Discrete-time Day-to-day Dynamic Congestion Pricing Scheme Considering Multiple Equilibria

Linghui Han[a], David Z.W. Wang[a,*], Hong K. Lo[b], Chengjuan Zhu[a], Xingju Cai[c]

[a]*School of Civil & Environmental Engineering, Nanyang Technological University, 50 Nanyang Avenue, 639798, Singapore*
[b]*Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China*
[c]*School of Mathematical Sciences, Jiangsu Key Labratory for NSLSCS, Nanjing Normal University, Nanjing 210023, China*

Abstract

In this study, we focus on the discrete-time day-to-day dynamic congestion pricing scheme which varies the toll on a day-to-day basis and aims to drive the traffic system to a given objective traffic equilibrium state. As is well known, due to the asymmetric nature of the travel cost functions, multiple equilibria exist. In this case, without external force, the traffic system cannot converge to the traffic equilibrium state as desired by traffic management through a day-to-day adjustment process if the initial traffic state does not fall into its attraction domain (Bie and Lo, 2010). Therefore, it is imperative for traffic management to propose a traffic control measure to ensure the desired traffic state can be achieved regardless of the initial traffic state. Previous studies on the day-to-day dynamic congestion pricing, either worked on continues-time day-to-day pricing scheme, or took the form of discrete-time day-to-day pricing scheme but did not guarantee the convergence to the desired objective traffic state for the cases when multiple traffic equilibria exist. Both are undesirable. This study aims to develop a discrete-time day-to-day pricing scheme so as to direct the traffic evolution to reach the desired equilibrium from any initial traffic state

*Corresponding author
Email addresses: `lhhan@ntu.edu.sg` (Linghui Han), `wangzhiwei@ntu.edu.sg` (David Z.W. Wang), `cehklo@ust.hk` (Hong K. Lo), `Cjzhu@ntu.edu.sg` (Chengjuan Zhu), `caixingju@njnu.edu.cn` (Xingju Cai)

when multiple traffic equilibria exist. Based on the very general formulation of day-to-day traffic dynamics model, we present a general formulation of such day-to-day pricing schemes and propose a method to obtain one specific road pricing scheme. Moreover, we present rigorous proofs and numerical tests to verify the proposed pricing scheme.

Keywords: Road pricing, Traffic dynamic control, Day-to-day traffic dynamics model, Multiple traffic equilibrium case

1. Introduction

Traffic congestion is one of the most challenging problems faced by many major cities, which has induced not only economic loss, but environmental pollution. It is believed that many of these costs can be prevented in principle, as they result from socially inefficient choices by individual drivers. To this end, economists have been advocating congestion pricing as an efficient way to reduce traffic congestion (Pigou, 1924; Knight, 1924; Yang and Huang, 2005). The traditional road pricing schemes are developed based on the static definition of traffic equilibrium, and the congestion pricing scheme is determined with the assumption that the desired traffic equilibrium state can eventually be achieved in a long run. However, as was demonstrated in Horowitz (1984) and Bie and Lo (2010), for a traffic system whose equilibrium solution was known to exist, depending on the dynamic route adjustment process, the system might still fail to converge to equilibrium. Therefore it is doubtful whether the traffic system can reach the desired traffic equilibrium state or not, from any feasible initial traffic state under the pricing scheme based on static equilibrium, especially when multiple traffic equilibrium states exist.

Having noticed the problem of the pricing scheme based on static equilibrium, researchers proposed dynamic road pricing scheme. In contrast to the fixed static pricing scheme, day-to-day dynamic pricing varies the toll on a day-to-day basis. Indeed, the advent of new technologies for the travel information collection and dissemination makes it more practically feasible to implement dynamic congestion pricing. However, as compared with the congestion pricing based on static equilibrium, the research works on day-to-day dynamic congestion pricing which varies the toll on a day-to-day basis are much less. Besides, most of existing day-to-day dynamic congestion pricing scheme are developed based on continuous time day-to-day traffic dynamic model, which is conventionally expressed by differential equations. Sandholm (2002) applied evolutionary game approach to study the day-to-day dynamic congestion pricing which can force traffic system to reach system optimal state. In their study, the dynamic congestion pricing can be viewed as generalizations the marginal travel cost, and the elastic and inelastic traffic demand are both considered. Friesz et al. (2004) proposed a disequilibrium day-to-day dynamic pricing scheme, which maximizes the net social benefit over the planning period, considering drivers' day-to-day behavior articulated in continuous time and taking the form of ordinary differential equations. Fang and Szeto (2006) developed a day-to-day dynamic congestion pricing strategies that can force the traffic system to evolve from the status quo to SO. Further, Yang et al. (2007) extended their work and proposed a steepest descent day-to-day dynamic toll which can force the traffic system to evolve from the status quo to SO, and minimizes the derivative of the total system cost with regard to day t or reduces the total system cost the most for each day. Tan et al. (2015) investigated day-to-day dynamic congestion pricing schemes to minimize the system cost and time, measured in monetary and time units, respectively, with the travelers' value-of-time heterogeneity. There are also some researches on the adaptive signal control based on traffic dynamic models. For more details, one can refer to Xiao and Lo (2014), Smith (2015) and Liu and Smith (2015). This study only focuses on the dynamic road pricing schemes.

Although continuous time day-to-day traffic dynamic models possess good mathematical properties in traffic evolution, Watling and Hazelton (2003) pointed out two major defects of continuous day-to-day approaches, and recommended that discrete-time versions of day-to-day traffic dynamic models should be more suitable to describe travelers' routing choice adjustment behavior, which is assumed to be repeated daily, in accordance with daily changes in traffic flows. In addition, continuous time dynamic congestion pricing is not appropriate for real application. In the existing literature, Guo et al. (2015a) is the first to develop discrete-time dynamic congestion pricing scheme. In their work, the target traffic state is only a restraint flow state bounded by a predetermined set of traffic flow, rather than a specific traffic flow state. Although the proposed dynamic pricing scheme of Guo et al. (2015a) can direct the traffic system to converge to the restraint stable traffic flow state, the convergence proof is based on the assumption of the uniqueness of the traffic equilibrium. However, the uniqueness assumption is often violated: for example, the asymmetric travel cost function (Watling, 1996), may induce multiple traffic equilibrium states. In this case, if the initial traffic state does not fall into the attraction domain of the given objective traffic equilibrium state, without external force, traffic system cannot reach the given objective traffic equilibrium state through a day-to-day routing adjustment process. Han et al. (2016) developed specific discrete-time day-to-day congestion pricing schemes for certain existing specific traffic dynamic models. By their dynamic pricing scheme, the traffic system can converge to the desired traffic state in finite “days”. However, Han et al. (2016) did not present a general formulation of dynamic pricing scheme which is applicable to the general day-to-day traffic dynamic models such as the formulation proposed by Guo et al. (2015b). This study aims to fill in the research gap by proposing a general discrete-time day-to-day dynamic congestion pricing scheme to explicitly consider the case when multiple traffic equilibria exist, which can drive the traffic system to reach a desired traffic equilibrium state through a route adjustment process as described by a general day-to-day traffic dynamic models regardless of the initial traffic state.

Specifically, in this study, we apply the very general day-to-day traffic dynamic model proposed by Guo et al. (2015b) to describe the dynamic route adjustment process. Indeed, this general day-to-day traffic dynamic model includes many existing models as particular cases. Then, we develop a general discrete-time day-to-day dynamic congestion pricing scheme to ensure that, from any feasible initial traffic state, traffic system can be directed to converge to the desired traffic state. Rigorous mathematical proof has been given to verify the model properties. Then, a specific dynamic road pricing scheme is proposed, and numerical tests are conducted to validate the pricing scheme.

This paper is organized as follows: the notations, the applied day-to-day traffic dynamic model, the problems of existing road pricing schemes are given in Section 2. In Section 3, we present a general formulation of dynamic road pricing scheme to achieve the aim of this study as well as the proof of convergence of the dynamic road pricing scheme. In Section 4, we demonstrate how to derive a specific dynamic road pricing scheme. The numerical tests are showed in Section 5. Finally, Section 6 presents the conclusions of this study and our future work.

2. Model Description

2.1. Notation

Let $G(N, A)$ denote a general traffic network with a set N of nodes and a set A of directed links. W is the set of OD pairs. R_w is the set of feasible acyclic routes between OD pair $w \in W$, and $R = \bigcup_{w \in W} R_w$. The cardinality of R_w is assumed to be finite. $\mathbf{d} = (d_w, w \in W)$ denotes the traffic demand column vector, whose element $d_w (\geq 0)$ is the travel demand between OD pair $w \in W$ and fixed. Let $f_{rw}^{(t)} (\geq 0)$ be the traffic flow on route $r \in R_w$ and $x_a^{(t)} (\geq 0)$ be the flow on link $a \in L$ at time t, and $\mathbf{f}^{(t)} = (f_{rw}^{(t)}, r \in R_w, w \in W)$, and $\mathbf{x}^{(t)} = (x_a^{(t)}, a \in A)$ denote the vectors of route flows and link flows, respectively. $\Delta = (\delta_{a,r}^w, a \in A, r \in R_w, w \in W)$ denotes the link-route incidence matrix, where $\delta_{a,r}^w = 1$ if route r uses link a and 0 otherwise. Let $\Lambda = (\lambda_{rw}, r \in R_w, w \in W)$ be the OD-route incidence matrix, where $\lambda_{rw} = 1$ if route r connects OD pair w and 0 otherwise. Naturally, we have $\mathbf{x}^{(t)} = \Delta \mathbf{f}^{(t)}$ and $\mathbf{d} = \Lambda \mathbf{f}^{(t)}$. $c_a(\mathbf{x})$ is the travel cost function of link $a \in A$, which is a continuous and positive function of the link flow vector $\mathbf{x}$, i.e., the link cost may be not separable. $\mathbf{c}(\mathbf{x}^{(t)}) = (c_a(\mathbf{x}^{(t)}), a \in A)$ is the corresponding link travel cost vector at time t. Let τ be the vector of link-specific toll, in which τ_a, $a \in A$ is the toll charge on link a. Finally, it is supposed that the traffic network each G is strongly connected. i.e., each OD pair is connected by at least one route in the network. Let the set $\Omega_{\mathbf{x}}$ be the feasible link flow and $\Omega_{\mathbf{f}}$ be the feasible path flow.

$$\Omega_{\mathbf{f}} = \{\mathbf{f} | \mathbf{d} = \Lambda \mathbf{f}, \mathbf{f} \geq 0\} \tag{2.1}$$

$$\Omega_{\mathbf{x}} = \{\mathbf{x} | \mathbf{x} = \Delta \mathbf{f}, \mathbf{f} \in \Omega_{\mathbf{f}}\} \tag{2.2}$$

In this study, it is assumed that, before departure from their origins on each day, each traveler has the complete information of network link flows, travel cost on the previous day and the intraday link tolls charged. Each traveler will try to minimize his or her travel cost when traveling from an origin to a destination, and the traffic state follows user equilibrium (UE) principle when the traffic system reaches a steady state.

2.2. Model of Day-to-day Traffic Dynamics under road pricing

The implementation process of discrete-time day-to-day dynamic congestion pricing, which controls the evolution of traffic state, can be described as in Figure 1. In the evolution process of traffic state, the traffic state $\mathbf{x}^{(t+1)}$ on day $t+1$ is determined by the traffic state $\mathbf{x}^{(t)}$ on day t and the road pricing $\tau^{(t+1)}$ on day $t+1$. Therefore, based on the very general day-to-day traffic dynamics model in Guo et al. (2015b), the day-to-day traffic dynamic adjustment process under the dynamic pricing can be expressed as follows:

$$\mathbf{x}^{(t+1)} = (1 - \alpha^{(t)})\mathbf{x}^{(t)} + \alpha^{(t)}\mathbf{y}^{(t)}, \ t = 0, 1, 2, \ldots, \tag{2.3}$$

where $0 < \alpha^{(t)} \leq 1$ is the parameter of step size or adjustment ratio of link flow, and $\mathbf{y}^{(t)} = \mathbf{y}(\mathbf{x}^t, \tau^{t+1})$ is the objective adjustment vector of link flow at time $t+1$ and satisfies

$$\mathbf{y}^{(t)} \begin{cases} \in \Psi^{(t)}, \ \text{if } \Psi^{(t)} \neq \emptyset, \\ = \mathbf{x}^{(t)}, \ \text{if } \Psi^{(t)} = \emptyset. \end{cases} \tag{2.4}$$

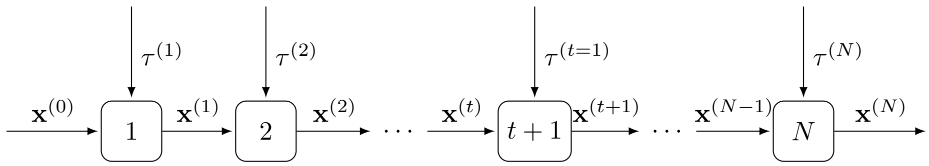

Figure 1: The dynamic adjustment of traffic state under dynamic pricing

where

$$\Psi^{(t)} = \{\mathbf{y}|\mathbf{y} \in \Omega_{\mathbf{x}}, (\mathbf{y} - \mathbf{x}^{(t)})^T(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) < 0\} \tag{2.5}$$

It should be noted that above model is a link based day-to-day traffic dynamics model and also belongs to the so-called rational behavior adjustment process (RBAP) of flows in a traffic network with fixed demand (Zhang et al., 2001). The day-to-day traffic dynamics model of RBAP, can be grouped into two classes: path-based and link-based models. The path-based dynamical models are built upon route flow variables, and traffic system reaches steady state when the route flows remain unchanged from day to day. As summarized by Yang and Zhang (2009), there exists five major categories of path based day-to-day dynamical model, i.e., the simplex gravity flow dynamics (Smith, 1983), proportional-switch adjustment process (Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003; Mounce, 2006; Mounce and Carey, 2011), network tatonnement process (Friesz et al., 1994; Jin, 2007; Guo and Huang, 2009), projected dynamical system (Zhang and Nagurney, 1996; Nagurney and Zhang, 1997), and evolutionary traffic dynamics (Sandholm, 2001; Yang, 2005). This study only discusses day-to-day dynamic road pricing scheme levied on links. So the link-based day-to-day traffic dynamics model is considered. The first link-based day-to-day traffic dynamics model is presented by He et al. (2010).

2.3. The problem of existing congestion pricing schemes

When multiple traffic equilibria exist, each traffic equilibrium state has its own attraction domain. If the initial traffic state does not lie in the attraction domain, then traffic system cannot automatically converge to desired objective equilibrium state through a day-to-day adjustment process without external force. In the following, a small example is given to illustrate this statement. The example consists of one OD pair connected with two links. The travel cost c_1 is $f_1 + 3f_2 + 1$, and $c_2 = 2f_1 + f_2 + 2$. The traffic demand is 2. The traffic flow of system optimal (SO) state is $(2, 0)$ which is an UE state under the toll $\tau = (2, 4)$. It can also be found that there are three user equilibrium states with toll charge of $\tau = (2, 4)$:

$$\mathbf{f}^I = (2, 0), \quad \mathbf{f}^{II} = (1/3, 5/3), \quad \mathbf{f}^{III} = (0, 2) \tag{2.6}$$

Only implementing the road pricing based on the static equilibrium, i.e., $(2, 4)$, the dynamic adjustment process of traffic system may not converge to SO. Figure.2 shows that the traffic system converges to $(0, 2)$ from the initial link flow $(0.15, 1.85)$ based on the traffic dynamics adjustment model of He et al. (2010), which is described as follows:

$$\mathbf{x}^{(t+1)} - \mathbf{x}^t = \alpha^{(t)}(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \tag{2.7}$$

where $\alpha^{(t)} = 1/(t + 1)$ is adjustment rate at time t, and

$$\mathbf{y}^{(t)} = \arg \min_{\mathbf{y^{(t)}} \in \Omega_{\mathbf{x}}} \lambda(\mathbf{c}^{(t)}(\mathbf{x}^{(t)}))^T y + (1 - \lambda)D(\mathbf{x}^{(t)} - \mathbf{y}) \tag{2.8}$$

where $0 < \lambda < 1$ and is set as 0.2 in this example. $D(\mathbf{x}^{(t)} - y)$ is a function which can measure the distance between $\mathbf{x}^{(t)}$ and $\mathbf{y}$. Here, it is taken as Euclidean distance $||\mathbf{x}^{(t)} - \mathbf{y}||^2$.

Figure (2) shows that the toll $\tau = (2, 4)$ cannot guarantee that the traffic system can converge to the desired SO state. In the following, we also use another simple example to demonstrate that the discrete-time day-to-day dynamic road pricing scheme of Guo et al. (2015a) cannot also ensure that a desired traffic state can be reached from any initial traffic state. The simple example has only one OD pair between which there are three links. The travel cost of each link is: $c_1 = f_1 + 3f_2 + 0.5$, $c_2 = 2f_1 + f_2 + 0.5$ and $c_3 = f_3 + 0.5$. Assume the traffic demand is 2 unit. There are three UE flow patterns under the toll

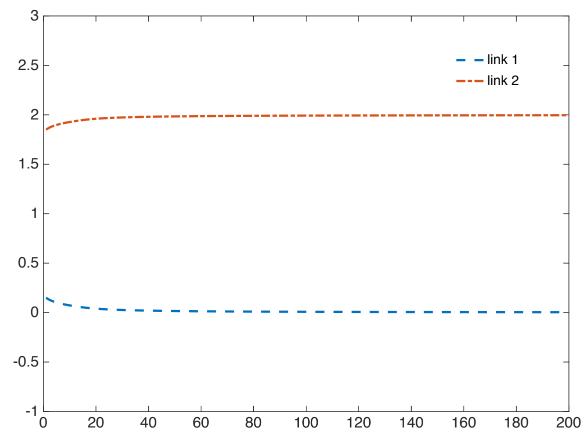

Figure 2: Flow evolution under SO toll

(0.5, 0.5, 0.5): $\mathbf{f}^{I} = (1,0,1)$,$\mathbf{f}^{II} = (0,1,1)$ and $\mathbf{f}^{III} = (1/2,1/4,5/4)$.Let the flow pattern $(0,1,1)$ be the objective equilibrium. The discrete-time day-to-day dynamic road pricing scheme of Guo et al. (2015a) is described as follows:

$$\tau^{(t+1)} = H(\mathbf{x}^{(t)}, \tau^{(t)}) \begin{cases} \in Q^{(t)}, \text{if } Q^{(t)} \neq \emptyset, \\ = \tau^{(t)}, \text{if } Q^{(t)} = \emptyset, \end{cases} \tag{2.9}$$

where

$$Q^{(t)} = \{\tau | \tau \geqslant 0, (\tau - \tau^{(t)})^T (\bar{\mathbf{x}} - \mathbf{x}^{(t)}) < 0\}. \tag{2.10}$$

and, $\bar{x}$ is the SO link flow vector. A specific formulation of Eq.(2.10) is the Walrasian scheme of Garcia et al. (2012):

$$\tau^{(t+1)} = [\tau^{(t)} + \rho^{(n)}(\mathbf{x}^{(t)} - \bar{\mathbf{x}})]_+, \quad t = 0, 1, 2, \dots \tag{2.11}$$

where $\rho^{(t)}$ is the sensitive parameter and determines the speed at which the tolls are updated. In this study, $\rho^{(t)} = 1/\sqrt{t+1}$ according the assumption in Garcia et al. (2012).

From Figure (3), one can observe that, from the initial feasible flow $(0,0,2)$, the traffic system cannot attain the objective equilibrium $(0,1,1)$ under the dynamic pricing scheme of Garcia et al. (2012). This indicates that the dynamic

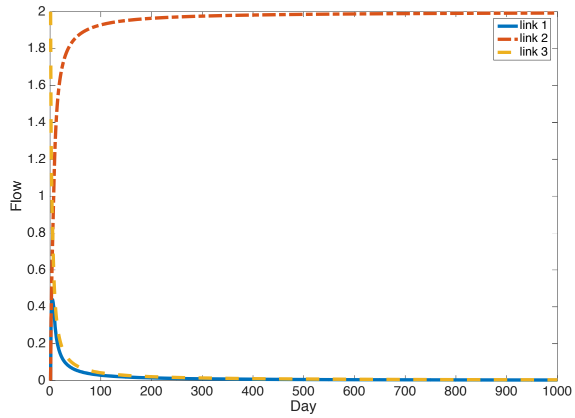

Figure 3: Flow evolution under existing dynamic pricing scheme

pricing scheme of Guo et al. (2015a) may not drive the traffic system to converge to the given objective equilibrium when multiple traffic equilibria exist.

3. The general discrete-time day-to-day dynamic congestion pricing scheme

3.1. Formulation of the general dynamic pricing scheme

To circumvent the problem of existing congestion pricing scheme mentioned in Section 2.3, the discrete-time day-to-day dynamic congestion pricing scheme proposed in this study must be able to guarantee convergence of the traffic dynamic adjustment process under the dynamic pricing scheme. Basically, under the dynamic pricing, the distance between $\mathbf{x}^{(t)}$ and the objective traffic state $\mathbf{x}^*$ would converge to 0. To achieve these goals, we develop the day-to-day discrete-time dynamic congestion pricing scheme as follows.

The link-based road pricing $\tau^{(t+1)} = \{\tau_a^{(t+1)}, a \in A\}$ $(t = 0, 1, 2, \dots)$ at time $t+1$ is a function with respect to the traffic state $\mathbf{x}^{(t)}$ at time t and satisfies the following condition:

$$\tau^{(t+1)} \begin{cases} \in \Upsilon^{(t+1)}, \text{if } \mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta), \\ = \tau^*, \text{if } \mathbf{x}^{(t)} \in \mathbf{N}(\mathbf{x}^*, \delta); \end{cases} \tag{3.1}$$

where

$$\Upsilon^{(t+1)} = \{\tau | \tau \geqslant 0, (\mathbf{c}(\mathbf{x}^{(t)}) + \tau)^T (\mathbf{x}^* - \mathbf{x}^{(t)}) < 0\}. \tag{3.2}$$

where $\mathbf{x}^*$ is the given objective traffic equilibrium. τ^* is the static road pricing scheme under which the objective traffic equilibrium state $\mathbf{x}^*$ is UE state. Specifically, $\mathbf{x}^*$ and τ^* satisfy the following variaional inequality formulation:

$$(\mathbf{c}(\mathbf{x}^*) + \tau^*)^T (\mathbf{x} - \mathbf{x}^*) \geqslant 0, \forall \mathbf{x} \in \Omega_{\mathbf{x}}. \tag{3.3}$$

From the traffic dynamic standpoint, an ideal target flow pattern $\mathbf{x}^*$ desired by traffic management authorities should be asymptotically stable, i.e., a temporary, small perturbation does not lead to a large deviation from the stable equilibrium and the equilibrium can recover itself within time. Readers can refer to Bie and Lo (2010) on the definition of asymptotical stability, by which $\mathbf{x}^*$ is asymptotically stable if and only if its attraction domain contains at least one of its neighborhoods.

The set $\mathbf{N}(\mathbf{x}^*, \delta) = \{\mathbf{x} \in \Omega_{\mathbf{x}} | \|\mathbf{x} - \mathbf{x}^*\| < \delta\}$ is a subset of the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$ of the objective traffic equilibrium $\mathbf{x}^*$ under road pricing τ^*. The definition of attraction domain can be found in Bie and Lo (2010) described by the following expression:

$$\mathbf{D}(\mathbf{x}^*, \tau^*) = \{\mathbf{x}^{(0)} \in \Omega_{\mathbf{x}} : \lim_{n \to +\infty} \mathbf{x}^{(n)} = \mathbf{x}^*\}.$$

That is, with fixed road pricing $\tau^{(t+1)} = \tau^*$ $(t = 0, 1, \dots)$, the traffic system will converge to the target equilibrium $\mathbf{x}^*$ from any feasible traffic state $\mathbf{x}$ belonging to the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$ of $\mathbf{x}^*$.

The road pricing $\tau^{(t+1)}$ at time $t + 1$ $(t = 0, 1, \dots)$ in Eq.(3.2) and Eq.(3.3) has two cases: The fist case is when the traffic state $\mathbf{x}^{(t)}$ does not belong to the neighborhood $\mathbf{N}(\mathbf{x}, \delta)$ which is a subset of the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$ of the target equilibrium $\mathbf{x}^*$ under road pricing τ^*. In this case, the traffic state $\mathbf{x}^{(t)}$ at time t may not be within the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$. Therefore, the road pricing $\tau^{(t+1)}$ at time $t+1$ make the target equilibrium $\mathbf{x}^*$ to be a feasible direction from $\mathbf{x}^{(t)}$ following Eq.(2.5) so that the traffic state $\mathbf{x}^{(t+1)}$ at time $t+1$ is closer to $\mathbf{x}^*$. Second, if the traffic state $\mathbf{x}^{(t)}$ at time t belongs to the neighborhood $\mathbf{N}(\mathbf{x}, \delta)$, then $\mathbf{x}^{(t)}$ must also be a element of the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$. According to the definition of the attraction domain $\mathbf{D}(\mathbf{x}^*, \tau^*)$, fixed the road pricing $\tau^{(t^{'}+1)} = \tau^*$ $(t^{'} \geq t)$ will drive the traffic system to converge to the target equilibrium $\mathbf{x}^*$.

In the next subsection, we will give a rigorous proof to verify that the dynamic pricing scheme can drive the traffic dynamic system in Eq.(2.3)-Eq.(2.5) to converge to the given objective traffic equilibrium state $\mathbf{x}^*$ from any feasible initial link flow $\mathbf{x}^{(0)} \in \Omega_{\mathbf{x}}$.

3.2. Convergence

Before presenting the proof of convergence of the traffic dynamic system in Eq.(2.3)-Eq.(2.5) under the dynamic pricing scheme in Eq.(3.1) and Eq.(3.2), three basic assumptions are firstly given:

Assumption 1. The step size parameter of link flow $\alpha^{(t)}$ $(t = 0, 1, \dots)$ in Eq.(2.3) satisfies

$$\sum_{t=0}^{+\infty} \alpha^{(t)} = +\infty, \sum_{t=0}^{+\infty} (\alpha^{(t)})^2 < +\infty. \tag{3.4}$$

Assumption 2. The function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ in Eq.(2.4) satisfies, $\forall \mathbf{x} \in \Omega_{\mathbf{x}} \backslash \{\mathbf{x}^*\}$,

$$(\mathbf{c}(\mathbf{x}) + \tau)^T (\mathbf{y} - \mathbf{x}) < -\|\mathbf{y} - \mathbf{x}\| \|\mathbf{c}(\mathbf{x}) + \tau\| \sqrt{1 - (\frac{(\mathbf{x} - \mathbf{x}^*)^T (\mathbf{c}(\mathbf{x}) + \tau)}{\|\mathbf{x} - \mathbf{x}^*\| \|\mathbf{c}(\mathbf{x}) + \tau\|})^2}. \tag{3.5}$$

Assumption 3. The function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ in Eq.(2.4) is continuous with respect to $\mathbf{x}$ and τ, and the function $\tau = \tau(\mathbf{x})$ in Eq.(3.2) is continuous with respect to $\mathbf{x} \in \Omega_{\mathbf{x}} / \mathbf{N}(\mathbf{x}^*, \delta)$.

As the traffic dynamic model of this study is the same as used in Guo et al. (2015a), we introduce the same assumption on the step sizes $\alpha^{(t)}$ in Assumption 1. The implication of Assumption 1 can be inferred from the proof of Theorem 3.1. From the proof of Theorem 3.1, it is required that the step sizes shrink slowly enough to ensure that the evolution process of the traffic dynamic system is able to approach the target equilibrium; on the other hand, the step sizes must reduce sufficiently fast so that one can ignore the second term in the right-hand side of Eq.(3.13) when evaluating the distance between the traffic state $\mathbf{x}^{(t+1)}$ and the objective equilibrium $\mathbf{x}^*$. In addition, Eq.(3.5) of Assumption 2 is the condition of Theorem 1 in Guo et al. (2015b) to prove the asymptotical stability of the fixed point of the link-based day-to-day traffic dynamic model. In Assumption 3, the continuity of the function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ is assumed. The reasonability of this assumption can be justified by the existing specific link-based day-to-day traffic dynamic models, such as the model of He et al. (2010), the link flow splitting model of Smith and Mounce (2011), and the link-based network tatonnement process and link-based projected dynamic model of Guo et al. (2015b). Specifically,from the results of Guo et al. (2015b), the function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ of the link flow splitting model of Smith and Mounce (2011) can be described as follows:

$$\mathbf{y} = \mathbf{M}^T \bar{\mathbf{x}},$$

where $\bar{\mathbf{x}}$ is

$$\bar{\mathbf{x}} = \sum_{(a,b)} x_a [\bar{c}_a(\mathbf{x}) - \bar{c}_b(\mathbf{x})]_+ \Upsilon_{ab},$$

where $[y]_+ = \max\{y, 0\}$. $\bar{\mathbf{M}}$ is a coefficient matrix with relation to the link flow splitting rate. Υ_{ab} is a vector with the dimension of $\|A\|$, in which $\gamma_{aba} = -1$ if links a and b lead away the same node; $\gamma_{abb} = 1$ if links a and b lead away the same node; $\gamma_{abq} = -1$ if links a and b do not lead away the same node, or $q \neq a$ and $q \neq b$. In addition, $\bar{c}(\mathbf{x}) = \{\bar{c}_a(\mathbf{x}), a \in A\} = \bar{\mathbf{M}}(\mathbf{c}(\mathbf{x}) + \tau)$. The function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ of the link-based dynamic model of He et al. (2010), the link-based network tatonnement process and link-based projected dynamic model of Guo et al. (2015b) can be described by a uniform formulation:

$$\mathbf{y} = \arg\min_{y\in\Omega} \lambda(\mathbf{c}(\mathbf{x}) + \tau)^T \mathbf{y} + \|\mathbf{y} - \mathbf{x}\|^2$$

On page 74 of their paper, Han and Du (2012) has presented that $\mathbf{y}$ is Lipschitz continuous with respect to $\mathbf{x}$ on the bounded set in $\Omega_{\mathbf{x}}$. Obviously, $\mathbf{y}$ is also continuous on the road pricing τ. A specific method to find the road pricing $\tau(\mathbf{x})$ for any $\mathbf{x} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$ which is continuous with respect to $\mathbf{x}$ in Eq.(3.2) is presented in Section 4.

To prove convergence of the traffic dynamic system in Eq.(2.3)-Eq.(2.5) under the dynamic road pricing scheme of this study, Lemma 2 of Guo et al. (2015b) is needed and presented as follows:

Lemma 3.1. *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$, $\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ *and* $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)$ *be three unit vectors, i.e,* $\|\mathbf{v}\| = 1$, $\|\bar{\mathbf{v}}\| = 1$ *and* $\|\tilde{\mathbf{v}}\| = 1$, *where* $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}$ *denotes the Euclidean norm. Moreover,* $\mathbf{v}^T\tilde{\mathbf{v}} > 0$ *and* $\bar{\mathbf{v}}^T\tilde{\mathbf{v}} > 0$. *Then, it holds that*

$$\mathbf{v}^T\bar{\mathbf{v}} \geqslant (\mathbf{v}^T\tilde{\mathbf{v}})(\bar{\mathbf{v}}^T\tilde{\mathbf{v}}) - \sqrt{1 - (\mathbf{v}^T\tilde{\mathbf{v}})^2}\sqrt{1 - (\bar{\mathbf{v}}^T\tilde{\mathbf{v}})^2}. \tag{3.6}$$

By Lemma 3.1 and Assumption 2, the following proposition can be obtained:

Proposition 3.1. *Under the dynamic pricing scheme in Eq.(3.1) and Eq.(3.2), if* $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}} \backslash \mathbf{N}(\mathbf{x}^*, \delta)$, *then* $\mathbf{x}^{(t)}$, $\mathbf{y}^{(t)} \in \Psi^{(t)}$, *and the objective traffic equilibrium state* $\mathbf{x}^*$ *have the following relation:*

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) < 0 \tag{3.7}$$

Proof. From Eq.(2.5) and Eq.(3.2), it is known that, for $\forall \mathbf{x} \in \Omega_{\mathbf{x}} \backslash \mathbf{N}(\mathbf{x}^*, \delta)$

$$-(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) > 0 \tag{3.8}$$

and

$$-(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{x}^* - \mathbf{x}^{(t)}) > 0 \tag{3.9}$$

Combining the Lemma 3.1 and above two inequalities, it can be found that

$$\begin{aligned}\frac{(\mathbf{y}^{(t)}-\mathbf{x}^{(t)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{y}^{(t)}-\mathbf{x}^{(t)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|} \geqslant &(\frac{-(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|}) \\ &\bullet (\frac{-(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|}) \\ &- \sqrt{1-(\frac{-(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|})^2} \\ &\bullet \sqrt{1-(\frac{-(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|})^2}.\end{aligned} \tag{3.10}$$

Applying Assumption 2 into the right hand side of the (3.10), it is obtained that

$$\begin{aligned}\frac{(\mathbf{y}^{(t)}-\mathbf{x}^{(t)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{y}^{(t)}-\mathbf{x}^{(t)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|} > &\sqrt{1-(\frac{(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|})^2} \\ &\bullet (\frac{(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|}) \\ &- \sqrt{1-(\frac{(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|})^2} \\ &\bullet (\frac{(\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)})^T(\mathbf{x}^*-\mathbf{x}^{(t)})}{\|\mathbf{c}(\mathbf{x}^{(t)})+\tau^{(t+1)}\|\|\mathbf{x}^*-\mathbf{x}^{(t)}\|}) = 0\end{aligned} \tag{3.11}$$

The proof is completed. □

Theorem 3.1. *With Assumption 1-Assumption 3, the dynamic congestion pricing scheme in Eq.(3.1) and Eq.(3.2) can direct the traffic dynamic system in Eq.(2.3)-Eq.(2.5) to eventually converge to the desired objective traffic equilibrium state* $\mathbf{x}^*$.

Proof. The distance $D(\mathbf{x}^{(t+1)}, \mathbf{x}^*)$ between $\mathbf{x}^{(t+1)}$ and the objective traffic equilibrium state $\mathbf{x}^*$ can be described by Euclidean norm:

$$D(\mathbf{x}^{(t+1)}, \mathbf{x}^*) = \|\mathbf{x}^{(t+1)} - \mathbf{x}^*\|^2. \tag{3.12}$$

Substituting Eq.(2.3) into Eq.(3.12), one can obtain that

$$\begin{aligned}D(\mathbf{x}^{(t+1)}, \mathbf{x}^*) = &||\mathbf{x}^{(t)} - \mathbf{x}^*||^2 + 2\alpha^{(t)}(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \\ &+ (\alpha^{(t)})^2\|(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}\|^2,\end{aligned}$$

i.e.,

$$
\begin{aligned}
D(\mathbf{x}^{(t+1)}, \mathbf{x}^*) - D(\mathbf{x}^{(t)}, \mathbf{x}^*) = 2\alpha^{(t)}(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \\
+ (\alpha^{(t)})^2 \|(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}\|^2.
\end{aligned}
\tag{3.13}
$$

From Eq.(3.13), it can be derived that

$$
\begin{aligned}
D(\mathbf{x}^{(t+1)}, \mathbf{x}^*) - D(\mathbf{x}^{(0)}, \mathbf{x}^*) = 2\sum_{n=0}^{t} \alpha^{(n)}(\mathbf{x}^{(n)} - \mathbf{x}^*)^T(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) \\
+ \sum_{n=0}^{t} (\alpha^{(n)})^2 \|(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}\|^2.
\end{aligned}
\tag{3.14}
$$

Since $\Omega_{\mathbf{x}}$ is a compact set, and $D(\mathbf{x}, \mathbf{x}^*)$ is continuous with respect to $\mathbf{x} \in \Omega_{\mathbf{x}}$, the distance function $D(\mathbf{x}, \mathbf{x}^*)$ $(\mathbf{x} \in \Omega_{\mathbf{x}})$ is bounded. Therefore, the right-hand side of Eq.(3.14) is bounded in $\Omega_{\mathbf{x}}$. As $||\mathbf{y} - \mathbf{x}||^2$ $(\mathbf{x}, \mathbf{y} \in \Omega_{\mathbf{x}})$ is also bounded, from Assumption 1, we have

$$
\sum_{n=0}^{+\infty} (\alpha^{(n)})^2 ||(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}||^2 < +\infty. \tag{3.15}
$$

Further,

$$
\sum_{n=0}^{+\infty} \alpha^{(n)}(\mathbf{x}^{(n)} - \mathbf{x}^*)^T(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) > -\infty. \tag{3.16}
$$

Next, we will prove that, under the dynamic pricing scheme, the trajectories of the traffic dynamic system will eventually enter the neighborhood $\mathbf{N}(\mathbf{x}^*, \delta)$ of $\mathbf{x}^*$ in Eq.(3.1) which is a subset of the attraction domain $\mathbf{B}(\mathbf{x}, \tau^*)$ of the objective equilibrium $\mathbf{x}^*$.

Let $\overline{N}_\delta(\mathbf{x}^*) = \{\mathbf{x} \in \Omega_{\mathbf{x}} | \|\mathbf{x} - \mathbf{x}^*\| \geqslant \delta\}$. It is easy to find that $\overline{N}_\delta(\mathbf{x}^*)$ is a compact set as the neighborhood $\mathbf{N}(\mathbf{x}^*, \delta)$ of the objective equilibrium $\mathbf{x}^*$ is the $\mathbf{N}(\mathbf{x}^*, \delta) = \{\mathbf{x} \in \Omega_{\mathbf{x}} | \|\mathbf{x} - \mathbf{x}^*\| < \delta\}$. Define function $Z(\mathbf{x}^{(t)})$:

$$
Z(\mathbf{x}^{(t)}) = (\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}). \tag{3.17}
$$

Then, from Proposition 3.1, we have $Z(\mathbf{x}^{(t)}) < 0$ for any $\mathbf{x}^{(t)} \in \overline{N}_\delta(\mathbf{x}^*)$. From Assumption 3, one can find that $\mathbf{y}^{(t)}$ is a continuous function on $\mathbf{x}^{(t)}$ as $\mathbf{y}^{(t)} = \mathbf{y}^{(t)}(\mathbf{x}^{(t)}, \tau^{(t+1)})$ is continuous on $\mathbf{x}^{(t)}$ and $\tau^{(t+1)}$, and the road pricing $\tau^{(t+1)} =$

$\tau^{(t+1)}(\mathbf{x}^{(t)})$ is a continuous function on $\mathbf{x}^{(t)}$. Thus, $Z(\mathbf{x}^{(t)})$ is continuous on $\mathbf{x}^{(t)}$. By Theorem 5.2.12 in Trench (2013), there is a constant $h < 0$ such that $Z(\mathbf{x}^{(t)}) \leqslant h < 0$ for $\forall \mathbf{x}^{(t)} \in \overline{N}_{\delta}(\mathbf{x}^*)$. As it is known that $0 < \alpha^{(t)} \leqslant 1$ in Assumption 1, we have

$$Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) = (\mathbf{x}^{(t)} - \mathbf{x}^*)^T (\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \leqslant \alpha^{(t)} Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}). \tag{3.18}$$

Under the dynamic pricing scheme Eq.(3.1) and Eq.(3.2), if the trajectories of traffic dynamical system Eq.(2.3)-Eq.(2.5) $\{\mathbf{x}^{(t)}, t = 0, 1, \dots\}$ do not enter $N_{\epsilon}(\mathbf{x}^*)$, then

$$\sum_{n=0}^{+\infty} \alpha^{(n)} (\mathbf{x}^{(n)} - \mathbf{x}^*)^T (\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) \leqslant \sum_{n=0}^{+\infty} \alpha^{(n)} h < -\infty \tag{3.19}$$

This contradicts with Eq.(3.16). Therefore, the trajectories $\{\mathbf{x}^{(t)}, t = 0, 1, \dots, \}$ of the traffic dynamical system Eq.(2.3)-Eq.(2.5) enter the neighborhood $N(\mathbf{x}^*, \delta)$ of the objective equilibrium $\mathbf{x}^*$.

Let t' denote the time when the trajectories $\{\mathbf{x}^{(t)}, t = 0, 1, \dots, \}$ first enter $\mathbf{N}(\mathbf{x}^*, \delta)$. As $\mathbf{N}(\mathbf{x}^*, \delta) \subset \mathbf{B}(\mathbf{x}^*, \tau^*)$, $\mathbf{x}^{t'}$ also belongs to the attraction domain $\mathbf{B}(\mathbf{x}^*, \tau^*)$ of the objective equilibrium $\mathbf{x}^*$. From the dynamic road pricing scheme in Eq.(3.1) and Eq.(3.2), the road pricing $\tau^{(t'+1)} = \tau^*$. From the definition of the attraction domain $\mathbf{B}(\mathbf{x}^*, \tau^*) = \{\mathbf{x}^{(0)} \in \Omega_{\mathbf{x}} | \lim_{n \to +\infty} \mathbf{x}^{(n)} = \mathbf{x}^*\}$ of $\mathbf{x}^*$ under road pricing τ^*, one can observe that the traffic state $\mathbf{x}^{(t)}$ $(t > t')$ of the trajectories of the traffic dynamic system after time t' will still belong to $\mathbf{B}(\mathbf{x}^*, \tau^*)$ of the objective equilibrium $\mathbf{x}^*$ under the road pricing τ^*. Therefore, after time t', the dynamic pricing $\tau^{(t+1)} = \tau^*$ $(t > t')$, and the trajectories $\{\mathbf{x}^{(t)}, t = t'+1, t'+2, \dots, \}$ of the traffic dynamical system Eq.(2.3)-Eq.(2.5) eventually converge to $\mathbf{x}^*$. The proof is completed.

□

As shown above,, we have presented a general formulation in Eq.(3.1) and Eq.(3.2) of the dynamic road pricing scheme, which can ensure that the traffic dynamic system in Eq.(2.3)-Eq.(2.5) can eventually converge to the given objective traffic equilibrium state $\mathbf{x}^*$. However, it is still necessary to determine a specific form of dynamic road pricing scheme to to specify how the general formulation in Eq. (3.1) and Eq. (3.2) can be applied in real practice.

4. A specific dynamic pricing scheme

In Section 3, we present a very general dynamic road pricing scheme, which can drive the traffic system to achieve the given objective traffic equilibrium state from any initial traffic state even when multiple traffic equilibrium states exist. Basically, there are many such dynamic road pricing schemes that fulfil the conditions in Section 3. To ensure a practical implementation, we have to design a specific dynamic pricing scheme that is readily used in practice. The specific dynamic pricing scheme is described as follows:

$$\tau^{(t+1)} \begin{cases} = \hat{\tau}^{(t)}/(2\beta^{(t)}), \text{ if } \mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta), \\ = \tau^*, \text{if } \mathbf{x}^{(t)} \in \mathbf{N}(\mathbf{x}^*, \delta), \end{cases} \tag{4.1}$$

where $\mathbf{x}^*$ is the given objective traffic equilibrium. $\mathbf{x}^*$ with the road pricing τ^* follow the formulation Eq.(3.3). To determine $\hat{\tau}^{(t)}$, we first formulate the following optimization problem Eq.(4.2) and use the Lagrange multiplier vector of the link flow constraint Eq.(4.2b) as the values of $\hat{\tau}^{(t)}$. It should be noted that, the formulated optimization problem in Eq.(4.2) has no physical meaning at all, and its mathematical construction is purely used to acquire the value of $\hat{\tau}^{(t)}$ in the specific dynamic pricing scheme designed in Eq.(4.1):

$$\min_{\mathbf{y}} \|\beta^{(t)}\mathbf{c}(\mathbf{x}^{(t)}) - (\mathbf{x}^{(t)} - \mathbf{y})\|^2, \tag{4.2a}$$

$$s.t. \quad y_a \leqslant x_a^*, a \in A, \tag{4.2b}$$

$$\sum_r f_w^r = d_w, w \in W, \tag{4.2c}$$

$$y_a = \sum_w \sum_r f_w^r \delta_{ar}^w, \tag{4.2d}$$

$$f_w^r \geqslant 0, r \in R_w, w \in W, \tag{4.2e}$$

where parameter $\beta^{(t)} > 0$.

In the following, we prove that the dynamic road pricing scheme in Eq.(4.1) can drive the traffic system to converge to the desired objective traffic equilibrium state from any initial feasible traffic state in Theorem 4.2. To achieve so, we firstly prove that the dynamic pricing scheme in Eq.(4.1) belongs to the general pricing scheme in Section 3, as is stated in Proposition 4.3. Before this conclusion can be verified, Proposition 4.1 and 4.2 are presented to describe the mathematical properties of the constructed optimization problems Eq.(4.2) which are used to produce the pricing scheme.

Proposition 4.1. *The optimization problem Eq.(4.2) has a unique feasible link flow vector* $\mathbf{x}^*$.

Proof. As the objective traffic state $\mathbf{x}^*$ in this study is a user equilibrium link flow pattern, and it can be described as the solution of a variational inequity in Eq.(3.3), i.e.,

$$(\mathbf{c}(\mathbf{x}^*) + \tau^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0, \quad \forall \mathbf{x} \in \Omega_{\mathbf{x}}$$

If there is a link flow $\mathbf{x}' \in \Omega_{\mathbf{x}}$ wherein $\mathbf{x}' \leqslant \mathbf{x}^*$ and $\mathbf{x}' \neq \mathbf{x}^*$, there exist $a' \in A$, which $\mathbf{x}'_{a'} < \mathbf{x}^*_{a'}$. Then

$$(\mathbf{c}(\mathbf{x}^*) + \tau^*)^T(\mathbf{x}' - \mathbf{x}^*) < 0,$$

and $\mathbf{x}^*$ will be not a user equilibrium state. So, the feasible domain of problem Eq.(4.2) has only one feasible flow $\mathbf{x}^*$, and the conclusion of Proposition 4.1 is proved. □

Based on Proposition 4.1, one can obtain that, under the road pricing $\tau^{(t+1)} = \hat{\tau}^{(t)}/(2\beta^{(t)})$ when $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$, the objective equilibrium $\mathbf{x}^*$ is the solution of optimization problem (4.3). The conclusion is stated as follows:

Proposition 4.2. *For* $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$*, under the road pricing* $\tau^{(t+1)}$ *in Eq.(4.1), the objective traffic equilibrium state* $\mathbf{x}^*$ *is the unique optimal solution of the following optimization problem,*

$$\min_{\mathbf{y}} \|\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{y})\|^2. \tag{4.3a}$$

$$s.t. \qquad \mathbf{y} \in \Omega_{\mathbf{x}}. \tag{4.3b}$$

Proof. Because problem in Eq.(4.3) is a strict convex optimization problem on the decision variable $\mathbf{y} \in \Omega_{\mathbf{x}}$, it has one unique optimal solution. Besides, its solution optimality conditions are also its KKT conditions. Let $\tilde{\mathbf{x}}$ be the optimal solution of Eq.(4.3), then

$$(\sum_{a \in A}(\hat{c}_a(\tilde{\mathbf{x}}) + \hat{\tau}_a^{(t)})\delta_{a,r}^w - \hat{\mu}_w)f_r^w = 0, w \in W, \tag{4.4a}$$

$$(\hat{c}_a(\tilde{\mathbf{x}}) + \hat{\tau}_a^{(t)})\delta_{a,r}^w - \hat{\mu}_w \geqslant 0, \tag{4.4b}$$

$$\tilde{x}_a = \sum_{w \in W}\sum_{r \in R_w} f_r^w \delta_{a,r}^w, a \in A, f_r^w \in \Omega_{\mathbf{f}}. \tag{4.4c}$$

where $\hat{c}_a(\tilde{\mathbf{x}}) = 2\beta^{(t)}(c_a(\mathbf{x}^{(t)}) - 2(\mathbf{x}^{(t)} - \tilde{\mathbf{x}}))$, and $\hat{\mu}_w$ is the Lagrange multiplier related to traffic demand constraint conditions between OD pair $w \in W$. It can also be found that the objective function of optimization problem Eq.(4.2) is strictly convex on link flow $\mathbf{y} \in \Omega_{\mathbf{x}}$. Based on Proposition 4.1, its solution optimality conditions can be expressed as:

$$(\sum_{a \in A}(\hat{c}_a(\mathbf{x}^*) + \hat{\tau}_a^{(t)})\delta_{a,r}^w - \mu_w)f_r^w = 0, w \in W, \tag{4.5a}$$

$$(\hat{c}_a(\mathbf{x}^*) + \hat{\tau}_a^{(t)})\delta_{a,r}^w - \mu_w \geqslant 0, \tag{4.5b}$$

$$\hat{\tau}_a^{(t)} \geqslant 0, a \in A, \tag{4.5c}$$

$$x_a^* = \sum_{w\in W}\sum_{r\in R_w} f_r^w \delta_{a,r}^w, a \in A, f_r^w \in \Omega_{\mathbf{f}}. \tag{4.5d}$$

where $\hat{c}_a(\mathbf{x}^*) = 2\beta^{(t)}(c_a(\mathbf{x}^{(t)}) - 2(\mathbf{x}^{(t)} - \mathbf{x}^*))$. μ_w $(w \in W)$ is the Lagrange multiplier of demand constraint between OD pair w in Eq.(4.2b).

Comparing Eq.(4.4) with Eq.(4.5), one can find that the solutions of the equations Eq.(4.5) are also the solutions of the equation (4.4). Therefore, $\mathbf{x}^*$ and μ_w $(w \in W)$ are also the solution of equation system in Eq.(4.4). Based on the uniqueness of optimal solution of problem Eq.(4.3), $\tilde{\mathbf{x}} = \mathbf{x}^*$. The proof is completed. □

Based on Proposition 4.2, we can prove that the dynamic pricing scheme in Eq.(4.1) belongs to the general pricing scheme in Section 3, i.e, for any $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^{\mathbf{t}}, \delta)$, the $\tau^{(t+1)}$ can ensure that $\mathbf{x}^*$ follows Eq.(2.5),

$$(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{x}^* - \mathbf{x}^{(t)}) < 0$$

Proposition 4.3. *When $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^{\mathbf{t}}, \delta)$, under the road pricing $\tau^{(t+1)}$ in Eq.(4.1), $\mathbf{x}^{(t)} - \mathbf{x}^*$ is the projection of $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$ onto the closed convex set $Q^{(t)}$, which is described as:*

$$Q^{(t)} = \{\mathbf{z} | \mathbf{z} = \mathbf{x}^{(t)} - \mathbf{y}, \mathbf{y} \in \Omega_{\mathbf{x}}\}. \tag{4.6}$$

And,

$$(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{x}^* - \mathbf{x}^{(t)}) < 0, \tag{4.7}$$

i.e., for $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^{\mathbf{t}}, \delta)$, $\mathbf{x}^* \in \Psi^{(t)}$ in Eq.(2.5).

Proof. As $\Omega_{\mathbf{x}}$ is a compact convex set, $Q^{(t)}$ is also a compact convex set. By the definition of projection and the set $Q^{(t)}$ in Eq.(4.6),

$$\begin{aligned} P_{Q^{(t)}}[\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})] &= \arg\min_{\tilde{\mathbf{y}} \in Q^{(t)}} ||\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - \tilde{\mathbf{y}}||^2 \\ &= \mathbf{x}^{(t)} - \arg\min_{\mathbf{y}\in\Omega_{\mathbf{x}}} ||\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{y})||^2. \end{aligned}$$

From Proposition 4.2,

$$\mathbf{x}^* = \arg\min_{\mathbf{y}\in\Omega_{\mathbf{x}}} ||\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{y})||^2.$$

Therefore,

$$\mathbf{x}^{(t)} - \mathbf{x}^* = P_{Q^{(t)}}[\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})].$$

And,

$$\begin{aligned} &\|\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{x}^*)\|^2 \\ < &\|\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{x}^{(t)})\|^2, \end{aligned}$$

thus,

$$(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{x}^* - \mathbf{x}^{(t)}) < 0.$$

The proof is completed. □

Based on the conclusions of Proposition 4.3, we can prove that, without Assumption 2, when $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^{\mathbf{t}}, \delta)$, the road pricing $\tau^{(t+1)}$ in Eq.(4.1) can ensure that $\mathbf{x}^{(t)}$, $\mathbf{y}^{(t)}$ and $\mathbf{x}^*$ also have the relation in Eq.(3.7).

Theorem 4.1. *Under the dynamic pricing scheme in Eq.(4.1), if* $\mathbf{x}^{(t)} \neq \mathbf{x}^*$ *at time* t*, then* $\mathbf{x}^{(t)}$, $\mathbf{y}^{(t)}$ *in Eq.(2.5), and the objective traffic equilibrium* $\mathbf{x}^*$ *have the following relation*

$$(\mathbf{x}^* - \mathbf{x}^{(t)})^T(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}) < 0. \tag{4.8}$$

Proof. From Proposition 4.3, when $\mathbf{x}^{(t)} \neq \mathbf{x}^*$, $\mathbf{x}^* \in \Psi^{(t)}$ in Eq.(2.5) under the road pricing $\tau^{(t+1)}$ in Eq.(4.1). From the definition $\Psi^{(t)}$ in Eq.(2.5), it can be found $\mathbf{x}^{(t)} \notin \Psi^{(t)}$. If $\Psi^{(t)} - \{\mathbf{x}^*\} = \emptyset$, then $\mathbf{x}^* = \mathbf{y}^{(t)}$, and

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) = (\mathbf{x}^* - \mathbf{x}^{(t)})^T(\mathbf{x}^{(t)} - \mathbf{x}^*) < 0. \tag{4.9}$$

From the definition of $Q^{(t)}$ in Eq.(4.6), one can find that original point $\mathbf{O} \in Q^{(t)}$. If $\Psi^{(t)} - \{\mathbf{x}^*\} \neq \emptyset$, for any $\hat{\mathbf{y}}^{(t)} \in \Psi^{(t)} - \{\mathbf{x}^*\}$, points $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ can make a plane or a line.

When they make a plane, let point $\hat{\mathbf{B}}$ be the projection of $\beta^{(t)}(\mathbf{c}(\mathbf{x}^t)+\tau^{(t+1)})$ onto the plane, and $\hat{Q}^{(t)} = \{\mathbf{z}|\mathbf{z} = \alpha_1\mathbf{O} + \alpha_2(\mathbf{x}^{(t)} - \mathbf{x}^*) + \alpha_3(\mathbf{x}^{(t)} - \hat{\mathbf{y}}), \sum_i \alpha_i =$

$1, \alpha_1, \alpha_2, \alpha_3 \geqslant 0\}$. It can be found that $\hat{Q}^{(t)} \subset Q^{(t)}$, and $\hat{Q}^{(t)}$ is a closed convex set. From Proposition 4.3, $\mathbf{x}^{(t)} - \mathbf{x}^*$ is the projection of $\beta^{(t)}(\mathbf{c}(\mathbf{x}^t) + \tau^{(t+1)})$ onto $Q^{(t)}$. Thus $\mathbf{x}^{(t)} - \mathbf{x}^*$ is also the projection of $\beta^{(t)}(\mathbf{c}(\mathbf{x}^t) + \tau^{(t+1)})$ onto $\hat{Q}^{(t)}$. Although $\mathbf{x}^{(t)} - \mathbf{x}^*$ may be not equal to $\hat{\mathbf{B}}$, $\|\hat{\mathbf{B}} - (\mathbf{x}^{(t)} - \mathbf{x}^*)\|$ is the minimum distance between point belonging to $\hat{Q}^{(t)}$ and $\hat{\mathbf{B}}$. Thus, when $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ make a plane, we can use Figure 4 to show the relation among $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}$.

In Figure 4 , as $\hat{\mathbf{y}}^{(t)} \in \Psi^{(t)}$ in Eq.(2.5), the angle $\gamma \in [0, \pi/2)$ between vectors $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$. In the same way, the angle $\zeta \in [0, \pi/2)$ between $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$ and $\mathbf{x}^{(t)} - \mathbf{x}^*$. Thus the angle between vectors $\mathbf{O}\hat{\mathbf{B}}$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ is a cute angle, and the angle between vectors $\mathbf{O}\hat{\mathbf{B}}$ and $\mathbf{x}^{(t)} - \mathbf{x}^*$ is also a cute angle. Further, the angle between vectors $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ is a cute angle. We obtain that, when $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ make a plane,

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\hat{\mathbf{y}}^{(t)} - \mathbf{x}^{(t)}) < 0. \tag{4.10}$$

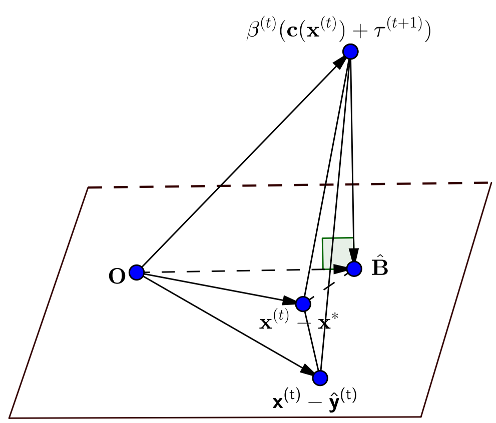

Figure 4: Relation among $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{x}}$

If points $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ are collinear, then $\mathbf{x}^{(t)}$, $\mathbf{x}^*$ and $\hat{\mathbf{y}}^{(t)}$ are collinear. As $\mathbf{x}^{(t)} \notin \Psi^{(t)}$, and $\Psi^{(t)}$ is a convex set,

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\hat{\mathbf{y}}^{(t)} - \mathbf{x}^{(t)}) < 0.$$

Therefore, for any $\mathbf{y} \in \Psi^{(t)}$,

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) > 0.$$

The proof is completed. □

Based on Theorem 4.1, the convergence of the traffic dynamic system under the specific dynamic road pricing scheme developed in this section can be described as follows:

Theorem 4.2. *With Assumption 1 and the assumption on the continuity of the function* $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ *with respect to* $\mathbf{x}$ *and* τ*, the dynamic road pricing scheme* $\tau^{(t+1)}$ $(t = 0, 1,)$ *in Eq.(4.1) can direct the traffic dynamic system Eq.(2.3)-Eq.(2.5) to converge to the objective traffic equilibrium state* $\mathbf{x}^*$ *from any feasible initial traffic state.*

Proof. From Theorem 4.1, the dynamic road pricing scheme $\tau^{(t+1)}$ $(t = 0, 1, \dots)$ in Eq.(4.1) satisfies Proposition 3.1. Next, we prove that the road pricing $\tau^{(t+1)}$ at time $t + 1$ $(t = 0, 1, \dots)$ in Eq.(4.1) is continuous on $\mathbf{x}^{(t)}$ when $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$.

From the proof of Proposition 4.2, one can find that, for any $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$, the road pricing $\tau^{(t+1)} \geq 0$ always exists and is the solution of equations (4.5), i.e.,

$$(\sum_{a \in A}(\hat{c}_a(\mathbf{x}^*) + 2\beta^{(t)}\tau_a^{(t+1)})\delta_{a,r}^w - \mu_w)f_r^w = 0, w \in W, \tag{4.11a}$$

$$(\hat{c}_a(\mathbf{x}^*) + 2\beta^{(t)}\tau_a^{(t+1)})\delta_{a,r}^w - \mu_w \geqslant 0, \tag{4.11b}$$

$$\tau_a^{(t+1)} \geqslant 0, a \in A, \tag{4.11c}$$

$$x_a^* = \sum_{w \in W} \sum_{r \in R_w} f_r^w \delta_{a,r}^w, a \in A, f_r^w \in \Omega_{\mathbf{f}}. \tag{4.11d}$$

where $\hat{c}_a(\mathbf{x}^*) = 2\beta^{(t)}(c_a(\mathbf{x}^{(t)}) - 2(\mathbf{x}^{(t)} - \mathbf{x}^*))$. As the objective equilibrium $\mathbf{x}^*$ is given, the corresponding path flow f_w^r $(r \in R_w, w \in W)$ making the link flow pattern $\mathbf{x}^*$ should be also known. Therefore, from Eq.(4.11a) and Eq.(4.11b), one can obtain the follow linear equations:

$$\sum_{a \in A} (\hat{c}_a(\mathbf{x}^*) + 2\beta^{(t)}\tau_a^{(t+1)})\delta_{a,r}^w - \mu_w = 0, w \in W, \tag{4.12}$$

where $f_w^r > 0$ $(r \in R_w,\ w \in W)$. The road pricing $\tau^{(t+1)}$ also follows linear equations Eq.(4.12). One can find that the coefficient matrix of the linear equations Eq.(4.12) is constant, and each of its solution is continuous on $\hat{c}_a(\mathbf{x}^*)$ which is continuous on $\mathbf{x}^{(t)}$. Therefore, the road pricing $\tau^{(t+1)}$ is continuous on $\mathbf{x}^{(t)}$. By the proof of Theorem 3.1, the conclusion of Theorem 4.2 is verified. The proof is completed.

□

5. Numerical test

In this section, some numerical tests are conducted to verify the validity of our dynamic road pricing scheme. The first test example only has one OD pair and three links. The travel cost functions of these three links are shown as below:

$$c_1 = f_1 + f_2 + 2, \quad c_2 = 2f_1 + f_2 + 1, \quad c_3 = f_3 + 6. \tag{5.1}$$

The traffic demand is 2 units. Its system optimal flow is $(0, 2, 0)$ which is also an UE state under the road pricing $\tau = (2, 2, 0)$. Under this pricing $\tau = (2, 2, 0)$, there are three traffic equilibrium states in this example:

$$\mathbf{f}^I = (2, 0, 0), \quad \mathbf{f}^{II} = (0, 2, 0), \quad \mathbf{f}^{III} = (1, 1, 0). \tag{5.2}$$

The link-based traffic dynamic model of He et al. (2010) is applied in this numerical tests, and its specific formulation has been presented in Eq.(2.7)-Eq.(2.8).

The parameter δ used to definition the neighborhood $\mathbf{N}(\mathbf{x}^*, \delta)$ in Eq.(3.1) is valued of 0.1

It can be found that, if we let $\beta^{(t)}$ in Eq.(4.1) be equal to $\frac{\lambda}{2(1-\lambda)}$ of Eq.(2.7), then the road pricing determined by Eq.(4.1) makes $\mathbf{y}^{(t)}$ ($t = 0, 1, \ldots,$) in Eq.(2.8) equal to the objective traffic equilibrium state $\mathbf{x}^*$. First, we set the SO traffic flow pattern $(0, 2, 0)$ to be the objective traffic state. Figure (5) shows the trajectories of flows on link 2 and link 3 under dynamic pricing scheme in Eq.(4.1) with $\lambda^{(t)} = \lambda = 0.6$ and $\alpha^{(t)} = 1/(t + 1)$. It can be found that all feasible initial traffic state points are driven towards the objective traffic state under the dynamic road pricing scheme. Figure (6) shows the evolution of tolls on link 1, 2 and 3 corresponding to the evolution of link flows with initial traffic state of (0.10,0,1.90).

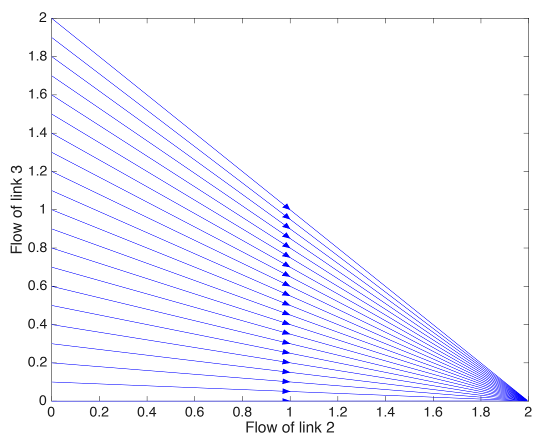

Figure 5: Trajectory of flows on link 2 and link 3

In above numerical tests, the objective traffic equilibrium state is set to be SO state. We can also use other traffic equilibrium state as the objective traffic state. For example, the other two traffic equilibrium states under the marginal cost road pricing, i.e.,$(2, 0, 0)$ and $(1, 1, 0)$ can also be set as the objective traffic state. Figure.(7) and Figure (8) show the trajectories of network flows. The values of parameters are the same as those in above numerical tests. One can

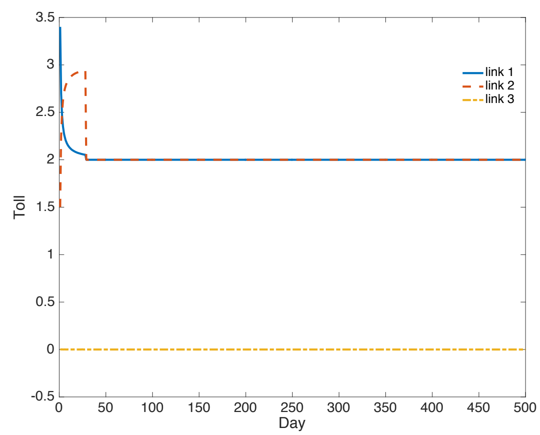

Figure 6: Evolution of road toll of adjusted pricing scheme

find that, under the dynamic road pricing scheme, traffic dynamic system also converges to the desired objective traffic equilibrium state.

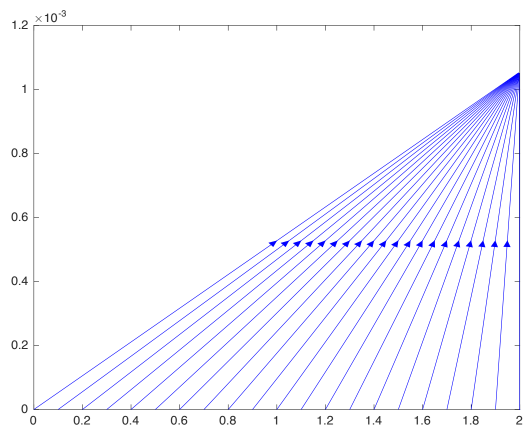

Figure 7: Trajectories of flows with objective flow $(2, 0, 0)$

One can find that the first example is limited in that route tolls and link tolls are the same. Therefore, a slightly larger example is presented in Figure 9. The second example has 10 links and two OD pairs: $OD(1, 8)$ and $OD(2, 8)$. The travel cost function for each link is; $c_1 = x_1 + 1$, $c_2 = x_2 + 1$, $c_3 = x_3 + 1$,

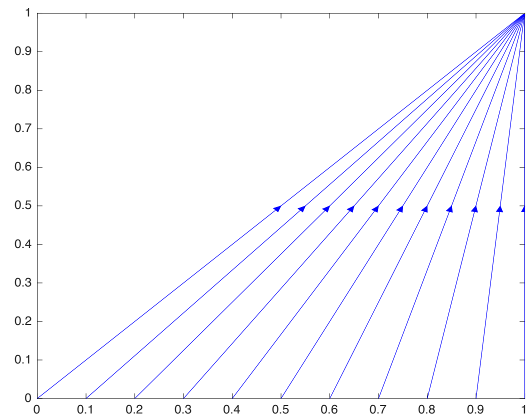

Figure 8: Trajectories of flows with objective flow $(1, 1, 0)$

$c_4 = x_4+1$, $c_5 = x_5+3x_6+1$, $c_6 = 2x_5+x_6+1$, $c_7 = x_7+3x_8+1$, $c_8 = 2x_7+x_8+1$, $c_9 = x_8 + 1$ and $c_{10} = x_{10} + 1$. The demand of these two OD pairs are both 1 unit. Link flow patterns $(1, 1, 1, 1, 1, 0, 1, 0, 1, 1)$ and $(1, 1, 1, 1, 0, 1, 0, 1, 1, 1)$ are both UE equilibrium flow patterns under the toll $(0, 0, 1, 1, 1, 2, 1, 2, 1, 1)$.

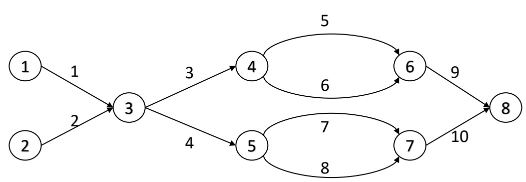

Figure 9: The second Example

Figure 10 presents the flow trajectories of link 5 and link 7 when the objective equilibrium is $(1, 1, 1, 1, 1, 0, 1, 0, 1, 1)$. Figure 11 shows the flow trajectories of link 6 and link 8, the objective equilibrium is $(1, 1, 1, 1, 0, 1, 0, 1, 1, 1)$. From Figure 10 and 11, one can find that, under the dynamic pricing scheme of this study, the traffic system of the second example can converge to the given objective equilibrium state.

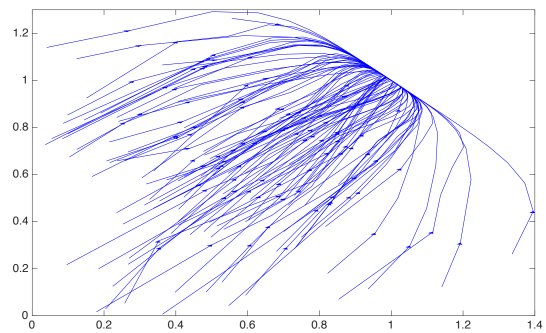

Figure 10: Flow trajectories of link 5 and link 7 for the objective equilibrium $(1, 1, 1, 1, 1, 0, 1, 0, 1, 1)$

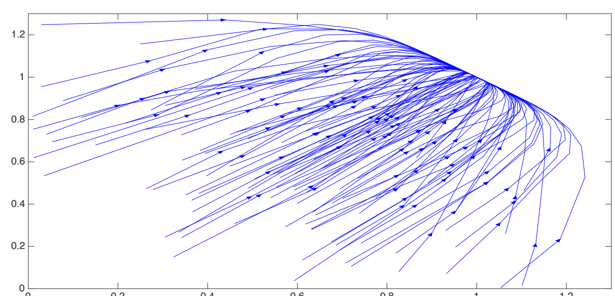

Figure 11: Flow trajectories of link 6 and link 8 for the objective equilibrium $(1, 1, 1, 1, 0, 1, 0, 1, 1, 1)$

6. Conclusions

In this study, we present a general discrete-time day-to-day dynamic road pricing scheme which varies the toll on a day-to-day basis and aims to drive the traffic system to a given objective traffic equilibrium state from any initial traffic state when multiple equilibria exist. As was pointed out in previous researches, a desired traffic equilibrium will not be achieved through a day-to-day adjustment process if the initial traffic state does not fall into its attraction domain. This study explicitly addresses this problem on how to apply a day-to-day dynamic pricing scheme to direct the traffic evolution towards the desired traffic state regardless of the initial traffic state and the attraction domain. We present vigorous proof and numerical tests to verify the validity of our dynamic road pricing scheme.

Nevertheless, from the analysis in this study, one can find that we propose a general form of dynamic pricing scheme whose specific formulation is not unique. Therefore, in future works, we will study how to develop specific dynamic road pricing considering more practical constraints, for example, what is the pricing scheme that incurs the minimum system travel cost during its implementation process before the desired traffic state is reached. In addition, we also want to develop dynamic road pricing or other dynamic control measures with consideration of more general assumptions on travelers' route choice behaviors.

Acknowledgements

The authors gratefully acknowledge stimulating discussions with Professor Han Deren of Nanjing Normal University, China. This research was partly funded by the Singapore Ministry of Education Academic Research Fund Tier 2 MOE2015-T2-2-076, General Research Fund #615712 of the Research Grants Council of the HKSAR Government, National Natural Science Foundation of China (71621001), and the China National Funds for Distinguished Young Scientists (71525002).

Appendix A. Dynamic pricing scheme in a network with elastic demand

In this appendix, we extend the dynamic pricing scheme in Section 3 to the traffic network with elastic demand. In the network, B_w denote the generalized travel cost or the inverse demand function between OD pair $w \in W$, which is a positive function of traffic demand vector $\mathbf{d}$, and $\mathbf{B}(\mathbf{d}) = (B_w(\mathbf{d}), w \in W)^T$ is the corresponding vector. Let $\bar{d}(\geq 0)$ be the potential or maximum demand between OD pair $w \in W$, and $\bar{\mathbf{d}} = (\bar{d}_w, w \in W)^T$. Let $\Omega_{\mathbf{x},\mathbf{d}}$ denote the set of feasible link flows and travel demands, i.e.,

$$\Omega_{\mathbf{x},\mathbf{d}} = \{(\mathbf{x}) = \Delta\mathbf{f}, \mathbf{d} = \Lambda\mathbf{f}, \mathbf{f} \geq 0, \mathbf{d} \leq \bar{\mathbf{d}}\}. \tag{A.1}$$

It is assumed that there is at least one route between each OD pair. Then, the feasible set $\Omega_{\mathbf{x},\mathbf{d}}$ is nonempty, compact and convex.

For elastic demand case, the day-to-day traffic dynamic model can be described as:

$$\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} + \alpha^{(t)}(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}); \tag{A.2}$$

$$\mathbf{d}^{(t+1)} = \mathbf{d}^{(t)} + \alpha^{(t)}(\mathbf{e}^{(t)} - \mathbf{x}^{(t)}); \tag{A.3}$$

where $(\mathbf{y}^{(t)}, \mathbf{e}^{t})$ is a function on $(\mathbf{x}^{t}, \mathbf{e}^{(t)})$ and road pricing $\tau^{(t+1)}$ and satisfies

$$(\mathbf{y}^{(t)}, \mathbf{e}^{(t)}) \begin{cases} \in \Psi^{(t)}, \Psi^{(t)} \neq \emptyset, \\ = (\mathbf{x}^{(t)}, \mathbf{d}^{(t)}), \Psi^{(t)} = \emptyset, \end{cases} \tag{A.4}$$

where

$$\begin{aligned} \Psi^{(t)} = \{(\mathbf{y}, \mathbf{e}) | (\mathbf{y}, \mathbf{e}) \in \Omega_{\mathbf{x},\mathbf{d}}, \\ (\mathbf{y} - \mathbf{x}^{(t)})^T(\mathbf{c}(\mathbf{x}^{t}) + \tau^{(t+1)}) - (\mathbf{e} - \mathbf{d}^{(t)})^T\mathbf{B}(\mathbf{d}^{t}) < 0\}. \end{aligned} \tag{A.5}$$

Let $(\mathbf{x}^*, \mathbf{d}^*)$ be the UE link flow pattern and traffic demand under the road pricing τ^*, i.e., $(\mathbf{x}^*, \mathbf{d}^*)$ is the solution of the following variational inequality:

$$(\mathbf{x} - \mathbf{x}^*)^T(\mathbf{c}(\mathbf{x}^*) + \tau^*) - (\mathbf{e} - \mathbf{d}^*)^T\mathbf{B}(\mathbf{d}^*) \geq 0, \forall(\mathbf{x}, \mathbf{e}) \in \Omega_{\mathbf{x},\mathbf{d}} \tag{A.6}$$

In similar manner, the dynamic pricing scheme in elastic demand case can be represented as:

$$\tau^{(t+1)} \begin{cases} \in \Upsilon^{(t+1)}, (\mathbf{x}^{(t)}, \mathbf{d}^{(t)}) \in \Omega_{\mathbf{x},\mathbf{d}}/\mathbf{N}((\mathbf{x}^*, \mathbf{d}^*), \delta), \\ = \tau^*, (\mathbf{x}^*, \mathbf{d}^*) \in \mathbf{N}((\mathbf{x}^*, \mathbf{d}^*), \delta), \end{cases} \tag{A.7}$$

where

$$\Upsilon^{(t+1)} = \{\tau | \tau \geq 0, (\mathbf{x}^* - \mathbf{x}^{(t)})^T (\mathbf{c}(\mathbf{x}^{(t)}) + \tau) - (\mathbf{d}^* - \mathbf{d}^{(t)})^T \mathbf{B}(\mathbf{d}^{(t)}) < 0\}. \tag{A.8}$$

$\mathbf{N}((\mathbf{x}^*, \mathbf{d}^*), \delta)$ is a neighborhood of $(\mathbf{x}^*, \mathbf{d}^*)$ and a subset of the attraction domain $\mathbf{B}(\mathbf{x}^*, \tau^*)$ of $(\mathbf{x}^*, \mathbf{d}^*)$.

To prove the convergence of the traffic dynamic system in Eq.(A.2)-Eq.(A.4) under the dynamic pricing scheme in Eq.(A.7), three assumptions are given:

Assumption 4. The step size parameter of link flow $\alpha^{(t)}$ $(t = 0, 1, \dots)$ in Eq.(A.2) satisfies

$$\sum_{t=0}^{+\infty} \alpha^{(t)} = +\infty, \sum_{t=0}^{+\infty} (\alpha^{(t)})^2 < +\infty. \tag{A.9}$$

Assumption 5. $(\mathbf{y}(\mathbf{x}, \mathbf{d}, \tau), \mathbf{e}(\mathbf{x}, \mathbf{d}, \tau))$ in Eq.(A.2) and Eq.(A.3) satisfies, $\forall (\mathbf{x}, \mathbf{b}) \in \Omega_{\mathbf{x},\mathbf{b}} \backslash \mathbf{N}(\mathbf{x}^*, \delta)$,

$$\begin{pmatrix} \mathbf{y}(\mathbf{x}, \mathbf{d}, \tau) - \mathbf{x} \\ \mathbf{e}(\mathbf{x}, \mathbf{d}, \tau) - \mathbf{d} \end{pmatrix}^T \begin{pmatrix} \mathbf{c}(\mathbf{x}) + \tau \\ -\mathbf{B}(\mathbf{d}) \end{pmatrix} < \left\| \begin{matrix} \mathbf{y}(\mathbf{x}, \mathbf{d}, \tau) - \mathbf{x} \\ \mathbf{e}(\mathbf{x}, \mathbf{d}, \tau) - \mathbf{d} \end{matrix} \right\| \left\| \begin{matrix} \mathbf{c}(\mathbf{x}) + \tau \\ -\mathbf{B}(\mathbf{d}) \end{matrix} \right\| \sqrt{1 - \left(\frac{\begin{pmatrix} \mathbf{x} - \mathbf{x}^* \\ \mathbf{d} - \mathbf{d}^* \end{pmatrix}^T \begin{pmatrix} \mathbf{c}(\mathbf{x}) + \tau \\ -\mathbf{B}(\mathbf{d}) \end{pmatrix}}{\left\| \begin{matrix} \mathbf{x} - \mathbf{x}^* \\ \mathbf{d} - \mathbf{d}^* \end{matrix} \right\| \left\| \begin{matrix} \mathbf{c}(\mathbf{x}) + \tau \\ -\mathbf{B}(\mathbf{d}) \end{matrix} \right\|} \right)^2} \tag{A.10}$$

Assumption 6. The function $(\mathbf{y}, \mathbf{e}) = \mathbf{y}(\mathbf{x}, \mathbf{d}, \tau)$ in Eq.(A.2) is continuous with respect to $\mathbf{x}$, $\mathbf{d}$ and τ, and the function $\tau = \tau(\mathbf{x})$ in Eq.(A.7) is continuous with respect to $\mathbf{x} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^*, \delta)$.

Combining Assumption 5 and Lemma 3.1, one can easily observe that, for any $(\mathbf{x}^{(t)}, \mathbf{d}^{(t)}) \in \Omega_{\mathbf{x},\mathbf{d}}/\mathbf{N}((\mathbf{x}^*, \mathbf{d}^*), \delta)$, $(\mathbf{x}^{(t)}, \mathbf{d}^{(t)})$, $(\mathbf{y}, \mathbf{e})$ and $(\mathbf{x}^*, \mathbf{d}^*)$ have the following relation:

$$(\mathbf{x}^{(t)} - \mathbf{x}^*)^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) + (\mathbf{d}^{(t)} - \mathbf{d}^*)^t(\mathbf{e}^{(t)} - \mathbf{d}^{(t)}) < 0 \tag{A.11}$$

Theorem Appendix A.1. *With Assumption 4-Assumption 6, the dynamic congestion pricing scheme in Eq.(A.2) and Eq.(A.3) can direct the traffic dynamic system in Eq.(A.7) to eventually converge to the desired objective traffic equilibrium state* $(\mathbf{x}^*, \mathbf{d}^*)$.

Proof. This theorem can be proved by applying the same method of Theorem 3.1.Therefore, we do not present the elaborated details here. □

References

Bie J, Lo HK. Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation. Transportation Research Part B: Methodological 2010;44(1):90–107.

Fang Y, Szeto WY. Day-to-day dynamic congestion pricing policies towards system optimal. In: First International Symposium on Dynamic Traffic Assignment. Leeds, United Kingdom; 2006. p. 266–75.

Friesz TL, Bernstein D, Kydes N. Dynamic congestion pricing in disequilibrium. Networks and Spatial Economics 2004;4(2):181–202.

Friesz TL, Bernstein D, Mehta NJ, Tobin RL, Ganjalizadeh S. Day-to-day dynamic network disequilibria and idealized traveler information systems. Operations Research 1994;42(6):1120–36.

Garcia A, Hong M, Barrera J. ?cap and trade? for congestion control. Dynamic Games and Applications 2012;2(3):280–93.

Guo RY, Huang HJ. Chaos and bifurcation in dynamical evolution process of traffic assignment with flow “mutation”. Chaos, Solitons & Fractals 2009;41(3):1150–7.

Guo RY, Yang H, Huang HJ, Tan Z. Day-to-day flow dynamics and congestion control. Transportation Science 2015a;.

Guo RY, Yang H, Huang HJ, Tan Z. Link-based day-to-day network traffic dynamics and equilibria. Transportation Research Part B: Methodological 2015b;71:248–60.

Han L, Du L. On a link-based day-to-day traffic assignment model. Transportation Research Part B: Methodological 2012;46(1):72–84.

Han LH, Wang DZW, Lo HK, Zhu CJ, Li MZF. The discrete-time day-to-day dynamic cogestion pricing scheme. Submitted to Transportation Science 2016;.

He X, Guo X, Liu HX. A link-based day-to-day traffic assignment model. Transportation Research Part B: Methodological 2010;44(4):597–608.

Horowitz JL. The stability of stochastic equilibrium in a two-link transportation network. Transportation Research Part B: Methodological 1984;18(1):13–28.

Huang HJ, Lam WH. Modeling and solving the dynamic user equilibrium route and departure time choice problem in network with queues. Transportation Research Part B: Methodological 2002;36(3):253–73.

Jin WL. A dynamical system model of the traffic assignment problem. Transportation Research Part B: Methodological 2007;41(1):32–48.

Knight FH. Some fallacies in the interpretation of social cost. The Quarterly Journal of Economics 1924;:582–606.

Liu R, Smith M. Route choice and traffic signal control: A study of the stability and instability of a new dynamical model of route choice and traffic signal control. Transportation Research Part B: Methodological 2015;77:123–45.

Mounce R. Convergence in a continuous dynamic queueing model for traffic networks. Transportation Research Part B: Methodological 2006;40(9):779–91.

Mounce R, Carey M. Route swapping in dynamic traffic networks. Transportation Research Part B: Methodological 2011;45(1):102–11.

Nagurney A, Zhang D. Projected dynamical systems in the formulation, stability analysis, and computation of fixed-demand traffic network equilibria. Transportation Science 1997;31(2):147–58.

Peeta S, Yang TH. Stability issues for dynamic traffic assignment. Automatica 2003;39(1):21–34.

Pigou AC. The economics of welfare. Transaction Publishers, 1924.

Sandholm WH. Potential games with continuous player sets. Journal of Economic Theory 2001;97(1):81–108.

Sandholm WH. Evolutionary implementation and congestion pricing. The Review of Economic Studies 2002;69(3):667–89.

Smith M. Traffic signal control and route choice: A new assignment and control model which designs signal timings. Transportation Research Part C: Emerging Technologies 2015;58:451–73.

Smith M, Mounce R. A splitting rate model of traffic re-routeing and traffic control. Transportation Research Part B: Methodological 2011;45(9):1389–409.

Smith M, Wisten M. A continuous day-to-day traffic assignment model and the existence of a continuous dynamic user equilibrium. Annals of Operations Research 1995;60(1):59–79.

Smith MJ. The existence and calculation of traffic equilibria. Transportation Research Part B: Methodological 1983;17(4):291–303.

Smith MJ. The stability of a dynamic model of traffic assignment-an application of a method of lyapunov. Transportation Science 1984;18(3):245–52.

Tan Z, Yang H, Guo RY. Dynamic congestion pricing with day-to-day flow evolution and user heterogeneity. Transportation Research Part C: Emerging Technologies 2015;61:87–105.

Trench WF. Introduction to real analysis 2013;.

Watling D. Asymmetric problems and stochastic process models of traffic assignment. Transportation Research Part B: Methodological 1996;30(5):339–57.

Watling D, Hazelton ML. The dynamics and equilibria of day-to-day assignment models. Networks and Spatial Economics 2003;3(3):349–70.

Xiao L, Lo HK. Combined route choice and adaptive traffic control in a day-to-day dynamical system. Networks and Spatial Economics 2014;:1–21.

Yang F. An evolutionary game theory approach to the day-to-day traffic dynamics 2005;.

Yang F, Yin Y, Lu J. Steepest descent day-to-day dynamic toll. Transportation Research Record: Journal of the Transportation Research Board 2007;2039(1):83–90.

Yang F, Zhang D. Day-to-day stationary link flow pattern. Transportation Research Part B: Methodological 2009;43(1):119–26.

Yang H, Huang HJ. Mathematical and economic theory of road pricing, 2005.

Zhang D, Nagurney A. On the local and global stability of a travel route choice adjustment process. Transportation Research Part B: Methodological 1996;30(4):245–62.

Zhang D, Nagurney A, Wu J. On the equivalence between stationary link flow patterns and traffic network equilibria. Transportation Research Part B: Methodological 2001;35(8):731–48.