Generalized GNSS Signal Carrier Tracking: Part I—Modeling and Analysis

Yang, Rong; Ling, Keck Voon; Poh, Eng-Kee; Morton, Yu

2017

Yang, R., Ling, K. V., Poh, E.-K., & Morton, Y. (2017). Generalized GNSS Signal Carrier Tracking: Part I—Modeling and Analysis. IEEE Transactions on Aerospace and Electronic Systems, 53(4), 1781-1797.

<https://hdl.handle.net/10356/85560>

<https://doi.org/10.1109/TAES.2017.2673998>

© 2017 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works. The published version is available at: [<http://dx.doi.org/10.1109/TAES.2017.2673998>].

Downloaded on 26 Apr 2023 18:23:46 SGT

Generalized GNSS Signal Carrier Tracking: Part I - Modeling and Analysis

Rong Yang, Keck-Voon Ling, Eng-Kee Poh, and Yu Morton

Abstract—This paper presents a generalized theoretical framework for carrier tracking of Global Navigation Satellite Systems signals by constructing a state space representation for the carrier tracking loop and applying control system design techniques to derive state feedback and state estimator gain matrices. Both phase locked loop and frequency locked loop are studied using this approach. Their performances are evaluated using the closed-form expressions in the presence of thermal noise, oscillator noise, and receiver platform dynamics. The proposed approach unified the phase tracking and frequency tracking within the same theoretical framework, thereby facilitating a systematic analysis of the effects of tracking loop design parameters under weak signal and highly dynamic signal environments.

Index Terms—GNSS receiver, state feedback control, phase locked loop, frequency locked loop.

I. INTRODUCTION

As the most fragile component of a Global Navigation Satellite Systems (GNSS) receiver, the carrier tracking loop has challenges when operates under challenging and stressful conditions, where the signal can experience deep fading, blockage, or high platform dynamics [1]. Optimization of carrier tracking loop holds the key to robust and accurate receiver signal processing under these challenging conditions, i.e., the phase lock loop (PLL) and frequency lock loop (FLL). This paper will shed new insight into the optimization of carrier tracking, facilitate the systematic analysis of tracking loop design parameters impact on receiver performance, and enable development of effective adaptive carrier tracking algorithms under challenging conditions.

When reviewing the traditional PLL design, it has been found that three filter design methods, i.e., proportional integral filter (PIF), Wiener filter (WF), and Kalman filter (KF), are most frequently used and investigated. The PIF design follows the transformation of an appropriate continuous-time (s -domain) analog filter to its corresponding z -domain representation of a discrete-time digital filter [2] [3]. In [4], the PIF-based digital PLL was presented, and the tracking error in the presence of thermal noise, oscillator noise, and dynamic stress was analyzed. A major limitation of the PIF-based design is that the s -domain to z -domain transformation is only valid when the product of the noise equivalent bandwidth and integration time is less than 0.5. To overcome this limitation, digital PLL, which is directly

designed in the z -domain based on the discrete-time input carrier phase, was proposed in [5]–[7]. The well-known, controlled-root method to determine the filter parameters in digital PLL was first proposed in [8] and subsequently employed in [5] and [6] for GNSS tracking loop design. The stability ranges of the discrete-time filter parameters and the corresponding tracking error variance were provided in [7]. However, reference [7] did not consider the oscillator noise effect, which could not be neglected under weak signal conditions. In [9]–[11], the PIF-based PLL was transformed into a state space tracking loop architecture for comparison with the KF-based tracking loop. However, the theoretical tracking accuracy of the resulting state space tracking loop architecture was not investigated.

The WF-based approach is based on the minimum mean square error (MMSE) criteria [12]. It separates the input signal from noise in the frequency domain and is known to have better tracking accuracy than the PIF design. Hence, it has been widely used in GNSS tracking loop applications [13]–[15]. References [13] [14] proposed a WF that tracked carrier signals with thermal and oscillator noises under static conditions, while in [15], a WF separated the carrier phase dynamics from thermal noise without including the oscillator effects.

Being the optimal filter in white Gaussian noise environments, KF is widely adopted in carrier tracking loop implementations [9]–[11] [16]–[18]. The signal models used in the KF technique consist of a state space model and a measurement model. The state space model represents the signal of a dynamic process driven by system noise, whereas the measurement model depicts measurements corrupted by thermal noise. There are two typical state space models, namely the error state model and the direct state model, which have both been used in KF-based PLL design [11]. PLLs based on these two models are equivalent and share the same architecture with the PIF-based PLL [9] [11]. By using these models, KF is able to provide the optimal estimates of an input signal corrupted by a white Gaussian noise.

Various efforts have been made to improve the carrier phase tracking loop performance with these filter designs in weak signal environments [4] [13] [16] [19] or for receivers on highly dynamic platforms [20]–[22], or in environments where both weak signals and high dynamics co-exist [23]. However, cycle-slips and potential loss of phase lock frequently occur in PLL due to its sensitivity in phase measurement. To address this issue, a FLL is often employed for the tracking of carrier signals. A FLL ignores absolute phase error and tracks relative phase rotations between the received signal and the local carrier replica [2] [24] [25]. Compared to a PLL, a FLL therefore can tolerate larger, sustained frequency errors and has more robust tracking

R. Yang, K. V. Ling, and E. K. Poh are with the Department of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.
E-mail: {YANG0356, EKVLING, EEKPOH}@ntu.edu.sg

Y. Morton is with Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO, USA.
E-mail: Jade.Morton@ColoState.Edu

performance in high dynamic environments.

The FLL design has often been discussed in the research literature as well. The conventional method that converts the analog FLL to the digital FLL to obtain the PIF parameters [2] is analogous to that of a traditional PLL design. Subsequently, KF is implemented for frequency tracking using measurements from the frequency discriminator output [10]. However, the measurement noise in FLL is not white [25]. The non-white noise-induced effects can have a significant impact on the overall FLL performance. Considering this issue, the controlled-root method [8] is applied in [25] to design the loop filter in the z -domain directly. However, the frequency error induced by the oscillator noise has not been considered in FLL design.

Intuitively, a higher fidelity mathematical model of the tracking loop should lead to better tracking loop design, and a model-based design has the potential to achieve better performance if the model is accurate. In this paper, we cast the design problem in the state space framework to leverage the state space design control techniques to provide analytic guidance to optimize PLL and FLL tracking loop parameters. Specifically, we take the state space approach and view the design of PLL and FLL from a state feedback and state estimator control system design perspective. This generalized tracking loop architecture unifies the PIF, WF, and KF design methods in PLL and also unifies PLL and FLL, enables in-depth comparative analysis of phase and frequency tracking and advanced design.

The main contribution of this paper is the analytical derivation of PLL and FLL tracking performances in the presence of thermal noise, oscillator effects, and receiver platform dynamics. We derived the closed-form solution of a third order WF gain matrix and verified its equivalence between WF gain and steady-state KF gain in PLL. These enable optimized design for GNSS phase and frequency tracking under challenging conditions. The details of design optimization and implementation will be presented in a companion paper [26].

The organization of the paper is as follows. In Section II, the general state space design process in the control system is provided. In Section III, the state space design is applied for PLL design. In Section IV, a similar design process is applied in FLL design. Section V provides the derivations of tracking accuracy, including carrier phase and frequency tracking error variance and dynamic stress error in the presence of thermal noise, oscillator effects, and receiver platform dynamics. Finally, Section VI concludes this paper.

II. GENERAL STATE SPACE DESIGN PROCESS IN CONTROL SYSTEM

Given a system model, the design of a control system involves the design of a controller $\mathbf{u}_k$ to control the plant to generate the required signals. It includes the following steps: i). Modeling a plant; ii). Designing and analyzing a controller for the plant; iii). Designing the state estimator.

The plant model for a discrete time system can be represented as:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{n}_k \quad (1)$$

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k \quad (2)$$

where

$\mathbf{x}_k \in \mathbb{R}^{n \times 1}$ is the state vector

$\mathbf{z}_k \in \mathbb{R}^{m \times 1}$ is the measurement vector

$\mathbf{n}_k \in \mathbb{R}^{n \times 1}$ is the system noise vector

$\mathbf{v}_k \in \mathbb{R}^{m \times 1}$ is the measurement noise vector

$\mathbf{A} \in \mathbb{R}^{n \times n}$ is the system transition matrix

$\mathbf{H} \in \mathbb{R}^{m \times n}$ is the measurement matrix

and $\mathbf{B}$ is an operator that maps the controller $\mathbf{u}_k$ to the plant. The matrix $\mathbf{B}$ should be chosen to ensure the pair $(\mathbf{A}, \mathbf{B})$ controllable; thus, the system state can transition from an arbitrary initial state to any desired state [27].

To design the controller $\mathbf{u}_k$, feedback strategy can be adopted and $\mathbf{u}_k$ is typically chosen as:

$$\mathbf{u}_k = \mathbf{K}\mathbf{x}_k \quad (3)$$

where $\mathbf{K}$ is the state feedback gain matrix. To ensure the stability of the control system, $\mathbf{K}$ should be designed to make eigenvalues of $(\mathbf{A} + \mathbf{B}\mathbf{K})$ less than 1. The pole placement approach can be adopted to make the eigenvalues of $(\mathbf{A} + \mathbf{B}\mathbf{K})$ at the desired locations, such as the origin in a typical control system design [27].

Finally, the state feedback is accomplished by using the actual state variables if the state variables are measurable. However, $\mathbf{x}_k$ may not be available in reality. In this case, we construct a state estimator (e.g. KF) as follows:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}(\mathbf{z}_{k+1} - \hat{\mathbf{z}}_{k+1}) \quad (4)$$

where

$$\hat{\mathbf{z}}_{k+1} = \mathbf{H}\hat{\mathbf{x}}_{k+1} \quad (5)$$

$$\mathbf{u}_k = \mathbf{K}\hat{\mathbf{x}}_k. \quad (6)$$

In this case, the estimated state variables $\hat{\mathbf{x}}_k$, instead of the actual state variables $\mathbf{x}_k$, are used to control the plant. It is noted that besides KF, many other filter techniques and estimation approaches, such as moving horizon estimation (MHE) [28], could be used to design the state estimator.

In summary, the state feedback control system design is characterized by three matrices, i.e., the plant input matrix $\mathbf{B}$, the state feedback gain matrix $\mathbf{K}$, and the state estimator gain matrix $\mathbf{L}$. The selection of matrix $\mathbf{B}$ is to ensure controllability, which is a necessary condition for placing the closed-loop pole locations [27]. The procedure for determining the state feedback gain matrix $\mathbf{K}$ is to select suitable locations for all closed-loop poles so that the effects of the disturbances can be minimized with a sufficient speed [27]. The design of state estimator gain matrix $\mathbf{L}$ is to extract the accurate state information from a noisy measurement for state feedback control. The overall system performance is determined by the choice of these three matrices. Hence, the design of the control system involves the design of $\mathbf{B}$, $\mathbf{K}$, and $\mathbf{L}$, and this design process can be applied for PLL and FLL design in GNSS receivers.

III. STATE SPACE DESIGN AND ANALYSIS FOR PLL

In this section, the general state space design process described in Section II is applied for PLL design and analysis. The signal models comprising state and measurement models for general dynamic scenarios are reviewed first, laying the foundation for a state feedback and state estimator design approach for GNSS carrier phase tracking. The PIF, WF, and KF are unified within this theoretical framework.

A. System Model

The following state space system model:

$$\mathbf{x}_{k+1} = \mathbf{A}_P \mathbf{x}_k + \mathbf{n}_k \quad (7)$$

$$\mathbf{z}_k = \mathbf{H}_P \mathbf{x}_k + \mathbf{v}_k \quad (8)$$

is widely applied for GNSS carrier signal tracking loop design [11], where the platform dynamics, oscillator noise, and thermal noise effects can be incorporated into these models. We use these models to design the generalized PLL in state space.

1) *State model*: The state vector $\mathbf{x}_k$, which describes the received carrier signal on dynamic platforms at the k^{th} epoch, can be expressed as a 2-state or a 3-state vector as follows:

$$2\text{-state} : \mathbf{x}_k = \begin{bmatrix} \varphi & \omega \end{bmatrix}_k^T \quad (9)$$

$$3\text{-state} : \mathbf{x}_k = \begin{bmatrix} \varphi & \omega & \dot{\omega} \end{bmatrix}_k^T \quad (10)$$

where φ_k is the initial fractional phase in rad, ω_k is carrier frequency in rad/s, and $\dot{\omega}_k$ is the frequency rate in rad/s².

The system transition matrix $\mathbf{A}_P$ has the following forms [10] [29]:

$$2\text{-state} : \mathbf{A}_P = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad (11)$$

$$3\text{-state} : \mathbf{A}_P = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \quad (12)$$

where T represents the integration time (typically 1ms for GPS L1 CA signals).

The vector $\mathbf{n}_k$, representing the system noise, typically includes the oscillator noise effect in the RF front-end and the random walk process due to the platform dynamics in GNSS applications [17]. Its covariance matrix $\mathbf{Q}_P$ are dependent on receiver platform dynamics [13] [17]:

$$2\text{-state} : \mathbf{Q}_P = \begin{bmatrix} \sigma_{\varphi}^2 & \sigma_{\omega\varphi}^2 \\ \sigma_{\omega\varphi}^2 & \sigma_{\omega}^2 \end{bmatrix} = (2\pi f_L)^2 \begin{bmatrix} T q_{\varphi} + \frac{T^3}{3} q_{\omega} & \frac{T^2}{2} q_{\omega} \\ \frac{T^2}{2} q_{\omega} & T q_{\omega} \end{bmatrix} \quad (13)$$

$$3\text{-state} : \mathbf{Q}_P = \begin{bmatrix} \sigma_{\varphi}^2 & \sigma_{\omega\varphi}^2 & \sigma_{\dot{\omega}\varphi}^2 \\ \sigma_{\omega\varphi}^2 & \sigma_{\omega}^2 & \sigma_{\omega\dot{\omega}}^2 \\ \sigma_{\dot{\omega}\varphi}^2 & \sigma_{\omega\dot{\omega}}^2 & \sigma_{\dot{\omega}}^2 \end{bmatrix} = (2\pi f_L)^2 \begin{bmatrix} T q_{\varphi} + \frac{T^3}{3} q_{\omega} + \frac{T^5}{20} \frac{q_a}{c^2} & \frac{T^2}{2} q_{\omega} + \frac{T^4}{8} \frac{q_a}{c^2} & \frac{T^3}{6} \frac{q_a}{c^2} \\ \frac{T^2}{2} q_{\omega} + \frac{T^4}{8} \frac{q_a}{c^2} & T q_{\omega} + \frac{T^3}{3} \frac{q_a}{c^2} & \frac{T^2}{2} \frac{q_a}{c^2} \\ \frac{T^3}{6} \frac{q_a}{c^2} & \frac{T^2}{2} \frac{q_a}{c^2} & T \frac{q_a}{c^2} \end{bmatrix} \quad (14)$$

In equations (13) and (14), f_L is the carrier frequency ($f_L = 1575.42\text{MHz}$ for the GPS L1 signal), q_{φ} and q_{ω} represent

the power spectral density of the carrier phase noise and carrier frequency noise due to the local oscillator instability, respectively. Given the oscillator h -parameters h_0 and h_{-2} , the spectral densities q_{φ} and q_{ω} are obtained as follows [13] [17]:

$$q_{\varphi} = \frac{h_0}{2} \quad (15)$$

$$q_{\omega} = 2\pi^2 h_{-2}. \quad (16)$$

The values of the oscillator h -parameters depend on the type of oscillator used in receivers. In GNSS applications, temperature-compensated oscillators (TCXO) and oven-controlled oscillators (OCXO) are commonly used [2] [30]. Two receiver front-ends, one with a low quality oscillator (LQO) similar to that of a TCXO, while the other has a high quality oscillator (HQO) similar to that of an OCXO, will be used in analysis and simulation validation in this paper. Their h -parameters are set as $h_0 = 1 \times 10^{-21}(\text{s}^2/\text{Hz})$, $h_{-2} = 2 \times 10^{-20}(1/\text{Hz})$ for the receiver front-end with LQO, and $h_0 = 6.4 \times 10^{-26}(\text{s}^2/\text{Hz})$, $h_{-2} = 4.3 \times 10^{-23}(1/\text{Hz})$ for the receiver front-end with HQO, respectively. These values are similar to typical values that can be found in [13] [17]. Additionally, the parameter q_a in equation (14) denotes the power spectral density of the random walk process due to the line-of-sight (LOS) platform acceleration with the unit of $(\text{m}^2/\text{s}^6)/\text{Hz}$ [17] and c is the speed of light ($3 \times 10^8 \text{m/s}$).

2) *Measurement model*: Although in real world applications, there are various error sources, such as interference [31], ionospheric scintillation [32], and time-correlated clock errors [33] that corrupt and distort carrier phase measurements, only white Gaussian thermal noise effects are considered in the analysis presented here, and the measurement noise is assumed to be uncorrelated with the system noise. The carrier phase θ_k of the received signal, which includes the average input signal phase $\bar{\theta}_k$ during the integration time T and the white Gaussian noise, v_k , can be described as:

$$\theta_k \triangleq \bar{\theta}_k + v_k = \mathbf{H}_P \mathbf{x}_k + v_k \quad (17)$$

where $\mathbf{H}_P$ is the measurement matrix [29]:

$$2\text{-state} : \mathbf{H}_P = \begin{bmatrix} 1 & \frac{T}{2} \end{bmatrix} \quad (18)$$

$$3\text{-state} : \mathbf{H}_P = \begin{bmatrix} 1 & \frac{T}{2} & \frac{T^2}{6} \end{bmatrix}. \quad (19)$$

Let the vector $\hat{\mathbf{x}}_k$ denote the local estimate of $\mathbf{x}_k$. The average phase $\hat{\theta}_k$ of a local generated signal can be expressed as:

$$\hat{\theta}_k \triangleq \mathbf{H}_P \hat{\mathbf{x}}_k. \quad (20)$$

Since θ_k cannot be directly measured, the phase error $\Delta\theta_k = \theta_k - \hat{\theta}_k$ is used as the measurement in real implementations. From (17) and (20), we have:

$$\Delta\theta_k = \mathbf{H}_P \Delta\mathbf{x}_k + v_k \quad (21)$$

where $\Delta\mathbf{x}_k = (\mathbf{x}_k - \hat{\mathbf{x}}_k)$. $\Delta\theta_k$ is the average phase error which contains all the estimated state errors, i.e., the initial phase error, the frequency error, and the frequency rate error if the frequency rate is considered. This average phase error can be obtained from

the phase discriminator,

$$\Delta\theta_k = \arctan\left(\frac{Q_k}{I_k}\right) \quad (22)$$

where I_k and Q_k are the correlations between the local replica signal and the incoming signal of in-phase and quadrature branch at the k^{th} interval. Generally, the two-quadrant and four-quadrant arctangent carrier phase discriminators are adopted in the generic receiver design for the signals with and without navigation message modulations, due to their linear characteristic in the range of -90° to 90° and -180° to 180° , respectively [2]. The noise variance σ_v^2 of v_k in an arctangent discriminator output is given as [4] [13]:

$$\mathbf{R} = \sigma_v^2 = E[v_k v_k^T] = \frac{1}{2TC/N_0} \left(1 + \frac{1}{2TC/N_0}\right) \quad (23)$$

The typical value of C/N_0 for nominal signal strength is above 40dB-Hz.

B. State Space Design For PLL

Fig.1 shows the block diagram of the closed phase tracking loop in the GNSS receiver.

Fig. 1: Generalized closed-loop phase tracking architecture in GNSS software receiver.

The local signal generated by the local reference generator can be expressed as:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_P \hat{\mathbf{x}}_k + \mathbf{B} \mathbf{u}_k. \quad (24)$$

Subtracting (24) from (7) and letting $\Delta \mathbf{x}_{k+1} = \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}$, we have the error state plant model as:

$$\Delta \mathbf{x}_{k+1} = \mathbf{A}_P \Delta \mathbf{x}_k - \mathbf{B} \mathbf{u}_k + \mathbf{n}_k \quad (25)$$

The objective of the tracking loop is to drive $\Delta \mathbf{x}_k$ towards zero. This can be achieved if we select

$$\mathbf{u}_k = \mathbf{K} \Delta \mathbf{x}_k \quad (26)$$

such that $\mathbf{A}_P - \mathbf{B} \mathbf{K}$ has all its eigenvalues within the stability region. Since, $\Delta \mathbf{x}_k$ is not directly measurable, but it has to be estimated from the phase error measurement $\Delta\theta_k$ (equation (21)). Equation (26) is replaced by

$$\mathbf{u}_k = \mathbf{K} \Delta \hat{\mathbf{x}}_k. \quad (27)$$

with a state estimator of the form

$$\Delta \hat{\mathbf{x}}_{k+1} = \mathbf{A}_P \Delta \hat{\mathbf{x}}_k - \mathbf{B} \mathbf{u}_k + \mathbf{L} (\Delta\theta_{k+1} - \mathbf{H}_P (\mathbf{A}_P \Delta \hat{\mathbf{x}}_k - \mathbf{B} \mathbf{u}_k)) \quad (28)$$

where $\mathbf{L}$ is the estimator gain matrix.

Substituting equation (27) into (28), the error estimation can be obtained as:

$$\Delta \hat{\mathbf{x}}_{k+1} = (\mathbf{I} - \mathbf{L} \mathbf{H}_P) (\mathbf{A}_P - \mathbf{B} \mathbf{K}) \Delta \hat{\mathbf{x}}_k + \mathbf{L} \Delta\theta_{k+1}. \quad (29)$$

Finally, this estimation is input to the local reference generator, and using equations (24) and (27) the local signal can be generated as:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_P \hat{\mathbf{x}}_k + \mathbf{B} \mathbf{K} \Delta \hat{\mathbf{x}}_k. \quad (30)$$

The above formulation shows that the state feedback tracking loop design is characterized by the three matrices, i.e., the plant input matrix $\mathbf{B}$, the state feedback gain matrix $\mathbf{K}$, and the state estimator gain matrix $\mathbf{L}$. The design of these matrices will be discussed next.

C. B and K design

$\mathbf{B}$ should satisfy the following condition to guarantee the system controllability [27]:

$$\text{rank}[\mathbf{B} | \mathbf{A}_P \mathbf{B} | \dots | \mathbf{A}_P^{n-1} \mathbf{B}] = n \quad (31)$$

where n is the state dimension.

To ensure the stability of the PLL, $\mathbf{B}$ and $\mathbf{K}$ should satisfy the following condition:

$$\text{eig}(\mathbf{A}_P - \mathbf{B} \mathbf{K}) < 1. \quad (32)$$

Clearly, $\mathbf{A}_P - \mathbf{B} \mathbf{K}$ with different eigenvalues will lead to different error dynamics (equation(29)), and hence different system responses. In this paper, we use the deadbeat design which placed all the eigenvalues of $\mathbf{A}_P - \mathbf{B} \mathbf{K}$ at the origin [27]. It enables fast transient time and minimal steady-state error in control system design. More importantly, it simplifies the error dynamic of (29), so that the corresponding state estimation update method is similar to those in the literature [10].

Appendix A provides a detailed analysis of the system controllability and stability requirements on the selection of $\mathbf{B}$ and $\mathbf{K}$ for GNSS carrier tracking. The selection of $\mathbf{B} = \mathbf{I}$ and $\mathbf{K} = \mathbf{A}_P$ meet the requirement. Substituting them into equations (29) and (30) leads to:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_P \hat{\mathbf{x}}_k + \mathbf{A}_P \mathbf{L} \Delta\theta_k \quad (33)$$

Equation (33) demonstrates that PIF, WF, and KF can be unified through the design of the estimator gain matrix $\mathbf{L}$.

D. Estimator Gain Matrix L

The estimator gain matrix $\mathbf{L}$ will be analyzed for PIF, WF, and KF designs. The PIF and WF can be classified as a traditional single-input single-output system based on the transfer function from scalar phase input to scalar phase output. KF is based on the state and the measurement models in state space, which can be classified as a model-based design. For PIF and WF, we obtained their steady-state gain matrix $\mathbf{L}$ through their corresponding closed-loop transfer function, while for KF, its gain matrix was derived according to the state and measurement models.

$$F_3(z) = \frac{(7T^2\gamma + 9T\beta + 6\alpha)z^2 + (-2T^2\gamma - 12T\beta - 12\alpha)z + T^2\gamma + 3T\beta + 6\alpha}{6z^3 + (7T^2\gamma + 9T\beta + 6\alpha - 18)z^2 + (-2T^2\gamma - 12T\beta - 12\alpha + 18)z + T^2\gamma + 3T\beta + 6\alpha - 6} \quad (38)$$

The closed-loop transfer function in the state feedback tracking loop with $\mathbf{B} = \mathbf{I}$, $\mathbf{K} = \mathbf{A}_P$ can be obtained as:

$$F(z) = \frac{\hat{\theta}_k}{\theta_k} = \frac{\mathbf{H}_P(z\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{A}_P\mathbf{L}}{1 + \mathbf{H}_P(z\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{A}_P\mathbf{L}}. \quad (34)$$

Let $\mathbf{L}$ take the following forms for the 2- and 3-state model, respectively:

$$2\text{-state} : \mathbf{L} = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T \quad (35)$$

$$3\text{-state} : \mathbf{L} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^T. \quad (36)$$

Their corresponding closed-loop transfer functions can be written as:

$$F_2(z) = \frac{(\alpha + \frac{3T}{2}\beta)z - (\alpha + \frac{T}{2}\beta)}{z^2 + (\alpha + \frac{3T}{2}\beta - 2)z + (-\alpha - \frac{T}{2}\beta + 1)} \quad (37)$$

and equation (38).

1) *Proportional integral filter*: PIF is based on the discretization of an analog PLL [2] [3], and both the practical and theoretical aspects of analog PLL and their performances under various conditions are well developed. The implementation of Fig. 1 using second-order and third-order analog PLL with PIF are shown in Fig. 2. Their loop parameters can be obtained by the predefined values of damping ratio ξ and noise equivalent bandwidth BN , as listed in Table I [2], where w_n is the natural frequency.

TABLE I: Loop Filter Characteristics

Loop order	Noise bandwidth $BN(\text{Hz})$	Typical values
First	$\frac{w_n}{4}$	$BN = 0.25w_n$
Second	$\frac{w_n(1+\xi^2)}{4\xi}$	$\xi = 0.707$ $BN \approx 0.53w_n$
Third	$\frac{w_n(a_nb_n^2 + a_n^2 - b_n)}{4(a_nb_n - 1)}$	$a_n = 1.1$ $b_n = 2.4$ $BN \approx 0.7845w_n$

The closed-loop transfer functions for the first-, second-, and third-order PLL can be obtained as:

$$H_{PIF1}(s) = \frac{w_n}{s + w_n} \quad (39)$$

$$H_{PIF2}(s) = \frac{2\xi w_n s + w_n^2}{s^2 + 2\xi w_n s + w_n^2} \quad (40)$$

$$H_{PIF3}(s) = \frac{b_n w_n s^2 + a_n w_n^2 s + w_n^3}{s^3 + b_n w_n s^2 + a_n w_n^2 s + w_n^3} \quad (41)$$

By using the forward Euler transformation $s = (z - 1)/T$ (valid only for $BN \cdot T \ll 1/2$ [5]), we discretize equations (39), (40), and (41) from the s -domain to the z -domain:

$$H_{PIF1}(z) = \frac{w_n T}{z + w_n T - 1} \quad (42)$$

Fig. 2: Analog PLL. (a) First order. (b) Second order. (c) Third order.

$$H_{PIF2}(z) = \frac{2\xi w_n T z + w_n^2 T^2 - 2\xi w_n T}{z^2 + (2\xi w_n T - 2)z + (w_n^2 T^2 - 2\xi w_n T + 1)} \quad (43)$$

Comparing (37) and (38) with (43) and (44), we obtain the estimator gain matrix $\mathbf{L}$ for the 2-state and 3-state PIF-based PLLs, respectively.

$$\mathbf{L}_{PIF2} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2\xi w_n T - \frac{3}{2}w_n^2 T^2 \\ w_n^2 T \end{bmatrix} \quad (45)$$

$$\mathbf{L}_{PIF3} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} (11w_n^3 T^3 - 9a_n w_n^2 T^2 + 6b_n w_n T)/6 \\ -2w_n^3 T^2 + a_n w_n^2 T \\ w_n^3 T \end{bmatrix}. \quad (46)$$

2) *Wiener filter*: A WF computes a statistical estimate of $\hat{\theta}_k$ of the input signal θ_k which is corrupted by additive noise v_k in the sense of minimizing $E[(\hat{\theta}_k - \theta_k)^2]$ [12] [13]. A WF transfer function is obtained by spectral factorization of the auto-correlation functions $S_{\hat{\theta}}$ and S_{θ} of the discrete signal $\hat{\theta}_k$ and θ_k in the z -domain.

$$S_{\theta}(z) = Y(z)Y(z^{-1}) \quad (47)$$

$$H_{WF}(z) = \frac{\hat{\theta}_k}{\theta_k} = \frac{\left[\frac{S_{\hat{\theta}}(z)}{Y(z^{-1})} \right]_+}{Y(z)} \quad (48)$$

where $[\]_+$ represents the part of the partial fraction expansion of its argument whose poles are inside the unit circle [12].

$$H_{PIF3}(z) = \frac{(b_n w_n T) z^2 + (a_n w_n^2 T^2 - 2b_n w_n T) z + (w_n^3 T^3 - a_n w_n^2 T^2 + b_n w_n T)}{z^3 + (b_n w_n T - 3) z^2 + (a_n w_n^2 T^2 - 2b_n w_n T + 3) z + (w_n^3 T^3 - a_n w_n^2 T^2 + b_n w_n T - 1)} \quad (44)$$

$$H_{WF3}(z) = \frac{(3 - z_0 - z_1 - z_2) z^2 + (z_0 z_1 + z_0 z_2 + z_1 z_2 - 3) z + 1 - z_0 z_1 z_2}{z^3 + (-z_0 - z_1 - z_2) z^2 + (z_0 z_1 + z_0 z_2 + z_1 z_2) z - z_0 z_1 z_2} \quad (50)$$

Appendix B shows the derivation of the second- and third-order WF transfer functions based on this process:

$$H_{WF2}(z) = \frac{(2 - z_0 - z_1) z + z_0 z_1 - 1}{z^2 - (z_0 + z_1) z + z_0 z_1} \quad (49)$$

where z_0 , z_1 and z_2 are determined by the signal and noise characteristics, and their expressions can be found in Appendix B.

Comparing (49) and (50) with the closed-loop transfer functions in equations (37) and (38), the estimator gain matrix $\mathbf{L}$ for the 2-state and 3-state WF-based PLLs, respectively can be obtained as:

$$\mathbf{L}_{WF2} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1+z_0+z_1-3z_0z_1}{(1-z_0)^2(1-z_1)} \\ \frac{2}{T} \end{bmatrix} \quad (51)$$

$$\mathbf{L}_{WF3} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac{(z_0+z_1+z_2)+2(z_0z_1+z_0z_2+z_1z_2)+2-11z_0z_1z_2}{6} \\ \frac{1-(z_0z_1+z_0z_2+z_1z_2)+2z_0z_1z_2}{T} \\ \frac{1+(z_0z_1+z_0z_2+z_1z_2)-z_0z_1z_2-(z_0+z_1+z_2)}{T^2} \end{bmatrix}. \quad (52)$$

3) *Kalman filter*: A KF estimates the error state $\Delta \hat{\mathbf{x}}_k$ based on the given measurement $\Delta \theta_k$ as shown in Fig.1. A KF estimator should be independent of the controller, and its matrix $\mathbf{L}$ should minimize the quadratic form $\chi^T E[(\Delta \mathbf{x}_k - \Delta \hat{\mathbf{x}}_k)(\Delta \mathbf{x}_k - \Delta \hat{\mathbf{x}}_k)^T] \chi$, where χ is an arbitrary $n \times 1$ vector. Referring to the error state plant model in equation (25), the measurement model in equation (21), and also the state estimator in equation (28), the estimation error can be obtained as:

$$\begin{aligned} \varepsilon_{k+1} &= \Delta \mathbf{x}_k - \Delta \hat{\mathbf{x}}_k \\ &= (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{H}_P) \mathbf{A}_P \varepsilon_k \\ &\quad + (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{H}_P) \mathbf{n}_k - \mathbf{L}_{k+1} \mathbf{v}_{k+1}. \end{aligned} \quad (53)$$

The error state estimation error's covariance matrix $\mathbf{W}_{k+1} = E[\varepsilon_{k+1} \varepsilon_{k+1}^T]$ is:

$$\begin{aligned} \mathbf{W}_{k+1} &= (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{H}_P) (\mathbf{A}_P \mathbf{W}_k \mathbf{A}_P^T + \mathbf{Q}_P) (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{H}_P)^T \\ &\quad + \mathbf{L}_{k+1} \mathbf{R} \mathbf{L}_{k+1}^T. \end{aligned} \quad (54)$$

Defining

$$\mathbf{N}_{k+1} = \mathbf{A}_P \mathbf{W}_k \mathbf{A}_P^T + \mathbf{Q}_P \quad (55)$$

and expanding equation (54), we have:

$$\begin{aligned} \mathbf{W}_{k+1} &= \mathbf{N}_{k+1} + \mathbf{G}_{k+1} (\mathbf{R} + \mathbf{H} \mathbf{N}_{k+1} \mathbf{H}^T) \mathbf{G}_{k+1}^T \\ &\quad - \mathbf{N}_{k+1} \mathbf{H}^T (\mathbf{R} + \mathbf{H} \mathbf{N}_{k+1} \mathbf{H}^T)^{-1} \mathbf{H} \mathbf{N}_{k+1} \end{aligned} \quad (56)$$

where

$$\mathbf{U}_{k+1} = \mathbf{L}_{k+1} - \mathbf{N}_{k+1} \mathbf{H}_P^T (\mathbf{R} + \mathbf{H}_P \mathbf{N}_{k+1} \mathbf{H}_P^T)^{-1} \mathbf{H}_P \mathbf{N}_{k+1}. \quad (57)$$

Hence the quadratic form $\chi^T \mathbf{W}_{k+1} \chi$ is a minimum when $\mathbf{U}_{k+1} = 0$. Thus, KF gain matrix can be obtained as:

$$\mathbf{L}_{k+1} = \mathbf{N}_{k+1} \mathbf{H}_P^T (\mathbf{R} + \mathbf{H}_P \mathbf{N}_{k+1} \mathbf{H}_P^T)^{-1} \quad (58)$$

and the corresponding $\mathbf{W}_{k+1}$ is:

$$\mathbf{W}_{k+1} = (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{H}_P) \mathbf{N}_{k+1}. \quad (59)$$

The above analysis shows that the KF gain $\mathbf{L}_{k+1}$ is the optimal gain matrix that minimizes $\chi^T \mathbf{W}_{k+1} \chi$ regardless of the matrices $\mathbf{B}$, $\mathbf{K}$ and the controller $\mathbf{u}_k$. However, the error state estimator is only a subsystem of the control system, as shown in Fig.1. To obtain the performance of the overall closed system with this error state estimator, we apply $\mathbf{B} = \mathbf{I}$, $\mathbf{K} = \mathbf{A}_P$ and $\mathbf{u}_k = \mathbf{K} \Delta \hat{\mathbf{x}}_k$ into equation (24) to obtain:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_P \hat{\mathbf{x}}_k + \mathbf{A}_P \Delta \hat{\mathbf{x}}_k. \quad (60)$$

The full state estimation variance can be derived from (7) and (60):

$$E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T] = \mathbf{A}_P \mathbf{W}_k \mathbf{A}_P^T + \mathbf{Q}_P = \mathbf{N}_{k+1}. \quad (61)$$

Equation (61) shows that a minimal $\mathbf{W}_k$ corresponds to a minimal quadratic form $\chi^T E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T] \chi$; therefore, the gain matrix $\mathbf{L}_{k+1}$ in equation (58) not only minimizes the error state estimation errors, but also the full state estimation errors. The gain matrix $\mathbf{L}_{k+1}$ is globally optimal for the overall feedback control loop. Since χ is an arbitrary $n \times 1$ vector, we may choose $\chi = \mathbf{H}_P^T$:

$$\begin{aligned} &\chi^T E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T] \chi \\ &= \mathbf{H} E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T] \mathbf{H}^T \\ &= E[(\hat{\theta}_k - \theta_k)^2]. \end{aligned} \quad (62)$$

As a result, with $\mathbf{B} = \mathbf{I}$, $\mathbf{K} = \mathbf{A}_P$ and $\mathbf{u}_k = \mathbf{K} \Delta \hat{\mathbf{x}}_k$ and the KF gain matrix $\mathbf{L}_{k+1}$, the overall closed PLL operates at the MMSE tracking performance.

In the KF implementation, the gain update process is operated in two stages; namely, the transient and the steady-state stage. Given an initial error covariance matrix $\mathbf{W}_0$, $\mathbf{L}_{k+1}$ will converge to the constant optimal gain matrix as the system transits from the transient-state to the steady-state with gain matrix $\mathbf{L}_{KF} = \lim_{k \rightarrow \infty} \mathbf{L}_{k+1}$. To obtain $\mathbf{L}_{KF}$, we first solve the following discrete algebraic Riccati equation (DARE) to get the steady-state covariance matrix $\mathbf{N}_\infty$

$$\mathbf{A}_P \mathbf{N}_\infty \mathbf{A}_P^T - \mathbf{A}_P \mathbf{N}_\infty \mathbf{H}_P^T (\mathbf{H}_P \mathbf{N}_\infty \mathbf{H}_P^T + \mathbf{R})^{-1} \mathbf{H}_P \mathbf{N}_\infty \mathbf{A}_P^T + \mathbf{Q}_P - \mathbf{N}_\infty = 0. \quad (63)$$

Substituting $\mathbf{N}_\infty$ into equation (58), we obtain the numerical

solutions of $\mathbf{L}_{KF}$.

4) *Discussion and analysis:* The gain matrices for PIF, WF, and KF provide powerful analysis tools for their corresponding designs and performances. In the PIF design, the values of $\mathbf{L}_{PIF}$ in equations (45) and (46) are primarily determined by the values of BN and T as shown in Table I. Adjusting the value of BN is the most effective way to obtain the desired performance for nominal signal tracking, i.e., increasing BN allows fast convergence in the transient-state and decreasing BN achieves better tracking accuracy in the steady-state. For weak signal tracking, T should be increased to compensate for the power loss through coherent or non-coherent accumulations [16] [17]; in this case BN should also be adjusted accordingly. In practice, the values of BN and T are typically chosen based on the empirical guidelines which do not provide rigorous analysis under different signal conditions.

In the WF design, $\mathbf{L}_{WF}$ is a function of the characteristic roots as shown in equations (51) and (52), where the locations of these roots are uniquely determined by the signal and noise characteristics. Since the value of $\mathbf{L}_{WF}$ is related to the signal model, with this optimal gain, the WF-based tracking loop could achieve MMSE tracking performance in the steady-state by design, which is superior compared to the heuristic nature of the PIF-based tracking loop.

In the KF design, $\mathbf{L}_{KF}$ is calculated from an iterative operation and determined by the system characteristic matrices $\mathbf{A}_P$, $\mathbf{H}_P$, $\mathbf{Q}_P$ and $\mathbf{R}$. With this gain the KF-based tracking loop could achieve the MMSE performance with both the error state and full state estimation.

Since WF and KF both generate optimal estimations to minimize $E[(\hat{\theta}_k - \hat{\theta}_k)^2]$, is it possible that $\mathbf{L}_{WF}$ and $\mathbf{L}_{KF}$ are equivalent? If they are, then we may use $\mathbf{L}_{WF}$ as the closed-form representation of steady-state $\mathbf{L}_{KF}$. To investigate the relationship between $\mathbf{L}_{WF}$ and $\mathbf{L}_{KF}$ and the impact of the systematic characteristic matrices $\mathbf{A}_P$, $\mathbf{H}_P$, $\mathbf{Q}_P$ and $\mathbf{R}$ on the gain matrices $\mathbf{L}_{WF}$ and $\mathbf{L}_{KF}$, we compare the analytical solutions of $\mathbf{L}_{WF}$ with numerical solutions of $\mathbf{L}_{KF}$ under various settings. In all calculations, we set $C/N_0 = 14\text{dB-Hz}$ and assume that either the data has been wiped off from the data channel or the pilot channel signals are used. The integration time T varies from 1ms to 1000ms. The LQO and HQO are considered and several values of the dynamic parameter q_a , i.e., $q_a = 0$ (m^2/s^6)/Hz, $q_a = 0.1(m^2/s^6)$ /Hz and $q_a = 10(m^2/s^6)$ /Hz, are used to represent the static, low dynamic, and high dynamic scenarios, respectively. The numerical solutions for $\mathbf{L}_{KF}$ (circles) and the analytical solutions (lines) for $\mathbf{L}_{WF}$ are plotted and depicted in Fig.3(a)-(c). The following observations are derived from the figures.

i). The values of $\mathbf{L}_{WF}$ perfectly match with those of $\mathbf{L}_{KF}$, demonstrating that the analytic expressions of the derived $\mathbf{L}_{WF}$ indeed mirrors the $\mathbf{L}_{KF}$ behavior.

ii). From the perspective of system noise, the values of α and β are higher for the receiver with LQO than those with HQO. This indicates that the system modelling error with a good quality oscillator is smaller than that of a poor quality oscillator. Therefore, more weight should be given to estimation from system prediction for receivers with HQOs.

Fig. 3: α , β , and γ as functions of integration time and oscillator qualities in (a) 2-state PLL under static conditions, (b) 3-state PLL under low dynamic conditions when $q_a = 0.1(m^2/s^6)$ /Hz, and (c) 3-state PLL under high dynamic conditions when $q_a = 10(m^2/s^6)$ /Hz.

iii). From the perspective of measurement noise, increasing the value of T effectively reduces the measurement noise, especially when the signal is weak. However, there is a limit to the effective range of T values. The figures show that as T increases from 1ms to 100ms, the loop gain parameters values are increased monotonically. For $T > 200$ ms, the gain parameters subsequently decrease before reaching a steady level. Therefore, the filter gain and the integration time are not always positively correlated. Moreover, long time integration will cause some problems. For example, the oscillator noise effect on the phase and frequency error is accumulated with a longer integration time, and the small errors of frequency estimation may lead to the signal correlation loss. Hence, it is better to combine an appropriate integration time with a well-designed filter to achieve equivalent or better tracking performance.

iv). From the perspective of signal dynamics, the dynamic parameter q_a has a greater influence on the value of γ than the values of α and β in $\mathbf{L}_{WF}$ and $\mathbf{L}_{KF}$. Fig.3(b) and (c) show that γ increases drastically as q_a increases, while α and β are less affected. For the special case $q_a = 0(m^2/s^6)/\text{Hz}$, comparison of the $\mathbf{Q}_P$ matrix in equations (13) and (14) shows that the 3-state model degrades to the 2-state model and γ is 0. Hence, γ determines the ability of the filter to track dynamic signals. Furthermore, the dynamic parameter q_a has a greater influence on the values of α and β for a receiver with a HQO than one with a LQO. For example, the maximum values of β in the receiver with LQO in static, low dynamic, and high dynamic scenarios, respectively are ~ 3.1 , ~ 4.1 , and ~ 9 , while, for the receiver with a HQO, the corresponding maximum values of β are ~ 0.6 , ~ 2.7 , and ~ 5.1 , respectively. As dynamic increases from static case in Fig.3(a) to high dynamic case in Fig.3(c), the value of β in the receiver with a LQO increases by a factor of 2. However, for the receiver with a HQO, there is a 7-fold increase. Therefore, the loop gains in the receivers with HQO are more sensitive to platform dynamics than that with LQO.

IV. STATE SPACE DESIGN FOR FLL

Frequency tracking is essentially differential carrier phase tracking that neglects absolute phase error and permits relative phase rotation between the received signal and the local carrier replica [2]. Hence, a FLL can be treated as a reduced-order PLL, where a first-order or a second-order FLL achieves the equivalent frequency tracking capability as a second-order or a third-order PLL, respectively. In this section, the general state space design control system process in Section II is applied for FLL design. The detailed FLL design and analysis follow the same approach as that of the generalized PLL in Section III.

A. System Model

Similar to the system model in PLL in Section III, we have the following system model for FLL:

$$\mathbf{x}_{k+1} = \mathbf{A}_F \mathbf{x}_k + \mathbf{n}_k \quad (64)$$

$$\mathbf{z}_k = \mathbf{H}_F \mathbf{x}_k + \mathbf{v}_k \quad (65)$$

1) *State model*: While PLLs replicate the exact phase and frequency of the incoming signal to enable carrier wipe-off from the incoming signals, FLLs perform the carrier wipe-off process by replicating the approximate frequency and allowing the replicate phase to rotate with respect to the incoming carrier signal [2]. Thus, only the estimation of the frequency ω_k and frequency rate $\dot{\omega}_k$ are required in FLL design. For this reason, we adopt the following state vectors:

$$1 - \text{state} : \mathbf{x}_k = \omega_k \quad (66)$$

$$2 - \text{state} : \mathbf{x}_k = \begin{bmatrix} \omega & \dot{\omega} \end{bmatrix}_k^T \quad (67)$$

to represent the input signals associated with typical dynamics of a GNSS receiver.

The corresponding reduced-order system transition matrix $\mathbf{A}_F$ in a FLL has the following forms:

$$1 - \text{state} : \mathbf{A}_F = 1 \quad (68)$$

$$2 - \text{state} : \mathbf{A}_F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad (69)$$

and the associated noise covariance matrix $\mathbf{Q}_F$ are:

$$1 - \text{state} : \mathbf{Q}_F = \sigma_\omega^2 = (2\pi f_L)^2 T q_\omega \quad (70)$$

$$2 - \text{state} : \mathbf{Q}_F = \begin{bmatrix} \sigma_\omega^2 & \sigma_{\omega\dot{\omega}}^2 \\ \sigma_{\omega\dot{\omega}}^2 & \sigma_{\dot{\omega}}^2 \end{bmatrix} = (2\pi f_L)^2 \begin{bmatrix} T q_\omega + \frac{T^3}{3} \frac{q_a}{c^2} & \frac{T^2}{2} \frac{q_a}{c^2} \\ \frac{T^2}{2} \frac{q_a}{c^2} & T \frac{q_a}{c^2} \end{bmatrix} \quad (71)$$

2) *Measurement model*: In a FLL, the measurement can be obtained through the following discriminator [2]:

$$\begin{aligned} \Delta\varpi_{k+1} &= \frac{1}{T} \arctan \left(\frac{I_k Q_{k+1} - I_{k+1} Q_k}{I_k I_{k+1} + Q_k Q_{k+1}} \right) \\ &= \frac{1}{T} (\Delta\theta_{k+1} - \Delta\theta_k) \end{aligned} \quad (72)$$

Generally, the two-quadrant and four-quadrant arctangent FLL discriminators are adopted in generic receiver design for signals with and without navigation message modulations [2] [25]. The frequency tracking of GPS receivers must be insensitive to 180° reversals in the I_k and Q_k signals. Therefore, the sample times of the I_k and Q_k signals should not straddle the data bit transitions [24]. The maximum integration time of a FLL is 10ms for GPS L1 signal to avoid the data bit transition for channels with data modulation. Substituting equation (21) into equation (72) (see derivation in Appendix C), we can obtain:

$$1 - \text{state} : \Delta\varpi_k = \Delta\omega_k + \frac{v_k - v_{k-1}}{T} \quad (73)$$

$$2 - \text{state} : \Delta\varpi_k = \Delta\omega_k + \frac{T}{2} \Delta\dot{\omega}_k + \frac{v_k - v_{k-1}}{T}. \quad (74)$$

Denoting the vector $\hat{\mathbf{x}}_k$ as the local estimation of $\mathbf{x}_k$ and $v_k = (v_k - v_{k-1})/T$, the frequency discriminator output $\Delta\varpi_k$ can be formulated as:

$$\Delta\varpi_k = \mathbf{H}_F (\mathbf{x}_k - \hat{\mathbf{x}}_k) + v_k \quad (75)$$

where $\mathbf{H}_F$ is the FLL measurement matrix:

$$1 - \text{state} : \mathbf{H}_F = 1 \quad (76)$$

$$2 - \text{state} : \mathbf{H}_F = \begin{bmatrix} 1 & \frac{T}{2} \end{bmatrix}. \quad (77)$$

B. State Space Design For FLL

Denote $\varpi_k \triangleq \hat{\varpi} + v_k \triangleq \mathbf{H}_F \mathbf{x}_k + v_k$ as the system input and $\hat{\varpi}_k \triangleq \mathbf{H}_F \hat{\mathbf{x}}_k$ as the system output in a FLL. The block diagram of the generalized FLL is the same as Fig.1 by replacing θ with ϖ , v with v , and $\mathbf{H}_P$ with $\mathbf{H}_F$.

Similar to the state space design for PLL, the FLL local reference generator is controlled by the controller $\mathbf{u}_k$:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_F \hat{\mathbf{x}}_k + \mathbf{B} \mathbf{u}_k \quad (78)$$

$$\mathbf{u}_k = \mathbf{K} \Delta \mathbf{x}_k \quad (79)$$

A state estimator needs to be constructed based on the measurement $\Delta \varpi_k$ as the true state information is not available:

$$\Delta \hat{\mathbf{x}}_k = \mathbf{A}_F \Delta \hat{\mathbf{x}}_{k-1} - \mathbf{B} \mathbf{u}_{k-1} + \mathbf{L}_F (\Delta \varpi_k - \mathbf{H}_F (\mathbf{A}_F \Delta \hat{\mathbf{x}}_{k-1} - \mathbf{B} \mathbf{u}_{k-1})) \quad (80)$$

where $\mathbf{L}_F$ is the estimator gain matrix in the FLL.

Following the analysis in Section III, we choose $\mathbf{B} = \mathbf{I}$ and $\mathbf{K} = \mathbf{A}_F$ to simplify the above error state estimation and the corresponding full state estimation can be obtained as:

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}_F \hat{\mathbf{x}}_k + \mathbf{A}_F \mathbf{L}_F \Delta \varpi_k. \quad (81)$$

C. Estimator Gain Matrix $\mathbf{L}_F$

The gain matrices for three PLL filter designs, i.e., PIF, WF, and KF are analyzed here. Under the white Gaussian noise assumption, $\mathbf{L}_{WF}$ and $\mathbf{L}_{KF}$ are both the optimal gain matrices that enable MMSE PLL performance. However, the measurement noise v_k in FLL is not white. KF is not an optimal filter in the sense of MMSE for a FLL, although it has been used in FLL design in [10], and it is also too complicated to obtain the closed-form $\mathbf{L}_{WF}$ for a FLL. Since the PIF is a model-free design approach that is immune to the noise characteristic, we only consider the PIF design in the generalized FLL analysis, and the gain matrix for a PIF is obtained by mapping the PIF-based FLL into the state space architecture through the equivalent closed-loop transfer function discussed in Section III.

The 1- and 2-state forms for $\mathbf{L}_F$ are:

$$1 - \text{state} : \mathbf{L}_F = \alpha \quad (82)$$

$$2 - \text{state} : \mathbf{L}_F = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T. \quad (83)$$

The corresponding closed-loop transfer function for a FLL is:

$$F(z) = \frac{\hat{\varpi}_k}{\varpi_k} = \frac{\mathbf{H}_F (z\mathbf{I} - \mathbf{A}_F)^{-1} \mathbf{A}_F \mathbf{L}_F}{1 + \mathbf{H}_F (z\mathbf{I} - \mathbf{A}_F)^{-1} \mathbf{A}_F \mathbf{L}_F} \quad (84)$$

with the 1-state and 2-state forms as:

$$1 - \text{state} : F(z) = \frac{\alpha}{z + \alpha - 1} \quad (85)$$

$$2 - \text{state} : F(z) = \frac{(\alpha + \frac{3T}{2}\beta)z - (\alpha + \frac{T}{2}\beta)}{z^2 + (\alpha + \frac{3T}{2}\beta - 2)z + (-\alpha - \frac{T}{2}\beta + 1)}. \quad (86)$$

Comparing equations (85) and (86) with that of equations (42) and (43) for the first- and second-order analog PLL, the gain matrix $\mathbf{L}_F$ for 1-state and 2-state generalized FLLs can be

obtained as:

$$1 - \text{state} : \mathbf{L}_F = w_f T \quad (87)$$

$$2 - \text{state} : \mathbf{L}_F = \begin{bmatrix} 2\xi w_f T - \frac{3}{2}w_f^2 T^2 \\ w_f^2 T \end{bmatrix}. \quad (88)$$

Here we denote w_f as natural frequency in FLL and can be calculated with the predefined value of the FLL noise equivalent bandwidth BW as shown in Table I.

Note that the FLL design presented here is based on predefined empirical values of loop parameters. It does not provide the rigorous design process for the loop parameters under different signal conditions, especially in harsh environments where the signal is both weak and on a highly dynamic platform.

V. CLOSED-FORM PERFORMANCE INDICATORS AND PERFORMANCE ANALYSIS

The most frequently used performance indicators for GNSS applications are tracking error variance and dynamic stress steady-state error. In this section, the closed-form expressions of these performance indicators for the state feedback/state estimator representation of PLL and FLL are derived. In addition, the 3-sigma rule will be applied to characterize the admissible range of the loop parameters.

A. Tracking error variance

Denote $\mathbf{P}_{k+1}$ as the tracking error covariance matrix at the $(k+1)^{\text{th}}$ epoch:

$$\mathbf{P}_{k+1} = E \left[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}) (\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T \right]. \quad (89)$$

1) *PLL error variance*: Substituting the received and local generated signal in equations (7) and (33) into (89), we have:

$$\mathbf{P}_{k+1} = \mathbf{A}_P (\mathbf{I} - \mathbf{L} \mathbf{H}_P) \mathbf{P}_k (\mathbf{I} - \mathbf{L} \mathbf{H}_P)^T \mathbf{A}_P^T + \mathbf{Q}_P + \mathbf{A}_P \mathbf{L} \mathbf{R} (\mathbf{A}_P \mathbf{L})^T \quad (90)$$

When k approaches infinity, the system reaches the steady-state, $\mathbf{P}_{k+1} = \mathbf{P}_k$. Denoting the steady-state error covariance as $\mathbf{P}_X$, for a given estimator gain matrix $\mathbf{L}$, the closed-form expression of $\mathbf{P}_X$ can be obtained by solving the equation (90). $\mathbf{L}$ can be $\mathbf{L}_{PIF}$ (equation (45), (46)), $\mathbf{L}_{WF}$ (equation (51), (52)) and $\mathbf{L}_{KF}$ (equation (58)), with their corresponding $\mathbf{P}_X$ being $\mathbf{P}_{PIF}$, $\mathbf{P}_{WF}$ and $\mathbf{P}_{KF}$. Note that $\mathbf{L}_{WF}$ is equivalent to $\mathbf{L}_{KF}$, and $\mathbf{P}_{WF}$ is identical to $\mathbf{P}_{KF}$ which converges to $\mathbf{N}_\infty$ (see equation (61)). For the 3-state tracking model, the main diagonal elements of $\mathbf{P}_X$ represent the instantaneous estimation error for initial carrier phase, frequency, and frequency rate, respectively. The remaining elements of $\mathbf{P}_X$ denote the error covariance between the state variables. The error variance of the overall average phase over an integration time T can be obtained as:

$$\mathbf{p}_{\bar{\theta}} = E \left[(\bar{\theta}_{k+1} - \hat{\theta}_{k+1})^2 \right] = \mathbf{H} \mathbf{E} \left[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}) (\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T \right] \mathbf{H}^T = \mathbf{H} \mathbf{P}_X \mathbf{H}^T. \quad (91)$$

Substituting equation (90) into (91), it can be seen that $\mathbf{p}_{\bar{\theta}}$ is related to system characteristics, such as oscillator noise, platform dynamics, signal C/N_0 , and signal frequency f_L . Equation

(91) also provides a quantitative relationship between the PLL tracking accuracy and design parameters, such as T or BN (if $\mathbf{L}_{PIF}$ is adopted).

2) *FLL error variance*: Substituting the received and local generated signals in equations (64) and (81) into (89), we have:

$$\begin{aligned} \mathbf{P}_{k+1} = & \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{P}_k (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T + 2\mathbf{A}_F \mathbf{L}_F \frac{\mathbf{R}}{T^2} (\mathbf{A}_F \mathbf{L}_F)^T \\ & + \mathbf{Q}_F - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{G}_k (\mathbf{A}_F \mathbf{L}_F)^T - \mathbf{A}_F \mathbf{L}_F \mathbf{G}_k^T (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T \end{aligned} \quad (92)$$

where $\mathbf{G}_k = E[\Delta \mathbf{x}_k \mathbf{v}_k^T]$ and $\mathbf{G}_k^T = E[\mathbf{v}_k \Delta \mathbf{x}_k^T]$ represent the correlation between the error state and non-white measurement noise.

At steady state, $\mathbf{P}_{k+1} = \mathbf{P}_k = \mathbf{P}_Y$ and $\mathbf{G}_{k+1} = \mathbf{G}_k = \mathbf{G}_Y$ (see derivation in Appendix D):

$$\mathbf{G}_Y = \frac{\mathbf{A}_F \mathbf{L}_F}{T^2} \sigma_v^2 \quad (93)$$

and

$$\begin{aligned} \mathbf{P}_Y = & \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{P}_k (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T + 2\mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} \\ & + \mathbf{Q}_F - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} \\ & - \mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T \frac{\sigma_v^2}{T^2}. \end{aligned} \quad (94)$$

Denoting $\mathbf{p}_{\bar{\omega}}$ as the overall average frequency error variance, we can get:

$$\mathbf{p}_{\bar{\omega}} = \mathbf{H}_F E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T] \mathbf{H}_F^T = \mathbf{H}_F \mathbf{P}_Y \mathbf{H}_F^T. \quad (95)$$

Substituting equation (94) into (95), it can be seen that $\mathbf{p}_{\bar{\omega}}$ is again related to system characteristics, such as oscillator noise, platform dynamics, signal C/N_0 , and signal frequency f_L . Like equation (91), equation (95) also provides a quantitative relationship between the FLL tracking accuracy and design parameters, such as T or BW .

B. Dynamic stress steady-state error

When the receiver platform acceleration $a_k \neq 0m/s^2$, a steady-state error will occur in the 2-state PLL and 1-state FLL. Similarly, when the receiver platform jerk $j_k \neq 0m/s^3$, a steady-state error will occur in the 3-state PLL and 2-state FLL. Here, we derived the dynamic stress error in the carrier tracking loops for the dynamic scenarios having constant acceleration or constant jerk. To reflect this dynamic, we add an input dynamic vector $\mathbf{d}_k$

$$\mathbf{d}_k = \begin{cases} a_k (m/s^2) & \text{for 2-state PLL or 1-state FLL} \\ j_k (m/s^3) & \text{for 3-state PLL or 2-state FLL} \end{cases} \quad (96)$$

to the system model [34] and obtain:

$$\mathbf{x}_{k+1} = \mathbf{A} \mathbf{x}_k + \mathbf{n}_k + \mathbf{M} \mathbf{d}_k \quad (97)$$

where $\mathbf{M}$ is an operator that maps the dynamic input vector $\mathbf{d}_k$ to the state vector.

Thus, the dynamic stress error at $(k+1)^{th}$ epoch can be

expressed as:

$$E(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}) = \mathbf{A} (\mathbf{I} - \mathbf{L} \mathbf{H}) E(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{M} \mathbf{d}_k. \quad (98)$$

1) *PLL dynamic stress steady-state error*: $\mathbf{M}$ for PLL has the following corresponding forms:

$$2\text{-state} : \mathbf{M}_P = \begin{bmatrix} \frac{2\pi f_L T^2}{c} & \frac{2\pi f_L T}{c} \end{bmatrix}^T \quad (99)$$

$$3\text{-state} : \mathbf{M}_P = \begin{bmatrix} \frac{2\pi f_L T^3}{c} & \frac{2\pi f_L T^2}{c} & \frac{2\pi f_L T}{c} \end{bmatrix}^T. \quad (100)$$

Denoting $\mathbf{e}_X = \lim_{k \rightarrow \infty} E(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})$ as the PLL state dynamic stress error, we have:

$$\mathbf{e}_X = [\mathbf{I} - \mathbf{A}_P (\mathbf{I} - \mathbf{L} \mathbf{H}_P)]^{-1} \mathbf{M}_P \mathbf{d}_k. \quad (101)$$

The steady-state dynamic stress phase error for a PLL can be expressed as:

$$\begin{aligned} \mathbf{e}_{\bar{\theta}} = & \lim_{k \rightarrow \infty} E(\bar{\theta}_{k+1} - \hat{\theta}_{k+1}) = \mathbf{H}_P \mathbf{e}_X \\ = & \mathbf{H}_P [\mathbf{I} - \mathbf{A}_P (\mathbf{I} - \mathbf{L} \mathbf{H}_P)]^{-1} \mathbf{M}_P \mathbf{d}_k \end{aligned} \quad (102)$$

$\mathbf{e}_{\bar{\theta}}$ is related to platform dynamics and signal frequency f_L . Equation (102) also provides the quantitative relationship between the dynamic stress error and design parameters, such as T or BN (if $\mathbf{L}_{PIF}$ is adopted).

2) *FLL dynamic stress steady-state error*: For FLL, $\mathbf{M}$ has the following corresponding forms:

$$1\text{-state} : \mathbf{M}_F = \frac{2\pi f_L T}{c} \quad (103)$$

$$2\text{-state} : \mathbf{M}_F = \begin{bmatrix} \frac{2\pi f_L T^2}{c} & \frac{2\pi f_L T}{c} \end{bmatrix}^T. \quad (104)$$

Denoting $\mathbf{e}_Y = \lim_{k \rightarrow \infty} E(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})$ as the FLL state dynamic stress error, we have:

$$\mathbf{e}_Y = [\mathbf{I} - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)]^{-1} \mathbf{M}_F \mathbf{d}_k. \quad (105)$$

Hence, the overall dynamic stress frequency error for a FLL can be expressed as:

$$\begin{aligned} \mathbf{e}_{\bar{\omega}} = & \lim_{k \rightarrow \infty} E(\bar{\omega}_{k+1} - \hat{\omega}_{k+1}) = \mathbf{H}_F \mathbf{e}_Y \\ = & \mathbf{H}_F [\mathbf{I} - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)]^{-1} \mathbf{M}_F \mathbf{d}_k. \end{aligned} \quad (106)$$

$\mathbf{e}_{\bar{\omega}}$ is also dependent on platform dynamics and signal frequency f_L . Equation (106) also provides the quantitative relationship between the dynamic stress error and design parameters, such as T or BW .

C. The 3-sigma rule

1) *PLL*: The 3-sigma phase jitter [2] can be expressed as:

$$3\sigma_{PLL} = 3\sqrt{\mathbf{p}_{\bar{\theta}}} + \mathbf{e}_{\bar{\theta}} \quad (107)$$

In traditional PLL design, a conservative rule of thumb requires that the 3-sigma phase jitter does not exceed one-fourth of the phase pull-in range of the PLL discriminator [2]. For the two-quadrant arctangent and four-quadrant phase discriminators in data and pilot channels, the tracking error should satisfy the following conditions, respectively [2]:

$$\sigma_{PLL} = \sqrt{\mathbf{p}_{\bar{\theta}}} + \frac{\mathbf{e}_{\bar{\theta}}}{3} \leq 15^\circ \text{ or } 30^\circ. \quad (108)$$

We can observe that σ_{PLL} is related to the users' dynamic, signal strength, oscillator quality, loop parameters, and signal frequency f_L . Therefore, to satisfy the above conditions, the loop parameters, such as T or BN , should be carefully designed to adapt to different environments to avoid loss of lock.

2) *FLL*: In traditional FLL design, a conservative rule of thumb is usually applied to determine the tracking threshold of FLL, which requires that the 3-sigma frequency jitter must not exceed one-fourth of the frequency pull-in range of the FLL discriminator [2]. Therefore, for the two-quadrant arctangent and four-quadrant frequency discriminators in data and pilot channels, the tracking error should satisfy the following conditions, respectively [2]:

$$\sigma_{FLL} = \sqrt{\mathbf{p}_{\bar{\omega}}} + \frac{\mathbf{e}_{\bar{\omega}}}{3} \leq \frac{1}{24T}(\text{Hz}) \text{ or } \frac{1}{12T}(\text{Hz}) \quad (109)$$

Equation (109) can be applied to characterize the admissible range of the loop parameters, such as T or BW , for FLL design implementation under various operating conditions, with varying signal strength, platform dynamics, and oscillator qualities. The maximum value of T is 10ms in GPS data channel to avoid the data bit transition. Equation (109) shows that as T increases from 1ms to 10ms, the threshold for the pilot channel decreases from 83.3Hz to 8.33Hz. A shorter integration time is also preferred because the threshold decreases as T increases.

In summary, σ_{PLL} and σ_{FLL} in equations (108) and (109) provide the admissible range of the loop parameters under various operating conditions having varying signal strength, platform dynamics, and oscillator qualities. If $\mathbf{d}_k$ is non-zero, there will be a bias, such as $\mathbf{e}_{\bar{\theta}}$ and $\mathbf{e}_{\bar{\omega}}$ that degrades the PLL and FLL's tracking performance. To avoid the performance degradation caused by the dynamic stress error, an additional state should be included in the state model to account for this unknown constant bias. For example, under the constant acceleration where $\mathbf{d}_k = a_k \neq 0\text{m/s}^2$, there will be a bias in 2-state PLL and 1-state FLL, while 3-state PLL and 2-state FLL are unbiased because the additional state $\dot{\omega}_k$ is used to estimate this bias. To further improve the tracking performance, the loop parameters that minimize $\mathbf{p}_{\bar{\theta}}$ and $\mathbf{p}_{\bar{\omega}}$ will be discussed in a separate companion paper [26].

VI. CONCLUSIONS

In this paper, we established a generalized GNSS signal carrier tracking loop architecture using a state space representation for carrier phase and carrier frequency tracking. A state space model and state feedback control system for general PLL and FLL are established. Three PLL design methods, PIF, WF and KF are presented and relationships between their corresponding state estimator gain matrices to parameters, such as the integration time, the loop bandwidth, dynamic parameter, and the oscillator h -parameters, are derived. Our analysis shows that the WF and KF designs are equivalent, where the WF state estimator gain is the equivalent closed form expression for the steady-state KF gain. In a generalized FLL design, the PIF-based gain matrix is obtained by mapping the PIF into the state space architecture through the equivalent closed-loop transfer function due to the non-white noise characteristic.

Performance metrics, such as phase and frequency tracking error variances and dynamic stress errors, were derived to evaluate the performance of various tracking loop designs, taking into consideration of thermal noise, oscillator noise, and platform dynamics. The generalized tracking loop architecture enables us to represent and compare the different tracking loop designs under a unified theoretical framework. The optimization of loop parameters using the state space representation will be discussed in a separate companion paper [26].

VII. ACKNOWLEDGEMENTS

Rong Yang is supported by a research scholarship from Nanyang Technological University.

Yu Morton's contribution is partially supported by an AFRL grant #FA8650-14-1735 at Colorado State University.

APPENDIX A

B AND K MATRICES DESIGN CONSIDERATIONS IN PLL

The control theoretic analysis of $\mathbf{B}$, $\mathbf{K}$ matrices design is presented in here. Taking the 2-state system as an example, there are six possible choices of $\mathbf{B}$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (A.1)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (A.2)$$

depending whether $\mathbf{u}_k$ is a scalar or a vector.

A straight-forward controllability analysis shows that only $\mathbf{B} = [1; 1]$ and $\mathbf{B} = [1, 0; 0, 1]$ are acceptable. This is because for $\mathbf{B} = [1; 0]$ and $[1, 0; 0, 0]$, the rank of $[\mathbf{B} | \mathbf{A}_p \mathbf{B}]$ is 1, which indicates uncontrollable systems. For $\mathbf{B} = [0; 1]$, $[1; 1]$, and $[0, 0; 0, 1]$, the rank of $[\mathbf{B} | \mathbf{A}_p \mathbf{B}]$ is 2 only when the value of T is nonzero, which is indeed the case for a GNSS tracking loop. However, for $\mathbf{B} = [0; 1]$ and $[0, 0; 0, 1]$, it is not possible for $\hat{\mathbf{x}}_k$ to transit from an arbitrary state to a desired state in one step. Finally, for $\mathbf{B} = [1, 0; 0, 1]$, $[\mathbf{B} | \mathbf{A}_p \mathbf{B}]$ is full rank, regardless of the value of T , and hence the system is completely controllable. Therefore, $\mathbf{B} = [1; 1]$ and $[1, 0; 0, 1]$ are the only candidates associated with controllable systems for the scalar and vector controller $\mathbf{u}_k$, respectively. It should be noted that indeed $\mathbf{B} = [1, 0; 0, 1]$ was adopted in [35] although a detailed explanation was not provided.

To ensure the closed-loop matrix $\mathbf{A}_p - \mathbf{B}\mathbf{K}$ has all its eigenvalues at the origin, the well-known pole placement approach [27] was applied to derive the matrix $\mathbf{K}$ as the following:

$$\mathbf{K} = \begin{cases} \begin{bmatrix} \frac{1}{T} & 2 - \frac{1}{T} \end{bmatrix}, & \text{for } \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{for } \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases} \quad (A.3)$$

and the corresponding closed-loop matrix is:

$$\mathbf{A}_P - \mathbf{B}\mathbf{K} = \begin{cases} \begin{bmatrix} 1 - \frac{1}{T} & T - 2 + \frac{1}{T} \\ -\frac{1}{T} & -1 + \frac{1}{T} \end{bmatrix}, & \text{for } \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{K} = \begin{bmatrix} \frac{1}{T} & 2 - \frac{1}{T} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{for } \mathbf{B} = \mathbf{I}, \text{ and } \mathbf{K} = \mathbf{A}_P \\ \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, & \text{for } \mathbf{B} = \mathbf{I}, \text{ and } \mathbf{K} = \mathbf{I} \end{cases} \quad (\text{A.4})$$

Equation (A.4) shows that for $\mathbf{K} = \begin{bmatrix} \frac{1}{T} & 2 - \frac{1}{T} \end{bmatrix}$ and $\mathbf{B} = [1; 1]$, the values of $\mathbf{K}$ and $\mathbf{A}_P - \mathbf{B}\mathbf{K}$ become very large since the typical value of T is 1ms in GNSS applications. Therefore, the error state is accumulated and amplified by this gain through equations (27) and (29). Such a situation should be avoided. According to (29) and (A.4), for $\mathbf{B} = \mathbf{I}$ and $\mathbf{K} = \mathbf{I}$, the error state estimation becomes a first-order Gauss-Markov process, where an inaccurate prior estimation will affect the present estimates. This effect will be eliminated in 2 sampling periods, since $(\mathbf{A}_P - \mathbf{B}\mathbf{K})^2 = 0$. For the case $\mathbf{B} = \mathbf{I}$ and $\mathbf{K} = \mathbf{A}_P$, equation (29) reduces to $\Delta \hat{\mathbf{x}}_{k+1} = \mathbf{L} \Delta \theta_{k+1}$, hence the integrator action in the estimator is eliminated and the estimation error will not be accumulated. Hence, the desired state could be achieved instantly. A similar conclusion can be derived for the 3-state case.

Although this paper focuses on a general design framework and fundamental performance analysis, practical implementation feasibility and constraints should still be taken into consideration for the proposed schemes. One such practical issue concerns the implementation of the local reference generator, which corresponds to the NCO in the traditional tracking loop. In a traditional PLL, the loop filter output is used for a NCO to update its carrier phase and carrier frequency (phase-rate) registers. In a PLL with rate-only feedback NCO, the NCO rate register is updated, but the NCO phase register is left untouched, while in a PLL with phase and phase-rate feedback NCO, both the rate and phase registers are updated, based on the single input from the loop filter output [5] [8] [36]. Therefore, if $\mathbf{u}_k$ is selected scalar variable and $\mathbf{B}$ is set to $[0; 1]$ or $[1; 1]$, the local reference generators effectively correspond to the rate-only feedback NCO or phase and rate feedback NCO, respectively. The analysis on controllability and stability presented in this paper leads to the recommendation of a two-input system, which requires carrier phase and phase-rate updated with respect to their corresponding corrections from the two-input variables. While the two-input system imposes certain limitations on practical implementations, such limitations are diminishing as the receiver design trend becomes more software-defined. The same argument applies to the multi-input NCO model challenges in the 3-state system. If we assume that the NCO is implemented in software, then the 3-state variables: $\hat{\varphi}$, $\hat{\omega}$, and $\hat{\dot{\omega}}$ can be used to generate the carrier signals. However, if traditional designs implement only the single-input NCOs where the phase and phase-rate registers for some practical reasons, then some conversions from multi-input to single-input, as well as some approximations between the frequency-rate ($\hat{\dot{\omega}}$) register, and

phase-rate register are needed.

APPENDIX B

WIENER FILTER TRANSFER FUNCTION DERIVATION IN PLL

For the 2-state model, the z transformation of the auto-correlation function of the input signal $\bar{\theta}_k$ can be expressed as:

$$S_{\bar{\theta}2}(z) = \frac{\frac{T^2}{4} z(z+1)^2 \sigma_\omega^2 - 4z(z-1)^2 \sigma_\varphi^2}{(z-1)^4}. \quad (\text{B.1})$$

From equation (17), we obtain:

$$S_\theta(z) = S_{\bar{\theta}}(z) + \sigma_v^2. \quad (\text{B.2})$$

Using spectral factorization [37],

$$S_\theta(z) = Y(z)Y(z^{-1}) \quad (\text{B.3})$$

We obtain:

$$Y_2(z) = \sigma_v \frac{(z - z_0)(z - z_1)}{(z - 1)^2}. \quad (\text{B.4})$$

Since z_0 and z_1 should be real numbers or complex conjugates, they have to take the following values:

$$z_i = \frac{(\gamma_i - \sqrt{\gamma_i^2 - 4})}{2}, i = 0, 1 \quad (\text{B.5})$$

where

$$\gamma_i = 2 + \left(\frac{\sigma_a^2}{2}\right) + (-1)^i \sqrt{\left(\frac{\sigma_a^2}{2}\right)^2 - \sigma_b^2}, i = 0, 1 \quad (\text{B.6})$$

and

$$\begin{cases} \sigma_a^2 = \frac{\sigma_\varphi^2}{\sigma_v^2} - \frac{T^2}{4} \frac{\sigma_\omega^2}{\sigma_v^2} \\ \sigma_b^2 = T^2 \frac{\sigma_\omega^2}{\sigma_v^2} \end{cases} \quad (\text{B.7})$$

Then, we can obtain the 2-state WF transfer function

$$H_{WF2}(z) = \frac{\left[\frac{S_{\bar{\theta}2}(z)}{Y_2(z^{-1})}\right]_+}{Y_2(z)} = 1 - \frac{\sigma_v}{Y_2(z)} = \frac{(2 - z_0 - z_1)z + z_0 z_1 - 1}{z^2 - (z_0 + z_1)z + z_0 z_1}. \quad (\text{B.8})$$

Similarly, z transformation of the auto-correlation functions of input signal θ_k for 3-state model can be written as:

$$S_{\theta3}(z) = \sigma_v^2 \frac{\left(z^6 - \left(6 + \frac{T^2}{3} \frac{\sigma_\omega^2}{\sigma_v^2} + \frac{T^4}{36} \frac{\sigma_\omega^2}{\sigma_v^2} + \frac{\sigma_\varphi^2}{\sigma_v^2} - \frac{T^2}{4} \frac{\sigma_\omega^2}{\sigma_v^2} \right) z^5 + \left(15 + 4 \frac{\sigma_\varphi^2}{\sigma_v^2} - \frac{2T^4}{9} \frac{\sigma_\omega^2}{\sigma_v^2} - \frac{2T^2}{3} \frac{\sigma_\omega^2}{\sigma_v^2} \right) z^4 - \left(20 + 6 \frac{\sigma_\varphi^2}{\sigma_v^2} + \frac{T^4}{2} \frac{\sigma_\omega^2}{\sigma_v^2} + \frac{T^2}{2} \frac{\sigma_\omega^2}{\sigma_v^2} - 2T^2 \frac{\sigma_\omega^2}{\sigma_v^2} \right) z^3 + \left(15 + 4 \frac{\sigma_\varphi^2}{\sigma_v^2} - \frac{2T^4}{9} \frac{\sigma_\omega^2}{\sigma_v^2} - \frac{2T^2}{3} \frac{\sigma_\omega^2}{\sigma_v^2} \right) z^2 - \left(6 + \frac{T^2}{3} \frac{\sigma_\omega^2}{\sigma_v^2} + \frac{T^4}{36} \frac{\sigma_\omega^2}{\sigma_v^2} + \frac{\sigma_\varphi^2}{\sigma_v^2} - \frac{T^2}{4} \frac{\sigma_\omega^2}{\sigma_v^2} \right) z + 1 \right)}{(z-1)^6}. \quad (\text{B.9})$$

According to equation (B.3), S_θ should be expressed as the

following form

$$S_\theta(z) = \sigma_v^2 \frac{(z - z_0)(z - z_1)(z - z_2) \left(z - \frac{1}{z_0}\right) \left(z - \frac{1}{z_1}\right) \left(z - \frac{1}{z_2}\right)}{(z - 1)^6}. \quad (\text{B.10})$$

Thus, $Y(z)$ can be written as:

$$Y(z) = \sigma_v \frac{(z - z_0)(z - z_1)(z - z_2)}{(z - 1)^3}. \quad (\text{B.11})$$

Expanding equation (B.10), we obtain:

$$S_\theta(z) = \sigma_v^2 \frac{\left\{ \left(z^2 - \left(z_0 + \frac{1}{z_0}\right)z + 1\right) \cdot \left(z^2 - \left(z_1 + \frac{1}{z_1}\right)z + 1\right) \cdot \left(z^2 - \left(z_2 + \frac{1}{z_2}\right)z + 1\right) \right\}}{(z - 1)^6}. \quad (\text{B.12})$$

Denoting r_0 , r_1 and r_2 as:

$$\begin{cases} r_0 = \left(z_0 + \frac{1}{z_0}\right) \\ r_1 = \left(z_1 + \frac{1}{z_1}\right) \\ r_2 = \left(z_2 + \frac{1}{z_2}\right) \end{cases} \quad (\text{B.13})$$

and by solving the quadratic equation, the roots of S_θ can be obtained as:

$$z_i = \frac{r_i \pm \sqrt{r_i^2 - 4}}{2}, i \in \{0, 1, 2\}. \quad (\text{B.14})$$

Among these 6 roots, only three of them are located inside the unit circle and the rest are located outside the unit circle. Their locations depend on r_i and r_i and can be solved by the following steps.

First, we substitute r_i into S_θ to obtain:

$$\begin{aligned} S_\theta(z) &= \sigma_v^2 \frac{(z^2 - r_0z + 1)(z^2 - r_1z + 1)(z^2 - r_2z + 1)}{(z - 1)^6} \\ &= \sigma_v^2 \frac{\left\{ \begin{aligned} &z^6 - (r_0 + r_1 + r_2)z^5 + (r_0r_1 + r_1r_2 + r_0r_2 + 3)z^4 \\ &- (r_0r_1r_2 + 2(r_0 + r_1 + r_2))z^3 \\ &+ (r_0r_1 + r_1r_2 + r_0r_2 + 3)z^2 - (r_0 + r_1 + r_2)z + 1 \end{aligned} \right\}}{(z - 1)^6}. \end{aligned} \quad (\text{B.15})$$

Then, by comparing the above equation to equation (B.9), we have:

$$\begin{cases} r_0 + r_1 + r_2 = 6 + \frac{T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2} + \frac{T^4}{36} \frac{\sigma_{\omega}^2}{\sigma_v^2} + \frac{\sigma_{\varphi}^2}{\sigma_v^2} - \frac{T^2}{4} \frac{\sigma_{\omega}^2}{\sigma_v^2} \\ r_0r_1 + r_1r_2 + r_0r_2 = 12 + 4 \frac{\sigma_{\varphi}^2}{\sigma_v^2} - \frac{2T^4}{9} \frac{\sigma_{\omega}^2}{\sigma_v^2} - \frac{2T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2} \\ r_0r_1r_2 = 8 + 4 \frac{\sigma_{\varphi}^2}{\sigma_v^2} + \frac{4T^4}{9} \frac{\sigma_{\omega}^2}{\sigma_v^2} + T^2 \frac{\sigma_{\omega}^2}{\sigma_v^2} - \frac{8T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2}. \end{cases} \quad (\text{B.16})$$

The above three equations indicate that r_0 , r_1 and r_2 are the three roots of $(z + r_0)(z + r_1)(z + r_2) = 0$. To solve this cubic equation the Cardano's Formula is applied and r_i can be expressed in the following forms:

$$r_0 = 2 + x + \sqrt[3]{y + \sqrt{y^2 + m^3}} + \sqrt[3]{y - \sqrt{y^2 + m^3}} \quad (\text{B.17})$$

$$r_1 = 2 + x + \frac{-1 + \sqrt{3}j}{2} \sqrt[3]{y + \sqrt{y^2 + m^3}} + \frac{-1 - \sqrt{3}j}{2} \sqrt[3]{y - \sqrt{y^2 + m^3}} \quad (\text{B.18})$$

$$r_2 = 2 + x + \frac{-1 - \sqrt{3}j}{2} \sqrt[3]{y + \sqrt{y^2 + m^3}} + \frac{-1 + \sqrt{3}j}{2} \sqrt[3]{y - \sqrt{y^2 + m^3}}. \quad (\text{B.19})$$

Then we have:

$$\begin{cases} r_0 + r_1 + r_2 = 6 + 3x \\ r_0r_1 + r_1r_2 + r_0r_2 = 12 + 12x + 3x^2 + 3m \\ r_0r_1r_2 = 8 + 12x + 2y + 6x^2 + x^3 + 3xm + 6m. \end{cases} \quad (\text{B.20})$$

Comparing equation (B.20) to equation (B.16), x , y and m can be determined as below:

$$\begin{cases} x = d/3 \\ y = 2d + \frac{1}{27}d^3 - f + \frac{2}{3}d^2 - \frac{1}{6}df + \frac{1}{2}g \\ m = (3f - 12d - d^2)/9 \end{cases} \quad (\text{B.21})$$

with

$$\begin{cases} d = \frac{T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2} + \frac{T^4}{36} \frac{\sigma_{\omega}^2}{\sigma_v^2} + \frac{\sigma_{\varphi}^2}{\sigma_v^2} - \frac{T^2}{4} \frac{\sigma_{\omega}^2}{\sigma_v^2} \\ g = 4 \frac{\sigma_{\varphi}^2}{\sigma_v^2} + \frac{4T^4}{9} \frac{\sigma_{\omega}^2}{\sigma_v^2} + T^2 \frac{\sigma_{\omega}^2}{\sigma_v^2} - \frac{8T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2} \\ f = 4 \frac{\sigma_{\varphi}^2}{\sigma_v^2} - \frac{2T^4}{9} \frac{\sigma_{\omega}^2}{\sigma_v^2} - \frac{2T^2}{3} \frac{\sigma_{\omega\varphi}^2}{\sigma_v^2}. \end{cases} \quad (\text{B.22})$$

Finally, z_0 , z_1 and z_2 are solved and can be written as:

$$z_i = \frac{r_i \pm \sqrt{r_i^2 - 4}}{2}, i = 0, 1, 2 \quad (\text{B.23})$$

Equation (B.23) has 6 roots due to the additional parameter q_a 's effect on the roots' locations. Among the 6 roots, only three of them are located inside the unit circle, and the rest are located outside circle. To keep the loop stable, z_0 , z_1 and z_2 should be the roots inside or on the unit circle.

Then, we could obtain the 3-state WF transfer function:

$$\begin{aligned} H_{WF3}(z) &= \frac{\left[\frac{S_{\theta 3}(z)}{Y_3(z^{-1})} \right]_+}{Y_3(z)} = 1 - \frac{\sigma_v}{Y_3(z)} \\ &= \frac{(3 - z_0 - z_1 - z_2)z^2 + (z_0z_1 + z_0z_2 + z_1z_2 - 3)z + 1 - z_0z_1z_2}{z^3 + (-z_0 - z_1 - z_2)z^2 + (z_0z_1 + z_0z_2 + z_1z_2)z - z_0z_1z_2}. \end{aligned} \quad (\text{B.24})$$

APPENDIX C

MEASUREMENT MODEL DERIVATION IN FLL

According to equations (18) and (21),

$$\begin{aligned} \Delta\theta_{k+1} - \Delta\theta_k &= \frac{T}{2} \Delta\omega_{k+1} + \Delta\varphi_{k+1} + v_{k+1} - \frac{T}{2} \Delta\omega_k - \Delta\varphi_k - v_k \\ &= \frac{T}{2} \Delta\omega_{k+1} + T\Delta\omega_k + \Delta\varphi_k + v_{k+1} - \frac{T}{2} \Delta\omega_k - \Delta\varphi_k - v_k \\ &= \frac{T}{2} (\Delta\omega_{k+1} + \Delta\omega_k) + v_{k+1} - v_k \end{aligned} \quad (\text{C.1})$$

Assuming that the frequency error will not change by a significant amount over one sample period, we have:

$$\Delta\theta_{k+1} - \Delta\theta_k \approx T\Delta\omega_{k+1} + v_{k+1} - v_k \quad (\text{C.2})$$

Similarly, according to equations (19) and (21)

$$\begin{aligned}
\Delta\theta_{k+1} - \Delta\theta_k &= \frac{T^2}{6}\Delta\dot{\omega}_{k+1} + \frac{T}{2}\Delta\omega_{k+1} + \Delta\varphi_{k+1} \\
&\quad + v_{k+1} - \frac{T^2}{6}\Delta\dot{\omega}_k - \frac{T}{2}\Delta\omega_k - \Delta\varphi_k - v_k \\
&= \frac{T^2}{6}\Delta\dot{\omega}_{k+1} + \frac{T}{2}\Delta\omega_{k+1} + \frac{T^2}{2}\Delta\dot{\omega}_k + T\Delta\omega_k + \Delta\varphi_k + v_{k+1} \\
&\quad - \frac{T^2}{6}\Delta\dot{\omega}_k - \frac{T}{2}\Delta\omega_k - \Delta\varphi_k - v_k \\
&= \left(\frac{T^2}{6}\Delta\dot{\omega}_{k+1} + \frac{T^2}{2}\Delta\dot{\omega}_k - \frac{T^2}{6}\Delta\dot{\omega}_k \right) \\
&\quad + \frac{T}{2}(\Delta\omega_{k+1} + \Delta\omega_k) + v_{k+1} - v_k
\end{aligned} \tag{C.3}$$

Again, assuming that the frequency rate error will not change by a significant amount over one sample period, we have:

$$\Delta\theta_{k+1} - \Delta\theta_k \approx \frac{T^2}{2}\Delta\dot{\omega}_{k+1} + T\Delta\omega_{k+1} + v_{k+1} - v_k \tag{C.4}$$

APPENDIX D

TRACKING ERROR COVARIANCE MATRIX DERIVATION IN FLL

Substituting the received and local generated signal in equations (64), (75) and (81) into the equation (89), we have:

$$\begin{aligned}
\mathbf{P}_{k+1} &= E \left[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}) (\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T \right] \\
&= E \left[\begin{pmatrix} \mathbf{A}_F \mathbf{x}_k + \mathbf{n}_k - \mathbf{A}_F \hat{\mathbf{x}}_k - \mathbf{A}_F \mathbf{L}_F \Delta \varpi_k \\ \mathbf{A}_F \mathbf{x}_k + \mathbf{n}_k - \mathbf{A}_F \hat{\mathbf{x}}_k - \mathbf{A}_F \mathbf{L}_F \Delta \varpi_k \end{pmatrix}^T \right] \\
&= E \left[\begin{pmatrix} \mathbf{A}_F \Delta \mathbf{x}_k + \mathbf{n}_k - \mathbf{A}_F \mathbf{L}_F (\mathbf{H}_F \Delta \mathbf{x}_k + v_k) \\ \mathbf{A}_F \Delta \mathbf{x}_k + \mathbf{n}_k - \mathbf{A}_F \mathbf{L}_F (\mathbf{H}_F \Delta \mathbf{x}_k + v_k) \end{pmatrix}^T \right] \\
&= \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) E [\Delta \mathbf{x}_k \Delta \mathbf{x}_k^T] (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T \\
&\quad + 2\mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) E [\Delta \mathbf{x}_k v_k^T] (\mathbf{A}_F \mathbf{L}_F)^T \\
&\quad + \mathbf{Q}_F - \mathbf{A}_F \mathbf{L}_F E [v_k \Delta \mathbf{x}_k^T] (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T
\end{aligned} \tag{D.1}$$

Let $\mathbf{G}_k$ represents the correlation between the error state and non-white measurement noise, we have $\mathbf{G}_k = E [\Delta \mathbf{x}_k v_k^T]$ and $\mathbf{G}_k^T = E [v_k \Delta \mathbf{x}_k^T]$. When k approaches infinity, $\mathbf{G}_{k+1} = \mathbf{G}_k = \mathbf{G}_Y$ is in a steady state. We expand $\mathbf{G}_Y$ as following steps:

$$\begin{aligned}
\mathbf{G}_Y &= E [\Delta \mathbf{x}_k v_k^T] \\
&= E \left[-(\mathbf{A}_F \hat{\mathbf{x}}_{k-1} + \mathbf{A}_F \mathbf{L}_F \Delta \varpi_{k-1}) \frac{(v_k - v_{k-1})}{T} \right] \\
&= E \left[-\mathbf{A}_F \mathbf{L}_F \frac{(v_{k-1} - v_{k-2})}{T} \frac{(v_k - v_{k-1})}{T} \right] \\
&= \mathbf{A}_F \mathbf{L}_F \frac{\sigma_v^2}{T^2}
\end{aligned} \tag{D.2}$$

Finally, we can obtain:

$$\begin{aligned}
\mathbf{P}_Y &= \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{P}_Y (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T + \mathbf{Q}_F + 2\mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} \\
&\quad + \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{G}_Y (\mathbf{A}_F \mathbf{L}_F)^T + \mathbf{A}_F \mathbf{L}_F \mathbf{G}_Y^T (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T \\
&= \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{P}_Y (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T + \mathbf{Q}_F + 2\mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} \\
&\quad - \mathbf{A}_F (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F) \mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T \frac{\sigma_v^2}{T^2} \\
&\quad - \mathbf{A}_F \mathbf{L}_F (\mathbf{A}_F \mathbf{L}_F)^T (\mathbf{I} - \mathbf{L}_F \mathbf{H}_F)^T \mathbf{A}_F^T \frac{\sigma_v^2}{T^2}
\end{aligned} \tag{D.3}$$

REFERENCES

- [1] J. A. López-Salcedo, D. Peral-Rosado, G. Seco-Granados, *et al.*, "Survey on robust carrier tracking techniques," *Communications Surveys and Tutorials, IEEE*, vol. 16, no. 2, pp. 670–688, 2014.
- [2] P. W. Ward, J. W. Betz, and C. J. Hegarty, "Satellite signal acquisition, tracking, and data demodulation," in *Understanding GPS principles and applications, Second Edition* (E. D. Kaplan and C. J. Hegarty, eds.), ch. 5, pp. 170–258, Artech house, 2005.
- [3] J. B.-Y. Tsui, *Fundamentals of Global Positioning System Receivers: A Software Approach, Second Edition*, ch. 8, pp. 165–190. John Wiley and Sons Ltd, 2005.
- [4] A. Razavi, D. Gebre-Egziabher, and D. M. Akos, "Carrier loop architectures for tracking weak GPS signals," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 44, no. 2, pp. 697–710, 2008.
- [5] P. L. Kazemi, *Development of new filter and tracking schemes for weak GPS signal tracking*. PhD thesis, University of Calgary, 2010.
- [6] D. Borio, N. Sokolova, and G. Lachapelle, "Memory discriminators for non-coherent integration in GNSS tracking loops," in *Proc. European Navigation Conf. ENC*, vol. 9, 2009.
- [7] W. Zhuang, "Performance analysis of GPS carrier phase observable," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 32, no. 2, pp. 754–767, 1996.
- [8] S. Stephens and J. Thomas, "Controlled-root formulation for digital phase-locked loops," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 31, no. 1, pp. 78–95, 1995.
- [9] C. Odriscoll and G. Lachapelle, "Comparison of traditional and Kalman filter based tracking architectures," in *Proceedings of European navigation conference*, pp. 3–6, 2009.
- [10] J.-H. Won, D. Dötterböck, and B. Eissfeller, "Performance comparison of different forms of Kalman filter approaches for a vector-based GNSS signal tracking loop," *Navigation*, vol. 57, no. 3, pp. 185–199, 2010.
- [11] J.-H. Won, T. Pany, and B. Eissfeller, "Characteristics of Kalman filters for GNSS signal tracking loop," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 48, no. 4, pp. 3671–3681, 2012.
- [12] B. D. Anderson and J. B. Moore, *Optimal filtering*, ch. 9, pp. 264–268. Courier Dover Publications, 2012.
- [13] J. T. Curran, G. Lachapelle, and C. C. Murphy, "Digital GNSS PLL design conditioned on thermal and oscillator phase noise," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 49, no. 1, pp. 687–687, 2013.
- [14] J. T. Curran, *Weak signal digital GNSS tracking algorithms*. PhD thesis, National University of Ireland, 2010.
- [15] P. L. Kazemi, "Optimum digital filters for GNSS tracking loops," in *Proceedings of international technical meeting of the Satellite Division of the Institute of Navigation (ION GNSS)*, Savannah, GA, pp. 16–19, 2008.
- [16] D. Peral-Rosado, J. A. López-Salcedo, G. Seco-Granados, J. M. López-Almansa, J. Cosmen, *et al.*, "Kalman filter-based architecture for robust and high-sensitivity tracking in GNSS receivers," in *Satellite Navigation Technologies and European Workshop on GNSS Signals and Signal Processing (NAVITEC), 2010 5th ESA Workshop on*, pp. 1–8, IEEE, 2010.
- [17] C. Odriscoll, M. G. Petovello, and G. Lachapelle, "Choosing the coherent integration time for Kalman filter-based carrier-phase tracking of GNSS signals," *GPS solutions*, vol. 15, no. 4, pp. 345–356, 2011.

- [18] X. Tang, G. Falco, E. Falletti, and L. Lo Presti, "Practical implementation and performance assessment of an extended Kalman filter-based signal tracking loop," in *Localization and GNSS (ICL-GNSS), 2013 International Conference on*, pp. 1–6, IEEE, 2013.
- [19] N. I. Ziedan, *GNSS receivers for weak signals*, ch. 5, pp. 155–202. Artech house, 2006.
- [20] P. Lian, G. Lachapelle, and C. Ma, "Improving tracking performance of PLL in high dynamics applications," in *ION NTM*, vol. 1, pp. 042–1, 2005.
- [21] K.-H. Kim, G.-I. Jee, and J.-H. Song, "Carrier tracking loop using the adaptive two-stage Kalman filter for high dynamic situations," *International Journal of Control Automation and Systems*, vol. 6, no. 6, pp. 948–953, 2008.
- [22] G. Yao, W. Wenqi, and H. Xiaofeng, "High dynamic carrier phase tracking based on adaptive Kalman filtering," in *Control and Decision Conference (CCDC), 2011 Chinese*, pp. 1245–1249, IEEE, 2011.
- [23] X. Wang, X. Ji, S. Feng, and V. Calmettes, "A high-sensitivity GPS receiver carrier-tracking loop design for high-dynamic applications," *GPS Solutions*, vol. 19, no. 2, pp. 225–236, 2015.
- [24] A. Van Dierendonck, "GPS receivers," in *Global Positioning System: Theory and Applications (volume One)* (B. W. Parkinson and J. J. Spilker, eds.), ch. 8, pp. 329–409, Aiaa, 1996.
- [25] J. T. Curran, G. Lachapelle, and C. C. Murphy, "Improving the design of frequency lock loops for GNSS receivers," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 48, no. 1, pp. 850–868, 2012.
- [26] R. Yang, Y. Morton, K. V. Ling, and E. K. Poh, "Generalized GNSS signal carrier tracking in challenging environments: Part II: Optimization and Implementation." to be submitted to the journal of IEEE Transactions on Aerospace and Electronic Systems.
- [27] K. Ogata, *Discrete-time control systems*, ch. 5–6, pp. 293–516. Prentice Hall Englewood Cliffs, NJ, 1995.
- [28] K. Ling and K. Lim, "Receding horizon recursive state estimation," *Automatic Control, IEEE Transactions on*, vol. 44, no. 9, pp. 1750–1753, 1999.
- [29] M. L. Psiaki and H. Jung, "Extended Kalman filter methods for tracking weak GPS signals," in *Proceedings of the 15th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GPS 2002)*, pp. 2539–2553, 2002.
- [30] P. Misra and P. Enge, *Global Positioning System: Signals, Measurements and Performance Second Edition*, ch. 12, pp. 467–498. Lincoln, MA: Ganga-Jamuna Press, 2006.
- [31] R. Landry and A. Renard, "Analysis of potential interference sources and assessment of present solutions for GPS/GNSS receivers," *4th Saint-Petersburg on INS*, pp. 1–13, 1997.
- [32] Y. Jiao, Y. T. Morton, S. Taylor, and W. Pelgrum, "Characterization of high-latitude ionospheric scintillation of GPS signals," *Radio Science*, vol. 48, no. 6, pp. 698–708, 2013.
- [33] M. G. Petovello, K. O'Keefe, G. Lachapelle, and M. E. Cannon, "Consideration of time-correlated errors in a Kalman filter applicable to GNSS," *Journal of Geodesy*, vol. 83, no. 1, pp. 51–56, 2009.
- [34] M. Lashley, *Modeling and performance analysis of GPS vector tracking algorithms*. PhD thesis, Auburn University, 2009.
- [35] M. Gómez Arias, *Adaptive Kalman Filter-Based Phase-Tracking in GNSS*. BSc thesis, Technische Universität München, 2010.
- [36] J. Thomas, *An analysis of digital phase-locked loops*. Pasadena, CA: Jet Propulsion Laboratory, 1989.
- [37] A. Patapoutian, "On phase-locked loops and Kalman filters," *Communications, IEEE Transactions on*, vol. 47, no. 5, pp. 670–672, 1999.

Rong Yang is a PhD student in Electrical and Electronic Engineering at Nanyang Technological University (NTU) from August 2012. The research topic during her PhD study was high dynamics and high sensitivity GNSS software receiver technology. She is currently a research scholar for the Global Positioning Systems Laboratory at Colorado State University (CSU) and works on signal processing techniques for atmospheric remote sensing application in challenging environments.

fast dynamic systems. In recently years, he is applying his research to multi-band GNSS software receiver, as well as navigation and positioning applications.

Dr. Keck-Voon LING is currently an Associate Professor at the School of Electrical and Electronic Engineering, Nanyang Technological University (NTU), Singapore. He obtained a BEng in Electrical Engineering from the National University of Singapore and a DPhil in Control Engineering at Oxford University. Dr Ling was a visiting researcher at the Department of Engineering, University of Cambridge. Dr LING's main research interests include Model Predictive Control (MPC) and Moving Horizon Estimation (MHE), their embedded implementation and applications to

Program Manager at DSO National Laboratories from 1994 to 2007. Presently, he is a Program Manager of Guidance, Navigation and Control program at DSO National Laboratories where he also holds the title of Distinguished Member of Technical Staff. Concurrently, he is also an adjunct Associate Professor at the School of EEE, NTU where he conducts postgraduate courses in computational studies and control and also supervises PhD students in the areas of control and signal processing.

Dr. Eng-Kee Poh graduated from the National University of Singapore with a Bachelor of Engineering (Electrical), First Class Honours, in 1986. Upon graduation, he joined DSO as a project engineer specialising in flight control design. In 1989, he was awarded the Defence Technology Training Award (DTTA) to pursue postgraduate studies at the University of Michigan, USA, where he completed his MSc (EE:Systems) and PhD (EE:Systems) in 1990 and 1993 respectively. Since then, he has assumed the appointments of Laboratory Head, Chief Engineer, Program Director and

new applications using satellite navigation technologies. She was the chair of Institute of Navigation Satellite Division from 2012 to 2014, chaired several international conferences, and serves on a number of other international conference committees and professional society technical committees. She also serves on several technical publications editorial boards. She is currently the technical editor of Navigation Systems for IEEE Transactions on Aerospace and Electronics Systems. She is a fellow of IEEE and the Institute of Navigation and recipient of the 2013 Institute of Navigation Thurlow Award. She also received GPS World's 2015 Leadership Award for her achievements in position, navigation, and timing services.

Dr. Yu Morton is an electrical engineering Professor at Colorado State University. She received a PhD in electrical engineering from Penn State and was a post doctor research fellow at the University of Michigan, Space Physics Research Laboratory. Prior to join CSU, she was a professor in the Department of Electrical and Computer Engineering at Miami University. Her research interests are advanced GNSS receiver algorithms for accurate and reliable operations in challenging environments, studies of the atmosphere using radar and satellite signals, and development of