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Greedy Pursuits Assisted Basis Pursuit for Reconstruction of Joint-Sparse Signals

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Abstract

Distributed Compressive Sensing (DCS) is an extension of compressive sensing from single measurement vector problem to Multiple Measurement Vectors (MMV) problem. In DCS, several reconstruction algorithms have been proposed to reconstruct the joint-sparse signal ensemble. However, most of them are designed for signal ensemble sharing common support. Since the assumption of common sparsity pattern is very restrictive, we are more interested in signal ensemble containing both common and innovation components. With a goal of proposing an MMV-type algorithm that is robust to outliers (absence of common sparsity pattern), we propose Greedy Pursuits Assisted Basis Pursuit for Multiple Measurement Vectors (GPABP-MMV). It employs modified basis pursuit and MMV versions of multiple greedy pursuits. We also formulate the exact reconstruction conditions and the reconstruction error bound for GPABP-MMV. GPABP-MMV is suitable for a variety of applications including time-sequence reconstruction of video frames, reconstruction of ECG signals, etc.

Keywords: Modified basis pursuit, multiple measurement vectors

1. Introduction

Compressive Sensing (CS) [1] ensures the reconstruction of a sparse signal $x \in \mathbb{R}^n$ from $m \ll n$ linear incoherent measurements of the form $y = \Phi x \in \mathbb{R}^m$ where $\Phi \in \mathbb{R}^{m \times n}$ is a known sensing matrix. CS reconstruction algorithms can be broadly classified as convex relaxation methods (such as Basis Pursuit (BP) [2]) and Greedy Pursuit (GP) algorithms (such as Orthogonal Matching Pursuit (OMP) [3] and Subspace Pursuit (SP) [4]). For

multiple signals, Single Measurement Vector (SMV) reconstruction problem can be extended to Multiple Measurement Vectors (MMV) problem [5]. The joint-sparse signal ensemble (otherwise known as correlated sparse signal ensemble) can be broadly classified as innovative joint-sparse and common joint-sparse. Let J denote the number of signals in the joint-sparse signal ensemble, $x_j \in \mathbb{R}^n$ denote the signal j where $j \in \{1, 2, \dots, J\}$, and K denote the number of non-zero elements in each x_j . In the innovative joint-sparse signal ensemble (otherwise known as Joint-Sparsity Model (JSM) - 1), each x_j consists of two components: a common sparse component that is present in all the signals, and a sparse innovation component that is unique to it [6]. In other words, x_j can be split as $x_j = z + z_j$ where z is the same for all of the x_j and z_j is the unique portion of x_j . Let K_c denote the number of non-zero elements in z . In the common joint-sparse signal ensemble (otherwise known as JSM-2), all the signals share a common sparsity pattern. In other words, each x_j is supported only on the same set of K non-zero locations but with different non-zero coefficients [6].

Each signal in the ensemble, x_j , is independently sensed using Φ such that $y_j = \Phi x_j \in \mathbb{R}^m$, and then the resulting measurements are transmitted. The data matrix for an ensemble of J signals (in the case of noiseless measurements) is given by,

$$\mathbf{Y} = \Phi \mathbf{X} \in \mathbb{R}^{m \times J} \quad (1)$$

where $\mathbf{Y} = [y_1, y_2, \dots, y_J]$ and $\mathbf{X} = [x_1, x_2, \dots, x_J] \in \mathbb{R}^{n \times J}$.

Motivation and Relation to Prior Work: For joint-sparse signals sharing a common sparsity pattern (i.e. common joint-sparse signals), several algorithms have been developed to reconstruct $\mathbf{X}$ from $\mathbf{Y}$, utilizing the common sparsity condition [7]-[12]. However, in most cases this knowledge does not exist and the assumption of common sparsity pattern for 100% of the data becomes far from ideal. Therefore, we are interested in innovative joint-sparse signals. In [13], a SMV-type solution was proposed for reconstructing innovative joint-sparse signals. It solves a single linear program involving concatenated measurement vectors and concatenated measurement matrices. Due to concatenations, dimension of the signal to be reconstructed increased from n to nJ , and therefore, the complexity of the algorithm is high. An MMV-type recovery method for innovative joint-sparse signals, the Texas Hold

'Em algorithm [14], separated the recovery of the common and innovation components into two stages. The measurements used to recover the common component are obtained by either averaging or concatenating measurements from all the sensors in the network. Furthermore, each innovation component is recovered locally at the corresponding sensor. The computational complexity of the Texas Hold 'Em algorithm is linear in J . However, due to the fact that the recovery of innovation components is based on the average or concatenated measurements, its achievable reconstruction accuracy is limited. To recover two correlated signals with partially disjoint supports, a modified Orthogonal Matching Pursuit (OMP) algorithm called Probability weighted OMP (P-OMP) was proposed [15]. However, it has a drawback: it requires prior probabilistic information on the signals supports. Such information may not be available in many practical applications and therefore, P-OMP cannot be applied in those applications. In [16], a Robust Multiple Sparse Bayesian Learning (R-MSBL) algorithm was proposed which captured the support set of the majority of data vectors. Though R-MSBL, unlike P-OMP, does not require any probabilistic information, the time it takes for reconstruction is a concern. This concern arises mainly due to the presence of innovation components.

Certain optimization-based algorithms attempted to recover $\mathbf{X}$ by mixed-norm minimization [17]. Mixed-norms exploit both sparsity and structure in the signal ensemble. Several mixed-norm based optimization algorithms have been designed to handle structured sparsity [18]-[19]. For data ensemble with simultaneous low-rank and joint-sparse structure, a CS recovery approach jointly regularizing the solutions with their nuclear norm and the $\ell_{2,1}$ -norm is proposed [9]. However, these mixed-norm based algorithms are not robust to innovation components present in x_j . Therefore, we seek an MMV-type algorithm that is robust to innovation components.

Contributions: In this article, we propose Greedy Pursuits Assisted Basis Pursuit for Multiple Measurement Vectors (GPABP-MMV), and formulate its exact reconstruction conditions and the reconstruction error bound. Through extensive simulations, we show that the GPABP-MMV is robust to outliers and it outperforms state-of-the-art MMV-type algorithms (such as R-MSBL and Texas Hold 'Em) in terms of reconstruction accuracy. GPABP-MMV

is suitable for applications wherein the signal ensemble follows innovative joint-sparse formulation with K_c significantly less than K . The rest of this paper is organized as follows. In section 2, we briefly discuss Greedy Pursuits Assisted Basis Pursuit (GPABP) [20]. In section 3, we propose and analyze GPABP-MMV. In section 4, we present the simulation results comparing the performance of GPABP-MMV to that of the R-MSBL, Texas Hold 'Em, etc. Section 5 concludes the paper.

2. Greedy Pursuits Assisted Basis Pursuit

Consider the SMV reconstruction problem (i.e. reconstruction of x from y). The actual support of x , $T \subset \{1, 2, \dots, n\}$, is defined as the set of indices i where $x(i)$ is non-zero. The partial support, $\Lambda \subset \{1, 2, \dots, n\}$, is defined as the set of indices i where $x(i)$ is estimated to be non-zero. In some CS reconstruction problems, the partial support is known (termed as Partially Known Support (PKS)) and it is used to run Modified Basis Pursuit (Mod-BP) [21]. In [20], it was shown that, if the partial support is not known, it can be derived using multiple GPs and then Mod-BP can be applied. The GPABP algorithm [20], given as algorithm 1, employs multiple GPs to form Λ . Let L denote the number of GPs involved in the formation of Λ . In fact, the L GPs are L different GPs. For example, if $L = 2$, OMP is chosen for GP_1 and SP is chosen for GP_2 . Therefore, L GPs can give up to L different solutions of the form

$$\hat{T}_i = GP_i(\Phi, y, K) \quad \forall i \in \{1, 2, \dots, L\}$$

where GP_i stands for i^{th} greedy pursuit and $\hat{T}_i$ is the support set estimated by GP_i . Then, the partial support is formed as follows,

$$\Lambda := \bigcap_{i=1}^L \hat{T}_i.$$

The need for L different GPs to derive Λ is as follows. Reconstruction performance of a GP varies and depends on the nature of the sparse signal [22]-[23]. For example, OMP performs better than SP for some types of signals and SP performs better than OMP for some other type of signals. If the statistical distribution of the non-zero values of the signal is known

a priori, the best recovery algorithm (for that type of signal) can be applied to derive the partial support. But in many practical scenarios, this prior knowledge is not available and hence, a fusion based approach (as in [24]) that utilizes several GPs is used to obtain the partial support.

Upon obtaining Λ , x is reconstructed from y using Mod-BP (termed Mod-CS in [21]). The Mod-BP problem (in the case of noiseless measurements) is formulated as

$$\hat{x} = \arg \min_{\tilde{x}} \|\tilde{x}_{\Lambda^c}\|_1 \text{ s.t. } \Phi \tilde{x} = y \quad (2)$$

where Λ^c is the set compliment of Λ , $\tilde{x}_{\Lambda^c}$ is the subset of $\tilde{x}$ formed by extracting the entries of $\tilde{x}$ corresponding to the indices in Λ^c , $\|\cdot\|_1$ stands for the vector ℓ_1 -norm and $\hat{x}$ is the reconstructed signal. Note that, in the case of noisy measurements, the constraint of the Mod-BP problem will be $\|\Phi \tilde{x} - y\|_2 \leq \|w\|_2$ where $w \in \mathbb{R}^m$ is an additive noise such that $y = \Phi x + w \in \mathbb{R}^m$.

Lemma 1 recalls the exact reconstruction conditions for GPABP (in the case of noiseless measurements) as given in [21] and lemma 2 recalls the reconstruction error bound for GPABP (in the case of noisy measurements) as given in [25]. The actual support of x , T , is split as $T = (\Lambda \cup \Delta) \setminus \Delta^e$ where $\Delta := T \setminus \Lambda$ is the part of the support missing in Λ and $\Delta^e := \Lambda \setminus T$ is the set of wrong locations in Λ . A schematic describing T , Λ , Δ , and Δ^e is given in figure 1. Consider a typical reconstruction example where $n = 50$, $T = \{5, 10, 15, 20, 25, 30, 35, 40\}$ and $\Lambda = \{5, 10, 15, 20, 25, 30, 45\}$. Then, $\Delta = \{35, 40\}$ and $\Delta^e = \{45\}$. The signal sparsity (i.e. the number of non-zero elements in x) is related to the signal support as $K = |T|$ where $|\cdot|$ stands for the cardinality of a set. The Restricted Isometry Constant (RIC), δ_S , for $\Phi \in \mathbb{R}^{m \times n}$ is as defined in [2].

Lemma 1 (Theorem 1 in [21]): Let $y = \Phi x \in \mathbb{R}^m$. If Λ obeys $T := (\Lambda \cup \Delta) \setminus \Delta^e$, then $\hat{x}$ is the unique minimizer of Mod-BP (in step 3 of algorithm 1) if

$$|\Delta| \leq |\Lambda| \text{ and } \delta_{K+|\Delta|+|\Delta^e|} < \frac{1}{5}.$$

Assuming $|\Delta| \leq |\Lambda|$, the worst case RIC requirement of GPABP is obtained.

Algorithm 1 GPABP [20]

Require: Φ , y , K and L .

1: *for* $i=1:L$; $\hat{T}_i = \text{GP}_i(\Phi, y, K)$; *end*

2: *Partial support:* $\Lambda := \bigcap_{i=1}^L \hat{T}_i$

3: Obtain $\hat{x}$ using Mod-BP by solving (2):

$$\hat{x} = \arg \min_{\tilde{x}} \|\tilde{x}_{\Lambda^c}\|_1 \text{ s.t. } \Phi \tilde{x} = y$$

Lemma 2 (Theorem 1 in [25]): Let $y = \Phi x + w \in \mathbb{R}^m$ such that $\|w\|_2 \leq \epsilon$. If $\delta_{K+|\Delta|+|\Delta^e|} < \sqrt{2} - 1$, then

$$\|x - \hat{x}\|_2 \leq B(K + |\Delta| + |\Delta^e|)\epsilon, \text{ where}$$

$$B(S) \triangleq \frac{4\sqrt{1+\delta_S}}{1 - (\sqrt{2} + 1)\delta_S}. \quad (3)$$

Substituting $\delta_{K+|\Delta|+|\Delta^e|} < \frac{1}{5}$ in (3) will give, $\|x - \hat{x}\|_2 \leq 8.472\epsilon$. The reconstruction is exact if $\epsilon = 0$.

Reconstruction conditions and error bound of Mod-BP revealed the fact that a good PKS will give a better reconstruction [21],[25]-[27]. The fact that the exact reconstruction conditions and the reconstruction error bound of GPABP depend on the accuracy of Λ can be verified from lemmas 1 and 2. Usually for a good PKS, the number of correct locations missing in the PKS will be much less than the number of non-zero elements (i.e. $|\Delta| \ll |T|$). Similarly, the number of wrong locations in the PKS will be much less than the number of non-zero elements (i.e. $|\Delta^e| \ll |T|$). On the other hand, the number of correct locations missing in the PKS will be approximately the same as the number of wrong locations in the PKS (i.e. $|\Delta| \approx |\Delta^e|$). Another PKS based convex relaxation method is the Weighted ℓ_1 -minimization ($\text{W}\ell_1$) [28]-[29]. The objective function of $\text{W}\ell_1$ (using Λ as PKS) is a weighted ℓ_1 -norm expressed as $\|\tilde{x}\|_{1,v} := \sum_i v(i)|\tilde{x}(i)|$ where $v(i) = \omega \in [0, 1)$ whenever $i \in \Lambda$ and $v(i) = 1$ otherwise. For $\text{W}\ell_1$, the sufficient condition that guarantees stable and robust

recovery is

$$\delta_{2K} < \frac{1}{\sqrt{2}(\omega + (1 - \omega)\sqrt{1 + \frac{|\Lambda|}{|T|} - 2\frac{|T \cap \Lambda|}{|T|}}) + 1}.$$

It was shown in [28] that, if at least 50% of the PKS is accurate (i.e. $\frac{|T \cap \Lambda|}{|\Lambda|} \geq 0.5$), then $W\ell_1$ is stable and robust under weaker sufficient conditions than the analogous conditions for standard ℓ_1 -minimization. Khajehnejad et al also studied a similar problem to $W\ell_1$ but they assumed a probabilistic prior on the support [29].

3. Greedy Pursuits Assisted Basis Pursuit for Multiple Measurement Vectors

Let us now consider the MMV reconstruction problem (i.e. reconstruction of $\mathbf{X}$ from $\mathbf{Y}$). For the reconstruction of joint-sparse signals (innovative joint-sparse signals in particular), we propose the GPABP-MMV algorithm in this section. The GPABP-MMV algorithm, given as algorithm 2, has two major parts: First, it extracts the common sparse locations using MMV versions of GPs (MMV-GPs) [30] and then, it sequentially reconstructs the signals using Mod-BP. Examples of MMV-GP include MMV-OMP, MMV-SP, etc.

As given in algorithm 2, first, L MMV-GPs are applied, and their support estimates are obtained. Then, as in GPABP, a *partial support*, Λ_1 , is formed by taking the intersection of all these support estimates. As MMV-GPs are joint reconstruction algorithms designed for common joint-sparse signals, it is more likely that Λ_1 contains most of the common sparse locations. Next, Λ_1 is used to compute the residual data matrix,

$$\mathbf{Y}_{res} = \mathbf{Y} - \Phi_{\Lambda_1} \Phi_{\Lambda_1}^\dagger \mathbf{Y},$$

where Φ_{Λ_1} denotes the sub-matrix obtained by extracting the columns of Φ corresponding to the indices in Λ_1 and $\Phi_{\Lambda_1}^\dagger$ denotes the pseudo-inverse of Φ_{Λ_1} . Then, the residual data matrix is used to compute the residual norm vector, $[y_{r,1}, y_{r,2}, \dots, y_{r,J}]$, where $y_{r,i}$ denotes the ℓ_2 -norm of the i^{th} column of $\mathbf{Y}_{res}$. Next, the residual norm vector is sorted in ascending order to form the set of sorted indices, φ , expressed as

$$\varphi = \text{sort-ascend}([y_{r,1}, y_{r,2}, \dots, y_{r,J}]) \quad (4)$$

where *sort-ascend* stands for sorting in ascending order. In other words, φ contains the order of signals with increasing residual vector norm. Then, the innovative joint-sparse signals are reconstructed sequentially by applying Mod-BP, in the order given by φ . For reconstructing the first signal, $x_{\varphi(1)}$, Λ_1 is used as the PKS. The Mod-BP problem (in the case of noiseless measurements) is formulated as

$$\hat{x}_{\varphi(i)} = \arg \min_{\tilde{x}} \|\tilde{x}_{\Lambda_i^c}\|_1 \text{ s.t. } \Phi \tilde{x} = y_{\varphi(i)}. \quad (5)$$

Note that, in the case of noisy measurements, the constraint of the Mod-BP problem will be $\|\Phi \tilde{x} - y_{\varphi(i)}\|_2 \leq \|w_{\varphi(i)}\|_2$ where $w_{\varphi(i)} \in \mathbb{R}^m$ is an additive noise such that $y_{\varphi(i)} = \Phi x_{\varphi(i)} + w_{\varphi(i)} \in \mathbb{R}^m$. For the reconstruction of subsequent signals, $x_{\varphi(i)}$ where $2 \leq i \leq J$, the PKS (Λ_i) is formed as follows,

$$\Lambda_i := \Lambda_{i-1} \cap \hat{T}_{\varphi(i-1)}$$

where $\hat{T}_{\varphi(i-1)}$ is the set of indices corresponding to the K largest magnitude entries in $\hat{x}_{\varphi(i-1)}$. Note that $\Lambda_i \subset \Lambda_{i-1}$. This step, with high probability, increases the accuracy of PKS which is crucial for Mod-BP. The purpose of sorting signals based on their residual vector norm is to reconstruct them in the order of effectiveness of the PKS.

For any signal $x_{\varphi(i)}$, its actual support $T_{\varphi(i)}$, can be expressed as $T_{\varphi(i)} = (\Lambda_i \cup \Delta_i) \setminus \Delta_i^e$ where $\Delta_i := T_{\varphi(i)} \setminus \Lambda_i$ and $\Delta_i^e := \Lambda_i \setminus T_{\varphi(i)}$. Note that, for any signal $x_{\varphi(i)}$, the number of non-zero elements is K (i.e. $|T_{\varphi(i)}| = K$). Following lemma gives the condition on Λ_i for which $\delta_{K+|\Delta_i|+|\Delta_i^e|}$ is a decreasing function of i .

Lemma 3: $\forall i \in [2, J]$, if the PKS Λ_i obeys

$$|T_{\varphi(i)} \cap (\Lambda_{i-1} \setminus \Lambda_i)| < \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}, \quad (6)$$

then, $\delta_{K+|\Delta_i|+|\Delta_i^e|}$ is a decreasing function of i , i.e.

$$\delta_{K+|\Delta_i|+|\Delta_i^e|} < \delta_{K+|\Delta_{i-1}|+|\Delta_{i-1}^e|}.$$

Proof: While reconstructing $x_{\varphi(i)}$, the size of the undetected portion of the support (misses) can be expressed as,

$$|\Delta_i| = |\Delta_{i-1}| + |T_{\varphi(i)} \cap (\Lambda_{i-1} \setminus \Lambda_i)|.$$

Algorithm 2 Proposed GPABP-MMV

Require: $\Phi, \mathbf{Y}, K, L$ and J

1: *for* $i=1:L$; $\hat{T}_i = \text{MMV-GP}_i(\Phi, \mathbf{Y}, K)$; *end*

2: *Partial support:* $\Lambda_1 := \bigcap_{i=1}^L \hat{T}_i$

3: Residual data matrix: $\mathbf{Y}_{res} = \mathbf{Y} - \Phi_{\Lambda_1} \Phi_{\Lambda_1}^\dagger \mathbf{Y}$

4: Residual norm vector: $[y_{r,1}, y_{r,2}, \dots, y_{r,J}]$

5: Sort the residual norm vector using (4):

$$\varphi = \text{sort-ascend}([y_{r,1}, y_{r,2}, \dots, y_{r,J}])$$

6: Initialize: $i = 1$

repeat

7: Obtain $\hat{x}_{\varphi(i)}$, the $\varphi(i)^{\text{th}}$ column of $\hat{\mathbf{X}}$, using Mod-BP by solving (5):

$$\hat{x}_{\varphi(i)} = \arg \min_{\tilde{x}} \|\tilde{x}_{\Lambda_i^c}\|_1 \text{ s.t. } \Phi \tilde{x} = y_{\varphi(i)}$$

8: Extract $\hat{T}_{\varphi(i)}$ using $\hat{x}_{\varphi(i)}$

9: Increment: $i = i + 1$

10: Update Λ_i : $\Lambda_i := \Lambda_{i-1} \cap \hat{T}_{\varphi(i-1)}$

until $i \leq J$

If $|T_{\varphi(i)} \cap (\Lambda_{i-1} \setminus \Lambda_i)| < \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}$ (i.e. there are more correct deletions than wrong deletions), the above expression becomes,

$$|\Delta_i| < |\Delta_{i-1}| + \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}. \quad (7)$$

On the other hand,

$$|\Delta_i^e| = |\Delta_{i-1}^e| - |T_{\varphi(i)}^c \cap (\Lambda_{i-1} \setminus \Lambda_i)|$$

where $T_{\varphi(i)}^c$ is the set compliment of $T_{\varphi(i)}$. As $|T_{\varphi(i)}^c \cap (\Lambda_{i-1} \setminus \Lambda_i)| > \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}$ is a consequence of $|T_{\varphi(i)} \cap (\Lambda_{i-1} \setminus \Lambda_i)| < \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}$, the above expression becomes,

$$|\Delta_i^e| < |\Delta_{i-1}^e| - \frac{|\Lambda_{i-1} \setminus \Lambda_i|}{2}. \quad (8)$$

Combining (7) and (8) gives,

$$|\Delta_i| + |\Delta_i^e| < |\Delta_{i-1}| + |\Delta_{i-1}^e|. \quad (9)$$

Since δ_S is a monotonically increasing function of S , using (9), it can be said that

$$\delta_{K+|\Delta_i|+|\Delta_i^e|} < \delta_{K+|\Delta_{i-1}|+|\Delta_{i-1}^e|},$$

or equivalently, $\delta_{K+|\Delta_i|+|\Delta_i^e|}$ is a decreasing function of i . $\square$

Using the result of lemma 3, theorem 1 gives the exact reconstruction conditions for GPABP-MMV (in the case of noiseless measurements) and theorem 2 gives its reconstruction error bound (in the case of noisy measurements).

Theorem 1 (GPABP-MMV exact reconstruction conditions): Let $\mathbf{Y} = \Phi\mathbf{X} \in \mathbb{R}^{m \times J}$. If $\Lambda_1 := (T_{\varphi(1)} \cup \Delta_1^e) \setminus \Delta_1$, and Λ_i obeys (6) $\forall i \in [2, J]$, then $\forall i \in [1, J]$, $\hat{x}_{\varphi(i)}$ is the unique minimizer of Mod-BP if

$$|\Delta_J| \leq |\Lambda_J| \text{ and } \delta_{K+|\Delta_1|+|\Delta_1^e|} < \frac{1}{5}. \quad (10)$$

Proof: Using the result of lemma 1, conditions for exact reconstruction of $x_{\varphi(i)}$ can be written as

$$|\Delta_i| \leq |\Lambda_i| \text{ and } \delta_{K+|\Delta_i|+|\Delta_i^e|} < \frac{1}{5}.$$

Since $\Lambda_i := \Lambda_{i-1} \cap \hat{T}_{\varphi(i-1)}$, Λ_i is a subset of Λ_{i-1} and therefore, $|\Lambda_i| \leq |\Lambda_{i-1}|$. In other words, $|\Lambda_i|$ is a non-increasing function of i . On the other hand, Δ_i obeys $|\Delta_i| = |\Delta_{i-1}| + |T_{\varphi(i)} \cap (\Lambda_{i-1} \setminus \Lambda_i)|$. This implies $|\Delta_i|$ is a non-decreasing function of i . Furthermore, if Λ_i obeys (6), as given by lemma 3, $\delta_{K+|\Delta_i|+|\Delta_i^e|}$ is a decreasing function of i . Therefore, exact reconstruction conditions for the entire signal ensemble can be expressed as

$$|\Delta_J| \leq |\Lambda_J| \text{ and } \delta_{K+|\Delta_1|+|\Delta_1^e|} < \frac{1}{5}. \quad \square$$

Theorem 2 (GPABP-MMV error bound): Let $\mathbf{Y} = \Phi\mathbf{X} + \mathbf{W} \in \mathbb{R}^{m \times J}$ where $\mathbf{W} = [w_1, w_2, \dots, w_J] \in \mathbb{R}^{m \times J}$ such that $\|w_i\|_2 \leq \epsilon \forall i \in [1, J]$. If Λ_i obeys (6) $\forall i \in [2, J]$, and $\delta_{K+|\Delta_1|+|\Delta_1^e|} < \sqrt{2} - 1$, then

$$\|\mathbf{X} - \hat{\mathbf{X}}\|_F \leq \sqrt{J}B(K + |\Delta_1| + |\Delta_1^e|)\epsilon,$$

where $\|\cdot\|_F$ denotes the Frobenius norm and $B(S)$ is as defined in (3).

Proof: Using the result of lemma 2, the reconstruction error bound for each signal, $x_{\varphi(i)}$, is given by

$$\|x_{\varphi(i)} - \hat{x}_{\varphi(i)}\|_2 \leq B(K + |\Delta_i| + |\Delta_i^e|)\epsilon,$$

where $B(S)$ is as defined in (3). The reconstruction error for the entire signal ensemble can be expressed as

$$\begin{aligned} \|\mathbf{X} - \hat{\mathbf{X}}\|_F &= \sqrt{\sum_{i=1}^J \|x_{\varphi(i)} - \hat{x}_{\varphi(i)}\|_2^2} \\ &\leq \sqrt{\sum_{i=1}^J [B(K + |\Delta_i| + |\Delta_i^e|)]^2 \epsilon^2} \end{aligned} \quad (11)$$

Since $\forall i \in [2, J]$, Λ_i obeys (6), using the result of lemma 3, the error bound can be written as

$$\begin{aligned} \|\mathbf{X} - \hat{\mathbf{X}}\|_F &\leq \sqrt{J[B(K + |\Delta_1| + |\Delta_1^e|)]^2 \epsilon^2} \\ &\leq \sqrt{J}B(K + |\Delta_1| + |\Delta_1^e|)\epsilon. \quad \square \end{aligned}$$

Features of GPABP-MMV: Mod-BP is chosen for the PKS based algorithm mainly due to its reconstruction accuracy. As Mod-BP has the provision of picking any atom regardless of its presence in PKS, GPABP-MMV will be robust to innovation components. Moreover, being sequential in nature, GPABP-MMV reconstruction will be stable even for large J also. GPABP-MMV does not require K_c to be known. Only K is required. K , being marginal statistics, can be estimated easily [31]. On the other hand, K_c , being joint statistics, cannot be estimated that easily. GPABP-MMV is designed for innovative joint-sparse signals and it can handle common joint-sparse signals as well.

Computational Complexity of GPABP-MMV: If the MMV-GP algorithms have a complexity $C(m, n, K, J)$ then the fusion framework has a complexity $L \times C(m, n, K, J)$. Therefore, if Mod-BP has a complexity $C'(m, n)$, the computational complexity of GPABP-MMV will be $L \times C(m, n, K, J) + J \times C'(m, n)$. Since $C(m, n, K, J) \ll C'(m, n)$, and typically

$L \leq J$, GPABP-MMV's complexity becomes $\approx J \times C'(m, n)$. Low computational complexity of MMV-GPs allows us to have a large L . However, the size of Λ_1 is non-increasing in L and therefore, a smaller L is preferred.

4. Simulation Results

In this section, we present experimental results of joint-sparse reconstruction using three methods: R-MSBL [16], Texas Hold 'Em [14] algorithm and the proposed GPABP-MMV. In addition to these three methods, reconstruction performance of GPABP-separate is also included. GPABP-separate refers to the sequential but independent reconstruction of each signal using GPABP. For both GPABP-separate and GPABP-MMV, we fixed $L = 2$. In the case of GPABP-separate, fusion step involves OMP and SP. For GPABP-MMV, analogous to OMP and SP used in GPABP-separate, we use MMV-OMP and MMV-SP algorithms [30]. For Texas Hold 'Em, the fraction of community measurements is fixed as 75, and BP is used for CS reconstruction. The *Sparselab* solver [32] is used for the implementation of BP in Texas Hold 'Em and the *cvx* solver [33] is used for the implementation of Mod-BP in GPABP-separate and GPABP-MMV. In the case of noiseless measurements, the constraint of the Mod-BP algorithm is $\Phi\tilde{x} = y_{\varphi(i)}$. In the case of noisy measurements, we do blind reconstruction (in terms of knowledge of measurement noise) by fixing the constraint of the Mod-BP algorithm as $\|\Phi\tilde{x} - y_{\varphi(i)}\|_2 \leq 0.001$.

Synthetic joint-sparse signals: For our experiments, we generated innovative joint-sparse signals (Gaussian sparse signals) of length $n = 250$. The sparsity (common + innovation components) of each signal is fixed as 20. In each of our experiments, 250 independent trials are performed to obtain the average results. For each trial, an $m \times n$ Gaussian random measurement matrix is generated. In our first experiment, we present the influence of the Common Support Ratio ($\text{CSR} = \frac{K_c}{K}$) on the reconstruction accuracy (expressed in terms of Mean Square Error ($\text{MSE} = \frac{1}{nJ} \sum_{i=1}^J \|x_i - \hat{x}_i\|_2^2$)). We fixed $m = 64$ and $J = 10$. By choosing different K_c values, we performed signal recovery using all four techniques. Average MSE plot is shown in figure 2. GPABP-MMV gives the least MSE among all the methods and is stable for low K_c values also. Figure 3 shows the reconstruction time (computation

time in MATLAB 7.12.0 running on a 64-bit Intel(R) CoreTM i5-2400 processor with 8 GB RAM) for the same experiment. For $\text{CSR} < 0.8$, GPABP-MMV clearly outperforms R-MSBL (GPABP-MMV's closest rival in terms of accuracy) in terms of reconstruction speed. The Texas Hold 'Em is the fastest among all the methods. However, its reconstruction accuracy is worse. In our next experiment, we present the influence of the number of signals on MSE. K_c and m are fixed as 15 and 64 respectively. Average MSE versus J plot is shown in figure 4. It can be inferred that, as J increases from 5 to 20, unlike in other algorithms, average MSE corresponding to GPABP-MMV always remains closer to zero. In particular, average MSE corresponding to R-MSBL is closer to zero until $J \leq 8$ and increases thereafter. Next, we present the influence of m on MSE. K_c and J are fixed as 15 and 10 respectively. Average MSE versus Measurement Ratio ($\text{MR} = \frac{m}{n}$) plot is shown in figure 5. It can be seen that GPABP-MMV gives the least MSE for $\text{MR} \geq 0.2$. In our next experiment, we present the influence of the measurement noise on MSE. K_c , m and J are fixed as 15, 64 and 10 respectively. We performed signal reconstruction for different values of Signal to Measurement Noise Ratio (SMNR). The SMNR is defined as follows,

$$\text{SMNR (in dB)} = 10 \log_{10} \frac{\|x_{\varphi(i)}\|_2^2}{\|w_{\varphi(i)}\|_2^2}$$

where $x_{\varphi(i)}$ is the signal and $w_{\varphi(i)}$ is the measurement noise added to the measurement vector $y_{\varphi(i)}$. The noise generated in our experiment is a Gaussian random vector and its variance depends on the chosen SMNR value. Average MSE versus SMNR plot is shown in figure 6. It can be inferred that GPABP-MMV gives the least MSE for all SMNRs in the range 0 dB to 60 dB. For all four algorithms, the MSE reduces rapidly when the SMNR increases from 0 dB to 20 dB, and there is not much reduction thereafter.

Compressible ECG signals: The leads (ECG signals) are extracted from records 100, 101, 102 and 103 from the MIT-BIH Arrhythmia database [34]. They are divided into chunks of $n = 900$ samples (with amplitudes ranging from 0 to 255). This choice of n is based on the periodicity of the signals obtained from their annotations. These signals will serve as ground truth for MSE computation. Discrete cosine transform is used to obtain sparse representations of the signals. Joint-sparse signal ensemble (similar to innovative

joint-sparse signals) is formed using J chunks of these sparse signals. We fixed $J = 50$ and $K = \lfloor \frac{m}{\log n} \rfloor$. For each signal in the ensemble, y_i is obtained using a Gaussian random measurement matrix and is corrupted by an additive noise such that its SMNR is 20 dB. Figure 7 shows the variation of average MSE as a function of MR (varying between 0.2 and 0.6). For any MR above 0.25, GPABP-MMV gives the least reconstruction error (among all four methods).

5. Conclusion

In this article, we have proposed GPABP-MMV, an MMV-type algorithm for reconstruction of innovative joint-sparse signals, in which the fusion step utilizes MMV versions of multiple GPs to form a common support, which in turn is used to reconstruct signals sequentially by applying Mod-BP. Experimental results show that the proposed GPABP-MMV is robust to the innovation components present in the innovative joint-sparse signals. Moreover, being sequential, GPABP-MMV reconstruction remains stable for large J . Experimental results show that, for any MR above 0.25, GPABP-MMV clearly outperforms state-of-the-art MMV-type innovative joint-sparse reconstruction algorithms (such as R-MSBL and Texas Hold 'Em) in terms of reconstruction accuracy.

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7. References

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Figure 1: Schematic of actual support (T) and partial support (Λ)

Figure 2: Synthetic signals (noiseless measurements): Average MSE versus CSR

Figure 3: Synthetic signals (noiseless measurements): Average reconstruction time (in sec) versus CSR

Figure 4: Synthetic signals (noiseless measurements): Average MSE versus J

Figure 5: Synthetic signals (noiseless measurements): Average MSE versus MR

Figure 6: Synthetic signals (noisy measurements): Average MSE versus SMNR

Figure 7: Compressible ECG signals (noisy measurements - SMNR = 20 dB): Average MSE versus MR