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Windowed Fourier ridges as a spatial carrier phase-shifting algorithm

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Abstract. The windowed Fourier ridges (WFR) algorithm is shown to be a phase-shifting algorithm for phase extraction from a carrier fringe pattern. However, the former only provides a phase estimation with a controllable phase error, whereas the latter pursues exact phase extraction. This link not only is interesting but also enhances the understanding of different phase extraction techniques. Advantages and disadvantages of the WFRs algorithm are discussed. © 2017 Society of Photo-Optical Instrumentation Engineers (SPIE) [DOI: 10.1117/1.OE.56.8.080501]

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1 Introduction

Fringe pattern analysis is critical for optical interferometric techniques and noninterferometric moiré and fringe projection techniques.^{1,2} “Pixelwise” phase shifting^{3,4} and “global” Fourier transform^{5,6} have been established as two pillar techniques, which look at the spatial and spectral properties of a fringe pattern, respectively. By considering both properties, “local” windowed Fourier transform (WFT) was developed as an “in-between” technique to process a fringe pattern patch by patch.^{7–9} WFT has been naturally considered as an extension of the Fourier transform technique—the former examines the Fourier spectrum of a patch of a fringe pattern at a time, whereas the latter examines that of the entire fringe pattern all at once. The Fourier spectrum of a patch of a fringe pattern is usually simpler for characterization, rendering WFT a more effective technique in many applications.¹⁰ However, the relationship between WFT and the phase-shifting technique remains unrevealed. In this letter, we show that WFT can also be seen as a phase-shifting algorithm. This link is useful in understanding and selecting a proper technique for effective fringe pattern analysis.

The content of this letter is structured as follows: Sec. 2 introduces a spatial carrier phase-shifting (SCPS) algorithm. Section 3 describes the windowed Fourier ridges (WFR) algorithm. Section 4 depicts WFR as a phase-shifting algorithm. Section 5 provides further discussions and finally, Sec. 6 concludes the letter.

2 Spatial Carrier Phase-Shifting Technique

The phase-shifting technique has been extensively used for fringe pattern analysis for over four decades and can be applied to temporally or spatially phase-shifted fringe patterns, or a spatial carrier fringe pattern.⁹ In the case of the temporal/spatial phase-shifting scenario, a phase-shifting algorithm is applied to extract the phase first, followed by WFT for phase enhancement such as denoising. Thus, WFT and the temporal/spatial phase-shifting techniques complement each other. However, in the case of the SCPS scenario [1, Ch. 5], since the phase can be extracted by either SCPS or WFT, a possible equivalence between them is yet to be explored.

A carrier fringe pattern can be written as

$$f(x, y) = a(x, y) + b(x, y) \cos[\varphi(x, y) + \omega_{cx}x], \quad (1)$$

where $a(x, y)$ is the background intensity; $b(x, y)$ is the fringe amplitude; $\varphi(x, y)$ is the phase field; and ω_{cx} is the global carrier frequency that applies to the entire fringe pattern and, without losing generality, is assumed to be horizontal. In order to extract the phase at any pixel (u, v) , adjacent pixels at $(u + i, v)$, $-n \leq i \leq n$ are also interrogated. These $N = 2n + 1$ pixels are further assumed to share the same background intensity, fringe amplitude, and phase, so that they are seen as phase-shifted pixels as

$$f(u + i, v) = a(u, v) + b(u, v) \cos[\phi(u, v) + \delta_i], \quad -n \leq i \leq n, \quad (2)$$

with the following respective phase and phase-shift values:

$$\phi(u, v) = \varphi(u, v) + \omega_{cx}u, \quad (3)$$

$$\delta_i = \omega_{cx}i, \quad -n \leq i \leq n. \quad (4)$$

The carrier frequency ω_{cx} has to be known either by adjusting it during the experiment or by estimating it from the fringe pattern. Subsequently, the phase can be extracted by the well-known least-squares fitting method.^{2,11}

To do so, Eq. (2) is rewritten as

$$f(u + i, v) = p_1 + p_2 \cos \delta_i + p_3 \sin \delta_i, \quad -n \leq i \leq n, \quad (5)$$

with

$$\mathbf{p} = [p_1, p_2, p_3]^T = \{a(u, v), b(u, v) \cos[\phi(u, v)], -b(u, v) \sin[\phi(u, v)]\}^T, \quad (6)$$

where the superscript T denotes vector transpose. The parameter vector $\mathbf{p}$ can be estimated in the least-squares sense by minimizing the following energy function:

$$E(\mathbf{p}) = \sum_{i=-n}^n [p_1 + p_2 \cos \delta_i + p_3 \sin \delta_i - f(u + i, v)]^2. \quad (7)$$

By letting $\partial E / \partial p_i = 0$, for $i = 1, 2, 3$, the following equations in the matrix form can be obtained as

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$$\mathbf{A}\mathbf{p} = \mathbf{B}, \quad (8)$$

where

$$\mathbf{A} = \begin{bmatrix} N & \sum_{i=-n}^n \cos \delta_i & \sum_{i=-n}^n \sin \delta_i \\ \sum_{i=-n}^n \cos \delta_i & \sum_{i=-n}^n \cos^2 \delta_i & \sum_{i=-n}^n \cos \delta_i \sin \delta_i \\ \sum_{i=-n}^n \sin \delta_i & \sum_{i=-n}^n \cos \delta_i \sin \delta_i & \sum_{i=-n}^n \sin^2 \delta_i \end{bmatrix}, \quad (9)$$

$$\mathbf{B} = \begin{bmatrix} \sum_{i=-n}^n f(u+i, v) & \sum_{i=-n}^n f(u+i, v) \cos \delta_i & \sum_{i=-n}^n f(u+i, v) \sin \delta_i \end{bmatrix}^T. \quad (10)$$

The parameter vector $\mathbf{p}$ can thus be obtained from Eq. (8) as $\mathbf{p} = \mathbf{A}^{-1}\mathbf{B}$, with which the phase can be calculated according to Eq. (6) as

$$\phi(u, v) = \tan^{-1}(-p_3/p_2). \quad (11)$$

The set of equations [Eqs. (8)–(11)] is the general form of the well-established least-squares-based phase-shifting algorithm, where the phase shifts δ_i are arbitrary known values.^{4,11}

As Eq. (4) depicts, the phase shifts are considered to be linear in our case. Thus, by using some trigonometric identities, Eq. (9) can be simplified as

$$\mathbf{A} = \begin{bmatrix} N & Nt_N(\omega_{cx}/2) & 0 \\ Nt_N(\omega_{cx}/2) & N/2 + Nt_N(\omega_{cx})/2 & 0 \\ 0 & 0 & N/2 - Nt_N(\omega_{cx})/2 \end{bmatrix}, \quad (12)$$

where $t_N(\alpha)$ is defined as

$$t_N(\alpha) = \frac{1}{N} \sum_{i=-n}^n \cos(2\alpha i) = \frac{\sin(N\alpha)}{N \sin(\alpha)}. \quad (13)$$

If we select $\omega_{cx} = 2\pi/N$,¹⁰ then $t_N(\omega_{cx}/2) = t_N(\omega_{cx}) = 0$, resulting in

$$\mathbf{A} = \begin{bmatrix} N & 0 & 0 \\ 0 & N/2 & 0 \\ 0 & 0 & N/2 \end{bmatrix}. \quad (14)$$

Obtaining the parameter vector $\mathbf{p}$ using Eq. (14) results in the extracted phase as

$$\phi(u, v) = \tan^{-1} \left[-\frac{\sum_{i=-n}^n f(u+i, v) \sin(\omega_{cx} i)}{\sum_{i=-n}^n f(u+i, v) \cos(\omega_{cx} i)} \right], \quad (15)$$

which has been a widely used phase-shifting algorithm.^{4,12}

3 Windowed Fourier Ridges Algorithm

Given a fringe pattern $f(x, y)$, its WFT is defined as

$$Sf(u, v; \xi_x, \xi_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g(x-u, y-v) \times \exp(-j\xi_x x - j\xi_y y) dx dy, \quad (16)$$

where (x, y) and (u, v) are the spatial coordinates, (ξ_x, ξ_y) is the frequency coordinate, and $g(x, y)$ is a window function. Based on the WFT spectrum $Sf(u, v; \xi_x, \xi_y)$, WFR algorithm and windowed Fourier filtering algorithm have been developed.^{7,8} The WFR algorithm is concerned in this letter and can be written as follows:

$$[\hat{\omega}_x(u, v), \hat{\omega}_y(u, v)] = \arg \max_{\xi_x, \xi_y} |Sf(u, v; \xi_x, \xi_y)|, \quad (17)$$

$$\hat{\phi}(u, v) = \angle Sf[u, v; \hat{\omega}_x(u, v), \hat{\omega}_y(u, v)] - \frac{1}{2} \arctan(\sigma_x^2 c_{xx}) - \frac{1}{2} \arctan(\sigma_y^2 \omega_{yy}), \quad (18)$$

where $\hat{\cdot}$ means estimation; ω_x and ω_y are the local frequencies along x and y directions, respectively; c_{xx} and c_{yy} are the curvatures of the phase along x and y directions, respectively; σ_x and σ_y represent window size in x and y directions, respectively; and $\angle(\cdot)$ takes the argument of a complex number. The last two terms on the right-hand side of Eq. (18) is for error compensation when the phase is nonlinear, i.e., $c_{xx} \neq 0$ and/or $c_{yy} \neq 0$.

4 Windowed Fourier Ridges Algorithm as a Spatial Carrier Phase-Shifting Algorithm

The previous two sections give a brief review of an SCPS algorithm and the WFR algorithm. In this section, we argue that the WFR algorithm can be seen as an SCPS algorithm, where the phase-shift values are searched as in Eq. (17) and the phase value is estimated by Eq. (18). To make it more explicit, we rewrite Eq. (16) with the following considerations: (i) the integration is changed to a summation, which is actually used in real computation; (ii) the window function is taken as a 1-D rectangular window centering at (u, v) with a width of $N = 2n + 1$ to consider just a few pixels in a row of fringe pattern as done in SCPS; (iii) the carrier frequency along y is ignored since only one row of fringe pattern is considered at a time. Equation (16) is then simplified to

$$Sf(u, v; \xi_x) = \sum_{i=-n}^n f(u+i, v) \exp[-j\xi_x(u+i)]. \quad (19)$$

As the phase is assumed to be constant in the phase-shifting technique, we have $c_{xx} = c_{yy} = 0$ and, consequently, the second and third terms on the right-hand side of Eq. (18) disappear. Substituting Eq. (19) into the first term on the right-hand side of Eq. (18), replacing ξ_x by $\hat{\omega}_x$ due to Eq. (17), and incorporating the relationship in Eq. (3) yields

$$\hat{\phi}(u, v) = \tan^{-1} \left[-\frac{\sum_{i=-n}^n f(u+i, v) \sin(\hat{\omega}_x i)}{\sum_{i=-n}^n f(u+i, v) \cos(\hat{\omega}_x i)} \right]. \quad (20)$$

Note that in Eq. (15), the global frequency ω_{cx} plays the role of $\hat{\omega}_x$ to deal with local pixels. Thus, Eq. (20) is almost exactly the same as Eq. (15), proving that WFR can indeed be seen an SCPS algorithm.

However, a difference does exist between Eqs. (15) and (20): the former returns $\phi(u, v)$, whereas the latter returns $\hat{\phi}(u, v)$. In other words, the SCPS algorithm provides the

true phase, whereas the WFR algorithm only gives an estimation. This is not surprising since in getting Eq. (15) we require $t_N(\omega_x/2) = t_N(\omega_x) = 0$, which is not satisfied for arbitrary phase-shift values, raising a question to the effectiveness of the WFR estimation. It is noticeable that $t_N(\alpha)$ decreases with N , and so for a large N , $t_N(\omega_x/2) \approx t_N(\omega_x) \approx 0$, rendering Eq. (20) a good estimator. The equivalence of WFR and SCPS can also be understood from the point of view of quadrature phase detection.⁴ We know that a spatial carrier fringe pattern has three spectrum components, one for phase extraction and the other two present undesired influences. Quadrature phase detection attempts to eliminate the undesired influences so as to extract the true phase value, whereas the WFR concentrates only on the fundamental component in the presence of other components, leading us to the estimation of the true phase.

It is thus necessary to analyze the error of the phase extracted by the WFR algorithm. The most straightforward examination is to find the estimated phase by substituting Eq. (2) into Eq. (20) and compare it with its nominal value. Note that in applying WFR, we usually suppress the background intensity $a(x, y)$ in advance to reduce its influence on phase extraction, and thus we assume $a(x, y) = 0$.⁹ Substituting Eq. (2) with $a(x, y) = 0$ into Eq. (20) yields the following WFR phase estimation:

$$\hat{\phi}(u, v) = \tan^{-1} \left\{ \frac{\sin[\phi(u, v)][1 - t_N(\omega_x)]}{\cos[\phi(u, v)][1 + t_N(\omega_x)]} \right\}, \quad (21)$$

where $\hat{\omega}_x$ has been replaced by ω_x because the former is a very good estimation of the latter.⁹ The phase estimation error, $\delta\phi(u, v) = \hat{\phi}(u, v) - \phi(u, v)$, can then be evaluated as follows:

$$\begin{aligned} \delta\phi(u, v) \approx \tan[\delta\phi(u, v)] &= \frac{\tan[\hat{\phi}(u, v)] - \tan[\phi(u, v)]}{1 + \tan[\hat{\phi}(u, v)] \tan[\phi(u, v)]} \\ &= \frac{-\sin(2\phi)t_N(\omega_x)}{1 + \cos(2\phi)t_N(\omega_x)}. \end{aligned} \quad (22)$$

By letting the derivative of $\delta\phi(u, v)$ be zero, the extreme points can be found at $\cos(2\phi) = -t_N(\omega_x)$ with the extreme error as $\delta\phi(u, v) = \frac{-\sin(2\phi)t_N(\omega_x)}{1 - t_N^2(\omega_x)}$, and thus we have the following maximum absolute phase error in a simple form:

$$|\delta\phi|_{\max} \approx |t_N(\omega_x)|. \quad (23)$$

This theoretical result has been verified to be consistent with our simulation results. From this result, we can easily compute $|\delta\phi|_{\max}$ with respect to N and ω_x . Figure 1(a) shows in white that $|\delta\phi|_{\max} \leq 2\pi/100$ and in black otherwise. The same is shown in Fig. 1(b) except that the threshold is changed to $2\pi/30$. It is obvious that the low WFR phase extraction error is easily achievable. In the mean time, it is also observed that a small N results in a large phase error.

5 Discussions

When WFR is considered as a phase-shifting algorithm, it cannot use a small N , which may be considered as a disadvantage. However, in practice, involving a large N , i.e., many pixels, increases the insensitivity to noise, which becomes an advantage. Another interesting point is that WFR uses the local frequency ω_x as the phase-shift value, whereas SCPS uses the global carrier frequency ω_{cx} . Thus SCPS is not suitable when the phase is locally changing. To deal with this problem, iterative algorithms have been proposed.^{13,14}

The above advantages of using WFR, together with others,⁹ are summarized and listed below

- i. the phase-shift and phase-value estimations are integrated as one WFR algorithm;
- ii. WFR does not require the phase to be locally constant;
- iii. WFR is insensitive to noise, and the estimated phase has already been denoised;
- iv. since the phase shift can be estimated with a very high accuracy, there is no phase-shift detuning error;
- v. WFR is insensitive to high-order harmonics;¹⁵
- vi. the window shape can be flexibly selected. For example, a Gaussian window is often used;
- vii. when the phase is quadratic, WFR gives an explicit way for error compensation, which is given as the second and third term in Eq. (18); and
- viii. WFR represented by Eqs. (17) and (18) is two-dimensional. It can be easily developed into one-dimensional or three-dimensional with little modification.

Fig. 1 WFR phase extraction error with respect to the phase-shifting steps and the carrier frequency. White indicates that the errors are less than (a) $2\pi/100$ and (b) $2\pi/30$.

Thus, whenever SCPS is used to process a fringe pattern, WFR can always be attempted for better results. Finally, it is worth mentioning that, WFR applying to a fringe sequence with a temporal carrier for dynamic measurement [9, Ch. 8] can be similarly linked to the temporal phase-shifting technique, although it is not emphasized in this letter.

6 Conclusions

In this letter, the WFR algorithm for fringe pattern analysis is shown to be a phase-shifting algorithm. Their difference is also exposed, where the former provides a phase estimation, whereas the latter gives the true value. Despite this difference, two seemingly completely different techniques are interestingly linked up. With this, Fourier transform, WFT, and phase-shifting technique can now be understood as a continuous spectrum in fringe pattern analysis.

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