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Alternative extrapolation-based symmetry boundary implementations for the axisymmetric lattice Boltzmann method

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In this study, alternative symmetry boundary implementations for the axisymmetric lattice Boltzmann (LB) method are proposed based on the nonequilibrium extrapolation and the direct extrapolation schemes. The proposed boundary schemes are directly implemented on the symmetry axis, and the postcollision distribution function and the macroscopic variables at the boundary nodes are extrapolated from the inner fluid nodes; thereby, the singularities arising at the symmetry axis ($r = 0$) during the collision and the macroscopic variable calculations are completely avoided. The accuracy of the present schemes is consistent with the well-established axisymmetric LB model. Moreover, in comparison with previous symmetry boundary schemes, the present implementations are slightly more accurate than the symmetry scheme by Guo *et al.* [Phys. Rev. E **79**, 046708 (2009)] and numerically more stable than the specular reflection-based schemes.

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I. INTRODUCTION

The axisymmetric computational fluid dynamics (CFD) models enable the reduction of the flows in the three-dimensional (3D) cylindrical system into the two-dimensional (2D) pseudo-Cartesian problems in the meridian plane, and thus have the advantage of significantly diminishing the computational cost and bypassing the complicated curve boundary treatments. As a promising alternative to the conventional CFD methods, the axisymmetric lattice Boltzmann (LB) method has advanced significantly in terms of model improvements and practical applications [1–25]. By inserting two source terms in the LB equation accounting for the axisymmetric mass and momentum conservation, the first axisymmetric LB model was proposed by Halliday *et al.* [25] and then further improved by Lee *et al.* [4] and Reis and Phillips [26] in accordance with the accurate axisymmetric Navier-Stokes (NS) equations. Despite the widespread applications of the Halliday *et al.* model and the associated improved models [2–5,25–28], these earlier axisymmetric LB models suffer from complicated source terms, specifically in that the second-order source term for the momentum conservation contains more than ten terms in the improved models [6]. Moreover, the improper forcing scheme adopted in these earlier models led to additional complex differential terms due to the discrete effects on the first-order source term discretization [9]. For the purpose of deriving more efficient axisymmetric LB models, appropriate forcing schemes free of the discrete lattice effects, such as the trapezium rule [9,29–31] (i.e., the implicit centered scheme [6,13]), are necessary for the discretization of the source terms. Furthermore, many efforts have been devoted to avoid the spatial velocity gradient calculations in the source term. Additional relaxation terms were innovatively introduced in

the LB equation by Li *et al.* [8,9], and thus the velocity gradients in the source terms were represented by the moments of the nonequilibrium (NE) part of the distribution. Efficient axisymmetric LB models were subsequently developed with the idea by Li *et al.* [9,13,16,19], whereby the source terms contain no velocity gradients and are much simpler than the earlier models. Summarily, the axisymmetric LB models discussed above were developed by directly adding the source terms to the standard LB equation, and thus the axisymmetric effects were realized at the macroscopic level. Alternatively, axisymmetric LB models were derived from the axisymmetric Boltzmann equation by Guo *et al.* [1], and the axisymmetric effects were represented at the distribution function level by introducing special distribution function definitions. The Guo *et al.* model [1] also has the advantage that the source term is simple and contains no velocity gradients, such that all the velocity components are described in a consistent fashion. The validation of these two kinds of efficient axisymmetric LB models has been well demonstrated, along with widespread applications [1,9,11,13,14,17,19,21,22,32].

In practical applications, attention should also be paid to the implementation of the symmetry boundary condition since the accuracy and stability of the axisymmetric CFD models are inevitably affected by the adopted boundary schemes. Notably, singularity arising at the symmetry axis $r = 0$ due to the terms containing r^{-1} should be avoided, where r is the radial coordinate in the meridian plane. In particular, the mathematical description of the symmetry boundary condition is given as

$$\left. \frac{\partial \varphi}{\partial r} \right|_{r=0} = 0, \quad \forall \varphi, \quad (1)$$

where φ is a macroscopic flow variable. Equation (1) is essentially a general Neumann boundary condition [1,16]. As for the axisymmetric LB simulations, the singularity at $r = 0$ can be avoided by locating the symmetry axis outside the

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FIG. 1. Schematic diagram for the symmetry boundary condition implementations: (a) for the schemes locating the symmetry axis outside the computational domain (i.e., $0 < r < R$), namely, the symmetry scheme by Guo *et al.* and the midgrid specular reflection scheme; (b) for schemes implemented directly on the symmetry axis, namely, the specular reflection coupled with the l'Hôpital's rule and the proposed extrapolation-based schemes.

computational domain of $0 < r < R$, as depicted in Fig. 1(a). Specifically, the first lattice line is set at $r = 0.5\delta x$ whereby δx is a spatial step. Based on such a mesh arrangement, the symmetry boundary scheme was proposed by Guo *et al.* [1], whereby a ghost lattice line is first introduced at $r = -0.5\delta x$, and the postcollision distribution functions f_{α}^{+} at the ghost

lattice line are derived by applying the symmetry condition

$$\begin{aligned} f_1^{+}(\mathbf{x}_A, t) &= f_3^{+}(\mathbf{x}_B, t), & f_5^{+}(\mathbf{x}_A, t) &= f_6^{+}(\mathbf{x}_B, t), \\ f_8^{+}(\mathbf{x}_A, t) &= f_7^{+}(\mathbf{x}_B, t), \end{aligned} \quad (2)$$

where $\mathbf{x} = (r, z)$ is the spatial coordinates in the meridian plane and t is time. Then, the unknown distribution functions at the boundary node B are obtained after the streaming step as

$$\begin{aligned} f_1(\mathbf{x}_B, t + \delta t) &= f_1^{+}(\mathbf{x}_A, t), & f_5(\mathbf{x}_B, t + \delta t) &= f_5^{+}(\mathbf{x}_A, t), \\ f_8(\mathbf{x}_B, t + \delta t) &= f_8^{+}(\mathbf{x}_A, t), \end{aligned} \quad (3)$$

where δt is a discrete time step, and thus $f_{\alpha}(\mathbf{x}, t + \delta t)$ denote the distribution function after streaming. Alternatively, Srivastava *et al.* applied the specular reflection scheme at the midgrid point (i.e., $r = 0.5\delta x$) in their axisymmetric multiphase model for implementing the symmetry boundary condition [15] as

$$\begin{aligned} f_1(\mathbf{x}_B, t + \delta t) &= f_3(\mathbf{x}_B, t + \delta t), & f_5(\mathbf{x}_B, t + \delta t) &= f_6(\mathbf{x}_B, t + \delta t), \\ f_8(\mathbf{x}_B, t + \delta t) &= f_7(\mathbf{x}_B, t + \delta t). \end{aligned} \quad (4)$$

In contrast to the former two boundary schemes locating the symmetry axis outside of the computational domain (i.e., $0 < r < R$), the specular reflection scheme is directly implemented at the symmetry axis in many axisymmetric LB applications, and the l'Hôpital's rule ($\lim_{r \rightarrow 0} \frac{\varphi}{r} = \frac{\partial \varphi}{\partial r}$) is introduced to remove the singularity at $r = 0$ [9,18,19,33]. Therefore, the collision and the macroscopic variables calculation at the symmetry axis should be implemented separately by assuming $\frac{\varphi}{r} = 0$. However, this kind of implementation cannot be extended to the Guo *et al.* model [1] since the distribution functions in the Guo *et al.* model explicitly involve the radial coordinate r , and the macroscopic variables computations are realized through $\frac{1}{r} \sum_{\alpha} f_{\alpha}$ and $\frac{1}{r} \sum_{\alpha} f_{\alpha} \xi_{\alpha i}$, such as $\rho = \frac{1}{r} \sum_{\alpha} f_{\alpha}$.

In this study, alternative extrapolation-based implementations for the symmetry boundary condition are proposed with the advantage that (i) the present implementations locate the first lattice line at the symmetry axis and no additional mesh arrangements are required; (ii) both the postcollision distribution functions and the macroscopic variables at the symmetry axis are directly extrapolated from the inner fluid nodes, and thus the singularity at $r = 0$ is avoided; and (iii) the present symmetry boundary implementations are applicable for both the Li *et al.* model [9] derived from the standard LB model and the Guo *et al.* model [1] with special distribution function definitions, based on the NE extrapolation and the direct extrapolation. Theoretical derivations demonstrate the accuracy of the present implementations, and numerical validations and comparisons with previous implementations discussed above are performed based on the Li *et al.* model [9]. It is shown that the proposed extrapolation-based schemes are qualified for the axisymmetric flow description in terms of the numerical accuracy, stability, and efficiency.

II. FORMULATIONS

In this part, the efficient axisymmetric LB model by Li *et al.* [9], in which the trapezium rule is adopted for the source terms discretization and an additional relaxation term is applied to represent the velocity gradients in the momentum source term, is first introduced. Then, the present extrapolation-based symmetry boundary implementations are derived and the theoretical accuracy demonstrated.

A. Axisymmetric LB model by Li *et al.* [9]

The LB equation in the Li *et al.* model, which is generally consistent with that of the standard LB equation but with additional relaxation terms for simplifying the source term, is given as

$$f_\alpha(\mathbf{x} + \xi_\alpha \delta t, t + \delta t) - f_\alpha(\mathbf{x}, t) = \frac{\delta t}{2} [\Omega_\alpha|_{(\mathbf{x}, t)} + \Omega_\alpha|_{(\mathbf{x} + \xi_\alpha \delta t, t + \delta t)}] + \frac{\delta t}{2} [S_\alpha|_{(\mathbf{x}, t)} + S_\alpha|_{(\mathbf{x} + \xi_\alpha \delta t, t + \delta t)}] - \delta t \frac{\xi_{\alpha r}}{r} (f_\alpha - f_\alpha^{(\text{eq})})|_{(\mathbf{x}, t)} \quad (5)$$

The discrete particle velocity vectors ξ_α from the D2Q9 model [34] are applied here for the evolution of velocity vector $\mathbf{u} = (u_r, u_z)$ in the meridian plane

$$\xi_\alpha = \begin{cases} (0, 0) & \alpha = 0 \\ c(\cos[(\alpha - 1)\pi/2], \sin[(\alpha - 1)\pi/2]) & \alpha = 1, 2, 3, 4, \\ \sqrt{2}c(\cos[(2\alpha - 1)\pi/4], \sin[(2\alpha - 1)\pi/4]) & \alpha = 5, 6, 7, 8 \end{cases} \quad (6)$$

with c the lattice speed relating to the gas constant R and fluid temperature T by $c = \sqrt{3RT}$ for isothermal flows. In addition, $\Omega_\alpha = -\frac{1}{\tau_f}[f_\alpha - f_\alpha^{(\text{eq})}]$ is the collision term whereby τ_f is the relaxation time in the continuous Boltzmann equation, and the equilibrium distribution function $f_\alpha^{(\text{eq})}$ in the Li *et al.* model accords exactly with the standard LB model

$$f_\alpha^{(\text{eq})} = \rho w_\alpha \left[1 + \frac{\xi_\alpha \cdot \mathbf{u}}{RT} + \frac{(\xi_\alpha \cdot \mathbf{u})^2}{2(RT)^2} - \frac{u^2}{2RT} \right], \quad (7)$$

where ρ is the fluid density, and w_α is the weight coefficients from the D2Q9 model [34] (namely, $w_0 = 4/9$, $w_{1,2,3,4} = 1/9$, $w_{5,6,7,8} = 1/36$). Additionally, S_α is the source term accounting for the axisymmetric effects and the external forcing effects

$$S_\alpha = \left[\frac{(\xi_{\alpha i} - u_i)\tilde{F}_i}{\rho RT} - \frac{u_r}{r} \right] f_\alpha^{(\text{eq})} + F_\alpha, \quad (8)$$

$$\tilde{F}_i = -\left(\frac{2\mu u_r}{r^2} - \frac{\rho u_\theta^2}{r} \right) \delta_{ir},$$

where u_θ is the azimuthal velocity component, and the external force term F_α accounting for $\mathbf{a} = (a_r, a_z)$ is defined as

$$F_\alpha = \rho w_\alpha \left[\frac{\mathbf{a} \cdot \xi_\alpha}{RT} + \frac{(\mathbf{a} \cdot \xi_\alpha)(\mathbf{u} \cdot \xi_\alpha)}{(RT)^2} - \frac{\mathbf{a} \cdot \mathbf{u}}{RT} \right]. \quad (9)$$

In order to remove the implicitness in Eq. (5), a reconstructed distribution is introduced as

$$\tilde{f}_\alpha = f_\alpha - \frac{\delta t}{2} (\Omega_\alpha + S_\alpha) \quad (10)$$

and thereby Eq. (5) can be rewritten as

$$\begin{aligned} & \tilde{f}_\alpha(\mathbf{x} + \xi_\alpha \delta t, t + \delta t) - \tilde{f}_\alpha(\mathbf{x}, t) \\ &= -\omega_\alpha [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] + \delta t \left(1 - \frac{\omega_\alpha}{2} \right) S_\alpha, \end{aligned} \quad (11)$$

where a discrete particle velocity ξ_α -dependent relaxation parameter is defined as

$$\omega_\alpha = \frac{(1 + \frac{\tau \delta t \xi_{\alpha r}}{r})}{(\tau + \frac{1}{2})} \quad (12)$$

with a dimensionless relaxation time $\tau = \tau_f / \delta t$. In accordance with the definition of the reconstructed distribution, the

macroscopic variable calculations are related to the source term as

$$u_i = \frac{\sum_\alpha \tilde{f}_\alpha \xi_{\alpha i} + \frac{\delta t}{2} \left(\frac{\rho_0 u_\theta^2}{r} \delta_{ir} + \rho a_i \right)}{\left[\sum_\alpha \tilde{f}_\alpha + \left(\frac{\delta t \rho_0 v}{r^2} \right) \delta_{ir} \right]}, \quad (13a)$$

$$\rho = \frac{\sum_\alpha \tilde{f}_\alpha}{\left(1 + \frac{\delta t u_r}{2r} \right)}. \quad (13b)$$

Additionally, the LB equation for the azimuthal velocity evolution is proposed in the Li *et al.* model for swirling axisymmetric flows as

$$\begin{aligned} & g_k(\mathbf{x} + \xi_k \delta t, t + \delta t) - g_k(\mathbf{x}, t) \\ &= -\omega_k [g_k(\mathbf{x}, t) - g_k^{(\text{eq})}(\mathbf{x}, t)] + \delta t G_k, \end{aligned} \quad (14)$$

which evolves on the D2Q4 lattice model [9] and the discrete particle velocity vectors are defined as

$$\xi_k = c(\cos[(k - 1)\pi/2], \sin[(k - 1)\pi/2]), \quad k = 1, 2, 3, 4 \quad (15)$$

with weight coefficient $w_k = 1/4$ and the lattice speed $c = \sqrt{2RT'}$. The ξ_k -dependent relaxation parameter ω_k is determined as

$$\omega_k = \frac{(1 + \frac{\tau_g \delta t \xi_{kr}}{r})}{(\tau_g + \frac{1}{2})}. \quad (16)$$

In addition, the equilibrium distribution function $g_k^{(\text{eq})}$ and the source term G_k are expressed as

$$g_k^{(\text{eq})} = \rho_0 u_\theta w_k \left(1 + \frac{\mathbf{u} \cdot \xi_k}{RT'} \right), \quad (17a)$$

$$G_k = w_k \left(-\frac{2\rho_0 u_\theta u_r}{r} - \frac{\mu u_\theta}{r^2} \right) \left(1 + \frac{\mathbf{u} \cdot \xi_k}{RT'} \right). \quad (17b)$$

It is noted that for the purpose of facilitating the macroscopic variable calculations, the trapezium rule is not applied for the source term discretization in Eq. (14), and the fluid

density is assumed to be a constant ρ_0 [9]. Therefore, the azimuthal velocity component is free of the source term G_k :

$$u_\theta = \frac{\sum_k g_k}{\rho_0}. \quad (18)$$

Applying the Chapman-Enskog (CE) analysis, macroscopic equations are derived from the LB equation in Eqs. (5) and (14) as

$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x_j} (\rho u_j) = -\frac{\rho u_r}{r}, \quad (19a)$$

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \frac{\mu}{r} \left(\frac{\partial u_i}{\partial r} + \frac{\partial u_r}{\partial x_i} \right) - \frac{\rho u_i u_r}{r} - \frac{2\mu u_i}{r^2} \delta_{ir}, \quad (19b)$$

$$\rho_0 \left[\frac{\partial u_\theta}{\partial t} + \frac{\partial}{\partial x_j} (u_j u_\theta) \right] = \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_\theta}{\partial x_j} \right) + \frac{\mu}{r} \frac{\partial u_\theta}{\partial r} - \frac{2\rho_0 u_\theta u_r}{r} - \frac{\mu u_\theta}{r^2} \quad (19c)$$

with the following kinematic viscosity definition:

$$\nu = \tau \delta t R T = \tau_g \delta t R T'. \quad (20)$$

B. Extrapolation-based symmetry boundary implementations

1. Nonequilibrium extrapolation-based scheme

The postcollision distribution function f_α^+ is derived from Eq. (11) as

$$\begin{aligned} f_\alpha^+(\mathbf{x}, t) &= \tilde{f}_\alpha(\mathbf{x}, t) - \omega_\alpha [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] + \delta t \left(1 - \frac{\omega_\alpha}{2} \right) S_\alpha \\ &= f_\alpha^{(\text{eq})}(\mathbf{x}, t) + (1 - \omega_\alpha) \tilde{f}_\alpha^{(\text{neq})}(\mathbf{x}, t) + \delta t \left(1 - \frac{\omega_\alpha}{2} \right) S_\alpha, \end{aligned} \quad (21)$$

where $\tilde{f}_\alpha^{(\text{neq})}(\mathbf{x}, t)$ is the nonequilibrium part of the distribution function, and the following relations are directly obtained from the CE analysis [8,9]:

$$\tilde{f}_\alpha^{(\text{neq})}(\mathbf{x}, t) = \tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t) \sim O(\delta t), \quad (22)$$

which leads to

$$f_\alpha^{+, \text{neq}}(\mathbf{x}, t) = f_\alpha^+(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t) \sim O(\delta t) \quad (23)$$

with $f_\alpha^{+, \text{neq}}$ the nonequilibrium part of the postcollision distribution function. Therefore, $f_\alpha^{+, \text{neq}}$ at the boundary node B can be approximated as

$$f_\alpha^{+, \text{neq}}(\mathbf{x}_B, t) = f_\alpha^{+, \text{neq}}(\mathbf{x}_C, t) + O(\delta x \delta t). \quad (24)$$

Secondly, applying the following second-order finite difference stencil to the first lattice line (i.e., the symmetry axis for the present implementation) gives

$$\frac{\partial \Phi}{\partial x} = \frac{-3\Phi_i + 4\Phi_{i+1} - \Phi_{i+2}}{2\delta x} + O(\delta x^2), \quad (25)$$

where the subscripts i , $i+1$, and $i+2$ are the lattice coordinates in the r direction. The macroscopic variables at the boundary nodes are extrapolated from the inner fluid nodes C and D [as demonstrated in Fig. 1(b)] in accordance with the Neumann constraint given in Eq. (1) as

$$\Phi_i = (4\Phi_{i+1} - \Phi_{i+2})/3 + O(\delta x^2). \quad (26)$$

Then, the equilibrium part of the distribution function is obtained with the approximated macroscopic variables

$$f_\alpha^{(\text{eq})}(\mathbf{x}_B, t) = f_\alpha^{(\text{eq})}(\rho_B^E, \mathbf{u}_B^E) + O(\delta x^2), \quad (27)$$

where ρ_B^E and $\mathbf{u}_B^E$ are, respectively, the fluid density and velocity vector extrapolated from the inner fluid nodes according to Eq. (26).

Therefore, the postcollision distribution function at the boundary node B is obtained by combining Eq. (27) with Eq. (24),

$$f_\alpha^+(\mathbf{x}_B, t) = f_\alpha^{(\text{eq})}(\rho_B^E, \mathbf{u}_B^E) + f_\alpha^{+, \text{neq}}(\mathbf{x}_C, t) + O(\delta x^2), \quad (28)$$

which is the NE extrapolation-based symmetry boundary implementation.

2. Direct extrapolation-based scheme

Alternatively, the postcollision distribution function at the symmetry axis can also be directly derived from the inner fluid nodes C and D by the second-order extrapolation in Eq. (26),

$$f_\alpha^+(\mathbf{x}_B, t) = [4f_\alpha^+(\mathbf{x}_C, t) - f_\alpha^+(\mathbf{x}_D, t)]/3 + O(\delta x^2). \quad (29)$$

Similarly, all the macroscopic variables at the boundary nodes are also extrapolated from the inner fluid nodes.

It is concluded from above derivations that (i) the proposed extrapolation-based symmetry boundary implementation is second-order accurate; (ii) the present implementations are performed directly on the symmetry axis but the singularity is removed since the collision and the macroscopic variable calculations are avoided at the symmetry axis in the present implementations; and (iii) the present implementations are more efficient and easier than the previous ones since no additional mesh arrangements and separate treatments (such as the l'Hôpital's rule for the collision and macroscopic variables calculation) at the symmetry axis are involved.

III. NUMERICAL RESULTS

In this part, based on the axisymmetric LB model by Li *et al.* [9], numerical validations are performed for the two proposed symmetry boundary implementations and comparisons with the previous symmetry boundary schemes are also carried

FIG. 2. Hagen-Poiseuille flow: (a) axial velocity (u_z/u_0) profile; and (b) error with respect to mesh resolution (N_r).

out. The error convergence is investigated using the Hagen-Poiseuille flow and the Womersley flow with analytical solutions, and the order of the numerical accuracy of the proposed boundary implementations is evaluated. Thereafter, the present symmetry boundary implementations are extended to more practical rotational and thermal flows. It is noted that, for the convenience of comparing the accuracy of the

FIG. 3. Womersley flow at $\text{Re} = 1200$, $\nu = 1/3$: axial velocity (u_z/u_0) profiles for (a) $\alpha = 8$, $R = 30$ and (b) $\alpha = 16$, $R = 80$; (c) error with respect to mesh resolution (N_r) when $\alpha = 8$.

symmetry boundary scheme considered here, the nonslip viscous boundary condition in the following tests is realized by the NE extrapolation scheme (NEES) [35,36]. Additionally, the dimensionless spatial step $\delta x = 1$ and time step $\delta t = 1$ are applied in the following validation tests with the lattice speed $c = \delta x/\delta t = 1$.

FIG. 4. Flow structure of the cylindrical cavity flow obtained with the $N_r = 101$ mesh resolution at (a) $R_A = 1.5$, $\text{Re} = 990$; (b) $R_A = 1.5$, $\text{Re} = 1290$; (c) $R_A = 2.5$, $\text{Re} = 1010$, and (d) $R_A = 2.5$, $\text{Re} = 2200$. For each subplot, the streamlines on the left half are results from the NE extrapolation-based scheme, and the right half from the direct extrapolation-based scheme.

A. Hagen-Poiseuille flow

The first test case is for flow in a pipe, driven by a constant force a_z , with the following analytical solution for the axial velocity component:

$$u_z(r) = u_0 \left(1 - \frac{r^2}{R^2} \right), \quad (30)$$

where u_0 is the peak axial velocity at $r = 0$ given by

$$u_0 = \frac{a_z R^2}{4\nu} \quad (31)$$

and R and ν are, respectively, the radius of the pipe and the kinematic viscosity, which corresponds to a Reynolds number (Re) of $\text{Re} = 2Ru_0/\nu$. In addition, the boundary conditions for the present test are as follows: the periodic condition in the z direction is applied at the inlet (i.e., $z = 0$) and outlet (i.e., $z = L$) of the pipe, and the outer surface of the pipe adopts the nonslip boundary condition:

$$r = R : u_r = u_z = 0. \quad (32)$$

Notably, convergent results are obtained from the two proposed extrapolation-based symmetry boundary implementations with the mesh resolution of $N_z \times N_r = 33 \times 17$ (where N_z and N_r represent the number of lattice nodes, respectively, in the z and r directions, and the pipe radius R is related to N_r by $R = (N_r - 1)\delta x$). Figure 2(a), which illustrates the dimensionless axial velocity (u_z/u_0) versus the dimensionless radius (r/R) for both extrapolation-based schemes and also the analytical solution, indicates good agreement among all three solutions. To further quantify the error involved, the following relative error definition is used:

$$E(u) = \frac{\|u_z^{\text{present}} - u_z^{\text{analytical}}\|_2}{\|u_z^{\text{analytical}}\|_2}, \quad (33)$$

where u_z^{present} and $u_z^{\text{analytical}}$ denote the numerical and analytical solution, respectively. Figure 2(b) shows the error [Eq. (33)] with respect to the number of lattice nodes in the radial direction (i.e., the mesh resolution N_r). It is clear that the accuracy of the present extrapolation-based symmetry schemes

FIG. 5. Axial velocity profiles along the symmetry axis of the cylindrical cavity flow obtained with the $N_r = 101$ mesh resolution at (a) $R_A = 1.5$, $\text{Re} = 990$; (b) $R_A = 1.5$, $\text{Re} = 1290$; (c) $R_A = 2.5$, $\text{Re} = 1010$, and (d) $R_A = 2.5$, $\text{Re} = 2200$.

is comparable to the previous boundary implementations, and the errors from the present two schemes become lower as the number of lattice nodes N_r (i.e., mesh resolution) increases. Moreover, the accuracy of the proposed symmetry boundary schemes in velocity gradients calculations is further validated by the following relative error definition based on $\frac{\partial u_z}{\partial r}$:

$$E\left(\frac{\partial u_z}{\partial r}\right) = \frac{\left\| \frac{\partial u_z^{\text{present}}}{\partial r} - \frac{\partial u_z^{\text{analytical}}}{\partial r} \right\|_2}{\left\| \frac{\partial u_z^{\text{analytical}}}{\partial r} \right\|_2}. \quad (34)$$

Again, as demonstrated in Fig. 2(c), the order of the accuracy of all the symmetry boundary schemes studied in evaluating the velocity gradients agrees with each other.

B. Time-dependent Womersley flow

If the constant driven force in the Hagen-Poiseuille flow follows the time-dependent relationship $a_z = \hat{G} \cos(\hat{\Omega}t)$, where $\hat{\Omega}$ and $T^p = 2\pi/\hat{\Omega}$ are, respectively, the angular frequency and period, the flow becomes the unsteady-state Womersley flow with the following analytical velocity distribution:

$$u_z(r, t) = \text{Re} \left\{ \frac{\hat{G}}{i\hat{\Omega}} \left[1 - \frac{J_0(r\varphi/R)}{J_0(\varphi)} \right] e^{i\hat{\Omega}t} \right\}, \quad (35)$$

where Re refers to the real part of the complex variable, $\varphi = \alpha(i-1)/\sqrt{2}$ (whereby i is the imaginary unit and $\alpha = R\sqrt{\hat{\Omega}}/\nu$ is the Womersley number), and J_0 is the zeroth-order Bessel function of the first type. The Reynolds number (Re) becomes $\text{Re} = 2Ru_0/\nu$ with $u_0 = \hat{G}R^2/4\nu$.

The velocity field is initially set to be zero and numerical results are obtained after $10T^p$ iterations. Figures 3(a) and 3(b) display the axial velocity distribution at various time instances (namely, $T^p/4$, $T^p/2$, $3T^p/4$, and T^p) for $\text{Re} = 1200$ and $\nu = 1/3$. While Fig. 3(a) is for $\alpha = 8$ and $N_r = 31$, Fig. 3(b) is for $\alpha = 16$ and $N_r = 81$. In both test cases, the two extrapolation-based schemes agree well with the analytical solution. Analogous to Eq. (33), for a quantitative comparison of the accuracy of the symmetry boundary schemes, a time-averaged global error (E_{ave}) can be calculated:

$$E_{\text{ave}} = \frac{1}{16} \sum_{n=1}^{16} E(nT/16), \quad (36)$$

where $E(t)$ is the time-dependent error

$$E(t) = \sqrt{\frac{1}{N_r} \sum_{x \in N_r} |u_z(t) - u_z^{\text{analytical}}(t)|^2}, \quad (37)$$

FIG. 6. Axial velocity profiles along the symmetry axis of the cylindrical cavity flow obtained with the $N_r = 51$ mesh resolution at (a) $R_A = 1.5$, $\text{Re} = 990$; (b) $R_A = 1.5$, $\text{Re} = 1290$, and (c) $R_A = 2.5$, $\text{Re} = 1010$.

where $u_z(t)$ and $u_z^{\text{analytical}}(t)$ are, respectively, the numerical results from the LB simulation and the analytical solutions in Eq. (35), and N_r is the number of lattice nodes in the r direction. Figure 3(c) depicts E_{ave} with respect to the mesh

FIG. 7. Axial velocity gradients ($\frac{\partial u_z}{\partial z}$) profiles along the symmetry axis of the cylindrical cavity flow obtained with the $N_r = 51$ mesh resolution at (a) $R_A = 1.5$, $\text{Re} = 990$; (b) $R_A = 1.5$, $\text{Re} = 1290$, and (c) $R_A = 2.5$, $\text{Re} = 1010$.

resolution (N_r) when $\text{Re} = 1200$ and $\alpha = 8$. It is clear that the average errors (E_{ave}) for all the schemes are similar and also decrease at very similar rates with respect to mesh resolution (N_r).

TABLE I. Maximum Re of the convergent solutions based on various symmetry boundary implementations for $N_r = 51$.

R_A	NE extrapolation	Direct extrapolation	Specular reflection ($r = 0$)	Specular reflection ($r = 0.5\delta x$)	Symmetry
1.5	2800	2800	1700	1700	2700
2.5	2800	2800	1700	1700	2500

C. Cylindrical cavity flow

The more practical cylindrical cavity flow is assessed for further comparing the accuracy and applicability of the symmetry boundary schemes developed here. The cylindrical cavity flow, driven by the rotating top lid at a constant

angular velocity Ω , is characterized by two dimensionless parameters, namely, the geometric aspect ratio $R_A = H/R$ (where H and R denote, respectively, the height and radius of the cylindrical cavity) and Reynolds number $\text{Re} = \Omega R^2/\nu$. The reference velocity $u_0 = \Omega R = 0.1$ (i.e., the maximum

FIG. 8. (a) Azimuthal velocity contours and (b) isotherms for the case of $\text{Ri} = 0$ and $N_r = 101$. The streamlines on the left half are results from the NE extrapolation-based scheme, and the right half from the direct extrapolation-based scheme.

FIG. 9. (a) Azimuthal velocity contours and (b) isotherms for the case of $\text{Ri} = 1$ and $N_r = 151$. The streamlines on the left half are results from the NE extrapolation-based scheme, and the right half from the direct extrapolation-based scheme.

azimuthal velocity component at the top lid) and the radius of the cylindrical cavity $R = (N_r - 1)\delta x$ are fixed once the mesh resolution N_r is given (in the present LB simulations the spatial step δx and the time step δt are set to be constant at 1), and the kinematic viscosity is adjusted for various Re values.

In addition, the boundary conditions, except for the symmetry condition at $r = 0$, are given as

$$\text{Top: } u_r = u_z = 0, \quad u_\theta = \Omega r, \quad z = H, \quad (38a)$$

$$\text{Bottom: } u_r = u_z = u_\theta = 0, \quad z = 0, \quad (38b)$$

$$\text{Outer cylindrical surface: } u_r = u_z = u_\theta = 0, \quad r = R. \quad (38c)$$

As reference solutions, the results obtained via the spectral element method (SEM), whose accuracy is well acknowledged, are adopted here.

Four test cases with varying pairs of (R_A, Re) and at a mesh resolution of $N_r = 101$ were evaluated, namely, (i) $R_A = 1.5$, $Re = 990$ [Fig. 4(a)], (ii) $R_A = 1.5$, $Re = 1290$ [Fig. 4(b)], (iii) $R_A = 2.5$, $Re = 1010$ [Fig. 4(c)], and (iv) $R_A = 2.5$, $Re = 2200$ [Fig. 4(d)]. Figure 4 illustrates the flow structures; for each subplot, the streamlines on the left half are results from the NE extrapolation-based scheme, and the right half from the direct extrapolation-based scheme. The similarity in the distribution of the flow structures with respect to the

centerline reflects the agreement between the two proposed symmetry boundary schemes. Moreover, both the location and strength of the recirculation regions (i.e., the breakdown bubble) along the symmetry axis reflected in Figs. 4(b) and 4(d), whereby the Reynolds numbers (Re) are higher, are in line with previous published results [1,9,13,14,37]. Furthermore, quantitative comparisons with the SEM reference solutions are conducted for the considered symmetry boundary schemes for the four test cases in Fig. 4. Figure 5 shows the axial velocity distribution (i.e., z/H versus u_z/u_0) along the symmetry axis (i.e., $r = 0$) for the two extrapolation-based symmetry boundary schemes developed here, two results from the specular reflection-based scheme at two different r positions, and the symmetry scheme by Guo *et al.* [1] and SEM. It is clear that (i) the results from the two extrapolation-based symmetry boundary schemes developed here are consistent with those from the specular reflection-based scheme (at both r positions) for all four test cases [Figs. 5(a)–5(d)]; (ii) the results from the two extrapolation-based symmetry boundary schemes and the specular reflection-based scheme agree well with the SEM solution for the first three cases [Figs. 5(a)–5(c)], but not the fourth [Fig. 5(d)]; (iii) the symmetry scheme by Guo *et al.* [1] is slightly less accurate than the other schemes for the first two test cases [Figs. 5(a) and 5(b)] in

FIG. 10. Axial profiles at $r/R = 0.8$ for the case of $N_r = 101$ and $Ri = 0$ of dimensionless velocity components, namely, (a) u_r/u_0 , (b) u_z/u_0 , and (c) u_θ/u_0 , and (d) dimensionless temperature (T^*).

the region near the bottom wall (i.e., $0.1 < z < 0.3$); and (iv) at the highest Re investigated of 2200 [Fig. 5(d)], all the considered boundary schemes cannot provide exact agreement with the SEM solution, which indicates that more refined mesh resolutions may be required. Moreover, for further differentiating the accuracy of the various symmetry schemes, velocity (u_z) and velocity gradients ($\frac{\partial u_z}{\partial z}$), profiles along the symmetry axis are compared under coarse mesh resolution, as shown in Figs. 6 and 7. It is demonstrated from Figs. 6(a), 6(b), 7(a), and 7(b) that the proposed symmetry schemes exhibit similar accuracy with the reflection-based schemes, and both are more accurate than the symmetry scheme in the region near the top ($0.8 < z/H < 1.0$) and bottom ($0 < z/H < 0.3$) walls.

Since greater mesh resolutions are needed for greater Re values, it is worth assessing the maximum Re possible at a fixed mesh resolution (N_r) for the various schemes. Accordingly, Table I lists the maximum Re possible for each scheme to ensure convergent solutions at a mesh resolution of $N_r = 51$ and two R_A values. Two observations are highlighted. Firstly, the maximum Re values for the two extrapolation-based schemes developed here are the same and higher than other schemes. Secondly, despite the similar accuracy with the other schemes depicted in Figs. 5–7, the specular reflection-based

schemes are inferior to other schemes in terms of numerical stability at higher Re values.

D. Swirling thermal flows in a cylindrical container

Lastly, the two proposed extrapolation-based symmetry boundary schemes are extended to thermal rotational flows in a cylindrical container for further validation, and the axisymmetric thermal LB model by Li *et al.* [8] is applied for solving the temperature field. The geometry of the cylindrical container is characterized by the aspect ratio $H/R = 2$, with H and R representing, respectively, the height and radius of the cylindrical container, and the boundary conditions for the present test are given as

$$\text{Top } (z = H): u_r = u_z = 0, \quad u_\theta = \Omega_t r, \quad T = T_h, \quad (39a)$$

$$\text{Bottom } (z = 0): u_r = u_z = 0, \quad u_\theta = \Omega_b r, \quad T = T_c, \quad (39b)$$

$$\text{Outer cylindrical surface } (r = R): u_r = u_z = u_\theta = 0,$$

$$\frac{\partial T}{\partial r} = 0, \quad (39c)$$

where Ω_t and Ω_b are the constant angular speed of the rotating top and bottom ends, respectively, of the container. In addition, the constant temperatures of T_h and T_c (whereby

FIG. 11. Axial profiles at $r/R = 0.8$ for the case of $N_r = 151$ and $Ri = 1$ of dimensionless velocity components, namely, (a) u_r/u_0 , (b) u_z/u_0 , and (c) u_θ/u_0 , and (d) dimensionless temperature (T^*).

$T_h > T_c$) are designated for the top and bottom walls, respectively, and the cylindrical surface is insulated. The present thermal flow test is characterized by four dimensionless parameters, namely, the Prandtl number $Pr = \nu/\lambda$ (where λ is the thermal diffusion coefficient), the Reynolds number $Re = \Omega_b R^2/\nu$, the ratio of the angular speeds of the top and bottom ends $s = \Omega_t/\Omega_b$, and the Richardson number $Ri = g\beta(T_h - T_c)/R\Omega_b^2$ (where g and β are, respectively, the gravitational acceleration and thermal-expansion coefficient). The first three parameters are fixed in the present test with the values of $Pr = 1.0$, $Re = 1000$ and $s = -1$, the Boussinesq approximation is invoked in the present test for $Ri = 1$, and the relevant body force $\rho a_z = -\rho g\beta(T - T_m)$ is added to the momentum equation, where $T_m = (T_i + T_o)/2$ is the reference temperature.

Figures 8 and 9 show the results for two test cases in terms of the azimuthal velocity contours and isotherms. Figure 8 is for the case of $Ri = 0$ at a mesh resolution of $N_r = 101$, while Fig. 9 is for the case of $Ri = 1$ at a mesh resolution of $N_r = 151$. Similar to Fig. 4, the results in the left half of each figure are from the NE extrapolation-based scheme, and the right half from the direct extrapolation-based scheme. It is clear from Figs. 8 and 9 that the results from both schemes are consistent with each other. When Ri increases from 0 (Fig. 8) to 1 (Fig. 9), the body force term ρa_z starts to take effect at $\rho a_z > 0$ for $z > H/2$ and $\rho a_z < 0$ for $z < H/2$, and causes the azimuthal velocity and the temperature distribution to be more confined near the top and bottom walls. These detailed flow structures accord well with the reported results in the literature [11,22,38]. Furthermore, the dimensionless velocity components (namely, u_r/u_0 , u_z/u_0 , and u_θ/u_0) and dimensionless temperature $T^* = (T - T_m)/(T_h - T_c)$ distribution along the vertical line $r/R = 0.8$ are plotted for a mesh resolution of $R = 100$ in Fig. 10 for the case of $Ri = 0$ and Fig. 11 for the case of $Ri = 1$. The results from the two extrapolation-based schemes developed here and that by Omi and Iwatsu [38] are compared. Figures 10 and 11 indicate that both the proposed symmetry boundary schemes provide quantitatively accurate results for the present thermal tests. In addition, the differences in profiles between Figs. 10

and 11 are due to the nonzero force for the case of $Ri = 1$ (Fig. 11), which causes the velocity and temperature profiles to become linear in the center of the container (i.e., near $z = 0.5H$).

IV. CONCLUSION

Two symmetry boundary implementations, based on the NE extrapolation and the direct extrapolation, are developed in this study. It can be concluded from the theoretical derivations and the numerical validations that the proposed extrapolation-based symmetry boundary implementations (i) are second-order accurate, as demonstrated in the theoretical derivations and also confirmed by the pipe flow tests with analytical solutions; (ii) enable the singularity arising from the collision and the macroscopic variable calculation step to be avoided, since the postcollision distribution functions, as well as the macroscopic variables, at the symmetry axis are extrapolated from the inner fluid nodes; and (iii) are implemented directly on the symmetry axis, so no additional mesh arrangements are required. The accuracy and applicability of the two proposed symmetry boundary implementations are well verified by various numerical tests, and further comparison with previous symmetry boundary schemes demonstrated that the present boundary implementations are slightly more accurate than the symmetry scheme by Guo *et al.* [1] and numerically more stable than the reflection-based scheme.

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