Skew-Symmetric Association Schemes With Two Classes and Strongly Regular Graphs of Type $L_{2n-1}(4n-1)$

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Abstract. A construction of a pair of strongly regular graphs Γ_n and Γ'_n of type $L_{2n-1}(4n-1)$ from a pair of skew-symmetric association schemes W, W' of order $4n-1$ is presented. Examples of graphs with the same parameters as Γ_n and Γ'_n, i.e., of type $L_{2n-1}(4n-1)$, were known only if $4n-1=p^s$, where p is a prime. The first new graph appearing in the series has parameters $(v,k,\lambda)=(225,98,45)$. A 4-vertex condition for relations of a skew-symmetric association scheme (very similar to one for the strongly regular graphs) is introduced and is proved to hold in any case. This has allowed us to check the 4-vertex condition for Γ_n and Γ'_n, thus to prove that Γ_n and Γ'_n are not rank three graphs if $n>2$.

0. Introduction

The main aim of this paper is to describe a construction of strongly regular graphs (SRG, for short) of type $L_{2n-1}(4n-1)$ from a pair of skew-symmetric Hadamard matrices of order $4n$ (Theorem 1 and Corollary 1). Examples of such graphs were known only if $4n-1=p^s$, p is a prime. But it is well-known that skew-symmetric Hadamard matrices exist for a much wider range of n, cf., e.g., Hall [9]. In fact, the first new SRG, which appears in our series, has parameters $(v,k,\lambda)=(225,98,45)$. In the catalog of SRGs with small v due to Brouwer [2] there are no examples of graphs with the latter set of parameters. Some statements equivalent to Theorem 1 and Corollary 1 were announced by the author in [6].

Questions concerning some further symmetry of our SRG are considered in the paper as well. There is the classification of rank three graphs due to Kantor and Liebler [13] and Liebeck [14], which uses the classification of finite simple groups. But we are able to prove (Theorem 2) using the so-called 4-vertex condition for an SRG, (see, e.g., Hestenes and Higman [10]) that our graphs are not rank three graphs if $n \geqslant 2$.

Among the many different combinatorial objects associated with a skew-symmetric Hadamard matrix H of order $4n$ (see Reid and Brown [17], Delsarte [4], Szekeres [18], Hall [9]) there is a 2-class association scheme $W=W(H)$ which we

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prefer to use for our purposes. To calculate 4-constants for our SRGs, we first introduce a 4-vertex condition for relations of $W(H)$ and prove that it holds for any $W(H)$. It seems that this fact (Theorem 3) is of some independent value.

The content of this paper is as follows. In Sect. 1 we give the terminology and our theorems. Section 2 contains proofs of Theorem 1 and Corollary 1. Section 3 contains a proof of Theorem 3, and Sect. 4 contains a proof of Theorem 2.

1. Terminology and Statement of Results

The terminology concerning *association schemes* is fairly standard, see, e.g., Bannai and Ito [1]. For an association scheme $A = (V, \{R_i\}_{0 \leqslant i \leqslant r})$, k_i and p_{ij}^k denote the valencies and the intersection numbers respectively. Any relation R_i of A forms a graph (or digraph) with vertex set V and edge set R_i. This allows us to speak about the *graph* R_i.

A *strongly regular graph* (SRG) is an irreflexive relation Γ in a symmetric association scheme $A = (V, \{I, \Gamma, \Delta\})$. As usual, $v = |V|$, $k = k_1$, $\lambda = p_{11}^1$, $\mu = p_{11}^2$ form the set of parameters (v, k, λ, μ) of Γ. See [1], [10], [11], and [3] for constructions and further properties of SRGs.

A *Hadamard matrix* H is a $4n \times 4n$ matrix, whose entries are ± 1, such that $HH^T = 4nI$ (I denotes the identity $n \times n$ matrix). It is *skew-symmetric* or *skew* if $H + H^T = 2I$. Any Hadamard matrix H can be converted to a *normalized* one, for which $h_{1i} = 1$ for $1 \leqslant i \leqslant 4n$. We assume below that an Hadamard matrix H is skew and normalized. We define the relation $T = T(H)$ on the set $V = \{2, \dots, 4n\}$ of numbers of H-lines by the following rule. For $2 \leqslant i, j \leqslant 4n$, let $(i, j) \in T$ if $h_{ij} = 1$ and $i \neq j$. Notice that T is a tournament on V. In fact, $W = W(H) = (V, \{I, T, T^t\})$ is an association scheme. It is easy to show, see Lemma 1 in Sect. 2, that the intersection numbers of W depend only on n. It is worth stressing that, starting from such T, we can obtain a skew Hadamard matrix. For further study of relations of W see, e.g., Ito [15].

Let $X = (V, \{R_i\}_{0 \leqslant i \leqslant r})$ be a commutative association scheme. The *symmetrization* of X (see [1]) is the association scheme $\widetilde{X} = (V, \{\widetilde{R}_i\}_{0 \leqslant i \leqslant s})$, where $\widetilde{R} = R \cup R'$ for any pair R, R' of mutually paired relations of X, and $\widetilde{R} = R$ if $R = R'$.

The *tensor product* of two association schemes $A = (X, \{R_i\}_{0 \leqslant i \leqslant s})$ and $B = (Y, \{Q_i\}_{0 \leqslant i \leqslant r})$ is the association scheme $A \otimes B = (X \times Y, \{F_{ij}\}_{0 \leqslant i \leqslant s, 0 \leqslant j \leqslant r})$, where the relations are defined by

$$F_{ij} = \{((a, c), (b, d)) \mid a, b \in X,\ c, d \in Y,\ (a, b) \in R_i,\ (c, d) \in Q_j\}.$$

This notion appeared in [16]. We can find in [16] the following formulae for the intersection numbers of $A \otimes B$:

$$P_{(bi),(cj)}^{(ak)} = p_{bc}^a q_{ij}^k,$$

where p_{bc}^a and q_{ij}^k are the intersection numbers of A and B, respectively.

An *amorphic cell* is an association scheme $A = (V, \{R_i\})$ such that any (possibly, trivial) fusion of irreflexive relations is an SRG. This notion was introduced by

Gol'fand and Klin [7]. For further results concerning this object see Ivanov [12] and a survey by Faradžev *et al.* [5]. The most remarkable theorems due to Ivanov [12] say that $|V| = m^2$ and all irreflexive R_i are graphs of Latin square type $L_{k_i}(m)$ or of negative Latin square type $NL_{k_i}(m)$, (series K.1 or K.5 from [11]).

We are now able to state our first result.

THEOREM 1. *Let H and H' be skew-symmetric Hadamard matrices of order $4n$, $S = T(H) \otimes T(H')$. Then $\widetilde{S} = (V, \{\Gamma_i\}_{0 \leqslant i \leqslant 4})$ (the symmetrization of S) is an amorphic cell with the following intersection numbers:*

$$k_1 = k_2 = 4n - 2, \qquad k_3 = k_4 = 8n^2 - 8n + 2,$$

$$(P^1_{ij}) = \begin{pmatrix} 4n-3 & 0 & 0 & 0 \\ 0 & 0 & 2n-1 & 2n-1 \\ 0 & 2n-1 & 4n^2-6n+2 & 4n^2-4n-1 \\ 0 & 2n-1 & 4n^2-4n-1 & 4n^2-6n+2 \end{pmatrix},$$

$$(P^2_{ij}) = \begin{pmatrix} 0 & 0 & 2n-1 & 2n-1 \\ 0 & 4n-3 & 0 & 0 \\ 2n-1 & 0 & 4n^2-6n+2 & 4n^2-4n-1 \\ 2n-1 & 0 & 4n^2-4n-1 & 4n^2-6n+2 \end{pmatrix},$$

$$(P^3_{ij}) = \begin{pmatrix} 0 & 1 & 2n-2 & 2n-1 \\ 1 & 0 & 2n-2 & 2n-1 \\ 2n-2 & 2n-2 & 4n^2-6n+3 & 4n^2-6n+2 \\ 2n-1 & 2n-1 & 4n^2-6n+2 & 4n^2-6n+2 \end{pmatrix},$$

$$(P^4_{ij}) = \begin{pmatrix} 1 & 0 & 2n-1 & 2n-2 \\ 0 & 1 & 2n-1 & 2n-2 \\ 2n-1 & 2n-1 & 4n^2-6n+2 & 4n^2-6n+2 \\ 2n-2 & 2n-2 & 4n^2-6n+2 & 4n^2-6n+3 \end{pmatrix}, \qquad 1 \leqslant i,j \leqslant 4.$$

For the numbering of the relations of $\widetilde{S}$ see Sect. 2.

This theorem immediately implies

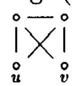
COROLLARY 1. *The graphs Γ_3 and Γ_4 of $\widetilde{S}$ are strongly regular of type L_{2n-1} $(4n-1)$.*

Remark 1. All the other graphs and fusions of graphs of $\widetilde{S}$ are known. These are graphs of type $L_g(4n-1)$, where $g = 1$, 2 or $2n$. In the former two cases such graphs are uniquely determined by parameters, cf. [3]. In the latter case there is a construction for such graphs starting from a skew Hadamard matrix of order $4n$, see [3] and [8].

Remark 2. The referee has pointed out that Corollary 1 can be proved by use of the so-called $(-1, +1, 0)$-adjacency matrices very quickly along the lines of the proof of Theorem 4.1 in [8]. However, we prefer our own way, which goes back to the initial discover of our new SRGs and which makes it possible to obtain further results, as well.

Our further aim is to settle the following question. Are the graphs in Corollary 1 rank 3 graphs? We use the approach due to [10], involving calculation of the following two parameters:

(1) the parameter $\alpha(u,v)$ for any edge (u,v) of an SRG Γ, which is equal to the number of distinct complete 4-vertex subgraphs through (u,v);

(2) the parameter $\beta(u,v)$ for any non-edge (u,v), which is equal to the number of distinct subgraphs of the shape 

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If $\alpha(u,v)$ depends on the particular choice of the edge (u,v) or $\beta(u,v)$ depends on the particular choice of the non-edge (u,v) then, of course, Γ is not a rank 3 graph. If it is not so, then we say that Γ satisfies the 4-*vertex condition*.

THEOREM 2. *The graphs* Γ_3 *and* Γ_4 *of* $\widetilde{S}$ *satisfy the* 4-*vertex condition iff* $n = 1$.

The proof of this theorem involves counting certain 4-vertex configurations passing through an arc of each tournament of $T(H)$ and $T(H')$. Namely, let $A = (V, \{I, T, T^t\})$ be the above-mentioned 2-class association scheme, $|V| = 4n - 1$. Fix an arc $(x,y) \in T$. Define the partition of $V \backslash \{x,y\}$ into the following sets:

$$\Delta_1 = \{z \in V \mid (x,z),(y,z) \in T\},$$

$$\Delta_2 = \{z \in V \mid (x,z),(z,y) \in T\},$$

$$\Delta_3 = \{z \in V \mid (z,x),(y,z) \in T\},$$

$$\Delta_4 = \{z \in V \mid (z,x),(z,y) \in T\}.$$

Each 4-vertex subgraph Ω of T through (x,y) is uniquely determined by a subset $\{a,b\} \subset V\backslash\{x,y\}$. We denote Ω by $E(a,b)$ if $(a,b) \in T$ and by $E(b,a)$ otherwise. Choose $(a,b) \in \Delta_i \times \Delta_j$, $a \neq b$. Then $E(a,b)$ and $E(c,d)$, $\{c,d\} \subset V\backslash\{x,y\}$, are isomorphic (as digraphs with one distinguished arc) iff $(c,d) \in \Delta_i \times \Delta_j$. Thus any $E(a,b)$, $(a,b) \in \Delta_i \times \Delta_j$, is isomorphic to a digraph E_{ij} with one distinguished arc. Note that E_{ij} and E_{kl} are isomorphic iff $i = k$, $j = l$. Therefore we can define $e_{ij}(x,y)$ to be the number of 4-vertex subgraphs through (x,y) which are isomorphic (in the afor-mentioned sense) to E_{ij}.

If for any $1 \leqslant i,j \leqslant 4$, the parameter $e_{ij}(x,y)$ does not depend on the particular choice of the arc (x,y), then we say that the 4-*vertex condition* holds for A. The idea of this condition seems to be very close to the 4-vertex condition for SRG, see e.g., [10]. But, contrary to the SRG case, we have

THEOREM 3. *For any* $A = (V, \{I, T, T^t\})$ *the* 4-*vertex condition holds. Moreover, the* e_{ij} *are as follows.*

$$(e_{ij}) = \frac{1}{2}\begin{pmatrix} (n-1)(n-2) & n(n-1) & n(n-1) & n(n-1) \\ (n-1)(n-2) & (n-1)(n-2) & n(n-1) & n(n-1) \\ n(n-2) & n(n-1) & n(n-1) & n(n-1) \\ (n-1)(n-2) & (n-1)(n-2) & n(n-1) & (n-1)(n-2) \end{pmatrix}.$$

2. Proof of Theorem 1

We shall calculate the intersection numbers of $\widetilde{S}$. Using these, we verify that $\widetilde{S}$ is an amorphic cell.

Before using the formulae for the intersection numbers of the tensor product and the formulae for the intersection numbers of symmetrization, we need the intersection numbers of $T(H)$, where H is a skew Hadamard matrix of order $4n$. We have

LEMMA 1. *The intersection numbers of* $T(H)$ *are as follows.*

$$(p^0_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2n-1 \\ 0 & 2n-1 & 0 \end{pmatrix},$$

$$(p^1_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & n-1 & n-1 \\ 0 & n-1 & n \end{pmatrix},$$

$$(p^2_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & n & n-1 \\ 1 & n-1 & n-1 \end{pmatrix}.$$

Proof. Using well-known relations between intersection numbers of an association scheme, see, e.g. [1], it is easy to produce a system of equations, in which the intersection numbers play the role of unknowns. In our case this system has a unique solution in non-negative integers. □

Let $S = T(H) \otimes T(H')$, where

$$T(H) = (V, \{I, T, T^t\}) = (V, \{R_0, R_1, R_2\}),$$

$$T(H') = (V, \{I, T', T'^t\}) = (V, \{Q_0, Q_1, Q_2\}).$$

This allows us to number the relations of S by the pairs (i, j), $i, j \in \{0, 1, 2\}$. The pairing of the relations of S is stated in the following

LEMMA 2. $S_{00} = S'_{00}$, $S_{02} = S'_{01}$, $S_{20} = S'_{10}$, $S_{22} = S'_{11}$, $S_{21} = S'_{12}$.

Proof. The argument consists of direct use of the definitions of the tensor product and S. Let us prove, e.g., that $S_{02} = S'_{01}$. Indeed, $S_{01} = \{((x,a),(y,b)) \mid x = y, (a,b) \in Q_1\}$. Then

$$\begin{aligned} S'_{01} &= \{((y,b),(x,a)) \mid x = y,\ (a,b) \in Q_1\} \\ &= \{((y,b),(x,a)) \mid x = y,\ (b,a) \in Q_2\} = S_{02}. \end{aligned}$$ □

The above lemma enables us to number the relations of $\widetilde{S}$ as follows: $\Gamma_0 = S_{00}$, $\Gamma_1 = S_{01} \cup S_{02}$, $\Gamma_2 = S_{10} \cup S_{20}$, $\Gamma_3 = S_{12} \cup S_{21}$, $\Gamma_4 = S_{11} \cup S_{22}$.

LEMMA 3. *The intersection numbers of $\widetilde{S}$ are as stated in Theorem* 1.

Proof. Using the formulae for the intersection numbers of the tensor product (see Sect. 1) and the formulae for the intersection numbers of the symmetrization (see [1]), we obtain the following formulae for these numbers of $\widetilde{S}$

$$P^{ab}_{cd,ef} = p^a_{ce}q^b_{df} + \delta(c,d)p^a_{de}q^b_{cf} + \delta(e,f)p^a_{cf}q^b_{de} + \delta(c,d)\delta(e,f)p^a_{df}q^b_{ce},$$

where $\delta(x,y) = 0$ if $S_{xy} = S'_{xy}$ and $\delta(x,y) = 1$ otherwise; p^a_{df} and q^b_{ce} are intersection numbers of $T(H)$ and $T(H')$.

Using this formulae, we perform direct computations. Lemma 1 gives us $p^k_{ij} = q^k_{ij}$. For example, $P^3_{12} = p^1_{01}p^1_{10} + p^1_{01}p^1_{20} + p^1_{02}p^1_{10} + p^1_{02}p^1_{20} = 1$.

LEMMA 4. *$\widetilde{S}$ is an amorphic cell.*

Proof. First, let us prove that any Γ_i is an SRG. It is sufficient to check that for $j \in \{1,\ldots,4\}\backslash\{i\}$ the parameter $\mu = \mu(\Gamma_i) = p^j_{ii}$ is independent on the particular choice of j. We see that it is so for any i. Next, we must prove that any $\Gamma_i \cup \Gamma_j$, $i \neq j$, $1 \leqslant i,j \leqslant 4$, is an SRG. It is sufficient to check the following two conditions: (1) the equality $\lambda = \lambda(\Gamma_i \cup \Gamma_j) = p^i_{ii} + 2p^i_{ij} + p^i_{jj} = p^j_{ii} + 2p^j_{ij} + p^j_{jj}$ holds, (2) for $k \in \{1,\ldots,4\}\backslash\{i,j\}$ the parameter $\mu = \mu(\Gamma_i \cup \Gamma_j) = p^k_{ii} + 2p^k_{ij} + p^k_{jj}$ is independent on the particular choice of k. The observation that the complement of an SRG is an SRG completes the proof. □

The proof of Theorem 1 is complete. We turn to

Proof of Corollary 1. It is sufficient to check (see [11]) series K.1, that the parameters of Γ_3 and Γ_4 are as follows for $g = 2n - 1$:

$$\begin{aligned} &v = (4n-1)^2, \qquad k = g(4n-2), \\ &\lambda = (g-1)(g-2) + 4n - 3, \qquad \mu = g(g-1). \end{aligned}$$

3. Proof of Theorem 3

We present the adjacency matrix A of the digraph T in the following form (Fig. 1), where $(x, y) \in T$ and Δ_i are the sets mentioned in Sect. 1.

$$A = \left|\begin{array}{c|c|c|c|c|c}
0 & 1 & 11\ldots\ldots1 & 11\ldots\ldots1 & 00\ldots\ldots0 & 00\ldots\ldots0 \\ \hline
0 & 0 & 11\ldots\ldots1 & 00\ldots\ldots0 & 11\ldots\ldots1 & 00\ldots\ldots0 \\ \hline
\begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & A_{11} & A_{12} & A_{13} & A_{14} \\ \hline
\begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}1\\ \vdots\\ 1\end{matrix} & A_{21} & A_{22} & A_{23} & A_{24} \\ \hline
\begin{matrix}1\\ \vdots\\ 1\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & A_{31} & A_{32} & A_{33} & A_{34} \\ \hline
\begin{matrix}1\\ \vdots\\ 1\end{matrix} & \begin{matrix}1\\ \vdots\\ 1\end{matrix} & A_{41} & A_{42} & A_{43} & A_{44}
\end{array}\right| \begin{array}{c} x \\ y \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array}$$

$$\begin{array}{cccccc}
x & y & \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 \\
1 & 1 & n-1 & n-1 & n & n-1
\end{array}$$

Figure 1.

The last line in this figure presents sizes of the blocks. By Lemma 1 these sizes are independent of the particular choice of (x, y). Observe that $e_{ij}(x, y) = e_{ij}$ is equal to the number of 1's in the block A_{ij}. Since T is a tournament, we have the following equalities:

$$e_{11} = e_{22} = e_{44} = \binom{n-1}{2}, \qquad e_{33} = \binom{n}{2},$$

$$e_{12} + e_{21} = e_{14} + e_{41} = e_{24} + e_{42} = (n-1)^2,$$

$$e_{13} + e_{31} = e_{23} + e_{32} = e_{34} + e_{43} = n(n-1).$$

Since any line of A contains exactly $2n - 1$ units, we obtain

$$\sum_i e_{1i} = (2n-1)(n-1), \qquad \sum_i e_{2i} = (2n-1)(n-1) - n + 1,$$

$$\sum_i e_{3i} = n(2n-1) - n, \qquad \sum_i e_{4i} = (2n-1)(n-1) - 2n + 2.$$

Denoting $e_{12} = d$, $e_{13} = b$, $e_{23} = c$, we have

$$(e_{ij}) = \begin{pmatrix}
\binom{n-1}{2} & d & b & 3\binom{n}{2} - d - b \\
(n-1)^2 - d & \binom{n-1}{2} & c & 3\binom{n}{2} + d - c \\
n(n-1) - b & n(n-1) - c & \binom{n}{2} & b + c - 3\binom{n}{2} \\
d + b - \binom{n-1}{2} & \binom{n-1}{2} - d + c & 3\binom{n}{2} - b - c & \binom{n-1}{2}
\end{pmatrix}.$$

Let us denote by y_k the vector of length k all of whose entries are equal to y. Observe that A is the 6×6 block matrix, whose blocks are shown in Fig. 1 by thin lines. Let us calculate $(A^2)_{x,\Delta_1}$. Lemma 1 gives

$$A^2 = p_{11}^1 A + p_{11}^2 A^t = (n-1)A + nA^t = (n-1)J + A^t.$$

Hence we have $(A^2)_{x,\Delta_1} = (n-1)_{n-1}$. On the other hand, $(A^2)_{x,\Delta_1} = 1_{n-1} + 1_{n-1}(A_{11} + A_{21})$. Therefore, $(n-2)_{n-1} = 1_{n-1}(A_{11} + A_{21})$, i.e., for any column j of the matrix $A_{11} + A_{21}$ we have $\sum_i (A_{11} + A_{21})_{ij} = n - 2$. This allows us to obtain

$$\sum_{i,j}(A_{11} + A_{21})_{ij} = \sum_{i,j}(A_{11})_{ij} + \sum_{i,j}(A_{21})_{ij} = e_{11} + e_{12} = (n-1)(n-2).$$

Hence $e_{12} = (n-1)^2 - d = \binom{n}{2}$, i.e., $d = \binom{n}{2}$.

Calculating $(A^2)_{y,\Delta_2}$ and $(A^2)_{y,\Delta_3}$, we similarly obtain c and b, respectively. The proof is complete.

4. Proof of Theorem 2

First, let us observe that if $n = 1$ then each of the graphs Γ_3 and Γ_4 are isomorphic to a disjoint union of complete subgraphs. Such SRGs are, of course, rank 3 graphs. Hence the 4-vertex condition holds in this case. In what follows we will prove that the 4-vertex condition does not hold in the case $n > 1$.

Let us start with a general observation. For an association scheme $A = (X, \{R_i\})$ let $E = (a, b) \in R_I$, where R_I is a relation of A. We denote by $M_A(E; i_1, \ldots, i_5)$ the number of distinct (unordered) pairs $\{c, d\} \subset V$ such that $(a, c) \in R_{i_1}$, $(a, d) \in R_{i_2}$, $(b, c) \in R_{i_3}$, $(b, d) \in R_{i_4}$, $(c, d) \in R_{i_5}$. For association schemes $A = (X, \{R_i\})$ and $B = (Y, \{Q_i\})$ let $C = A \otimes B = (X \times Y, \{F_{ij}\})$ be their tensor product. As above, for $E = ((a, p), (b, q)) \in F_{IJ}$, where F_{IJ} is a relation of C, we denote by $M_C(E; i_1j_1, \ldots, i_5j_5)$ the number of distinct (unordered) pairs $\{(c, s), (d, t)\} \subseteq X \times Y$ such that $((a, p), (c, s)) \in F_{i_1j_1}$, *etc.*, as is depicted on the following Fig. 2.

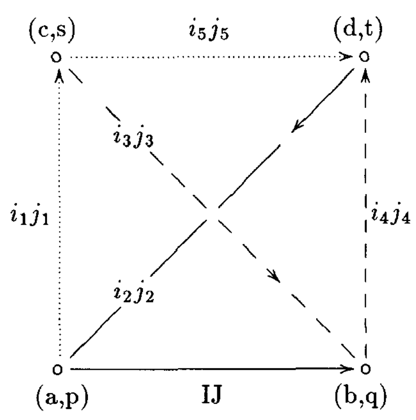

Figure 2.

Note that in this figure we can see source configurations $\mathcal{A}$ and $\mathcal{B}$ with the vertex sets $\{a, b, c, d\} \subseteq X$ and $\{p, q, s, t\} \subseteq Y$, respectively. In general, $\mathcal{A}$ and (or) $\mathcal{B}$ may have $\leqslant 3$ vertices, e.g., if $a = b$. The following lemma is a direct consequence of the definition of the tensor product.

LEMMA 5. *For* $E = ((a, p), (b, q)))$ *we have*

$$M_C(E; i_1j_1, \ldots, i_5j_5) = M_A((a, b); i_1, \ldots, i_5) \cdot M_B((p, q); j_1, \ldots, j_5).$$

By the latter lemma the calculation of the 4-vertex constants α and β for Γ_3 and Γ_4 can be reduced to corresponding calculations in the relations of $T(H)$ and $T(H')$. Much of the results of the latter calculations are contained in Theorem 3. The next lemma, in fact, completes this calculations. For $W = T(H) = (V, \{I, T, T^t\}) = (V, \{R_0, R_1, R_2\})$ let us denote

$$m_1 = M_W\big((x, x); 1, 1, 1, 1, 1\big), \qquad m_2 = M_W\big((x, x); 1, 2, 1, 2, 1\big),$$

$$m_3 = M_W\big((x, x); 2, 1, 2, 1, 1\big), \qquad m_4 = M_W\big((x, x); 2, 2, 2, 2, 1\big).$$

LEMMA 6. $m_1 = m_3 = m_4 = (n - 1)(2n - 1)$, $m_2 = n(2n - 1)$.

Proof. For example, we calculate m_1. Observe that m_1 denotes the number of the following configurations through x.

$$\begin{tikzcd} x \arrow[r] \arrow[d] & p \arrow[dl] \\ q & \end{tikzcd}$$

(Note that T is a tournament, $\quad p, q \in V$).

Fix $p \in V$. The number of such vertices q that $\{x, p, q\}$ forms a configuration we are looking for is $p^1_{12} = n - 1$ (cf. Lemma 1). The number of different appropriate vertices p is $p^0_{12} = 2n - 1$. Hence $m_1 = (n - 1)(2n - 1)$. Another m_i can be determined by the same way. □

We can now turn to the calculation of $\beta(x, y)$ for Γ_3 and Γ_4. Recall that $\widetilde{S}$ is the symmetrization of $S = (V \times V, \{S_{ij}\})$ (we use the same numbering of the relations of S as in Sect. 2). In these terms the graphs complementary to Γ_3 and Γ_4 can be described as the fusions $\bigcup S_{ij}$, where $0 \leqslant i, j \leqslant 2$, $(ij) \neq (00)$ and for Γ_3 we must take $i \neq j$, whereas for Γ_4 we must take $\{i, j\} \neq \{1, 2\}$.

Consider the graph Γ_3. Suppose first that $E = ((a, p), (b, p)) \in S_{10}$. It is easy to check that

$$\beta(E) = M_S(E; 12, 12, 12, 12, 12) + M_S(E; 12, 12, 21, 21, 12)$$
$$+ M_S(E; 21, 21, 12, 12, 12) + M_S(E; 21, 21, 21, 21, 12).$$

Using Lemmas 5, 6, and Theorem 3, we determine

$$\beta(E) = e_{11}m_4 + e_{14}m_2 + e_{41}m_3 + e_{44}m_1 = (n-1)(2n-1)(4n^2 - 9n + 6)/2.$$

Next, we fix $E' = ((a,p),(b,q)) \in S_{11}$. In this case we obtain $\beta(E') = (n-1)^2(4n^2 - 6n + 3)$. Since $\beta(E) \neq \beta(E')$ for $n > 1$, the 4-vertex condition for Γ_3 does not hold.

We consider the nonedges of Γ_4. Similarly, we find that $\beta(E) = (n-1)(2n-1)(4n^2 - 9n + 6)/2$ for $E = ((a,p),(b,p)) \in S_{10}$ and $\beta(E') = (n-1)^2(4n^2 - 6n + 3)$ for $E' = ((a,p),(b,q)) \in S_{12}$. Since $\beta(E) \neq \beta(E')$, this completes the proof.

Remark 2. Using this technique, we can obtain the parameter α for both Γ_3 and Γ_4. It is equal to $2(n-1)^2(2n^2 - 3n + 3)$.

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