes BK)\delta(k) + (I_N \otimes BK)\tilde{\xi}(k). \tag{16}$$

Let $E(k) = (P \otimes I_n)^T (I_N \otimes BK)\tilde{\xi}(k)$ and partition into two parts $E(k) = [E_1^T(k), E_2^T(k)]^T$ where $E_1(k) \in \mathbb{R}^n$ is a vector consisting of the first n elements of $E(k)$. Then, following a similar line in the proof of Theorem 3.1 leads to that

$$\tilde{\delta}_2(k+1) = \text{diag}(A - \lambda_2 BK, \ldots, A - \lambda_N BK)\tilde{\delta}_2(k) + E_2(k). \tag{17}$$

Necessity: By (1), we obtain that

$$\xi_i(k+1) = \sum_{j\in\mathcal{N}_i} a_{ij}(X_j(k+1) - X_i(k+1)) = A\xi_i(k) + BK\sum_{j\in\mathcal{N}_i} a_{ij}(\widehat{\xi}_j(k) - \widehat{\xi}_i(k)). \quad (18)$$

Together with (5), the error dynamic of $\widetilde{\xi}_i(k)$ is written by $\widetilde{\xi}_i(k+1) = (A - LC)\widetilde{\xi}_i(k)$. Assume that the multi-agent systems in (1) reaches a consensus under the *dynamic* protocol on (5), it follows that

$$\lim_{k\to\infty} \|\widetilde{\xi}_i(k)\| = \lim_{k\to\infty} \|\widehat{\xi}_i(k) - \xi_i(k)\|$$
$$\leq \lim_{k\to\infty} \|\widehat{\xi}_i(k)\| + \lim_{k\to\infty} \|\xi_i(k)\|$$
$$\leq \|K\| \sum_{j\in\mathcal{N}_i} a_{ij} \lim_{k\to\infty} \|X_j(k) - X_i(k)\| = 0, \forall i \in \mathcal{N}. \quad (19)$$

Thus, it follows that $\rho(A - LC) < 1$. This implies that (C, A) is detectable.

Now, we consider a special case that the initial estimate of $\xi_i(0), \forall i \in \mathcal{N}$ is perfect. By the error dynamic of $\widetilde{\xi}_i(k)$, it is easy to see that $\widetilde{\xi}_i(k) = 0, \forall i \in \mathcal{N}$, which further implies that $E_2(k) = 0, \forall k \in \mathbb{N}$. In light of (17), it immediately follows that $\rho\left(A - \lambda_j BK\right) < 1, \forall j \in \{2, \dots, N\}$. The rest of the necessity follows from Lemma 3.2 of [26].

Sufficiency: Since (A, B) is stabilizable, there exists a positive definite solution P to the algebraic Riccati inequality (13). In view of [26], the proposed control gain K can simultaneously stabilize the stabilizable pairs $(A, \lambda_j B), \forall j \in \{2, \dots, N\}$, i.e.,

$$\varrho \triangleq \max_{j\in\{2,\dots,N\}} \rho(A - \lambda_j BK) < 1. \quad (20)$$

In addition, the observer gain L will make the estimation error asymptotically converge to zero, i.e., $\lim_{k\to\infty} \tilde{\xi}_i(k) = 0$, which further infers that $\lim_{k\to\infty}\|E_2(k)\| = 0$. Denote $J(k) = \text{diag}\,(A - \lambda_2 BK, \dots, A - \lambda_N BK)$, it follows from (17) that $\tilde{\delta}_2(k+1) = J(K)^{k+1}\tilde{\delta}_2(0) + \sum_{i=0}^{k} J(K)^{k-i} E_2(i)$. Select a positive ϵ such that $\epsilon < \frac{1-\varrho}{\|J(K)\|}$ and $\eta = \varrho + \epsilon\|J(K)\| < 1$. By Lemma 3.2, it follows that $\|J(K)^k\| \leq M\eta^k$. Thus, we obtain that $\left\|\tilde{\delta}_2(k+1)\right\| \leq M(\eta^{k+1} + \sum_{i=0}^{k} \eta^{k-i} \|E_2(i)\|)$. Consider an auxiliary system as follows $z_{k+1} = \eta z_k + \|E_2(k)\|, z_0 = 1$. In view of Lemma 3.1, we have that $\lim_{k\to\infty} z_k = \frac{\lim_{k\to\infty}\|E_2(k)\|}{1-\eta} = 0$. By induction, it is clear that $z_k = \eta^{k+1} + \sum_{i=0}^{k} \eta^{k-i} \|E_2(i)\|$. Hence, we have proved that

$$\lim_{k\to\infty} \|\widetilde{\delta}_2(k)\| = 0. \quad (21)$$

Together with the fact that $\tilde{\delta}_1(k) = 0, \forall k \in \mathbb{N}$, it follows that $\lim_{k\to\infty}\|\delta(k)\| = 0$. Thus, we get that $\lim_{k\to\infty}\left\|X_i(k) - X_j(k)\right\| = 0, \forall i, j \in \mathcal{N}$, which further implies that $\lim_{k\to\infty}\|\xi_i(k)\| = 0, \forall i \in \mathcal{N}$. Moreover, the following statements are straightforward:

$$\lim_{k\to\infty}\|\widehat{\xi}_i(k)\| = \lim_{k\to\infty}\|\xi_i(k)\| + \lim_{k\to\infty}\|\widetilde{\xi}_i(k)\| = 0. \quad (22)$$

The proof is completed.

IV. SPECIAL CASE: DOUBLE-INTEGRATOR SYSTEMS

Consider discrete-time double-integrator systems for each agent as follows:

$$\begin{cases} x_i(k+1) &= x_i(k) + hv_i(k), \\ v_i(k+1) &= v_i(k) + hu_i(k) \end{cases}, \forall i \in \mathcal{N}, \quad (23)$$

where $x_i(k) \in \mathbb{R}$ and $v_i(k) \in \mathbb{R}$ respectively correspond to the position and velocity of agent i at time kh. $u_i(k) \in \mathbb{R}$ is the control input.

A. One-step relative position feedback

Consider the situation that each agent does not know its position in a global coordinate system but can measure its position relative to neighboring agents. One may attempt to reach a consensus by adopting control protocol as follows:

$$u_i(k) = \gamma \sum_{j=1}^{N} a_{ij}(x_j(k) - x_i(k)), \quad \gamma \in \mathbb{R}. \quad (24)$$

Intuitively, the above control protocol only uses relative position information, it may not be able to drive the multi-agent system to reach a consensus.

Theorem 4.1: The second-order multi-agent systems (23) can not reach a consensus under the control protocol (24) for any undirected communication graph.

Proof: In view of (17), it can be similarly established that

$$\widetilde{\delta}_j(k+1) = (A - \lambda_j\gamma BC)\widetilde{\delta}_j(k), \forall j \in \{2, \ldots, N\}. \quad (25)$$

It is straightforward that

$$\det(zI_2 - (A - \lambda_j\gamma BC)) = z^2 - 2z + 1 + \lambda_j h^2\gamma. \quad (26)$$

Together with (25), we can not guarantee that for any finite initial state $\xi_0(k), \lim_{k\to\infty}\|\tilde{\delta}(k)\| \neq 0$. This complete the proof.

B. Two-step relative position feedback

The relative velocity can be accessed by using the relative position information with one-step delay. For example, $v_j(k-1) = \frac{x_j(k)-x_j(k-1)}{h}$. Thus, we study the following control protocol. Let $x_j\ (k) = 0, \forall k < 0$,

$$
\begin{aligned}
u_i(k) \\
&= \sum_{j=1}^{N} a_{ij}\left[\gamma_0(x_j(k)-x_i(k))+\gamma_1(v_j(k-1)-v_i(k-1))\right], \\
&\triangleq \sum_{j=1}^{N} a_{ij}\left[\alpha(x_j(k)-x_i(k))+\beta(x_j(k-1)-x_i(k-1))\right].
\end{aligned} \tag{27}
$$

This above protocol requires each agent to store one-step relative position feedback. Even under such a simple protocol, a connected graph is also sufficient to guarantee to reach a consensus.

Theorem 4.2: Given an *undirected* communication graph $\mathcal{G}$, the second-order multi-agent systems (23) are consensusable under the control protocol (27) *if and only if* the communication graph is connected. Moreover, if this condition holds, (α, β) in the protocol of (27) can be chosen from the set

$$
\Omega_c \triangleq \left\{(\alpha,\beta) \,|\, \max\{-\frac{1}{h^2}, -\frac{1}{\lambda_N h^2}\} < \beta < 0, \; \alpha = -\frac{\lambda_N h^2 \beta^2 + 3\beta}{2}\right\}. \tag{28}
$$

Proof: Similarly, we obtain that $\forall j \in \{2, \dots, N\}$,

$$
\tilde{\delta}_j(k+1) = (A - \alpha\lambda_j BC)\tilde{\delta}_j(k) - \beta\lambda_j BC\tilde{\delta}_j(k-1). \tag{29}
$$

Let $\Delta_j(k) = [\tilde{\delta}_j^T(k-1), \tilde{\delta}_j^T(k)]$, where $\tilde{\delta}_j(k) = 0, \forall k < 0$. In view of (29), the dynamical equation of $\Delta_j(k)$ is expressed by

$$
\begin{aligned}
\Delta_j(k+1) &= \begin{bmatrix} 0 & I_2 \\ -\beta\lambda_j BC & A-\alpha\lambda_j BC \end{bmatrix}\Delta_j(k) \\
&\triangleq M_j(\alpha,\beta)\Delta_j(k), \forall j \in \{2,\dots,N\}.
\end{aligned} \tag{30}
$$

Thus, the necessary and sufficient condition for the multi-agent system (23) to reach a consensus is that $\rho(M_j(\alpha,\beta)) < 1, \forall j \in \{2,\dots,N\}$.

Necessity: If the communication graph is not connected, it immediately follows that $\lambda_2 = 0$, which implies that $\rho(M_2(\alpha,\beta)) = 1, \forall \alpha,\beta \in \mathbb{R}$. In view of (30), we can't guarantee that $\lim_{k\to\infty}\|\Delta_j(k)\| = 0, \forall \Delta_j(0) \in \mathbb{R}^4$. This contradicts the definition of consensusability of the multi-agent systems.

Sufficiency: We show that for any connected graph and any $(\alpha,\beta) \in \Omega_c$, it holds that $\rho(M_j(\alpha,\beta)) < 1, \forall j \in \{2,\dots,N\}$. It is easy to compute that

$$
\begin{aligned}
&\det(zI_4 - M_j(\alpha,\beta)) \\
&= z\left(z^3 - 2z^2 + (1+\lambda_j h^2\alpha)z + \lambda_j h^2\beta\right).
\end{aligned} \tag{31}
$$

Let the polynomial $f(z) = z^3 - 2z^2 + (1 + x)z + y$. By using the Jury's test stability criterion, it can be shown that all roots of $f(z)$ are inside the unit circle *if and only if* $(x, y) \in \Omega$, where Ω is the *stability region* and defined as follows:

$$\Omega \triangleq \{(x, y) | -y < x < -y^2 - 2y\}. \qquad (32)$$

Finally, it can be verified that for any $(\alpha, \beta) \in \Omega_c$, $(\lambda_j\, \alpha h^2,\ \lambda_j\, \beta h^2) \in \Omega, \forall j \in \{2, \ldots, N\}$. Together with (31), the proof is completed.

V. AN ILLUSTRATIVE EXAMPLE

In this section, a simulation example is included to validate the theoretic results in Section IV. We consider a team of four vehicles with an undirected graph $\mathcal{G}$ shown in Fig 2. The adjacency matrix is selected as (0, 1)-weighted symmetric matrix. The configuration variables are initialized as $x(0) = [15\ \ 30\ \ 0\ -15]^T$ and $v(0) = [4\ \ 4\ \ 2\ \ 3]^T$. Let the sampling interval h = 0.25 s and the control gain $[\alpha\ \beta] = [2.5\ -2] \in \Omega_c$. It shows in Fig 3 and 4 that a consensus is reached for all vehicles.

VI. CONCLUSION

Motivated by the constraint that the agent can only get the relative output feedback for distributed control design, we have studied the joint effects of the agent dynamic and the communication graph on the consensusability with respect to an admissible control protocol. The distinct feature of our results lies in the precise quantification of their effects on consensusability. A simple control protocol has been proposed to reach a consensus for the discrete-time double integrator multi-agent systems. A simulation example was studied to verify our theoretic results.

REFERENCES

[1] P. Antsaklis and J. Baillieul, "Special issue on technology of networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 5–8, 2007.

[2] P. Antsaklis and A. Michel, *Linear systems*. Birkhauser, 2006.

[3] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Physical review letters*, vol. 89, no. 5, p. 054101, 2002.

[4] J. Cortés and F. Bullo, "Coordination and geometric optimization via distributed dynamical systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 5, pp. 1543–1574, 2006.

[5] Z. Duan, G. Chen, and L. Huang, "Complex network synchronizability: Analysis and control," *Physical Review E*, vol. 76, no. 5, p. 56103, 2007.

[6] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[7] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 2, pp. 298–305, 1973.

[8] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[9] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[10] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[11] T. Li, M. Fu, L. Xie, and J. Zhang, "Distributed Consensus with Limited Communication Data Rate," *to appear in IEEE Transactions on Automatic Control*, 2010.

[12] T. Li and J. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *to appear in IEEE Transactions on Automatic Control*, 2010.

[13] N. Lynch, *Distributed algorithms*. Morgan Kaufmann, 1996.

[14] C. Ma and J. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1263–1268, 2010.

[15] P. Minero, M. Franceschetti, S. Dey, and G. Nair, "Data rate theorem for stabilization over time-varying feedback channels," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 243–255, 2009.

[16] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and Cooperationin Networked Multi-Agent Systems," *Proceedings of the IEEE*, vol. 95, no. 1, 2007.

[17] R. Olfati-Saber and R. Murray, “Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks,” *Proc. 42nd IEEE Conference on Decision and Control*, vol. 2, pp. 2022–2028, 2003.

[18] ——, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[19] W. Ren and R. Beard, “Consensus seeking in multiagent systems under dynamically changing interaction topologies,” *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[20] L. Scardovi and R. Sepulchre, “Synchronization in networks of identical linear systems,” *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.

[21] J. Seo, H. Shim, and J. Back, “Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach,” *Automatica*, vol. 45, no. 11, pp. 2659–2664, 2009.

[22] V. Solo, “One step ahead adaptive controller with slowly time-varying parameters,” Dept. ECE, John Hopkins University, Baltimore, Tech. Rep., 1991.

[23] K. You, W. Su, M. Fu, and L. Xie, “Attainability of the minimum data rate for stabilization of linear systems via logarithmic quantization,” *provisionally accepted by Automatica*, February 2010.

[24] K. You and L. Xie, “Minimum data rate for mean square stabilizability of linear systems with Markovian packet losses,” *to appear in IEEE Transactions on Automatic Control*, 2010.

[25] ——, “Minimum data rate for mean square stabilization of discrete LTI systems over lossy channels,” *to appear in IEEE Transactions on Automatic Control*, 2010.

[26] ——, “Network topology and communication data rate for consensusability of discrete-time multi-agent systems,” *submitted to IEEE Transactions on Automatic Control*, 2010.

List of Figures

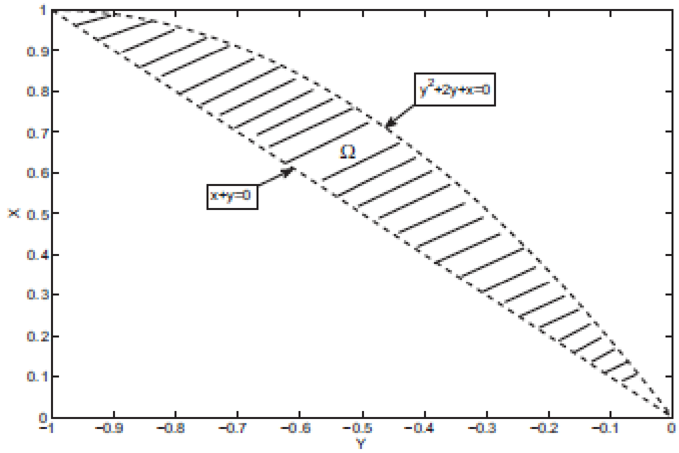


Fig. 1

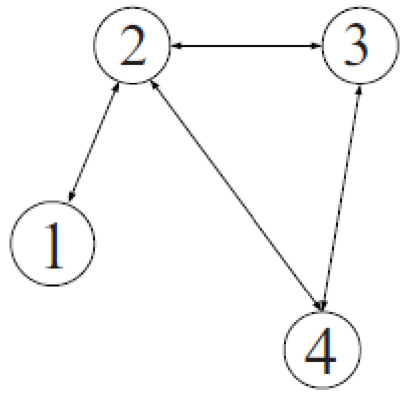


Fig. 2

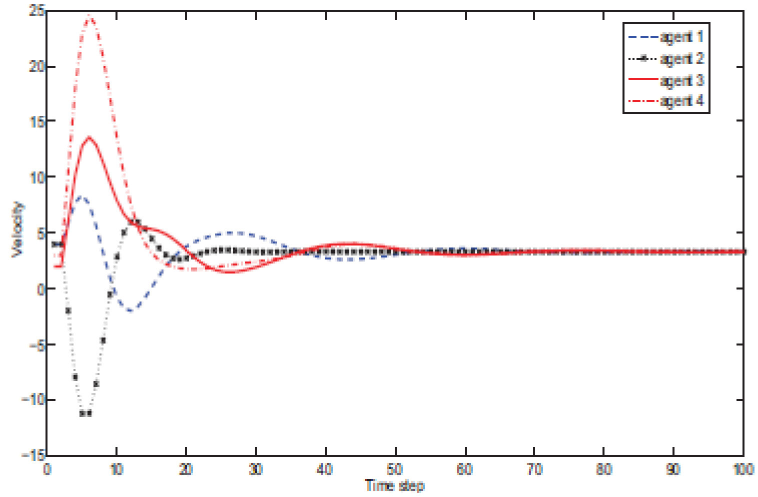
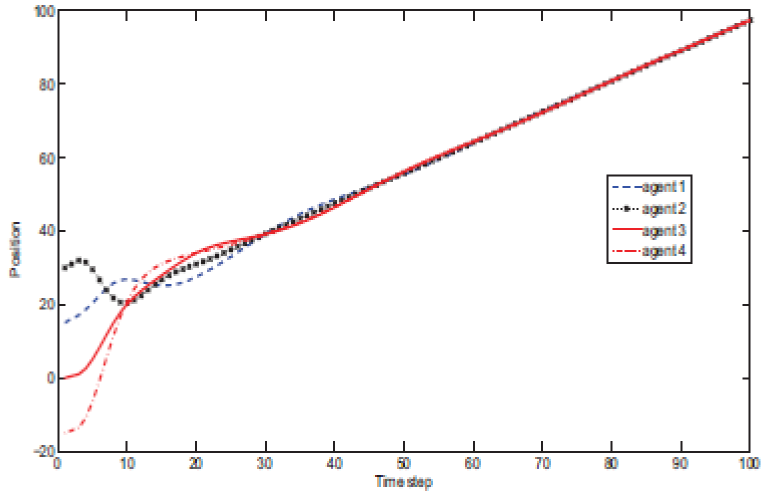


Fig. 3



100
80
60
40
20
0
-20
0
10
20
30
40
50
60
70
80
90
100
Time step
Position
agent 1
agent 2
agent 3
agent 4

Fig. 4