Output feedback control for uncertain nonlinear systems with input quantization

Xing, Lantao; Wen, Changyun; Zhu, Yang; Su, Hongye; Liu, Zhitao

2015

Xing, L., Wen, C., Zhu, Y., Su, H., & Liu, Z. (2016). Output feedback control for uncertain nonlinear systems with input quantization. *Automatica*, 65, 191-202.

<https://hdl.handle.net/10356/85729>

<https://doi.org/10.1016/j.automatica.2015.11.028>

© 2015 Elsevier. This is the author created version of a work that has been peer reviewed and accepted for publication by *Automatica*, Elsevier. It incorporates referee's comments but changes resulting from the publishing process, such as copyediting, structural formatting, may not be reflected in this document. The published version is available at: [<http://dx.doi.org/10.1016/j.automatica.2015.11.028>].

Downloaded on 26 Apr 2023 20:56:27 SGT

Output Feedback Control for Uncertain Nonlinear Systems with Input Quantization [★]

Lantao Xing^a, Changyun Wen^b, Yang Zhu^a, Hongye Su^{a,★}, Zhitao Liu^a

^aState Key Laboratory of Industrial Control Technology, Zhejiang University, Hangzhou, P.R.China

^bSchool of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.

Abstract

In this paper, we propose a new adaptive output-feedback tracking control scheme for a class of uncertain nonlinear systems with input quantized by a newly-proposed quantizer. This quantizer is a combination of a logarithmic (or a hysteresis) quantizer and a uniform quantizer, and it has the advantages of both logarithmic and uniform quantizers in ensuring reducible communication expenses and acceptable quantization errors for better system performances. Compared with existing results in adaptive control, the proposed scheme provides a way to relax certain restrictive conditions, in addition to solving the problem of adaptive output-feedback control with input quantization. It is shown that the designed adaptive controller ensures global boundedness of all the signals in the closed-loop system and enables the tracking error to exponentially converge towards a compact set which is adjustable.

Key words: Output feedback, Uncertain nonlinear system, Input quantization, Sector-bounded quantizers

1 Introduction

Quantization can be seen as a map from continuous signals to discrete finite sets. Recently, a great deal of attention has been paid to the study of quantization problems because of its theoretical and practical importance in modern engineering such as hybrid systems, discrete-event systems, digital control systems and control with information constraints, see [1–4]. For systems with information quantization, a continuous control input or state is quantized by a quantizer which results in an inevitable quantization error. Thus the effects of the quantization error to the performances of the closed system, especially system stability, need to be carefully and clearly studied.

Up to now, quantized control for systems with uncertainties has been mainly considered based on robust approaches. Reference [5] studied the coarsest quantizer

for single-input-single-output (SISO) linear systems and proved that the coarsest quantizer should follow a logarithmic law. This result was extended to nonlinear systems in [6] and the ideas of using robust control Lyapunov function to get robust quantized controllers were developed. In [7], the coarsest quantizer was taken as a sector bound uncertainty to study quantized control with quadratic stability and H_2 and H_∞ performance criteria were developed. The quantized control problem with input and output quantization was also considered in [8]. In [9], robust stabilization problem for linear discrete-time systems via a limited communication channel was addressed. More results on robust quantized control could be found in [10–16].

Besides robust control, adaptive control is another important approach to deal with system uncertainties as it can provide on-line estimation of unknown parameters. In [17] and [18], adaptive control with quantized input signals for linear systems is reported. In [19], adaptive quantized control for nonlinear systems is considered. It is noted that a sector bounded property on quantization errors is used to establish the results. However, the resulting stability conditions depend on control signals, which is hardly checkable in advance as control signals are unavailable before the designed controller is put in operation. In [20], a backstepping-based adaptive con-

[★] This work is partially supported by the 111 Project (B07031); National Basic Research Program of China (973 Program 2013CB035406); National Natural Science Foundation of China (NSFC: 61134007, 61203015). Corresponding Author: Hongye Su

Email addresses: xinglantao1989@126.com (Lantao Xing), ecywen@ntu.edu.sg (Changyun Wen), hysu@iipc.zju.edu.cn (Hongye Su).

trol scheme is presented for a class of strict-feedback uncertain systems with input quantization. Although the proposed design method can avoid stability conditions depending on control input and make the stabilization error arbitrarily small, the nonlinearities of the system to be controlled should satisfy global Lipschitz conditions with known Lipschitz constants and their partial derivatives to be bounded. Recently in [21], such a restriction was relaxed by using the sector bound property of the logarithmic quantizers. However, the unknown parameters are only contained in the last nonlinear function of the system. In addition, to the best of our knowledge, all the existing results in adaptive quantized control are based on state feedback. Therefore, it is also important to investigate output-feedback quantized control, besides relaxing the conditions mentioned above.

It should be noted that all the mentioned results above consider either logarithmic quantizer or the extended hysteresis quantizer. The main reason is that in network control, the logarithmic quantizer (or hysteresis quantizer) can largely reduce communication burden when the amplitude of the input signal is large and decrease quantization error when the amplitude is small, due to its varying quantization levels compared to a uniform quantizer. However, such quantizers have certain disadvantages. Their structure determines that the quantization level becomes coarser as the magnitude of the signal gets bigger (away from the origin). Inevitably, excessively large amplitude signals will result in very large quantization errors, which may be unbounded. Actually this is unnecessary just for decreasing communication cost, since such large quantization errors may degrade the performance of the system or even result in instability. For example in tracking control, if the reference signal requires the control signal to be big, the resulting large quantization errors would result in large tracking errors.

In this paper, we solve the problems mentioned above by proposing a new adaptive output-feedback control scheme for a class of uncertain nonlinear systems with input quantization. Firstly, we propose a new quantizer which is a combination of a logarithmic (or hysteresis) quantizer and a uniform quantizer. More specifically, when the magnitude of the control signal is smaller than a threshold specified by designer, the proposed quantizer is a logarithmic (or hysteresis) quantizer; on the other hand, when the magnitude of the control signal is larger than the threshold, the quantizer becomes a uniform quantizer. In this way, the coarseness of the quantization level remains unchanged after the logarithmic quantizer is replaced by a uniform quantizer. Clearly such a quantizer has the advantages of both logarithmic and uniform quantizers in ensuring reducible communication expenses and acceptable quantization errors for better system performances. This is illustrated in the example in Section 2. Then based on this quantizer, a method to design the state observers is given. Compared with the currently

available design schemes without input quantization, the state estimation error can only be ensured to converge to a bounded set independent of the control signal u , instead of zero. With our proposed adaptive controller, we successfully compress the effects of the quantization error on the final tracking error into a bounded set that can be decreased by choosing suitable design parameters. With our newly proposed scheme and quantizer, we are able to remove the assumptions imposed in [20] that the nonlinearities of the system to be controlled should satisfy global Lipschitz conditions with known Lipschitz constants and their partial derivatives to be bounded. In addition, in contrast to [17] and [19], the established stability conditions do not depend on control signals, either. It is shown that the designed adaptive controller in this paper ensures global boundedness of all the signals in the closed-loop system and enables the tracking error to exponentially converge towards a compact set which is adjustable.

The remaining part of this paper is organized as follows. Section 2 first describes the existing sector-bounded quantizers, then a new quantizer is proposed to overcome the disadvantage of the existing logarithmic (or hysteresis) quantizer. An example is given to show the effectiveness of the newly-proposed quantizer. Section 3 describes the system to be controlled and also gives the control objective. Section 4 presents the adaptive controller designed based on backstepping technique and analyses the stability and tracking performance of the closed-loop system. A simulation example is given in Section 5 to illustrate the control scheme and verify the established theoretical results. Finally Section 6 concludes the paper.

2 Quantizer Description

2.1 Sector-bounded Quantizer

Let $\Delta_q = q(u) - u$, denoting the quantization error. A sector-bounded quantizer is a quantizer with its quantization error satisfying the following sector bound property introduced in [22].

$$|\Delta_q| \leq \delta|u| + (1 - \delta)d, \quad (1)$$

where $0 \leq \delta < 1$ and d are known parameters of quantizers to be described below. Based on [7] and [22], most practical quantizers belong to such a class with logarithmic quantizer, hysteresis quantizer, and a uniform quantizer being typical examples.

2.1.1 Logarithmic Quantizer

In this paper, the logarithmic quantizer described in [13] is considered. It is modelled as

$$q_l(u) = \begin{cases} u_i & \frac{u_i}{1+\delta} < u \leq \frac{u_i}{1-\delta} \\ 0 & 0 \leq u < \frac{d}{1+\delta} \\ -q(-u) & u < 0 \end{cases} \quad (2)$$

where $u_i = \rho^{(1-i)}d$ with $i = 1, 2, \dots$. The parameters $d > 0$ and $0 < \rho < 1$, $\delta = \frac{1-\rho}{1+\rho}$ determine the quantization density of $q(u)$. $q(u)$ is in the set $U = \{0, \pm u_i\}$. $u_{min} = \frac{d}{1+\delta}$ determines the size of the deadzone for $q(u)$. For this quantizer, some remarks about its range, number of quantization levels and quantization density are given in [15].

2.1.2 Hysteresis Quantizer

In this paper, the hysteresis quantizer employed is described in the following form, similar to [20].

$$q_h(u(t)) = \begin{cases} u_i \text{sgn}(u) & \frac{u_i}{1+\delta} < |u| \leq u_i, \dot{u} < 0, \text{ or} \\ & u_i < |u| \leq \frac{u_i}{1-\delta}, \dot{u} > 0 \\ u_i(1+\delta)\text{sgn}(u) & u_i < |u| \leq \frac{u_i}{1-\delta}, \dot{u} < 0, \text{ or} \\ & \frac{u_i}{1-\delta} < |u| \leq \frac{u_i(1+\delta)}{1-\delta}, \dot{u} > 0 \\ 0 & 0 \leq |u| < \frac{d}{1+\delta}, \dot{u} < 0, \text{ or} \\ & \frac{d}{1+\delta} \leq |u| \leq d, \dot{u} > 0 \\ q(u(t^-)) & \dot{u} = 0 \end{cases} \quad (3)$$

where $i = 1, 2, \dots$. The parameters $d > 0$ and $0 < \rho < 1$ determine the quantization density of $q(u(t))$, $\delta = \frac{1-\rho}{1+\rho}$, $u_i = \rho^{(1-i)}d$. $q(u)$ is in the set $U = \{0, \pm u_i, \pm u_i(1+\delta)\}$ and $q(u(t^-))$ denotes the status prior to $q(u(t))$. For this kind of quantizer, some detailed discussions about its parameters and hysteresis mechanism can be found in [15], [20].

2.1.3 A Uniform Quantizer

There are different kinds of uniform quantizers satisfying the sector-bounded property (1). In this paper, we introduce the uniform quantizer proposed in [23] as an example because it has certain extended forms as discussed in Remark 2. It has a relatively simple form and is described as:

$$q_u(u) = \lfloor \frac{u}{\varpi} + \frac{1}{2} \rfloor \varpi \quad (4)$$

where $\lfloor a \rfloor$ denotes the greatest integer that is less than or equal to a , $\varpi > 0$ is the parameter of the uniform

quantizer and determines the quantization density. Detailed discussions about this quantizers can be found in [23]. For this kind of quantizer, the following inequality is satisfied:

$$|\Delta_q| \leq \frac{\varpi}{2} \quad (5)$$

Clearly, the logarithmic quantizer and the hysteresis quantizer satisfy property (1) with $0 < \delta < 1$ based on [22]. For the uniform quantizer, the sector bound property is also true with $\delta = 0$ and $d = \frac{\varpi}{2}$.

Remark 1: The hysteresis quantizer can be considered as an extension of the logarithmic quantizer as it can be seen as a combination of two logarithmic quantizers with the same coarseness but different quantized values. Thus control law which adopts a logarithmic quantizer can be also applied to a hysteresis quantizer, and vice versa. In the rest part of the paper, we only consider the logarithmic quantizer for simplicity. However, with the additional quantized values, a hysteresis quantizer can avoid chattering phenomenon. The detailed discussions can be found in [19].

Remark 2: There are many other forms of uniform quantizers which satisfy the sector-bounded property (1). For example, the symmetric uniform quantizer $q(u(t)) = (\lfloor \frac{u}{\varpi} \rfloor + \frac{1}{2})\varpi$ used in [24] and the asymmetric uniform quantizer $q(u(t)) = \lfloor \frac{u}{\varpi} \rfloor \varpi$ adopted in [25]. It should be noted that the uniform quantizers can also have its extended hysteresis forms to avoid chattering phenomenon, as shown in [23].

2.2 A New Quantizer

As pointed out in the Introduction, for a logarithmic quantizer, the quantization level becomes coarser as the magnitude of the signal gets bigger (away from the origin), which results in unnecessary large quantization error. To overcome this problem, we propose a new quantizer combining a logarithmic quantizer and a uniform quantizer defined as below (Note that this quantizer can be easily extended to a combination of a hysteresis quantizer and a uniform quantizer by replacing $q_l(\cdot)$ with $q_h(\cdot)$):

$$q_s(u) = \begin{cases} q_l(u_{th}) + \lfloor \frac{u-u_{th}}{\varpi} + \kappa \rfloor \varpi & |u| \geq u_{th} \\ q_l(u) & |u| < u_{th} \end{cases} \quad (6)$$

where $\kappa = 1$ if $q_l(u_{th}) < u_{th}$ and $\kappa = 0$ if $q_l(u_{th}) \geq u_{th}$, u_{th} is a positive constant specified by the designer denoting the threshold to switch between the logarithmic and uniform quantizer, and $\varpi = |q_l(u_{th}) - u_{th}|$ is a parameter for the uniform quantizer.

Fig. 1. The new quantizer

For this new quantizer, we have

$$|\Delta_q| \leq \begin{cases} \varpi & |u| \geq u_{th} \\ \delta|u| + (1-\delta)d & |u| < u_{th} \end{cases} \quad (7)$$

Clearly, Δ_q for this new quantizer is always bounded for any u . In addition, with a suitable choice of large u_{th} , low communication cost is still maintained. Thus the proposed quantizer has the advantages of both logarithmic and uniform quantizers.

Remark 3: With this new quantizer, we can guarantee that the quantization error when $u > u_{th}$ remains the same as that of the logarithmic quantizer when $u = u_{th}$. This can be considered as that a saturation level is introduced to the quantization error of the traditional logarithmic quantizer. Note that u_{th} is a user-defined parameter denoting the trade-off between system performances and communication burden, and it can be chosen according to the practical situations.

Now we give an example of quantizer (6) by choosing u_{th} as one of the switching points of the logarithmic quantizer, i.e. $u_{th} = \frac{u_i}{1-\delta}$, where u_i is given in (2) describing the logarithmic quantizer level set U . As a result, we can get $\varpi = \frac{u_i}{1-\delta} - u_i$. This new quantizer is illustrated in Fig.1. In the rest of the paper, we will use this quantizer.

To better illustrate the advantages of the new quantizer, we compare the respective tracking performances achieved with the new quantizer and the logarithmic quantizer by considering the following simple system:

$$\dot{x} = q(u) \quad (8)$$

Fig. 2. Tracking performance

The control objective is to make x track the reference signal $r(t) = 200 \sin(5t)$. Based on the results in [10] where condition (1) is used, the control signal with quantization by a logarithmic quantizer is given as

$$u = \frac{-x + r + \dot{r} - (1-\delta)d \tanh(\frac{x-r}{\varepsilon})}{1 + \delta \tanh(\frac{(x-r)u}{\varepsilon})} \quad (9)$$

where δ , d , ε are design parameters. Since the newly-proposed quantizer also satisfies (1), the control in (9) is still applicable in achieving the tracking objective. For simulation, we choose the logarithmic quantizer parameters $\delta = 0.2$ and $d = 0.02$, $u_i = 67$, $\varepsilon = 10$, respectively. Thus, we can obtain $u_{th} = 83$ and $\varpi = 16$ for the new quantizer. Fig.2 illustrates the tracking performances, while Fig.3 shows the designed input $u(t)$ and the quantized signal $q(u(t))$. It is observed that the newly-proposed quantizer achieves better tracking performance as shown in Fig.2, while its quantization density is almost the same as that of the logarithmic quantizer as seen in Fig.3.

3 Problem Statement

The following class of uncertain nonlinear systems is considered.

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1^T(y)\theta + \varphi_1(y) \\ &\vdots \\ \dot{x}_{\zeta-1} &= x_{\zeta} + \phi_{\zeta-1}^T(y)\theta + \varphi_{\zeta-1}(y) \\ \dot{x}_{\zeta} &= x_{\zeta+1} + \phi_{\zeta}^T(y)\theta + \varphi_{\zeta}(y) + b_m q_s(u) \\ &\vdots \end{aligned}$$

Fig. 3. Control u and $q(u)$

$$\begin{aligned}\dot{x}_{n-1} &= x_n + \phi_{n-1}^T(y)\theta + \varphi_{n-1}(y) + b_1 q_s(u) \\ \dot{x}_n &= \phi_n^T(y)\theta + \varphi_n(y) + b_0 q_s(u) \\ y &= x_1(t)\end{aligned}\quad (10)$$

where $x_1, \dots, x_n, y$ and u are system states, output and control input signal to be designed. $\theta \in \mathbb{R}^r$ and $b_m, \dots, b_0$ are unknown constants. $\phi_i(y) \in \mathbb{R}^r$ and $\varphi_i(y) \in \mathbb{R}^1$, $i=1, \dots, n$ are known smooth nonlinear functions. $q_s(u)$ is the quantized input signal and take quantized values.

The objective of this paper is to propose a control design scheme which can make the output $y = x_1(t)$ track a reference signal $r(t)$ with the input quantized by a quantizer. For the mentioned reason above, the newly-proposed quantizer $q_s(u)$ is utilized in this paper. Similar to the existing literature in controlling systems with input quantization such as [6], [12], [17], [19] and [20], we assume that the existence and uniqueness of a solution forward in time are satisfied for such a class of nonlinear systems.

For the development of the control laws, the following assumptions are made.

Assumption 1: The sign of b_m is known.

Assumption 2: The relative degree $\zeta = n - m$ is known and the system is minimum phase, i.e. the polynomial $B(s) = b_m s^m + \dots + b_1 s + b_0$ is Hurwitz.

Assumption 3: The reference signal y_r and its first ζ th order derivatives are piecewise continuous, known and bounded.

4 Controller Design

4.1 State Estimation Filters

In order to design the desired adaptive output-feedback control law, we rewrite system (10) in the following form

$$\dot{x} = Ax + \Phi(y)\theta + \Psi(y) + \begin{pmatrix} 0 \\ b \end{pmatrix} q_s(u) \quad (11)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \Psi(y) = \begin{pmatrix} \varphi_1(y) \\ \vdots \\ \varphi_n(y) \end{pmatrix} \quad (12)$$

$$\Phi(y) = \begin{pmatrix} \phi_1^T(y) \\ \vdots \\ \phi_n^T(y) \end{pmatrix} \quad b = \begin{pmatrix} b_m \\ \vdots \\ b_0 \end{pmatrix} \quad (13)$$

Since x is unavailable and only the output y is measurable, we need to design filters to estimate x . With the input signal u quantized by a sector bounded quantizer, the filters are designed as follows:

$$\dot{\xi} = A_0 \xi + ky + \Psi(y) \quad (14)$$

$$\dot{\Xi}^T = A_0 \Xi^T + \Phi(y) \quad (15)$$

$$\dot{\lambda} = A_0 \lambda + e_n u \quad (16)$$

$$\nu_i = A_0^i \lambda, \quad i = 0, 1, \dots, m \quad (17)$$

where e_i ($i=1, \dots, n$) is a row vector with the i th entry being 1 and the others being 0, $k = [k_1, \dots, k_n]^T$ such that all eigenvalues of $A_0 = A - ke_1^T$ are at some desired stable locations.

The state estimations are given by

$$\hat{x}(t) = \xi + \Xi^T \theta + \sum_{i=0}^m b_i \nu_i \quad (18)$$

Then we get the derivative of $\hat{x}$ as

$$\dot{\hat{x}}(t) = A_0 \hat{x} + ky + \Phi(y)\theta + \Psi(y) + \begin{pmatrix} 0 \\ b \end{pmatrix} u \quad (19)$$

The state estimation error is defined as

$$\epsilon = x(t) - \hat{x}(t) \quad (20)$$

Remark 4: Note that all states of the filters in (14) - (17) are available for feedback. However the estimated states given in (18) are still unknown due to unknown parameters and thus cannot be utilized in controller design. Instead, they will be used for analysis of the resulting closed loop system. The following Lemma shows a property of the proposed state observers.

Lemma 1: For system (10), the proposed filters (14)-(17) guarantee that the state estimation error is always bounded for all $t > 0$, regardless of the control signal.

Proof: From (11) and (18) - (20), the state estimation error satisfies

$$\begin{aligned} \dot{\epsilon} &= A_0 \epsilon + \begin{pmatrix} 0 \\ b \end{pmatrix} (q_s(u) - u) \\ &= A_0 \epsilon + B \Delta_q \end{aligned} \quad (21)$$

where $B = \begin{pmatrix} 0 \\ b \end{pmatrix}$ and $\Delta_q = q_s(u) - u$.

Since A_0 is Hurwitz, there must exist a positive definite matrix P satisfying $A_0^T P + P A_0 \leq -I$. By considering the Lyapunov function $V_\epsilon = \frac{1}{2} \epsilon^T P \epsilon$ for system (21), we can obtain

$$\begin{aligned} \dot{V}_\epsilon &= \frac{1}{2} \epsilon^T (A_0^T P + P A_0) \epsilon + \epsilon^T P B \Delta_q \\ &\leq \left(-\frac{1}{2} + \frac{\|P\|}{4\alpha}\right) \|\epsilon\|^2 + \alpha \|P\| \|B\|^2 \Delta_q^2 \\ &\leq -\frac{\beta}{\lambda_{\max}(P)} V_\epsilon + \alpha \|P\| \|B\|^2 \Delta_q^2 \end{aligned} \quad (22)$$

where $\alpha > 0$ satisfying $\beta = \frac{1}{2} - \frac{\|P\|}{4\alpha} > 0$. Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the the biggest and smallest eigenvalues of the matrix P , respectively. Then we have

$$\begin{aligned} \lambda_{\min}(P) \|\epsilon\|^2 &\leq V_\epsilon \leq e^{-\frac{\beta}{\lambda_{\max}(P)} t} V_\epsilon(0) \\ &+ \frac{\alpha \|P\| \|B\|^2 \lambda_{\max}(P) \Delta_q^2}{\beta} (1 - e^{-\frac{\beta}{\lambda_{\max}(P)} t}) \end{aligned} \quad (23)$$

So the size of the state estimation error $\|\epsilon\|$ converges exponentially towards a bounded compact set $\Omega_\epsilon = \{\epsilon | \|\epsilon\|^2 \leq \frac{\alpha \|P\| \|B\|^2 \lambda_{\max}(P) \Delta_q^2}{\beta \lambda_{\min}(P)}\}$ at a rate of $\frac{\beta}{\lambda_{\max}(P)}$. Note that the size of Ω_ϵ is determined by the quantization error Δ_q . Since Δ_q is always bounded from (7), the state estimation error is always bounded, independent of the control signal u . ■

4.2 Design of Adaptive Controller

In this section, we will design an adaptive controller based on backstepping technique with tuning functions, which involves ζ recursive steps.

Let ϵ_2 , $v_{i,2}$, ξ_2 and Ξ_2 denote the second entries of ϵ , v_i , ξ and Ξ respectively and $v_{m,i}$ denote the i th entry of v_m . Based on the design procedure in [26], we have

$$\dot{y} = b_m v_{m,2} + \xi_2 + \varphi_1(y) + \bar{w}^T \Theta + \epsilon_2 \quad (24)$$

$$\dot{v}_{m,i} = v_{m,i+1} - k_i v_{m,1}, \quad i = 2, 3, \dots, \zeta - 1 \quad (25)$$

$$\dot{v}_{m,\zeta} = v_{m,\zeta+1} - k_\zeta v_{m,1} + u \quad (26)$$

where

$$\Theta = [b_m, \dots, b_0, \theta^T]^T \quad (27)$$

$$w = [\nu_{m,2}, \nu_{m-1,2}, \dots, \nu_{0,2}, \Xi_2 + \phi_1^T]^T \quad (28)$$

$$\bar{w} = [0, \nu_{m-1,2}, \dots, \nu_{0,2}, \Xi_2 + \phi_1^T]^T \quad (29)$$

Then we take the change of coordinates

$$z_1 = y - y_r \quad (30)$$

$$z_i = v_{m,i} - \alpha_{i-1} - \hat{\rho} y_r^{(i-1)} \quad i = 2, \dots, \zeta \quad (31)$$

where $\hat{\rho}$ is the estimation of $\rho = \frac{1}{b_m}$, and α_{i-1} is the virtual control at step $i-1$. The ζ design steps are summarized as follows by following the recursive backstepping procedure, see for example [26].

Step 1: Select the virtual control law α_1 as

$$\alpha_1 = \hat{\rho} \bar{\alpha}_1 \quad (32)$$

$$\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \varphi_1(y) - \bar{w} \hat{\Theta} \quad (33)$$

where c_1 and d_1 are positive design parameters. Then we have

$$\begin{aligned} \dot{z}_1 &= -(c_1 + d_1) z_1 + \epsilon_2 + (w - \hat{\rho}(\dot{y}_r + \bar{\alpha}_1) e_1)^T \tilde{\Theta} \\ &\quad - b_m(\dot{y}_r + \bar{\alpha}_1) \tilde{\rho} + \hat{b}_m z_2 \end{aligned} \quad (34)$$

Choose the Lyapunov function V_1 as

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta} + \frac{|b_m|}{2\gamma} \tilde{\rho}^2 \quad (35)$$

The updating law of $\hat{\rho}$ is chosen as

$$\dot{\hat{\rho}} = -\gamma \text{sign}(b_m)(\dot{y}_r + \bar{\alpha}_1) z_1 - \sigma_1 \hat{\rho} \quad (36)$$

where γ and σ_1 are positive constants and Γ^{-1} is a positive definite matrix. Define

$$\tau_1 = (w - \hat{\rho}(\dot{y}_r + \bar{\alpha}_1) e_1) z_1 \quad (37)$$

Then we get

$$\begin{aligned} \dot{V}_1 \leq & -c_1 z_1^2 + \hat{b}_m z_1 z_2 + \tilde{\Theta}^T (\tau_1 - \Gamma^{-1} \dot{\hat{\Theta}}) \\ & + \frac{|b_m|}{\gamma} \sigma_1 \tilde{\rho} \hat{\rho} + \frac{\epsilon^T \epsilon}{4d_1} \end{aligned} \quad (38)$$

Step 2: We choose the second virtual control law α_2 and the tuning function as

$$\alpha_2 = -\hat{b}_m z_1 - (c_2 + d_2 (\frac{\partial \alpha_1}{\partial y})^2) z_2 + \beta_2 + \frac{\partial \alpha_1}{\partial \hat{\Theta}} \Gamma \tau_2 \quad (39)$$

$$\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y} w z_2 \quad (40)$$

where c_2 and d_2 are positive constants and

$$\begin{aligned} \beta_2 = & \frac{\partial \alpha_1}{\partial y} (\xi_2 + \psi_1 + \omega^T \hat{\Theta}) + k_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r \\ & + (\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\rho}} \dot{\hat{\rho}} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_1}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1})) \\ & + \frac{\partial \alpha_1}{\partial \xi} (A_0 \xi + k y + \Psi(y)) \\ & + \frac{\partial \alpha_1}{\partial \Xi^T} (A_0 \Xi^T + \Phi(y)) \end{aligned} \quad (41)$$

Choose Lyapunov function as

$$V_2 = V_1 + \frac{1}{2} z_2^2 \quad (42)$$

Then we obtain

$$\begin{aligned} \dot{V}_2 \leq & -\sum_{i=1}^2 c_i z_i^2 + z_2 z_3 + \tilde{\Theta}^T (\tau_2 - \Gamma^{-1} \dot{\hat{\Theta}}) + \frac{|b_m|}{\gamma} \sigma_1 \tilde{\rho} \hat{\rho} \\ & + \frac{\partial \alpha_1}{\partial \hat{\Theta}} (\Gamma \tau_2 - \dot{\hat{\Theta}}) + (\frac{1}{4d_1} + \frac{1}{4d_2}) \epsilon^T \epsilon \end{aligned} \quad (43)$$

Step i(i=3,...,ζ): Choose the virtual control law and the tuning function as

$$\begin{aligned} \alpha_i = & -z_{i-1} - [c_i + d_i (\frac{\partial \alpha_{i-1}}{\partial y})^2] z_i + \beta_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\Theta}} \Gamma \tau_i \\ & - (\sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\Theta}}) \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega, \quad i = 3, \dots, \zeta \end{aligned} \quad (44)$$

where c_i and d_i are positive constants and

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y} w z_i \quad (45)$$

$$\begin{aligned} \beta_i = & \frac{\partial \alpha_{i-1}}{\partial y} (\xi_2 + \psi_1 + \omega^T \hat{\Theta}) + k_i v_{m,1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}} y_r^{(j)} \\ & + (y_r^{(i-1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\rho}} \dot{\hat{\rho}} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1})) \end{aligned}$$

Fig. 4. The closed-loop system

$$\begin{aligned} & + \frac{\partial \alpha_{i-1}}{\partial \xi} (A_0 \xi + k y + \Psi(y)) \\ & + \frac{\partial \alpha_{i-1}}{\partial \Xi^T} (A_0 \Xi^T + \Phi(y)) \end{aligned} \quad (46)$$

In the last design step, the adaptive controller and the parameter updating law are finally obtained as

$$u = \alpha_\zeta - v_{m,\zeta+1} + \hat{\rho} y_r^{(\zeta)} \quad (47)$$

$$\dot{\hat{\Theta}} = \Gamma \tau_\zeta - \sigma_2 \Gamma \hat{\Theta} \quad (48)$$

where σ_2 is a positive design parameter and is chosen based on the σ -modification scheme proposed in [27]. A block diagram of the resulting closed loop systems is given in Fig.4.

Remark 5: In contrast to standard adaptive backstepping design in [26], the term $\sigma_1 \hat{\rho}$ in (36) and $\sigma_2 \Gamma \hat{\Theta}$ in (48) are introduced to handle the effects of the bounded state estimation error for guaranteeing the boundedness of all the closed-loop signals as shown later in Theorem 1.

4.3 Stability Analysis

We now analyze the designed controller and establish the stability of the closed-loop system and its tracking performance, as stated in the following theorem.

Theorem 1: Consider the closed-loop system consisting of uncertain system (10) with the input signal u quantized by the newly-proposed quantizer (6), the filters (14)-(17), the control law (47) and parameter updating law (36), (48). All the signals of the closed-loop system are globally bounded and the tracking error $y - y_r$ will exponentially converge towards a set which is adjustable by choosing suitable design parameters.

Proof: We choose the Lyapunov function V_ζ as

$$V_\zeta = \sum_{i=1}^{\zeta} \frac{1}{2} z_i^2 + \frac{1}{2} \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta} + \frac{|b_m|}{2\gamma} \tilde{\rho}^2 \quad (49)$$

Then we get

$$\begin{aligned} \dot{V}_\zeta &= \sum_{i=1}^{\zeta} z_i \dot{z}_i - \tilde{\Theta}^T \Gamma^{-1} \dot{\tilde{\Theta}} - \frac{b_m}{\gamma} \tilde{\rho} \dot{\tilde{\rho}} \\ &\leq - \sum_{i=1}^{\zeta} c_i z_i^2 + \frac{\sigma_1 |b_m|}{\gamma} \tilde{\rho} \dot{\tilde{\rho}} + \sigma_2 \tilde{\Theta}^T \dot{\tilde{\Theta}} + \sum_{i=1}^{\zeta} \frac{1}{4d_i} \epsilon^T \epsilon \end{aligned} \quad (50)$$

Using Young's inequalities $\tilde{\theta}^T \hat{\theta} \leq -\frac{1}{2} \tilde{\theta}^T \tilde{\theta} + \frac{1}{2} \theta^T \theta$ gives

$$\begin{aligned} \dot{V}_\zeta &\leq - \sum_{i=1}^{\zeta} c_i z_i^2 - \frac{\sigma_1 |b_m|}{2\gamma} \tilde{\rho}^2 + \frac{\sigma_1 |b_m|}{2\gamma} \rho^2 - \frac{\sigma_2}{2} \tilde{\Theta}^T \tilde{\Theta} \\ &\quad + \frac{\sigma_2}{2} \Theta^T \Theta + \sum_{i=1}^{\zeta} \frac{\alpha \|P\| \|B\|^2 \lambda_{\max}(P) \Delta_q^2}{4d_i \beta \lambda_{\min}(P)} \end{aligned} \quad (51)$$

Let $\mu = \min\{2c_1, \dots, 2c_\rho, \sigma_1, \frac{\sigma_2}{\lambda_{\max}(\Gamma^{-1})}\}$ where $\lambda_{\max}(\Gamma^{-1})$ is the maximum eigenvalue of Γ^{-1} and $\Delta = \frac{\sigma_1 |b_m|}{2\gamma} \rho^2 + \frac{\sigma_2}{2} \Theta^T \Theta + \sum_{i=1}^{\zeta} \frac{\alpha \|P\| \|B\|^2 \lambda_{\max}(P) \Delta_q^2}{4d_i \beta \lambda_{\min}(P)}$ which is bounded from Lemma 1. Then we can obtain

$$\dot{V}_\zeta \leq -\mu V_\zeta + \Delta \quad (52)$$

From (52) we have

$$\begin{aligned} V_\zeta(t) &\leq e^{-\mu t} V_\zeta(0) + (\Delta/\mu)(1 - e^{-\mu t}) \\ &\leq V_\zeta(0) + \Delta/\mu \end{aligned} \quad (53)$$

By (53) we get V_ζ is uniformly bounded. Thus z_i , $\hat{\Theta}$ and $\hat{\rho}$ are bounded. Since z_1 and y_r are bounded, y is also bounded. From (14) and (15), ξ and Ξ are bounded because A_0 is Hurwitz. Now we prove the boundedness of λ , then the boundedness of x follows from the boundedness of ϵ , ξ , Ξ , and λ . From (16) we obtain

$$\lambda_i = \frac{s^{i-1} + k_1 s^{i-2} + \dots + k_{i-1}}{K(s)} u \quad (54)$$

where $i = 1, \dots, n$, $K(s) = s^n + k_1 s^{n-1} + \dots + k_n$. Meanwhile, the system model gives

$$\frac{d^n y}{dt^n} - \sum_{i=1}^n \frac{d^{n-i}}{dt^{n-i}} [\varphi_i(y) + \phi_i(y)^T \theta] = \sum_{i=0}^m b_i \frac{d^i}{dt^i} q(u) \quad (55)$$

Let $o = q(u) - u$, then o must be bounded. Noting that $\sum_{i=0}^m b_i \frac{d^i}{dt^i} q(u) = B(s)(u + o)$ and substituting (55) into

(54) we get

$$\begin{aligned} \lambda_i &= \frac{s^{i-1} + k_1 s^{i-2} + \dots + k_{i-1}}{B(s)K(s)} \left(\frac{d^n y}{dt^n} \right. \\ &\quad \left. - \sum_{i=1}^n \frac{d^{n-i}}{dt^{n-i}} [\varphi_i(y) + \phi_i(y)^T \theta] - o \right) \end{aligned} \quad (56)$$

Since $\varphi(y)$ and $\phi(y)$ are smooth, y and o are bounded, then from Assumption 2, $\lambda_1, \dots, \lambda_{m+1}$ are bounded. By following a recursive analysis similar to [?], we can obtain that λ_i for all $i = m+2, \dots, n$ are bounded and therefore x is also bounded. Since the right hand side of (47) is a function of y , ξ , Ξ and λ , then the control signal u is bounded from Lemma 2. Therefore the boundedness of all the closed-loop signals is guaranteed.

Moreover, from the definition of z_1 , V_ζ and (53), we obtain that the tracking error $e = z_1$ satisfies

$$\begin{aligned} \frac{1}{2} e^2 &= \frac{1}{2} z_1^2 \leq \frac{1}{2} \sum_{i=1}^{\zeta} z_i^2 \leq V_\zeta(t) \\ &\leq e^{-\mu t} V_\zeta(0) + (\Delta/\mu)(1 - e^{-\mu t}) \end{aligned} \quad (57)$$

So e^2 is bounded by a function that converges exponentially towards a compact set $\Omega = \{e | e^2 \leq 2\Delta/\mu = 2 \times \frac{\sigma_1 |b_m|}{2\gamma} \rho^2 + \frac{\sigma_2}{2} \Theta^T \Theta + \sum_{i=1}^{\zeta} \frac{\alpha \|P\| \|B\|^2 \lambda_{\max}(P) \Delta_q^2}{4d_i \beta \lambda_{\min}(P)}\}$ at a rate of μ . As can be seen, the quantization error Δ_q and the vector Θ of all the unknown system parameters are both included in Ω , so they will affect the tracking error. But the size of Ω can be reduced by choosing suitable design parameters. Specifically, when the convergence speed μ , the parameters of quantizer (6) are set, the size of Ω can be reduced by decreasing $\lambda_{\max}(\Gamma^{-1})$ and $\frac{\sigma_1}{\gamma}$ or increasing d_i . ■

Remark 6: In [20], the quantizer $q(u)$ is decomposed into a linear part $u(t)$ and a nonlinear part $d(t)$. To compensate for the nonlinear part $d(t)$, the nonlinearities of the system to be controlled should satisfy global Lipschitz conditions with known Lipschitz constants and their partial derivatives to be bounded. In [17] and [19], the sector bound property is utilized to handle the quantization error. However, this property involves the control signal to be designed and thus the established stability conditions depend on control signals. In this paper, with the newly-proposed quantizer and the state observer, the effects of the quantization error on the tracking error are successfully compressed into a bounded set Ω , and the size of Ω is adjustable by choosing suitable design parameters.

5 Simulation Results

For simulation, the following system is considered which is similar to the simulation example in [28] except that

Fig. 5. State estimation performance with Quantizer 1

the control input u is quantized.

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta y^3 \\ \dot{x}_2 &= b_0 q_s(u) \end{aligned} \quad (58)$$

where $q_s(u)$ is the quantized input by quantizer (6) and θ , b_0 are the unknown parameters. Two cases are considered for the quantizer: One is a quantizer combining a hysteresis quantizer and a uniform quantizer, named Quantizer 1; The other one is a combination of a logarithmic quantizer and a uniform quantizer, called Quantizer 2. The parameters of the hysteresis quantizer and the logarithmic quantizer are $d = 0.02$, $\delta = 0.2$, $u_i = 8.8$, respectively. Then the parameters $u_{th} = 11$ and $\varpi = 2.2$ are set for both cases. For the sake of simulation, we choose $\theta = 2$, $b_0 = 1$, $x_1(0) = 0$, $x_2(0) = 0.8$, and all the other initial conditions are set to be 0. We set the convergence rate $\mu = 0.5$, $c_1 = 100$ and $c_2 = 120$, so the size of the ultimate tracking error bound Λ is mainly determined by $\lambda_{max}(\Gamma^{-1})$, σ_1 , σ_2 , γ and d_1 , d_2 . For comparison, we choose the following two sets of parameters: (1). $\Gamma^{-1} = \text{diag}[10, 10]$, $\sigma_1 = 5$, $\sigma_2 = 5$, $\gamma = 0.1$, $d_1 = 2$, $d_2 = 1$; (2). $\Gamma^{-1} = \text{diag}[1, 1]$, $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\gamma = 0.1$, $d_1 = 20$, $d_2 = 10$.

For the closed loop system with Quantizer 1, Fig.5 shows the state estimation performances, while Fig.6 shows the tracking performance of $y(t)$ and Fig.7 shows the input u and the quantized signal $q(u(t))$, respectively. On the other hand, for the system with Quantizer 2, Figs.8-10 illustrate the respective results.

By comparing the corresponding trajectories in the figures, it is observed that the ultimate tracking error bound can be reduced by decreasing $\lambda_{max}(\Gamma^{-1})$ and $\frac{\sigma_1}{\gamma}$

Fig. 6. Tracking performance with Quantizer 1

Fig. 7. Control u and $q(u)$ with Quantizer 1

or increasing d_i , which is consistent with the established theoretical results.

6 Conclusions

In this paper, we propose a new adaptive output-feedback control scheme for a class of uncertain nonlinear systems whose input is quantized by a newly-proposed quantizer. This new quantizer has the advantages of both logarithmic and uniform quantizers in ensuring reducible communication expenses and acceptable quantization errors for better system performances.

Fig. 8. State estimation performance with Quantizer 2

Fig. 10. Control u and $q(u)$ with Quantizer 2

Fig. 9. Tracking performance with Quantizer 2

By the proposed design scheme in this paper, the nonlinearities of the controlled system are not required to be global Lipschitz; Moreover, the established stability conditions do not depend on the control signals. It is shown that the designed adaptive controller ensures global boundedness of all the signals in the closed-loop system and enables the tracking error to exponentially converge towards a compact set which is adjustable.

References

- [1] H. Ishii and B. A. Francis, *Limited data rate in control systems with networks*, vol. 275. Springer Science & Business Media, 2002.
- [2] S. Tatikonda and S. Mitter, “Control under communication constraints,” *Automatic Control, IEEE Transactions on*, vol. 49, no. 7, pp. 1056–1068, 2004.
- [3] J. Lunze, B. Nixdorf, and J. Schröder, “Deterministic discrete-event representations of linear continuous-variable systems,” *Automatica*, vol. 35, no. 3, pp. 395–406, 1999.
- [4] W. S. Wong and R. W. Brockett, “Systems with finite communication bandwidth constraints. ii. stabilization with limited information feedback,” *Automatic Control, IEEE Transactions on*, vol. 44, no. 5, pp. 1049–1053, 1999.
- [5] N. Elia and S. K. Mitter, “Stabilization of linear systems with limited information,” *Automatic Control, IEEE Transactions on*, vol. 46, no. 9, pp. 1384–1400, 2001.
- [6] J. Liu and N. Elia, “Quantized feedback stabilization of non-linear affine systems,” *International Journal of Control*, vol. 77, no. 3, pp. 239–249, 2004.
- [7] M. Fu and L. Xie, “The sector bound approach to quantized feedback control,” *Automatic Control, IEEE Transactions on*, vol. 50, no. 11, pp. 1698–1711, 2005.
- [8] D. F. Coutinho, M. Fu, and C. I. E. de Souza, “Input and output quantized feedback linear systems,” *Automatic Control, IEEE Transactions on*, vol. 55, no. 3, pp. 761–766, 2010.
- [9] V. N. Phat, J. Jiang, A. V. Savkin, and I. R. Petersen, “Robust stabilization of linear uncertain discrete-time systems via a limited capacity communication channel,” *Systems & control letters*, vol. 53, no. 5, pp. 347–360, 2004.
- [10] L. Xing, C. Wen, H. Su, Z. Liu, and J. Cai, “Robust control for a class of uncertain nonlinear systems with input quantization,” *International Journal of Robust and Nonlinear Control*, 2015.
- [11] G. N. Nair and R. J. Evans, “Stabilizability of stochastic linear systems with finite feedback data rates,” *SIAM Journal*

- on *Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.
- [12] D. Liberzon and J. P. Hespanha, “Stabilization of nonlinear systems with limited information feedback,” *Automatic Control, IEEE Transactions on*, vol. 50, no. 6, pp. 910–915, 2005.
 - [13] Z. Yang, Y. Hong, Z.-P. Jiang, and X. Wang, “Quantized feedback stabilization of hybrid impulsive control systems,” in *CDC*, pp. 3903–3908, 2009.
 - [14] C. De Persis, “Robust stabilization of nonlinear systems by quantized and ternary control,” *Systems & Control Letters*, vol. 58, no. 8, pp. 602–608, 2009.
 - [15] C. De Persis and F. Mazenc, “Stability of quantized time-delay nonlinear systems: a lyapunov–krasowskii-functional approach,” *Mathematics of Control, Signals, and Systems*, vol. 21, no. 4, pp. 337–370, 2010.
 - [16] T. Liu, Z.-P. Jiang, and D. J. Hill, “Quantized stabilization of strict-feedback nonlinear systems based on iss cyclic-small-gain theorem,” *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 24, no. 1, pp. 75–110, 2012.
 - [17] T. Hayakawa, H. Ishii, and K. Tsumura, “Adaptive quantized control for linear uncertain discrete-time systems,” *Automatica*, vol. 45, no. 3, pp. 692–700, 2009.
 - [18] H. Sun, N. Hovakimyan, and T. Basar, “L₁ adaptive controller for systems with input quantization,” in *American Control Conference (ACC), 2010*, pp. 253–258, IEEE, 2010.
 - [19] T. Hayakawa, H. Ishii, and K. Tsumura, “Adaptive quantized control for nonlinear uncertain systems,” *Systems & control letters*, vol. 58, no. 9, pp. 625–632, 2009.
 - [20] J. Zhou, C. Wen, and G. Yang, “Adaptive backstepping stabilization of nonlinear uncertain systems with quantized input signal,” *Automatic Control, IEEE Transactions on*, vol. 59, no. 2, pp. 460–464, 2014.
 - [21] L. Xing, C. Wen, H. Su, J. Cai, and L. Wang, “A new adaptive control scheme for uncertain nonlinear systems with quantized input signal,” *Journal of the Franklin Institute*, 2015.
 - [22] T. Liu, Z.-P. Jiang, and D. J. Hill, “A sector bound approach to feedback control of nonlinear systems with state quantization,” *Automatica*, vol. 48, no. 1, pp. 145–152, 2012.
 - [23] F. Ceragioli, C. De Persis, and P. Frasca, “Discontinuities and hysteresis in quantized average consensus,” *Automatica*, vol. 47, no. 9, pp. 1916–1928, 2011.
 - [24] R. Carli, F. Bullo, and S. Zampieri, “Quantized average consensus via dynamic coding/decoding schemes,” *International Journal of Robust and Nonlinear Control*, vol. 20, no. 2, pp. 156–175, 2010.
 - [25] K. H. Johansson, A. Speranzon, and S. Zampieri, “On quantization and communication topologies in multi-vehicle rendezvous,” in *16th Triennial World Congress of International Federation of Automatic Control, IFAC 2005; Prague; Czech Republic; 3 July 2005 through 8 July 2005*, 2005.
 - [26] J. Zhou and C. Wen, *Adaptive backstepping control of uncertain systems: Nonsmooth nonlinearities, interactions or time-variations*. Springer, 2008.
 - [27] P. A. Ioannou, “Adaptive systems with reduced models,” *Lecture notes in control and information sciences*, 1983.
 - [28] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and adaptive control design*. John Wiley & Sons, Inc., 1995.