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Optimum Placement of Phasor Measurement Units in Power Systems

Lu Sun, Tengpeng Chen, Xuebing Chen, W.K. Ho, KV Ling, K.J. Tseng and Gehan A. J. Amaratunga

Abstract—In this paper, we proposed to evaluate the optimal PMU placement based on the Sum of Variance of the robust estimators. Variance is traditionally used as a measure of the quality of estimates. In this paper, after satisfying the requirements of the minimum number of PMUs, maximum measurement redundancy and observability, the placements obtained are further distinguished on the basis of the variance of the estimated states. Both the Weighted Least Squares (WLS) and robust estimators are considered. Examples on the IEEE 30-bus system using Least Absolute Value (LAV) estimation are given. The covariance of the state estimates from the WLS can be calculated using existing formulas. However, there is no equivalent formula for the robust LAV estimator. A formula is derived in this paper using influence function to approximately calculate the covariance matrix of the state estimates from the robust LAV estimator. The optimum placement i.e. the placement with the smallest sum of variance can be selected.

Index Terms—Smart Grid; Phasor Measurement Unit; Sensor Placement; Robust Estimation; Least Absolute Value.

NOMENCLATURE

The notations used in the paper are summarized below for easy reference.

x	true state vector
n	length of x
$z(k)$	measurement vector obtained by Phasor Measurement Units at k
m	length of $z(k)$
σ_i	standard deviation of $z_i(k)$
H	measurement matrix
$\epsilon(k)$	measurement noise at k

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V_i^r	real part of the voltage phasor of Bus i
V_i^{im}	imaginary part of the voltage phasor of Bus i
J	cost function
$\rho(\cdot)$	ρ -function
N	batch size of measurements used in the state estimation
$\hat{x}$	state estimate
$e_i(k)$	measurement residual $e_i(k) = z_i(k) - H_i \hat{x}$
Ψ	Ψ -function
$\bar{H}$	collective of H defined by (5)
$w_i(k)$	weight associated with $e_i(k)$ defined by (6)
W	weight matrix defined by (7)
Z	collective of $z(k)$ defined by (8)
E	collective of $e_i(k)$ defined by (9)
α_i, a_i, b_i, r_i	parameters of the Multiple-Segment estimator given in (11)
$IF(\cdot)$	Influence Function
$f(\epsilon)$	joint Probability Density Function of the measurement noise
x_0	operating point
Ω, Λ	intermediate value in (16)
$Cov(\cdot)$	covariance of a vector
$tr(\cdot)$	trace of a matrix
SV	sum of variance

I. INTRODUCTION

Power System State Estimation (PSSE) is traditionally a nonlinear problem [1]–[4]. However, when the Phasor Measurement Unit (PMU) is used to take measurements, it can be formulated as a linear problem [5] and application of linear system analysis is now possible. Together with its other advantages of fast sampling frequency and low measurement error led to the increasing use of PMU in power system state estimation [6]–[12].

Given the fact that PMU is able to provide measurements that are linearly related to system states, it is possible to formulate state estimation as a linear problem as long as the entire system is fully observed by PMUs. The simplest way to ensure full observability and maximum measurement redundancy is to install PMUs on all the buses. However, this is hardly possible in practice due to the high cost of a PMU. Even though the cost of PMUs has been decreasing in the past years, logistic, economic and design aspects limit the massive installation of PMUs on all buses in the system [13]. In addition, the communication network that links the massive number of PMUs to the control center can be expensive as well

[14]–[17]. Therefore, Optimal PMU Placement (OPP) problem has been formulated as a constrained optimization problem where the objective function is to achieve an observable system with the minimum number of PMUs placed at strategic bus locations [17]–[23]. More studies on OPP can be found in [24]–[36].

Minimizing the number of PMUs alone can give unsatisfactory state estimation because of poor or no measurement redundancy [19], [23]. In such cases, the quality of the estimation is likely to be affected by noise [37]. The System Observability Redundancy Index (SORI) is proposed in [17], [38] to evaluate the multiple solutions obtained by minimizing the number of PMUs. Placements with the highest value of SORI are selected, thus achieving the designs which maximize the measurement redundancy given the minimum number of PMUs. However, this method does not produce a unique solution, and the quality or estimation uncertainty of the state estimate is not taken into account.

In the literature, the covariance of the state estimation error is used as a measure of estimation uncertainty [39]–[42]. In [42] and [43], the covariance of the state estimation error of a Weighted Least Squares estimator and a Kalman Filter is used respectively to evaluate the optimality of PMU placement while guaranteeing the observability of the system. Weighted Least Squares estimator and Kalman Filter are popular in PSSE because their estimation error covariances can be computed analytically [1], [37]. These non-robust estimators give good performance when measurement noise follows the Gaussian distribution, but can be adversely affected by outliers producing biased estimation results [37].

Robust estimators have been proposed in power system state estimation to reject measurement outliers [1], [2], [5], [44]. In this paper, we propose to evaluate the optimal PMU placement based on the sum of variance of not only the non-robust estimators but also the robust estimators. More specifically, given the minimum number of PMUs to guarantee the observability of the system, the sum of variance is then used to further distinguish among the multiple placement solutions obtained by maximizing SORI. The placement with the minimum sum of variance is then selected. The IEEE 30-bus system and the LAV robust estimator are used to illustrate our proposed method.

By implementing Optimal PMU Placement (OPP) based on minimization of the Sum of Variance of the estimates, the state estimation uncertainty is lowered and its precision is improved. The higher precision of the state estimates enables more accurate supervision of the bus voltage phasors, which is a prerequisite to better control of the power system voltages.

The rest of the paper is organized as follows. The WLS and the robust estimation algorithms are given in Section II. A formula is derived to approximately calculate the state estimate variance for the robust estimators in Section III. Examples are given in Section IV, and conclusion in Section V.

II. ROBUST ESTIMATION

When measurements are taken by PMUs, the model is linear and given by [5]

$$z(k) = Hx(k) + \epsilon(k) \quad (1)$$

where

$$\begin{aligned} x(k) &= [V_1^r(k), \dots, V_B^r(k), V_1^{im}(k), \dots, V_B^{im}(k)]^T \in \mathbb{R}^{n \times 1} \\ z(k) &= [z_1(k), \dots, z_m(k)]^T \\ \epsilon(k) &= [\epsilon_1(k), \dots, \epsilon_m(k)]^T \end{aligned}$$

The state $x(k)$ is independent of the sampling instance k since a steady-state problem is considered. In the sequel, x instead of $x(k)$ will be used. V_i^r and V_i^{im} are the real and imaginary part of the voltage phasor of Bus i respectively while B is the total number of buses and $n = 2B$. The measurement vector and measurement matrix are given by $z(k)$ and H respectively. There are two parameters, the standard deviation of $\epsilon(k)$ and H , in the model of equation (1). The standard deviation of $\epsilon_i(k)$ depends on the measurement accuracy of the associated PMU, usually provided by the manufacturer. H is the measurement matrix which depends on the placement of the PMUs. The derivation of H is given in [7]. The measurement noise $\epsilon_i(k)$ is assumed to be an independent zero-mean Gaussian random variables.

Estimation of the state $\hat{x}$ can be obtained as follows. Define cost function J as

$$J = \sum_{i=1}^m \sum_{k=1}^N \rho(e_i(k)) \quad (2)$$

where $e_i(k) = z_i(k) - H_i \hat{x}$ is the measurement residual and $\rho(\cdot)$, a nonlinear function. N is the batch size of measurements used. The state estimate $\hat{x}$ can be obtained by minimizing J .

Differentiate J in (2) w.r.t. x gives

$$\begin{aligned} \frac{\partial J}{\partial x} = \Psi &= \frac{\partial J}{\partial e_i(k)} \frac{\partial e_i(k)}{\partial x} \\ &= - \sum_{i=1}^m \sum_{k=1}^N w_i(k) (e_i(k)) H_i^T \\ &= -\bar{H}^T W (Z - \bar{H}x) \\ &= -\bar{H}^T W E \end{aligned} \quad (3)$$

where

$$\bar{H} = [H^T, \dots, H^T]^T \in \mathbb{R}^{Nm \times n} \quad (5)$$

$$w_i(k) = \frac{\partial \rho(e_i(k))}{\partial e_i(k)} \frac{1}{e_i(k)} \quad (6)$$

$$W = \text{diag}(w_1(1), \dots, w_m(1), \dots, w_1(N), \dots, w_m(N)) \in \mathbb{R}^{Nm \times Nm} \quad (7)$$

$$Z = [z(1)^T, \dots, z(N)^T]^T \in \mathbb{R}^{Nm \times 1} \quad (8)$$

$$E = [e(1)^T, \dots, e(N)^T]^T \quad (9)$$

The state estimate $\hat{x}$ is the solution to the equation

$$\Psi = 0 \quad (10)$$

This method is applicable to common robust estimators such as the Multiple-Segment (MS), Quadratic-Linear (QL), Quadratic-Constant (QC), Square-Root (SR), Least Absolute Value (LAV) as well as the well-known Weighted Least Squares (WLS) estimator [1].

Considering the MS estimator, the cost function $\rho(e_i(k))$ in (2) is given by [37]

$$\rho(e_i(k)) = \begin{cases} \frac{(e_i(k))^2}{2\alpha_i^2} & |e_i(k)| \leq a_i\alpha_i \\ \frac{a_i|e_i(k)|}{\alpha_i} - \frac{a_i^2}{2} & a_i\alpha_i < |e_i(k)| \leq b_i\alpha_i \\ -\frac{a_i(r_i\alpha_i - |e_i(k)|)^2}{2(r_i - b_i)\alpha_i^2} + \frac{1}{2}a_ir_i + \frac{1}{2}a_ib_i - \frac{1}{2}a_i^2 & b_i\alpha_i < |e_i(k)| \leq r_i\alpha_i \\ \frac{1}{2}(a_ir_i + a_ib_i - a_i^2) & r_i\alpha_i < |e_i(k)| \end{cases} \quad (11)$$

where a_i, b_i and r_i are thresholds selected by the user satisfying $0 \leq a_i \leq b_i \leq r_i$.

Its associated $w_i(k)$ in (7) is

$$w_i(k) = \begin{cases} \frac{1}{\alpha_i^2} & |e_i(k)| \leq a_i\alpha_i \\ \frac{a_i}{\alpha_i|e_i(k)|} & a_i\alpha_i < |e_i(k)| \leq b_i\alpha_i \\ \frac{a_i(r_i\alpha_i - |e_i(k)|)}{(r_i - b_i)\alpha_i^2|e_i(k)|} & b_i\alpha_i < |e_i(k)| \leq r_i\alpha_i \\ 0 & r_i\alpha_i < |e_i(k)| \end{cases} \quad (12)$$

Note that the QL, QC, WLS and LAV estimators can all be regarded as special cases of the MS estimator [1]. It reduces to the QL estimator when $b_i \rightarrow \infty$, the QC estimator when $r_i \rightarrow b_i$ and $b_i \rightarrow a_i$, the WLS estimator when $a_i \rightarrow \infty$, the LAV estimator when $\alpha_i \rightarrow 0$, $a_i = \alpha_i$, $b_i \rightarrow \infty$ and $b_i\alpha_i \rightarrow \infty$. Figure 1 is an example of the cost function of the different estimators. In Figure 1, the choice of α_i, a_i, b_i and r_i are 1, 3, 4 and 5 respectively, where α_i is the standard deviation of the measurement noise.

III. VARIANCE OF THE STATE ESTIMATES

This section summarizes the Influence Function (IF) approach in [44] to derive the variance of robust estimators. It serves as a necessary background for the derivation, in Section IV, of the variance of the LAV robust estimator which is not given in [44].

The definition of the IF is given in [37], [45] as

$$\begin{aligned} \text{IF}(\epsilon) &= - \left[\int_{-\infty}^{\infty} \frac{\partial \Psi}{\partial x} f(\epsilon) d\epsilon \right]^{-1} \Psi \\ &\approx \hat{x} - x_0 \end{aligned} \quad (13)$$

where $\epsilon = [\epsilon(1)^T, \dots, \epsilon(N)^T]^T$, $f(\epsilon)$ is the associated joint probability density function, and x_0 the operating point of the system. Equation (13) can be interpreted intuitively as follows.

Fig. 1. $\rho(e_i(k))$ for MS, QL, QC, WLS and LAV estimators.

The first-order Taylor series expansion of $g(x)$ about the operating point $(x_0, g(x_0))$ is given by

$$g(x_1) \approx g(x_0) + \left. \frac{dg(x)}{dx} \right|_{x=x_0} (x_1 - x_0) \quad (14)$$

Let $x_1 = \hat{x}$, $g(x_0) = \Psi$, $g(x_1) = 0$. Taking the expectation of $\frac{d\Psi}{dx}$ gives (13). A formal derivation of the influence function is given in [37].

Using (13), the covariance of the state estimates can be approximated as

$$\begin{aligned} \text{Cov}(\hat{x}) &= \text{Cov}(\hat{x} - x_0) \\ &\approx \int_{-\infty}^{\infty} \text{IF}(\epsilon) \text{IF}(\epsilon)^T f(\epsilon) d\epsilon \\ &= \left[\int_{-\infty}^{\infty} \frac{\partial \Psi}{\partial x} f(\epsilon) d\epsilon \right]^{-1} \left[\int_{-\infty}^{\infty} \Psi \Psi^T f(\epsilon) d\epsilon \right] \\ &\quad \times \left[\int_{-\infty}^{\infty} \frac{\partial \Psi}{\partial x} f(\epsilon) d\epsilon \right]^{-T} \end{aligned} \quad (15)$$

Replacing E with ϵ in (4) gives $\Psi = \bar{H}W\epsilon$. Equations (1), (5) and (8) give $Z = \bar{H}x + \epsilon$. Substituting them into (15) gives

$$\begin{aligned} \text{Cov}(\hat{x}) &\approx \left[\bar{H}^T \left(\int_{-\infty}^{\infty} \frac{\partial W\epsilon}{\partial \epsilon} f(\epsilon) d\epsilon \right) \bar{H} \right]^{-1} \\ &\quad \times \left[\bar{H}^T \left(\int_{-\infty}^{\infty} W\epsilon \epsilon^T W f(\epsilon) d\epsilon \right) \bar{H} \right] \\ &\quad \times \left[\bar{H}^T \left(\int_{-\infty}^{\infty} \frac{\partial W\epsilon}{\partial \epsilon} f(\epsilon) d\epsilon \right) \bar{H} \right]^{-T} \\ &= [\bar{H}^T \Omega \bar{H}]^{-1} [\bar{H}^T \Lambda \bar{H}] [\bar{H}^T \Omega \bar{H}]^{-T} \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Omega &= \int_{-\infty}^{\infty} \frac{\partial W\epsilon}{\partial \epsilon} f(\epsilon) d\epsilon \\ \Lambda &= \int_{-\infty}^{\infty} W\epsilon \epsilon^T W f(\epsilon) d\epsilon \end{aligned}$$

Assuming stationary independent measurement noise,

$$\begin{aligned}\Omega &= \text{diag}(\Omega_1(1), \dots, \Omega_m(1), \dots, \Omega_1(N), \dots, \Omega_m(N)) \\ \Omega_i(k) &= \int_{-\infty}^{\infty} \frac{\partial w_i(k) \epsilon_i(k)}{\partial \epsilon_i(k)} f_{i,k}(\epsilon_i(k)) d\epsilon_i(k)\end{aligned}\quad (17)$$

$$\begin{aligned}\Lambda &= \text{diag}(\Lambda_1(1), \dots, \Lambda_m(1), \dots, \Lambda_1(N), \dots, \Lambda_m(N)) \\ \Lambda_i(k) &= \int_{-\infty}^{\infty} W_i(k)^2 \epsilon_i(k)^2 f_{i,k}(\epsilon_i(k)) d\epsilon_i(k)\end{aligned}\quad (18)$$

Substituting (12) into (17) and (18), $\Omega_i(k)$ and $\Lambda_i(k)$ can be obtained as follows.

$$\Omega_i(k) = \frac{1}{\alpha_i^2} \int_{-a_i \alpha_i}^{a_i \alpha_i} f_i(\epsilon_i) d\epsilon_i - \frac{2}{\alpha_i^2} \int_{b_i \alpha_i}^{r_i \alpha_i} \frac{a_i f_i(\epsilon_i)}{r_i - b_i} d\epsilon_i \quad (19)$$

$$\begin{aligned}\Lambda_i(k) &= \int_{-a_i \alpha_i}^{a_i \alpha_i} \frac{\epsilon_i^2}{\alpha_i^4} f_i(\epsilon_i) d\epsilon_i \\ &+ 2 \left(\int_{a_i \alpha_i}^{b_i \alpha_i} \frac{a_i^2}{\alpha_i^2} f_i(\epsilon_i) d\epsilon_i \right. \\ &\left. + \int_{b_i \alpha_i}^{r_i \alpha_i} \frac{a_i^2 (r_i \alpha_i - \epsilon_i)^2}{\alpha_i^4 (r_i - b_i)^2} f_i(\epsilon_i) d\epsilon_i \right)\end{aligned}\quad (20)$$

IV. EXAMPLES

In this paper, simulations on the IEEE 30-bus system are used to demonstrate the Optimal PMU Placement based on the minimization of the Sum of Variance for both Weighted Least Squares (WLS) and Least Absolute Value (LAV) estimation. The IEEE 30-bus system model is widely used in the studies of power system design and analysis [17], [22], [46]. Consider the 3 placements (A, B and C) in [38] which were obtained by maximizing the SORI while minimizing the PMU number on the IEEE 30-bus system. All the placements require ten PMUs and have the same SORI. Figure 2 shows the positions of the PMUs. Bus Number 2, 4, 6, 9, 10, 12, 15, 25 and 27 are nine PMU positions common to all the placements. Bus Number 18, 19 and 20 are unique to Placements A, B and C respectively. Since the SORI is the same for all the placements, using it alone is unable to distinguish among the three placements. Here we use the variance of the state estimate to pick the optimum placement.

The standard Weighted Least Squares estimator is given in Section IV-A. The increasingly popular LAV estimator [5] chosen among the robust estimators is given in Section IV-B to illustrate the calculations. Monte-Carlo experiments with 10,000 runs are used to verify the proposed formula (26). Monte-Carlo experiments are widely used in power system design and analysis to verify the effectiveness of methodologies, especially to test the statistical performance of the power system [3], [47], [48].

The simulations are performed using MATLAB 2016a on a Windows 10 computer configured with Intel[®] Core[™], CPU i7-4790, 3.60GHz and 8GB RAM.

Fig. 2. The PMU positions obtained by maximizing SORI.

Red Star: Common PMU position;
Yellow Star: PMU position unique to Placement A;
Blue Star: PMU position unique to Placement B;
Green Star: PMU position unique Placement C.

A. WLS Estimation

The state estimate and variance are well-known and given in [1] as

$$\hat{x} = (\bar{H}^T W \bar{H})^{-1} \bar{H} W Z \quad (21)$$

$$\text{Cov}(\hat{x}) = (\bar{H}^T W \bar{H})^{-1} \quad (22)$$

where

$$W = \text{diag} \left(\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_m^2}, \dots, \frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_m^2} \right)$$

The sum of variance (SV) is the trace of the covariance matrix

$$\text{SV} = \text{tr} \left((\bar{H}^T W \bar{H})^{-1} \right) \quad (23)$$

For the ten PMUs, let one have smaller measurement standard deviation of $\sigma_s = 0.0025$ and the rest, bigger standard deviation of $\sigma_b = 0.005$. For a batch size of $N = 5$, the SV of WLS estimation obtained from (23) for Placements A, B, C are shown in Table I Columns (b), (f) and (j) respectively. For instance, consider Row (1) Column (b) as an example. If the PMU with σ_s is placed on Bus 2 of Placement A and the nine PMUs with σ_b on the other Buses (4, 6, 9, 10, 12, 15, 18, 25, 27) of Placement A, then a SV of 47.04×10^{-6} is obtained. The smallest and largest SV are 37.08×10^{-6} and 47.48×10^{-6} respectively. The smallest SV is obtained in Row (12) Column (f). In other words, optimality defined as minimum variance is obtained when the PMU with σ_s is on Bus 27 and the rest of the nine PMUs with σ_b on Buses 2, 4, 6, 9, 10, 12, 15, 19 and 25.

TABLE I
THE SUM OF VARIANCE (SV) FOR DIFFERENT PLACEMENT OF THE 10 PMUS.

Row Number	Bus Location of PMU with σ_s (a)	Placement A (Bus: 2, 4, 6, 9, 10, 12, 15, 18, 25, 27)				Placement B (Bus: 2, 4, 6, 9, 10, 12, 15, 19, 25, 27)				Placement C (Bus: 2, 4, 6, 9, 10, 12, 15, 20, 25, 27)			
		WLS (b)	LAV (c) (d) (e)			WLS (f)	LAV (g) (h) (i)			WLS (j)	LAV (k) (l) (m)		
(1)	2	47.04	76.50	73.98	3.4%	45.82	74.57	72.74	2.5%	46.86	76.21	73.61	3.5%
(2)	4	46.49	75.78	73.62	2.9%	45.58	74.33	72.37	2.7%	46.32	75.51	73.38	2.9%
(3)	6	46.62	75.95	73.13	3.9%	45.56	74.27	72.57	2.3%	46.42	75.64	73.09	3.5%
(4)	9	46.55	75.80	73.05	3.8%	45.51	74.16	72.25	2.7%	46.40	75.57	73.86	2.3%
(5)	10	46.51	75.67	73.14	3.5%	45.28	73.78	71.36	3.4%	46.54	75.74	72.95	3.8%
(6)	12	45.76	74.62	72.21	3.3%	44.88	73.20	71.86	1.9%	45.54	74.25	72.09	3.0%
(7)	15	46.62	75.76	73.34	3.3%	45.19	73.57	71.10	3.5%	46.13	74.97	72.52	3.4%
(8)	18	47.48	76.94	74.79	2.9%	—	—	—	—	—	—	—	—
(9)	19	—	—	—	—	46.35	75.17	72.95	3.0%	—	—	—	—
(10)	20	—	—	—	—	—	—	—	—	47.29	76.73	75.03	2.3%
(11)	25	44.60	72.14	70.32	2.6%	43.37	70.20	68.33	2.7%	44.42	71.85	69.30	3.2%
(12)	27	38.30	62.76	61.71	1.7%	37.08	60.83	59.31	2.6%	38.12	62.47	60.92	2.5%

Variance Unit: $\times 10^{-6}$.

This result is counter-intuitive because, as shown in Table II, a PMU on Bus 6 can take the most number of measurements. Hence one would expect that a good PMU with small measurement standard deviation, σ_s , should be put on Bus 6 to give the greatest reduction in the sum of variance. This is further explained using Table III. In Table III, Column (a) is the state index. Column (b) is the variance of the state estimates using WLS when PMUs of standard deviation $\sigma_b = 0.005$ are used (Configuration 1). Column (c) is the variance of the state estimates when the PMU on bus 6 is replaced by the PMU with $\sigma_s = 0.0025$ (Configuration 2) while Column (d) is the variance of the state estimates when the PMU on bus 27 is replaced by the PMU with $\sigma_s = 0.0025$ (Configuration 3). The improvements in the variances are given in Columns (e) and (f) respectively.

The results in Table III are plotted in Figure 3. It can be seen from the figure that the variances do not change much when the PMU on bus 6 is replaced. However, when the PMU on bus 27 is replaced, the variances of the states V_{24} , V_{25} , V_{26} , V_{27} , V_{29} and V_{30} are reduced significantly. Thus unless the actual variance is calculated it is not intuitively clear where to place the PMUs to reduce the overall variance.

TABLE II
THE NUMBER OF MEASUREMENTS THAT CAN BE TAKEN ON EACH BUS.

Bus Number	Number of Measurements
2	10
4	10
6	16
9	8
10	14
12	12
15	10
18	6
19	6
20	6
25	8
27	10

TABLE III
THE VARIANCE V_n^r (REAL) + V_n^i (IMAGINARY) OF THE STATE ESTIMATE FOR PLACEMENT B WITH WLS ESTIMATOR.

State V_n	All PMUs with σ_b (b)	Replacement of PMU with σ_b by PMU with σ_s on		Improvement	
		Bus 6 (c)	Bus 27 (d)	Bus 6 (e)=(c)-(b)	Bus 27 (f)=(d)-(b)
V_1	1.189	0.888	0.953	0.301	0.236
V_2	1.156	0.854	0.919	0.302	0.237
V_3	1.108	0.834	0.869	0.276	0.240
V_4	1.093	0.817	0.853	0.276	0.240
V_5	1.562	1.262	1.327	0.299	0.235
V_6	1.091	0.816	0.849	0.275	0.242
V_7	1.164	0.833	0.922	0.331	0.241
V_8	1.109	0.820	0.868	0.289	0.242
V_9	1.154	0.855	0.934	0.299	0.220
V_{10}	1.170	0.878	0.953	0.292	0.217
V_{11}	1.587	1.287	1.367	0.299	0.220
V_{12}	1.196	0.957	0.984	0.239	0.212
V_{13}	1.392	1.153	1.180	0.239	0.212
V_{14}	1.613	1.377	1.403	0.236	0.210
V_{15}	1.234	1.002	1.027	0.232	0.207
V_{16}	1.680	1.441	1.468	0.239	0.212
V_{17}	1.252	0.960	1.035	0.292	0.217
V_{18}	1.385	1.153	1.184	0.232	0.201
V_{19}	1.352	1.115	1.153	0.237	0.199
V_{20}	1.350	1.108	1.149	0.241	0.201
V_{21}	1.238	0.946	1.022	0.292	0.217
V_{22}	1.447	1.155	1.231	0.292	0.217
V_{23}	1.742	1.510	1.535	0.232	0.207
V_{24}	3.418	3.252	2.449	0.166	0.969
V_{25}	1.979	1.813	1.010	0.166	0.969
V_{26}	4.070	3.904	3.101	0.166	0.969
V_{27}	1.805	1.630	0.920	0.175	0.885
V_{28}	1.108	0.817	0.854	0.292	0.254
V_{29}	4.013	3.838	1.472	0.175	2.541
V_{30}	6.463	6.288	2.085	0.175	4.378
SV	53.12	45.56	37.08	7.557	16.04

Variance Unit: $\times 10^{-6}$.

B. LAV Estimation

To prevent the large bias caused by measurement outliers, many robust estimators have been proposed [5], [6], [49]. The LAV robust estimator is computationally efficient and suitable for handling simultaneous multiple measurement outliers [7],

Fig. 3. Comparison of the variance of each state estimate.

Configuration 1: All PMUs with $\sigma_b = 0.005$. (Black)

Configuration 2: PMU on bus 6 is replaced by the PMU with $\sigma_s = 0.0025$. (Shaded)

Configuration 3: PMU on bus 27 is replaced by the PMU with $\sigma_s = 0.0025$. (White)

[10], [49]. The LAV algorithm is discussed elsewhere [1], [5], here we only give the equations for the calculation of the covariance.

For the LAV estimator, equation (11) reduces to $\rho(e_i(k)) = |e_i(k)|$ and (2) becomes

$$J = \sum_{i=1}^m \sum_{k=1}^N |e_i(k)| \quad (24)$$

The LAV estimator is a special case of the MS estimator with $\alpha_i \rightarrow 0$, $a_i = \alpha_i$, $b_i \rightarrow \infty$, $b_i \alpha_i \rightarrow \infty$. Equation (12) now gives

$$w_i(k) = \frac{1}{|e_i(k)|} \quad (25)$$

and the state estimate, $\hat{x}$, is obtained by solving (10).

Substituting $w_i(k) = \frac{1}{|e_i(k)|}$ into (25) and $a_i = \alpha_i$, $a_i \rightarrow 0$, $b_i \rightarrow \infty$, $b_i \alpha_i \rightarrow \infty$ into (19) and (20) give

$$\begin{aligned} \Omega_i(k) &= \frac{1}{\alpha_i^2} \int_{-a_i \alpha_i}^{a_i \alpha_i} f_i(\epsilon_i) d\epsilon_i = \sqrt{\frac{2}{\pi \sigma_i^2}} \\ \Lambda_i(k) &= \int_{-a_i \alpha_i}^{a_i \alpha_i} \frac{\epsilon_i^2}{\alpha_i^4} f_i(\epsilon_i) d\epsilon_i + 2 \int_{a_i \alpha_i}^{b_i \alpha_i} \frac{a_i^2}{\alpha_i^2} f_i(\epsilon_i) d\epsilon_i = 1 \end{aligned}$$

where $f(\epsilon_i)$ is the zero-mean Gaussian probability density function

$$f(\epsilon_i) = \frac{1}{\sqrt{2\sigma_i^2\pi}} \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right)$$

Equation (16) reduces to

$$\text{Cov}(\hat{x}) \approx [\bar{H}^T \Omega \bar{H}]^{-1} [\bar{H}^T \bar{H}] [\bar{H}^T \Omega \bar{H}]^{-T} \quad (26)$$

where

$$\Omega = \sqrt{\frac{2}{\pi}} \text{diag} \left(\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_m}, \dots, \frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_m} \right) \quad (27)$$

The SV is now given as

$$\text{SV} \approx \text{tr} \left([\bar{H}^T \Omega \bar{H}]^{-1} [\bar{H}^T \bar{H}] [\bar{H}^T \Omega \bar{H}]^{-T} \right) \quad (28)$$

and can be used as a criterion to evaluate the performance of a LAV estimator. A flowchart is given in Figure 4 to illustrate the key steps of the proposed OPP method based on minimization of the SV.

Note that the SV for the QL ($b_i \rightarrow \infty$) and QC ($r_i \rightarrow b_i, b_i \rightarrow a_i$) robust estimators can likewise be obtained from (16).

The same three placements (A, B, C) and ten PMUs (nine with σ_b and one with σ_s) in the WLS estimation are used. For a batch size of $N = 5$, the SV obtained from (28) when the PMU with σ_s is placed on different buses are also given in Table I Columns (c), (g) and (k). The smallest and largest SV are 60.83×10^{-6} and 76.94×10^{-6} respectively. The smallest SV is obtained in Row (12) Column (g). In other words, optimality defined as minimum variance is obtained when the PMU with σ_s is on Bus 27 and the rest of the nine PMUs with σ_b on Buses 2, 4, 6, 9, 10, 12, 15, 19 and 25. Monte-Carlo experiments with 10,000 runs are given in Table I Columns (d), (h), (l) to verify the variance of the state estimation calculated using (28). The relative error of the result between the formula (28) and the Monte-Carlo experiment are given in Columns (e), (i), (m). The largest relative error in Table I is 3.9% which should be acceptable in practice.

It should also be noted that (23) and (28) are general variance formulas for the WLS and LAV estimators and their uses are not restricted to PMUs placement problems. For

Fig. 4. Key steps of the proposed OPP method based on minimization of the SV.

example, if a specified SV is given then (23) and (28) can be used to find the quality and/or quantity of PMUs to satisfy the specification. Although in the examples, (23) and (28) give the same PMUs placements for the largest and smallest SV, this is not true in general. Although Monte-Carlo experiments can be used to find the variance, this is not a feasible approach in practice because of the computational load. For example in the current IEEE 30-bus system example it took a i7 8G RAM computer with Matlab 2016a, 120 seconds for 1,000 Monte-Carlo experiment runs to obtain 1 variance result compared with 0.01 second for (28). If all ten PMUs have different values of σ then there will be $3 \times 10!$ combinations. To calculate the SV for all possible placements will then take about 15,000 days for Monte-Carlo experiments but only 1 day for (28).

V. CONCLUSION

In this paper, the variance of the state estimate is included as an objective in the PMU placement problem. Therefore not only the minimum number of PMU and maximum measurement redundancy but also the minimum state estimate variance is obtained. The proposed method is able to distinguish the optimal placement among those with the same SORI, or when different measurement standard deviations are considered for

different measurements. The placement problem for PMUs with the WLS estimator and robust estimators are considered. In the current work, the formula is derived for steady state robust estimations only. Therefore, the use of the formula is restricted to static power systems [1]. This can be extended to dynamic power systems [50].

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