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A NOVEL METHOD FOR ESTABLISHING SOLUTIONS TO NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The paper presents a novel method that allows one to establish solutions of some non-linear ordinary differential equations, which contain products of the unknown functions and/or its derivatives. This method can be especially useful when one needs to address the problem of the solution existence for such equations. In particular, the method was used in this work for illustrative purposes, to find some solutions to the Blasius equation.

INTRODUCTION

No general methods are available for solving non-linear ordinary and partial differential equations [1]. The solutions that lead to analytical solutions of these equations or even to reducing the given equations to simpler problems are few [2], [3].

In particular, the non-linear Navier-Stokes partial differential equation remains unsolved until nowadays [4]. The Navier-Stokes equation, together with the continuity equation, is the governing equation of fluid motion valid as long as the continuum hypothesis holds. For a wide variety of flow phenomena encountered in science and engineering, the assumption of constant properties is adequate. However, even when simplified by this assumption the Navier-Stokes equation still presents a challenge, namely the nonlinear convective term, which makes the solving procedure very cumbersome, if possible at all [5]. Moreover, even if the Navier-Stokes equation is simplified to its boundary layer versions (e.g., the Blasius equation), no analytical solutions are still to be established.

This paper presents a novel method that allows one to establish solutions of some non-linear ordinary differential equations, which contain products of the unknown functions and/or its derivatives. This method can be especially useful when one needs to address the problem of the solution existence for such equations. In particular, the method was used in this work for illustrative purposes, to find some solutions to the Blasius equation.

THE METHOD

Solutions to most of linear ordinary differential equations are established by assuming a proportionality of the solution to the exponential function, that is

$$f(x) = ae^{nx} \quad (1)$$

where a and n are to be found from the characteristic equation that arises upon substituting (1) into a given differential equation.

In the case of non-linear differential equations, this method is rarely used, because the resulting characteristic equations are themselves non-linear. Instead, the so-called power series method is used, in which the unknown function is assumed in the form

$$f(x) = \sum_i a_i x^i \quad (2)$$

Then, solutions to a given differential equation are established from finding the coefficients a_i upon performing direct substitution of (2) into the given differential equation.

The method, proposed in this paper, uses the very same idea that is used for solving linear ordinary differential equations. Then, the unknown function is assumed in the form (1), so that all of its higher order derivatives become proportional to the function itself. Similarly, assuming the relation between the unknown function and its first derivative in the form,

$$f' = af^n \quad (3)$$

should lead to a certain class of solutions to non-linear ordinary differential equations.

With the use of assumption (3), **all** higher order derivatives of the unknown function f become proportional to the unknown function itself and are given in the form

$$f^{(m)} = a^m f^{mn-(m-1)} \prod_{k=1}^{m-1} [kn - (k-1)] \quad (4)$$

Hence, upon substituting (4) into the given non-linear ordinary differential equation, the latter reduces to an algebraic characteristic equation and, consequently, if the resulting characteristic equation does have solutions a and n , the solution to the given non-linear equation exists and is the solution to the linear ordinary differential equation given by (3). That is,

$$f(x) = [a(1-n)(x+C)]^{1-n} \quad (5)$$

where C is an arbitrary constant.

Note that, in the case of non-linear ordinary differential equations, the assumption, given by (1), can lead to additional solutions, if a solution to the resulting characteristic equation exists.

VALIDATION OF THE METHOD

Consider the Blasius equation in the form

$$f''' + ff'' = 0 \quad (6)$$

with the boundary conditions

$$f(0) = 0; f'(0) = 0; f'(\infty) = 1 \quad (7)$$

Assume

$$f' = af^n \quad (8)$$

where a and n are arbitrary.

Then

$$f'' = a n f^{n-1} f' = a^2 n f^{2n-1} \quad (9)$$

$$f''' = a^3 n(2n-1) f^{3n-2} \quad (10)$$

and

$$f f'' = a^2 n f^{2n} \quad (11)$$

Upon substituting (10) and (11) into the Blasius equation (6) and dividing both of its parts by $a^2 n$, because neither a nor n can be zero, the latter becomes

$$a(2n-1) f^{3n-2} + f^{2n} = 0 \quad (12)$$

From which $n = 2$ and $a = -1/3$, because $f \neq 0$ identically.

Hence, at least one of the solutions to the Blasius equation coincides with the solution of

$$f' = -\frac{1}{3} f^2 \quad (13)$$

that is

$$f(x) = \frac{3}{x + C} \quad (14)$$

where C is an arbitrary constant.

The fact that equation (14) is one of possible solutions to the Blasius equation can be easily verified by direct substitution of (14) into (6).

It is clear that the solution (14) does not satisfy the boundary conditions (7), but merely establishes the fact that at least one solution to the Blasius equation exists and can be written in a closed form.

Note that the power series method, applied to the Blasius equation and its boundary conditions given by (7), yields

$$f(x) = \sum_{m=0}^{\infty} \left(-\frac{1}{2}\right)^m \frac{a_2^{m+1} C_m}{(3m+2)!} x^{3m+2} \quad (15)$$

where $C_0 = 1$, $C_1 = 1$, $C_2 = 11$, $C_3 = 375$, $C_4 = 27897$, $C_5 = 3817137$, and $a_2 = 0.33206$.

Note also that the assumption in the form given by (1), in the case of the Blasius equation, leads to the characteristic equation in the form

$$e^n + n = 0 \quad (16)$$

This, in turn, leads to another closed-form solution of the Blasius equation, namely:

$$f(x) = e^{-W(1)x} \quad (17)$$

where $W(1) \approx 0.567143$ is known as the analytic continuation of the product logarithmic function.

CONCLUSIONS

The paper discussed a novel method that allows one to establish solutions of some non-linear ordinary differential equations, which contain products of the unknown functions and/or its derivatives. This method can be especially useful when one needs to address the problem of the solution existence for such equations. In particular, the method was used in this work for illustrative purposes, to find some solutions to the Blasius equation.

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