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Forcing scheme analysis for the axisymmetric lattice Boltzmann method under incompressible limitLiangqi Zhang,¹ Shiliang Yang,¹ Zhong Zeng,^{2,3} Jie Chen,³ Linmao Yin,² and Jia Wei Chew^{1,4,*}¹*School of Chemical and Biomedical Engineering, Nanyang Technological University, Singapore 637459, Singapore*²*Department of Engineering Mechanics, College of Aerospace Engineering, Chongqing University, Chongqing 400044, People's Republic of China*³*State Key Laboratory of Coal Mine Disaster Dynamics and Control, Chongqing University, Chongqing 400044, People's Republic of China*⁴*Singapore Membrane Technology Center, Nanyang Environment and Water Research Institute, Nanyang Technological University, Singapore 637141, Singapore*

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Because the standard lattice Boltzmann (LB) method is proposed for Cartesian Navier-Stokes (NS) equations, additional source terms are necessary in the axisymmetric LB method for representing the axisymmetric effects. Therefore, the accuracy and applicability of the axisymmetric LB models depend on the forcing schemes adopted for discretization of the source terms. In this study, three forcing schemes, namely, the trapezium rule based scheme, the direct forcing scheme, and the semi-implicit centered scheme, are analyzed theoretically by investigating their derived macroscopic equations in the diffusive scale. Particularly, the finite difference interpretation of the standard LB method is extended to the LB equations with source terms, and then the accuracy of different forcing schemes is evaluated for the axisymmetric LB method. Theoretical analysis indicates that the discrete lattice effects arising from the direct forcing scheme are part of the truncation error terms and thus would not affect the overall accuracy of the standard LB method with general force term (i.e., only the source terms in the momentum equation are considered), but lead to incorrect macroscopic equations for the axisymmetric LB models. On the other hand, the trapezium rule based scheme and the semi-implicit centered scheme both have the advantage of avoiding the discrete lattice effects and recovering the correct macroscopic equations. Numerical tests applied for validating the theoretical analysis show that both the numerical stability and the accuracy of the axisymmetric LB simulations are affected by the direct forcing scheme, which indicate that forcing schemes free of the discrete lattice effects are necessary for the axisymmetric LB method.

DOI: [10.1103/PhysRevE.95.043311](https://doi.org/10.1103/PhysRevE.95.043311)**I. INTRODUCTION**

Axisymmetric computational fluid dynamics (CFD) models have the advantage of saving the computational resources by reducing the flows in three-dimensional (3D) cylindrical coordinate system to their simplified counterparts in the two-dimensional (2D) pseudo-Cartesian coordinate system with macroscopic Navier-Stokes (NS) equations of the following form:

$$\frac{\partial u_j}{\partial x_j} + \frac{u_r}{r} = 0, \quad (1a)$$

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = & -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[v \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \\ & + \frac{v}{r} \left(\frac{\partial u_i}{\partial r} + \frac{\partial u_r}{\partial x_i} \right) - \frac{2v u_r}{r^2} \delta_{ir} - \frac{u_i u_r}{r}, \end{aligned} \quad (1b)$$

where i, j indicate, respectively, the r, z coordinate components, and u_i denotes the velocity components in the meridian plane $\mathbf{u} = (u_r, u_z)$. p and v are, respectively, the fluid pressure and the kinematic viscosity [1–9]. The macroscopic equations in Eq. (1) can also be solved within the standard lattice Boltzmann (LB) framework by regarding the second term in Eq. (1a) and the last three terms at the right-hand side of Eq. (1b) as, respectively, the mass and momentum

source terms due to axisymmetry. Moreover, if the rotation effects are involved, another LB equation for another set of the distribution functions should be included to recover the macroscopic equation for the azimuthal velocity component u_θ ,

$$\frac{\partial u_\theta}{\partial t} + u_j \frac{\partial u_\theta}{\partial x_j} = v \frac{\partial^2 u_\theta}{\partial x_j \partial x_j} - \frac{v}{r} \left[\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} \right] - \frac{u_\theta u_r}{r}, \quad (2)$$

whereby Eq. (2) is a general convection diffusion equation with source terms.

Incorporating the source terms into the standard LB equation, the first axisymmetric LB model was proposed by Halliday *et al.* [10], and then modified by Lee *et al.* [11] and Reis and Phillips [12,13] for recovering the accurate macroscopic equations for axisymmetric flows. Although the Halliday *et al.* model and the subsequently improved models have achieved wide applications [10–18], these early models suffered from complicated source terms. Specifically, more than ten terms were included in the second order source terms [accounting for the momentum source terms in Eq. (1b)] of the improved Halliday *et al.* model, and complex finite differential calculations were necessary which would further complicate the axisymmetric LB models [11–13,19]. Therefore, successive axisymmetric LB models were proposed for simplifying the source terms. Starting from the vorticity stream equation, an alternative axisymmetric LB model was proposed by Chen *et al.* [20] and then validated by axisymmetric thermal flows [21]. However, the Chen *et al.* model was rendered impractical by its inefficient evolution processes since a Poisson equation

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had to be solved at each time step and the boundary condition for the vorticity was hard to determine. Afterwards, the finite difference calculations in the momentum source terms were represented by an additional relaxation term in the Li *et al.* model [22,23], and similar implementation was also introduced in the axisymmetric models for thermal flows [24] and low Mach number compressible flows [4]. It is worth noting that the axisymmetric models discussed above were developed within the framework of the standard LB model, and the axisymmetric effects were realized at the macroscopic level by introducing the source terms in Eq. (1) into the LB equation. Alternatively, another axisymmetric LB model was directly derived from the axisymmetric Boltzmann equation by Guo *et al.* [25] with the introduction of a new distribution function definition, and thus the axisymmetric effects were represented at the distribution function level. The main advantage of the Guo *et al.* model [25] was its simple source terms since the mass source terms, inertial terms, and the velocity gradients due to the axisymmetry in Eq. (1) were recovered naturally by the novel distribution function definition and thus avoided in the source terms at the distribution function level. Therefore, the Guo *et al.* model won great popularity and was successfully extended to axisymmetric thermal flows [26–28] and multiphase flows [5].

Since the source terms are inevitable when extending the LB method to axisymmetric flows, the accuracy and the applicability of the axisymmetric LB models would be affected by the forcing scheme adopted for discretizing the source terms. The direct forcing scheme in Eq. (3), obtained by time integration along a characteristic line [29,30], was applied in the early axisymmetric LB models, such as the Halliday *et al.* model [10] and its improved models [11–13], and the Chen *et al.* model [20,21]. It has been demonstrated by the Chapman-Enskog (CE) analysis that spurious terms were found in the macroscopic equations derived from the direct forcing scheme based discretization, and such error terms were reported as the discrete lattice effects [31]. Accordingly, it was claimed by Li *et al.* [22] that the discrete lattice effects from the direct forcing scheme made the source terms in Halliday *et al.* model [10] and the improved models [11–13] complicated. Particularly, complex differential terms were involved in the second order source terms for removing such errors terms and recovering the accurate macroscopic equations, but the complex terms would introduce some additional errors and affect the numerical stability of the axisymmetric model. For the purpose of avoiding the discrete lattice effects, the trapezium rule based scheme was proposed by He *et al.* [32,33] and Guo *et al.* [31], which gave accurate macroscopic equations. Analogously, the spurious differential source terms in the Halliday *et al.* model were removed by the trapezium rule based discretization, and the resulting model was efficiently applied to axisymmetric multiphase flows [15]. In addition, as an approximate implementation of the trapezium rule based scheme, a semi-implicit centered scheme was applied by Zhou [19,34] with the advantage of removing the implicitness caused by the trapezium rule based discretization and making the calculations of the macroscopic variables independent of the source terms. This was appealing for the axisymmetric LB models for multiphase flows [5,15,35,36] since the complicated source terms, consisting

of the axisymmetric effect source terms and interfacial force terms, would lead to complex calculations of macroscopic variables.

In this study, instead of the CE expansion analysis in Guo *et al.* [31], the finite difference interpretation of the standard LB method developed by Junk [37,38] is adopted to investigate the discrete lattice effects arising in various forcing schemes. The finite difference discretization framework is firstly extended to the LB equation with source terms, and the accuracy of the direct forcing scheme for the implementation of the general external force (i.e., only the momentum source terms are considered) is evaluated. Then, the discrete lattice effects arising from three forcing schemes, namely, the trapezium rule based scheme, the direct forcing scheme, and the semi-implicit centered scheme, are analyzed within the framework of the finite difference discretization for the axisymmetric LB models by Guo *et al.* [25] and Li *et al.* [22]. It is well acknowledged that the discrete lattice effects caused by the direct forcing scheme can be viewed as part of the truncation errors and would not affect the order of the overall accuracy of the standard LB method with general force term, but leads to incorrect macroscopic equations within the axisymmetric LB models framework. On the other hand, accurate macroscopic equations are recovered from the other two forcing scheme based discretization methods (i.e., the trapezium rule and the semi-implicit centered scheme), which remove the discrete lattice effects. Numerical validations for the various forcing schemes considered are performed based on the axisymmetric LB models by Guo *et al.* [25] and Li *et al.* [22], respectively, and the numerical results compared with the theoretical analysis.

II. FORMULATIONS

In this part, the finite difference interpretation of the LB method is firstly introduced to evaluate the discrete lattice effects caused by the direct forcing scheme for the standard LB model. Then, the derived theoretical framework is extended to the axisymmetric LB models, and the accuracy of three forcing schemes (namely, trapezium rule based scheme, direct forcing scheme, and semi-implicit centered scheme) for representing the axisymmetric effects is then investigated based on the kinetic theory based axisymmetric LB model by Guo *et al.* [25]. Moreover, a similar analysis is also performed within the framework of the axisymmetric LB model by Li *et al.* [22] in the Appendix.

A. Finite difference interpretation of the standard LB method

The LB equation with the direct forcing scheme is given as

$$\begin{aligned} f_\alpha(\mathbf{x} + \xi_\alpha \delta t, t + \delta t) - f_\alpha(\mathbf{x}, t) \\ = -\frac{1}{\tau} [f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] + \delta t S_\alpha(\mathbf{x}, t), \end{aligned} \quad (3)$$

where f_α and $f_\alpha^{(\text{eq})}$ are, respectively, the density distribution function and its equilibrium form, and the discrete particle

velocity ξ_α in the D2Q9 model [39] is defined as

$$\xi_\alpha = \begin{cases} (0,0) & \alpha = 0 \\ c\{\cos[(\alpha-1)\pi/2], \sin[(\alpha-1)\pi/2]\} & \alpha = 1,2,3,4, \\ \sqrt{2}c\{\cos[(2\alpha-1)\pi/4], \sin[(2\alpha-1)\pi/4]\} & \alpha = 5,6,7,8 \end{cases} \quad (4)$$

with weight coefficients $w_0 = 4/9$, $w_{1,2,3,4} = 1/9$, and $w_{5,6,7,8} = 1/36$. In the standard LB model, the equilibrium distribution function and the external force source term are expressed as

$$f_\alpha^{(\text{eq})} = \rho w_\alpha \left[1 + \frac{\mathbf{u} \cdot \xi_\alpha}{RT} + \frac{(\mathbf{u} \cdot \xi_\alpha)^2}{2(RT)^2} - \frac{u^2}{2RT} \right], \quad (5)$$

$$S_\alpha = \frac{(\xi_\alpha - \mathbf{u}) \cdot \mathbf{a}}{RT} f_\alpha^{(\text{eq})}, \quad (6)$$

where R is the ideal gas constant and T is the fluid temperature, $RT = c_s^2$ is constant for isothermal flows, and $\mathbf{a}$ denotes the external force. The fluid density ρ and velocity $\mathbf{u}$ are calculated as

$$\rho = \sum_\alpha f_\alpha, \quad u_i = \frac{1}{\rho} \sum_\alpha f_\alpha \xi_{\alpha i}, \quad (7)$$

and the dimensionless relaxation time τ is related to the kinematic viscosity ν as

$$\nu = \left(\tau - \frac{1}{2}\right) RT \delta t. \quad (8)$$

In the following, the macroscopic equations are derived from Eq. (3) within the framework of finite difference discretization of the LB method, and the diffusive scaling system [i.e., the Navier-Stokes (NS) scale] is firstly introduced in comparison with the Boltzmann equation (BE) scale. In both scaling systems, the physical length L is adopted as the representative space scale, but the velocity scales differ in that the lattice speed c (which relates to the sound speed c_s by $c_s = c/\sqrt{3}$) is the typical speed for the BE scale, whereas the macroscopic speed U serves as the velocity scale in the NS scale. Under the incompressible limit, the Mach number $\text{Ma} = U/c_s$ adheres to the following relationship:

$$U/c_s \sim U/c = \epsilon \sim \delta x/L, \quad (9)$$

where δx and $\Delta x = \delta x/L$ are, respectively, the spatial step and its scaled form. The incompressible condition in Eq. (9) leads to different time scales $T_{\text{BE}} = \epsilon T_{\text{NS}}$, whereby $T_{\text{BE}} = L/c$ and $T_{\text{NS}} = L/U$ are, respectively, the typical time for the BE and NS scales. Then, the scaled time steps for these two scaling systems are obtained and related to the scaled spatial step as

$$\Delta t_{\text{BE}} = \frac{\delta t}{T_{\text{BE}}} = \frac{\delta x}{L} = \Delta x, \quad (10a)$$

$$\Delta t_{\text{NS}} = \frac{\delta t}{T_{\text{NS}}} = \epsilon \frac{\delta t}{T_{\text{BE}}} = \epsilon \frac{\delta x}{L} = \Delta x^2, \quad (10b)$$

where δt is the time step. Applying the diffusive scaling system, the scaled quantities are defined as $\hat{\mathbf{x}} = \mathbf{x}/L$, $\mathbf{e}_\alpha = \xi_\alpha/c$, $\hat{t} = t/T_{\text{NS}}$ and the scaled distribution function as $\hat{f}_\alpha(\hat{\mathbf{x}}, \mathbf{e}_\alpha, \hat{t}) = f_\alpha(L\hat{\mathbf{x}}, \xi_\alpha, T_{\text{NS}}\hat{t})$, while the scaled density $\hat{\rho} = \rho(L\hat{\mathbf{x}}, T_{\text{NS}}\hat{t})$

and velocity $\hat{\mathbf{u}} = \mathbf{u}/U = \mathbf{u}/\epsilon$ can be analogously defined (with $\epsilon = U/c$ and $c = \delta x/\delta t = 1$ in the present work). Thus the scaled LB equation is derived from Eq. (3) as

$$\begin{aligned} & \hat{f}_\alpha(\hat{\mathbf{x}} + \epsilon \mathbf{e}_\alpha, \mathbf{e}_\alpha, \hat{t} + \epsilon^2) - \hat{f}_\alpha(\hat{\mathbf{x}}, \mathbf{e}_\alpha, \hat{t}) \\ &= -\frac{1}{\tau} \left\{ \hat{f}_\alpha(\hat{\mathbf{x}}, \mathbf{e}_\alpha, \hat{t}) - f_\alpha^{(\text{eq})}[\hat{\rho}(\hat{\mathbf{x}}, \hat{t}), \epsilon \hat{\mathbf{u}}(\hat{\mathbf{x}}, \hat{t}); \mathbf{e}_\alpha] \right\} + \hat{S}_\alpha(\hat{\mathbf{x}}, \hat{t}), \end{aligned} \quad (11a)$$

with the scaled source term

$$\hat{S}_\alpha = 3\epsilon^3(e_{\alpha k} - \epsilon \hat{u}_k) \hat{a}_k f_\alpha^{(\text{eq})}(\hat{\rho}, \epsilon \hat{\mathbf{u}}). \quad (11b)$$

Eliminating the hat superscripts ($\hat{\cdot}$) to make the present derivations concise, Eq. (11) is further rewritten as

$$\begin{aligned} & f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t + \Delta t) - f_\alpha(\mathbf{x}, t) \\ &= -\frac{1}{\tau} \left\{ f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}[\rho, \epsilon \mathbf{u}(\mathbf{x}, t)] \right\} + S_\alpha(\mathbf{x}, t), \end{aligned} \quad (12)$$

$$S_\alpha = 3\epsilon^3(e_{\alpha k} - \epsilon u_k) a_k f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}). \quad (13)$$

Then, with the following definition of the scaled orthogonal basis,

$$Q_0(\mathbf{e}_\alpha) = 1, \quad (14a)$$

$$Q_1(\mathbf{e}_\alpha) = e_{\alpha x}/\epsilon, \quad (14b)$$

$$Q_2(\mathbf{e}_\alpha) = e_{\alpha y}/\epsilon, \quad (14c)$$

$$Q_3(\mathbf{e}_\alpha) = (e_{\alpha x}^2 - \frac{1}{3})/\epsilon^2, \quad (14d)$$

$$Q_4(\mathbf{e}_\alpha) = e_{\alpha x} e_{\alpha y}/\epsilon^2, \quad (14e)$$

$$Q_5(\mathbf{e}_\alpha) = (e_{\alpha y}^2 - \frac{1}{3})/\epsilon^2, \quad (14f)$$

$$Q_6(\mathbf{e}_\alpha) = (3|e_\alpha|^2 - 4)e_{\alpha x}/\epsilon^3, \quad (14g)$$

$$Q_7(\mathbf{e}_\alpha) = (3|e_\alpha|^2 - 4)e_{\alpha y}/\epsilon^3, \quad (14h)$$

$$Q_8(\mathbf{e}_\alpha) = (9|e_\alpha|^4 - 15|e_\alpha|^2 + 2)/\epsilon^4, \quad (14i)$$

which follow the orthogonality condition as

$$\langle Q_i(\mathbf{e}_\alpha) Q_j(\mathbf{e}_\alpha) w_\alpha \rangle = \sum_\alpha Q_i(\mathbf{e}_\alpha) Q_j(\mathbf{e}_\alpha) w_\alpha = 0 \quad \text{for } i \neq j, \quad (15)$$

a transformation matrix $\mathbf{Q} = (Q_0, Q_1, \dots, Q_8)^\dagger$ forms with † denoting the transpose operator. Thus a linear invertible mapping between the distribution function vector $|f\rangle = (f_0, f_1, \dots, f_8)^\dagger$ and the moment vector $\mathbf{M} = (M_0, M_1, \dots, M_8)^\dagger$ is developed as

$$\mathbf{M} = \mathbf{Q}|f\rangle = (\langle Q_0 f \rangle, \langle Q_1 f \rangle, \dots, \langle Q_8 f \rangle)^\dagger, \quad (16a)$$

$$|f\rangle = \mathbf{Q}^{-1} \mathbf{M} = \sum_{j=0}^8 \frac{M_j Q_j(\mathbf{e}_\alpha)}{\langle Q_j^2(\mathbf{e}_\alpha) w_\alpha \rangle} w_\alpha. \quad (16b)$$

Thus the moment vectors for the equilibrium distribution function and the general external force term in the above standard LB model are derived as

$$\mathbf{M}^{(\text{eq})} = (\rho, \rho u_x, \rho u_y, \rho u_x u_x, \rho u_x u_y, \rho u_y u_y, M_6^{(\text{eq})}, M_7^{(\text{eq})}, M_8^{(\text{eq})})^\dagger, \quad (17a)$$

$$\mathbf{M}^S = [0, \epsilon^2 \rho a_x, \epsilon^2 \rho a_y, \epsilon^2 2\rho u_x a_x, \epsilon^2 \rho(u_x a_y + u_y a_x), \epsilon^2 2\rho u_y a_y, M_6^S, M_7^S, M_8^S]^\dagger. \quad (17b)$$

Moreover, based on the linear mapping defined in Eq. (16), the evolution equation for the moment component M_i is obtained as

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2(\mathbf{e}_\alpha) w_\alpha \rangle} \left\langle w_\alpha Q_i(\mathbf{e}_\alpha) Q_j(\mathbf{e}_\alpha) \left[M_j^n - \frac{1}{\tau} M_j^{(\text{neq}),n} + M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle. \quad (18)$$

where the upper index n denotes the time step. Considering a general expression reduced from Eq. (18),

$$\langle \psi(\mathbf{x} - \mathbf{e}_\alpha \Delta x) P(\mathbf{e}_\alpha) w_\alpha \rangle, \quad (19)$$

expansions of Eq. (19) lead to the finite difference interpretation of the LB method [37,38]. Firstly, taking advantage of the symmetry of the D2Q9 lattice model, Eq. (19) is rewritten as

$$\langle \psi(\mathbf{x} - \mathbf{e}_\alpha \Delta x) P(\mathbf{e}_\alpha) w_\alpha \rangle = \sum_{\alpha} \psi(\mathbf{x} - \mathbf{e}_\alpha \Delta x) P(\mathbf{e}_\alpha) w_\alpha = \sum_{\alpha} \psi(\mathbf{x} + \mathbf{e}_\alpha \Delta x) P(-\mathbf{e}_\alpha) w_\alpha = \sum_{k,l=-1}^1 \beta_{kl} \psi(\mathbf{x}_{i+k, j+l}), \quad (20a)$$

with

$$\beta_{kl} = \begin{bmatrix} \beta_{-11} & \beta_{01} & \beta_{11} \\ \beta_{-10} & \beta_{00} & \beta_{10} \\ \beta_{-1-1} & \beta_{0-1} & \beta_{1-1} \end{bmatrix} = \begin{bmatrix} P(\mathbf{e}_8)w_8 & P(\mathbf{e}_4)w_4 & P(\mathbf{e}_7)w_7 \\ P(\mathbf{e}_1)w_1 & P(\mathbf{e}_0)w_0 & P(\mathbf{e}_3)w_3 \\ P(\mathbf{e}_5)w_5 & P(\mathbf{e}_2)w_2 & P(\mathbf{e}_6)w_6 \end{bmatrix}, \quad (20b)$$

and $\mathbf{x}_{ij} = (i\Delta x, j\Delta x)$ represents the coordinates of the lattice nodes in the regular Cartesian discrete grid system. It is worth noting that Eq. (20) essentially defines the finite difference stencils resulting from Eq. (19). On the other hand, the Taylor expansion of $\psi(\mathbf{x} - \mathbf{e}_\alpha \Delta x)$ leads to the spatial gradient operators pertaining to the finite difference stencil given in Eq. (20):

$$\begin{aligned} \langle \psi(\mathbf{x} - \mathbf{e}_\alpha \Delta x) P(\mathbf{e}_\alpha) w_\alpha \rangle &= \psi(\mathbf{x}) \langle P(\mathbf{e}_\alpha) w_\alpha \rangle - \epsilon \langle \mathbf{e}_{\alpha i} P(\mathbf{e}_\alpha) w_\alpha \rangle \frac{\partial \psi}{\partial x_i} + \frac{\epsilon^2}{2} \langle \mathbf{e}_{\alpha i} \mathbf{e}_{\alpha j} P(\mathbf{e}_\alpha) w_\alpha \rangle \frac{\partial^2 \psi}{\partial x_i \partial x_j} \\ &\quad - \frac{\epsilon^3}{6} \langle \mathbf{e}_{\alpha i} \mathbf{e}_{\alpha j} \mathbf{e}_{\alpha k} P(\mathbf{e}_\alpha) w_\alpha \rangle \frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k} + O(\epsilon^4). \end{aligned} \quad (21)$$

In the following, the finite difference properties in terms of the finite difference stencils and the approximate gradient operator, derived from a typical polynomial $P(\mathbf{e}_\alpha)$, are provided as

$$P(\mathbf{e}_\alpha) = 1 \Leftrightarrow \beta_{kl} = \frac{1}{36} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix} \Leftrightarrow 1 + \frac{\epsilon^2}{6} \Delta + O(\epsilon^4), \quad (22)$$

$$P(\mathbf{e}_\alpha) = -\frac{3e_{\alpha x}}{\Delta x} \Leftrightarrow \beta_{kl} = \frac{1}{12\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} \Leftrightarrow \frac{\partial}{\partial x} + O(\epsilon^2), \quad (23a)$$

$$P(\mathbf{e}_\alpha) = -\frac{3e_{\alpha y}}{\Delta x} \Leftrightarrow \beta_{kl} = \frac{1}{12\Delta x} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & -4 & -1 \end{bmatrix} \Leftrightarrow \frac{\partial}{\partial y} + O(\epsilon^2), \quad (23b)$$

$$P(\mathbf{e}_\alpha) = -\frac{9e_{\alpha x}e_{\alpha y}^2}{\Delta x} \Leftrightarrow \beta_{kl} = \frac{1}{4\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Leftrightarrow \frac{\tilde{\partial}}{\partial x} + O(\epsilon^2), \quad (24a)$$

$$P(\mathbf{e}_\alpha) = -\frac{9e_{\alpha x}^2e_{\alpha y}}{\Delta x} \Leftrightarrow \beta_{kl} = \frac{1}{4\Delta x} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \Leftrightarrow \frac{\tilde{\partial}}{\partial y} + O(\epsilon^2), \quad (24b)$$

$$P(\mathbf{e}_\alpha) = \frac{9}{\Delta x^2} \left(e_{\alpha x}^2 - \frac{1}{3} \right) \leftrightarrow \beta_{kl} = \frac{1}{6\Delta x^2} \begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix} \leftrightarrow \frac{\partial^{*2}}{\partial x^2} + O(\epsilon^2), \quad (25a)$$

$$P(\mathbf{e}_\alpha) = \frac{9}{\Delta x^2} \left(e_{\alpha y}^2 - \frac{1}{3} \right) \leftrightarrow \beta_{kl} = \frac{1}{6\Delta x^2} \begin{bmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 1 & 4 & 1 \end{bmatrix} \leftrightarrow \frac{\partial^{*2}}{\partial y^2} + O(\epsilon^2), \quad (25b)$$

$$P(\mathbf{e}_\alpha) = \frac{9e_{\alpha x}e_{\alpha y}}{\Delta x^2} \leftrightarrow \beta_{kl} = \frac{1}{4\Delta x^2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \leftrightarrow \frac{\partial^{*2}}{\partial x \partial y} + O(\epsilon^2). \quad (25c)$$

In particular, with the following relationships,

$$\psi = M_j^n - \frac{1}{\tau} M_j^{(\text{neq}),n} + M_j^{S,n}, \quad (26a)$$

$$P(\mathbf{e}_\alpha) = \frac{Q_i(\mathbf{e}_\alpha) Q_j(\mathbf{e}_\alpha)}{\langle Q_j^2(\mathbf{e}_\alpha) w_\alpha \rangle}, \quad (26a)$$

the evolution equation for the moment component in Eq. (18) is further expanded as

$$\frac{M_0^{n+1} - M_0^n}{\Delta t} = \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial x} M_1^n + \frac{\partial}{\partial y} M_2^n \right) + O(\epsilon^2), \quad (27a)$$

$$\begin{aligned} \frac{M_1^{n+1} - M_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial x} M_0^n + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial x} \left(\frac{\partial^*}{\partial x} M_1^n + \frac{\partial^*}{\partial y} M_2^n \right) - \frac{\partial}{\partial x} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_3^{(\text{neq}),n} \right] \\ & - \frac{\tilde{\partial}}{\partial y} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] + O(\epsilon^2), \end{aligned} \quad (27b)$$

$$\begin{aligned} \frac{M_2^{n+1} - M_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial y} M_0^n + \frac{1}{3} \frac{\partial^*}{\partial y} \left(\frac{\partial^*}{\partial x} M_1^n + \frac{\partial^*}{\partial y} M_2^n \right) + \frac{\Delta}{6} M_2^n + \frac{M_2^{S,n}}{\epsilon^2} - \frac{\tilde{\partial}}{\partial x} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] \\ & - \frac{\partial}{\partial y} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_5^{(\text{neq}),n} \right] + O(\epsilon^2), \end{aligned} \quad (27c)$$

$$M_3^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_x)}{\partial x} + O(\epsilon^2), \quad (27d)$$

$$M_4^{(\text{neq})} = -\frac{\tau}{3} \left[\frac{\tilde{\partial}(\rho u_x)}{\partial y} + \frac{\tilde{\partial}(\rho u_y)}{\partial x} \right] + O(\epsilon^2), \quad (27e)$$

$$M_5^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_y)}{\partial y} + O(\epsilon^2). \quad (27f)$$

Introducing the fluid density decomposition $\rho = \bar{\rho}(1 + 3\epsilon^2 p)$ under the incompressible limit, with p and $\bar{\rho}$, respectively, the order one function and the constant part of the density, the accurate incompressible NS equations with the general external force term are obtained as

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad (28a)$$

$$\frac{\partial u_x}{\partial t} + \frac{\partial}{\partial x} (u_x u_x) + \frac{\tilde{\partial}}{\partial y} (u_x u_y) = -\frac{\partial p}{\partial x} + a_x + \frac{\Delta}{6} u_x + \frac{1}{3} (\tau - 1) \left(\frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\tilde{\partial}}{\partial y} \frac{\tilde{\partial}}{\partial y} \right) u_x, \quad (28b)$$

$$\frac{\partial u_y}{\partial t} + \frac{\tilde{\partial}}{\partial x} (u_x u_y) + \frac{\partial}{\partial y} (u_y u_y) = -\frac{\partial p}{\partial y} + a_y + \frac{\Delta}{6} u_y + \frac{1}{3} (\tau - 1) \left(\frac{\tilde{\partial}}{\partial x} \frac{\tilde{\partial}}{\partial x} + \frac{\partial}{\partial y} \frac{\partial}{\partial y} \right) u_y, \quad (28c)$$

and the corresponding finite difference interpretation is well demonstrated by the finite difference stencil definitions in Eq. (20). Therefore, the discrete lattice effects from the direct forcing scheme, as reported by Guo *et al.* [31], are essentially part of the truncation errors of the standard LB model with an additional external force, and therefore the overall accuracy would not be affected.

B. Discrete lattice effects analysis within the framework of the kinetic theory based axisymmetric LB model by Guo et al. [25]

1. Kinetic theory based axisymmetric LB model by Guo et al. [25]

The LB equations for the Guo *et al.* model [25] are given as

$$\tilde{f}_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \delta t, t + \delta t) - \tilde{f}_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] + \delta t \left(1 - \frac{1}{2\tau}\right) S_\alpha(\mathbf{x}, t), \quad (29a)$$

$$\tilde{g}_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \delta t, t + \delta t) - \tilde{g}_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [\tilde{g}_\alpha(\mathbf{x}, t) - g_\alpha^{(\text{eq})}(\mathbf{x}, t)] + \delta t \left(1 - \frac{1}{2\tau}\right) G_\alpha(\mathbf{x}, t), \quad (29b)$$

where $\tilde{f}_\alpha$ and $\tilde{g}_\alpha$ are, respectively, the density distribution function for the velocity vector in the meridian plane $\mathbf{u} = (u_r, u_z)$ and the azimuthal velocity u_θ . Moreover, the trapezium rule based scheme is applied in the Guo *et al.* model [25] for discretizing the axisymmetric source terms with the following reconstruction of the distribution function:

$$\tilde{f}_\alpha = f_\alpha - \frac{\delta t}{2} S_\alpha, \quad (30a)$$

$$\tilde{g}_\alpha = g_\alpha - \frac{\delta t}{2} G_\alpha. \quad (30b)$$

And the equilibrium distribution function and the source terms in Eq. (29) are given as

$$f_\alpha^{(\text{eq})} = r\rho w_\alpha \left[1 + \frac{\boldsymbol{\xi}_\alpha \cdot \mathbf{u}}{RT} + \frac{(\boldsymbol{\xi}_\alpha \cdot \mathbf{u})^2}{2(RT)^2} - \frac{\mathbf{u}^2}{2RT} \right], \quad (31a)$$

$$S_\alpha = \left(RT + u_\theta^2 - \frac{2\nu u_r}{r} \right) \frac{\xi_{\alpha r} - u_r}{RT} \frac{f_\alpha^{(\text{eq})}}{r}, \quad (31b)$$

$$g_\alpha^{(\text{eq})} = r u_\theta f_\alpha^{(\text{eq})}, \quad (32a)$$

$$G_\alpha = (u_\theta^2 + 3RT) \frac{\xi_{\alpha r} - u_r}{RT} \frac{g_\alpha^{(\text{eq})}}{r}. \quad (32b)$$

Lastly, from the definition of the reconstructed distribution function in Eq. (30), the computing formulas for the macroscopic variables in the Guo *et al.* model [25] are derived as

$$\rho = \frac{1}{r} \sum_\alpha \tilde{f}_\alpha, \quad \rho u_\theta = \frac{1}{r^2} \sum_\alpha \tilde{g}_\alpha, \quad (33a)$$

$$u_i = \frac{r}{\rho(r^2 + \nu \delta t \delta_{ir})} \left[\sum_\alpha \tilde{f}_\alpha \xi_{\alpha i} + \frac{\delta t}{2} \rho (RT + u_\theta^2) \delta_{ir} \right]. \quad (33b)$$

2. Finite difference interpretation of the Guo et al. model [25]

Excluding the rotation effects, the scaled LB equation of the Guo *et al.* model [25] is given as

$$\tilde{f}_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t + \Delta t) - \tilde{f}_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] + \left(1 - \frac{1}{2\tau}\right) S_\alpha(\mathbf{x}, t), \quad (34)$$

with the scaled equilibrium distribution function and source terms,

$$f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}) = r\rho w_\alpha \left[1 + 3(\mathbf{e}_\alpha \cdot \epsilon \mathbf{u}) + \frac{9(\mathbf{e}_\alpha \cdot \epsilon \mathbf{u})^2}{2} - \frac{3\epsilon^2 \mathbf{u}^2}{2} \right], \quad (35a)$$

$$S_\alpha = \epsilon \left[1 - \frac{\epsilon^2 (2\tau - 1) u_r}{r} \right] \frac{e_{\alpha r} - \epsilon u_r}{r} f_\alpha^{(\text{eq})}. \quad (35b)$$

Accordingly, the known moment vectors are determined as

$$\mathbf{M}^{(\text{eq})} = (r\rho, r\rho u_r, r\rho u_z, r\rho u_r u_r, r\rho u_r u_z, r\rho u_z u_z, M_6^{(\text{eq})}, M_7^{(\text{eq})}, M_8^{(\text{eq})})^\dagger, \quad (36a)$$

$$\mathbf{M}^S = \left\{ 0, \frac{\rho}{3} \left[1 - \frac{\epsilon^2 (2\tau - 1) u_r}{r} \right], 0, \frac{2\rho u_r}{3}, \frac{\rho u_z}{3}, 0, M_6^S, M_7^S, M_8^S \right\}^\dagger, \quad (36b)$$

and then the evolution of the reconstructed moment component $\tilde{M}_i = M_i - \frac{1}{2}M_i^S$ is derived as

$$\tilde{M}_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{[Q_j^2 w_\alpha]} \left\langle w_\alpha Q_i Q_j \left[\tilde{M}_j^n - \frac{1}{\tau} \tilde{M}_j^{(\text{neq}),n} + \left(1 - \frac{1}{2\tau}\right) M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle. \quad (37)$$

Applying the equilibrium moments and the moment vector for the source term in Eq. (36), the macroscopic equations for each moment component are derived as

$$\frac{\tilde{M}_0^{n+1} - \tilde{M}_0^n}{\Delta t} = \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (38a)$$

$$\begin{aligned} \frac{\tilde{M}_1^{n+1} - \tilde{M}_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} M_0^n + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_3^{(\text{neq}),n} + \left(1 - \frac{1}{2\tau}\right) M_3^{S,n} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_4^{(\text{neq}),n} + \left(1 - \frac{1}{2\tau}\right) M_4^{S,n} \right] + O(\epsilon^2), \end{aligned} \quad (38b)$$

$$\begin{aligned} \frac{\tilde{M}_2^{n+1} - \tilde{M}_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} M_0^n + \frac{\Delta}{6} M_2^n - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_4^{(\text{neq}),n} + \left(1 - \frac{1}{2\tau}\right) M_4^{S,n} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_5^{(\text{neq}),n} \right] + O(\epsilon^2), \end{aligned} \quad (38c)$$

$$\tilde{M}_3^{(\text{neq})} = -\frac{2\tau}{3} \frac{r \partial(\rho u_r)}{\partial r} - \frac{\rho u_r}{3} + O(\epsilon^2), \quad (38d)$$

$$\tilde{M}_4^{(\text{neq})} = -\frac{\tau}{3} r \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] - \frac{\rho u_z}{6} + O(\epsilon^2), \quad (38e)$$

$$\tilde{M}_5^{(\text{neq})} = -\frac{2\tau}{3} \frac{r \partial(\rho u_z)}{\partial z} + O(\epsilon^2), \quad (38f)$$

which lead to the accurate NS equations for incompressible axisymmetric flows,

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0, \quad (39a)$$

$$\frac{\partial u_r}{\partial t} + \frac{\partial(u_r u_r)}{\partial r} + \frac{\tilde{\partial}(u_r u_z)}{\partial z} + \frac{u_r u_r}{r} = -\frac{\partial p}{\partial r} + \frac{1}{6} \Delta u_r + \frac{1}{3}(\tau - 1) \left(\frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{\tilde{\partial}}{\partial z} \frac{\tilde{\partial}}{\partial z} \right) u_r + \frac{1}{3r} \left(\tau - \frac{1}{2} \right) \frac{\partial u_r}{\partial r} - \frac{1}{3} \left(\tau - \frac{1}{2} \right) \frac{u_r}{r^2}, \quad (39b)$$

$$\frac{\partial u_z}{\partial t} + \frac{\tilde{\partial}(u_r u_z)}{\partial r} + \frac{\partial(u_z u_z)}{\partial z} + \frac{u_r u_z}{r} = -\frac{\partial p}{\partial z} + \frac{1}{6} \Delta u_z + \frac{1}{3}(\tau - 1) \left(\frac{\tilde{\partial}}{\partial r} \frac{\tilde{\partial}}{\partial r} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) u_z + \frac{1}{3r} \left(\tau - \frac{1}{2} \right) \frac{\tilde{\partial} u_z}{\partial r}. \quad (39c)$$

Therefore, in accordance with the CE analysis by Guo *et al.* [31], present derivations demonstrate that the trapezium rule based scheme would lead to the accurate macroscopic equations, and thus the Guo *et al.* model [25] is free of the discrete lattice effects.

3. Direct forcing scheme

Substituting the known moment vectors given in Eq. (36) from the Guo *et al.* model [25], the moment equations in Eq. (27), pertaining to the direct forcing scheme based LB equation [i.e., Eq. (12)], are rewritten in the pseudo-Cartesian meridian plane as

$$\frac{M_0^{n+1} - M_0^n}{\Delta t} = \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (40a)$$

$$\begin{aligned} \frac{M_1^{n+1} - M_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} M_0^n + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial r} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) M_3^{(\text{neq}),n} + M_3^{S,n} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) M_4^{(\text{neq}),n} + M_4^{S,n} \right] + O(\epsilon^2), \end{aligned} \quad (40b)$$

$$\begin{aligned} \frac{M_2^{n+1} - M_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} M_0^n + \frac{1}{3} \frac{\partial^*}{\partial z} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) + \frac{\Delta}{6} M_2^n - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) M_4^{(\text{neq}),n} + M_4^{S,n} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) M_5^{(\text{neq}),n} \right] + O(\epsilon^2), \end{aligned} \quad (40c)$$

$$M_3^{(\text{neq})} = -\frac{2\tau}{3}r \frac{\partial(\rho u_r)}{\partial r} + O(\epsilon^2), \quad (40d)$$

$$M_4^{(\text{neq})} = -\frac{\tau}{3}r \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] + O(\epsilon^2), \quad (40e)$$

$$M_5^{(\text{neq})} = -\frac{2\tau}{3}r \frac{\partial(\rho u_z)}{\partial z} + O(\epsilon^2), \quad (40f)$$

and thus the macroscopic equations recovered from the direct forcing scheme are obtained as

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0, \quad (41a)$$

$$\frac{\partial u_r}{\partial t} + \frac{\partial(u_r u_r)}{\partial r} + \frac{\tilde{\partial}(u_r u_z)}{\partial z} + \frac{u_r u_r}{r} = -\frac{\partial p}{\partial r} + \frac{1}{6}\Delta u_r + \frac{1}{3}(\tau - 1) \left(\frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{\tilde{\partial}}{\partial z} \frac{\tilde{\partial}}{\partial z} \right) u_r + \frac{1}{3r}(\tau - 1) \frac{\partial u_r}{\partial r} - \frac{1}{3}(\tau - 1) \frac{u_r}{r^2}, \quad (41b)$$

$$\frac{\partial u_z}{\partial t} + \frac{\tilde{\partial}(u_r u_z)}{\partial r} + \frac{\partial(u_z u_z)}{\partial z} + \frac{u_r u_z}{r} = -\frac{\partial p}{\partial z} + \frac{1}{6}\Delta u_z + \frac{1}{3}(\tau - 1) \left(\frac{\tilde{\partial}}{\partial r} \frac{\tilde{\partial}}{\partial r} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) u_z + \frac{1}{3r}(\tau - 1) \frac{\tilde{\partial} u_z}{\partial r}. \quad (41c)$$

Clearly, the derived macroscopic equations suffer from inconsistent kinematic viscosity definition: $\nu = \frac{1}{3}(\tau - \frac{1}{2})$ for the main Laplace viscosity terms, while $\nu' = \frac{1}{3}(\tau - 1)$ in the viscous source terms. Specifically, the discrete lattice effects arising in the direct forcing scheme make the derived axisymmetric macroscopic equations incorrect. However, since the distribution function reconstruction in Eq. (30) is avoided here, thus the direct forcing scheme has the advantage of easy calculations of the macroscopic variables,

$$\rho = \frac{1}{r} \sum_{\alpha} f_{\alpha}, \quad u_i = \frac{1}{r\rho} \sum_{\alpha} f_{\alpha} \xi_{\alpha i}. \quad (42)$$

4. Semi-implicit centered scheme

The trapezium rule based discretization of the source term can be approximately implemented as

$$\frac{1}{2}[S_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha}\Delta x, t + \Delta t) + S_{\alpha}(\mathbf{x}, t)] \approx \frac{1}{2}[S_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha}\Delta x, t) + S_{\alpha}(\mathbf{x}, t)], \quad (43)$$

which is the semi-implicit centered scheme adopted in the axisymmetric LB model by Zhou [19,34]. Thus the reconstructed distribution function is avoided, and the simplified macroscopic variable calculations in Eq. (42) are maintained by the semi-implicit centered scheme. With Eq. (43), the scaled LB equation in Eq. (34) is approximated as

$$f_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha}\Delta x, t + \Delta t) - f_{\alpha}(\mathbf{x}, t) = -\frac{1}{\tau}[f_{\alpha}(\mathbf{x}, t) - f_{\alpha}^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] + \frac{1}{2}[S_{\alpha}(\mathbf{x} + \mathbf{e}_{\alpha}\Delta x, t) + S_{\alpha}(\mathbf{x}, t)]. \quad (44)$$

The corresponding moment component evolution equation is derived as

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_{\alpha} \rangle} \left\langle w_{\alpha} Q_i Q_j \left[M_j^n - \frac{1}{\tau} M_j^{(\text{neq}),n} + \frac{1}{2} M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_{\alpha}\Delta x) \right\rangle + \frac{1}{2} M_i^{S,n}(\mathbf{x}), \quad (45)$$

and the following macroscopic equations are obtained as

$$\frac{M_0^{n+1} - M_0^n}{\Delta t} = \frac{1}{6}\Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (46a)$$

$$\begin{aligned} \frac{M_1^{n+1} - M_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} M_0^n + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial r} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_3^{(\text{neq}),n} + \frac{M_3^{S,n}}{2} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} + \frac{M_4^{S,n}}{2} \right] + O(\epsilon^2), \end{aligned} \quad (46b)$$

$$\begin{aligned} \frac{M_2^{n+1} - M_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} M_0^n + \frac{1}{3} \frac{\partial^*}{\partial z} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) + \frac{\Delta}{6} M_2^n - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} + \frac{M_4^{S,n}}{2} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_5^{(\text{neq}),n} \right] + O(\epsilon^2), \end{aligned} \quad (46c)$$

$$M_3^{(\text{neq})} = -\frac{2\tau}{3}r \frac{\partial(\rho u_r)}{\partial r} + O(\epsilon^2), \quad (46d)$$

$$M_4^{(\text{neq})} = -\frac{\tau}{3}r \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] + O(\epsilon^2), \quad (46e)$$

$$M_5^{(\text{neq})} = -\frac{2\tau}{3}r \frac{\partial(\rho u_z)}{\partial z} + O(\epsilon^2). \quad (46f)$$

With the moment vectors given in Eq. (36), the incompressible NS equations for axisymmetric flows are also accurately recovered from the semi-implicit centered scheme based LB equation in Eq. (43), and the adopted finite difference stencils are consistent with Eq. (39). In other words, the trapezium rule based discretization, as well as its approximated implementation (i.e., the semi-implicit centered scheme), is able to remove the discrete lattice effects arising from the direct forcing scheme. Therefore, the semi-implicit centered scheme is viewed as a compromised alternative since the overall accuracy of the axisymmetric LB model and the simple calculations of the macroscopic variables are preserved. Furthermore, a similar analysis is also performed within the framework of the axisymmetric LB model by Li *et al.* [22] in the Appendix, and consistent conclusions are obtained.

Therefore, it is concluded from the theoretical analysis that (i) both the axisymmetric LB models considered are free of the discrete lattice effects since the trapezium rule based scheme is adopted in their original model definitions; (ii) the direct forcing scheme would lead to incorrect macroscopic equations for both axisymmetric LB models considered; (iii) the semi-implicit centered scheme, as an approximated implementation of the trapezium rule based scheme, provides accurate macroscopic equations and preserves the simple calculations of the macroscopic variables, and can be regarded as a compromised alternative. Moreover, comparisons of the detailed derivations for the direct forcing scheme demonstrate that the discrete lattice effects within the Guo *et al.* [25] model

framework are caused by the improper discretization for the second order moment components of the source terms which directly contribute to the constitutive relations, while the origin of the discrete lattice effects in the framework of the Li *et al.* model [22] lie in the incorrect treatments of the mass source terms M_0^S . Thus the discrete lattice effects arise differently for the considered axisymmetric LB models, and would therefore affect the axisymmetric LB models in different ways.

FIG. 1. Axial velocity profile of the Hagen-Poiseuille flow obtained by using various forcing schemes within the Guo *et al.* model [25] framework.

FIG. 2. Axial velocity profile (a), convergence of the velocity errors with respect to the mesh resolution (b) for the Hagen-Poiseuille flow obtained by using various forcing schemes within the Li *et al.* model [22] framework.

FIG. 3. Flow structures of the cylindrical cavity flow with $R_A = 1.5$ and $\text{Re} = 200$ obtained based on the Guo *et al.* model [25] with various forcing schemes. The left panel of each subplot reflects each forcing scheme: (a) the direct forcing scheme A (i.e., the source terms in the LB equation for $\mathbf{u}$ discretized by the direct forcing scheme, while the trapezium rule is applied for the evolution of the swirling velocity u_θ); (b) the direct forcing scheme B (i.e., source terms for both distribution functions are discretized by the direct forcing scheme); (c) the semi-implicit centered scheme (which is only implemented for the LB equation for $\mathbf{u}$ in the meridian plane while the source terms for the evolution of distribution for azimuthal velocity component u_θ are discretized by the trapezium rule). The right panel of each subplot shows the reference results from the original Guo *et al.* model [25].

III. NUMERICAL RESULTS

As a validation for the theoretical analysis, discrete lattice effects arising from the various forcing schemes are practically evaluated in the following two axisymmetric flow tests, namely, the Hagen-Poiseuille flow and the cylindrical cavity flow. Both the symmetry condition and the nonslip condition are realized by the nonequilibrium extrapolation boundary scheme by Guo *et al.* [40,41]. Specifically, the unknown macroscopic variables and the nonequilibrium part of the postcollision distribution function at the boundary nodes is extrapolated from the inner fluid nodes after each collision step; then the postcollision distribution function at the boundary nodes is obtained. Since the calculation of the macroscopic variables and the collision steps are not implemented, the singularity at the symmetry axis $r = 0$ is avoided.

A. Hagen-Poiseuille flow

The Hagen-Poiseuille pipe flow, driven by a constant axial force a_z , adheres to the following analytical solution:

$$u_z(r) = u_0 \left(1 - \frac{r^2}{R^2} \right), \quad (47)$$

where u_0 is the peak axial velocity at $r = 0$ and is related to the constant driven force a_z by

$$u_0 = a_z R^2 / 4\nu, \quad (48)$$

where R and ν are, respectively, the radius of the pipe and the kinematic viscosity. The only dimensionless parameter, Reynolds number Re , is defined as $\text{Re} = 2Ru_0/\nu$. Except for the symmetry condition at $r = 0$ and the nonslip condition ($u_r = u_z = 0$) at the cylindrical surface $r = R$, periodic condition in the z direction is applied for the two ends of the pipe.

FIG. 4. Flow structures of the cylindrical cavity flow with $R_A = 2.5$ and $\text{Re} = 200$ obtained based on the Guo *et al.* model [25] with various forcing schemes. The left panel of each subplot reflects each forcing scheme: (a) the direct forcing scheme A (i.e., the source terms in the LB equation for $\mathbf{u}$ discretized by the direct forcing scheme, while the trapezium rule is applied for the evolution of the swirling velocity u_θ); (b) the direct forcing scheme B (i.e., source terms for both distribution functions are discretized by the direct forcing scheme); (c) the semi-implicit centered scheme (which is only implemented for the LB equation for $\mathbf{u}$ in the meridian plane while the source terms for the evolution of distribution for azimuthal velocity component u_θ are discretized by the trapezium rule). The right panel of each subplot shows the reference results from the original Guo *et al.* model [25].

Figure 1 plots the velocity profile from various forcing schemes under the framework of the axisymmetric model by Guo *et al.* [25]. Clearly, the direct forcing scheme leads to inaccurate velocity distribution due to the inconsistent viscosity definition by the discrete lattice effects, whereas results from the trapezium rule based scheme and the semi-implicit centered scheme accord well with the analytical solution. Similarly, within the framework of the Li *et al.* model [22], the accuracy of the various forcing schemes is compared with the present flow test. As demonstrated in Fig. 2(a), the velocity profiles from all three forcing schemes agree well with the analytical solution. Furthermore, comparisons of the convergence rates are performed with the following relative error definition:

$$E(u) = \frac{\|u_z - u_z^{\text{analytical}}\|_2}{\|u_z^{\text{analytical}}\|_2}, \quad (49)$$

where $u_z^{\text{analytical}}$ denotes the analytical solution in Eq. (47). It is demonstrated by Fig. 2(b) that the relative velocity errors from all three forcing schemes also agree with each other, and it appears that the discrete lattice effects vanish under the framework of the Li *et al.* model [22]. As discussed in the theoretical analysis, the discrete lattice effects from the direct forcing scheme function differently for different axisymmetric LB models. Within the framework of the Guo *et al.* model [25] discrete lattice errors are related to $\frac{\partial}{\partial r}(\frac{u_r}{r})$ in Eq. (41b) and $\frac{\partial u_z}{\partial r}$ in Eq. (41c); on the other hand, for the Li *et al.* model [22], the error terms are proportional to $\frac{\partial}{\partial r}(\frac{u_r}{r})$ in Eq. (A16b) and $\frac{\partial}{\partial z}(\frac{u_r}{r})$ in Eq. (A16c). However, in the present pipe flow test, the radial velocity uniformly equals 0, i.e., $u_r = 0$; thus the discrete lattice errors by the direct forcing scheme cannot be reflected within the Li *et al.* model [22] framework. Moreover, it is found that the discrete lattice effects would affect the numerical stability of the Guo *et al.* model [25] based simulation; therefore, the test case with lower $Re = 10$ (i.e., larger viscosity) is applied for the validation based on the Guo *et al.* model [25] in Fig. 1, whereas the $Re = 40$ case is adopted for the Li *et al.* model [22] based test as in Fig. 2.

B. Cylindrical cavity flow

The dimensionless parameters for the cylindrical cavity flow are the aspect ratio $R_A = H/R$ (whereby H and R are the height and radius, respectively, of the cylinder) and the Reynolds number $Re = \Omega R^2/\nu$ (whereby Ω and ν are the constant angular velocity of the top lid and the kinematic viscosity, respectively). The symmetry condition is applied at $r = 0$, nonslip boundary conditions at the bottom (i.e., $u_r = u_z = u_\theta = 0$ at $z = 0$) and cylindrical surface (i.e., $u_r = u_z = 0$ at $r = R$), and $u_\theta = \Omega r$ for the top lid ($z = H$). Additionally, the quantitative results from the spectral element method (SEM) are introduced as the reference solution.

Within the Guo *et al.* model [25] framework, simulations on the two test cases of $(R_A, Re) = (1.5, 200)$ and $(2.5, 200)$ are performed, with the flow structures obtained from the various forcing schemes presented in Figs. 3 and 4, respectively. Three forcing schemes are presented on the left panel of each subplot for comparison against the trapezium rule based scheme on the right panel. Figures 3(a) and 4(a) show the “direct forcing

scheme A” in the left panels, which represents the axisymmetric model with the source terms in the LB equation for $\mathbf{u}$ discretized by the direct forcing scheme, while the trapezium rule is applied for the evolution of the swirling velocity u_θ . In addition, Figs. 3(b) and 4(b) present the “direct forcing scheme B” in the left panels, which stands for the model wherein the source terms for both distribution functions are discretized by the direct forcing scheme. Finally, Figs. 3(c) and 4(c) display the model results obtained using the semi-implicit centered scheme in the left panels, which is only implemented for the LB equation for $\mathbf{u}$ in the meridian plane while the source terms for the evolution of distribution for azimuthal velocity component u_θ are discretized by the trapezium rule. Figures 3(a), 3(b), 4(a), and 4(b) show that the flow structures in the left and right panels are different, which indicates that the flow structures cannot be accurately obtained from the direct forcing scheme based model. On the other hand, the semi-implicit centered scheme reflected in Figs. 3(c) and 4(c) is able to accurately

FIG. 5. Axial velocity profile along the symmetry axis obtained by using various forcing schemes within the Guo *et al.* model [25] framework: (a) for the case of $R_A = 1.5$ and $Re = 200$; (b) for the case of $R_A = 2.5$ and $Re = 200$.

FIG. 6. Axial velocity profile along the symmetry axis obtained by using various forcing schemes within the Li *et al.* model [22] framework for test cases of (R_A, Re) equaling (a) (1.5, 990), (b) (1.5, 1290), (c) (2.5, 1010), and (d) (2.5, 2200).

account for the flow structures. Besides, the numerical stability of the present simulations is significantly affected by the adopted forcing scheme since convergent simulations cannot be obtained for cases with Re greater than 300 with the direct forcing scheme and the semi-implicit centered scheme.

Furthermore, quantitative comparisons for the axial velocity distribution on the symmetry axis ($r = 0$) are performed for the forcing schemes in Fig. 5 for the same cases of $(R_A, Re) = (1.5, 200)$ and $(2.5, 200)$. Similarly, the axial velocity distributions from the direct forcing scheme (i.e., direct forcing schemes A and B) deviate significantly from the reference SEM results. Notably, the good agreements between the results from the other two forcing schemes (i.e., trapezium rule and centered schemes) and the SEM reference solutions confirm that discrete lattice effects are effectively removed by both the trapezium rule and the semi-implicit centered scheme, which accords well with our theoretical analysis.

Aside from the Guo *et al.* model [25], the impact of the discrete lattice effects is also assessed under the Li *et al.*

model [22] framework. The results of the various forcing schemes (namely, trapezium rule, direct forcing scheme B, and centered scheme) and the reference SEM data are compared using the test cases of (R_A, Re) of (1.5, 990), (1.5, 1290), (2.5, 1010), and (2.5, 2200) in Fig. 6. Negligible differences in the trends indicate that the axial velocity distributions agree quantitatively with the reference solution from the SEM. Moreover, in contrast to the Guo *et al.* model [25], Fig. 6 indicates that the numerical stability of the Li *et al.* model [22] based simulations is not as sensitive to the variation of the adopted forcing schemes.

However, further comparisons of the pressure profiles in Fig. 7 obtained within the Li *et al.* model framework [22] reveal the discrete lattice effects arising from the direct forcing scheme B. The left panels show the profiles obtained with the direct forcing scheme, while the right panels are the reference results obtained with the trapezium rule based scheme. Figure 7 shows that the pressure contours obtained with the direct forcing scheme deviate from that obtained with the trapezium

FIG. 7. Pressure contours obtained based on the Li *et al.* model [22] for the test cases of (R_A, Re) equaling (a) (1.5, 990), (b) (1.5, 1290), (c) (2.5, 1010), and (d) (2.5, 2200). The left panels show the profiles obtained with the direct forcing scheme B, while the right panels are the reference results obtained from the original Li *et al.* model [22].

rule in the region near the top lid, and such discrepancies worsen with the increase of Re . It is reasonable to attribute such disagreements in the pressure distribution to the discrete lattice effects from the direct forcing scheme, since the error term in the recovered macroscopic equations [i.e., Eqs. (A16b) and (A16c)] is related to the spatial gradients of the radial velocity component u_r , which are larger near the top lid.

Moreover, in comparison with the reference pressure distributions in the right panels in Fig. 7, the pressure contours from the semi-implicit centered scheme are demonstrated in Fig. 8, and the consistent pressure distributions as well as the accurate axial velocity profile (as in Fig. 6) confirm that the semi-implicit centered scheme is also helpful in avoiding the discrete lattice effects within the Li *et al.* model [22] framework.

IV. CONCLUSIONS

In this study, the discrete lattice effects arising in the discretization of the source terms in the axisymmetric LB models are investigated under the incompressible limit. The finite difference interpretation for the LB method [37,38] is introduced to theoretically analyze the discrete lattice effects from the trapezium rule scheme, the direct forcing scheme, and the semi-implicit centered scheme. It is concluded from the theoretical analysis that the discrete lattice effects from

the direct forcing scheme are part of the truncation error and thus would not affect the overall accuracy of the standard LB method with general force (i.e., only momentum source terms are considered), but would lead to incorrect macroscopic equations for the axisymmetric LB models. Particularly, inconsistent kinematic viscosity definitions exist between the main Laplace viscosity term and the viscous velocity gradients in the source terms. Such discrete lattice effects from the direct forcing scheme can be removed by the trapezium rule based scheme which is just adopted in the axisymmetric LB models considered in this work; therefore, both the Guo *et al.* model [25] and the Li *et al.* model [22], in their original forms are free of the discrete lattice effects. Moreover, the semi-implicit centered scheme, as an approximated implementation of the trapezium rule based scheme, can also recover the accurate macroscopic equations within the framework of both axisymmetric LB models considered. Thus the semi-implicit centered scheme can be a compromised alternative with the advantage of avoiding the discrete lattice effects and maintaining the simple macroscopic variables calculations.

Numerical results confirm the theoretical analysis: (i) within the Guo *et al.* model [25] framework, the direct forcing scheme makes the flow structures inaccurate and significantly affects the numerical stability of the simulations; (ii) based on the Li *et al.* model [22], the pressure contour results are influenced by the direct forcing scheme; (iii) the trapezium rule

FIG. 8. Pressure contours obtained from the semi-implicit centered scheme within the Li *et al.* model [22] framework for the test cases of (R_A, Re) equaling (a) (1.5, 990), (b) (1.5, 1290), (c) (2.5, 1010), and (d) (2.5, 2200).

scheme and the semi-implicit centered scheme are capable of removing the discrete lattice effects, but the latter would render the Guo *et al.* model [25] impractical due to poor numerical stability.

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APPENDIX: DISCRETE LATTICE EFFECTS ANALYSIS WITHIN THE Li *et al.* [22] AXISYMMETRIC LB MODEL FRAMEWORK

1. Li *et al.* model [22]

The LB equations in the Li *et al.* model [22] are given as

$$\begin{aligned} \tilde{f}_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \delta t, t + \delta t) - \tilde{f}_\alpha(\mathbf{x}, t) = & -\frac{1}{\tau} [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] - \frac{(2\tau - 1)\xi_{\alpha r} \delta t}{2\tau r} [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\mathbf{x}, t)] \\ & + \delta t \left\{ 1 - \frac{1}{2} \left[\frac{1}{\tau} + \frac{(2\tau - 1)\xi_{\alpha r} \delta t}{2\tau r} \right] \right\} S_\alpha(\mathbf{x}, t), \end{aligned} \quad (\text{A1a})$$

$$g_k(\mathbf{x} + \boldsymbol{\xi}_k \delta t, t + \delta t) - g_k(\mathbf{x}, t) = -\frac{1}{\tau_g} [g_k(\mathbf{x}, t) - g_k^{(\text{eq})}(\mathbf{x}, t)] - \frac{(2\tau_g - 1)\xi_{kr} \delta t}{2\tau_g r} [g_k(\mathbf{x}, t) - g_k^{(\text{eq})}(\mathbf{x}, t)] + \delta t G_k, \quad (\text{A1b})$$

which introduce additional relaxation terms for recovering the velocity gradients in the momentum source terms. It is noted that the trapezium rule based forcing scheme is applied in Eq. (A1a) with the same distribution function reconstruction in Eq. (30a),

while the direct forcing scheme is adopted in Eq. (A1b) for facilitating the macroscopic variables calculations. Besides, the distribution function for the azimuthal velocity component in Eq. (A1b) evolves on the D2Q4 lattice model,

$$\xi_k = c\{\cos[(k-1)\pi/2], \sin[(k-1)\pi/2]\} \quad k = 1, 2, 3, 4, \quad (\text{A2})$$

with the corresponding weight coefficient $w_k = 1/4$ and the lattice speed $c = \sqrt{2RT'}$. Differing from the kinetic theory based model by Guo *et al.* [25], the Li *et al.* model [22] is developed under the framework of the standard LB method, and the consistent equilibrium distribution function as in Eq. (5) is adopted in Eq. (A1a). Besides, the remaining unknown equilibrium distribution function and the source terms in the Li *et al.* model [22] are defined as

$$S_\alpha = \left[\frac{(\xi_{\alpha i} - u_i)\tilde{F}_i}{\rho RT} - \frac{u_r}{r} \right] f_\alpha^{(\text{eq})}, \quad \tilde{F}_i = -\left(\frac{2\mu u_r}{r^2} - \frac{\rho u_\theta^2}{r} \right) \delta_{ir}, \quad (\text{A3})$$

$$g_k^{(\text{eq})} = \rho_0 u_\theta w_k \left(1 + \frac{\mathbf{u} \cdot \xi_k}{RT'} \right), \quad (\text{A4a})$$

$$G_k = w_k \left(-\frac{2\rho_0 u_\theta u_r}{r} - \frac{\mu u_\theta}{r^2} \right) \left(1 + \frac{\mathbf{u} \cdot \xi_k}{RT'} \right), \quad (\text{A4b})$$

where a constant density ρ_0 is applied in the azimuthal velocity evolution. Moreover, the macroscopic variables in the Li *et al.* model [22] are determined as

$$u_\theta = \sum_k g_k / \rho_0, \quad (\text{A5a})$$

$$u_i = \frac{\sum_\alpha \tilde{f}_\alpha \xi_{\alpha i} + \frac{\delta t}{2} \left(\frac{\rho_0 u_\theta^2}{r} \delta_{ir} \right)}{\sum_\alpha \tilde{f}_\alpha + (\delta t \rho_0 v / r^2) \delta_{ir}}, \quad (\text{A5b})$$

$$\rho = \sum_\alpha \tilde{f}_\alpha / \left(1 + \frac{\delta t}{2} \frac{u_r}{r} \right), \quad (\text{A5c})$$

and the relaxation parameter in Eq. (A1b) τ_g is related to the kinematic viscosity as

$$\nu = \left(\tau_g - \frac{1}{2} \right) RT' \delta t. \quad (\text{A6})$$

2. Finite difference interpretation of the Li *et al.* model [22]

Similar analysis with Sec. II B 2 is herein extended to the Li *et al.* model [22] without the rotation effects, and Eq. (A1a) is scaled as

$$\begin{aligned} \tilde{f}_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t + \Delta t) - \tilde{f}_\alpha(\mathbf{x}, t) = & -\frac{1}{\tau} [\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] - \frac{(2\tau - 1)e_{\alpha r} \epsilon}{2\tau r} \left[\tilde{f}_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t)) + \frac{1}{2} S_\alpha(\mathbf{x}, t) \right] \\ & + \left(1 - \frac{1}{2\tau} \right) S_\alpha(\mathbf{x}, t), \end{aligned} \quad (\text{A7})$$

with the scaled equilibrium distribution function and source terms

$$f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}) = \rho w_\alpha \left[1 + 3(\mathbf{e}_\alpha \cdot \epsilon \mathbf{u}) + \frac{9(\mathbf{e}_\alpha \cdot \epsilon \mathbf{u})^2}{2} - \frac{3\epsilon^2 \mathbf{u}^2}{2} \right], \quad (\text{A8a})$$

$$S_\alpha = \left[-\frac{\epsilon^3 (e_{\alpha r} - \epsilon u_r)(2\tau - 1)u_r}{r^2} - \frac{\epsilon^2 u_r}{r} \right] f_\alpha^{(\text{eq})}, \quad (\text{A8b})$$

which lead to the known moment vectors for the Li *et al.* model [22] as

$$\mathbf{M}^{(\text{eq})} = (\rho, \rho u_r, \rho u_z, \rho u_r u_r, \rho u_r u_z, \rho u_z u_z, M_6^{(\text{eq})}, M_7^{(\text{eq})}, M_8^{(\text{eq})})^\dagger, \quad (\text{A9a})$$

$$\mathbf{M}^S = \left[-\frac{\epsilon^2 \rho u_r}{r}, -\frac{\epsilon^2 (2\tau - 1) \rho u_r}{3r} - \frac{\epsilon^2 \rho u_r u_r}{r}, -\frac{\epsilon^2 \rho u_r u_z}{r}, 0, 0, 0, M_6^S, M_7^S, M_8^S \right]^\dagger. \quad (\text{A9b})$$

The evolution equation for moment component $\tilde{M}_i$ is derived from Eq. (A7) as

$$\begin{aligned}\tilde{M}_i^{n+1}(\mathbf{x}) = & \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j \left[\tilde{M}_j^n - \frac{1}{\tau} \tilde{M}_j^{(\text{neq}),n} + \left(1 - \frac{1}{2\tau}\right) M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle \\ & - \frac{(2\tau - 1)}{2\tau r} \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j e_{\alpha r} \epsilon \left[\tilde{M}_j^{(\text{neq}),n} + \frac{1}{2} M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle.\end{aligned}\quad (\text{A10})$$

In particular, before relating Eq. (A10) with the macroscopic equations, additional finite difference stencils pertaining to $P'(\mathbf{e}_\alpha) = \frac{Q_i(\mathbf{e}_\alpha) Q_j(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_j^2(\mathbf{e}_\alpha) w_\alpha \rangle}$ are obtained as

$$P'_{13}(\mathbf{e}_\alpha) = \frac{Q_1(\mathbf{e}_\alpha) Q_3(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_3^2(\mathbf{e}_\alpha) w_\alpha \rangle} = 3\epsilon^2 e_{\alpha r}^2 \leftrightarrow \epsilon^2 + O(\epsilon^4), \quad (\text{A11a})$$

$$P'_{14}(\mathbf{e}_\alpha) = \frac{Q_1(\mathbf{e}_\alpha) Q_4(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_4^2(\mathbf{e}_\alpha) w_\alpha \rangle} = 9\epsilon^2 e_{\alpha r} e_{\alpha z} \leftrightarrow O(\epsilon^4), \quad (\text{A11b})$$

$$P'_{15}(\mathbf{e}_\alpha) = \frac{Q_1(\mathbf{e}_\alpha) Q_5(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_5^2(\mathbf{e}_\alpha) w_\alpha \rangle} = \frac{9\epsilon^2}{2} \left(e_{\alpha r}^2 e_{\alpha z}^2 - \frac{1}{3} e_{\alpha r}^2 \right) \leftrightarrow O(\epsilon^4), \quad (\text{A11c})$$

$$P'_{23}(\mathbf{e}_\alpha) = \frac{Q_2(\mathbf{e}_\alpha) Q_3(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_3^2(\mathbf{e}_\alpha) w_\alpha \rangle} = 3\epsilon^2 e_{\alpha r} e_{\alpha z} \leftrightarrow O(\epsilon^4), \quad (\text{A11d})$$

$$P'_{24}(\mathbf{e}_\alpha) = \frac{Q_2(\mathbf{e}_\alpha) Q_4(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_4^2(\mathbf{e}_\alpha) w_\alpha \rangle} = 9\epsilon^2 e_{\alpha r}^2 e_{\alpha z}^2 \leftrightarrow \epsilon^2 + O(\epsilon^4), \quad (\text{A11e})$$

$$P'_{25}(\mathbf{e}_\alpha) = \frac{Q_2(\mathbf{e}_\alpha) Q_5(\mathbf{e}_\alpha) e_{\alpha r} \epsilon}{\langle Q_5^2(\mathbf{e}_\alpha) w_\alpha \rangle} = 3\epsilon^2 e_{\alpha r} e_{\alpha z} \leftrightarrow O(\epsilon^4). \quad (\text{A11f})$$

Applying the finite difference stencil definitions in Eqs. (20) and (A11), macroscopic relations derived from Eq. (A10) are

$$\frac{\tilde{M}_0^{n+1} - \tilde{M}_0^n}{\Delta t} = \frac{M_0^{S,n}}{\epsilon^2} + \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (\text{A12a})$$

$$\begin{aligned}\frac{\tilde{M}_1^{n+1} - \tilde{M}_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} \left(M_0^n + \frac{M_0^{S,n}}{2} \right) + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial r} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_3^{(\text{neq}),n} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_4^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} \tilde{M}_3^{(\text{neq}),n} + O(\epsilon^2),\end{aligned}\quad (\text{A12b})$$

$$\begin{aligned}\frac{\tilde{M}_2^{n+1} - \tilde{M}_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} \left(M_0^n + \frac{M_0^{S,n}}{2} \right) + \frac{M_2^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_2^n + \frac{1}{3} \frac{\partial^*}{\partial z} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_4^{(\text{neq}),n} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau}\right) \tilde{M}_5^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} \tilde{M}_4^{(\text{neq}),n} + O(\epsilon^2),\end{aligned}\quad (\text{A12c})$$

$$\tilde{M}_3^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_r)}{\partial r} + O(\epsilon^2), \quad (\text{A12d})$$

$$\tilde{M}_4^{(\text{neq})} = -\frac{\tau}{3} \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] + O(\epsilon^2), \quad (\text{A12e})$$

$$\tilde{M}_5^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_z)}{\partial z} + O(\epsilon^2). \quad (\text{A12f})$$

Thus the standard axisymmetric NS equations for incompressible flows are accurately obtained by applying the moment vector definitions in Eq. (A9),

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0, \quad (\text{A13a})$$

$$\frac{\partial u_r}{\partial t} + \frac{\partial(u_r u_r)}{\partial r} + \frac{\tilde{\partial}(u_r u_z)}{\partial z} = -\frac{\partial p}{\partial r} + \frac{\Delta u_r}{6} + \frac{1}{3}(\tau - 1) \left(\frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{\tilde{\partial}}{\partial z} \frac{\tilde{\partial}}{\partial z} \right) u_r + \frac{1}{3r} \left(\tau - \frac{1}{2} \right) \frac{\partial u_r}{\partial r} - \frac{1}{3} \left(\tau - \frac{1}{2} \right) \frac{u_r}{r^2} - \frac{u_r u_r}{r}, \quad (\text{A13b})$$

$$\frac{\partial u_z}{\partial t} + \frac{\tilde{\partial}(u_r u_z)}{\partial r} + \frac{\partial(u_z u_z)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\Delta u_z}{6} + \frac{1}{3}(\tau - 1) \left(\frac{\tilde{\partial}}{\partial r} \frac{\tilde{\partial}}{\partial r} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) u_z + \frac{(\tau - 1/2)}{3r} \frac{\tilde{\partial} u_z}{\partial r} - \frac{u_r u_z}{r}, \quad (\text{A13c})$$

with consistent kinematic viscosity definition $\nu = \frac{1}{3}(\tau - \frac{1}{2})$, and therefore the original Li *et al.* model [22] is also free of the discrete lattice effects due to the application of the trapezium rule based scheme.

3. Direct forcing scheme

Within the Li *et al.* model [22] framework, the scaled LB equation in Eq. (A7) is transformed by applying the direct forcing scheme as

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] - \frac{(2\tau - 1)e_{\alpha r} \epsilon}{2\tau r} [f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] + S_\alpha(\mathbf{x}, t), \quad (\text{A14a})$$

and the resulting moment component evolution equation is

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j \left[M_j^n - \frac{1}{\tau} M_j^{(\text{neq}),n} + M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle - \frac{(2\tau - 1)}{2\tau r} \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j e_{\alpha r} \epsilon M_j^{(\text{neq}),n} (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle, \quad (\text{A14b})$$

which leads to the macroscopic equation for each moment component,

$$\frac{M_0^{n+1} - M_0^n}{\Delta t} = \frac{M_0^{S,n}}{\epsilon^2} + \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (\text{A15a})$$

$$\begin{aligned} \frac{M_1^{n+1} - M_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} (M_0^n + M_0^{S,n}) + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial r} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_3^{(\text{neq}),n} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} M_3^{(\text{neq}),n} + O(\epsilon^2), \end{aligned} \quad (\text{A15b})$$

$$\begin{aligned} \frac{M_2^{n+1} - M_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} (M_0^n + M_0^{S,n}) + \frac{M_2^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_2^n + \frac{1}{3} \frac{\partial^*}{\partial z} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_5^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} M_4^{(\text{neq}),n} + O(\epsilon^2), \end{aligned} \quad (\text{A15c})$$

$$M_3^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_r)}{\partial r} + O(\epsilon^2), \quad (\text{A15d})$$

$$M_4^{(\text{neq})} = -\frac{\tau}{3} \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] + O(\epsilon^2), \quad (\text{A15e})$$

$$M_5^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_z)}{\partial z} + O(\epsilon^2). \quad (\text{A15f})$$

The axisymmetric NS equations recovered from the direct forcing scheme are derived within the Li *et al.* model [22] framework as

$$\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0, \quad (\text{A16a})$$

$$\frac{\partial u_r}{\partial t} + \frac{\partial(u_r u_r)}{\partial r} + \frac{\tilde{\partial}(u_r u_z)}{\partial z} = -\frac{\partial p}{\partial r} + \frac{\Delta u_r}{6} + \frac{1}{3}(\tau - 1) \left(\frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{\tilde{\partial}}{\partial z} \frac{\tilde{\partial}}{\partial z} \right) u_r + \frac{\tau}{3r} \frac{\partial u_r}{\partial r} - \frac{\tau u_r}{3r^2} - \frac{u_r u_r}{r}, \quad (\text{A16b})$$

$$\frac{\partial u_z}{\partial t} + \frac{\tilde{\partial}(u_r u_z)}{\partial r} + \frac{\partial(u_z u_z)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\Delta u_z}{6} + \frac{1}{3}(\tau - 1) \left(\frac{\tilde{\partial}}{\partial r} \frac{\tilde{\partial}}{\partial r} + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \right) u_z + \frac{(\tau - 1/2)}{3r} \frac{\tilde{\partial} u_z}{\partial r} + \frac{1}{6r} \frac{\tilde{\partial} u_r}{\partial z} - \frac{u_r u_z}{r}, \quad (\text{A16c})$$

which also suffer from inconsistent kinematic viscosity definitions. That is, the discrete lattice effect from the direct forcing scheme also leads to inaccurate macroscopic equations within the Li *et al.* model [22] framework.

4. Semi-implicit centered scheme

The semi-implicit centered scheme is also validated by the Li *et al.* model [22], and the approximate implementation in Eq. (43) is applied to Eq. (A7) as

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} [f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] - \frac{(2\tau - 1)e_{\alpha r} \epsilon}{2\tau r} [f_\alpha(\mathbf{x}, t) - f_\alpha^{(\text{eq})}(\rho, \epsilon \mathbf{u}(\mathbf{x}, t))] + \frac{1}{2} [S_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta x, t) + S_\alpha(\mathbf{x}, t)], \quad (\text{A17a})$$

which leads to

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j \left[M_j^n - \frac{1}{\tau} M_j^{(\text{neq}),n} + \frac{1}{2} M_j^{S,n} \right] (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle - \frac{(2\tau - 1)}{2\tau r} \sum_{j=0}^8 \frac{1}{\langle Q_j^2 w_\alpha \rangle} \left\langle w_\alpha Q_i Q_j e_{\alpha r} \epsilon M_j^{(\text{neq}),n} (\mathbf{x} - \mathbf{e}_\alpha \Delta x) \right\rangle + \frac{1}{2} M_i^n(\mathbf{x}), \quad (\text{A17b})$$

Macroscopic equations for each moment component are

$$\frac{M_0^{n+1} - M_0^n}{\Delta t} = \frac{M_0^{S,n}}{\epsilon^2} + \frac{1}{6} \Delta M_0^n - \left(\frac{\partial}{\partial r} M_1^n + \frac{\partial}{\partial z} M_2^n \right) + O(\epsilon^2), \quad (\text{A18a})$$

$$\begin{aligned} \frac{M_1^{n+1} - M_1^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial r} \left(M_0^n + \frac{M_0^{S,n}}{2} \right) + \frac{M_1^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_1^n + \frac{1}{3} \frac{\partial^*}{\partial r} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\partial}{\partial r} \left[M_3^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_3^{(\text{neq}),n} \right] \\ & - \frac{\tilde{\partial}}{\partial z} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} M_3^{(\text{neq}),n} + O(\epsilon^2), \end{aligned} \quad (\text{A18b})$$

$$\begin{aligned} \frac{M_2^{n+1} - M_2^n}{\Delta t} = & -\frac{1}{3\epsilon^2} \frac{\partial}{\partial z} \left(M_0^n + \frac{M_0^{S,n}}{2} \right) + \frac{M_2^{S,n}}{\epsilon^2} + \frac{\Delta}{6} M_2^n + \frac{1}{3} \frac{\partial^*}{\partial z} \left(\frac{\partial^*}{\partial r} M_1^n + \frac{\partial^*}{\partial z} M_2^n \right) - \frac{\tilde{\partial}}{\partial r} \left[M_4^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_4^{(\text{neq}),n} \right] \\ & - \frac{\partial}{\partial z} \left[M_5^{(\text{eq}),n} + \left(1 - \frac{1}{\tau} \right) M_5^{(\text{neq}),n} \right] - \frac{(2\tau - 1)}{2\tau r} M_4^{(\text{neq}),n} + O(\epsilon^2), \end{aligned} \quad (\text{A18c})$$

$$M_3^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_r)}{\partial r} + O(\epsilon^2), \quad (\text{A18d})$$

$$M_4^{(\text{neq})} = -\frac{\tau}{3} \left[\frac{\tilde{\partial}(\rho u_r)}{\partial z} + \frac{\tilde{\partial}(\rho u_z)}{\partial r} \right] + O(\epsilon^2), \quad (\text{A18e})$$

$$M_5^{(\text{neq})} = -\frac{2\tau}{3} \frac{\partial(\rho u_z)}{\partial z} + O(\epsilon^2). \quad (\text{A18f})$$

The axisymmetric NS equations for incompressible flows with consistent kinematic viscosity definition [i.e., Eq. (A13)] can also be derived from Eq. (A17). Therefore, the discrete lattice effects from the direct forcing scheme are avoided by the semi-implicit centered scheme, which accords well with the analysis based on the kinetic theory based axisymmetric LB model by Guo *et al.* [25]. Moreover, since the distribution function reconstruction arising in the trapezium rule based scheme is avoided by the approximation in Eq. (43), the simple macroscopic variable calculations as in Eq. (7) are maintained for the Li *et al.* model [22] by the semi-implicit centered scheme.

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