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Time-Consistent Mean-Variance Portfolio Selection with Only Risky Assets¹

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Abstract

Time consistency and optimal diversification criteria are popular in the dynamic portfolio construction in practice. This paper is devoted to the exact analytical solution of the time-consistent mean-variance portfolio selection with assets that can be all risky in a continuous-time economy, of which the time-consistent global minimum-variance portfolio selection is a special case. Our solution generalizes the studies with a risk-free asset in the sense that one of the risky assets can be set as risk-free. By applying the extended dynamic programming, we manage to derive the exact analytical solution of the time-consistent mean-variance strategy with risky assets via the solution of an Abel differential equation. To stabilize the solution, we derive an analytical expansion for the Abel differential equation with any desired accuracy. In addition, we derive the statistical properties of the optimal strategy and prove a separation theorem. Moreover, we establish the links of time-consistent strategy with pre-commitment and myopic strategies and investigate the curse of dimensionality on the time-consistent strategies. We show that under the low-dimensional setting, the intertemporal hedging demands are significant; however, under the high-dimensional setting, the time-consistent strategies are approximately equivalent to myopic strategies, in the presence of estimation risk. Empirical studies are conducted to illustrate and verify our results.

JEL Classification: C61, C68, C72, D92

Key words: Time-consistent strategy; Dynamic global minimum-variance strategy; Extended dynamic programming; Abel's differential equation; Curse of dimensionality

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1. Introduction

Modern portfolio theory, pioneered by Markowitz (1952), is recognized as a cornerstone theory of finance. In his seminal paper, Markowitz quantified the expected return and the risk of a portfolio by its mean and variance, respectively, and found the best balance between them. His single-period mean-variance (MV) framework has inspired the development of multi-period and continuous-time portfolio choices. Li and Ng (2000) and Zhou and Li (2000) adopted the embedding techniques to get rid of the non-separability of the variance operator and obtained the optimal dynamic MV portfolios under multi-period and continuous-time settings, respectively. Since then, many works extended the MV framework; see Lim and Zhou (2002) for the MV framework with random parameters, Chiu and Li (2006) for asset and liability management, and Chiu and Wong (2011) for dynamic allocation on co-integrated assets.

In recent years, there is a growing literature investigating time-consistent investment policies, whose sub-policies are still optimal for the associated sub-problems. The time-consistency property is found important in investments, but the dynamic portfolios in Li and Ng (2000) and Zhou and Li (2000), referred as pre-commitment policies, fail to satisfy the time consistency due to the time inconsistency of variance operator; see (5). As stated in Basak and Chabakauri (2010), the pre-commitment investor may find it optimal to deviate from her policy while the time-consistent investor has no incentive to do so. In their paper, a tractable framework for time-consistent MV analysis is firstly established. Wu and Chen (2015) extended their analysis to regime-switching risk aversions. Recently, Björk et al. (2017) extended the framework of Basak and Chabakauri (2010) to general Markovian time-inconsistent stochastic control problems with the interpretation of Nash equilibrium concepts and mathematically proved a verification theorem for the stochastic control problem. Loosely speaking, a time-consistent strategy can be heuristically described as the equilibrium strategy of a non-cooperative game played by the future incarnations of the investor. Time-consistent portfolios are dynamically optimal and empirically outperform pre-commitment portfolios in terms of out-of-sample Sharpe ratio, partially because the enforcement of time consistency restricts the feasible strategy set and thus protects the investment policies from the effects of huge estimation errors. We refer to Jagannathan and Ma (2003) for the reasons why constrained

portfolios have better out-of-sample performance, and Pun and Wong (2016); Chiu et al. (2017); Pun and Wong (2017) for the empirical studies of dynamic portfolios, especially for the poor empirical performance of pre-commitment policies.

As discussed in Yao et al. (2014), it is necessary to consider an economy with only risk assets. The fundamental reason is that a risk-free asset hardly exists in reality and in a relatively long investment horizon due to the stochastic nature of real interest rates and the inflation risk (see Viceira (2012)). In addition, the empirical evidence in DeMiguel et al. (2009) shows that the static global minimum-variance (GMV) strategy with only risky assets (deduced in Markowitz (1952)) performs the best in terms of out-of-sample performance among all estimated theoretically optimal strategies. The inclusion of a nominal risk-free asset into the market will simplify the portfolio selection problem but degenerates the GMV strategy to a bank deposit strategy with zero risk, which is not favorable to investors. From this empirical perspective, it is interesting to study MV strategy with only risky assets, which will deduce the desirable GMV strategy. However, for dynamic setting, the existing literature only provide pre-committment policies of GMV strategy (see Li and Ng (2000) for multi-period and Yao et al. (2014) for continuous-time settings), to the best of our knowledge. The time-consistent policies of MV or GMV strategy with only risky assets, which are of more interests to investors, had not been obtained explicitly. Hence, we aim to fill this gap and investigate the properties of the time-consistent MV strategies in a general market setting with only risky assets as in Yao et al. (2014) and Zhang et al. (2017).

In fact, an analytical challenge is posed to the determination of the time-consistent MV strategy in the case of no risk-free asset, because the portfolio's wealth process, which embeds the self-financing constraint, will unavoidably involve state-dependent parameters in diffusion terms (see equation (2) below). Such a state-dependency feature complicates the problem as in Björk et al. (2014) and the existing studies facing similar concerns end up with an unsolved integral equation. Even though the existence and uniqueness of the solution to the integral equation are proved in those studies, one hardly acquires the properties of the solution. Another challenge is to clarify the relationships among the three popular dynamic portfolio constructions, namely myopic, pre-commitment, and time-consistent policies. Myopic polices optimize the single-period objective in each sub-period of the whole time horizon and they do not take into account the future optimal-

ity. Myopic policies are analogous to pre-commitment policies with constraints. Mathematically, myopic policies are suboptimal, while pre-commitment policies are global optimal. From some empirical results, we have seen certain similarity between myopic and time-consistent strategies and they are even identical in the market with a risk-free asset (see Basak and Chabakauri (2010)). Some natural questions could be: would that be still true for the market with only risky assets? What factor(s) drive them to be equivalent? The key to answering these questions is to obtain the explicit solution to time-consistent MV strategy with only risky assets.

In this paper, we solve the time-consistent MV problem in a possibly incomplete market with only risky assets. Consequently, the time-consistent GMV strategy is obtained. The resulting time-consistent strategies depend on the current wealth (see equation (19) below), which is economically meaningful, in contrast with that in Basak and Chabakauri (2010). We consider multiple assets in a continuous-time Black-Scholes economy with time-varying parameters. Our methodology of characterizing the optimal dynamic MV policies is based on the extended dynamic programming techniques, developed in Björk et al. (2017). It is noteworthy that this paper gets rid of the difficulty of the unsolvable integral equation in time-inconsistency problems by using differential equation and analytical expansion approaches. We manage to derive an analytical solution of the time-consistent MV portfolios via the solution of an Abel differential equation. In addition, we study the statistical properties of the time-consistent MV portfolios and prove a separation theorem for time-consistent MV portfolios. Moreover, we compare and establish the links of time-consistent portfolio with other two popular dynamic portfolios (myopic and pre-commitment portfolios). In particular, we identify the intertemporal hedging term as the difference between time-consistent and myopic portfolios, and conduct statistical analysis on this term, especially when the number of assets is relatively large (i.e. under the curse of dimensionality). Using the same datasets used in DeMiguel et al. (2009) and high-dimensional daily stock prices data, we illustrate and verify our theoretical results for the time-consistent MV and GMV strategies.

The contribution of this paper is threefold: first, we provide an exact analytical solution of time-consistent MV and GMV strategies with only risky assets. To the best of our knowledge, this paper is the first to use the Abel differential equation approach to solve financial problems. Second, we perform analytical expansions for the Abel differential equation and error analysis,

which help to derive the properties of time-consistent controls under the framework of Björk et al. (2017) in general. Third, we derive the properties of time-consistent strategies and examine the high-dimensionality effects on time-consistent strategies. We theoretically and empirically show that when the number of assets is small, the intertemporal hedging demands are significant and time-consistent GMV strategy yields lower empirical risk (see Table 2 below); however, when the number of assets is large, the time-consistent strategies are approximately equivalent to myopic strategies. This finding provides insight and inspiration into the studies of high-dimensional portfolio selection.

The remainder of this paper is organized as follows. Section 2 presents the model and the game-theoretic formulation of MV portfolio selection problems. It introduces the time-consistency problem in detail and an extended Hamilton-Jacobi-Bellman (HJB) equation characterizing the time-consistent (equilibrium) strategy. Section 3 solves the extended HJB equation analytically and performs mathematical analysis of the associated Abel differential equation. Section 4 derives the properties of the time-consistent MV strategy and its links with myopic and pre-commitment strategies. The curse of dimensionality (or high-dimensionality effect) is examined in Section 4.4. Section 5 presents empirical results to illustrate our theoretical results. Finally, Section 6 concludes. To simplify the presentation, some long proofs are put into Appendix A.

2. Game-Theoretic Formulation of Mean-Variance Portfolio Selection Problems

2.1. The Model

We consider a continuous-time, frictionless, Markovian financial market, where p risky assets (stocks) are traded with a finite horizon $[0, T]$. The dynamics of asset prices $S(t) = (S_1(t), \dots, S_p(t))'$ are defined on a filtered physical probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated by a m -dimensional standard Brownian motion $B(t) = (B_1(t), \dots, B_m(t))'$, in which $m \geq p$. Only when $m = p$, the market is complete. We are not restricted to a specific value of m , which allows the market to be incomplete. Hereafter, the transpose of a vector or matrix a is denoted by a' . We postulate the asset price process of the i th asset S_i evolves as follows:

$$dS_i(t)/S_i(t) = \mu_i(t)dt + \sigma_i(t)'dB(t), \quad S_i(0) = s_i > 0 \quad (1)$$

where $\mu_i(t)$ is the appreciation rate and $\sigma_i(t) = (\sigma_{i1}(t), \dots, \sigma_{im}(t))' \in \mathbb{R}^m$ characterizes the volatility of the i th asset return at time $t \in [0, T]$ for $i = 1, \dots, p$.

To facilitate the later analysis, we introduce the following notations. We denote the mean vector and the covariance matrix of asset returns respectively by $\mu(t) := (\mu_1(t), \mu_2(t), \dots, \mu_p(t))'$ and $\Sigma(t) := \sigma(t)\sigma(t)'$, where $\sigma(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_p(t))' \in \mathbb{R}^{p \times m}$ is the volatility matrix of asset returns. Moreover, we denote the mean vector and covariance matrix for returns of all assets except the first by $\tilde{\mu}(t) := (\mu_2(t), \dots, \mu_p(t))'$ and $\tilde{\Sigma}(t) := \tilde{\sigma}(t)\tilde{\sigma}(t)'$, respectively, where $\tilde{\sigma}(t) = (\sigma_2(t), \dots, \sigma_p(t))' \in \mathbb{R}^{(p-1) \times m}$.

Denote by $\mathbf{u} = \{u(t)\}_{t \in [0, T]} := \{(u_1(t), \dots, u_p(t))'\}_{t \in [0, T]}$ the investor's trading strategy, by $u_i(t)$ the money amount the investor invested in the i th asset at time t , $i = 0, 1, \dots, p$, and by $X(t)$ her portfolio wealth at time t . Then, we have the self-financing constraint $X(t) = \sum_{i=1}^p u_i(t)$, which implies $u_1(t) = X(t) - \sum_{i=2}^p u_i(t)$. As a result, the trading strategy $\mathbf{u}$ is characterized by $\tilde{u}(t) = (u_2(t), \dots, u_p(t))' \in \mathbb{R}^{p-1}$, $t \in [0, T]$. The first asset acts like a risk-free asset in the portfolio optimization literature and the choice of the “risk-free asset” is irrelevant to the trading strategy, which will be verified in the subsequent sections. By the self-financing principle, the dynamic of the wealth process $\{X^{\mathbf{u}}(t)\}$ with the trading strategy $\mathbf{u}$ is given by

$$\begin{aligned} dX^{\mathbf{u}}(t) &= [(X^{\mathbf{u}}(t) - \tilde{u}(t)'\tilde{\mathbf{1}})\mu_1(t) + \tilde{u}(t)'\tilde{\mu}(t)]dt + [(X^{\mathbf{u}}(t) - \tilde{u}(t)'\tilde{\mathbf{1}})\sigma_1(t)' + \tilde{u}(t)'\tilde{\sigma}(t)]dB(t) \\ &= [X^{\mathbf{u}}(t)\mu_1(t) + \tilde{u}(t)'\beta(t)]dt + [X^{\mathbf{u}}(t)\sigma_1(t)' + \tilde{u}(t)'\delta(t)]dB(t), \end{aligned} \quad (2)$$

where $\tilde{\mathbf{1}} = (1, \dots, 1)' \in \mathbb{R}^{p-1}$, $\beta(t) = \tilde{\mu}(t) - \tilde{\mathbf{1}}\mu_1(t) \in \mathbb{R}^{p-1}$ is the excess return vector, and $\delta(t) = \tilde{\sigma}(t) - \tilde{\mathbf{1}}\sigma_1(t)' \in \mathbb{R}^{(p-1) \times m}$ is the excess volatility matrix with respect to the first asset. We denote by $\Gamma(t) := \delta(t)\delta(t)'$ the corresponding excess covariance matrix. If the first asset is a risk-free asset (bank) with a deterministic interest rate $r_f(t)$, we have $\mu_1(t) = r_f(t)$, $\sigma_1(t)$ is a zero vector, and $\Gamma(t) = \tilde{\Sigma}(t)$. Hence, the analysis of this paper nests the presence of a risk-free asset in the market as a special case. Throughout this paper, we make the following assumption.

Assumption 2.1. For any $t \in [0, T]$,

- if the first asset is risk-free, i.e. $\sigma_1(t) = 0$, then we assume $\Gamma(t) = \tilde{\Sigma}(t)$ is positive-definite;

- if the first asset is risky, i.e. $\sigma_1(t) \neq 0$, then we assume $\Sigma(t)$ is positive-definite. We remark here that in this case, $\Gamma(t)$ is also positive-definite.

We employ the mean-variance (MV) criterion for the investment problem. Specifically, the investor aims to find a dynamic trading strategy $\mathbf{u} = \{u(t)\}_{t \in [0, T]}$ that minimizes the portfolio's risk for a given expected return at the terminal time T . Using the method of Lagrange multiplier, we consider the following maximization problem:

$$\max_{\mathbf{u}} \{ \lambda \mathbb{E}[X^{\mathbf{u}}(T)] - \text{Var}(X^{\mathbf{u}}(T)) \} \quad \text{subject to the wealth process (2),} \quad (3)$$

where $\lambda \geq 0$ measures the risk tolerance of the investor. When $\lambda = 0$, the MV criterion becomes global minimum-variance (GMV) criterion, which seeks for a portfolio with the global minimum risk. The trading strategies, which attain the optimum in MV (resp. GMV) criterion, are called MV (resp. GMV) strategies. We are interested in the market where there are only risky assets because when there is a risk-free asset in the market, the GMV strategy is trivial as we just need to invest all our money into the risk-free asset, which will result in zero risk under both static and dynamic settings. In this paper, we primarily deduce the results for the MV criterion, which can be reduced to the results for the GMV criterion by sending λ to zero.

The time inconsistency of the variance term in the MV criterion makes the conventional dynamic programming inapplicable. Hence, we deduce a dynamically optimal MV strategy by using a game-theoretic formulation and an extended dynamic programming theory developed in Björk et al. (2017). The resulting optimal strategy is a time-consistent policy in that the investor optimally chooses the trading strategy taking into account the future optimality.

2.2. The Equilibrium Value Function

In this section, we formulate the dynamic MV portfolio optimization problem using the dynamic programming and game-theoretic arguments. Let $\mathcal{X}$ and $\mathcal{U}$ be the measurable state and control spaces, respectively. For a fixed $(t, x) \in [0, T] \times \mathcal{X}$ and a fixed trading strategy $\mathbf{u} \in \mathcal{U}$, we consider the reward function:

$$J(t, x, \mathbf{u}) = \lambda \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)] - \text{Var}_{t,x}(X^{\mathbf{u}}(T)), \quad (4)$$

where $\mathbb{E}_{t,x}[\cdot]$ and $\text{Var}_{t,x}(\cdot)$ denote the conditional expectation and conditional variance given the filtration $\mathcal{F}_t$ and $X(t) = x$, respectively. Noteworthy is that the problem of maximizing $J(t, x, \mathbf{u})$ in (4) is time-inconsistent, since we have, for $s > t$ and $X(s) = y$,

$$\begin{aligned}
& J(t, x, \mathbf{u}) \\
&= \mathbb{E}_{t,x}[\lambda X^{\mathbf{u}}(T) - X^{\mathbf{u}}(T)^2] + (\mathbb{E}_{t,x}[X^{\mathbf{u}}(T)])^2 \\
&= \mathbb{E}_{t,x}[\mathbb{E}_{s,y}[\lambda X^{\mathbf{u}}(T) - X^{\mathbf{u}}(T)^2] + (\mathbb{E}_{s,y}[X^{\mathbf{u}}(T)])^2] - \mathbb{E}_{t,x}[(\mathbb{E}_{s,y}[X^{\mathbf{u}}(T)])^2] + (\mathbb{E}_{t,x}[X^{\mathbf{u}}(T)])^2 \\
&= \mathbb{E}_{t,x}[J(s, y, \mathbf{u})] - \mathbb{E}_{t,x}[(\mathbb{E}_{s,y}[X^{\mathbf{u}}(T)])^2] + (\mathbb{E}_{t,x}[X^{\mathbf{u}}(T)])^2 \\
&\neq \mathbb{E}_{t,x}[J(s, y, \mathbf{u})].
\end{aligned} \tag{5}$$

Hence, the Bellman optimality principle does not hold and the conventional Hamilton-Jacobi-Bellman (HJB) equation fails to characterize the optimal control (trading strategy); see Yong and Zhou (1999). The time inconsistency of the preferences also implies that the trading strategy $\mathbf{u}$ that maximizes $J(t, x, \mathbf{u})$ will no longer maximize $J(s, y, \mathbf{u})$ at the later time point.

Following the game-theoretic framework in Björk et al. (2017), we look for the optimal time-consistent trading strategy via Nash subgame perfect equilibrium points. The rough picture of the game is that for any $t \in [0, T]$, the “player t ” (an incarnation of the investor) chooses the optimal strategy $u^*(t)$ given that every “player s ” uses the strategy $u^*(s)$ for $s > t$. Economically, the equilibrium strategy is time-consistent in making decision; see Basak and Chabakauri (2010). We introduce the mathematical definition for equilibrium trading strategy and equilibrium value function as follows.

Definition 2.1. Consider a trading strategy $\mathbf{u}^* = \{u^*(t)\}_{t \in [0, T]}$. Given a fixed $\hat{u} \in \mathcal{U}$, a fixed $\Delta t \in \mathbb{R}^+$, and for any initial point (t, x) , define another trading strategy $\mathbf{u}_{\Delta t} = \{u_{\Delta t}(t)\}_{t \in [0, T]}$ by

$$u_{\Delta t}(s) = \begin{cases} \hat{u}, & \text{for } t \leq s < t + \Delta t, \ x \in \mathcal{X}, \\ u^*(s), & \text{for } t + \Delta t \leq s \leq T, \ x \in \mathcal{X}. \end{cases}$$

If

$$\liminf_{\Delta t \rightarrow 0} \frac{J(t, x, \mathbf{u}^*) - J(t, x, \mathbf{u}_{\Delta t})}{\Delta t} \geq 0,$$

for all $\hat{u} \in \mathcal{U}$, then $\mathbf{u}^*$ is called an equilibrium trading strategy. The corresponding reward function

is called the equilibrium value function V given by

$$V(t, x) = J(t, x, \mathbf{u}^*).$$

Our aim is to find the analytical solutions to the Nash equilibrium: equilibrium trading strategy $\mathbf{u}^*$ and equilibrium value function V . As shown in Björk et al. (2017), the solutions of $\mathbf{u}^*$ and V are characterized by an extended HJB system using the dynamic programming arguments. We adopt the following definition of the extend HJB system and its associated verification theorem in Björk et al. (2017).

Definition 2.2. The extended HJB system for the Nash equilibrium is defined as follows.

$$\sup_{\mathbf{u} \in \mathcal{U}} \{ \mathcal{A}^{\mathbf{u}} V(t, x) - [x^2 \varsigma_1(t) + 2x\tilde{u}(t)' \phi(t) + \tilde{u}(t)' \Gamma(t) \tilde{u}(t)] (\partial_x g)^2 \} = 0, \quad t \in [0, T], \quad (6)$$

$$\mathcal{A}^{\mathbf{u}^*} g(t, x) = 0, \quad t \in [0, T], \quad (7)$$

$$V(T, x) = \lambda x,$$

$$g(T, x) = x,$$

where $\varsigma_1(t) = \sigma_1(t)' \sigma_1(t)$ is the variance of the first asset return, $\phi(t) = \tilde{\sigma}(t) \sigma_1(t) - \mathbf{1} \varsigma_1(t)$, $\mathbf{1} = (1, \dots, 1)' \in \mathbb{R}^p$, $\partial_x g = \partial g / \partial x$, $\mathbf{u}^*$ in (7) realizes the supremum in (6), and $\mathcal{A}^{\mathbf{u}}$ is the infinitesimal generator of (2) associated with $\mathbf{u}$ given by

$$\mathcal{A}^{\mathbf{u}} = \frac{\partial}{\partial t} + [x\mu_1(t) + \tilde{u}(t)' \beta(t)] \frac{\partial}{\partial x} + \frac{1}{2} [x^2 \varsigma_1(t) + 2x\tilde{u}(t)' \phi(t) + \tilde{u}(t)' \Gamma(t) \tilde{u}(t)] \frac{\partial^2}{\partial x^2}.$$

Theorem 2.1 (Verification theorem, Theorem 5.2 in Björk et al. (2017)). Assume that V and g solve the extended HJB system (6) and (7) and $\mathbf{u}^*$ realizes the supremum in (6). Then $\mathbf{u}^*$ is an equilibrium trading strategy and V is the corresponding equilibrium value function. Moreover, $g(t, x) = \mathbb{E}_{t,x}[X^{\mathbf{u}^*}(T)]$ is the terminal reward anticipated at time t .

We call the equilibrium trading strategy $\mathbf{u}^*$ as time-consistent MV strategy. The main contributions of this paper is to solve the extended HJB system (6)-(7) and analyze the properties of the time-consistent MV strategies.

3. Time-Consistent Mean-Variance Strategies

3.1. Analytical Solution to the Extended HJB System

In this section, we solve the extended HJB system of equations (6)-(7). The proofs of Propositions 3.1-3.3 are rather long and are put into Appendix A. The following proposition summarizes our main results.

Proposition 3.1. The extend HJB system of equations (6) and (7) admits the analytical solutions:

$$V(t, x) = x^2 k_2(t, T) + x k_1(t, T) + k_0(t, T), \quad (8)$$

$$g(t, x) = x e_1(t, T) + \frac{2}{\lambda} k_0(t, T), \quad (9)$$

where

$$k_0(t, T) = \frac{\lambda^2}{4} \int_t^T \frac{b(\tau)}{e_2(\tau, T)} d\tau, \quad k_1(t, T) = \lambda e_1(t, T), \quad k_2(t, T) = e_1^2(t, T)(1 - e_2(t, T)), \quad (10)$$

$$e_1(t, T) = \exp \left\{ \int_t^T \left[c(\tau) + \left(\frac{1}{e_2(\tau, T)} - 1 \right) b(\tau) \right] d\tau \right\}, \quad (11)$$

$$b(t) = \beta(t)' \Gamma(t)^{-1} \beta(t), \quad c(t) = \mu_1(t) - \phi(t)' \Gamma(t)^{-1} \beta(t),$$

and $e_2(t, T)$ solves the Abel differential equation of the second kind:

$$e_2(t, T) \frac{\partial e_2(t, T)}{\partial t} = -e_2^2(t, T) a(t) - (1 - e_2(t, T))^2 b(t) \quad (12)$$

with $e_2(T, T) = 1$, in which $a(t) = \varsigma_1(t) - \phi(t)' \Gamma(t)^{-1} \phi(t)$. The time-consistent trading strategy $\mathbf{u}^* = \{u^*(t) = (u_1^*(t), \tilde{u}^*(t)')'\}_{t \in [0, T]}$, which realizes the supremum in (6), is given by

$$\tilde{u}^*(t) = \Gamma(t)^{-1} \left(\frac{\beta(t)}{e_2(t, T)} - \beta(t) - \phi(t) \right) x + \Gamma(t)^{-1} \beta(t) \frac{\lambda}{2e_1(t, T)e_2(t, T)} \quad (13)$$

and $u_1^*(t) = x - \tilde{\mathbf{1}}' \tilde{u}^*(t)$.

From the proof, we also obtain the integral equation for $e_2(t, T)$:

$$e_2(t, T) = \exp \left\{ \int_t^T \left[a(\tau) + \left(\frac{1}{e_2(\tau, T)} - 1 \right)^2 b(\tau) \right] d\tau \right\}, \quad (14)$$

which implies $e_2(t, T) \geq 1$ by noting that

$$b(t) > 0 \text{ and } a(t) = \sigma_1(t)' (I - \delta(t)' (\delta(t) \delta(t)')^{-1} \delta(t)) \sigma_1(t) \geq 0$$

since $\Gamma(t)^{-1}$ is positive-definite under Assumption 2.1 and $I - \delta(t)'(\delta(t)\delta(t)')^{-1}\delta(t)$ is symmetric idempotent. The similar integral equation is discussed in Björk et al. (2014), which proved the existence and uniqueness of its solution. However, it could only be solved numerically. This paper investigates the function $e_2(t, T)$ using the differential equation approach, which will facilitate our analysis and interpretation of the time-consistent MV strategies.

The equation (12) is the Abel differential equation of the second kind for $e_2(t, T)$, which is considered unsolvable in general. However, some restricted forms of (12) admit general solutions; see Polyanin and Zaitsev (2003). For our financial applications, we are interested in the following two special cases, where the exact analytical solutions of $e_2(t, T)$ and the optimal time-consistent trading strategy (13) can be obtained.

1. If $a(t) \equiv 0$, then $e_2(t, T) = 1$ for $t \in [0, T]$.
2. If $a(t) \equiv a > 0$ and $b(t) \equiv b > 0$ are both constants, then (12) is a separable equation and $e_2(t, T)$ can be solved analytically: $e_2(t, T) = l^{-1}(l(1) - T + t)$, where $l^{-1}(\cdot)$ is the inverse function of $l(\cdot)$ and

$$l(x) = \frac{1}{2(a+b)} \left[2\sqrt{\frac{b}{a}} \arctan \left(\frac{(a+b)x - b}{\sqrt{ab}} \right) + \log(ax^2 + b(x-1)^2) \right]. \quad (15)$$

We shall perform the mathematical analysis of the Abel differential equation (12) under a general setting in the subsequent subsection. Here, we remark that when the first asset is risk-free, i.e. $\sigma_1(t) = 0$, it is easy to see that $a(t) \equiv 0$. In this case, we can deduce the time-consistent MV strategy as follows, which is consistent with the results in Basak and Chabakauri (2010).

Corollary 3.1. When the first asset is risk-free, i.e. $\mu_1(t) = r_f(t)$, $\sigma_1(t) = 0$, we have

$$a(t) = 0, \quad b(t) = \beta(t)' \tilde{\Sigma}(t)^{-1} \beta(t), \quad c(t) = r_f(t), \quad e_2(t, T) = 1, \quad e_1(t, T) = e^{\int_t^T r_f(\tau) d\tau}, \quad (16)$$

$$\begin{aligned} g(t, x) &= e^{\int_t^T r_f(\tau) d\tau} x + \frac{\lambda}{2} \int_t^T b(\tau) d\tau, \quad V(t, x) = \lambda e^{\int_t^T r_f(\tau) d\tau} x + \frac{\lambda^2}{4} \int_t^T b(\tau) d\tau, \\ \tilde{u}^*(t) &= \frac{\lambda e^{-\int_t^T r_f(\tau) d\tau}}{2} \tilde{\Sigma}(t)^{-1} \beta(t), \quad u_1^*(t) = x - \tilde{\mathbf{1}}' \tilde{u}^*(t), \end{aligned}$$

where $\beta(t) = \tilde{\mu}(t) - \tilde{\mathbf{1}} r_f(t)$.

The proof of Corollary 3.1 is straightforward and hence omitted. Since our results in this paper mainly depend on functions a, b, c, e_1, e_2 , most results can be reduced to the special case of including a risk-free asset in the financial market, with the substitution by those functions in (16).

Recall that when we solved for the optimal strategy, we embedded the self-financing constraint $\sum_{i=1}^p u_i(t) = X(t)$ into the wealth process and the first asset plays a role like a risk-free asset. In fact, we can express $\mathbf{u}^* = \{u^*(t)\}_{t \in [0, T]}$ in (13) in terms of $\mu(t)$ and $\Sigma(t)$. As a result, we verify the independence of time-consistent trading strategy on the choice of the “risk-free asset.” This technique can be applied for solving an (extended) dynamic programming problem with a linear constraint. The following proposition rewrites the key elements in (13) in terms of $\mu(t)$ and $\Sigma(t)$.

Proposition 3.2. The functions $\Gamma(t)^{-1}\beta(t)$, $\Gamma(t)^{-1}\phi(t)$, $a(t)$, $b(t)$, and $c(t)$ can be expressed in terms of $\mu(t)$ and $\Sigma(t)$ as follows:

$$\Gamma(t)^{-1}\phi(t) = -\tilde{I} \frac{\Sigma(t)^{-1}\mathbf{1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \quad \Gamma(t)^{-1}\beta(t) = \tilde{I} \left(\Sigma(t)^{-1} - \frac{\Sigma(t)^{-1}\mathbf{1}\mathbf{1}'\Sigma(t)^{-1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} \right) \mu(t), \quad (17)$$

$$a(t) = \frac{1}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \quad b(t) = \mu(t)'\Sigma(t)^{-1}\mu(t) - \frac{(\mathbf{1}'\Sigma(t)^{-1}\mu(t))^2}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \quad c(t) = \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \quad (18)$$

where $\tilde{I} = [0_{p-1}; I_{p-1}] \in \mathbb{R}^{(p-1) \times p}$ is a $(p-1)$ -identity matrix with an additional zero vector at the first column and $0_{p-1} = (0, \dots, 0)' \in \mathbb{R}^{(p-1)}$.

The parameter functions $a(t)$, $b(t)$, $c(t)$ in (18) are exactly the three statistics for single-period MV efficient set, which characterize the hyperbolic equation for the efficient frontier. Readers are referred to Jobson (1991) for the distributional properties of these three statistics.

Corollary 3.2. The time-consistent trading strategy $\mathbf{u}^* = \{u^*(t)\}_{t \in [0, T]}$ can be expressed as

$$\begin{aligned} u^*(t) &= w(t)x + \lambda f(t), \\ w(t) &= \Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} \right) \left(\frac{1}{e_2(t, T)} - 1 \right) + \frac{\Sigma(t)^{-1}\mathbf{1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \\ f(t) &= \frac{1}{2e_1(t, T)e_2(t, T)} \Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} \right), \end{aligned} \quad (19)$$

where $e_1(t, T)$ is given in (11), $e_2(t, T)$ solves the Abel differential equation of the second kind (12), and $a(t)$, $b(t)$, $c(t)$ are given in (18).

Proof. Using Proposition 3.2 and noting that $\mathbf{1}'u^*(t) = x$, simple calculation yields that $\tilde{u}^*(t) = \tilde{I}u^*(t)$ and $u_1^*(t) = x - \tilde{\mathbf{1}}'\tilde{u}^*(t) = x - \tilde{\mathbf{1}}'\tilde{I}u^*(t) = \varepsilon_1' u^*(t)$. The result follows. $\square$

Remark: By (19), we immediately obtain the time-consistent GMV strategy by sending λ to zero:

$$u_{GMV}^*(t) = w(t)x = \left[\Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} \right) \left(\frac{1}{e_2(t, T)} - 1 \right) + \frac{\Sigma(t)^{-1}\mathbf{1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} \right] x.$$

The analytical results for time-consistent MV strategies, presented in Proposition 3.1 and Corollary 3.2, are complete when $1/e_2(t, T)$ is solved. In the next subsection, we analyze the Abel differential equation (12) under a general setting and provide a handy proxy of $1/e_2(t, T)$.

3.2. Analysis of the Abel Differential Equation

For the arbitrary coefficient functions $a(t)$ and $b(t)$, several constructions of the exact solution of the Abel differential equation are recently proposed in Panayotounakos and Zarpoutis (2011); Rostami (2015) and references therein. However, they require a number of functional transformations and the reliability of the constructed analytical solutions is questionable.

Alternatively, we apply the analytical expansion approach for the Abel differential equation. Since we only need $1/e_2(t, T)$ in our strategies (19), we provide an asymptotic solution of $1/e_2(t, T)$ up to an arbitrary order in the following proposition.

Proposition 3.3. Assume that $a(t)$ and $b(t)$ are infinitely differentiable in t and admit Taylor's expansions $a(t) = \sum_{i=0}^{\infty} a_i(t - T)^i$ and $b(t) = \sum_{i=0}^{\infty} b_i(t - T)^i$, respectively. For any fixed $M \in \mathbb{N}$,

$$\frac{1}{e_2(t, T)} = \sum_{m=0}^M r_m(t - T)^m + \mathcal{O}((t - T)^{M+1}), \quad (20)$$

where $r_0 = 1$ and

$$\begin{aligned} r_m &= \frac{1}{m} \left[\sum_{i=0}^{m-1} a_i r_{m-1-i} + \sum_{\substack{i,j,k; \\ i+j+k=m-1}} b_i r_j d_k \right] \text{ for } m \geq 1, \quad d_0 = d_1 = 0, \\ d_k &= \frac{2}{k} \sum_{i=1}^{k-1} (k-i) r_{k-i} r_i \text{ for } k \geq 2. \end{aligned}$$

Proposition 3.3 allows us to use the M th order approximation $R^{[M]}(t, T) := \sum_{m=0}^M r_m(t-T)^m$ as the proxy of $1/e_2(t, T)$, with any desired accuracy. As shown in our numerical studies, the third order approximation has already provided satisfactory accuracy. Here, we provide the explicit expressions of r_m , $m = 0, 1, 2, 3$:

$$\begin{aligned} r_0 &= 1, \quad r_1 = a_0 = a(T), \quad r_2 = \frac{a_0^2 + a_1}{2} = \frac{a^2(T) + a^{(1)}(T)}{2}, \\ r_3 &= \frac{1}{3} \left[\frac{1}{2} a_0(a_0^2 + 2a_0b_0 + 3a_1) + a_2 \right] = \frac{a^3(T) + 2a^2(T)b(T) + 3a(T)a^{(1)}(T) + a^{(2)}(T)}{6}. \end{aligned}$$

It can be seen that the parameter $b(t)$ takes effects on $1/e_2(t, T)$ only through the third and subsequent correction terms. From this perspective, we can infer that $1 - 1/e_2(t, T)$ is dominated by the magnitude of $a(t)$. We will conduct the asymptotic analysis for $e_2(t, T)$ in Section 4.4.

4. Properties of Time-Consistent Mean-Variance Strategies

The explicit form of time-consistent MV strategies (19) allows us to study their properties thoroughly. We also consider high-dimensional statistical analysis in Section 4.4.

4.1. Statistical Properties

In this subsection, we derive the statistical properties of the terminal wealth of time-consistent MV portfolios and identify a separation theorem for the time-consistent MV strategies.

We provide the explicit expressions of the optimal terminal wealth process and its first two moment statistics. The following proposition summarizes the results.

Proposition 4.1. The terminal wealth and its mean and variance of the time-consistent MV portfolio (19) are given by:

$$\begin{aligned} X^{\mathbf{u}^*}(T) &= \Phi(t, T)X^{\mathbf{u}^*}(t) + \lambda \int_t^T \Phi(\tau, T)f(\tau)'[\mu(\tau) - \Sigma(\tau)w(\tau)]d\tau \\ &\quad + \lambda \int_t^T \Phi(\tau, T)f(\tau)'\sigma(\tau)dB(\tau), \end{aligned} \tag{21}$$

$$\mathbb{E}_{t,x}[X^{\mathbf{u}^*}(T)] = xe_1(t, T) + \frac{\lambda}{2} \int_t^T \frac{b(\tau)}{e_2(\tau, T)}d\tau, \tag{22}$$

$$\text{Var}_{t,x}(X^{\mathbf{u}^*}(T)) = x^2e_1^2(t, T)(e_2(t, T) - 1) + \frac{\lambda^2}{4} \int_t^T \frac{b(\tau)}{e_2(\tau, T)}d\tau, \tag{23}$$

for $t \in [0, T]$, where a, b, c, e_1, e_2, w, f are the same as in Corollary 3.2 and

$$\Phi(t, T) = \exp \left\{ \int_t^T \left(w(\tau)' \mu(\tau) - \frac{w(\tau)' \Sigma(\tau) w(\tau)}{2} \right) d\tau + \int_t^T w(\tau)' \sigma(\tau) dB(\tau) \right\}.$$

Proof. (22) and (23) are from the proof of Proposition 3.1. Substituting (19) into the portfolio's wealth process yields a linear stochastic differential equation (SDE) for $\{X^{\mathbf{u}^*}(t)\}_{t \in [0, T]}$:

$$dX^{\mathbf{u}^*}(t) = [X^{\mathbf{u}^*}(t)w(t)'\mu(t) + \lambda f(t)'\mu(t)]dt + [X^{\mathbf{u}^*}(t)w(t)'\sigma(t) + \lambda f(t)'\sigma(t)]dB(t).$$

It is easy to solve for this linear SDE and obtain (21). $\square$

We now establish a separation theorem for the time-consistent MV strategies. Specifically, we prove that given two different time-consistent MV strategies, any time-consistent MV strategy can be duplicated by a convex combination of these two. The separation theorem is significant from a theoretical and empirical standpoint and it implies that two products (funds) can provide a complete investment service if one is interested in time-consistent MV strategy. This result is analogous to the famous two-fund separation theorem.

Before we prove the separation theorem, we first notice that the time-consistent MV strategy (19) is characterized by λ . To emphasize the λ dependence, we write $u^*(t) = u^*(t; \lambda)$.

Theorem 4.1. Let $u^*(t; \lambda_1)$ and $u^*(t; \lambda_2)$ be two different time-consistent MV strategies characterized by $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, respectively. For any $\kappa \in \mathbb{R}$ such that $\kappa\lambda_1 + (1 - \kappa)\lambda_2 \geq 0$, then $\kappa u^*(t; \lambda_1) + (1 - \kappa)u^*(t; \lambda_2)$ is also a time-consistent MV strategy. Moreover, for any $\lambda \geq 0$, there exists $\kappa \in \mathbb{R}$ such that $u^*(t; \lambda) = \kappa u^*(t; \lambda_1) + (1 - \kappa)u^*(t; \lambda_2)$.

Proof. One simple and important observation is that $u^*(t; \lambda)$ is linear in λ . Hence,

$$\kappa u^*(t; \lambda_1) + (1 - \kappa)u^*(t; \lambda_2) = w(t)x + (\kappa\lambda_1 + (1 - \kappa)\lambda_2)f(t) = u^*(t; \kappa\lambda_1 + (1 - \kappa)\lambda_2).$$

For any $\lambda > 0$, we can pick $\kappa = \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2}$ such that $\kappa\lambda_1 + (1 - \kappa)\lambda_2 = \lambda$ and by the equation above, we have $u^*(t; \lambda) = \kappa u^*(t; \lambda_1) + (1 - \kappa)u^*(t; \lambda_2)$. $\square$

Theorem 4.1 implies that investing in any two existing time-consistent MV portfolios with certain weights is equivalent to investing in all p assets, no matter what degree of the risk-tolerance of the investor is. It is practically meaningful as it can save the transaction cost.

4.2. Link with Pre-Commitment Strategies

In this subsection, we establish the link between time-consistent and pre-commitment MV strategies. The pre-commitment MV strategy $\{u_{pc}^*(t)\}_{t \in [0, T]}$ is determined by minimizing $\text{Var}(X^{\mathbf{u}}(T))$ subject to $\mathbb{E}[X^{\mathbf{u}}(T)] = z$, where z is the pre-specified reward level, or equivalently by maximizing the initial reward function $J(0, x, \mathbf{u})$ given in (4). It is shown in Zhou and Li (2000) that the pre-commitment MV problem can be embedded into an auxiliary linear quadratic problem:

$$\min_{\mathbf{u}} \mathbb{E}[(X^{\mathbf{u}}(T) - Y)^2], \quad (24)$$

where Y is interpreted as investment target. This auxiliary problem is also called target-based MV problem, which is solved in Yao et al. (2014) for the case with only risky assets. However, their solution depends on the ambiguous choice of “risk-free asset” and can be simplified with Proposition 3.2. We rewrite their results in terms of $\mu(t)$ and $\Sigma(t)$ in the following proposition while the proof is straightforward and hence omitted.

Corollary 4.1. The target-based MV problem (24) admits the optimal strategy

$$\begin{aligned} u_{tb}^*(t; Y) = & \left[\frac{\Sigma(t)^{-1} \mathbf{1}}{\mathbf{1}' \Sigma(t)^{-1} \mathbf{1}} - \Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}' \Sigma(t)^{-1} \mu(t)}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \right) \right] x \\ & + \Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}' \Sigma(t)^{-1} \mu(t)}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \right) \frac{L(t)Y}{K(t)}, \quad \text{for } t \in [0, T], \end{aligned} \quad (25)$$

where we recall $a(t), b(t), c(t)$ are given in (18) and

$$L(t) = \exp \left(\int_t^T (c(\tau) - b(\tau)) d\tau \right), \quad K(t) = \exp \left(\int_t^T (a(\tau) + 2c(\tau) - b(\tau)) d\tau \right).$$

Moreover, the pre-commitment MV strategy is given by $u_{pc}^*(t; z) = u_{tb}^*(t; Y^*)$, where

$$Y^* = \frac{z - X(0)L(0)}{M(0) - 1}, \quad M(t) = e^{-\int_t^T (a(\tau) + b(\tau)) d\tau} \left[1 + \int_t^T a(\tau) e^{\int_t^\tau (a(v) + b(v)) dv} d\tau \right].$$

From Corollary 4.1, we can rewrite the necessary and sufficient condition for GMV being zero in Yao et al. (2014) as $a(t) = 0$ for $t \in [0, T]$. In the case of $a(t) \equiv 0$, $e_2(t, T) \equiv 1$ and from (23), the time-consistent GMV is also zero. Hence, it is interesting to investigate under which situation $a(t) = 0$ or $e_2(t, T) = 1$. We analyze it in Section 4.4.

As in Cong and Oosterlee (2016), if the target-based strategies (25) coincide with the time-consistent strategies (19), we can deduce an implied investment target process Y_t^{imp} . Simple calculation yields that

$$Y_t^{imp} = \frac{K(t)}{L(t)e_2(t, T)} \left(x + \frac{\lambda}{2e_1(t, T)} \right). \quad (26)$$

Hence, the pre-commitment MV strategy can be interpreted as an investment with a fixed target Y^* while the time-consistent MV strategy is with a varying target Y_t^{imp} .

When the first asset is risk-free, analogous to Corollary 3.1, (26) is reduced to

$$Y_t^{imp} = xe^{\int_t^T r_f(\tau) d\tau} + \frac{\lambda}{2},$$

which extends the finding in Cong and Oosterlee (2016) to continuous-time setting.

The significance of the result (26) and the link between pre-commitment and time-consistent strategies is that it facilitates the numerical computation of time-consistent MV strategy with portfolio constraints and encourages a hybrid strategy that may perform better than pre-commitment and time-consistent strategies as in Cong and Oosterlee (2016). However, it is beyond the scope of this paper and remains as future research.

4.3. Link with Myopic Strategies

In this subsection, we establish the link between time-consistent and myopic MV strategies. Recall that a myopic investor determines her optimal strategy without taking into account the future optimality. Given the mean vector $\mu(t)$ and the covariance matrix $\Sigma(t)$ of the p assets returns at time t , the optimal myopic MV strategy $u_{my}^*(t)$ at time t maximizes the objective:

$$J_{my}(t, x; \lambda_{my}) = \lambda_{my} u(t)' \mu(t) - u(t)' \Sigma(t) u(t)$$

subject to $u(t)' \mathbf{1} = x$. The constrained maximization problem above can be easily solved by the Lagrangian method and $u_{my}^*(t)$ is given by

$$u_{my}^*(t; \lambda_{my}) = \frac{\Sigma(t)^{-1} \mathbf{1}}{\mathbf{1}' \Sigma(t)^{-1} \mathbf{1}} x + \frac{\lambda_{my}}{2} \Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}' \Sigma(t)^{-1} \mu(t)}{\mathbf{1}' \Sigma(t)^{-1} \mathbf{1}} \right). \quad (27)$$

Comparing (19) and (27), we can obtain a decomposition of $u^*(t)$:

$$u^*(t) = u_{hedge}^*(t) + u_{my}^*(t; \lambda_{my}^*)$$

with $\lambda_{my}^* = \lambda/e_1(t, T)$, where λ_{my}^* is interpreted as discounted risk tolerance and

$$u_{hedge}^*(t) = -\Sigma(t)^{-1} \left(\mu(t) - \mathbf{1} \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \right) \left(1 - \frac{1}{e_2(t, T)} \right) \left(x + \frac{\lambda}{2e_1(t, T)} \right). \quad (28)$$

We remark here that $\mathbf{1}'u_{hedge}^*(t) = 0$; thus u_{hedge}^* can be viewed as a self-financing portfolio with zero initial wealth. In general, $u_{hedge}^*(t)$ is non-zero and it captures the intertemporal hedging demands, which are significant when there is no risk-free asset no matter whether the market is complete or not. By (28), the intertemporal hedging demands are driven by a factor $1 - 1/e_2(t, T)$. If $e_2(t, T) = 1$, then $u_{hedge}^*(t)$ is a zero vector and thus the hedging demands are vanished and the time-consistent strategy is reduced to myopic strategy. Therefore, the value of $e_2(t, T)$ plays an important role in the distinction between time-consistent and myopic strategies.

Based on the discussion after Proposition 3.1, we reach a well-known result that when there is a risk-free asset ($e_2(t, T) = 1$), the time-consistent MV strategy is equivalent to the myopic MV strategy and in this case,

$$\tilde{u}^*(t) = \tilde{u}_{my}^*(t; \lambda_{my}^*) := \frac{\lambda_{my}^*}{2} \tilde{\Sigma}(t)^{-1} (\tilde{\mu}(t) - \tilde{\mathbf{1}}r_f(t)) \text{ with } \lambda_{my}^* = \lambda e^{-\int_t^T r_f(\tau) d\tau}.$$

Next, we shall turn to investigate a sufficient condition for $e_2(t, T) = 1$ from a statistical and economical perspective.

4.4. High-Dimensionality Effects on the Time-Consistent Strategies

The main finding in this subsection is that when there are estimation errors with the curse of dimensionality, the sample estimate of $a(t)/b(t)$ is close to zero, which will lead the estimate of $e_2(t, T)$ close to one for $t \in [0, T]$. As a result, the time-consistent MV strategy is asymptotically equivalent to the myopic MV strategy based on the discussion in Section 4.3. The proofs of the following Proposition 4.2 and Corollary 4.2 can be found in Appendix A.

To simplify matters, we consider the parametric functions $\mu(t) = \mu$ and $\Sigma(t) = \Sigma$ are constants and they are estimated by the sample mean vector $\hat{\mu}_n$ and the sample covariance matrix $\hat{\Sigma}_n$, computed with n samples of asset returns $\{s^{(1)}, \dots, s^{(n)}\}$, where $n > p$ and $s^{(i)} \in \mathbb{R}^p$ follows normal distribution with mean μ and covariance matrix Σ for $i = 1, \dots, n$. More specifically, we use

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n s^{(i)} \text{ and } \hat{\Sigma}_n = \frac{1}{n-1} \sum_{i=1}^n (s^{(i)} - \hat{\mu}_n)(s^{(i)} - \hat{\mu}_n)',$$

to estimate μ and Σ , respectively, $\hat{a} := 1/(\mathbf{1}'\hat{\Sigma}_n^{-1}\mathbf{1})$ to estimate $a(t) = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$, $\hat{\Theta} := \hat{\mu}_n'\hat{\Sigma}_n^{-1}\hat{\mu}_n$ to estimate $\Theta := \mu'\Sigma^{-1}\mu$, and $\hat{b} := \hat{\Theta} - \hat{a}(\mathbf{1}'\hat{\Sigma}_n^{-1}\hat{\mu}_n)^2$ to estimate $b(t)$.

We first examine how the magnitudes of $\hat{a}$ and $\hat{b}$ affect the solution of (12). The following proposition gives the bounds of $\hat{e}_2(t, T) - 1$, where $\hat{e}_2(t, T)$ satisfies (12) with $a(t)$ and $b(t)$ replaced by $\hat{a}$ and $\hat{b}$, respectively.

Proposition 4.2. Given that $\hat{a} \geq 0$, $\hat{b} \geq 0$, we have

$$0 \leq \hat{e}_2(t, T) - 1 \leq \sqrt{\frac{\hat{a}}{\hat{b}}} \exp\{-(\hat{a} + \hat{b})(T - t)\} \leq \sqrt{\frac{\hat{a}}{\hat{b}}}.$$

To further investigate the upper bound of $\hat{e}_2(t, T) - 1$, we derive the asymptotic properties of $\hat{a}$, $\hat{b}$, and $\hat{a}/\hat{b}$ under the high-dimensional setting that p is comparable to n , i.e. $\lim_{p, n \rightarrow \infty} p/n = \rho \in (0, 1)$. To this end, we introduce the following corollary of Theorems 4.1 and 4.4 in El Karoui (2010) for our application.

Corollary 4.2. Assume that $p/n \rightarrow \rho \in (0, 1)$ as p and n tend to infinity. Then we have

1. $\hat{a}/a \rightarrow 1 - \rho$ in probability.
2. Suppose that Θ stays bounded away from 0 and infinity and

$$\frac{\mathbf{1}'\Sigma^{-1}\mu}{(\mu + \mathbf{1})'\Sigma^{-1}(\mu + \mathbf{1})} \text{ and } \frac{\mathbf{1}'\Sigma^{-1}\mu}{(\mu - \mathbf{1})'\Sigma^{-1}(\mu - \mathbf{1})}$$

are bounded away from 0. Then

$$\hat{b} = \frac{1}{1 - \rho}b + \frac{\rho}{1 - \rho} + o_{\mathbb{P}}\left(1 \vee \frac{(\mathbf{1}'\Sigma^{-1}\mu)^2}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\right).$$

3. Under the same assumptions in the second claim, we have

$$\frac{\hat{a}}{\hat{b}} \bigg/ \frac{a}{b} \leq (1 - \rho)^2 \quad \text{in probability.}$$

Proposition 4.2 and Corollary 4.2 reveal that either $p/n \approx 1$ (high-dimensionality effect) or $a/b \approx 0$ leads to the estimate $\hat{a}/\hat{b}$ converging to zero in probability. Noting that investors may consider a large pool of stocks or assets worldwide, for example, $p > 500$, but the sample size (or the length of estimation window) n cannot be too large in order to preserve the stationarity of the

time-series data. Thus n is usually comparable to p . It is also economically meaningful that when p tends to infinity, $a = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1}) \rightarrow 0$. Noting that a is the variance of the single-period GMV portfolio. When there are infinite assets available for the purpose of hedging, the GMV portfolio is perfectly diversified and the resulting portfolio's risk tends to zero intuitively. Mathematically, we can see this with the assumption that the eigenvalues of Σ stay bounded away from zero and infinity. Let $\bar{\lambda}$ and $\underline{\lambda}$ be the largest and the smallest eigenvalues of Σ . Then we have

$$\bar{\lambda}^{-1}p \leq \mathbf{1}'\Sigma^{-1}\mathbf{1} \leq \underline{\lambda}^{-1}p \implies \mathbf{1}'\Sigma^{-1}\mathbf{1} = \mathcal{O}(p) \implies \frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} = \mathcal{O}\left(\frac{1}{p}\right).$$

Together with the findings in Section 4.3, we conclude that under the high-dimensional setting, the estimated time-consistent MV strategies are approximately equivalent to myopic MV strategies. This finding provides a new insight into the studies on high-dimensional portfolio selection.

5. Illustrations with Empirical Data

In this section, we illustrate our results with the parameters calibrated to six empirical datasets, including five low-dimensional datasets used in DeMiguel et al. (2009) and one high-dimensional dataset of S&P 500 index components (total 481 stocks with complete data) from January 3, 2012 to December 30, 2016 (five years daily data). The datasets can be found in Mendeley Data; see Pun (2017). The objectives of our empirical studies are to 1) examine the empirical estimates of a , b , and a/b and the accuracy of the asymptotic solution (20); and 2) evaluate the empirical performance of time-consistent and myopic GMV strategies, while the naive equally weighted strategy, which invests in all the assets with the same weights $1/p$, serves as a benchmark.

5.1. Empirical Estimates for Time-Consistent Strategies

We use a rolling-sample approach to obtain a series of estimates for time-consistent trading strategies. Specifically, given a dataset with sample size N , we choose an estimation window of length $n = 120$ months (for low-dimensional monthly data) or $n = 504$ days (for high-dimensional daily data). Then the parameters a and b needed to implement a trading strategy at time $t_i = n + i$, denoted by $\hat{a}_i$ and $\hat{b}_i$, are estimated using the previous n data, for $i = 1, \dots, N - n$. Hence, we

	Inter'l $p = 9$	S&P sectors $p = 11$	FF+1-factor $p = 21$	S&P Comp $p = 100$	S&P Comp $p = 300$	S&P Comp $p = 481$
$\hat{a}$	0.0171 (0.0012)	0.0111 (0.0017)	0.0100 (0.0021)	2.815e-04 (0.585e-04)	9.343e-05 (2.212e-05)	6.971e-06 (2.771e-06)
$\hat{b}$	0.9570 (0.3786)	1.8266 (0.2650)	5.3283 (2.4842)	1.7875 (0.2697)	8.4094 (1.5866)	97.641 (41.033)
$\hat{a}/\hat{b}$	0.0208 (0.0085)	0.0063 (0.0015)	0.0024 (0.0013)	1.632e-04 (0.518e-04)	1.173e-05 (4.381e-06)	8.510e-08 (5.293e-08)

Table 1: Means and standard deviations (in parentheses) of estimates of $\hat{a}$, $\hat{b}$, and $\hat{a}/\hat{b}$ for empirical data.

can obtain two time series of estimates: $\{\hat{a}_i\}_{i=1}^{N-n}$ and $\{\hat{b}_i\}_{i=1}^{N-n}$. Table 1 presents the means and standard deviations of estimates of $\hat{a}$, $\hat{b}$, and $\hat{a}/\hat{b}$ for empirical data with different dimensions.

In Table 1, “Inter'l” stands for dataset of 9 international portfolios, “S&P sectors” stands for dataset of 11 S&P sector portfolios, “FF+1-factor” stands for dataset of twenty size- and book-to-market portfolios with the US equity market portfolio, and “S&P Comp” stands for the high-dimensional daily dataset of S&P 500 index components, where we consider randomly chosen subsets of dimensions $p = 100, 300, 481$.

It can be seen that $\hat{a}$'s are generally small and are decreasing in p while $\hat{b}$'s are generally increasing in p , which is consistent with Corollary 4.2. As a result, we can see that $\hat{a}/\hat{b}$ tends to zero when p increases, and thus $\hat{e}_2(t, T)$ is close to one under the high-dimensional setting, based on the discussion in Section 4.4. As the intertemporal hedging demands in (28) are characterized by $1 - 1/\hat{e}_2(t, T)$, we conclude that the time-consistent MV strategy is asymptotically equivalent to the myopic MV strategy.

Using the empirical estimates of a and b in Table 1, we examine the accuracy of the M th order approximation $R^{[M]}(t, T) := \sum_{m=0}^M r_m(t - T)^m$ in (20) for $M = 0, 1, 2, 3$. Moreover, we set $T = 3$. As claimed before, $\hat{e}_2(t, T)$ is close to one (the zeroth-order approximation) under the high-dimensional setting. For the illustration purpose, we only use the parameters estimated from the low-dimensional datasets of “S&P sectors” and “FF+1-factor”.

Figure 1: The left panel uses the parameters $(a, b) = (0.0111, 1.8266)$ estimated from “S&P sectors”; the right panel uses the parameters $(a, b) = (0.01, 5.3283)$ estimated from “FF+1-factor”. In both plots, the blue dashed curve is $R(t)$, while $R^{[1]}(t)$, $R^{[2]}(t)$, and $R^{[3]}(t)$ are colored in red, green, and purple, respectively.

Figure 1 presents the $R(t)$ and $R^{[M]}(t)$, $M = 0, 1, 2, 3$ in the same plot for two empirical parameter sets. We can see that the third order approximation $R^{[3]}(t)$ has already provided a satisfactory accuracy. We remark here that if the high-dimensional dataset ($p = 481$) is used, then we find that $|\hat{e}_2(t, T) - 1| \leq 7.139\text{e-}04$ by Proposition 4.2.

5.2. Empirical Studies of Dynamic Trading Strategies

We adopt the same datasets and methodology in DeMiguel et al. (2009) to evaluate the out-of-sample performance of time-consistent and myopic GMV strategies. We do not consider MV strategies because DeMiguel et al. (2009) shows that the poor estimate of the mean will lead to the worst performance of sample-based MV strategies, which is still inevitable in the dynamic setting.

We set the investment horizon as three months and investor can rebalance her portfolio at the beginnings of first, second, and third months. The parameters needed to implement a dynamic trading strategy are estimated using the latest $n = 120$ months data prior to each investment. Using the rolling-sample approach as in DeMiguel et al. (2009) results in a series of $(N - n)/3$ monthly out-of-sample returns generated by each dynamic strategy. Then we compute the standard deviation of the time series of monthly out-of-sample returns, which quantifies the portfolio’s risk, as the performance measure because we adopted GMV criterion for the strategies in comparison.

In Table 2, “EW” stands for equally weighted strategy, “MY-GMV” stands for myopic GMV

Strategy	Inter'l $p = 9$	S&P sectors $p = 11$	Industry $p = 11$	FF+1-factor $p = 21$	FF+3-factor $p = 23$
EW	6.7717	5.5385	6.0059	8.2191	7.6361
MY-GMV	6.4895	5.2556	5.0777	5.7760	1.2788
TC-GMV	6.4837	5.2506	5.0824	5.7749	1.2784

Table 2: Portfolio risks (in percentage) for empirical data.

strategy, “TC-GMV” stands for time-consistent GMV strategy, “Industry” stands for dataset of 11 industry portfolios, “FF+3-factor” stands for dataset “FF+1-factor” plus SMB and HML portfolios proposed in Fama and French (1993). All the numbers in Table 2 are presented in percentage.

Although time-consistent GMV strategies are subject to the estimation error of the mean, they generally yield lower portfolio risks compared to myopic GMV strategies (only for the industry portfolios, MY-GMV has lower risk). Equally weighted strategies have higher risk than dynamic GMV strategies in all datasets. We can also observe the trend that when the number of assets p increases, the difference between time-consistent and myopic GMV strategies decreases, which is consistent with our theory. Therefore, we conclude that if the number of assets to be invested is small, time-consistent GMV strategy is the best trading strategy in practice, but if the number of assets is large, it may not be worth considering intertemporal hedging demands and myopic GMV strategy performs as good as time-consistent GMV strategy.

6. Conclusion

The literature of modern portfolio theory usually assumes a risk-free asset exists in the market. This paper investigates a general market, where it can be incomplete and the assets can be all risky. Under this setting, we solve the time-consistent MV problems using the Abel differential equation approach. As its use is new to the literature of quantitative finance, we perform mathematical analysis on the Abel differential equation with time-varying coefficients, including analytical expansions (Proposition 3.3) and error analysis (Proposition 4.2). That enables us to

study the properties of time-consistent MV strategy and its links with myopic and pre-commitment MV strategies.

This paper also contributes to the empirical aspects of economic control. First, we study the effects of estimation errors on the time-consistent MV strategies, especially when the number of assets p is large. Second, we conduct empirical studies to examine the time-consistent and myopic GMV strategies. We theoretically and empirically show that time-consistent GMV strategies are superior to myopic GMV strategies when p is small, but the superiority is diminished when p is large. It reveals that the intertemporal hedging demands are not significant in high dimensions.

Future research may consider more sophisticated asset price models, such as regime-switching model as in Zhang et al. (2017), or introduce robustness into the time-consistent GMV problems as in Fouque et al. (2016) and examine whether the robustness effects differ from that on myopic GMV strategies.

A. Proofs

Proof of Proposition 3.1. We first rewrite the equation (6) as follows.

$$\begin{aligned} & \partial_t V + x\mu_1(t)\partial_x V + \frac{\partial_{xx}V - 2(\partial_x g)^2}{2}x^2\varsigma_1(t) \\ & + \sup_{\mathbf{u} \in \mathcal{U}} \left\{ \tilde{u}(t)' \beta(t) \partial_x V + \frac{\partial_{xx}V - 2(\partial_x g)^2}{2} [2x\tilde{u}(t)' \phi(t) + \tilde{u}(t)' \Gamma(t) \tilde{u}(t)] \right\} = 0, \end{aligned} \quad (29)$$

where $\partial_{xx}V = \partial^2 V / \partial x^2$. Maximizing the quadratic form of $\tilde{u}(t)$ in the above supremum problem yields the representation of the optimal trading strategy $\tilde{u}^*(t)$:

$$\tilde{u}^*(t) = -\Gamma(t)^{-1} \left(\beta(t) \frac{\partial_x V}{\partial_{xx}V - 2(\partial_x g)^2} + \phi x \right). \quad (30)$$

Hence, we take an ansatz of $\tilde{u}^*(t)$ of the form $\tilde{u}^*(t) = \tilde{w}(t)x + \tilde{f}(t)$, where $\tilde{w}(t), \tilde{f}(t) \in \mathbb{R}^{p-1}$ are functions of only t which are to be determined.

Let $m_{t,x}(s) := \mathbb{E}_{t,x}[X^{\mathbf{u}^*}(s)]$, $n_{t,x}(s) := \mathbb{E}_{t,x}[(X^{\mathbf{u}^*}(s))^2]$ be the first two moments of the equilibrium wealth process, respectively. For fixed $(t, x) \in [0, T] \times \mathcal{X}$ and $s \in [t, T]$, by (2), Itô's lemma, and taking conditional expectation $\mathbb{E}_{t,x}[\cdot]$ on the both sides of SDEs driving $X^{\mathbf{u}^*}(s)$ and

$(X^{\mathbf{u}^*}(s))^2$, we can obtain the ordinary differential equations for $m_{t,x}(s)$ and $n_{t,x}(s)$ as follows:

$$\begin{aligned} dm_{t,x}(s)/ds &= m_{t,x}(s)[\mu_1(s) + \tilde{w}(s)'\beta(s)] + \tilde{f}(s)'\beta(s), \\ dn_{t,x}(s)/ds &= n_{t,x}(s)[\varsigma_1(s) + 2\tilde{w}(s)'\phi(s) + \tilde{w}(s)'\Gamma(s)\tilde{w}(s) + 2(\mu_1(s) + \tilde{w}(s)'\beta(s))] \\ &\quad + 2m_{t,x}(s)\tilde{f}(s)'[\beta(s) + \phi(s) + \Gamma(s)\tilde{w}(s)] + \tilde{f}(s)'\Gamma(s)\tilde{f}(s). \end{aligned}$$

Hence, let

$$e_1(t, s) = \exp \left\{ \int_t^s (\mu_1(\tau) + \tilde{w}(\tau)'\beta(\tau)) d\tau \right\}, \quad (31)$$

$$e_2(t, s) = \exp \left\{ \int_t^s [\varsigma_1(\tau) + 2\tilde{w}(\tau)'\phi(\tau) + \tilde{w}(\tau)'\Gamma(\tau)\tilde{w}(\tau)] d\tau \right\}. \quad (32)$$

We have

$$\begin{aligned} m_{t,x}(s) &= xe_1(t, s) + \int_t^s e_1(\tau, s)\tilde{f}(\tau)'\beta(\tau) d\tau, \\ n_{t,x}(s) &= x^2e_1^2(t, s)e_2(t, s) + 2x \int_t^s e_1^2(\tau, s)e_2(\tau, s)e_1(t, \tau)\tilde{f}(\tau)'[\beta(\tau) + \phi(\tau) + \Gamma(\tau)\tilde{w}(\tau)] d\tau \\ &\quad + 2 \int_t^s e_1^2(\tau, s)e_2(\tau, s)\tilde{f}(\tau)'[\beta(\tau) + \phi(\tau) + \Gamma(\tau)\tilde{w}(\tau)] \int_t^\tau e_1(v, \tau)\tilde{f}(v)'\beta(v) dv d\tau \\ &\quad + \int_t^s e_1^2(\tau, s)e_2(\tau, s)\tilde{f}(\tau)'\Gamma(\tau)\tilde{f}(\tau) d\tau. \end{aligned}$$

Denote by $h(t, x) = \text{Var}_{t,x}(X^{\mathbf{u}^*}(T))$ the terminal wealth's variance anticipated at time t . Then we have

$$\begin{aligned} g(t, x) &= m_{t,x}(T) = xe_1(t, T) + \int_t^T e_1(\tau, T)\tilde{f}(\tau)'\beta(\tau) d\tau, \\ h(t, x) &= n_{t,x}(T) - (m_{t,x}(T))^2 \\ &= x^2e_1^2(t, T)(e_2(t, T) - 1) - x \left[2e_1(t, T) \int_t^T e_1(\tau, T)\tilde{f}(\tau)'\beta(\tau) d\tau \right. \\ &\quad \left. - 2 \int_t^T e_1^2(\tau, T)e_2(\tau, T)e_1(t, \tau)\tilde{f}(\tau)'[\beta(\tau) + \phi(\tau) + \Gamma(\tau)\tilde{w}(\tau)] d\tau \right] \\ &\quad + 2 \int_t^T e_1^2(\tau, T)e_2(\tau, T)\tilde{f}(\tau)'[\beta(\tau) + \phi(\tau) + \Gamma(\tau)\tilde{w}(\tau)] \int_t^\tau e_1(v, \tau)\tilde{f}(v)'\beta(v) dv d\tau \\ &\quad + \int_t^T e_1^2(\tau, T)e_2(\tau, T)\tilde{f}(\tau)'\Gamma(\tau)\tilde{f}(\tau) d\tau - \left(\int_t^T e_1(\tau, T)\tilde{f}(\tau)'\beta(\tau) d\tau \right)^2. \end{aligned} \quad (33)$$

It can be seen that $V(t, x) = \lambda g(t, x) - h(t, x)$ is a quadratic function of x and it admits the form (8), where $k_0(t, T)$ is derived below, $k_2(t, T) = e_1^2(t, T)(1 - e_2(t, T))$, and

$$k_1(t, T) = \lambda e_1(t, T) + 2e_1(t, T) \int_t^T e_1(\tau, T)\tilde{f}(\tau)'\beta(\tau) d\tau$$

$$-2 \int_t^T e_1^2(\tau, T) e_2(\tau, T) e_1(t, \tau) \tilde{f}(\tau)' [\beta(\tau) + \phi(\tau) + \Gamma(\tau) \tilde{w}(\tau)] d\tau. \quad (34)$$

Hence, by (30), we have

$$\tilde{u}^*(t) = \Gamma(t)^{-1} \left(\frac{\beta(t)}{e_2(t, T)} - \beta(t) - \phi(t) \right) x + \Gamma(t)^{-1} \beta(t) \frac{k_1(t, T)}{2e_1^2(t, T) e_2(t, T)}.$$

Therefore,

$$\tilde{w}(t) = \Gamma(t)^{-1} \left(\frac{\beta(t)}{e_2(t, T)} - \beta(t) - \phi(t) \right), \quad \tilde{f}(t) = \Gamma(t)^{-1} \beta(t) \frac{k_1(t, T)}{2e_1^2(t, T) e_2(t, T)}. \quad (35)$$

Substituting (35) into (34), we have $k_1(t, T) = \lambda e_1(t, T)$ and thus

$$\tilde{f}(t) = \Gamma(t)^{-1} \beta(t) \frac{\lambda}{2e_1(t, T) e_2(t, T)}.$$

Moreover, substituting (8), (33), and (35) into (29), we have

$$\begin{aligned} x^2 \left(\partial_t k_2 + 2k_2(\mu_1 - \phi' \Gamma^{-1} \beta) + (k_1 - e_1^2)(\varsigma_1 - \phi' \Gamma^{-1} \phi) - \frac{k_2^2}{k_2 - e_1^2} \beta' \Gamma^{-1} \beta \right) \\ + x \left(\partial_t k_1 + k_1(\mu_1 - \phi' \Gamma^{-1} \beta) - \frac{k_1 k_2}{k_2 - e_1^2} \beta' \Gamma^{-1} \beta \right) \\ + \partial_t k_0 - \frac{k_1^2}{4(k_2 - e_1^2)} \beta' \Gamma^{-1} \beta = 0, \end{aligned} \quad (36)$$

where we suppressed the arguments t and (t, T) to ease the notational burden. Noting that $k_2 = e_1^2(1 - e_2)$, $k_1 = \lambda e_1$, $\partial_t k_2 = 2e_1(1 - e_2)(\partial_t e_1) - e_1^2(\partial_t e_2)$, (36) yields that

$$0 = \partial_t e_1(t, T) + e_1(t, T) c(t) + \frac{e_1(t, T)(1 - e_2(t, T))}{e_2(t, T)} b(t), \quad (37)$$

$$\begin{aligned} e_2(t, T)(\partial_t e_2(t, T)) &= -e_2^2(t, T) a(t) - (1 - e_2(t, T))^2 b(t), \\ 0 &= \partial_t k_0(t, T) + \frac{\lambda^2}{4} \frac{b(t)}{e_2(t, T)} \end{aligned} \quad (38)$$

with $e_1(T, T) = 1$, $e_2(T, T) = 1$, and $k_0(T, T) = 0$, where $a(t)$, $b(t)$, and $c(t)$ are given in this proposition. Substituting $\tilde{w}$ in (35) into (32) gives the integral equation for $e_2(t, T)$ (14). Solving (37) yields the expression of e_1 :

$$e_1(t, T) = \exp \left\{ \int_t^T \left[c(\tau) + \left(\frac{1}{e_2(\tau, T)} - 1 \right) b(\tau) \right] d\tau \right\}.$$

By (38), we have

$$k_0(t, T) = \frac{\lambda^2}{4} \int_t^T \frac{b(\tau)}{e_2(\tau, T)} d\tau.$$

Then the results follow. □

Proof of Proposition 3.2. We introduce some notations: $\varepsilon_1 = (1, 0, \dots, 0)' \in \mathbb{R}^p$, $\tilde{E} = \tilde{I} - \tilde{\mathbf{1}}\varepsilon_1' \in \mathbb{R}^{(p-1) \times p}$. Then we can rewrite the parameters as follows:

$$\Gamma(t) = \tilde{E}\Sigma(t)\tilde{E}', \quad \beta(t) = \tilde{E}\mu(t), \quad \phi = \tilde{E}\Sigma(t)\varepsilon_1, \quad \varsigma_1(t) = \varepsilon_1'\Sigma(t)\varepsilon_1, \quad \mu_1(t) = \varepsilon_1'\mu(t).$$

Noting that $\tilde{I}'\tilde{I} = I_p - \varepsilon_1\varepsilon_1'$, $\tilde{I}'\tilde{\mathbf{1}} = \mathbf{1} - \varepsilon_1$, $\tilde{E}'\tilde{I} = I_p - \varepsilon_1\mathbf{1}'$, $\tilde{E}\mathbf{1} = 0_{p-1}$. Hence, we have

$$\begin{aligned} \Gamma(t)\tilde{I}\Sigma(t)^{-1}\mathbf{1} &= \tilde{E}\Sigma(t)(I_p - \varepsilon_1\mathbf{1}')\Sigma(t)^{-1}\mathbf{1} = -\phi(t)\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}, \\ \Gamma(t)\tilde{I}\Sigma(t)^{-1}\mu(t) &= \tilde{E}\Sigma(t)(I_p - \varepsilon_1\mathbf{1}')\Sigma(t)^{-1}\mu(t) = \beta(t) - \phi(t)\mathbf{1}'\Sigma(t)^{-1}\mu(t), \end{aligned}$$

which yield the two equalities in (17). Using (17), we obtain

$$\begin{aligned} a(t) &= \varepsilon_1'\Sigma(t)\varepsilon_1 + \varepsilon_1'\Sigma(t)\tilde{E}'\tilde{I}\frac{\Sigma(t)^{-1}\mathbf{1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} = \frac{\varepsilon_1'\mathbf{1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}, \\ b(t) &= \mu(t)'\tilde{E}'\tilde{I}\left(\Sigma(t)^{-1} - \frac{\Sigma(t)^{-1}\mathbf{1}\mathbf{1}'\Sigma(t)^{-1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}\right)\mu(t) = \mu(t)'\left(\Sigma(t)^{-1} - \frac{\Sigma(t)^{-1}\mathbf{1}\mathbf{1}'\Sigma(t)^{-1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}\right)\mu(t), \\ c(t) &= \varepsilon_1'\mu(t) + \frac{\mathbf{1}'\Sigma(t)^{-1}}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}\tilde{I}'\tilde{E}\mu(t) = \frac{\mathbf{1}'\Sigma(t)^{-1}\mu(t)}{\mathbf{1}'\Sigma(t)^{-1}\mathbf{1}}. \end{aligned}$$

□

Proof of Proposition 3.3. Let $R(t) := 1/e_2(t, T)$ for a fixed T . By (12), we obtain the Abel differential equation of the first kind for $R(t)$:

$$\frac{dR(t)}{dt} = a(t)R(t) + b(t)R(t)(R(t) - 1)^2, \quad R(T) = 1. \quad (39)$$

By noting that $dR(t)/dt = F(R(t))$ with a smooth function $F(\cdot)$, $R(t)$ is infinitely differentiable at $t = T$. Using the Taylor expansion, we immediately have the expression (20) with $r_m = R^{(m)}(T)/m!$, where $R^{(m)}(T)$ denotes the m th derivative of R evaluated at $t = T$. Noting that $a_i = a^{(i)}(T)/i!$ and $b_i = b^{(i)}(T)/i!$. For $m \geq 1$, differentiating (39) m times yields

$$R^{(m+1)}(t) = \sum_{i=0}^m \frac{m!}{i!(m-i)!} a^{(i)}(t) R^{(m-i)}(t) + \sum_{\substack{i,j,k; \\ i+j+k=m}} \frac{m!}{i!j!k!} b^{(i)}(t) R^{(j)}(t) [(R(t) - 1)^2]^{(k)}, \quad (40)$$

$$\begin{aligned} [(R(t) - 1)^2]^{(1)} &= 2R^{(1)}(t)(R(t) - 1), \\ [(R(t) - 1)^2]^{(k+1)} &= 2 \left[R^{(k+1)}(t)(R(t) - 1) + \sum_{i=1}^k \frac{k!}{i!(k-i)!} R^{(k-i+1)}(t) R^{(i)}(t) \right], \end{aligned}$$

which lead to the recursive relations of $\{r_m\}_{m \geq 0}$ and $\{d_k\}_{k \geq 0}$, where $d_k = [(R(t) - 1)^2]^{(k)}|_{t=T}/k!$. $\square$

Proof of Proposition 4.2. When $a(t)$ and $b(t)$ are both estimated by constants $\hat{a}$ and $\hat{b}$, $\hat{e}_2 := \hat{e}_2(t, T)$ for a fixed $t \in [0, T]$ is solved from an implicit equation: $\hat{l}(\hat{e}_2) = \hat{l}(1) - (T - t)$, where $\hat{l}(\cdot)$ is given by $l(\cdot)$ in (15) with a and b replaced by $\hat{a}$ and $\hat{b}$ respectively. Hence, we have

$$\frac{2}{\hat{\eta}} \arctan \left(\left(\hat{\eta} + \frac{1}{\hat{\eta}} \right) (\hat{e}_2 - 1) + \hat{\eta} \right) + \log \left(\hat{a} \hat{e}_2^2 + \hat{b} (\hat{e}_2 - 1)^2 \right) = \frac{2}{\hat{\eta}} \arctan (\hat{\eta}) + \log (\hat{a}) - 2(\hat{a} + \hat{b})(T - t),$$

where $\hat{\eta} = \sqrt{\hat{a}/\hat{b}}$. Noting that $\hat{a} \geq 0$, $\hat{b} \geq 0$, $\hat{\eta} \geq 0$, and $\hat{e}_2 \geq 1$, we have

$$\begin{aligned} & \log(\hat{b}(\hat{e}_2 - 1)^2) \\ & \leq \log(\hat{a} \hat{e}_2^2 + \hat{b} (\hat{e}_2 - 1)^2) \\ & = \frac{2}{\hat{\eta}} \left[\arctan(\hat{\eta}) - \arctan \left(\left(\hat{\eta} + \frac{1}{\hat{\eta}} \right) (\hat{e}_2 - 1) + \hat{\eta} \right) \right] + \log(\hat{a}) - 2(\hat{a} + \hat{b})(T - t) \\ & \leq \log(\hat{a}) - 2(\hat{a} + \hat{b})(T - t). \end{aligned}$$

Hence, $\hat{e}_2(t, T) - 1 \leq e^{-(\hat{a} + \hat{b})(T - t)} \sqrt{\hat{a}/\hat{b}} \leq \sqrt{\hat{a}/\hat{b}}$. $\square$

Proof of Corollary 4.2. By Theorem 4.1 in El Karoui (2010) and continuous mapping theorem, we immediately have the first claim. Under the assumptions in the second claim, by Theorem 4.4 in El Karoui (2010), we have

$$\hat{\Theta} = \frac{1}{1 - \rho} \Theta + \frac{\rho}{1 - \rho} + o_{\mathbb{P}}(1).$$

Similarly to the proof of Theorem 4.4, we have

$$\begin{aligned} \frac{(\mathbf{1}' \hat{\Sigma}_n^{-1} \hat{\mu}_n)^2}{\mathbf{1}' \hat{\Sigma}_n^{-1} \mathbf{1}} &= \frac{(\mathbf{1}' \hat{\Sigma}_n^{-1} \hat{\mu}_n)^2}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \frac{\mathbf{1}' \Sigma^{-1} \mathbf{1}}{\mathbf{1}' \hat{\Sigma}_n^{-1} \mathbf{1}} \\ &= \left[\frac{1}{(1 - \rho)^2} \frac{(\mathbf{1}' \Sigma^{-1} \mu)^2}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} + o_{\mathbb{P}} \left(1 \vee \frac{(\mathbf{1}' \Sigma^{-1} \mu)^2}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \right) \right] [1 - \rho + o_{\mathbb{P}}(1)] \\ &= \frac{1}{1 - \rho} \frac{(\mathbf{1}' \Sigma^{-1} \mu)^2}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} + o_{\mathbb{P}} \left(1 \vee \frac{(\mathbf{1}' \Sigma^{-1} \mu)^2}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \right). \end{aligned}$$

Hence, we proved the second claim. The third claim is straightforwardly concluded from the first two claims. $\square$

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