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Coherence resonance and stochastic bifurcation behaviors of simplified standing-wave thermoacoustic systems

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In this work, noise-induced motions (i.e., external fluctuations) in two modelled standing-wave thermoacoustic systems are studied when these systems are close to the deterministic stability boundary. These systems include (1) open-open (i.e., Rijke-type) and (2) closed-open boundary conditions. It is found from the smooth transitions of the stationary probability density function that the thermoacoustic system is destabilized via stochastic P bifurcation, as the external noise intensity is continuously increased. In addition, the increased noise intensity can shift the hysteresis region, which makes the system more prone to quasi-periodic oscillations, but also reduces the hysteresis area. The noise-induced coherence motions are observed numerically in the open-open system, which is denoted by the occurrence of a bell-shaped signal to noise ratio (SNR). The SNR is shown to be applicable as a precursor. It becomes larger and the optimal noise intensity is decreased as the modelled thermoacoustic system approaches the critical bifurcation point. In addition, coherence resonance is observed in the closed-open system. To validate the findings, experimental studies are conducted on an open-open Rijke tube. Good qualitative agreements are obtained. The present study shed lights on the stochastic and coherence behaviors of the standing-wave thermoacoustic systems with different boundary conditions. © 2019 Acoustical Society of America.

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I. INTRODUCTION

Self-excited thermoacoustic oscillations occur in many engine systems such as thermoacoustic heat engines,^{1–3} industrial furnaces,⁴ gas turbines,^{5,6} and rocket motors.⁷ The phenomenon is a manifestation of feedback coupling between acoustic disturbances and heat release fluctuation.⁸ Heat release fluctuation can generate acoustic waves, and part of the reflected waves from the boundary can enter the combustion zone and enhance the unsteady heat release.^{9,10} In order to achieve a good understanding of the mechanism in the thermoacoustic system,^{11,12} predicting the process of transition to instability is significantly important. Generally, the onset of self-sustained thermoacoustic oscillations occurs via a supercritical (soft) or subcritical Hopf bifurcation,^{13,14} as critical operation parameters are varied. Typically, the state resulting from Hopf bifurcation¹⁵ in a standing-wave thermoacoustic system¹⁶ is a limit cycle at a frequency close to one of the acoustic resonance frequencies of the system. However, more complicated dynamics such as quasi-periodic, chaotic, or intermittent states can also appear at the onset, due to subsequent bifurcations of the limit cycle.¹⁷ Transition of a thermoacoustic system from steady operation (i.e., no coherent oscillations) to limit cycle oscillations is detrimental to

practical power generation and propulsion systems^{5,18} but desirable to thermoacoustic engines or cooling systems.¹⁹

Flow fluctuations²⁰ originating from aerodynamic noise,²¹ turbulence, and random modulations in the operating parameters are typically associated with heat release process. Noise therefore, is inevitably present in thermoacoustic systems. To a certain extent, noise can play a beneficial role. For example, Boujo *et al.*²² confirmed their proposed analytical model and approaches, by using the stochasticity and comparing the prediction with some measured data. Noiray and Schuermans²³ make use of the noise perturbed oscillation data to obtain the linear growth rate in the supercritical bifurcation case. Compared with the effect of noise in supercritical thermoacoustic systems, it plays a significantly important role, when the thermoacoustic system is in the hysteresis loop of subcritical bifurcation. First, large-amplitude stochastic noise can trigger self-sustained thermoacoustic oscillations and lead to a transition to instability, even when the system is linearly stable.^{24–26} Second, increasing noise intensity can decrease the hysteresis width.²⁷ Last, noise can induce two interesting physics phenomena, such as coherence motions and stochastic bifurcations in thermoacoustic systems, which has been observed in the experimental measurements.^{28,29}

Coherence resonance (CR)^{30,31} is known to incur noise-induced enhancement of deterministic dynamics in some nonlinear systems. The dynamic response of the nonlinear

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system shows a resonance-like behavior: an optimum response results from a particular intermediate level of noise intensity. CR is a variant of a similar phenomenon known as stochastic resonance (SR),^{32,33} where an external weak deterministic signal can be enhanced by the presence of the noise. SR and similar counterintuitive effects of the noise have been investigated in both experiments and numerical studies in a variety of nonlinear systems such as biological and chemical systems, lasers, electronic circuits, and even geological studies. Characterizing instability in a stochastic dynamic system is challenging. There are mainly two different ways³⁴ to classify the stochastic bifurcation types. One is related with stochastic D-bifurcation, and the other is with stochastic P-bifurcation. Stochastic D-bifurcation is characterized by the sign change of the maximum Lyapunov exponent, while stochastic P bifurcation is characterized by a qualitative change in the probability distribution of the system response.

Stochastic dynamic behaviors of a thermoacoustic system has been recently studied on a gas turbine combustor.²⁸ Such dynamics theory is used to explain random transitions from deterministic thermoacoustic modes to longitudinal instability. A stochastic model of an annular thermoacoustic combustor was derived in previous work²³ to study the probabilities of two azimuthal wave components as a function of the system asymmetry, random source intensity, and thermoacoustic interaction strength. On a lab-scale Rijke-type thermoacoustic system, Zalluhoglu *et al.*¹⁰ experimentally studied the secondary dynamic mode being triggered during the mitigation of thermoacoustic instability via a Helmholtz resonator. There are few investigations^{13,23,28,29} focusing on the role of random noise just before the bistable region for a given critical parameter such as the axial location of the heat source or total heat release rate. This partially motivated the present work.

In this work, a noisy nonlinear thermoacoustic model is proposed to study the effects of white Gaussian noise on the dynamic behaviors of standing-wave thermoacoustic systems. Two different configurations of standing-wave thermoacoustic systems are considered. One is an open-open (Rijke-type) thermoacoustic system^{35,36} and the other is a Sondhauss-like system with closed-open boundary conditions.³⁷ These modelled systems and governing equations are described in Sec. II. The Rijke-type thermoacoustic system is studied first, as described in Sec. III A. The stability of the stochastic thermoacoustic system is examined analytically, when the system is near the subcritical Hopf bifurcation point. The noise-induced coherence motion in the Rijke-type thermoacoustic system is then numerically studied, as the noise intensity is varied. After evaluating the Rijke-type thermoacoustic system, the noise-induced stochastic stability and coherence motions in the Sondhauss-type thermoacoustic system are investigated. This is described in Sec. III B. Experimental studies are then conducted on an open-open thermoacoustic system to validate the theoretical findings and to shed light on noise-induced stochastic behaviors. This is described in Sec. IV. By comparing the effects of the noisy fluctuations on the open- and closed-open modelled thermoacoustic systems, certain common features are obtained and summarized in Sec. V.

II. DESCRIPTION OF THEORETICAL MODELS

Two different physical configurations of simplified standing-wave thermoacoustic systems are considered, as shown schematically in Fig. 1. One is a Rijke-type system with both inlet and outlet acoustically open as shown in Fig. 1(a). The other is a Sondhauss-like system as shown in Fig. 1(b), in which one end is acoustically closed and the other is acoustically open. L denotes the total length of the tube and it is set to 1.0 m. $\tilde{x}_f$ is the axial location of the heat source.

The dimensional momentum and energy equations for the acoustic perturbations in the simplified 1D thermoacoustic system are given as⁵

$$\bar{\rho} \frac{\partial \tilde{u}'}{\partial t} + \frac{\partial \tilde{p}'}{\partial \tilde{x}} = 0, \quad (1)$$

$$\frac{\partial \tilde{p}'}{\partial t} + \gamma \bar{p} \frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1) \tilde{Q}'(x, t) \delta(\tilde{x} - \tilde{x}_f), \quad (2)$$

where the mean density, mean pressure, and the specific heat ratio, $\bar{\rho} = 1.2 \text{ kg/m}^3$, $\bar{p} = 1.01325 \times 10^5 \text{ Pa}$, and $\gamma = 1.4$, respectively, are given. $\tilde{p}'$ and $\tilde{u}'$ denote the acoustic pressure and the acoustic velocity. $\tilde{x}$ denotes the axial distance from the upstream end. $\delta(\cdot)$ is the Dirac delta function. $\delta(z) = +\infty$, when $z = 0$, or 0 when $z \neq 0$, and $\int_{-\infty}^{+\infty} \delta(z) dz = 1.0$. $\tilde{Q}'$ denotes unsteady heat release rate from the heat source. Here, it is assumed that the effects of the mean flow and mean temperature gradient in the tube are negligible. As confirmed by the experimental measurements on lab-scale Rijke tube, the mean flow due to the small fuel (hydrocarbon) flow rate and convection effect is typically much smaller than the speed of sound. Neglecting the mean flow velocity is thus reasonable.³⁸ These assumptions have been widely applied in theoretical models of lab-scale Rijke-type thermoacoustic systems.³⁹

In order to make our analysis to be more generalized and to predict stochastic resonance behaviors,^{40–42} Eqs. (1) and (2) are converted in a dimensionless form by introducing the reference scales as given,

$$\tilde{u}' = \bar{u} u', \quad \tilde{p}' = \bar{p} p', \quad \tilde{Q}' = \bar{Q} Q', \quad \tilde{x} = Lx,$$

FIG. 1. (Color online) Schematics of simplified 1D thermoacoustic systems: (a) open-open (Rijke-like), (b) closed-open (Sondhauss-like).

$$\tilde{t} = (L/\bar{c})t, \quad \delta(\tilde{x} - \tilde{x}_f) = \delta(x - x_f)/L, \quad (3)$$

where $\bar{c}$ denotes the sound speed. $\bar{u}$ and $\bar{Q}$ denote the mean flow velocity and mean heat release rate. Following the previous work,⁴³ these parameters are assumed to be $\bar{c} = 400.0$ m/s, $\bar{u} = 0.3$ m/s, and $\bar{Q} = K \bar{p} \bar{c}$, (W), where K is the mean heat release coefficient.

Differentiating non-dimensional Eq. (1) with respect to x and non-dimensional Eq. (2) with respect to time t , and eliminating the cross terms (i.e., the products of differentials) lead to

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = (\gamma - 1)K \frac{\partial Q'}{\partial t} \delta(\tilde{x} - \tilde{x}_f), \quad (4)$$

where $K \equiv \bar{Q}/\bar{p}\bar{c}$ and it is assumed to denote a dimensionless heating power. Galerkin series expansion^{26,36,38} are used to express the acoustic disturbances here, so that the behaviors of the standing-wave thermoacoustic system can be studied using nonlinear dynamic theory. Here, it is assumed that the unsteady pressure oscillations consist of a single dominant acoustic mode.^{24,43} This assumption is justified by several experimental^{36,43} and theoretical investigations.^{44–46} The dimensionless acoustic pressure is then given by $p'(x, t) = \varphi(kx)\eta(t)$. $\eta(t)$ denotes the time-dependent variable. $\varphi(kx)$ is k th acoustic mode function and its specific form is determined by the boundary condition.

Substituting the Galerkin expansion form of pressure perturbations into Eq. (4) leads to

$$(\ddot{\eta} + k^2\eta)\varphi(kx) = (\gamma - 1)K \frac{\partial Q'}{\partial t} \delta(x - x_f), \quad (5)$$

where overdots denote the second time derivative. It is worth noting that the basis function $\varphi(kx)$ satisfies the orthogonality condition, i.e.,

$$\int_0^1 \varphi(kx)\varphi(mx)dx = \begin{cases} 0, & k \neq m, \\ 1/2 & k = m. \end{cases} \quad (6)$$

Note that the integral lower and upper limit in Eq. (6) are 0 and 1, since x is non-dimensional (normalized with the combustor length L).

Multiplying Eq. (5) with $\varphi(mx)$ and integrating the term from $x = 0$ to $x = 1$ gives

$$\ddot{\eta} + k^2\eta = 2K(\gamma - 1)\dot{Q}'_f\varphi(kx_f). \quad (7)$$

Here, m/π is a non-zero integer.

Due to heat conduction and viscosity in the acoustic boundary layer and sound radiation at both ends of the standing-wave tube,²⁶ the acoustic loss/damping effect needs to be considered.⁴³ Thus Eq. (7) can be revised as

$$\ddot{\eta} + 2\xi k\dot{\eta} + k^2\eta = 2K(\gamma - 1)\dot{Q}'_f\varphi(kx_f), \quad (8)$$

where ξ denotes the overall damping coefficient.²⁶

Considering the fact that only a small amount of the total heat release is used to generate acoustic oscillations,⁸ and the damping effect is limited, a small parameter ϵ is introduced into the two forcing terms, as given by

$$\ddot{\eta} + 2\epsilon\xi k\dot{\eta} + k^2\eta = 2\epsilon K(\gamma - 1)\dot{Q}'_f\varphi(kx_f). \quad (9)$$

For simplicity, ϵ is set to be the same for both the damping and the unsteady heat release terms. ϵ involved in the right-hand side of Eq. (9) denotes the monopole-like efficient heat-to-sound conversion.^{5,8} In physics, it means that only a very small part of total heat release is used to generate sound. ϵ involved in the damping term characterizes acoustic losses²² and the irreversibility of the heat-to-sound conversion, which is quite small in nature. This justifies why ϵ is applied to both sides of Eq. (8).

Following Ref. 46, a fifth-order polynomial model is used to characterize the unsteady heat release rate as

$$Q' = b_1p' + b_2p'^3 + b_3p'^5, \quad (10)$$

where the coefficients b_1 , b_2 , and b_3 are constant. A proper selection of these coefficients can capture unsteady heat release response to acoustic disturbances in a qualitative way: the unsteady heat release will grow linearly with the increase of the acoustic forcing initially and then saturate at large-amplitude acoustic forcing.^{29,47}

Substituting Eq. (10) into Eq. (9) and introducing external noise $N(t)$ (Ref. 48) lead to

$$\ddot{\eta} + 2\epsilon\dot{\eta}(\alpha_1 + \alpha_2\eta^2 + \alpha_3\eta^4) + k^2\eta = N(t), \quad (11)$$

where $\alpha_1 = \xi k - b_1K(\gamma - 1)\varphi^2(kx_f)$, $\alpha_2 = -3b_2K(\gamma - 1)\varphi^4(kx_f)$, $\alpha_3 = -5b_3K(\gamma - 1)\varphi^6(kx_f)$. $N(t)$ denotes Gaussian white noise. It has the following properties: $\langle N(t) \rangle = 0$, $\langle N(t)N(t + \tau) \rangle = D\delta(\tau)$. $\langle . \rangle$ is the expectation operator, and D is the noise intensity. By applying stochastic averaging^{28,40} to approximate the nonlinear stochastic system, a pair of slowly varying process is needed. The following transformations are implemented:

$$\eta(t) = A(t)\cos\theta, \quad \dot{\eta}(t) = -kA(t)\sin\theta, \quad \theta = kt + \phi(t), \quad (12)$$

where $A(t)$ denotes the amplitude of the system response and $\phi(t)$ denotes the phase, where both of the two variables change slowly, because ϵ is small. Equation (11) can be reformulated into the following pair of ordinary differential equations (ODEs) in terms of A and ϕ :

$$\begin{aligned} \dot{A} = & -2\epsilon A \sin^2\theta(\alpha_1 + \alpha_2 A^2 \cos^2\theta + \alpha_3 A^4 \cos^4\theta) \\ & - \frac{\sin\theta}{k}N(t), \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\phi} = & -2\epsilon \sin\theta \cos\theta(\alpha_1 + \alpha_2 A^2 \cos^2\theta + \alpha_3 A^4 \cos^4\theta) \\ & - \frac{\cos\theta}{Ak}N(t). \end{aligned} \quad (14)$$

Due to the fact that ϵ is small, (A, ϕ) can be approximated as a joint Markov processes.⁴⁹ Using stochastic averaging,⁵⁰ the corresponding Ito equation⁵¹ can be obtained as

$$dA = -\left(\epsilon\alpha_1 A + \frac{\epsilon\alpha_2 A^3}{4} + \frac{\epsilon\alpha_3 A^5}{8}\right)dt + \sqrt{\frac{D}{2k^2}}dW_1, \quad (15)$$

$$d\phi = \sqrt{\frac{D}{2k^2A^2}}dW_2, \quad (16)$$

where dW_1 and dW_2 denote two independent Wiener process (Brownian motion).⁵² It should be noted that the amplitude $A(t)$ does not depend on ϕ in Eq. (15), so an 1D probability density for A is developed for convenience, rather than a joint density for A and ϕ , when $D \neq 0$. The transition density function $P(A, t)$ of the 1D Markov process A is governed by the following Fokker-Planck-Kolmogorov (FPK) equation:⁵¹

$$\frac{\partial P(A, t)}{\partial t} = \frac{\partial}{\partial A} \left\{ \left(\varepsilon \alpha_1 A + \frac{\varepsilon \alpha_2 A^3}{4} + \frac{\varepsilon \alpha_3 A^5}{8} - \frac{D}{4k^2 A} \right) P(A, t) \right\} + \frac{D}{4k^2} \frac{\partial^2 P(A, t)}{\partial A^2}. \quad (17)$$

The stationary density function $P_s(A)$ can be obtained by setting $\partial P(A, t)/\partial t = 0$, as given by

$$P_s(A) = \mathcal{N} \exp \{ \varepsilon k^2 A^2 (\alpha_3 A^4 + 3\alpha_2 A^2 + 24\alpha_1) / 12D \}, \quad (18)$$

where $\mathcal{N}$ is a normalization constant, so that $\int_{-\infty}^{\infty} P_s(A) dA = 1$. In addition, the peaks of $P_s(A)$ are determined by the roots of the following equation:

$$f(A_e) = \alpha_3 A_e^6 + 2\alpha_2 A_e^4 + 8\alpha_1 A_e^2 - \frac{2D}{\varepsilon k^2} = 0, \quad (19)$$

where A_e is the amplitude corresponding to the extrema of the amplitude density distribution.

III. RESULTS AND DISCUSSION

A. Open-open standing-wave thermoacoustic system

The dynamic behavior of the open-open (Rijke-type) thermoacoustic system are evaluated and discussed first. Figure 2 shows the bifurcation diagram, which is calculated from the modelled Rijke-type thermoacoustic system, in terms of the mean heat release coefficient K and the amplitude A . It can be seen that in the absence of stochasticity, i.e., the noise intensity $D = 0$, there is a fold bifurcation and an Andronov-Hopf bifurcation.⁵³ Furthermore, the bi-stable behavior is clearly observed in the shaded region, where a stable steady state and a stable limit cycle coexist.

Generally, the variance $\sigma^2(A)$ is used to measure the intensity of unsteady flow or acoustic disturbances. In order to describe the transition process more clearly, the variance of the system response $\sigma^2(A)$ is introduced in the stationary probability density distribution $P_s(A)$ by considering their product⁵⁴ as shown in Fig. 3. It can be seen that $P_s(A)\sigma^2(A)$ has only one local maximum in the vicinity of zero for a small intensified noise (e.g., $D = 5 \times 10^{-3}$). In this case, the nonlinear effect can be neglected. When the noise intensity is increased (e.g., $D = 1.2 \times 10^{-2}$), bimodal distribution characterized with two peaks can be clearly observed. A transition from the unimodal to the bimodal distributions occurs. This indicates that the thermoacoustic system

FIG. 2. (Color online) Calculated deterministic bifurcation diagram in the phase plane of (K, A) for the simplified 1D Rijke-type thermoacoustic system, as $b_1 = 1$, $b_2 = 1.2$, $b_3 = -3$, $k = \pi$, $x_f = 1/4$, $D = 0$. s-s: steady state, l-c: limit cycle.

oscillates between two local regions. In this bimodal regime, the nonlinearity of the thermoacoustic system becomes increasingly important.

Further increasing the noise intensity (e.g., $D = 0.03$) leads to the thermoacoustic system being “locked” in a large amplitude limit cycle. Moreover, the peak at the local small amplitude disappears. Now the amplitude distribution becomes unimodal again. It is found that the stochastic P-bifurcation occurs twice, as the noise intensity D is increased. For comparison, numerical results obtained by integrating Eq. (11) via Heun scheme⁵² are also illustrated in Fig. 3, as denoted by the black dots on the surface plot. Generally, Eqs. (15) and (16) cannot completely capture the properties of the thermoacoustic system governed by Eq. (11) in the presence of a large intensified noise. However, in the present work, the amplitude distributions predicted by using Eqs. (18) and (11) agree well as shown in Fig. 3. This reasonable agreement confirms that the analytical model is applicable to study the stochastic dynamics of the simplified open-open (Rijke-type) thermoacoustic system in the given noise range of interest.

Considering the significant variation in the probability density distribution induced by the Gaussian white noise, the noise intensity D is taken as a bifurcation parameter.^{40,46} The stochastic bifurcation diagram of the noisy thermoacoustic system is shown in Fig. 4. It reveals that there is a similar trend as the experimental measurements.²⁹ Note that the graph is obtained by evaluating the dependence of the number of the roots of Eq. (19) on the mean heat release coefficient K and the noise intensity D .

Curve l_1 denotes the critical boundary about the presence of the peak of $P_s(A)$ with a larger amplitude A . Below l_1 , the peak with a large amplitude A does not exist. Curve l_2 corresponds to the critical curve for the other peak with a smaller amplitude A . Above l_2 , the peak with a small amplitude A is disappeared. The shaded region in red color represents the bimodal probability distribution and the outside region denotes a unimodal distribution. Apparently, the

FIG. 3. (Color online) Stationary probability distribution $P_s(A)\sigma^2(A)$ varied with the amplitude A and the noise intensity D , as $K=2.475$, $b_1=1$, $b_2=1.2$, $b_3=-3$, $k=\pi$, $x_f=1/4$.

transition either from the outside region to the shaded region or in the opposite direction indicates that stochastic P-bifurcation occurs. The vertical dashed lines stand for the bistable boundary for the deterministic oscillator. It can be seen from Fig. 4 that the bimodal region for $D \neq 0$ and bistable region for $D=0$ do not overlap. Furthermore, the bimodal region shifts to a decreased mean heat release coefficient K and becomes narrower with increased noise intensity D . Further increasing D (e.g., $D > 0.02$) leads to the stochastic P bifurcation being unobserved for any given K .

FIG. 4. (Color online) Stochastic bifurcation diagram in the phase plane of (K, D) for the simplified 1D Rijke-type thermoacoustic system, as $b_1=1$, $b_2=1.2$, $b_3=-3$, $k=\pi$, $x_f=1/4$.

Thus, the bimodal region does not appear anymore. This reveals that adding “moderately” intensified noise would expand the unstable regime and leads to the thermoacoustic system being more prone to self-sustained instability.

With the previous physical findings obtained, the coherence motions in the Rijke-type thermoacoustic system are studied. The power spectrum density of acoustic pressure is applied to study the noise effect in frequency domain. It has been known^{31,55} that noise excitation of a self-sustained oscillator can lead to the well-known effect of washing the spectral line out⁵⁶ if the system is far from the critical bifurcation point. Therefore, the thermoacoustic system near the onset of the subcritical bifurcation is the only focus. Figure 5(a) shows the power spectrum of acoustic pressure at three different intensified noises, as $K=2.475$. It can be seen that there is a small-amplitude peak as $D=0$, where the thermoacoustic system is deterministic. If the noise intensity is increased further (i.e., $D=0.012$), the maximum power spectrum is increased, i.e., the amplitude of the peak is increased and the peak becomes more pronounced. This indicates that strong coherent motion is triggered by the added white Gaussian noise. Further increasing noise to $D=1$ gives rise to the peak being so flattened (i.e., lower gradient) that it is difficult to distinguish the peak from the background noise. In such conditions, the thermoacoustic system is dominated by the noise only.

As an alternative measure to quantify the noise-induced response, signal-to-noise ratio (SNR) is introduced. It is defined as³⁰

$$SNR = HQ = H\omega_p / \Delta\omega, \quad (20)$$

FIG. 5. (Color online) The simplified 1D Rijke-type thermoacoustic system with $b_1 = 1$, $b_2 = 1.2$, $b_3 = -3$, $k = \pi$, $x_f = 1/4$: (a) Power spectrum density of the normalized acoustic pressure with respect to the mean pressure $\bar{p}$ at three different noise intensities D , as $K = 2.475$; (b) power spectrum density, as $K = 2.475$, $D = 0.01$; (c) a bell-shaped SNR, as $K = 2.475$; (d) the role of precursor by SNR, as the mean heat release coefficient K is set to three different values.

where H denotes the peak value of the power spectrum, as illustrated in Fig. 5(b). ω_p is the corresponding peak frequency. $\Delta\omega$ is the width of the peak at the height $h_1 = e^{-1/2}H$. Thus, $\omega_p/\Delta\omega$ corresponds to the relative width of the peak. It is often used as a quality factor Q . Figure 5(c) shows SNR of the Rijke-type thermoacoustic system with respect to the noise intensity D . It can be seen that SNR is maximized, when the noise involves an “optimum” intensity D . A resonance-like behavior can be clearly observed. This reveals that coherence resonance occurs. Figure 5(d) shows the variation of SNR with respect to the noise intensity, as the mean heat release coefficient K is set to three different values. Comparing SNR at different heating powers reveals that the optimal noise intensity is decreased and the maximum SNR is increased, as the mean heat release coefficient K is approaching the critical bifurcation point $K = 2.495$. This is consistent with the experimental finding as reported in Ref. 28.

In order to shed more light on the noise-induced motions of the Rijke-type system, the frequency width $\Delta\omega$ and the peak H are also investigated. This is illustrated in Fig. 6. It can be seen that a distinguished hysteresis zone is present as shown in Fig. 6(a). When the noise is present, SNR obtained from the modeled Rijke-type thermoacoustic system [see

Fig. 6(b)] is shaped like a bell. Figure 6(c) shows that the width of the power spectrum $\Delta\omega$ is decreased first and then increased, as the noise intensity is increased. Moreover, there is a minimum $\Delta\omega$. However, the height of the power spectrum H exhibits a distinguished peak with increased D as shown in Fig. 6(d).

With the subcritical transition of the modeled standing-wave thermoacoustic system validated, the stochastic properties of the Rijke-type thermoacoustic system with a supercritical bifurcation are then numerically studied. The bifurcation diagram of the modeled system with respect to the axial location of the heat source x_f is shown in Fig. 7(a). Figure 7(b) shows that coherence resonance can also occur, when the transition to instability is supercritical. Different from the case with the subcritical bifurcation, the width $\Delta\omega$ and the peak height H are found to increase monotonically with the increased noise intensity D , as illustrated in Figs. 7(c) and 7(d). In addition, the height H is found to “saturate” at a more intensified noise. Comparing Figs. 6(a)–6(d) and Figs. 7(a)–7(d) reveals that the noise-induced coherence in the Rijke-type thermoacoustic system may involve obviously different behaviors, depending on whether the system loses its stability via a subcritical or supercritical bifurcation.

FIG. 6. (Color online) (a)–(d) Numerical prediction for the simplified 1D Rijke tube, as $b_1 = 1$, $b_2 = 1.2$, $b_3 = -3$, $k = \pi$, $K = 2.75$: (a) the bifurcation diagram in term of the heat location x_f , as $D = 0$, s-s: steady state, l-c: limit cycle; (b)–(d) SNR, the frequency width $\Delta\omega$, and the peak height of power spectrum H with respect to D , as $x_f = 0.23$.

B. Closed-open standing-wave thermoacoustic system

Now let us consider the effects of white Gaussian noise on the closed-open (Sondhauss-like) thermoacoustic system with one end closed and the other open. Note that such closed-open boundary conditions are widely applied to design standing-wave thermoacoustic heat engines.^{1,57} In this physical configuration, the developed model can also be applied by using the basis function $\varphi(kx) = \cos(kx)$. Figure 8(a) shows that a sub-critical bifurcation occurs in the simplified 1D closed-open (Sondhauss-like) thermoacoustic system, as the axial location of the heat source x_f is varied. Different from the open-open (Rijke-type) thermoacoustic system, the stability regime in the Sondhauss-like system is concentrated at the central zone in the axial direction. If the heat source is placed near the closed end, thermoacoustic oscillations are more prone to occur.

In the presence of stochasticity, the stationary probability density distribution $P_s(A)\sigma^2(A)$ is given by the contour plot of Fig. 8(b). It can be seen that the first transition from the unimodal mode to bimodal mode occurs because the noise intensity is small. Further increasing noise intensity leads to the bimodal mode being reverted back to a unimodal mode, although the new unimodal mode no longer corresponds to steady state, but a large-amplitude limit cycle. The two transitions denote the occurrence of stochastic P-bifurcations. Figure 8(c) illustrates that the shaded bimodal zone becomes narrower, as the noise intensity is increased. Simultaneously, the bimodal zone is found to be shifted towards the right half plane. This indicates the enlargement of instability margin. Finally, a resonance-like coherence resonance is also observed in such Sondhauss-like (closed-open) thermoacoustic system, as shown in Fig. 8(d).

IV. EXPERIMENTAL STUDIES

In order to validate the theoretical findings obtained in the open-open thermoacoustic systems, experimental studies are conducted on a Rijke tube^{38,39} with three loudspeakers implemented. The experimental setup is shown in Fig. 9. The Rijke tube is made of silica glass. Both ends are acoustically open. The overall axial length of the tube is 750 mm and the inner diameter is 50 mm. A Bunsen burner with a diameter of 30 mm and a height of 250 mm are applied to provide a premixed propane-fueled flame. It is enclosed in the bottom half of the Rijke tube (see Fig. 9). The burner is anchored on a workbench. The fuel flow rate ϕ_f is set to 250 ml/min. The loudspeakers (Vintage Altec Lansing 414-16E, 12 in.) are connected to a power amplifier (CROWN D-75A) and driven by a digital function generator (SONY Tektronix AFG320) to provide white Gaussian noise. The acoustic pressure is measured by using a B& K microphone (TYPE 4958). Data are acquired with a sampling rate of 10 000 samples/s and logged by National Instruments (NI-PXIE1062Q chassis, PXI4498 card). The pressure sensor is placed at 35 cm away from the bottom open end of the Rijke tube. A semi-infinite technique^{7,36} is used to obtain thermal insulation without distortion from acoustic reflections. The microphone was calibrated with a pistonphone (B&K Pistonphone 4228).

Figure 10 summarizes the noise-induced nonlinear dynamics of the Rijke tube tested in the laboratory. Figure 10(a) shows the experimentally measured bifurcation diagram with the flame location x_f normalized with the total length L of the tube. It can be seen that a distinguished hysteresis zone is present. Such hysteresis^{23,24} is also predicted by the model as shown in Fig. 6(a). It is worth noting that

FIG. 7. (Color online) Numerical predictions for the simplified 1D Rijke tube, as $b_1 = 2$, $b_2 = -2.2$, $b_3 = 0$, $k = \pi$, $K = 2.75$: (a) the bifurcation diagram in terms of the heater location x_f , as $D = 0$; (b)–(d) SNR, $\Delta\omega$, and H with respect to D , as $x_f = 0.15$.

the experimental critical point for transition to instability^{58,59} does not overlap the model prediction. This may be due to the differences between the model and experiment. For example, natural convection and higher temperature gradients are present in the tested Rijke rig, but not modelled. When Gaussian noise is supplied, a bell-shaped SNR is observed in Fig. 10(b). Similar behavior is also found in Fig. 6(b). Figure 10(c) illustrates that the width of the power spectrum $\Delta\omega$ is decreased first and then increased, as the noise intensity is increased. Furthermore, there is a minimum $\Delta\omega$. However, the height of H exhibits a distinguished peak with increased D . These reasonable agreements confirm the theoretical findings for the noise-induced nonlinear dynamics behaviors of the simplified Rijke-type standing-wave system.^{26,48}

V. DISCUSSION AND CONCLUSIONS

In this work, the effects of noisy flow fluctuations (i.e., unsteady flow characterized with Gaussian disturbances) on self-excited nonlinear standing-wave thermoacoustic systems were systematically investigated. Two different physical configurations of simplified 1D standing-wave thermoacoustic systems are considered. One is the Rijke-

type thermoacoustic system with both ends acoustically open; the other is closed-open standing-wave system. For this, a noisy standing-wave thermoacoustic model is developed. As the external noise is considered, it is found that the stochastic behaviors of the thermoacoustic system can be better described by stochastic P-bifurcation, as the external noise intensity is varied. The transitions from the unimodal mode to the bimodal mode are clearly observed. If the noise intensity is chosen as a bifurcation parameter (it refers to a parameter whose value change leads to the occurrence of a bifurcation), the noise is found to shift the hysteresis region and reduce the hysteresis area. Thus, the external white Gaussian noise leads to the thermoacoustic system being more prone to quasi-periodic oscillations. Meanwhile, the noise-induced coherence motions are also examined. Bell-shaped SNR is clearly observed in both types of thermoacoustic systems. This indicates that coherence resonances occur in these standing-wave thermoacoustic systems. In addition, SNR is found to become larger and the optimal noise intensity is decreased to a smaller value, when the system approaches the critical bifurcation point. This property can be used as a precursor to the Hopf bifurcation in standing-wave thermoacoustic systems. Furthermore, the

FIG. 8. (Color online) The simplified 1D closed-open (Sondhauss-like) standing-wave thermoacoustic system: (a) the deterministic bifurcation diagram in the phase plane of (x_f, A) , s-s: steady state, l-c: limit cycle; (b) the calculated contour of $P_s(A)\sigma^2(A)$ in the plane of (A, D) ; (c) stochastic P-bifurcation diagram in the phase plane of (x_f, D) ; (d) variation of SNR with noise intensity D , as $b_1 = 1$, $b_2 = 1.2$, $b_3 = -3$, $k = \pi/2$, $x_f = 1/4$.

FIG. 9. (Color online) Schematic of the experimental setup of the Rijke tube with a Bunsen burner confined and three loudspeakers implemented.

noise-induced motion is also examined in the modeled open-open thermoacoustic systems with a supercritical bifurcation.

To validate the theoretical findings obtained in the open-open systems, experimental tests are conducted on a lab-scale Rijke tube with loudspeakers implemented. Transition to instability is found to be subcritical. Comparing with the theoretical results reveals that good qualitative agreements are obtained in terms of SNR, the peak height of power spectrum H and the width of the frequency $\Delta\omega$. The present results show that the characteristics of coherence resonance in standing-wave thermoacoustic systems depend on whether the system loses stability via a subcritical or supercritical bifurcation. In general, the key findings from the two simplified systems are summarized as

- Effect of external noisy disturbances is studied in Rijke- and Sondhauss-type thermoacoustic systems.
- Coherence resonance and stochastic bifurcations characteristics are identified.
- Experimental tests are conducted on a Rijke-type thermoacoustic system for comparison.
- Some common features between Rijke- and Sondhauss-type systems are identified.

FIG. 10. (Color online) Experimental measurement on a lab-scale Rijke tube, as the fuel flow rate ϕ_f is set to 250 ml/min, (a) measured bifurcation diagram in terms of x_f/L , as $D=0$, (b)–(d) variation of SNR, $\Delta\omega$, and H with the noise intensity D , as $x_f=0.06$.

- SNR is identified as a precursor for self-excited thermoacoustic oscillations.

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