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Probabilistic Key Predistribution in Mobile Networks Resilient to Node-Capture Attacks

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Abstract—We present a comprehensive analysis on connectivity and resilience of secure sensor networks under the widely studied q -composite key predistribution scheme. For network connectivity which ensures that any two sensors can find a path in between for secure communication, we derive the conditions to guarantee connectivity in consideration of (i) node-capture attacks, where the adversary may capture a set of sensors and compromise keys in their memory; (ii) sensor mobility, meaning that sensors can move around so that the network topology may change over time; (iii) physical transmission constraints, under which two sensors have to be within each other's transmission range for communication; (iv) the boundary effect of network fields; and (v) link unreliability, meaning that links are allowed to be unreliable. In contrast, many prior connectivity analyses of secure sensor networks often ignore the above issues. For resilience, although limited studies have presented formal analysis, it is often assumed that the adversary captures a random set of sensors, whereas this paper allows the adversary to capture an arbitrary set of sensors. We present conditions to ensure unassailability and unsplitability in secure sensor networks under the q -composite scheme. Unassailability ensures that an adversary capturing any set consisting of a negligible fraction of sensors can compromise only a negligible fraction of communication links although the adversary may compromise communications between non-captured nodes which happen to use keys that are shared by captured nodes. Unsplitability means that when a negligible fraction of sensors are captured, almost all of the remaining nodes are still securely connected. Based on the results of connectivity, unassailability and unsplitability, we provide useful guidelines for the design of secure sensor networks.

Index Terms—Security, key predistribution, wireless sensor networks, connectivity, random graphs.

I. INTRODUCTION

A. Background and Motivation

Random key predistribution schemes have been widely recognized as appropriate solutions to secure communications in resource-constrained wireless sensor networks [1]–[6]. The idea of randomly assigning cryptographic keys to sensors

before deployment was proposed in the seminal work of Eschenauer and Gligor [2]. The Eschenauer–Gligor (EG) scheme [2] works as follows. To secure a sensor network with n nodes, in the key predistribution phase, the scheme uses a large *key pool* comprising P_n cryptographic keys to select K_n distinct keys *uniformly at random* for each sensor. These K_n keys form the *key ring* of a sensor, and are inserted into the sensor's memory. After sensors are deployed, two sensors can securely communicate over an existing link if and only if their key rings share at least one key. Common keys are found via neighbor discovery strategies [2]. We let P_n and K_n be functions of n for generality, with the natural condition $1 \leq K_n \leq P_n$.

Based on the EG scheme [2], Chan *et al.* [1] propose the q -composite key predistribution scheme as an extension. In such q -composite scheme, two sensors establish a secure link in between if and only if their key rings have at least q key(s) in common, where $1 \leq q \leq K_n$. Obviously, the q -composite scheme with $q = 1$ is the same as the EG scheme. The q -composite scheme with $q \geq 2$ performs better than the EG scheme in terms of resilience against small-scale node-capture attacks while trading off heightened vulnerability in the presence of large-scale node-capture attacks. The q -composite scheme has been widely investigated in the literature over the last decade [7]–[13].

For wireless sensor networks with probabilistic key predistribution, due to the analytical complexity, most work in the literature on connectivity and resilience (explained below) considers only static networks with few exceptions. In this paper, we consider mobile networks in addition to static networks. An example application of probabilistic key predistribution to mobile networks beyond static sensor networks is frequency hopping [14], which is a classic approach for transmitting wireless signals by switching a carrier among different frequency channels. In this application, each node uniformly and independently selects K_n secret seeds out of a secret pool consisting of P_n secret seeds, and two nodes communicate with each other via frequency hopping only if they share at least certain number of secret seeds.

Although random key predistribution schemes enable secure communications in wireless sensor networks, they are often not explicitly designed to defend against node-capture attacks, which sensor networks deployed in hostile environments are often subject to. The resilience of random key predistribution schemes to node-capture attacks is often analyzed informally, and existing results are often obtained under full visibility without addressing physical transmission constraints, where the full visibility model assumes that any two sensors have a direct communication link in between. We present results on the resilience of the q -composite scheme not only under full

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visibility, but also under physical transmission constraints in which two nodes have to be within a certain distance from each other to communicate.

A metric proposed by Chan *et al.* [1] to address the resilience to node-capture attacks is the probability (denoted by $p_{\text{compromised}}$) that capturing nodes enables an adversary to compromise communications between non-captured nodes. Such probability $p_{\text{compromised}}$ is non-zero in the EG scheme and its q -composite version because non-captured nodes may happen to use keys that are also shared by captured nodes [15], [16]. Clearly, a random key predistribution scheme is perfectly resilient to sensor-capture attacks if $p_{\text{compromised}}$ is always zero. This implies that only communications between a captured node and its direct neighbors are compromised in a perfectly resilient scheme.

The EG scheme and its q -composite version can be easily extended to become perfectly resilient in the Chan *et al.* [1] sense [17], [18]. For instance, after the neighbor discovery phase of the EG scheme ends, in case that two neighboring nodes identified by ID_a and ID_b discover that they share key k_{ab} , they can each use a cryptographic hash function $\text{hash}(\cdot)$ to compute a new shared key $K_{ab} = \text{hash}(ID_a || ID_b || k_{ab})$ for $a < b$, and erase the old key k_{ab} . Since K_{ab} is statistically unique up to the birthday bounds, the EG scheme becomes perfectly resilient. Similarly, the q -composite scheme can become perfectly resilient if the hash operation includes the uniquely ordered q keys instead of the single key k_{ab} .

In mobile sensor networks employing the EG scheme or its q -composite version, the key hashing process above is not applicable, since sensors' neighbor sets change over time as sensors move around. In static sensor networks with the key hashing process, there is still a neighbor discovery phase where perfect resilience cannot hold. Therefore, it is of interest and significance to analyze the resilience of q -composite scheme.

In addition to analyzing the resilience of the q -composite scheme, we also investigate connectivity properties of secure sensor networks under the q -composite scheme, when the adversary may capture a set of sensors. Connectivity means that any two sensors can find a path in between for secure communication. Our connectivity analysis considers many different settings: the network can be static or mobile; we consider the case of no node capture as well as the case of node-capture attacks; limited visibility models considered include the disk model, the link unreliability model, and their combination; under the disk model, we consider both the case ignoring the boundary effect of the network fields as well as the case considering the boundary effect.

B. Problems

1) Connectivity:

Connectivity is a fundamental property that networks are often designed to have.

Definition of Connectivity. A network is connected if each node can find at least one path to reach another node.

For secure sensor networks with the q -composite scheme, we obtain the conditions to have the networks connected; i.e., the conditions that any two sensors can find a path in between for secure communication.

2) Unsplittability:

Another metric to characterize resiliency is unsplittability proposed by Pietro *et al.* [16]. The motivation is that even if a negligible (i.e., $o(1)$)¹ fraction of nodes are compromised, almost all of the remaining nodes are still securely connected. Formally, unsplittability is defined as follows.

Definition of Unsplittability. A secure sensor network is unsplittable if with high probability an adversary that has captured an arbitrary set of $o(n)$ sensors cannot partition the network into two chunks, which both have linear sizes of sensors (i.e., $\Theta(n)$ sensors), and either are isolated from each other or only have compromised communications in between (i.e., keys used for communications between the two chunks are all compromised by the adversary).

From the definition above, for a unsplittable secure sensor network, even if $n_1 = o(n)$ nodes are compromised, we still have a securely connected network consisting of $n - n_2$ nodes, where n_2 can also be written as $o(n)$ despite $n_2 \geq n_1$. The reason for $n_2 \geq n_1$ is that capturing nodes enables an adversary to compromise communications between non-captured nodes which happen to use keys that are also shared by captured nodes.

For secure sensor networks with the q -composite scheme, we explore the conditions to have the networks unsplittable.

3) Unassailability:

We borrow the notion of unassailability by Mei *et al.* [19] and present the following definition. Briefly, a secure sensor network is unassailable if $p_{\text{compromised}}$ is $o(1)$ after the adversary has captured $o(n)$ sensors.

Definition of Unassailability. A secure sensor network is unassailable if an adversary that has captured an arbitrary set of $o(n)$ sensors can only compromise $o(1)$ fraction of communication links in the rest of the network; in other words, an adversary has to capture a constant fraction of sensors to compromise a constant fraction of communication links.

For secure sensor networks with the q -composite scheme, we derive the conditions to have the networks unassailable.

Many prior studies have analyzed node-captured attacks, but they often consider that the adversary *randomly* capture sensors. However, in practice, the adversary may tune its node-capture strategy according to its goals. To analyze unassailability, we analyze *arbitrary* node-capture attacks, in which the adversary can capture an arbitrary set of sensors. The choice of the captured nodes depends on the adversary's goal and also physical limitations (e.g., some sensors may be easier to be captured than others).

¹All limits are taken with $n \rightarrow \infty$, where n is the number of nodes in a network. The standard asymptotic notation $o(\cdot)$, $O(\cdot)$, $\omega(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$, $\sim$ are used. Given two positive sequences f_n and g_n , we have

- 1) $f_n = o(g_n)$ means $\lim_{n \rightarrow \infty} \frac{f_n}{g_n} = 0$.
- 2) $f_n = O(g_n)$ means that there exist positive constants c_1 and N_1 such that $f_n \leq c_1 g_n$ for all $n \geq N_1$.
- 3) $f_n = \omega(g_n)$ means $\lim_{n \rightarrow \infty} \frac{f_n}{g_n} = \infty$.
- 4) $f_n = \Omega(g_n)$ means that there exist positive constants c_2 and N_2 such that $f_n \geq c_2 g_n$ for all $n \geq N_2$.
- 5) $f_n = \Theta(g_n)$ means that there exist positive constants c_3, c_4 and N_3 such that $c_3 g_n \leq f_n \leq c_4 g_n$ for all $n \geq N_3$.
- 6) $f_n \sim g_n$ means that $\lim_{n \rightarrow \infty} \frac{f_n}{g_n} = 1$; i.e., f_n and g_n are asymptotically equivalent.

Symbols	Meanings
P_n	the size of the key pool
K_n	the number of keys assigned to each sensor before deployment
q	the minimum number of keys that two sensors need to share for secure communication over an existing link
r_n	the distance that two sensors have to be within each other for communication under the disk model
t_n	the probability that each link is active when links are allowed to be unreliable

TABLE I
SYMBOLS AND THEIR MEANINGS.

C. Contributions

Our work presents a rigorous and comprehensive analysis on resilience and connectivity of secure sensor networks under the q -composite key scheme. In particular, we investigate conditions to have unassailability and unsplitability. For network connectivity, we derive the conditions to guarantee connectivity in consideration of (i) node-capture attacks, where the adversary may capture a set of sensors and compromise keys in their memory; (ii) sensor mobility, meaning that sensors can move around so that the network topology may change over time; (iii) physical transmission constraints, under which two sensors have to be within each other's transmission range for communication; (iv) the boundary effect of network fields; and (v) link unreliability, meaning that links are allowed to be unreliable.

Summary of Results. We summarize our results as follows. Note that all results are in the asymptotic sense.

- **Connectivity:** We establish the following results for connectivity in secure sensor networks employing the q -composite scheme. Note that r_n denotes the distance that two sensors have to be within each other for communication under the disk model, and t_n denotes the probability that each link is active when links are allowed to be unreliable.

Connectivity of static networks:

- A static secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect is connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- A static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect is connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- A static secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect is connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^* := \max\{1 + \ln \frac{P_n}{K_n^2} / \ln n, 4 \ln \frac{P_n}{K_n^2} / \ln n\}$.
- A static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect is connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > c_n^\# := \max\{1 + (q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n}) / \ln n, 4(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n}) / \ln n\}$.

Connectivity of mobile networks:

- A mobile secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect and under the i.i.d. mobility model is connected for at least $n^{c-1-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- A mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect and under the i.i.d. mobility model is connected for at least $n^{c-1-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- A mobile secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect and under the i.i.d. mobility model is connected for at least $n^{c-c_n^*-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^*$ with c_n^* defined above.
- A mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect and under the i.i.d. mobility model is connected for at least $n^{c-c_n^\#-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > c_n^\#$ with $c_n^\#$ defined above.

Connectivity under node capture: To obtain the connectivity results for an n -size network after the adversary has captured a random set of m nodes out of all n nodes, we just replace $\ln n$ with $\ln(n-m)$, $\frac{\ln n}{n}$ with $\frac{\ln(n-m)}{n-m}$, and n with $(n-m)$ in the above connectivity results (note that we do not replace the “ n ” in K_n , P_n , r_n , t_n).

Unassailability:

- A secure sensor network with the q -composite scheme under any communication model is *unassailable* if $P_n/K_n = \Omega(n)$.

Unsplitability:

- A secure sensor network with the q -composite scheme under full visibility is *unsplitable* if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for any constant $c > 1$, $K_n = \omega(\ln n)$ and $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$.
- A static secure sensor network with the q -composite scheme under the disk model on some area $\mathcal{A}$ is *splitable* if $r_n = o(\sqrt{\mathcal{A}})$.

D. Roadmap

The rest of the paper is organized as follows. Section II reviews related work. In Section III, we describe the system model. Section IV presents the main results. We provide several useful lemmas in Section V. Afterwards, we establish the theorems in Section VI. We explain simulation results in Section VII. Finally, Section VIII concludes the paper.

II. RELATED WORK

Unassailability. Mei *et al.* [19] introduce the notion of unassailability. However, their work is different from ours in the following aspects: they address the EG scheme whereas we consider the more general q -composite scheme; they assume that the adversary knows the key rings of all nodes whereas we do not have such assumption; they enforce the strong assumption that whenever the adversary has compromised one shared key between two sensors, the secure link between the two sensors is compromised even though that compromised key is not used for the secure link. For the EG scheme, Di Pietro *et al.* [15], [16] have presented conditions to ensure that an adversary has to capture a constant fraction of sensors in order to compromise a constant fraction of communication links. Similar results for the q -composite scheme have been obtained by Zhao [17], [18]. However, the difference between [15]–[18] and this paper is that [15]–[18] consider an adversary *randomly* capturing sensors, while this paper investigates the more practical case of *arbitrary* node-capture attacks, in which the adversary can capture an arbitrary set of sensors. In addition to the EG scheme and the q -composite scheme, the random pairwise key predistribution scheme of Chan *et al.* [1] has also received much attention. The unassailability of the pairwise scheme has recently been studied by Yağan and Makowski [20]. The key predistribution schemes by Liu and Ning [12] and Du *et al.* [3] exhibit the threshold behavior in terms of resilience against node-capture attacks. Techniques to improve the resilience against node capture have been studied in [21]–[26].

Unsplittability. For the EG scheme, its unsplittability has been studied by [16]. Note that our unsplittability result is for the more general q -composite scheme. Yet, even when we apply our general result to the EG scheme, the obtained result will be stronger than that of [16]. Specifically, since our unsplittability condition for the q -composite scheme uses $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for a constant $c > 1$, our unsplittability result applying to the EG scheme (i.e., the q -composite scheme with $q = 1$) uses $\frac{K_n^2}{P_n} \geq \frac{c \ln n}{n}$ for a constant $c > 1$. In contrast, the unsplittability result of the EG scheme in [16] requires $\frac{K_n^2}{P_n} \geq \frac{c \ln n}{n}$ for a constant $c > 17$. For the random pairwise key predistribution scheme of Chan *et al.* [1], Yağan and Makowski [20] have recently studied its unsplittability.

Connectivity. Although connectivity of secure sensor networks has been widely investigated in the literature, many prior studies has one or more than one of the following limitations: much research [1], [4], [6] ignores either real-world transmission constraints between sensors or mobility of sensors because analyzing the key predistribution scheme, transmission constraints, and node mobility together makes the analysis very challenging; several researches considering transmission constraints obtain quite weak results [10], [27]–[29] or apply to limited settings [17] (e.g., the case ignoring the boundary effect of the network fields); the proof techniques [30] lack formality. Below we discuss some related work in detail.

For a secure sensor network with the q -composite scheme under the full visibility, its connectivity has been studied in

[10], [30], while its k -connectivity has been considered in [29], where k -connectivity ensures connectivity despite the failure of any $k-1$ nodes [31], [32]. However, the proof techniques in [30] lack formality (specifically [30, Equation (6.93)]) and are different from those in this paper. The results of connectivity in [29] and k -connectivity in [10] address only the narrow and impractical range of $P_n = o(n^{\frac{1}{q}} (\ln n)^{-\frac{3}{5q}})$, as explained below. In contrast, from condition set Λ in (6), our Theorem 1 considers a more practical range of $P_n = \omega(n \ln n)$. We present more details below.

We explain that the result of [10] for connectivity is not applicable to practical sensor networks, whereas our result is applicable. The limitation of [10] is that both K_n and P_n are required to be quite small in [10] so they will not satisfy the resiliency requirement $\frac{P_n}{K_n} = \Omega(n)$. More specifically, [10, Equation (2)] enforces

$$\left[(q+2) \binom{K_n}{q}^5 (\ln \ln n)^2 \right]^{\frac{3K_n-q}{3(K_n-q)}} \leq (\ln n)^{1-\gamma} \quad \text{for a constant } \gamma \in (0, 1), \quad (1)$$

where the notation s and d in [10] correspond to q and K_n here.

In (1), the exponent $\frac{3K_n-q}{3(K_n-q)}$ is greater than 1 given $K_n > q$, since q is a constant and does not scale with n (q is often less than 10), while K_n controls the number of keys on each sensor and is at least a two-digit number for a network comprising thousands of sensors. In fact, if $K_n < q$, the q -composite scheme is meaningless because each sensor has just K_n keys so it is impossible for two sensors to share q keys if $K_n < q$. Furthermore, in (1), the base is greater than 1 for all sufficiently large n owing to $\lim_{n \rightarrow \infty} \ln \ln n = \infty$. Given the above, we can cancel out the exponent in (1); i.e., (1) implies

$$(q+2) \binom{K_n}{q}^5 (\ln \ln n)^2 \leq (\ln n)^{1-\gamma} \quad \text{for a constant } \gamma \in (0, 1). \quad (2)$$

In view of $\binom{K_n}{q} = \frac{1}{q!} \prod_{i=0}^{q-1} (K_n - i) \geq \frac{1}{q!} (K_n - q)^q$, we use (2) to derive

$$K_n \leq q + \sqrt[q]{q!} \cdot \sqrt[q]{\frac{1}{q+2} \cdot \frac{(\ln n)^{1-\gamma}}{(\ln \ln n)^2}} \quad \text{for a constant } \gamma \in (0, 1), \quad (3)$$

which along with the fact that q is a constant further induces

$$K_n = O \left(\sqrt[q]{\frac{(\ln n)^{1-\gamma}}{(\ln \ln n)^2}} \right) = o \left((\ln n)^{\frac{1}{5q}} \right). \quad (4)$$

To ensure connectivity of a secure sensor network resulted from the q -composite key predistribution scheme, with p_q denoting the edge probability, both [10] and our work obtain that it is necessary to have $p_q = \Omega(\frac{\ln n}{n})$ or equivalently $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} = \Omega(\frac{\ln n}{n})$, since it holds that $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$ is an asymptotic expression of p_q from (18) later. The requirement

$\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^{2q}} = \Omega\left(\frac{\ln n}{n}\right)$ and the condition (4) derived from [10] together imply

$$P_n = \frac{K_n^2}{\sqrt[q]{q!} \times \Omega\left(\frac{\ln n}{n}\right)} = o\left(n^{\frac{1}{q}}(\ln n)^{-\frac{3}{5q}}\right). \quad (5)$$

However, we explain that (5) is not applicable to practical sensor networks. This paper for general q and Di Pietro *et al.* [15], [16] for $q = 1$ have both proved that the condition $\frac{P_n}{K_n} = \Omega(n)$ is needed to ensure reasonable network resiliency against node-capture attacks in the sense that an adversary capturing sensors at random has to obtain at least a constant fraction of nodes of the network in order to compromise a constant fraction of secure links. The condition $\frac{P_n}{K_n} = \Omega(n)$ with $K_n \geq 1$ clearly implies $P_n = \Omega(n)$, which does not hold for any P_n satisfying (5). Hence, we have explained that the result of [10] cannot be used to design secure sensor networks in practice. In contrast, from condition set Λ in (6), our theorems apply to a more practical range of $P_n = \omega(n \ln n)$. Hence, our results apply to real-world secure sensor networks and provide useful design guidelines.

For connectivity analysis of secure sensor networks taking into transmission constraints, for the EG scheme, Krzywdziński and Rybarczyk [28], and Krishnan *et al.* [27] recently show upper bounds $\frac{8 \ln n}{n}$ and $\frac{2\pi \ln n}{n}$ for the critical threshold that the edge probability (i.e., the probability of a secure link between two sensors) takes for connectivity, while [14] proves the exact threshold as $\frac{\ln n}{n}$. For the q -composite scheme with general q , [17] has presented connectivity results under transmission constraints. However, the following issues addressed in this paper are not tackled by [17]: the boundary effect of the network fields, the mobility of sensors, the combination of link unreliability and the transmission constraints modeled by the disk model (to be explained in Section III-A2). We note that there are many connectivity studies [4], [5], [33]–[36] considering only link unreliability without the disk model (node mobility and node-capture attacks are not considered in these references).

III. SYSTEM MODELS

All of our studied networks employ the q -composite scheme. Clearly, our analysis also applies to the basic Eschenauer–Gligor scheme which has $q = 1$. We first discuss the communication models and mobility, and then detail the studied networks.

A. Communication Models

We consider the following communication models.

1) Full Visibility:

The full visibility model assumes that any two sensors have a direct communication link in between. Under such model, a secure link exists between two sensors if and only if they share at least q keys.

2) Disk Model:

In the disk model, each node's transmission area is a disk with a transmission radius r_n , with r_n being a function of n for generality. Two nodes have to be within r_n (their distance

is at most r_n) for direct communication. As for the node distribution, the same as much previous work [27], [28], [37], [38], we consider that the n nodes are independently and uniformly deployed in some network region $\mathcal{A}$. We let the region $\mathcal{A}$ be either a torus $\mathcal{T}$ or a square $\mathcal{S}$, each with a unit area. The unit torus $\mathcal{T}$ eliminates the boundary effect (a node on $\mathcal{T}$ “exits” the area from one side appears as reentering from the opposite side). The boundary effect of the unit square $\mathcal{S}$ is that a circle with radius r_n centered a point near the boundary of $\mathcal{S}$ may have a part falling outside of $\mathcal{S}$, so a node close to one side and another node close to the opposite side may not have an edge in between on the square $\mathcal{S}$, but may have an edge in between on the torus $\mathcal{T}$ because of wrap-around connections of torus topology.

3) Link Unreliability:

Communication links between nodes may not be available due to the presence of physical barriers between nodes or because of harsh environmental conditions severely impairing transmission. To model unreliable links, each link is either *active* with probability t_n or *inactive* with probability $(1 - t_n)$, where t_n is a function of n for generality.

Table I summarizes the symbols and their meanings.

B. Mobility

In addition to static sensor networks, we also consider mobile sensor networks which are initialized in the same way as static sensor networks. After initialization, sensors may move around. On either the torus or the square, we consider the i.i.d. mobility model, where each node independently picks a new location uniformly at random at the beginning of a time slot and stays at the location in the rest of the time slot. This i.i.d. mobility model clearly yields a uniform node distribution at each time slot. For each mobile network that we consider, given a uniform node distribution at each time slot, we can view the mobile network at a single time slot as an instance of the corresponding static network. A future work is to consider other mobility models [39]–[42].

C. Studied Networks

We summarize our studied networks in Table II. As presented, we consider the following nine secure sensor networks:

- a secure sensor network with the q -composite scheme under full visibility (the network can be either static or mobile),
- a static secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect,
- a static secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect,
- a static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect,
- a static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect,

Settings			Notation for networks
full visibility (static/mobile secure sensor networks)			network ^{full visibility}
static secure sensor networks	disk model	torus	static-network ^{disk model unit torus}
		square	static-network ^{disk model unit square}
	disk model w/ unreliable links	torus	static-network ^{disk model unit torus unreliable links}
		square	static-network ^{disk model unit square unreliable links}
mobile secure sensor networks	disk model	torus	mobile-network ^{disk model unit torus}
		square	mobile-network ^{disk model unit square}
	disk model w/ unreliable links	torus	mobile-network ^{disk model unit torus unreliable links}
		square	mobile-network ^{disk model unit square unreliable links}

TABLE II

THE STUDIED SECURE SENSOR NETWORKS AND THEIR NOTATION, WHERE “w/” IS SHORT FOR “WITH”. ALL NETWORKS EMPLOY THE q -COMPOSITE RANDOM KEY PREDISTRIBUTION SCHEME.

- a mobile secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect,
- a mobile secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect,
- a mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect, and
- a mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect.

Throughout the paper, q is an arbitrary positive integer and does not scale with n , the number of nodes in the sensor network. In addition, $\ln$ is the natural logarithm function, the base of which is e .

IV. MAIN RESULTS

We present the main results below.

A. Connectivity of Secure Sensor Networks

We establish connectivity results for the nine settings of secure sensor networks in Table II. A network is connected if each node can find at least one path to securely communicate with another node. The results are summarized in Table III, and they will be presented as theorems in detail.

All results given in Table III rely a set Λ of conditions, defined as follows:

$$\text{Condition set } \Lambda : \begin{cases} K_n = \omega(\ln n), \\ K_n = o\left(\min\left\{\sqrt{P_n}, \frac{P_n}{n}\right\}\right), \\ r_n = o(1). \end{cases} \quad (6)$$

We explain that all conditions in set Λ are practical. Note that K_n is the number of keys assigned to each sensor before deployment. In real-world implementations, K_n is often larger [4], [8] than $\ln n$, so $K_n = \omega(\ln n)$ follows. As concrete examples, we have $\ln 1000 \approx 6.9$, $\ln 5000 \approx 8.5$ and $\ln 10000 \approx 9.2$. Since K_n is much smaller compared to both n and P_n due to constrained memory and computational resources of

sensors [1], [2], [4], then $\frac{K_n^2}{P_n} = o(1)$ and $\frac{K_n}{P_n} = o\left(\frac{1}{n}\right)$ are practical, yielding $K_n = o\left(\min\left\{\sqrt{P_n}, \frac{P_n}{n}\right\}\right)$. Note that $K_n = \omega(\ln n)$ and $K_n = o\left(\min\left\{\sqrt{P_n}, \frac{P_n}{n}\right\}\right)$ together imply $P_n = \omega(n \ln n)$, which is also practical since P_n is larger than n [1], [2], [16]. As examples, we have $1000 \ln 1000 \approx 6907$, $5000 \ln 5000 \approx 42585$ and $10000 \ln 10000 \approx 92103$. Finally, because we consider a unit area of network region and there are n nodes in this region, the condition $r_n = o(1)$ is also practical.

To look at the conditions related to $\frac{\ln n}{n}$ in Table III, we first explain that for each network in Table III, the left hand side of each condition related to $\frac{\ln n}{n}$ in the result is asymptotically equivalent to the edge probability of the network, where the edge probability is the probability that two nodes have an edge in between for direct communication, and is defined for each time slot for mobile networks (note that all nodes are symmetric in each network). Specifically, we have the following under condition set Λ :

- ① $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$ is asymptotically equivalent to the edge probability of network^{full visibility},
- ② $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ is a common asymptotic value of the edge probabilities of networks static-network^{disk model unit torus}, static-network^{disk model unit square}, mobile-network^{disk model unit torus}, and mobile-network^{disk model unit square} and
- ③ $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ is a common asymptotic value of the edge probabilities of static-network^{disk model unit torus unreliable links}, static-network^{disk model unit square unreliable links}, mobile-network^{disk model unit torus unreliable links}, and mobile-network^{disk model unit square unreliable links}.

Before explaining the above results ①–③, we first present a useful result (i.e., Lemma 1) and its proof.

Lemma 1. *Let $\mu_n(\mathcal{A})$ be the probability that two nodes are within distance r_n when they are independently and uniformly distributed in a network region $\mathcal{A}$. Then we have:*

- When $\mathcal{A}$ is a torus $\mathcal{T}$ of unit area, $\mu_n(\mathcal{A})$ becomes $\mu_n(\mathcal{T})$, which is given by

$$\mu_n(\mathcal{T}) = \pi r_n^2, \text{ for } r_n \leq \frac{1}{2}. \quad (7)$$

- When $\mathcal{A}$ is a square $\mathcal{S}$ of unit area, $\mu_n(\mathcal{A})$ becomes $\mu_n(\mathcal{S})$, which satisfies

$$(1 - 2r_n)^2 \cdot \pi r_n^2 \leq \mu_n(\mathcal{S}) \leq \pi r_n^2, \text{ for } r_n \leq \frac{1}{2}, \quad (8)$$

and

$$\mu_n(\mathcal{S}) \sim \pi r_n^2, \text{ for } r_n = o(1), \quad (9)$$

where $\mu_n(\mathcal{S}) \sim \pi r_n^2$ means $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{S})}{\pi r_n^2} = 1$ (i.e., $\mu_n(\mathcal{S})$ is asymptotically equivalent to πr_n^2), and $r_n = o(1)$ means $\lim_{n \rightarrow \infty} r_n = 0$.

Proof of Lemma 1:

Our goal is to analyze $\mu_n(\mathcal{A})$, which denotes the probability that two nodes are within distance r_n when they are independently and uniformly deployed in a network region $\mathcal{A}$. We let v_x and v_y denote the two nodes, and write their distance as distance(v_x, v_y).

Settings			Connectivity results
static/mobile secure sensor networks under full visibility			connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for any constant $c > 1$
static secure sensor networks	disk model	torus	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$
		square	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^*$
	disk model w/ unreliable links	torus	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > 1$
		square	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > c_n^\#$
mobile secure sensor networks	disk model	torus	connected for at least $n^{c-1-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$
		square	connected for at least $n^{c-c_n^*-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^*$
	disk model w/ unreliable links	torus	connected for at least $n^{c-1-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > 1$
		square	connected for at least $n^{c-c_n^\#-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for any constant $c > c_n^\#$

TABLE III

DIFFERENT SETTINGS OF SECURE SENSOR NETWORKS AND THEIR CONNECTIVITY RESULTS, WHERE “W/” IS SHORT FOR “WITH”. NOTE THAT c_n^* IS $\max \left\{ 1 + (\ln \frac{P_n}{K_n^2}) / \ln n, 4(\ln \frac{P_n}{K_n^2}) / \ln n \right\}$, $c_n^\#$ IS $c_n^\#$ IS $\max \left\{ 1 + (q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n}) / \ln n, 4(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n}) / \ln n \right\}$.

Proving (7) for $\mu_n(\mathcal{T})$ on the unit torus $\mathcal{T}$: We consider the case of the unit torus $\mathcal{T}$ here. Note that $\mu_n(\mathcal{T})$ means $\mathbb{P}[\text{distance}(v_x, v_y) \leq r_n]$ when nodes v_x and v_y are independently and uniformly deployed in the unit torus $\mathcal{T}$. Given any location of node v_x , the event $\text{distance}(v_x, v_y) \leq r_n$ happens if and only if the random position of node v_y is located in the circle centered at node v_x 's position with a radius of r_n ; this clearly happens with probability πr_n^2 since the above circle is completely inside the torus $\mathcal{T}$ due to the following three reasons:

- we consider $r_n \leq \frac{1}{2}$ in (7);
- the torus $\mathcal{T}$ has no boundary (if we think the torus as a square, then a point “exits” the torus area from one side appears as reentering from the opposite side),
- the circle centered at node v_x 's position with a radius of r_n has an area of πr_n^2 , and the torus $\mathcal{T}$ has an unit area.

Therefore, given any location of v_x , the event $\text{distance}(v_x, v_y) \leq r_n$ on the unit torus $\mathcal{T}$ occurs with probability πr_n^2 when node v_y is uniformly deployed in the unit torus $\mathcal{T}$. Now we consider that node v_x is also uniformly distributed in the unit torus $\mathcal{T}$, and clearly $\text{distance}(v_x, v_y) \leq r_n$ still happens with probability πr_n^2 . Hence, we have proved $\mu_n(\mathcal{T}) = \pi r_n^2$, for $r_n \leq \frac{1}{2}$; i.e., (7) holds.

Proving (8) for $\mu_n(\mathcal{S})$ on the unit square $\mathcal{S}$: We consider the case of the unit square $\mathcal{S}$ here. Note that $\mu_n(\mathcal{S})$ means $\mathbb{P}[\text{distance}(v_x, v_y) \leq r_n]$ when nodes v_x and v_y are independently and uniformly deployed in the unit square $\mathcal{S}$. Given any location of v_x , the event $\text{distance}(v_x, v_y) \leq r_n$ happens if and only if the random position of node v_y is located in the circle centered at node v_x 's position with a radius of r_n .

We let C denote the center of the unit square $\mathcal{S}$. Then for $r_n \leq \frac{1}{2}$, we define $\mathcal{Z}$ as a square centered at C with side length $1 - 2r_n$, as illustrated in Figure 1. If node v_x 's location is inside $\mathcal{Z}$, we denote it by $v_x \in \mathcal{Z}$.

On the one hand, whenever $v_x \in \mathcal{Z}$, the circle centered at node v_x 's position with a radius of r_n is completely inside the square $\mathcal{S}$, as shown in Figure 1, so the random position of

node v_y belongs to this circle with probability πr_n^2 . Hence, given any location of v_x which is inside $\mathcal{Z}$, the conditional probability that v_x and v_y is πr_n^2 , when we consider the uniform distribution of v_y 's position. Then the conditional probability of $\text{distance}(v_x, v_y) \leq r_n$ given $v_x \in \mathcal{Z}$ is πr_n^2 , for $r_n \leq \frac{1}{2}$. Formally, we obtain

$$\mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \mid (v_x \in \mathcal{Z})] = \pi r_n^2, \text{ for } r_n \leq \frac{1}{2}. \quad (10)$$

On the other hand, whenever $v_x \notin \mathcal{Z}$, the circle centered at node v_x 's position with a radius of r_n may or may not be completely inside the square $\mathcal{S}$, so the probability that the random position of node v_y belongs to the intersection of this circle and the square $\mathcal{S}$ is no greater than πr_n^2 . Hence, given any location of v_x which is outside of $\mathcal{Z}$, the conditional probability that v_x and v_y is at most πr_n^2 , when we consider the uniform distribution of v_y 's position. Then the conditional probability of $\text{distance}(v_x, v_y) \leq r_n$ given $v_x \notin \mathcal{Z}$ is no greater than πr_n^2 , for $r_n \leq \frac{1}{2}$. Formally, we have

$$\mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \mid (v_x \notin \mathcal{Z})] \leq \pi r_n^2, \text{ for } r_n \leq \frac{1}{2}. \quad (11)$$

Recall that $\mu_n(\mathcal{S})$ denotes $\text{distance}(v_x, v_y) \leq r_n$ on the unit square $\mathcal{S}$. Clearly, from the law of total probability, we derive

$$\begin{aligned} \mu_n(\mathcal{S}) &= \mathbb{P}[\text{distance}(v_x, v_y) \leq r_n] \\ &= \mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \cap (v_x \in \mathcal{Z})] \\ &\quad + \mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \cap (v_x \notin \mathcal{Z})] \\ &= \mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \mid (v_x \in \mathcal{Z})] \times \mathbb{P}[v_x \in \mathcal{Z}] \\ &\quad + \mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \mid (v_x \notin \mathcal{Z})] \times \mathbb{P}[v_x \notin \mathcal{Z}]. \end{aligned} \quad (12)$$

Ignoring the term after the plus sign in (12), we further have

$$\mu_n(\mathcal{S}) \geq \mathbb{P}[(\text{distance}(v_x, v_y) \leq r_n) \mid (v_x \in \mathcal{Z})] \times \mathbb{P}[v_x \in \mathcal{Z}]. \quad (13)$$

Fig. 1. For the position of a node v_x satisfying the condition that v_x 's distance to each edge of the unit square is at least r_n with $r_n \leq \frac{1}{2}$, given the position of v_x , the conditional probability that another node v_y falls in v_x 's transmission area is πr_n^2 , since the circle centered at node v_x 's position with a radius of r_n is completely inside the square.

Note that v_x is uniformly distributed on the square $\mathcal{S}$, whose area denoted by $|\mathcal{S}|$ is given by $|\mathcal{S}| = 1$. The square $\mathcal{Z}$ has a side length of $1 - 2r_n$, so its area denoted by $|\mathcal{Z}|$ equals $|\mathcal{Z}| = (1 - 2r_n)^2$. Hence, we obtain

$$\mathbb{P}[v_x \in \mathcal{Z}] = \frac{|\mathcal{Z}|}{|\mathcal{S}|} = \frac{(1 - 2r_n)^2}{1} = (1 - 2r_n)^2, \text{ for } r_n \leq \frac{1}{2}, \quad (14)$$

and

$$\mathbb{P}[v_x \notin \mathcal{Z}] = 1 - \mathbb{P}[v_x \in \mathcal{Z}] = 1 - (1 - 2r_n)^2, \text{ for } r_n \leq \frac{1}{2}. \quad (15)$$

Using (10) and (14) in (13), we get

$$\mu_n(\mathcal{S}) \geq \pi r_n^2 \times (1 - 2r_n)^2, \text{ for } r_n \leq \frac{1}{2}. \quad (16)$$

Applying (10) (11) (14) and (15) to (12), we establish

$$\begin{aligned} \mu_n(\mathcal{S}) &\leq \pi r_n^2 \times (1 - 2r_n)^2 + \pi r_n^2 \times [1 - (1 - 2r_n)^2] \\ &\leq \pi r_n^2, \text{ for } r_n \leq \frac{1}{2}. \end{aligned} \quad (17)$$

Summarizing (16) and (17), we have $(1 - 2r_n)^2 \cdot \pi r_n^2 \leq \mu_n(\mathcal{S}) \leq \pi r_n^2$, for $r_n \leq \frac{1}{2}$; i.e., (8) is proved.

Proving (9) for $\mu_n(\mathcal{S})$ on the unit square $\mathcal{S}$: Recall that for two positive sequences x_n and y_n , the relation $x_n \sim y_n$ means $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = 1$. From (8), it follows that $(1 - 2r_n)^2 \leq \frac{\mu_n(\mathcal{S})}{\pi r_n^2} \leq 1$, for $r_n \leq \frac{1}{2}$. Clearly, given $r_n = o(1)$, we have $\lim_{n \rightarrow \infty} (1 - 2r_n)^2 = 1$ and also have $r_n \leq \frac{1}{2}$ for all n sufficiently large. Summarizing the above results, we clearly obtain $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{S})}{\pi r_n^2} = 1$; i.e., $\mu_n(\mathcal{S}) \sim \pi r_n^2$. Hence, (9) is proved. ■

We now show results ①–③ of Page 6.

We first show the above result ①. By [43, Lemma 1], if $K_n = \omega(1)$ and $K_n = o(\sqrt{P_n})$, then

$$p_q \sim \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}. \quad (18)$$

The asymptotic equivalence (18) above holds for network^{full visibility} under condition set Λ in (6) on Page 6, since we have $K_n = \omega(1)$ and $K_n = o(\sqrt{P_n})$ from Λ .

We then explain the result ②. With $\mu_n(\mathcal{T})$ (resp., $\mu_n(\mathcal{S})$) denoting the probability that two nodes are within distance r_n

when they are independently and uniformly distributed on the unit torus $\mathcal{T}$ (resp., the unit square $\mathcal{S}$), Lemma 1 above shows $\mu_n(\mathcal{T}) = \pi r_n^2$, for $r_n \leq \frac{1}{2}$, and $\mu_n(\mathcal{S}) \sim \pi r_n^2$, for $r_n = o(1)$. We can use Lemma 1 because our condition set Λ in (6) on Page 6 has the condition $r_n = o(1)$, which implies $r_n \leq \frac{1}{2}$ for all n sufficiently large. In words, Lemma 1 says that the probability that two nodes are within distance r_n is given by πr_n^2 on the unit torus $\mathcal{T}$, and is asymptotically equivalent to πr_n^2 on the unit square $\mathcal{S}$. In each of static-network^{disk model}_{unit torus} and static-network^{disk model}_{unit square}, since two nodes have an edge in between if and only if they are within distance r_n and also they share at least q keys, then under condition set Λ , the edge probability is asymptotically equivalent to $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ from $\mu_n(\mathcal{T}) = \pi r_n^2$, $\mu_n(\mathcal{S}) \sim \pi r_n^2$ and $p_q \sim \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$ in (18). Furthermore, at each time slot, mobile networks mobile-network^{disk model}_{unit torus} and mobile-network^{disk model}_{unit square} can be viewed as instances of static networks static-network^{disk model}_{unit torus} and static-network^{disk model}_{unit square}, respectively. In view of the above, the result ② is proved.

We now show the result ③. Compared with the networks in the result ②, the networks in the result ③ also consider link unreliability, in which each link is allowed to be inactive with probability t_n . Multiplying the term $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ with t_n , we obtain $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$, which is a common asymptotic value of the edge probabilities of the networks in the result ③. Therefore, the result ③ follows.

We have shown the results ① ② and ③ on the asymptotic values of the edge probabilities of the networks in Table III. For each network in Table III, let edge-prob denote the used asymptotic value of its edge probability; namely, edge-prob is $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$ for network^{full visibility} in the result ①, is $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ for the four networks in the result ②, and is $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ for the four networks in the result ③. As given in Table III, the conditions related to $\frac{\ln n}{n}$ can be summarized as follows. For network^{full visibility} and the four networks on the unit torus, we have the condition that edge-prob $\geq \frac{c \ln n}{n}$ for any constant $c > 1$. For the four networks on the unit square, we have the condition that edge-prob $\geq \frac{c \ln n}{n}$ for any constant $c > \tilde{c}_n$,

where $\tilde{c}_n$ is given by

$$\begin{aligned} \tilde{c}_n &:= \max \left\{ 1 + \left(\ln [(\widetilde{\text{edge-prob}})^{-1}] \right) / \ln n, \right. \\ &\quad \left. 4 \left(\ln [(\widetilde{\text{edge-prob}})^{-1}] \right) / \ln n \right\} \\ &= \begin{cases} 1 + \left(\ln [(\widetilde{\text{edge-prob}})^{-1}] \right) / \ln n, & \text{if } \widetilde{\text{edge-prob}} = \tilde{\Omega}(n^{-1/3}), \\ 4 \left(\ln [(\widetilde{\text{edge-prob}})^{-1}] \right) / \ln n, & \text{if } \widetilde{\text{edge-prob}} = \tilde{O}(n^{-1/3}). \end{cases} \end{aligned} \quad (19)$$

In (19) above, we write a quantity x_n as $\tilde{\Omega}(n^\alpha)$ for some constant α if $x_n \geq n^{\alpha-\epsilon_1}$ for all n sufficiently large, where ϵ_1 is an arbitrary positive constant. Similarly, a quantity x_n is written as $\tilde{O}(n^\alpha)$ for some constant α if $x_n \leq n^{\alpha+\epsilon_2}$ for all n sufficiently large, where ϵ_2 is an arbitrary positive constant.

The difference between the condition $c > 1$ for networks on the torus, and the condition $c > \tilde{c}_n$ for networks on the square is resulted from the boundary effect of the square.

From (19), we observe a phase transition of $\tilde{c}_n$ when edge-prob or the edge probability asymptotically changes from $\tilde{\Omega}(n^{-1/3})$ to $\tilde{O}(n^{-1/3})$, or vice versa. The intuition is that to compute the expected number of isolated nodes, in integrating all possible points in unit square $\mathcal{S}$ with the boundary effect, different areas on $\mathcal{S}$ contribute to the dominant part of the integral depending on the edge probability, where a node is non-isolated if it has a link with at least another node, and a node is isolated if it has no link with any other node.

We now present the connectivity results in Table III as more understandable theorems, which are all proved in Section VI. We give first the results of secure sensor networks under full visibility, then the results of static secure sensor networks, and finally those of mobile secure sensor networks. As mentioned before in Section III for the system model, all studied networks employ the q -composite scheme.

1) Connectivity of Secure Sensor Networks with the q -Composite Scheme under Full Visibility:

Theorem 1 below on connectivity is for the full visibility case. Its proof is provided in Section VI-A.

Theorem 1 (Connectivity of network^{full visibility}). *Under condition set Λ in (6) on Page 6, a secure sensor network with the q -composite scheme under full visibility is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

As explained in the result ① in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$ is an asymptotic value of the edge probability of network^{full visibility}. Although Theorem 1 can also be derived using the results in [30], the proof techniques in [30] lack formality (specifically [30, Equation (6.93)]) and are different from those in this paper. In addition, Bloznelis and Rybarczyk [29] (resp., Bloznelis and Łuczak [10]) have recently investigated connectivity (resp., k -connectivity) of network^{full visibility}, but both results after a rewriting address only the narrow and impractical range of $P_n = o(n^{\frac{1}{q}} (\ln n)^{-\frac{3}{5q}})$. In contrast, from condition set Λ in (6), our Theorem 1 considers a more

practical range of $P_n = \omega(n \ln n)$. More details can be found in Section II for related work.

2) Connectivity of Static Secure Sensor Networks with the q -Composite Scheme:

Theorems 2–5 present connectivity results of static secure sensor networks. Their proofs are deferred to Section VI-B.

a) Connectivity under the Disk Model on the Torus without the Boundary Effect:

Theorem 2 (Connectivity of static-network^{disk model unit torus}). *Under condition set Λ in (6) on Page 6, a static secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

As given in the result ② in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ is asymptotically equivalent to the edge probability of static-network^{disk model unit torus}.

b) Connectivity with Unreliable Links under the Disk Model on the Torus without the Boundary Effect:

Theorem 3 (Connectivity of static-network^{disk model unit torus unreliable links}). *Under condition set Λ in (6) on Page 6, a static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

As explained in the result ③ in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ is an asymptotic value of the edge probability of static-network^{disk model unit torus unreliable links}.

c) Connectivity under the Disk Model on the Square with the Boundary Effect:

Theorem 4 (Connectivity of static-network^{disk model unit square}). *Under condition set Λ in (6) on Page 6, a static secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is some constant satisfying for all n sufficiently large that*

$$c > c_n^* := \max \left\{ 1 + \left(\ln \frac{P_n}{K_n^2} \right) / \ln n, 4 \left(\ln \frac{P_n}{K_n^2} \right) / \ln n \right\}. \quad (20)$$

As given in the result ④ in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ is asymptotically equivalent to the edge probability of static-network^{disk model unit square}.

d) Connectivity with Unreliable Links under the Disk Model on the Square with the Boundary Effect:

Theorem 5 (Connectivity of static-network^{disk model unit square unreliable links}). *Under condition set Λ in (6) on Page 6, a static secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$*

for all n sufficiently large, where c is some constant satisfying for all n sufficiently large that

$$c > c_n^\# := \max \left\{ 1 + \left(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n} \right) / \ln n, \right. \\ \left. 4 \left(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n} \right) / \ln n \right\}. \quad (21)$$

As explained in the result ③ in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ is an asymptotic value of the edge probability of static-network_{unit square}^{disk model}.

3) *Connectivity of Mobile Secure Sensor Networks with the q-Composite Scheme:*

Theorems 6–9 below present connectivity results of mobile secure sensor networks. Their proofs are deferred to Section VI-C. As explained in Section III for the system model, at each time slot, each mobile network can be viewed as an instance of its corresponding static network, and thus its edge probability defined for each time slot also equals that of the corresponding static network.

a) *Connectivity under the Disk Model on the Torus without the Boundary Effect:*

Theorem 6 (Connectivity of mobile-network_{unit torus}^{disk model}). *Under condition set Λ in (6) on Page 6, a mobile secure sensor network with the q -composite scheme under the disk model on the unit torus without the boundary effect and under the i.i.d. mobility model is connected with high probability for at least $n^{c-1-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

As given in the result ② in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ is asymptotically equivalent to the edge probability of mobile-network_{unit torus}^{disk model}. Also, from Theorem 2, the conditions in Theorem 6 also ensure that the corresponding static network static-network_{unit torus}^{disk model} is connected with high probability.

b) *Connectivity with Unreliable Links under the Disk Model on the Torus without the Boundary Effect:*

Theorem 7 (Connectivity of mobile-network_{unit torus}^{disk model}_{unreliable links}). *Under condition set Λ in (6) on Page 6, a mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit torus without the boundary effect and under the i.i.d. mobility model is connected with high probability for at least $n^{c-1-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

As explained in the result ③ in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ is an asymptotic value of the edge probability of mobile-network_{unit torus}^{disk model}_{unreliable links}. Moreover, from Theorem 3, the conditions in Theorem 7 also ensure that the corresponding static network static-network_{unit torus}^{disk model}_{unreliable links} is connected with high probability.

c) *Connectivity under the Disk Model on the Square with the Boundary Effect:*

Theorem 8 (Connectivity of mobile-network_{unit square}^{disk model}). *Under condition set Λ in (6) on Page 6, a mobile secure sensor network with the q -composite scheme under the disk model on the unit square with the boundary effect and under the i.i.d. mobility model is connected with high probability for at least $n^{c-c_n^*-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is some constant satisfying $c > c_n^*$ for all n sufficiently large, with c_n^* defined in (20).*

As given in the result ② in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2$ is asymptotically equivalent to the edge probability of mobile-network_{unit square}^{disk model}. In addition, from Theorem 4, the conditions in Theorem 8 also ensure that the corresponding static network static-network_{unit square}^{disk model} is connected with high probability.

d) *Connectivity with Unreliable Links under the Disk Model on the Square with the Boundary Effect:*

Theorem 9 (Connectivity of mobile-network_{unit square}^{disk model}_{unreliable links}). *Under condition set Λ in (6) on Page 6, a mobile secure sensor network with the q -composite scheme and unreliable links under the disk model on the unit square with the boundary effect and under the i.i.d. mobility model is connected with high probability for at least $n^{c-c_n^\#-\epsilon}$ consecutive time slots from the beginning for an arbitrary positive constant ϵ , if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is some constant satisfying $c > c_n^\#$ for all n sufficiently large, with $c_n^\#$ defined in (21).*

As explained in the result ③ in Section IV-A, $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n$ is an asymptotic value of the edge probability of mobile-network_{unit square}^{disk model}_{unreliable links}. Furthermore, from Theorem 5, the conditions in Theorem 9 also ensure that the corresponding static network static-network_{unit square}^{disk model}_{unreliable links} is connected with high probability.

4) *Connectivity under Random Node-Capture Attacks:* We present below the connectivity results under random node-capture attacks. After the adversary has captured some random set of m nodes, the remaining network is statistically equivalent to a network initially with $(n - m)$ nodes in the absence of node capture. Hence, by replacing $\ln n$ with $\ln(n - m)$, replacing $\frac{\ln n}{n}$ with $\frac{\ln(n - m)}{n - m}$ and n with $(n - m)$ in the connectivity results in Table III, we obtain the connectivity results in Table IV for an n -size network after the adversary has captured a random set of m nodes out of all n nodes. Note that we do not replace the “ n ” in K_n , P_n , r_n , t_n .

B. Unsplittability of Secure Sensor Networks with the q -Composite Scheme

Recall the definition of unsplittability in Section I-B. A secure sensor network is unsplittable if with high probability an adversary that has captured an arbitrary set of $o(n)$ sensors cannot partition the network into two chunks, which both have linear sizes of sensors, and either are isolated from each other or only have compromised communications in between.

Settings			Connectivity results under a random capture of m nodes
static/mobile secure sensor networks under full visibility			connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > 1$
static secure sensor networks	disk model	torus	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > 1$
		square	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > c_n^*$
	disk model w/ unreliable links	torus	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > 1$
		square	connected if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > c_n^\#$
mobile secure sensor networks	disk model	torus	connected for at least $(n-m)^{c-1-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > 1$
		square	connected for at least $(n-m)^{c-c_n^*-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > c_n^*$
	disk model w/ unreliable links	torus	connected for at least $(n-m)^{c-1-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > 1$
		square	connected for at least $(n-m)^{c-c_n^\#-\epsilon}$ consecutive time slots if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \cdot t_n \geq \frac{c \ln(n-m)}{n-m}$ for any constant $c > c_n^\#$

TABLE IV

DIFFERENT SETTINGS OF SECURE SENSOR NETWORKS AND THEIR CONNECTIVITY RESULTS UNDER A RANDOM CAPTURE OF m NODES, WHERE “W/” IS SHORT FOR “WITH”. NOTE THAT c_n^* IS $\max \left\{ 1 + \left(\ln \frac{P_n}{K_n^2} \right) / [\ln(n-m)], 4 \left(\ln \frac{P_n}{K_n^2} \right) / [\ln(n-m)] \right\}$, $c_n^\#$ IS

$$\max \left\{ 1 + \left(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n} \right) / [\ln(n-m)], 4 \left(q \ln \frac{P_n}{K_n^2} + \ln \frac{1}{t_n} \right) / [\ln(n-m)] \right\}$$

Fig. 2. An adversary that has captured all $o(n)$ nodes in the shaded region can partition the network into two chunks $\mathcal{A}_1$ and $\mathcal{A}_2$, which both have linear sizes of sensors with high probability and are isolated from each other. We let ℓ be a constant with $0 < \ell < 1$, and consider $r_n = o(1)$.

1) Unsplittability under Full Visibility:

Theorem 10. A secure sensor network with the q -composite scheme under full visibility is unsplittable with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for any constant $c > 1$, $K_n = \omega(\ln n)$ and $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$.

By Theorem 1, the conditions in Theorem 10 also ensure that a secure sensor network with the q -composite scheme under full visibility is connected with high probability. Intuitively, these conditions to guarantee connectivity in the absence of node capture can also ensure that when $o(n)$ nodes are compromised, almost all of the remaining nodes are still securely connected. More formally, the idea to show Theorem 1 is that even if $n_1 = o(n)$ nodes are compromised, we still have a securely connected network consisting of $n - n_2$ nodes, where n_2 can also be written as $o(n)$ despite $n_2 \geq n_1$. The reason for $n_2 \geq n_1$ is that capturing nodes enables an adversary to compromise communications between non-captured nodes which happen to use keys that are also shared by captured nodes.

2) Unsplittability under the Disk Model:

Theorem 11. A static secure sensor network with the q -composite scheme under the disk model on the unit torus or unit square is splittable with high probability if $r_n = o(1)$.

To explain Theorem 11, Figure 2 illustrates that an adversary capturing all $o(n)$ nodes in the shaded region can partition the network into two chunks $\mathcal{A}_1$ and $\mathcal{A}_2$, which both have linear sizes of sensors with high probability and are isolated from each other. We present the specific details below for an intuitive understanding, and provide the technical proofs in Section VI-E. First, under $r_n = o(1)$, we show that the number of nodes in the shaded region (i.e., a rectangle of width $2r_n$ and length 1) contains $o(n)$ nodes with high probability since the area of this region is also $o(1)$. Furthermore, because ℓ in Figure 2 is a constant with $0 < \ell < 1$, the areas of chunks $\mathcal{A}_1$ and $\mathcal{A}_2$ are both constants, which enable us to prove that each chunk has $\Theta(n)$ nodes (i.e., a constant fraction of the n nodes) with high probability. Finally, it is clear that any node in region $\mathcal{A}_1$ and any node in region $\mathcal{A}_2$ have a distance at least $2r_n$, so the node set in $\mathcal{A}_1$ and the node set in $\mathcal{A}_2$ have no edge in between.

C. Unassailability of Secure Sensor Networks with the q -Composite Scheme

Many prior studies have analyzed node-captured attacks, but they often consider that the adversary *randomly* captures sensors. However, in practice, the adversary may tune its node-capture strategy according to its goal. To analyze unassailability, we consider *arbitrary* node-capture attacks, in which the adversary can capture an arbitrary set of sensors. Recalling the definition of unassailability in Section I-B, a secure sensor network is unassailable if an adversary that has captured an *arbitrary* set of $o(n)$ sensors can compromise only $o(1)$ fraction of communication links in the rest of the network; in other words, an adversary has to capture a constant fraction of sensors to compromise a constant fraction of communication links. For the q -composite scheme, previous

studies have considered random node-capture attacks, but not arbitrary node-capture attacks formally. Hence, no prior work has rigorously established the parameter conditions under which secure sensor networks with the q -composite scheme are unassailable. Our Theorem 12 below presents the condition for unassailability of the q -composite scheme.

Theorem 12. *A secure sensor network with the q -composite scheme under any communication model is unassailable with high probability under $P_n/K_n = \Omega(n)$.*

Under a different setting, a result similar to Theorem 12 is given in [17], [18], but the result of [17], [18] is not for unassailability which by definition considers arbitrary node-capture attacks. Specifically, [17], [18] considers only *random* node capture where the adversary captures a random set of m sensors, whereas this paper addresses *arbitrary* node capture where the adversary captures an arbitrary set of m sensors. The choice of m nodes depends on the adversary's goal and also physical limitations (e.g., some sensors may be easier to be captured than others). Theorem 12 is proved in Section VI-F.

We check the practicality of $P_n/K_n = \Omega(n)$ in Theorem 12. Recall that the key ring size K_n denotes the number of keys in each sensor's memory and P_n means the key pool size which the keys on each sensor are selected from. In real-world sensor networks implementations, K_n is several orders of magnitude smaller than n due to limited memory and computational capability of sensors, and P_n is larger than n [1], [2], [4]. Thus, condition $P_n/K_n = \Omega(n)$ is practical.

To find the optimal q that defends against node-capture attacks, we will investigate how $p_{\text{compromised}}$ changes with respect to q . Since $p_{\text{compromised}}$ depends on the capture strategy, to study how $p_{\text{compromised}}$ varies with respect to q , below we consider the special case where the adversary captures a *random* set of m sensors. This has been studied in [17], but here we will improve the results of [17]. We state the research question more precisely as follows: given n , if we fix the key ring size K_n and the key-setup probability p_q , and vary q , the required amount of key overlap, how does the probability $p_{\text{compromised}}$ change asymptotically with respect to q ?

To answer the above question, we first explain the key-setup probability p_q below. In the q -composite scheme, noting that all sensors' key rings are random variables following the same probability distribution, we define the key-setup probability p_q as the probability that the key rings of two sensors have at least q keys in common. More specifically, with $\{v_1, v_2, \dots, v_n\}$ denoting the set of n sensors in the network, and with R_ℓ denoting the key ring of sensor v_ℓ for $\ell = 1, 2, \dots, n$, then we have the same value of $\mathbb{P}[|R_i \cap R_j| \geq q]$ for each pair of different i, j since each R_ℓ is constructed by selecting K_n keys uniformly at random from the key pool $\mathcal{P}_n$ of P_n keys. This value is denoted by p_q . Similarly, given u , we have the same value of $\mathbb{P}[|R_i \cap R_j| = u]$ for each pair of different i, j so we define ρ_u as $\mathbb{P}[|R_i \cap R_j| = u]$; i.e., ρ_u is the probability that two nodes v_i and v_j share u keys exactly in their key rings. Then we have $p_q = \mathbb{P}[|R_i \cap R_j| \geq q] = \sum_{u=q}^{K_n} \mathbb{P}[|R_i \cap R_j| = u] = \sum_{u=q}^{K_n} \rho_u$, where ρ_u denoting $\mathbb{P}[|R_i \cap R_j| = u]$ can be

computed as follows. After R_i comprising K_n keys is chosen from the key pool $\mathcal{P}_n$ of P_n keys, to have $|R_i \cap R_j| = u$ with R_j of K_n keys choosing from $\mathcal{P}_n$, we need to select u keys from R_i and $(K_n - u)$ keys from $\mathcal{P}_n \setminus R_i$ to construct R_j , which gives $\binom{K_n}{u} \binom{P_n - K_n}{K_n - u}$ possibilities of R_j . Without $|R_i \cap R_j| = u$, we have $\binom{P_n}{K_n}$ possible ways to find K_n keys from $\mathcal{P}_n$ to form R_j . Given the above, we have

$$\rho_u = \mathbb{P}[|R_i \cap R_j| = u] = \frac{\binom{K_n}{u} \binom{P_n - K_n}{K_n - u}}{\binom{P_n}{K_n}} \quad (22)$$

and

$$p_q = \mathbb{P}[|R_i \cap R_j| \geq q] = \sum_{u=q}^{K_n} \frac{\binom{K_n}{u} \binom{P_n - K_n}{K_n - u}}{\binom{P_n}{K_n}}. \quad (23)$$

From (23), p_q is a function of the parameters K_n, P_n and q . Hence, to fix p_q , the key pool size P_n needs to decrease as q increases.

We now present Theorem 13 below, which gives the optimal q that defends against node-capture attacks.

Theorem 13. *In a secure sensor network with the q -composite scheme under any communication model, if the key ring size K_n and the key-setup probability p_q are both fixed, and conditions $K_n = \omega(1)$ and $K_n = o(\sqrt{P_n})$ hold, after an adversary has captured m sensors with $m = o(\sqrt{\frac{P_n}{K_n^2}})$, then the fraction of communications compromised in the rest of the network asymptotically achieves its minimum*

- at $q = 1$ if $\frac{K_n}{m} < 2$,
- at both $q = 1$ and $q = 2$ if $\frac{K_n}{m} = 2$,
- at $q = \lfloor K_n/m \rfloor$ if $\frac{K_n}{m} > 2$ and K_n/m is not an integer, and
- at both $q = K_n/m$ and $q = (K_n/m) - 1$ if K_n/m is an integer greater than 2.

Theorem 13 improves the result of [17], and is proved in Section VI-G.

D. Design guidelines for secure sensor networks

Based on our theorems above, we can provide design guidelines of secure sensor networks to achieve desired resilience and connectivity. We first take static-network_{unit torus}^{disk model} as an example.

From Theorems 2 and 12, for static-network_{unit torus}^{disk model}, to ensure i) the network is resilient in the sense that an adversary capturing sensors at random has to obtain at least a constant fraction of nodes of the network in order to compromise a constant fraction of secure links, and ii) the network is connected with high probability, we can enforce $P_n/K_n = \Omega(n)$ and $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ with a constant $c > 1$, under condition set Λ in (6) on Page 6. We recall that Λ includes $K_n = \omega(\ln n)$, $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$, and $r_n = o(1)$. Given the above, we can set $K_n = c_1(\ln n)^{1+\epsilon_1}$ for some c_1 and small ϵ_1 , $P_n = c_2 n(\ln n)^{1+\epsilon_2}$ for some c_2 and $\epsilon_2 > \epsilon_1$. Substituting these K_n and P_n to the above $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$, we can set r_n such that $\pi r_n^2 \geq \frac{q!}{c} \cdot \frac{c_2^2 q}{c_1^{2q}} \cdot (\ln n)^{1-q(1+2\epsilon_1-\epsilon_2)} n^{q-1}$.

For static-network^{disk model}_{unit torus}^{unreliable links}, we can still use K_n and P_n above, while we set r_n such that $\pi r_n^2 \cdot t_n \geq \frac{q!}{c} \cdot \frac{c_2^q}{c_1^{2q}} \cdot (\ln n)^{1-q(1+2\epsilon_1-\epsilon_2)} n^{q-1}$, where t_n is the probability of a link being active.

For other networks, similarly we can use our theorems to present guidelines for setting the network parameters.

V. USEFUL LEMMAS

A. Coupling between random graphs

We will couple different random graphs together. The idea is converting a problem of one random graph to the corresponding problem in another random graph, in order to solve the original problem. Formally, a coupling [28], [44]–[47] of two random graphs G_1 and G_2 means a probability space on which random graphs G'_1 and G'_2 are defined such that G'_1 and G'_2 have the same distributions as G_1 and G_2 , respectively. For notation brevity, we simply say G_1 is a spanning subgraph² (resp., spanning supergraph) of G_2 if G'_1 is a spanning subgraph (resp., spanning supergraph) of G'_2 .

Using Rybarczyk's notation [44], we write

$$G_1 \succeq G_2 \quad (\text{resp., } G_1 \succeq_{1-o(1)} G_2) \quad (24)$$

if there exists a coupling under which G_2 is a spanning subgraph of G_1 with probability 1 (resp., $1 - o(1)$).

Lemma 2 (Rybarczyk [44]). *For two random graphs G_1 and G_2 , the following results hold for any monotone increasing graph property $\mathcal{I}$.*

- If $G_1 \succeq G_2$, then $\mathbb{P}[G_1 \text{ has } \mathcal{I}] \geq \mathbb{P}[G_2 \text{ has } \mathcal{I}]$.
- If $G_1 \succeq_{1-o(1)} G_2$, then

$$\mathbb{P}[G_1 \text{ has } \mathcal{I}] \geq \mathbb{P}[G_2 \text{ has } \mathcal{I}] - o(1).$$

Lemma 3 (Rybarczyk [44]). *For three random graphs G_1 , G_2 and G_3 , if $G_1 \succeq_{1-o(1)} G_2$, then $G_1 \cap G_3 \succeq_{1-o(1)} G_2 \cap G_3$.*

We use $G_q(n, K_n, P_n)$ to represent the graph model induced by the q -composite key predistribution scheme. In graph $G_q(n, K_n, P_n)$ defined on n nodes, each node independently selects K_n different keys uniformly at random from the same pool of P_n distinct keys, and two nodes establish an edge in between if and only if they have at least q keys in common. Also, an Erdős–Rényi graph [48] $G_{ER}(n, p_n)$ is defined on a set of n nodes such that any two nodes establish an edge in between independently with probability p_n . Lemma 4 below relates graph $G_q(n, K_n, P_n)$ with an Erdős–Rényi graph.

Lemma 4. *If $K_n = \omega(\max\{\ln n, \frac{\sqrt{P_n}}{n}\})$ and $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$, then*

$$G_q(n, K_n, P_n) \succeq_{1-o(1)} G_{ER}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)]). \quad (25)$$

To prove Lemma 4, we introduce an auxiliary graph called the *binomial q -intersection graph* $H_q(n, s_n, P_n)$ [7], [9], [49], which can be defined on n nodes by the following process. There exists a key pool of size P_n . Each key in the pool

is added to each node *independently* with probability s_n . After each node obtains a set of keys, two nodes establish an edge in between if and only if they share at least q keys. Clearly, the only difference between binomial q -intersection graph $H_q(n, s_n, P_n)$ and uniform q -intersection graph $G_q(n, K_n, P_n)$ is that in the former, the number of keys assigned to each node obeys a binomial distribution with P_n as the number of trials, and with s_n as the success probability in each trial, while in the latter graph, such number equals K_n with probability 1.

In [36], we prove Lemma 4 by using Lemmas 5 and 6 below.

Lemma 5. *If $K_n = \omega(\ln n)$ and $K_n = o(\sqrt{P_n})$, with s_n set by*

$$s_n = \frac{K_n}{P_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right), \quad (26)$$

then it holds that

$$G_q(n, K_n, P_n) \succeq_{1-o(1)} H_q(n, s_n, P_n). \quad (27)$$

Lemma 6. *If $s_n P_n = \omega(\ln n)$, $n s_n = o(1)$, $P_n s_n^2 = o(1)$ and $n^2 s_n^2 P_n = \omega(1)$, then there exists some p_n satisfying*

$$p_n = \frac{(P_n s_n^2)^q}{q!} \cdot [1 - o(1)] \quad (28)$$

such that Erdős–Rényi graph $G_{ER}(n, p_n)$ [48] obeys

$$H_q(n, s_n, P_n) \succeq_{1-o(1)} G_{ER}(n, p_n). \quad (29)$$

B. Useful results in the literature

Lemma 7 (Erdős and Rényi [48]). *An Erdős–Rényi graph $G_{ER}(n, p_n)$ is connected with high probability if $p_n \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.*

The following two lemmas present topological properties (including connectivity) of graph $\mathbb{G}(n, p_n, r_n, \mathcal{A})$. Graph $\mathbb{G}(n, p_n, r_n, \mathcal{A})$ is constructed as follows. Let n nodes be uniformly and independently deployed in some network area $\mathcal{A}$, which is either a unit torus $\mathcal{T}$ or a unit square $\mathcal{S}$. First, two nodes need to have a distance of no greater than r_n for having an edge in between, which models the transmission constraints of nodes in wireless networks. Then each edge between two nodes is preserved with probability p_n and deleted with probability $1 - p_n$, to model the link unreliability of wireless links.

Lemma 8 (Penrose [50]). *The following results hold.*

- Under $r_n = o(1)$, graph $\mathbb{G}(n, p_n, r_n, \mathcal{T})$ is connected with high probability if $p_n \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant with $c > 1$.
- Under $r_n = o(1)$, graph $\mathbb{G}(n, p_n, r_n, \mathcal{S})$ is connected with high probability if $p_n \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for all n sufficiently large, where c is a constant satisfying $c > c_n^*$, with c_n^* defined in (20).

Lemma 9 (Penrose [50]).

- The following results hold in graph $\mathbb{G}(n, p_n, r_n, \mathcal{T})$ under $r_n = o(1)$.
 - The expected number of isolated nodes in graph $\mathbb{G}(n, p_n, r_n, \mathcal{T})$ is asymptotically equivalent to $n e^{-\pi r_n^2 p_n}$.

²A graph G_a is a spanning subgraph (resp., spanning supergraph) of a graph G_b if G_a and G_b have the same node set, and the edge set of G_a is a subset (resp., superset) of the edge set of G_b .

- The probability of graph $\mathbb{G}(n, p_n, r_n, \mathcal{T})$ being disconnected is asymptotically equivalent to $1 - e^{-\lambda_1}$, where λ_1 denotes the expected number of isolated nodes in graph $\mathbb{G}(n, p_n, r_n, \mathcal{T})$.
- The following results hold in graph $\mathbb{G}(n, p_n, r_n, \mathcal{S})$ under $r_n = o(1)$.
 - The expected number of isolated nodes in graph $\mathbb{G}(n, p_n, r_n, \mathcal{S})$ is asymptotically equivalent to $ne^{-\pi r_n^2 p_n n / \tilde{c}_n}$, where $\tilde{c}_n := \max \{1 + (\ln \frac{1}{p_n}) / \ln n, 4(\ln \frac{1}{p_n}) / \ln n\}$.
 - The probability of graph $\mathbb{G}(n, p_n, r_n, \mathcal{S})$ being disconnected is asymptotically equivalent to $1 - e^{-\lambda_2}$, where λ_2 denotes the expected number of isolated nodes in graph $\mathbb{G}(n, p_n, r_n, \mathcal{S})$.

VI. ESTABLISHING THE THEOREMS

A. Establishing Theorem 1

We denote the graph topology of network^{full visibility} by $G_q(n, K_n, P_n)$, which is known as a kind of random intersection graph [7], [43] in the literature. Graph $G_q(n, K_n, P_n)$ is constructed on a node set with size n as follows. Each node is independently assigned a set of K_n different keys, selected uniformly at random from a pool of P_n keys. An edge exists between two nodes if and only if they have at least q keys in common.

From Lemma 4, there exists a coupling under which $G_q(n, K_n, P_n)$ is an spanning supergraph of an Erdős-Rényi graph $G_{ER}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)])$ with high probability, where an Erdős-Rényi graph [48] $G_{ER}(n, p_n)$ is defined on a set of n nodes such that any two nodes establish an edge in between independently with probability p_n . Since connectivity is a monotone increasing graph property³, we obtain from Lemmas 2 and 4 that

$$\begin{aligned} \mathbb{P}[G_q(n, K_n, P_n) \text{ is connected.}] \\ \geq \mathbb{P}[G_{ER}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)]) \text{ is connected.}] - o(1). \end{aligned} \quad (30)$$

Given condition $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$ for all n sufficiently large with constant $c > 1$, we use Lemma 7 and have

$$\mathbb{P}[G_{ER}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)]) \text{ is connected.}] \geq 1 - o(1), \quad (31)$$

which is substituted into (30) to derive

$$\mathbb{P}[G_q(n, K_n, P_n) \text{ is connected.}] \geq 1 - o(1). \quad (32)$$

Hence, $G_q(n, K_n, P_n)$ is connected with high probability; namely, network^{full visibility} is connected with high probability.

Although the above results can also be derived using the results in [30], the proof techniques in [30] lack formality (specifically [30, Equation (6.93)]) and are different from the above proof. In addition, Bloznelis and Rybarczyk [29] (resp., Bloznelis and Łuczak [10]) have recently investigated connectivity (resp., k -connectivity) of network^{full visibility}, but

³A graph property is called monotone increasing if it holds under the addition of edges in a graph.

both results after a rewriting address only the narrow and impractical range of $P_n = o(n^{\frac{1}{q}} (\ln n)^{-\frac{3}{5q}})$. In contrast, from condition set Λ in (6), our Theorem 1 considers a more practical range of $P_n = \omega(n \ln n)$. More details can be found in Section II for related work.

B. Establishing Theorems 2–5

To establish Theorems 2–5, we use an approach similar to that for proving Theorem 1. Specifically, we will show that the graph topology of each network in Theorems 2–5 is an spanning supergraph of some graph $\mathcal{G}$ with high probability, where graph $\mathcal{G}$ is connected with high probability under the corresponding conditions. Since connectivity is a monotone increasing graph property, we will complete proving Theorems 2–5 by the help of Lemma 2.

For each network on an area $\mathcal{A}$ in Theorems 2–5 ($\mathcal{A}$ is the unit torus $\mathcal{T}$ or the unit square $\mathcal{S}$), we will show that its topology is an spanning supergraph of some graph $\mathbb{G}(n, p_n, r_n, \mathcal{A})$ with high probability. Graph $\mathbb{G}(n, p_n, r_n, \mathcal{A})$ is constructed as follows. Let n nodes be uniformly and independently deployed in some network area $\mathcal{A}$, which is either a unit torus $\mathcal{T}$ or a unit square $\mathcal{S}$. First, two nodes need to have a distance of no greater than r_n for having an edge in between, which models the transmission constraints of nodes in wireless networks. Then each edge between two nodes is preserved with probability p_n and deleted with probability $1 - p_n$, to model the link unreliability of wireless links.

The disk model induces a so-called random geometric graph $G_{RGG}(n, r_n, \mathbb{A})$, which is defined as follows. Let n nodes be uniformly and independently deployed in a network area $\mathbb{A}$. An edge exists between two nodes if and only if their distance is no greater than r_n . With each link inactive with probability t_n , the link unreliability yields an Erdős-Rényi graph $G_{ER}(n, t_n)$. Then it is clear that graph $\mathbb{G}(n, p_n, r_n, \mathcal{A})$ is the intersection⁴ of random geometric graph $G_{RGG}(n, r_n, \mathbb{A})$ and Erdős-Rényi graph $G_{ER}(n, p_n)$.

Lemma 10. *The following results hold under*

$K_n = \omega(\max\{\ln n, \frac{\sqrt{P_n}}{n}\})$ and $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$.

- The graph topology of static-network^{disk model unit torus} is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T})$ with high probability.
- The graph topology of static-network^{disk model unit torus unreliable links} is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{T})$ with high probability.
- The graph topology of static-network^{disk model unit square} is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{S})$ with high probability.
- The graph topology of static-network^{disk model unit square unreliable links} is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{S})$ with high probability.

⁴With two graphs G_1 and G_2 defined on the same node set, two nodes have an edge in between in the intersection $G_1 \cap G_2$ if and only if these two nodes have an edge in G_1 and also have an edge in G_2 .

1) Proving Theorems 2–5 by Lemmas 8 and 10:

First, we use Lemma 8 to obtain the connectivity results of different graphs $\mathbb{G}(n, p_n, r_n, \mathcal{A})$ in Lemma 10. Note that if $p_n \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for some constant $c > 1$ (resp., $c > c_n^*$), then $p_n \cdot \pi r_n^2 \cdot [1 - o(1)] \geq \frac{c' \ln n}{n}$ for all n sufficiently large, where c' is any constant satisfying $1 < c' < c$ (resp., $c_n^* < c' < c$). Then from Lemma 8, under $r_n = o(1)$, we have the following results.

- Graph $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T})$ is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- Graph $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{T})$ is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > 1$.
- Graph $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{S})$ is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^*$.
- Graph $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{S})$ is connected with high probability if $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot \pi r_n^2 \geq \frac{c \ln n}{n}$ for any constant $c > c_n^\#$.

Together with Lemma 10 (we will explain that conditions in Lemma 10 all hold), the above results complete proving Theorems 2–5, since any spanning supergraph of a graph that is connected with high probability is also connected with high probability.

We now show that conditions in Lemma 10 all hold. First, condition set Λ in (6) on Page 6 hold in Theorems 2–5, so we have $K_n = \omega(\ln n)$, $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$ and $r_n = o(1)$. Therefore, all conditions in Lemma 10 will hold once we prove $K_n = \omega(\frac{\sqrt{P_n}}{n})$. Note that for each of Theorems 2–5, we always have the condition $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 = \omega(\frac{\ln n}{n})$, which along with $r_n = o(1)$ yields $\frac{K_n^{2q}}{P_n^q} = \omega((\frac{\ln n}{n})^{1/q}) = \omega(\frac{1}{n^2})$ and thus $K_n = \omega(\frac{\sqrt{P_n}}{n})$. ■

2) Proof of Lemma 10:

We consider the graph topologies of the networks in Lemma 10 (i.e., those in Theorems 2–5).

First, the q -composite key predistribution scheme is modeled by graph $G_q(n, K_n, P_n)$. The disk model induces the so-called random geometric graph $G_{RGG}(n, r_n, \mathbb{A})$. Finally, in the presence of link unreliability, since each link is preserved with probability t_n , the underlying graph of link unreliability is an Erdős–Rényi graph $G_{ER}(n, t_n)$. Therefore, it is straightforward to see that the graph topologies of the networks in Lemma 10 can be viewed as the intersections of different random graphs. Specifically, we obtain the following:

- The graph topology of network static-network^{disk model unit torus} is $G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{T})$.
- The graph topology of network static-network^{disk model unit torus unreliable links} is $G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{T}) \cap G_{ER}(n, t_n)$.
- The graph topology of network static-network^{disk model unit square} is $G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{S})$.
- The graph topology of network static-network^{disk model unit square unreliable links} is $G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{S}) \cap G_{ER}(n, t_n)$.

From Lemmas 3 and 4, it follows that

$$G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{T}) \succeq_{1-o(1)} \mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T}). \quad (33)$$

Similarly, we have

$$G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{T}) \cap G_{ER}(n, t_n) \succeq_{1-o(1)} \mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{T}), \quad (34)$$

$$G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{S}) \succeq_{1-o(1)} \mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{S}), \quad (35)$$

and

$$G_q(n, K_n, P_n) \cap G_{RGG}(n, r_n, \mathcal{S}) \cap G_{ER}(n, t_n) \succeq_{1-o(1)} \mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{S}). \quad (36)$$

Then the proof of Lemma 10 is completed. ■

C. Establishing Theorems 6–9

On either the torus or the square, we consider i.i.d. mobility model. Therefore, for all mobile networks that we consider, the overall node distribution at each time slot is uniform, although the position of each particular node may change over time. Then we can view each mobile network at a single time slot as the corresponding static network. Then we obtain the following Lemma 11 from Lemma 10.

Lemma 11. *The following results hold under $K_n = \omega(\max\{\ln n, \frac{\sqrt{P_n}}{n}\})$ and $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$.*

- *The graph topology of mobile-network^{disk model unit torus} at any time slot i is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T})$ with high probability.*
- *The graph topology of mobile-network^{disk model unit square} at any time slot i is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{S})$ with high probability.*
- *The graph topology of mobile-network^{disk model unit torus unreliable links} at any time slot i is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{T})$ with high probability.*
- *The graph topology of mobile-network^{disk model unit square unreliable links} at any time slot i is a spanning supergraph of $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot t_n \cdot [1 - o(1)], r_n, \mathcal{S})$ with high probability.*

We now use Lemma 11 to demonstrate Theorems 6–9. We first detail the proof of Theorem 6.

We define $\mathcal{C}_i$ as the event that network mobile-network^{disk model unit torus} at time slot i is connected.

Let λ denote the expected number of isolated nodes in graph $\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T})$. Then from Lemma 9 below, we have

$$\lambda \sim n e^{-\pi r_n^2 \cdot \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot n}. \quad (37)$$

Theorem 6 has the following condition:

$$\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot \pi r_n^2 \geq \frac{c \ln n}{n} \text{ for some constant } c > 1, \quad (38)$$

which yields

$$ne^{-\pi r_n^2 \cdot \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot n} \leq ne^{-c \ln n} = n^{1-c}. \quad (39)$$

From (37) (39) and $c > 1$, we obtain

$$\lambda \leq n^{1-c} \cdot [1 + o(1)], \quad (40)$$

and

$$\lambda = o(1). \quad (41)$$

From Lemma 9, we also have

$$\begin{aligned} & \mathbb{P}\left[\mathbb{G}(n, \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \cdot [1 - o(1)], r_n, \mathcal{T}) \text{ is not connected.}\right] \\ &= (1 - e^{-\lambda}) \cdot [1 \pm o(1)] \\ &\leq 2(1 - e^{-\lambda}), \text{ for all } n \text{ sufficiently large.} \end{aligned} \quad (42)$$

Then from (42) and Lemma 11, it follows for all n sufficiently large that

$$\mathbb{P}[\mathcal{C}_i] \leq 2(1 - e^{-\lambda}), \text{ for all } n \text{ sufficiently large,} \quad (43)$$

yielding

$$\mathbb{P}[\mathcal{C}_i] \geq 1 - 2(1 - e^{-\lambda}) \geq 1 - 2\lambda, \text{ for all } n \text{ sufficiently large.} \quad (44)$$

where the last step uses the inequality $1 - e^{-x} \leq x$ for any real x .

We now bound the probability of event $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_T$ (i.e., the event that mobile-network_{unit torus}^{disk model} is connected for at least T consecutive time slots from the beginning. Since we consider the i.i.d. mobility model, the events $\mathcal{C}_1, \dots, \mathcal{C}_T$ are independent. Then we have

$$\begin{aligned} & \mathbb{P}[\mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_T] \\ &= \mathbb{P}[\mathcal{C}_1] \mathbb{P}[\mathcal{C}_2] \dots \mathbb{P}[\mathcal{C}_T] \\ &\geq (1 - 2\lambda)^T, \end{aligned} \quad (45)$$

where the last step uses (44).

To evaluate $(1 - 2\lambda)^T$, we apply [31, Fact 2] which considers the Taylor series expansion and have

$$(1 - 2\lambda)^T \geq 1 - 2\lambda T + \frac{1}{2}(2\lambda)^2 T^2. \quad (46)$$

For $T = n^{c-1-\epsilon}$, we use (40) to obtain

$$\lambda T \leq n^{1-c} \cdot [1 + o(1)] \cdot n^{c-1-\epsilon} = o(1), \quad (47)$$

which is substituted into (46) to induce

$$(1 - 2\lambda)^T \geq 1 - o(1). \quad (48)$$

Applying (48) to (45), we establish

$$\mathbb{P}[\mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_T] \rightarrow 1 - o(1). \quad (49)$$

In view of the fact that probability $\mathbb{P}[\mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_T]$ is at most 1, we obtain from (49) that

$$\mathbb{P}[\mathcal{C}_1 \cap \mathcal{C}_2 \cap \dots \cap \mathcal{C}_T] \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (50)$$

Therefore, network mobile-network_{unit torus}^{disk model} is connected for at least $T = n^{c-1-\epsilon}$ consecutive time slots from the beginning.

The proofs of Theorems 7–9 using Lemma 10 are similar to that of the above proof of Theorem 6. ■

D. Proof of Theorem 10

We will prove Theorem 10 by establishing the following: after an adversary captures $m = o(n)$ nodes, with high probability there exist $n - o(n)$ non-captured nodes which form a connected graph using only the non-compromised keys. In particular, we will show with high probability that there exist $L = n - o(n)$ non-captured nodes satisfying that ① each of these L nodes has at least $(1 - \alpha)K_n$ keys that are not compromised by the adversary, where $\alpha = 1 - c^{-\frac{1}{6q}}$, and ② these L nodes form a connected graph with each node using its $(1 - \alpha)K_n$ non-compromised keys to discover neighbors.

We use τ to denote the number of keys compromised by an adversary that has captured $m = o(n)$ nodes. Clearly, $\tau \leq mK_n$ holds. Note that we have $K_n = o(\frac{P_n}{n})$ from condition $K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$. Let X be the number of a non-captured node's keys that are compromised by the adversary. Since each node has K_n keys selected from a P_n -size pool, the expected value of X is $K_n \cdot \tau/P_n$. From $\tau \leq mK_n$, $K_n = o(\frac{P_n}{n})$ and $m = o(n)$, it holds that

$$\mathbb{E}[X] = K_n \cdot \tau/P_n = o(K_n). \quad (51)$$

We define δ_n as the probability that a non-captured node has at least αK_n keys that are compromised by the adversary. From Markov's inequality, it follows that

$$\delta_n = \mathbb{P}[X \geq \alpha K_n] \leq \frac{\mathbb{E}[X]}{\alpha K_n} \leq o(\alpha^{-1}) = o(1). \quad (52)$$

Let Y be the number of non-captured nodes, each of which has at least $(1 - \alpha)K_n$ keys that are not compromised by the adversary. For two different non-captured nodes x and y , independence exists between how the key ring of node x is compromised by the adversary, and how the key ring of node y is compromised by the adversary. Then it is clear that Y follows a binomial distribution with parameters $n - m$ (the number of trials) and $1 - \delta_n$ (the success probability in each trial). From Hoeffding's inequality, for any β_n , we have

$$\mathbb{P}[Y \leq (1 - \delta_n - \beta_n)(n - m)] \leq e^{-2\beta_n^2(n - m)}. \quad (53)$$

Setting $\beta_n = n^{-\frac{1}{3}}$, we obtain from (53) and $m = o(n)$ that

$$\mathbb{P}[Y \leq (1 - \delta_n - n^{-\frac{1}{3}})(n - m)] \leq e^{-2n^{-\frac{2}{3}}(n - m)} = o(1). \quad (54)$$

Therefore, with L denoting $(1 - \delta_n - n^{-\frac{1}{3}})(n - m)$, we get

$$\mathbb{P}[Y \geq L] \geq 1 - o(1). \quad (55)$$

From (52) and $m = o(n)$, it holds that

$$L = (1 - \delta_n - n^{-\frac{1}{3}})(n - m) = n - o(n). \quad (56)$$

Clearly, (55) and (56) above prove the result ① mentioned at the beginning of the proof; i.e., each of these $L = n - o(n)$ nodes has at least $(1 - \alpha)K_n$ keys that are not compromised by the adversary.

Now we show the result ②. We use Theorem 1 to show that the L nodes form a connected graph with each node using its $(1 - \alpha)K_n$ non-compromised keys to discover neighbors. From Theorem 1, we need to have (i) $(1 - \alpha)K_n = \omega(\ln n)$ and $(1 - \alpha)K_n = o(\min\{\sqrt{P_n}, \frac{P_n}{n}\})$, which both clearly hold

from the conditions of Theorem 10, and (ii) $\frac{1}{q!} \cdot \frac{[(1-\alpha)K_n]^{2q}}{P_n^q} \geq \tilde{c} \cdot \frac{\ln L}{L}$ for some constant $\tilde{c} > 1$, which we prove below.

As mentioned, α equals $1 - c^{-\frac{1}{6q}}$. We choose this value of α since it leads to

$$\frac{1}{q!} \cdot \frac{[(1-\alpha)K_n]^{2q}}{P_n^q} = c^{-\frac{1}{3}} \cdot \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}. \quad (57)$$

From (56), we have $L \sim n$ so that $\frac{\ln L}{L} \sim \frac{\ln n}{n}$, which along with $c > 1$ yields for all n sufficiently large that

$$\frac{\ln L}{L} \leq c^{\frac{1}{3}} \cdot \frac{\ln n}{n}. \quad (58)$$

Then from (57) (58) and condition $\frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q} \geq \frac{c \ln n}{n}$, we obtain for all n sufficiently large that

$$\frac{1}{q!} \cdot \frac{[(1-\alpha)K_n]^{2q}}{P_n^q} \geq c^{-\frac{1}{3}} \cdot c \frac{\ln n}{n} \geq c^{\frac{1}{3}} \cdot \frac{\ln L}{L}. \quad (59)$$

Thus we can choose constant $\tilde{c} = c^{\frac{1}{3}} > 1$ to have $\frac{1}{q!} \cdot \frac{[(1-\alpha)K_n]^{2q}}{P_n^q} \geq \tilde{c} \cdot \frac{\ln L}{L}$ for all n sufficiently large. Then as explained above, result ② is proved from Theorem 1.

In view of results ① and ②, after an adversary captures $m = o(n)$ nodes, with high probability, there exist $n - o(n)$ non-captured nodes, which form a connected graph using only the non-compromised keys. Therefore, with high probability, an adversary that has captured an arbitrary set of $o(n)$ sensors cannot partition the network into two chunks, which both have linear sizes of sensors, and either are isolated from each other or only have compromised communications in between. Then the studied secure sensor network with the q -composite scheme under full visibility is unsplitable. ■

E. Proof of Theorem 11

We show that under the disk model, an adversary that has captured some set of $o(n)$ sensors can partition the network into two chunks, which both have linear sizes of sensors with high probability and are isolated from each other.

As shown in Figure 3-(a) (Figure 3-(b) is explained later), the adversary captures all nodes in the shaded region $\mathcal{A}_0$, which has a width $2r_n$ and a length 1. The region $\mathcal{A}_1$ on the left side of the shaded region has a width ℓ and a length 1, while the region $\mathcal{A}_2$ on the right side of the shaded area has a width $1 - \ell - 2r_n$ and a length 1, where ℓ is a constant with $0 < \ell < 1$.

With $|\mathcal{A}_i|$ denoting the area of $\mathcal{A}_i$ for each $i = 0, 1, 2$, we have $|\mathcal{A}_0| = 2r_n = o(1)$, $|\mathcal{A}_1| = \ell = \Theta(1)$ and $|\mathcal{A}_2| = 1 - \ell - 2r_n = \Theta(1)$. Therefore, to show the number of nodes in $\mathcal{A}_0$ is $o(n)$ with high probability, and the numbers of nodes in $\mathcal{A}_1$ and $\mathcal{A}_2$ are both $\Theta(n)$ with high probability, it suffices to prove the following: for some region $\tilde{\mathcal{A}}$ as a sub-field of the whole network region $\mathcal{A}$, with $|\tilde{\mathcal{A}}|$ denoting the area of $\tilde{\mathcal{A}}$ and with $\tilde{N}$ denoting the number of nodes in region $\tilde{\mathcal{A}}$, we have ① if $|\tilde{\mathcal{A}}| = o(1)$, then $\tilde{N} = o(n)$ holds with high probability, and ② if $|\tilde{\mathcal{A}}| = \Theta(1)$, then $\tilde{N} = \Theta(n)$ holds with high probability.

Since all n nodes are uniformly distributed in the whole network region $\mathcal{A}$ with area 1 ($\mathcal{A}$ is a unit torus or a unit square), then $\tilde{N}$ (the number of nodes in the region $\tilde{\mathcal{A}}$) follows a binomial distribution with parameter n as the number of

Fig. 3. An adversary captures all nodes in the shaded region.

trials, and parameter $|\tilde{\mathcal{A}}|$ as the success probability in each trial. For any $\nu_n > 0$, we use Hoeffding's inequality to have

$$\mathbb{P}[(|\tilde{\mathcal{A}}| - \nu_n)n \leq \tilde{N} \leq (|\tilde{\mathcal{A}}| + \nu_n)n] \geq 1 - 2e^{-2\nu_n^2 n}. \quad (60)$$

Setting $\nu_n = n^{-\frac{1}{3}}$, we have $2e^{-2\nu_n^2 n} = o(1)$. Then

$$\mathbb{P}[(|\tilde{\mathcal{A}}| - \nu_n)n \leq \tilde{N} \leq (|\tilde{\mathcal{A}}| + \nu_n)n] \geq 1 - o(1). \quad (61)$$

Then we have

- if $|\tilde{\mathcal{A}}| = o(1)$, then $\tilde{N} \leq (|\tilde{\mathcal{A}}| + \nu_n)n = o(n)$ holds with high probability.
- if $|\tilde{\mathcal{A}}| = \Theta(1)$, it holds with high probability that $(|\tilde{\mathcal{A}}| - \nu_n)n \leq \tilde{N} \leq (|\tilde{\mathcal{A}}| + \nu_n)n$. From $|\tilde{\mathcal{A}}| = \Theta(1)$ and $\nu_n = n^{-\frac{1}{3}}$, both $|\tilde{\mathcal{A}}| - \nu_n$ and $|\tilde{\mathcal{A}}| + \nu_n$ are $\Theta(1)$. Therefore, $\tilde{N} = \Theta(n)$ follows with high probability.

Therefore, the results ① and ② above are proved. Returning to Figure 3-(a), as explained before, in view of $|\mathcal{A}_0| = o(1)$, $|\mathcal{A}_1| = \Theta(1)$ and $|\mathcal{A}_2| = \Theta(1)$, we obtain the following: with N_i denoting the number of nodes in region $\mathcal{A}_i$ for each $i = 0, 1, 2$, we have $N_0 = o(n)$ with high probability, $N_1 = \Theta(n)$ with high probability, and $N_2 = \Theta(n)$ with high probability. Then $\{N_0 = o(n)\} \cap \{N_1 = \Theta(n)\} \cap \{N_2 = \Theta(n)\}$ occurs with probability $1 - o(1)$ by the union bound. Hence, $\{N_0 = o(n)\} \cap \{N_1 = \Theta(n)\} \cap \{N_2 = \Theta(n)\}$ holds with high probability.

From Figure 3-(a), it is clear that any node in region $\mathcal{A}_1$ and any node in region $\mathcal{A}_2$ have a distance at least $2r_n$, so the node set in region $\mathcal{A}_1$ and the node set in region $\mathcal{A}_2$ have no edge in between. Hence, with high probability, an adversary that has captured $o(n)$ nodes in region N_0 partitions the network into two chunks, which both have linear sizes of sensors and are isolated from each other. Then the studied static secure sensor network with the q -composite scheme under the disk model on the unit torus or unit square is splittable. ■

Remarks:

- The adversary may split the network into multiple chunks instead of just two, as illustrated in Figure 3-(b).
- A future direction is to extend the result to more general topology (here we consider a torus or square).

F. Proof of Theorem 12

We will prove Theorem 12 by showing that under $P_n/K_n = \Omega(n)$, an adversary that has captured an arbitrary set of $o(n)$

sensors can only compromise $o(1)$ fraction of communication links in the rest of the network.

Suppose the adversary has captured some set of m nodes. The choice of m nodes depends on the adversary's goal and also physical limitations (e.g., some sensors may be easier to be captured than others). Let E_τ be the event that the m captured nodes have τ different keys in their key rings in total. It is clear that $K_n \leq \tau \leq \min\{mK_n, P_n\}$. Recall that $p_{\text{compromised}}$ is the fraction of compromised communications among non-captured nodes ($p_{\text{compromised}}$ is also the probability that the secure link between two non-captured nodes is compromised). Conditioning on event E_τ , let $p_{\text{compromised}}(\tau)$ be the fraction of compromised communications among non-captured nodes (conditioning on event E_τ , $p_{\text{compromised}}(\tau)$ is also the probability that the secure link between two non-captured nodes is compromised). By the law of total probability, we have

$$p_{\text{compromised}} = \sum_{\tau=K_n}^{\min\{mK_n, P_n\}} \{\mathbb{P}[E_\tau] \cdot p_{\text{compromised}}(\tau)\}. \quad (62)$$

The adversary's node capture strategy determines $\mathbb{P}[E_\tau]$. With $K_n \leq \tau \leq \min\{mK_n, P_n\}$, it holds that $\sum_{K_n}^{\min\{mK_n, P_n\}} \mathbb{P}[E_\tau] = 1$. Clearly, as τ increases, $p_{\text{compromised}}(\tau)$ increases (of course, if τ reaches P_n , $p_{\text{compromised}}(\tau)$ becomes 1 and can not increase anymore). With $K_n \leq \tau \leq \min\{mK_n, P_n\}$, then $p_{\text{compromised}}(\tau)$ achieves its maximum when $\tau = \min\{mK_n, P_n\}$. Therefore, for an adversary that has captured *any* set of m sensors, it is simple to derive from (62) that

$$p_{\text{compromised}} \leq p_{\text{compromised}}(\min\{mK_n, P_n\}). \quad (63)$$

Under $P_n/K_n = \Omega(n)$ and $m = o(n)$, it holds that

$$mK_n = o(P_n). \quad (64)$$

Then we have $mK_n \leq P_n$ and thus $\min\{mK_n, P_n\} = mK_n$ for all n sufficiently large, which is used in (63) to derive

$$p_{\text{compromised}} \leq p_{\text{compromised}}(mK_n). \quad (65)$$

In addition, to maximize $p_{\text{compromised}}$ as $p_{\text{compromised}}(mK_n)$, the adversary can capture some set of m sensors which have mK_n different keys in their key rings in total. Therefore, given (65), the proof of Theorem 12 is completed once we demonstrate $p_{\text{compromised}}(mK_n) = o(1)$.

By Yum and Lee [51], it follows that

$$p_{\text{compromised}}(\tau) = \sum_{u=q}^{K_n} \left[\frac{\binom{\tau}{u}}{\binom{P_n}{u}} \cdot \frac{\rho_u}{p_q} \right], \quad (66)$$

where the expressions of ρ_u and p_q are given in (22) and (23).

To compute $p_{\text{compromised}}(mK_n)$, we look at $p_{\text{compromised}}(\tau)$ given in (66). Then we bound the term $\binom{\tau}{u} / \binom{P_n}{u}$ in (66). With $q \leq u \leq K_n$, we obtain

$$\binom{\tau}{u} / \binom{P_n}{u} \leq \frac{\tau^u / (u!)}{(P_n - u)^u / (u!)} \leq \frac{\tau^q}{(P_n - K_n)^q}, \quad (67)$$

which along with $\sum_{u=q}^{K_n} \rho_u = p_q$ is used in (66) to yield

$$p_{\text{compromised}}(\tau) \leq \sum_{u=q}^{K_n} \left[\frac{\tau^q}{(P_n - K_n)^q} \cdot \frac{\rho_u}{p_q} \right] = \frac{\tau^q}{(P_n - K_n)^q}. \quad (68)$$

Setting τ as mK_n , we have

$$p_{\text{compromised}}(mK_n) \leq \frac{(mK_n)^q}{(P_n - K_n)^q} = o(1), \quad (69)$$

where the last step uses $mK_n = o(P_n - K_n)$, which holds from (64). ■

G. Proof of Theorem 13

Note that different from the case of Theorem 12, in Theorem 13 here, we do not have $mK_n \leq P_n$ for all n sufficiently large. Hence, we can have (63) only, instead of having (65).

When the adversary captures m nodes *randomly*, $\mathbb{P}[E_\tau]$ is non-zero for all τ satisfying $K_n \leq \tau \leq \min\{mK_n, P_n\}$. Under this *random* attack, the computation of $p_{\text{compromised}}$ is not that straightforward. Yet, [17] presents an asymptotic expression of $p_{\text{compromised}}$ as Lemma 12 below.

Lemma 12. *If the adversary captures m nodes randomly with $m = o(\sqrt{\frac{P_n}{K_n^2}})$, then $p_{\text{compromised}}$ is asymptotically equivalent to $(\frac{mK_n}{P_n})^q$ under $K_n = \omega(1)$ and $K_n = o(\sqrt{P_n})$.*

From Lemma 12, it holds that

$$p_{\text{compromised}} \sim \left(\frac{mK_n}{P_n}\right)^q. \quad (70)$$

Under $K_n = \omega(1)$ and $K_n = o(\sqrt{P_n})$, as given in (18), we have $p_q \sim \frac{1}{q!} \cdot \frac{K_n^{2q}}{P_n^q}$, which with (70) above induces

$$\frac{p_{\text{compromised}}}{p_q} \sim \left(\frac{mK_n}{P_n}\right)^q \cdot q! \cdot \frac{P_n^q}{K_n^{2q}} = q! \left(\frac{m}{K_n}\right)^q = f(q), \quad (71)$$

where $f(q)$ is defined by

$$f(q) = q! \left(\frac{m}{K_n}\right)^q. \quad (72)$$

We now analyze how $f(q)$ changes as q varies, to understand the asymptotic behavior of $p_{\text{compromised}}$. From (72), we compute $f(q+1)/f(q)$ through

$$\frac{f(q+1)}{f(q)} = \frac{m(q+1)}{K_n}. \quad (73)$$

Then using (73) to analyze the monotonicity of $f(q)$, we finally establish Theorem 13.

VII. SIMULATION RESULTS

We now provide simulation results to confirm the theoretical findings. All plots in the paper are obtained from simulation. Specifically, in each plot, the data points are obtained by averaging over 500 independent network samples.

A. Simulation Results for Connectivity

We present below simulation results of connectivity in secure sensor networks to confirm the theorems. Figure 4 depicts the probability that network static-network_{unit torous}^{disk model} is connected with $q = 2$. We vary the key ring size K , and set $n = 2000$, $P = 5000$, and $r = 0.2, 0.3$. For each pair (K, r) , we generate 500 independent samples of $\mathbb{G}_q(n, K, P, r, T)$ and record the count that the obtained graph has connectivity. Then we derive the empirical probabilities after dividing the counts by 500. Similarly, in Figure 5, we plot the probability that network static-network_{unit square}^{disk model}_{unreliable links} is connected after an adversary captures 10 random nodes, for $q = 2$, $n = 1000, 900, 800$,

Fig. 4. A plot of the empirical probability that network static-network_{disk model}^{unit square} is connected as a function of the key ring size K with $q = 2$, $n = 2000$, $P = 5000$, and $r = 0.2, 0.3$.

Fig. 5. A plot of the empirical probability that network static-network_{disk model}^{unit square} after the random capture of 10 nodes is connected as a function of the key ring size K with $q = 2$, $n = 1000, 900, 800$, $P = 5000$, $r = 0.3$ and $t = 0.9$.

$P = 5000$, $r = 0.3$ and $t = 0.9$. As illustrated in both figures, we observe the transitional behavior of connectivity. Moreover, for both Figure 4 and 5, substituting the parameter values into the corresponding theorems of this paper, we can confirm the correctness of the theorems.

Figure 6 presents a plot of the empirical probability that mobile network mobile-network_{disk model}^{unit square} is connected for at least T consecutive time slots as a function of the key ring size with $q = 2$, $n = 1000$, $P = 6000$, $r = 0.25$ and $K = 44, 50, 60$. We consider the i.i.d. mobility model, where each node independently picks a new location uniformly at random at the beginning of a time slot and stays at the location in the rest of the time slot. At each time slot, we check whether the network is connected. We generate 500 independent samples and record the count that mobile-network_{disk model}^{unit square} is connected for at least T consecutive time slots. Then we derive the empirical probabilities after dividing the counts by 500. As illustrated in Figure 6, incrementing K increases the probability that mobile network mobile-network_{disk model}^{unit square} is connected for at least T consecutive time slots.

B. Simulation Results for Unassailability

We provide simulation plots below for a comparison with the theoretical results above.

Fig. 6. A plot of the empirical probability that mobile network mobile-network_{disk model}^{unit square} is connected for at least T consecutive time slots as a function of the key ring size with $q = 2$, $n = 1000$, $P = 6000$, $r = 0.25$ and $K = 44, 50, 60$.

Fig. 7. A plot of probability $p_{\text{compromised}}$ of link compromise with respect to q , under $K = 40$, $p_q = 0.1$ and $m = 15, 40$. To ensure a fixed p_q , the key pool size P needs to decrease as q increases.

1) *Varying q with key ring size and key-setup prob. fixed:* We first fix the key ring size and the key-setup probability, and vary q to see how the probability $p_{\text{compromised}}$ of link compromise changes.

Figure 7 depicts probability $p_{\text{compromised}}$ with respect to q . We fix the key ring size at $K = 40$, and the key-setup probability at $p_q = 0.1$, and set the number m of captured nodes as 15 or 40. As illustrated, for $m = 15$, as q increases, $p_{\text{compromised}}$ first decreases and then increases; and for $m = 40$, as q increases, $p_{\text{compromised}}$ always increases. These are in agreement with the analytical results. In particular, for $K = 40$ and $m = 15$, given $\frac{K}{m} = \frac{40}{15} \approx 2.67$ and Theorem 13, $p_{\text{compromised}}$ takes its minimum at $q = 2$, in accordance with the fact that in Figure 7, $p_{\text{compromised}}$ for $m = 15$ achieves its minimum at $q = 2$. For $K = 40$ and $m = 40$, both both theoretical and experimental results show that $p_{\text{compromised}}$ takes the minimum at $q = 2$.

In Figure 8, we plot the relation between q and the number m of sensors that the adversary needs to capture to compromise a link between two non-captured nodes with probability $p_{\text{compromised}}$. We also fix the key ring size at $K = 40$, and the key-setup probability at $p_q = 0.1$, and set $p_{\text{compromised}}$ as

Fig. 8. A plot under varying q for the number m of nodes that the adversary has to capture to compromise a link between two non-captured nodes with probability $p_{\text{compromised}}$. The parameters are $K = 40$, $p_q = 0.1$ and $p_{\text{compromised}} = 0.03, 0.1$. To fix p_q , the key pool size P decreases as q increases.

Fig. 9. A plot of probability $p_{\text{compromised}}$ of link compromise with respect to m , under $K = 50$, $P = 10000$ and $q = 2, 3$. Note that the key-setup probability here is different for different q .

0.03 or 0.1. The curves on how m changes with respect to q also confirm the analytical results as explained below. As mentioned, for small m , as q increases, $p_{\text{compromised}}$ asymptotically first decreases and then increases. Hence, if we fix a small $p_{\text{compromised}}$, as q increases, the required m should first increase and then decrease. This clearly is validated by the curve of $p_{\text{compromised}} = 0.03$ in Figure 8. Similarly, the theoretical result shows that for large m , as q increases, $p_{\text{compromised}}$ asymptotically always increases. Thus, if we fix a large $p_{\text{compromised}}$, as q increases, the required m should always decrease. This is validated by the curve of $p_{\text{compromised}} = 0.1$ in Figure 8.

2) Varying m with key ring size and key pool size fixed:

We now fix both the key ring size and the key pool size, and vary m (the number of nodes captured by the adversary) to see how the probability $p_{\text{compromised}}$ of link compromise changes with respect to m .

Figure 9 depicts probability $p_{\text{compromised}}$ with respect to m . We fix the key ring size at $K = 50$, and the key pool size at $P = 10000$, and set q (the required amount of key overlap) as 2 or 3. Clearly, for each q , we observe that $p_{\text{compromised}}$

increases as m increases. In addition, as we see, for very small m , probability $p_{\text{compromised}}$ under $q = 2$ is larger than that under $q = 3$. We explain below that this confirms the theoretical result. By analysis, for small m and small q , as q increases, $p_{\text{compromised}}$ asymptotically decreases. Therefore, for small m , $p_{\text{compromised}}$ under $q = 2$ is asymptotically larger than that under $q = 3$, which is in agreement with Figure 9. By analysis, for large m , as q increases, $p_{\text{compromised}}$ asymptotically increases. This is validated since we see in Figure 9 that for large m , $p_{\text{compromised}}$ under $q = 2$ is small than that under $q = 3$.

VIII. CONCLUSION

In wireless sensor networks, the q -composite key predistribution scheme is a widely studied mechanism to secure communications. Despite considerable studies on secure sensor networks under the q -composite scheme, most of them ignore one or more than one of the following aspects: node-capture attacks, sensor mobility, physical transmission constraints, the boundary effect of network fields, and link unreliability, whereas we consider all of them in this paper and present conditions to ensure secure connectivity. Moreover, few researches on the q -composite scheme formally analyze the scheme's resilience to arbitrary node-capture attacks considered in this paper, where the adversary can capture an arbitrary set of sensors. In the presence of arbitrary node-capture attacks, we derive conditions to ensure unassailability and unsplittability in secure sensor networks under the q -composite scheme. This paper presents a rigorous and comprehensive analysis to provide useful guidelines for connectivity and resilience design of secure sensor networks.

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