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FLUCTUATIONS OF LINEAR EIGENVALUES STATISTICS FOR WIGNER MATRICES: EDGE CASE

GUANGMING PAN, SHAOCHEN WANG, AND WANG ZHOU

ABSTRACT. In this note, we consider the fluctuation theorem for $X_{f_n}^{(n)} := \sum_{i=1}^n f(\lambda_i)I(\lambda_i \geq \theta_n)$, where $\lambda_i, i = 1, \dots, n$ are eigenvalues from a Wigner matrix and $\theta_n \rightarrow 2^-$. We prove that in the edge case $X_{f_n}^{(n)}$ behaves like the counting function of Wigner matrix. Our results can be viewed as a complement of Bao, Pan and Zhou [J. Stat. Physics. Vol. 150, Issue 1, 88–129(2013)].

1. INTRODUCTION AND MAIN RESULTS

The complex Wigner Ensemble is defined as a family of $n \times n$ random Hermitian matrices M_n of the form

$$W_n = \frac{1}{\sqrt{n}} M_n = \frac{1}{\sqrt{n}} \{\xi_{jk}\}_{j,k=1}^n,$$

in which $\xi_{ll} \in \mathbb{R}, 1 \leq l \leq n$, $\xi_{jk} = \bar{\xi}_{kj} \in \mathbb{C}, 1 \leq j < k \leq n$, and $\{\xi_{ll}, \xi_{jk}; 1 \leq l \leq n, 1 \leq j < k \leq n\}$ is a collection of independent variables such that $\mathbb{E}\xi_{ll} = \mathbb{E}\xi_{jk} = 0$, $\mathbb{E}|\xi_{jk}|^2 = 1$ and $\mathbb{E}\xi_{ll}^2 = \sigma^2 < \infty$. Moreover, if the entries are Gaussian distributed, i.e.

$$\xi_{ll} \sim N(0, 1)_{\mathbb{R}}, \quad 1 \leq l \leq n, \quad \xi_{jk} \sim N(0, 1)_{\mathbb{C}}, \quad 1 \leq j < k \leq n,$$

where $N(0, 1)_{\mathbb{R}}$ (resp. $N(0, 1)_{\mathbb{C}}$) represents the standard real (resp. complex) Gaussian distribution, we say that W_n is a Gaussian Unitary Ensemble (GUE).

For a Wigner matrix W_n , we denote its ordered eigenvalues by $\lambda_1(W_n) \leq \lambda_2(W_n) \leq \dots \leq \lambda_n(W_n)$. The famous semi-circle law says that the empirical spectral distribution (ESD) of W_n defined by

$$F^{W_n}(x) =: \frac{1}{n} \sum_{i=1}^n I(\lambda_i(W_n) \leq x)$$

converges weakly to the semi-circle law ρ_{sc} , where

$$\rho_{sc}(dx) = \frac{1}{2\pi^2} \sqrt{4 - x^2} dx, \quad -2 \leq x \leq 2.$$

This means that for any bounded and continuous function f ,

$$\frac{1}{n} \sum_{i=1}^n f(\lambda_i(W_n)) \rightarrow \int f(x) \rho_{sc}(dx)$$

in probability as $n \rightarrow \infty$. One can refer to Bai and Silverstein ([2]) as one of standard references for random matrix theory. Note that the above result describes the global limiting

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behavior of eigenvalues of Wigner Ensemble, which can be seen as a universal result corresponding to the classical law of large numbers (LLN) for sum of independent random variables.

A natural question is to consider the fluctuation problem for $\sum_{i=1}^n f(\lambda_i(W_n))$. This question has been studied extensively in the literature in different settings, for example Bai, Wang and Zhou ([3]), Lytova and Pastur ([12]), Pastur ([13]), Shcherbina ([14]), Soshnikov ([15]). For convenience, we denote

$$X_f^{(n)}(W_n) = \sum_{i=1}^n f(\lambda_i(W_n)).$$

$X_f^{(n)}(W_n)$ is usually referred to as the global linear eigenvalue statistic of Wigner matrices with test function f . A remarkable work on this topic is due to Lytova and Pastur ([12]). Particularly for GUE, Lytova and Pastur showed that for any bounded test function f with bounded derivatives, one has

$$X_f^{(n)}(W_n) - \mathbb{E}X_f^{(n)}(W_n) \xrightarrow{d} N(0, V_f), \quad (1.1)$$

where $\xrightarrow{d}$ means convergence in distribution and

$$V_f = \frac{1}{4\pi^2} \int_{-2}^2 \int_{-2}^2 \left(\frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right)^2 \frac{4 - \lambda\mu}{\sqrt{4 - \lambda^2} \sqrt{4 - \mu^2}} d\lambda d\mu. \quad (1.2)$$

Moreover, under more smoothness assumptions on the test function f , they extend the above result to general Wigner matrices. It is remarkable that there is no normalizing constant in the convergence in (1.1). This is mainly caused by the repulsion properties of the eigenvalues ([9]).

In a recent paper by Bao, Pan and Zhou ([4]), they extend the above results to the case $f_u(x) = g(x)I(x \geq u)$, where g is a smooth function and $u \in (-2, 2)$ is a fixed number. They obtain that

$$X_{f_u}^{(n)}(W_n) - \mathbb{E}X_{f_u}^{(n)}(W_n) \xrightarrow{d} N(0, V_{f_u}), \quad (1.3)$$

where

$$V_{f_u} = \frac{1}{4\pi^2} \int_u^2 \int_u^2 \left(\frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right)^2 \frac{4 - \lambda\mu}{\sqrt{4 - \lambda^2} \sqrt{4 - \mu^2}} d\lambda d\mu. \quad (1.4)$$

They also establish the weak convergence of corresponding processes by considering u as a parameter in the space $\mathbb{D}$, **i.e., the space of càdlàg functions endowed with Skorohod's topology**. Their approach is based on the idea of Lytova and Pastur ([12]) and comparison theorems for linear eigenvalue statistics. Note that they require that $u \in (-2, 2)$, i.e., u lies in the bulk region. But sometimes, this condition may be not fulfilled in practice, such as in the principle components analysis (cf. Anderson, [1]). For example, when we analyze the principle components of a Wigner matrix (say 1000×1000 entries), maybe we will only need the first 10 eigenvalues. In this case u is approximately equal to 1.96. So it is interesting to study the edge regime where $X_{f_u}^{(n)}(W_n)$ behaves completely different from the bulk case.

The main purpose of this paper is to study the fluctuation theorem of $X_{f_u}^{(n)}(W_n)$ when u is close to the edge. Denote $f_n(x) = f(x)I(x \geq \theta_n)$ and write

$$X_{f_n}^{(n)}(W_n) = \sum_{i=1}^n f(\lambda_i(W_n))I(\lambda_i(W_n) \geq \theta_n)$$

i.e., the sum of eigenvalues bigger than θ_n , where f is a real-valued measurable function. **Additionally, we assume that f is continuously differentiable in a small neighborhood of 2.** It is well known that the eigenvalues of Wigner matrices lie in $(-2 - \delta, 2 + \delta)$ with probability one for any $\delta > 0$. Thus, in our following discussion, without loss of generality, we may and will assume that f is supported on $[-3, 3]$. For the Wigner matrix W_n , we assume the following basic condition throughout the paper.

Condition $\mathbf{C}_0$: We say that a complex Wigner matrix W_n obeys Condition $\mathbf{C}_0$ if $\{\xi_{ll}, \operatorname{Re}\xi_{jk}, \operatorname{Im}\xi_{jk}; 1 \leq l \leq n, 1 \leq j < k \leq n\}$ is a collection of independent variables whose distributions are all supported on at least three points, and we have the exponential decay condition on the elements in the sense that

$$\mathbb{P}(|\xi_{jk}| \geq t^C) \leq e^{-t}$$

holds for all $t \geq C'$ with some positive constants C, C' (independent of l, j, k, n).

It is easy to check that any GUE matrix satisfies Condition $\mathbf{C}_0$. Our main results of this paper are the following strong law of large numbers (SLLN) of $X_{f_n}^{(n)}(W_n)$ for GUE and central limit theorem (CLT) of $X_{f_n}^{(n)}(W_n)$ for general Wigner matrix.

THEOREM 1.1 (SLLN). Let $(\theta_n)_{n \geq 1}$ be a sequence of numbers such that $-2 < \theta_n < 2$ and

$$\theta_n \rightarrow 2^-, \quad n(2 - \theta_n)^{3/2} \rightarrow \infty, \quad (1.5)$$

as $n \rightarrow \infty$. Assume that W_n is drawn from GUE and let f be a continuously differentiable function in a small neighborhood of 2. We have

$$\lim_{n \rightarrow \infty} \frac{1}{n(2 - \theta_n)^{3/2}} X_{f_n}^{(n)}(W_n) = \frac{2}{3\pi} f(2) \quad (1.6)$$

almost surely provided that

$$\sum_{n=1}^{\infty} \frac{\log[n(2 - \theta_n)^{3/2}]}{n^2(2 - \theta_n)^3} < +\infty. \quad (1.7)$$

Moreover, (1.6) also holds for Wigner matrix W_n whose off-diagonal entries and diagonal entries match the corresponding entries of a GUE matrix up to order 2 and order 4 respectively (see Section 3 for definition of matching orders).

THEOREM 1.2 (CLT). Assume (1.5) holds and let f be a continuously differentiable function in a small neighborhood of 2. Suppose $f(2) \neq 0$ and

$$\frac{n(2 - \theta_n)^{5/2}}{\sqrt{\log[n(2 - \theta_n)^{3/2}]}} \rightarrow 0 \quad (1.8)$$

as $n \rightarrow \infty$. Then for any GUE matrix W_n ,

$$\frac{X_{f_n}^{(n)}(W_n) - \frac{2}{3\pi} f(2) n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} \rightarrow N(0, f^2(2)) \quad (1.9)$$

in distribution, as $n \rightarrow \infty$.

Moreover, (1.9) also holds for Wigner matrix W_n whose off-diagonal entries and diagonal entries match the corresponding entries of a GUE matrix up to order 2 and order 4.

Particularly, if $f(x) \equiv 1$, we recover the CLT for the eigenvalue counting function of GUE and Wigner matrix (cf. [6], [7]), see Lemma 2.1 for more details. Moreover, if $f(x) = x$, then the above theorem reads

$$\sum_{i=1}^n \lambda_i(W_n) I(\lambda_i(W_n) \geq \theta_n) \approx \frac{4}{3\pi} n(2 - \theta_n)^{3/2} + N\left(0, \frac{2}{\pi^2} \log[n(2 - \theta_n)^{3/2}]\right), \quad (1.10)$$

which gives the approximation of partial sum of eigenvalues in the edge case.

REMARK 1. Condition (1.5) is required in the Gaussian fluctuation of eigenvalue counting function of GUE in the edge case, see Gustavsson ([11]) for details. We believe that Condition (1.8) is necessary in the CLT for the eigenvalue counting function of GUE. *This Condition can be removed if we only use the expectation and variance of $X_{f_n}^{(n)}$ as the centering and scaling constants in the CLT.* Condition (1.7) is needed when we use Borel-Cantelli's lemma to prove the SLLN. This condition can be dropped if we only need the weak law of large numbers, see Section 2 for more details.

REMARK 2. The assumption $f(2) \neq 0$ can be *replaced* by a more general condition. For example, we can assume that there exists some $k \in \mathbb{N}$ so that $f^{(k)}(2) \neq 0$ and $f^{(j)}(2) = 0, j = 1, 2, \dots, k-1$. Then in this case the SLLN reads

$$\lim_{n \rightarrow \infty} \frac{1}{n(2 - \theta_n)^{(3+2k)/2}} X_{f_n}^{(n)}(W_n) = \frac{2^{k+1}}{3\pi k!} f^{(k)}(2) \quad (1.11)$$

almost surely. The CLT also holds in a similar way.

Throughout the paper, the constant C may change from line to line and we also use $X_{f_n}^{(n)}$ (or X_{f_n}) and λ_i instead of $X_{f_n}^{(n)}(W_n)$ (resp. $\lambda_i(W_n)$) if there is no confusion.

The paper is organized as follows. The proofs of main theorems are given in Section 2 (GUE case) and Section 3 (Wigner case). Our proofs of SLLN are mainly based on the computation of mean and variance of $X_{f_n}(W_n)$. We derive the mean and variance of $X_{f_n}(W_n)$ via a comparison with known results for the eigenvalue counting function. As far as we know, this comparison with counting function is new. The proofs of CLT for Wigner matrices are based on the localization results of eigenvalues, but in the GUE case, we can also provide another approach based on the estimates of mean and variance for $X_{f_n}(W_n)$.

2. THE SLLN AND CLT OF $X_{f_n}^{(n)}$ FOR GUE: EDGE CASE

In this section, we provide the proofs in the GUE case. So throughout this section, W_n is a GUE. Our analysis is based on the following crucial observation

$$X_{f_n}^{(n)} = \sum_{i=1}^n f(\lambda_i) I(\lambda_i \geq \theta_n) = \sum_{i=1}^n [f(\lambda_i) - f(2)] I(\lambda_i \geq \theta_n) + f(2) N_{I_n},$$

where $I_n = [\theta_n, \infty)$, N_{I_n} means the number of eigenvalues in the interval I_n . In the following we will prove that $X_{f_n}^{(n)} \approx f(2) N_{I_n}$ and the remainder term can be neglected.

We first review some orthogonal polynomial structure of GUE. If W_n is drawn from GUE, the joint distribution of non-ordered eigenvalues of $\sqrt{n}W_n$ (i.e., M_n) has the following joint probability density function (cf. Deift [8]).

$$\rho_{n,n}(x_1, \dots, x_n) = \frac{1}{n!} \prod_{1 \leq j < k \leq n} |x_j - x_k|^2 e^{-\sum_{j=1}^n x_j^2/2}.$$

For the point process $x_1, \dots, x_n$, the k -point correlation function $\rho_{k,n}$ has the well known determinantal structure

$$\begin{aligned} \rho_{k,n}(x_1, \dots, x_k) &= \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} \rho_{n,n}(x_1, \dots, x_n) dx_{k+1} \dots dx_n \\ &= \det(K_n(x_l, x_j))_{l,j=1}^k. \end{aligned}$$

Here $K_n(x, y)$ is the kernel function given by

$$K_n(x, y) = \sum_{l=0}^{n-1} H_l(x) H_l(y) e^{-\frac{1}{4}(x^2+y^2)} = \sum_{i=0}^{n-1} \psi_l(x) \psi_l(y),$$

where $H_l(x)$ is the l -th orthonormalized Hermite polynomial with respect to the weight function $e^{-x^2/2}$ and $\psi_l(x)$ is the corresponding oscillator wave function. By the famous Christoffel-Darboux formula, one has for $x \neq y$

$$K_n(x, y) = \sqrt{n} \frac{\psi_n(x) \psi_{n-1}(y) - \psi_{n-1}(x) \psi_n(y)}{x - y}$$

and for $x = y$ by l'Hôpital's rule,

$$K_n(x, x) = n \psi_{n-1}^2(x) - \sqrt{n(n-1)} \psi_{n-2}(x) \psi_n(x). \quad (2.1)$$

Using the notation

$$\mathcal{K}_n(x, y) = \sqrt{n} K_n(\sqrt{n}x, \sqrt{n}y),$$

one has the following explicit formulas of expectation and variance of $X_{f_n}^{(n)}$ for GUE matrix W_n .

$$\mathbb{E} X_{f_n}^{(n)}(W_n) = \int_{\mathbb{R}} f_n(x) \mathcal{K}_n(x, x) dx \quad (2.2)$$

and

$$\text{Var} X_{f_n}^{(n)}(W_n) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f_n(x) - f_n(y))^2 \mathcal{K}_n^2(x, y) dx dy. \quad (2.3)$$

Equations (2.2) and (2.3) is our starting point of analysis.

Next, we recall the CLT for the eigenvalue counting function of GUE, cf. Dallaporta ([6]), Dallaporta and Vu ([7]).

LEMMA 2.1. *Let W_n be a GUE matrix, set $I_n = [\theta_n, \infty)$, where θ_n satisfies (1.5). Then*

$$\mathbb{E}N_{I_n}(W_n) = \frac{2}{3\pi}n(2 - \theta_n)^{3/2} + O(1).$$

and

$$\text{Var}(N_{I_n}(W_n)) = \frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}](1 + \eta_n)$$

where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, $N_{I_n}(W_n)$ satisfies the following CLT

$$\frac{N_{I_n}(W_n) - \frac{2}{3\pi}n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} \rightarrow N(0, 1) \quad (2.4)$$

in distribution as $n \rightarrow \infty$.

For $X_{f_n}^{(n)}$, we have the following result.

LEMMA 2.2. *As $n \rightarrow \infty$,*

$$\mathbb{E}X_{f_n}^{(n)} = \frac{2}{3\pi}f(2)n(2 - \theta_n)^{3/2} + \max\left\{O(1)n(2 - \theta_n)^{5/2}, O(1)\right\} \quad (2.5)$$

and

$$\text{Var}X_{f_n}^{(n)} = O(1)\|f_n\|_\infty^2 \log[n(2 - \theta_n)^{3/2}]. \quad (2.6)$$

Moreover

$$\text{Var}\left(\sum_{i=1}^n [f(\lambda_i) - f(2)]I(\lambda_i \geq \theta_n)\right) = o(1)\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]. \quad (2.7)$$

Proof. By (2.2), for any fixed $\delta > 0$, we can write

$$\begin{aligned} \mathbb{E}X_{f_n}^{(n)} &= \int_{\theta_n}^2 f(u)\mathcal{K}_n(u, u)du + \int_2^{2+\delta} f(u)\mathcal{K}_n(u, u)du + \int_{2+\delta}^\infty f(u)\mathcal{K}_n(u, u)du \\ &:= m_1 + e_1 + e_2. \end{aligned} \quad (2.8)$$

Since $\mathcal{K}_n(x, x)$ is nonnegative, we have for some $\xi_n \in (\theta_n, 2)$ by applying the mean value theorem,

$$m_1 = f(\xi_n) \int_{\theta_n}^2 \mathcal{K}_n(u, u)du = \{f(2) + O(1)(2 - \theta_n)\} \int_{\theta_n}^2 \mathcal{K}_n(u, u)du, \quad (2.9)$$

where in the second equality, we have used the Taylor expansion of $f(\xi_n)$, and $O(1)$ is a constant not depending on n .

Using the known result for the expectation of counting function of GUE in the interval $(\theta_n, 2)$, we derive that

$$m_1 = \frac{2}{3\pi}f(2)n(2 - \theta_n)^{3/2} + O(1)n(2 - \theta_n)^{5/2}. \quad (2.10)$$

For the remainder term in (2.8), by the fact that the Airy function and its derivative decay to zero with an exponentially speed outside $(\theta_n, 2)$, we can easily conclude that $e_1 = O(1)$ and

$e_2 = o(1)$ (or one can refer to the proof of Lemma 2.2 in Gustavsson[11] for more details). Combining these estimates together, we establish that

$$\mathbb{E} X_{f_n}^{(n)} = \frac{2}{3\pi} f(2) n(2 - \theta_n)^{3/2} + \max \left\{ O(1) n(2 - \theta_n)^{5/2}, O(1) \right\}. \quad (2.11)$$

We now turn to the proof of variance estimation (2.7). Note that

$$\sum_{i=1}^n [f(x_i) - f(2)] I(x_i \geq \theta_n) := \sum_{i=1}^n g(x_i) I(x_i \geq \theta_n) := \sum_{i=1}^n g_n(x_i) = X_{g_n}^{(n)},$$

where $g(x) = f(x) - f(2)$ and $g_n(x) = (f(x) - f(2)) I(x \geq \theta_n)$. By virtue of (2.3),

$$\text{Var} X_{g_n}^{(n)} = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (g_n(x) - g_n(y))^2 \mathcal{K}_n^2(x, y) dx dy. \quad (2.12)$$

Consider the region

$$\Omega_1 = \left\{ (x, y) \mid \theta_n \leq x \leq \theta_n + \frac{2 - \theta_n}{r(n)}, \quad 2 - \frac{2 - \theta_n}{r(n)} \leq y \leq \theta_n - \frac{1}{n(2 - \theta_n)^{1/2}} \right\}.$$

where $r(n)$ is a function tending slowly to infinity as $n \rightarrow \infty$. Note that from Gustavsson ([11], below (5.4), page 170),

$$\text{Var}(N_{I_n}(W_n)) = \frac{1}{2} \iint_{\Omega_1} \mathcal{K}_n^2(x, y) dx dy + O(1). \quad (2.13)$$

Thus

$$\begin{aligned} \text{Var} X_{g_n}^{(n)} &= \frac{1}{2} \left(\iint_{\Omega_1} + \iint_{\mathbb{R}^2 \setminus \Omega_1} \right) (g_n(x) - g_n(y))^2 \mathcal{K}_n^2(x, y) dx dy \\ &= \frac{1}{2} \iint_{\Omega_1} (g_n(x) - g_n(y))^2 \mathcal{K}_n^2(x, y) dx dy + O(1) \iint_{\mathbb{R}^2 \setminus \Omega_1} \mathcal{K}_n^2(x, y) dx dy \\ &= \frac{1}{2} \iint_{\Omega_1} (g_n(x) - g_n(y))^2 \mathcal{K}_n^2(x, y) dx dy + O(1). \end{aligned} \quad (2.14)$$

Note that on Ω_1 , $g_n(y) \equiv 0$ and

$$g_n(x) = (f(x) - f(2)) I(x \geq \theta_n) = O(1)(x - 2) I(x \geq \theta_n) = O(\theta_n - 2).$$

Combining this and Lemma 2.1 gives that

$$\frac{1}{2} \iint_{\Omega_1} (g_n(x) - g_n(y))^2 \mathcal{K}_n^2(x, y) dx dy = o(1) \log[n(2 - \theta_n)^{3/2}] \quad (2.15)$$

since $2 - \theta_n \rightarrow 0$ as $n \rightarrow \infty$. This proves (2.7). The proof of (2.6) is simpler, since by the variance estimation in Lemma 2.1

$$\text{Var} X_{f_n}^{(n)} = \frac{1 + o(1)}{2} \iint_{\Omega_1} (f_n(x) - f_n(y))^2 \mathcal{K}_n^2(x, y) dx dy = O(1) \|f_n\|_{\infty}^2 \log[n(2 - \theta_n)^{3/2}].$$

□

Now we are ready to prove Theorem 1.1 in the GUE case.

Proof of Theorem 1.1. Let $\tau > 0$ be any fixed constant. By virtue of Lemma 2.2 we know for large enough n 's,

$$\mathbb{P}\left(\frac{1}{n(2-\theta_n)^{3/2}} \left|X_{f_n}^{(n)} - \mathbb{E}X_{f_n}^{(n)}\right| \geq \tau\right) \leq C_\tau \frac{\log[n(2-\theta_n)^{3/2}]}{n^2(2-\theta_n)^3}. \quad (2.16)$$

By Borel-Cantelli's lemma and the assumption (1.7), we obtain that

$$\frac{1}{n(2-\theta_n)^{3/2}} \left|X_{f_n}^{(n)} - \mathbb{E}X_{f_n}^{(n)}\right| \longrightarrow 0$$

almost surely. The proof of SLLN is complete by using Lemma 2.2, the triangle inequality, the fact that $2-\theta_n \rightarrow 0$ and $n(2-\theta_n)^{3/2} \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Next, we prove Theorem 1.2 in the GUE case.

Proof of Theorem 1.2. By Slutsky's theorem, and the CLT for N_{I_n} in Lemma 2.1, the proof can be reduced to show that

$$\frac{\sum_{i=1}^n [f(x_i) - f(2)]I(x_i \geq \theta_n)}{\sqrt{\frac{1}{2\pi^2} \log[n(2-\theta_n)^{3/2}]}} \quad (2.17)$$

converges to zero in probability. By Lemmas 2.1 and 2.2,

$$\mathbb{E}\left(\sum_{i=1}^n [f(x_i) - f(2)]I(x_i \geq \theta_n)\right) = \mathbb{E}X_{f_n}^{(n)} - f(2)\mathbb{E}N_{I_n}(W_n) = \max\left\{O(1)n(2-\theta_n)^{5/2}, O(1)\right\}$$

Thus by Chebyshev's inequality and the fact that the $\log[n(2-\theta_n)^{3/2}] \rightarrow \infty$ and

$$\frac{\max\left\{O(1)n(2-\theta_n)^{5/2}, O(1)\right\}}{\sqrt{\frac{1}{2\pi^2} \log[n(2-\theta_n)^{3/2}]}} = o(1), \quad (2.18)$$

we only need to show

$$\text{Var}\left(X_{f_n}^{(n)} - f(2)N_{I_n}(W_n)\right) = o\left(\frac{1}{2\pi^2} \log[n(2-\theta_n)^{3/2}]\right)$$

in order to prove (2.17). This is already established in Lemma 2.2. $\square$

3. THE SLLN AND CLT OF $X_{f_n}^{(n)}$ FOR WIGNER MATRICES: EDGE CASE

In this section, we extend the results in the previous section to Wigner matrices. Our approach is based on Erdős, Yau and Yin's ([10]) strong rigidity results of eigenvalues and CLT for the eigenvalue counting function of Wigner matrices. As a starting point, we give the definition of matching moments.

Say that two complex random variables ξ and ξ' match to order k if

$$\mathbb{E}\left[\Re(\xi)^m \Im(\xi)^l\right] = \mathbb{E}\left[\Re(\xi')^m \Im(\xi')^l\right]$$

for all $m, l \geq 0$ such that $m + l \leq k$.

We recall the CLT for the eigenvalue counting function of Wigner matrices, cf. Dallaporta ([6]). The approach is based on Tao and Vu's Four Moment Theorem (cf. [16], [17]).

LEMMA 3.1. *Let M_n be a random Hermitian matrix whose entries satisfy Condition $\mathbf{C}_0$. Set $W_n = \frac{1}{\sqrt{n}}M_n$ and $I_n = [\theta_n, \infty)$, where θ_n satisfies (1.5). Suppose that the off-diagonal entries and diagonal entries of W_n match the corresponding entries of GUE up to order 2 and order 4 respectively. Then*

$$\mathbb{E}N_{I_n}(W_n) = \frac{2}{3\pi}n(2 - \theta_n)^{3/2} + O(1).$$

and

$$\text{Var}(N_{I_n}(W_n)) = \frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}](1 + \eta_n)$$

where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, $N_{I_n}(W_n)$ satisfies the following CLT

$$\frac{N_{I_n}(W_n) - \frac{2}{3\pi}n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} \rightarrow N(0, 1) \quad (3.1)$$

in distribution as $n \rightarrow \infty$.

Erdős, Yau and Yin [10] obtain the strong localization property of eigenvalues of Wigner matrices which provides the crucial step in the following analysis.

LEMMA 3.2. *Let M_n be a random Hermitian matrix whose entries satisfy Condition $\mathbf{C}_0$. There is a constant $C > 0$ such that, for any $i \in \{1, \dots, n\}$,*

$$\mathbb{P}\left(|\lambda_i - \gamma_i| \geq (\log n)^{C \log \log n} \min(i, n - i + 1)^{-1/3} n^{-2/3}\right) \leq n^{-3},$$

where γ_i is defined as

$$\frac{i}{n} = \frac{1}{2\pi} \int_{-2}^{\gamma_i} \sqrt{4 - x^2} dx.$$

Now we can prove

LEMMA 3.3.

$$\mathbb{E}X_{f_n}(W_n) = \frac{2f(2)}{3\pi}n(2 - \theta_n)^{3/2} + o(1) \sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]} \quad (3.2)$$

and

$$\text{Var} X_{f_n}(W_n) = \frac{1}{2\pi^2} f^2(2) \log[n(2 - \theta_n)^{3/2}](1 + \eta_n) \quad (3.3)$$

where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We first prove (3.2). Note that

$$\mathbb{E}X_{f_n}(W_n) = \mathbb{E}[X_{f_n}(W_n) - f(2)N_{I_n}(W_n)] + f(2)\mathbb{E}N_{I_n}(W_n).$$

By Lemma 3.1, we only need to prove

$$\sum_{i=1}^n \mathbb{E}[f(\lambda_i) - f(2)I(\lambda_i \geq \theta_n)] = o(1) \sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]} \quad (3.4)$$

Denote the sets

$$A_i = \left\{ |\lambda_i - \gamma_i| \leq (\log n)^{C \log \log n} \min(i, n - i + 1)^{-1/3} n^{-2/3} \right\}, \quad i = 1, 2, \dots, n.$$

and

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}|f(\lambda_i) - f(2)|I(\lambda_i \geq \theta_n) &= \sum_{i=1}^n \mathbb{E}|f(\lambda_i) - f(2)|I(\lambda_i \geq \theta_n)I(A_i) \\ &\quad + \sum_{i=1}^n \mathbb{E}|f(\lambda_i) - f(2)|I(\lambda_i \geq \theta_n)I(A_i^c) := I_1 + I_2. \end{aligned} \quad (3.5)$$

Obviously $I_2 \leq n \cdot n^{-3} = o(1)$ by Lemma 3.2. We write

$$I_1 \leq \sum_{i=1}^n \mathbb{E}|f(\lambda_i) - f(\gamma_i)|I(\lambda_i \geq \theta_n)I(A_i) + \sum_{i=1}^n \mathbb{E}|f(\gamma_i) - f(2)|I(\lambda_i \geq \theta_n)I(A_i) := I_{11} + I_{12}. \quad (3.6)$$

Next observe that the summation in I_{11}, I_{12} only has N_{I_n} non-vanishing terms. Thus

$$\begin{aligned} |I_{11}| &\leq \sum_{i=1}^n \mathbb{E}|f(\lambda_i) - f(\gamma_i)|I(\lambda_i \geq \theta_n)I(A_i) \\ &\leq O(1) \sum_{i=1}^n \mathbb{E}|\lambda_i - \gamma_i|I(\lambda_i \geq \theta_n)I(A_i) \\ &\leq O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^{N_{I_n}} \min\{i, n-i+1\}^{-1/3} n^{-2/3}. \end{aligned} \quad (3.7)$$

We further split the right hand side of (3.7) as follows:

$$\begin{aligned} |I_{11}| &\leq O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^{N_{I_n}} \min\{i, n-i+1\}^{-1/3} n^{-2/3} I(N_{I_n} \leq n/2) \\ &\quad + O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^{N_{I_n}} \min\{i, n-i+1\}^{-1/3} n^{-2/3} I(N_{I_n} > n/2) \\ &\leq O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^{N_{I_n}} i^{-1/3} n^{-2/3} \\ &\quad + O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^n \min\{i, n-i+1\}^{-1/3} n^{-2/3} I(N_{I_n} > n/2) \end{aligned} \quad (3.8)$$

since $N_{I_n} \leq n$. From Lemma 3.1

$$\mathbb{P}(N_{I_n} > n/2) \leq \mathbb{P}(N_{I_n} - \mathbb{E}N_{I_n} > n/4) \leq \frac{C}{n^2} \log[n(2 - \theta_n)^{3/2}] \leq \frac{C}{n^{1+c}}$$

for some $0 < c < 1$.

Note that

$$\sum_{i=1}^{N_{I_n}} i^{-1/3} \leq \sum_{i=1}^{N_{I_n}} \int_{i-1}^i x^{-1/3} dx = CN_{I_n}^{2/3}$$

and the summand in (3.8) can be bounded by

$$\begin{aligned} O(1)(\log n)^{C \log \log n} \mathbb{E} \sum_{i=1}^n \min\{i, n-i+1\}^{-1/3} n^{-2/3} I(N_{I_n} > n/2) \\ \leq O(1)(\log n)^{C \log \log n} \mathbb{P}(N_{I_n} > n/2) \\ \leq O(1)(\log n)^{C \log \log n} \frac{1}{n^{1+c}} \end{aligned}$$

which is $o(1)$ as $n \rightarrow \infty$. Combining all these together, we have shown that

$$I_{11} \leq O(1)(\log n)^{C \log \log n} n^{-2/3} \mathbb{E} N_{I_n}^{2/3} + o(1). \quad (3.9)$$

For I_{12} , since on the set $A_i \cap \{\lambda_i \geq \theta_n\}$,

$$|f(2) - f(\gamma_i)| = O(1)|2 - \gamma_i| \leq O(1)|2 - \theta_n| + O(1)|\lambda_i - \gamma_i|.$$

Using a similar analysis to the I_{11} case leads to

$$I_{12} \leq O(1)(2 - \theta_n) \mathbb{E} N_{I_n} + O(1)(\log n)^{C \log \log n} n^{-2/3} \mathbb{E} N_{I_n}^{2/3}. \quad (3.10)$$

Combining (3.9) and (3.10), we get

$$\begin{aligned} \sum_{i=1}^n \mathbb{E} |f(\lambda_i) - f(2)| I(\lambda_i \geq \theta_n) &\leq O(1)(2 - \theta_n) \mathbb{E} N_{I_n} + O(1)(\log n)^{C \log \log n} n^{-2/3} \mathbb{E} N_{I_n}^{2/3} + o(1) \\ &:= II_1 + II_2 + o(1). \end{aligned}$$

By Lemma 3.1 and assumption (1.8), it is easy to check that

$$II_1 = o(1) \sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}.$$

The proof of (3.2) is complete if we show that the above is still true for II_2 . In fact, by using the Jensen inequality and Lemma 3.1 for $\mathbb{E} N_{I_n}$, we have

$$II_2 \leq n^\delta n^{-2/3} (\mathbb{E} N_{I_n})^{2/3} \leq C n^\delta (2 - \theta_n) \quad (3.11)$$

for some sufficiently small $0 < \delta < 1/5$ and large n . By (1.8), $n^2(2 - \theta_n)^5 \ll (\log n)^5$. This implies

$$n^\delta (2 - \theta_n) \ll n^\delta \frac{\log n}{n^{2/5}} \rightarrow 0,$$

by our choice of δ .

Next, we turn to the prove of (3.3). By Lemma 3.1, we only need to prove $\text{Var}(X_{f_n}(W_n)) = \text{Var}(f(2)N_{I_n}(W_n)) + o(1)$. It suffices to show that

$$e_v = \mathbb{E} \left(\sum_{i=1}^n [f(\lambda_i) - f(2)] I(\lambda_i \geq \theta_n) \right)^2 = o(1). \quad (3.12)$$

Write

$$\begin{aligned} e_v &= \mathbb{E} \left(\sum_{i=1}^n [f(\lambda_i) - f(2)] I(\lambda_i \geq \theta_n) \right)^2 I(A) + \mathbb{E} \left(\sum_{i=1}^n [f(\lambda_i) - f(2)] I(\lambda_i \geq \theta_n) \right)^2 I(A^c) \\ &:= e_v(1) + e_v(2) \end{aligned} \quad (3.13)$$

where $A = A_1 \cap \cdots \cap A_n$. Since $\mathbb{P}(A^c) \leq \sum \mathbb{P}(A_i^c) \leq n^{-2}$,

$$e_v(2) \leq \frac{1}{n^2} O(1) \mathbb{E} N_{I_n}^2(W_n) = o(1). \quad (3.14)$$

For $e_v(1)$, by using the simple inequality $(x+y)^2 \leq 2(x^2+y^2)$ and triangle equality, we have

$$\begin{aligned} e_v(1) &\leq 2\mathbb{E} \left(\sum_{i=1}^n |f(\lambda_i) - f(\gamma_i)| I(\lambda_i \geq \theta_n) \right)^2 I(A) \\ &\quad + 2\mathbb{E} \left(\sum_{i=1}^n |f(\gamma_i) - f(2)| I(\lambda_i \geq \theta_n) \right)^2 I(A) := e_v(11) + e_v(12). \end{aligned}$$

For $e_v(11)$, with similar analysis as in (3.7),

$$e_v(11) \leq 2(\log n)^{2C \log \log n} n^{-4/3} \mathbb{E} N_{I_n}^{4/3} \leq C n^\delta (2 - \theta_n)^2, \quad (3.15)$$

for some sufficiently small $0 < \delta < 1/5$ and large n 's, where in the second inequality we have used Lemma 3.1 and the fact that

$$(\mathbb{E} N_{I_n}^{4/3})^{3/4} \leq (\mathbb{E} N_{I_n}^2)^{1/2} \leq \left[\text{Var}(N_{I_n}) + (\mathbb{E} N_{I_n})^2 \right]^{1/2} \leq C n (2 - \theta_n)^{3/2}. \quad (3.16)$$

With same reason as (3.11), now it is easy to see $e_v(11) = o(1)$, since $2 - \theta_n \rightarrow 0$. The proof of $e_v(12) = o(1)$ is similar as the proof of mean of $\mathbb{E} X_{f_n}(W_n)$, we omit the trivial details. This ends the whole proof. $\square$

Now we can give the proof of our main results for the general Wigner matrices.

Proof of Theorem 1.1. Let $\tau > 0$ be any fixed constant. By virtue of Lemma 3.3, we know for large enough n ,

$$\mathbb{P} \left(\frac{1}{n(2 - \theta_n)^{3/2}} \left| X_{f_n}^{(n)} - \mathbb{E} X_{f_n}^{(n)} \right| \geq \tau \right) \leq C_\tau \frac{\log[n(2 - \theta_n)^{3/2}]}{n^2(2 - \theta_n)^3}. \quad (3.17)$$

By Borel-Cantelli's lemma and the assumption (1.7), we obtain that

$$\frac{1}{n(2 - \theta_n)^{3/2}} \left| X_{f_n}^{(n)} - \mathbb{E} X_{f_n}^{(n)} \right| \rightarrow 0$$

almost surely. The proof of SLLN is complete by using Lemma 3.3, the triangle inequality, the fact that $2 - \theta_n \rightarrow 0$ and $n(2 - \theta_n)^{3/2} \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Proof of Theorem 1.2. By Lemma 3.1 and Slutsky's theorem, we only need to prove

$$\frac{X_{f_n}^{(n)}(W_n) - \frac{2}{3\pi} f(2) n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} = f(2) \frac{N_{I_n}(W_n) - \frac{2}{3\pi} n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} + o_p(1). \quad (3.18)$$

Observe that

$$\text{L.H.S. of (3.18)} = \frac{f(2) N_{I_n}(W_n) - \frac{2}{3\pi} f(2) n(2 - \theta_n)^{3/2}}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}} + \frac{\sum_{i=1}^n (f(\lambda_i) - f(2)) I(\lambda_i \geq \theta_n)}{\sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]}}.$$

As a consequence, (3.18) can be reduced to proving that

$$\sum_{i=1}^n |f(\lambda_i) - f(2)| I(\lambda_i \geq \theta_n) = o_p(1) \sqrt{\frac{1}{2\pi^2} \log[n(2 - \theta_n)^{3/2}]} \quad (3.19)$$

In fact, we have proved a stronger result in (3.4). This concludes the proof Theorem 1.2 in the general Wigner case. $\square$

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