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Efficient and fair system states in dynamic transportation networks

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Abstract

This paper sets out to model an efficient and fair transportation system accounting for both departure time choice and route choice of a general multi-OD network within a dynamic traffic assignment environment. Firstly, a bi-level optimization formulation is introduced based on the link-based traffic flow model. The upper level of the formulation minimizes the total system travel time, whereas the lower level captures traffic flow propagation and the user equilibrium constraints. Then the bi-level formulation is relaxed to a linear programming formulation that produces a lower bound of an efficient and fair system state. An efficient iterative algorithm is proposed to obtain the exact solution. It only requires solving one linear program in one iteration. Further, it is shown that the number of iterations is bounded, and the output traffic flow pattern is efficient and fair. Finally, two numerical cases (including a single OD network and a multi-OD network) are conducted to demonstrate the performance of the algorithm. The results consistently show that the departure rate pattern generated from the algorithm leads to an efficient and fair system state, and the algorithm converges within two iterations across all test scenarios.

Keywords: Linear programming; Efficient and fair; System optimal; User equilibrium; Route choice; Departure time choice

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1. Introduction

Advances in intelligent transportation systems (ITS) promise to enable advanced traveler information systems (ATIS) and the related travel guidance (e.g., route choice, departure time choice) systems for individual travelers. Particularly, with the recent development of information and communication technology, especially app-based technologies, the personalized guidance customized to the need of individual drivers is rapidly becoming a reality. Personalized traffic guidance system aims to provide travelers with real-time traffic information and reliable guidance on route selection and departure time choice. However, individual users may not comply with the guidance if the guidance is not efficient and fair. It is likely that the continued growth and acceptance of app-based navigation systems will hinge upon the capability of such devices to accurately predict future system states and consistently provide reliable guidance to travelers.

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As an increasing number of travelers start using traveler guidance systems, there is a pressing need for navigation devices to provide reliable (both efficient and fair) guidance to travelers. The personalized travel guidance systems are envisaged as a tool that will assist travelers to make efficient travel decisions by providing network state information both pre-trip and en-route. This paradigm of personalized guidance raises many fundamental research questions in terms of fairness and efficiency, well beyond the traditional traffic flow modeling and estimation. A key fundamental research question in a dynamic transportation network is: can we come up with the traffic guidance that leads to an efficient and fair system state which is close to system optimal (i.e., efficient) but does not make any individual user worse off (i.e., fair)? To answer this question, we need to solve a problem that yields a solution that is at user equilibrium but also at system optimum.

1.1. Related work

The study of this paper is in the context of dynamic traffic assignment (DTA). DTA plays an important role in transportation network modeling since it provides a clear representation of time-varying conditions in traffic networks. Based on the behavioral assumption of individual user's travel decision making, DTA problems can be classified into two categories: the dynamic system optimal problem and the dynamic user equilibrium problem. (1) For the dynamic system optimal (DSO) problem, Merchant and Nemhauser (1978a,b) modeled the traffic flow propagation with a nonlinear exit-flow function that is also non-convex. Carey (1987) reformulated the M-N model (Merchant and Nemhauser, 1978a,b) as a convex nonlinear program by relaxing the non-convex exit-flow function with a set of inequity constraints. Then Carey and Srinivasan (1993) proposed a marginal cost approach to solve the non-linear optimization problem of DSO. But the analysis is based on a point queue model. Various approaches (Ghali and Smith, 1995; Peeta and Mahmassani, 1995; Qian et al., 2012; Doan and Ukkusuri, 2015) based on the physical queue model have been developed to compute the path marginal cost. Shen and Zhang (2014) showed that the minimal system cost is the same and one common solution is shared among different traffic flow model-based DSO formulations. From a different perspective, (Friesz et al., 1989; Chow, 2009) formulated the DSO problem as a continuous time optimal control problem. Ma et al. (2014) extended the formulation to capture the queue spill-back effect for single-destination networks. (2) For the dynamic user equilibrium (DUE) problem, Wie et al. (1990) formulated the problem as an equivalent continuous time optimal control problem. Friesz et al. (1993) firstly proposed a variational inequality formulation considering both simultaneous route choice and departure time decisions. (Lo and Szeto, 2002; Szeto and Lo, 2004) incorporated the cell-based model into the DUE problem using the variational inequality approach. Waller and Ziliaskopoulos (2006) efficiently solved the dynamic user optimal problem embedding cell transmission model. (Han et al., 2011; Ukkusuri et al., 2012) formulated the cell-based DUE problem using complementarity theory. Besides the cell-based approach, the link-based DUE problem has also been well studied in the literature. Different formulations have been developed to solve the link-based DUE problem, including nonlinear complementarity (Wie et al., 2002; Ban et al., 2008; Di et al., 2013) and variational inequality (Long et al., 2013; Friesz et al., 2013; Han et al., 2015a; Gentile, 2016).

However, most of the above models are nonlinear and intractable. Solving a nonlinear DSO or DUE problems usually requires a specialized algorithm. From a different direction, to avoid the nonlinearity of the problem, a handful of researchers investigated in

developing DTA models based on linear programming (Ziliaskopoulos, 2000; Peeta and Ziliaskopoulos, 2001; Ukkusuri and Waller, 2008; Nie, 2011; Zhu and Ukkusuri, 2013). (1) For the DSO problem, it is relatively easier to apply LP compared with the DUE problem, as the construction of the objective function in DSO is more straightforward and carries clear physical meaning. The seminal work is done by Ziliaskopoulos (2000). In (Ziliaskopoulos, 2000), the nonlinear cell-based SODTA formulation is transformed into a linear programming problem, whereby the nonlinear constraints and the bi-level structure are relaxed with a set of linear inequalities. One limitation due to the linear relaxation is known as the holding back problem (Peeta and Ziliaskopoulos, 2001). Nie (2011) proposed a variant of the M-N model (Merchant and Nemhauser, 1978a,b) using a piecewise linear exit-flow function, and verified that Ho’s algorithm (Ho, 1980) can be applied to eliminate the unnecessary holding flows in the network. Most recently, Zhu and Ukkusuri (2013) resolved the holding-back problem by introducing penalty terms to the objective function of the linearized formulation. (2) For the DUE problem, it is usually formulated as complementarity problem or variational inequality (VI) problem (Friesz et al., 1993; Ukkusuri et al., 2012; Han et al., 2015b). If DUE is formulated as an optimization problem, it is hard to interpret the physical meaning of the objective function and it requires justification that the solution satisfies the equilibrium condition. In the literature, (Ukkusuri and Waller, 2008) seems to be the only paper that formulates DUE using LP. In the formulation, a Mt vector is associated with the arrival flows in the objective function to provide incentives to the traffic flow to arrive the destination as soon as possible. One limitation of the formulation lies in the difficulty of obtaining a proper Mt vector for a large network.

There is limited literature that formulates a problem that yields a system state that satisfies user equilibrium and is a global optimum. Most of the literature is within the context of static traffic assignment. In congestion games, Koutsoupias and Papadimitriou (1999) firstly brought out the problem of price of anarchy (POA) - the maximum ratio between the system outcome of a user equilibrium solution and the system optimum solution. Roughgarden and Tardos (2002) showed that POA is bounded by $4/3$ with a linear link performance function in static traffic assignment. A number of researchers have investigated the combined UE and SO problem through tolling intervention. Friesz et al. (2007) formulated the dynamic optimal toll problem with equilibrium constraints for day-to-day and within-day traffic assignment. Han et al. (2008a,b) proved that in a transportation system with linear and nonlinear asymmetric cost functions, the POA is lower with toll intervention than that without toll intervention. (Song et al., 2009; Lawphongpanich and Yin, 2010; Guo and Yang, 2010) studied the problem of finding Pareto-improving tolls to improve the social benefit while no user is worse off in terms of travel cost measured through tolling intervention. Ban and Liu (2009) further formulated the dynamic second-best toll pricing problem as a bi-level problem which is solved by adopting a relaxation scheme. (Zhong et al., 2012; Friesz et al., 2013; Ma et al., 2015) expanded the analysis to different dynamic externalities such as marginal cost, pollution charge, and emissions.

It is worthwhile to note that solving the combined problem of system optimum and user equilibrium is a challenging task even in the case of static traffic assignment. (Song et al., 2009; Lawphongpanich and Yin, 2010) formulated the Pareto-improving problem as a mathematical program with complementarity constraints (MPCC). (Correa et al., 2004, 2007) formulated the problem as minimizing the maximum latency of flows in networks with congestion. It is showed that in static traffic assignment, even in a single OD network

with a linear link cost function, the combined system optimum and user equilibrium problem is NP-hard. Jahn et al. (2005) firstly proposed a constrained system-optimum model that guarantees user fairness, and developed a constrained shortest path algorithm to solve the optimization problem. However, the model and the algorithm are limited to the case of static traffic assignment. It is foreseeable that the combined problem of system optimum and user equilibrium in the context of dynamic traffic assignment is even more challenging. This paper, to our best knowledge, is the first attempt in the literature to address the combined DSO and DUE problem using a linear programming approach.

1.2. Contributions of the paper

This paper sets out to model a transportation system that is both efficient (from the system’s perspective) and fair (from an individual user’s perspective) within a dynamic traffic assignment environment. The practical significance of this paper is to generate reliable (both efficient and fair) traffic guidance (e.g., dynamic route choices/departure time choices) to travelers. The research contributions of the paper are summarized as below.

- We propose a bi-level optimization formulation based on the link transmission model (Yperman, 2007; Zhu and Ukkusuri, 2015). At the upper level, the objective is to optimize the total system travel time of the traffic flow at the network level (efficient). At the lower level, the objective is to propagate traffic flows in the network and equilibrate the travel cost of individual users (fair). The modeling framework is capable of accounting for dynamic departure time and route choice at the network level.
- The bi-level optimization formulation is relaxed to a linear programming formulation where the nonlinear constraints are replaced by a set of linear inequality constraints. The linear programming (LP) relaxation produces a lower bound of an efficient and fair system state.
- Based on the lower bound of the LP relaxation, an efficient iterative algorithm is proposed to obtain the exact solution of the departure rate pattern that leads to an efficient and fair system state. Every iteration, the algorithm solves one linear program and produces a new and tighter lower bound on the efficient and fair system state. It is proved that the number of iterations of the algorithm is bounded and the travel time cost of different departure rates under the same OD pair are the same.
- Two numerical case studies including a single OD network (Ziliaskopoulos, 2000) and a multi-OD (Nguyen and Dupuis, 1984) network are conducted to demonstrate the performance of the algorithm. The results consistently confirm that the traffic flow pattern is efficient and fair. Moreover, it takes no more than two iterations for the algorithm to converge for all the test scenarios.

The rest of the paper is structured as below. Section 2 is devoted to the bi-level optimization formulation of an efficient and fair transportation network. Section 3 relaxes the bi-level formulation to a linear programming formulation and introduces an efficient algorithm to obtain the solution of an efficient and fair system state. Section 4 conducts numerical studies on two test networks including a single OD network and a multi-OD network to demonstrate the performance of the algorithm. Finally, Section 5 concludes the paper and discusses future research directions of this study.

Notations:

Sets

$\mathbf{Z}$	:	Set of all links
$\mathbf{Z}_{G,S,O,D,M}$	:	Set of origin, destination, ordinary, diverging, or merging links, respectively
$\mathbf{E}$	:	Set of all connectors
$\mathbf{E}_{O,D,M}$	:	Set of ordinary, diverging, or merging link connectors, respectively
$\Gamma^{-1}(i)$	:	Set of predecessors of link i
$\Gamma(i)$	:	Set of successors of link i
$\mathbf{U}$	:	Set of all OD pairs
$\mathbf{P}^{o,d}$	:	Set of all paths of OD (o,d)

Parameters

δ	:	Size of the time step
T	:	Entire time horizon
T_d	:	Time horizon of departure time choice, $T_d \leq T$
N_i	:	Jam density of link i
Q	:	Maximum out flow of an OD pair at any departure time
S_i^t	:	Inflow or outflow capacity of link i at time t
W_i	:	Shock wave speed of link i
V_i	:	Free flow speed of link i
$D^{o,d}$	:	Total demand of OD pair (o,d) during the departure period
L_i	:	Length of link i
ζ	:	A sufficiently small constant
ε	:	Tolerance of fairness

Variables

$x_i^{o,d,p}(\bar{t}, t)$	:	Occupancy (number of vehicles) of link i at time t on path p departed at time $\bar{t}$ of OD (o,d)
$y_{i,j}^{o,d,p}(\bar{t}, t)$	:	Flow from link i to j at time t on path p departed at time $\bar{t}$ of OD (o,d)
$x_i(t)$	:	Aggregate occupancy of link i at time t
$y_{i,j}(t)$	:	Aggregate flow from link i to j at time t
$r^{o,d}(p, \bar{t})$	:	The departure rate at departure time $\bar{t}$ of path p of OD (o,d), $\bar{t} \in [1, T_d]$. Note that $r^{o,d,p}(\bar{t}) = \sum_{\forall t} y_{o,j}^{o,d,p}(\bar{t}, t)$, $o \in \mathbf{Z}_G, j \in \Gamma(o)$
$c^{o,d}(p, \bar{t})$	:	The travel cost of departure rate $r^{o,d,p}(\bar{t})$ at departure time $\bar{t}$ of path p of OD (o,d), $\bar{t} \in [1, T_d]$
$l^{o,d}(p, \bar{t})$	:	The average travel time of the departure rate at departure time $\bar{t}$ of path p of OD (o,d), $\bar{t} \in [1, T_d]$
$l_{avg}^{o,d}(k)$	:	The average travel time of all departure rates of OD (o,d) at the k iteration

2. Formulation of an efficient and fair transportation network

2.1. Preliminary settings

Assumption 1. The departure rate of all paths at any departure time under the same OD pair is non-negative and no greater than Q , i.e.,

$$0 \leq \sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) \leq Q, \forall (o, d) \in \mathbf{U}, \bar{t} \in [0, T_d] \quad (1)$$

and the total demand $D^{o,d}$ is a positive integer multiple of Q , i.e., $\frac{D^{o,d}}{Q} \in \mathbb{Z}_+$.

Assumption 2. The outflow capacity of any link at any time is a nonnegative integer multiple of Q , i.e., $\forall i \in \mathbf{Z}, \forall t \in [0, T], \frac{S_i^t}{Q} \in \{0, \mathbb{Z}_+\}$.

Remark 1.

a) In Assumption 1, we consider that there is a limit to the total departure rate of an OD pair, denoted by Q . In practice, Q can be translated as the saturation flow rate. Assumption 1 and 2 shall be the base of the proof of Lemma 1 in the later section.

b) In Assumption 2, the outflow capacity of a link is allowed to be zero in order to capture the traffic realism of capacity drops in the real world. For example, if the end of the link is a signalized intersection, the capacity becomes zero when the traffic light is red. For another example, the outflow capacity of a link may become zero for a while when an accident occurs.

Definition 1. *Efficient and fair system state.* An efficient and fair system state is defined as the system state that: (1) the total system travel cost (total system travel time in this paper) is minimum under the condition that, (2) traffic flows are at User Equilibrium within a dynamic traffic assignment environment. The traffic information guidance, or a traffic departure rate pattern, or a traffic flow pattern, is said to be efficient and fair if it leads to an efficient and fair system state.

2.2. A bi-level optimization formulation

In the field of dynamic congestion pricing, a bi-level optimization framework is commonly realized in the literature whereby the system benefit is maximized in the upper level while the individual cost is minimized and equilibrated in the lower level (Friesz et al., 2007; Ban and Liu, 2009; Friesz et al., 2013). Similarly, a bi-level optimization framework is adopted to model the efficient and fair system states in a transportation network.

In this study, we apply the link-based traffic flow model (Yperman, 2007; Zhu and Ukkusuri, 2015) to propagate traffic flows in the network. The link-based traffic flow model is a discrete-time model where the time space is discretized into time steps. Other types of traffic flow model are also readily applicable, as long as the traffic flow model can be formulated in a linear form (e.g., Ziliaskopoulos (2000); Zhu and Ukkusuri (2013, 2015)). However, it is worth to note that the methodology of this paper may not apply to the continuous-time formulation. As from (Friesz et al., 1989; Chow, 2009; Ma et al., 2014; Jin, 2015), the continuous-time DTA formulation is different from discrete-time formulation.

In the discrete time link-based traffic flow model, the number of vehicles within the link is defined as the occupancy of the link, denoted as $x_i(t), i \in \mathbf{Z}$. The number of vehicle

moving from one link to another is defined as the traffic flow of the link connector, denoted as $y_{i,j}(t)$, $(i, j) \in \mathbf{E}$. The traffic flow propagation is realized by updating temporal values of $x_i(t)$, $i \in \mathbf{Z}$ and $y_{i,j}(t)$, $(i, j) \in \mathbf{E}$ in the network.

The bi-level optimization formulation of an efficient and fair network (BOEFN):

$$\min z_1 = \sum_{t=1}^T \sum_{i \in \mathbf{Z} \setminus \{\mathbf{Z}_G, \mathbf{Z}_S\}} x_i(t) \quad (2)$$

Subject to:

Demand satisfaction constraint:

$$\sum_{\bar{t}=1}^{T_d} \sum_{p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) = D^{o,d} \quad (3)$$

Aggregation of link occupancy and outflow constraints:

$$x_i(t) = \sum_{(o,d) \in \mathbf{U}} \sum_{p \in \mathbf{P}^{o,d}} \sum_{\bar{t}=1}^{T_d} x_i^{o,d,p}(\bar{t}, t), \forall i \in \mathbf{Z} \quad (4)$$

$$y_{i,j}(t) = \sum_{(o,d) \in \mathbf{U}} \sum_{p \in \mathbf{P}^{o,d}} \sum_{\bar{t}=1}^{T_d} y_{i,j}^{o,d,p}(\bar{t}, t), \forall (i, j) \in \mathbf{E} \quad (5)$$

Link occupancy constraints:

Source links, $\forall i \in \mathbf{Z}_G, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$:

$$x_i^{o,d,p}(\bar{t}, t) = x_i^{o,d,p}(\bar{t}, t-1) - \sum_{j \in \Gamma(i)} y_{i,j}^{o,d,p}(\bar{t}, t-1) \quad (6)$$

Sink links, $\forall i \in \mathbf{Z}_S, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$:

$$x_i^{o,d,p}(\bar{t}, t) = x_i^{o,d,p}(\bar{t}, t-1) + \sum_{k \in \Gamma^{-1}(i)} y_{k,i}^{o,d,p}(\bar{t}, t-1) \quad (7)$$

Ordinary links, $\forall i \in \mathbf{Z}_O, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$:

$$x_i^{o,d,p}(\bar{t}, t) = x_i^{o,d,p}(\bar{t}, t-1) + y_{k,i}^{o,d,p}(\bar{t}, t-1) - y_{i,j}^{o,d,p}(\bar{t}, t-1), k \in \Gamma^{-1}(i), j \in \Gamma(i) \quad (8)$$

Merging links, $\forall i \in \mathbf{Z}_M, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$:

$$x_i^{o,d,p}(\bar{t}, t) = x_i^{o,d,p}(\bar{t}, t-1) + \sum_{k \in \Gamma^{-1}(i)} y_{k,i}^{o,d,p}(\bar{t}, t-1) - y_{i,j}^{o,d,p}(\bar{t}, t-1), j \in \Gamma(i) \quad (9)$$

Diverging links, $\forall i \in \mathbf{Z}_D, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$:

$$x_i^{o,d,p}(\bar{t}, t) = x_i^{o,d,p}(\bar{t}, t-1) + y_{k,i}^{o,d,p}(\bar{t}, t-1) - \sum_{j \in \Gamma(i)} y_{i,j}^{o,d,p}(\bar{t}, t-1), k \in \Gamma^{-1}(i) \quad (10)$$

Link connector constraints:

Ordinary link connectors, $\forall (i, j) \in \mathbf{E}_O, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]$

$$\max y_{i,j}^{o,d,p}(\bar{t}, t) \quad (11)$$

$$\begin{cases} y_{i,j}^{o,d,p}(\bar{t}, t) & \leq x_i^{o,d,p}(\bar{t}, t + 1 - \frac{L_i}{V_i}) - \sum_{k=t+1-\frac{L_i}{V_i}}^{t-1} y_{i,j}^{o,d,p}(\bar{t}, k) \\ y_{i,j}(t) & \leq S_i^t \\ y_{i,j}(t) & \leq S_j^t \\ y_{i,j}(t) & \leq N_j - x_j(t) - \sum_{k=t+1-\frac{L_i}{W_i}}^{t-1} y_{j,l}(k) \end{cases} \quad (12)$$

Diverging link connectors, $\forall i \in \mathbf{Z}_D, p \in \mathbf{P}^{o,d}$

$$\max y_{i,j}^{o,d,p}(\bar{t}, t) \quad (13)$$

$$\begin{cases} \sum_{j \in \Gamma(i)} y_{i,j}^{o,d,p}(\bar{t}, t) & \leq x_i^{o,d,p}(\bar{t}, t + 1 - \frac{L_i}{V_i}) - \sum_{k=t+1-\frac{L_i}{V_i}}^{t-1} y_{i,j}^{o,d,p}(\bar{t}, k) \\ \sum_{j \in \Gamma(i)} y_{i,j}(t) & \leq S_i^t \\ y_{i,j}(t) & \leq S_j^t, j \in \Gamma(i) \\ y_{i,j}(t) & \leq N_j - x_j(t) - \sum_{k=t+1-\frac{L_i}{W_i}}^{t-1} y_{j,l}(k), j \in \Gamma(i) \end{cases} \quad (14)$$

Merging link connectors, $\forall j \in \mathbf{Z}_M, p \in \mathbf{P}^{o,d}$

$$\max y_{i,j}^{o,d,p}(\bar{t}, t) \quad (15)$$

$$\begin{cases} y_{i,j}^{o,d,p}(\bar{t}, t) & \leq x_i^{o,d,p}(\bar{t}, t + 1 - \frac{L_i}{V_i}) - \sum_{k=t+1-\frac{L_i}{V_i}}^{t-1} y_{i,j}^{o,d,p}(\bar{t}, k), i \in \Gamma^{-1}(j) \\ \sum_{i \in \Gamma^{-1}(j)} y_{i,j}(t) & \leq S_j^t \\ y_{i,j}(t) & \leq S_i^t, i \in \Gamma^{-1}(j) \\ \sum_{i \in \Gamma^{-1}(j)} y_{i,j}(t) & \leq N_j - x_j(t) - \sum_{k=t+1-\frac{L_i}{W_i}}^{t-1} y_{j,l}(k), l \in \Gamma(j) \end{cases} \quad (16)$$

User equilibrium constraints:

To compute the path travel time, if $r^{o,d}(p, \bar{t}) > 0, p \in \mathbf{P}^{o,d}, o \in \mathbf{Z}_G, d \in \mathbf{Z}_S, \bar{t} \in [1, T_d]$, then:

$$l^{o,d}(p, \bar{t}) = \frac{\sum_{t=1}^T \sum_{i \in \Gamma^{-1}(d)} t \cdot y_{i,d}^{o,d,p}(\bar{t}, t) - \sum_{t=1}^{T_d} \sum_{j \in \Gamma(o)} t \cdot y_{o,j}^{o,d,p}(\bar{t}, t)}{r^{o,d}(p, \bar{t})} \quad (17)$$

User equilibrium condition, $\forall p \in \mathbf{P}^{o,d}, \bar{t} \in [1, T_d]$:

$$0 \leq r^{o,d}(p, \bar{t}) \perp l^{o,d}(p, \bar{t}) - l_*^{o,d} \geq 0 \quad (18)$$

Nonnegative constraints: $y_{i,j}^{o,d,p}(\bar{t}, t) \geq 0, i \in \mathbf{Z}, j \in \Gamma(i), (o, d) \in \mathbf{U}, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d], t \in [0, T]$.

□

Remark 2. The traffic flow updating procedure of link connectors, i.e., (12) - (16) is consistent with the flow update procedure of the LTM model (Yperman, 2007).

Proof. A similar proof can be found at Zhu and Ukkusuri (2015). □

Remark 3. The BOEFN formulation belongs to the class of mathematical programming with equilibrium constraints (MPEC). Especially, the feasible set of the formulation is nonlinear and non-convex due to the bi-level optimization structure and the complementarity constraints of (18). Constraint (18) is mathematically equal to: $r^{o,d}(p, \bar{t}) \geq 0$, $l^{o,d}(p, \bar{t}) - l_*^{o,d} \geq 0$, and $r^{o,d}(p, \bar{t})(l^{o,d}(p, \bar{t}) - l_*^{o,d}) = 0$, where $l_*^{o,d}$ represents the equilibrium travel time of OD (o,d). It essentially indicates that if $r^{o,d}(p, \bar{t}) > 0$, $l^{o,d}(p, \bar{t}) = l_*^{o,d}$; if $l^{o,d}(p, \bar{t}) > l_*^{o,d}$, $r^{o,d}(p, \bar{t}) = 0$. In other words, constraint (18) captures the user equilibrium condition for users of different paths and departure time choices under the same OD: if the departure rate of path p and time $\bar{t}$ is positive, the associated travel time is equal to the equilibrium cost; otherwise, if the travel time of path p departed at time $\bar{t}$ is greater than the equilibrium cost, the associated departure rate is zero.

Remark 4. Equation (17) computes the average travel time of the departure rate of path p departed at time $\bar{t}$. It is computed by dividing the difference between the accumulative arrival travel time and the accumulative departure travel time over the departure rate. In (17), $\sum_{t=1}^{T_d} \sum_{j \in \Gamma(o)} t \cdot y_{o,j}^{o,d,p}(\bar{t}, t)$ represents the accumulative time of the traffic departs at origin o , and $\sum_{t=1}^T \sum_{i \in \Gamma^{-1}(s)} t \cdot y_{i,s}^{o,d,p}(\bar{t}, t)$ represents the accumulative time this traffic arrives at destination d . Similar approach to compute path travel time can also be seen in (Szeto and Lo, 2004; Ukkusuri et al., 2012). Figure 1 presents an example to demonstrate the idea. In this example, supposing 2 units of traffic flow depart at time 1 from O to D. One unit arrives at time 7, and the other unit arrives at time 9. Thus the average travel time is $(7 + 9 - 2)/2 = 7$.

Figure 1: A demonstrative example of average path travel time calculation

3. On computing an efficient and fair system state

3.1. A linear programming relaxation

Similar to the relaxation scheme in (Ziliaskopoulos, 2000; Zhu and Ukkusuri, 2015), we relax the nonlinear constraints of the bi-level optimization program with a set of linear inequities. The linear relaxation formulation is presented as below. It is worth to note

that the linear relaxation does not produce the solution of an efficient and fair network, but provides a lower bound of the solution. We will present the algorithm of computing the traffic flow pattern for an efficient and fair system state later in Section 3.

Firstly, to account for the unused routes, we presume that each route will contain at least a sufficiently small packet of traffic flow, ζ . It is captured through the following constraint:

$$r^{o,d}(p, t) \geq \zeta, \forall p \in \mathbf{P}^{o,d}, t \in [0, T_d] \quad (19)$$

where ζ is a sufficiently small constant.

In one OD pair, the total number of available routes is: $T_d|\mathbf{P}^{o,d}|$, where $|\cdot|$ denotes the number of elements in the set. From Lemma 1, the number of used routes is $\frac{D^{o,d}}{Q}$, hence the number of unused routes is $T_d|\mathbf{P}^{o,d}| - \frac{D^{o,d}}{Q}$. Relatively, the demand satisfaction constraint (3) is changed to:

$$\sum_{\bar{t}=1}^{T_d} \sum_{p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) = D^{o,d} + \left(T_d|\mathbf{P}^{o,d}| - \frac{D^{o,d}}{Q} \right) \zeta \quad (20)$$

Next, the linear programming relaxation of an efficient and fair network is presented as below:

$$\min z_2 = \sum_{\forall(o,d)} \sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t}) \quad (21)$$

Subject to:

Demand satisfaction constraint: (19), (20).

Aggregation of link occupancy and outflow: (4), (5)

Link occupancy constraints: (6), (7), (8), (9), (10)

Link connector constraints: (12), (14), (16)

Constraints from Assumption 1: (1)

User equilibrium constraints, $\forall p \in \mathbf{P}^{o,d}, t \in [1, T_d]$:

$$c^{o,d}(p, \bar{t}) = \sum_{t=1}^T \sum_{i \in \Gamma^{-1}(d)} t \cdot y_{i,d}^{o,d,p}(\bar{t}, t) - \sum_{t=1}^{T_d} \sum_{j \in \Gamma(o)} t \cdot y_{o,j}^{o,d,p}(\bar{t}, t) \quad (22)$$

$$\sum_{\forall p \in \mathbf{P}^{o,d}} \frac{c^{o,d}(p, \bar{t})}{Q} D^{o,d} \leq \sum_{\forall p \in \mathbf{P}^{o,d}} \sum_{\bar{t}=1}^{T_d} c^{o,d}(p, \bar{t}) \quad (23)$$

$$\frac{c^{o,d}(p, \bar{t})}{\zeta} D^{o,d} \geq \sum_{\forall p \in \mathbf{P}^{o,d}} \sum_{\bar{t}=1}^{T_d} c^{o,d}(p, \bar{t}) \quad (24)$$

Nonnegative constraints: $y_{i,j}^{o,d,p}(\bar{t}, t) \geq 0, i \in \mathbf{Z}, j \in \Gamma(i), (o, d) \in \mathbf{U}, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d], t \in [0, T]$.

□

Remark 5. It is easy to show that the LP relaxation is feasible. One feasible solution can be constructed as follows. Initiate the variable $r^{o,d}(p, \bar{t})$ and $y_{i,j}^{o,d,p}(\bar{t}, t)$ such that Eq. (1), (19), and (20) are satisfied and $c^{o,d}(p, \bar{t}) = 0$ (implying Eq. (22)-(24) are satisfied). And let all the other variables be zeros, then Eq. (4)-(5), (6)-(10), (12), (14), and (16) will be satisfied (as they are a set of linear inequalities with variables being zeros).

Lemma 1. In the LP relaxation, The accumulative departure rate of all paths $p \in \mathbf{P}^{o,d}$ at any departure time $\bar{t}$ is either negligible (extremely small, approaching zero as a limit) or Q , i.e., $\sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) \in \{0, Q\}, \bar{t} \in [0, T_d]$.

Proof. From Assumption 1, $0 \leq \sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) \leq Q$. From Assumption 2, the bottleneck of the network is either zero or greater than Q , and the objective is to minimize the total system travel time. Thus, if there is used route(s) within departure time $\bar{t}$, $\sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) = Q$. If there is no used route within time $\bar{t}$, $\sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) = |\mathbf{P}^{o,d}| \zeta$. Note that we can easily choose a sufficiently small ζ such that $|\mathbf{P}^{o,d}| \zeta$ is negligible. In all, $\sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) \in \{0, Q\}, \bar{t} \in [0, T_d]$. $\square$

Lemma 2. Let $c_*^{o,d}(p, \bar{t})$ denote the solution of the LP relaxation. If $\sum_{\forall p \in \mathbf{P}^{o,d}} l_*^{o,d}(p, \bar{t}) > |\mathbf{P}^{o,d}| \zeta$, then $\sum_{\forall p \in \mathbf{P}^{o,d}} c_*^{o,d}(p, \bar{t}) = c_*, \forall \bar{t} \in [0, T_d]$.

Proof. Let $\bar{c}_*^{o,d}(p, \bar{t})$ denote the $c_*^{o,d}(p, \bar{t})$ that is positive. Then, from (23), we have:

$$\begin{aligned} \sum_{\forall p \in \mathbf{P}^{o,d}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{Q} &\leq \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{D^{o,d}} + O(\zeta) \\ &\leq \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{D^{o,d}} \end{aligned} \quad (25)$$

where $O(\zeta)$ denotes the travel cost of the presumed sufficiently small traffic ζ on the unused routes. We can always fix ζ with a sufficiently small value such that $O(\zeta)$ is negligible.

Moreover, define $\gamma^{o,d}(\bar{t}) \geq 0$, such that (25) can be rewritten as:

$$\sum_{\forall p \in \mathbf{P}^{o,d}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{Q} = \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{D^{o,d}} - \gamma^{o,d}(\bar{t}) \quad (26)$$

where $0 \leq \gamma^{o,d}(\bar{t}) \leq \sum_{\forall p \in \mathbf{P}^{o,d}} \frac{c_*^{o,d}(p, \bar{t})}{Q}, \bar{t} \in [0, T_d]$.

Sum up both sides of (26) over $\bar{t}$, we have:

$$\sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{Q} = \frac{D^{o,d}}{Q} \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{D^{o,d}} - \sum_{\forall \bar{t}} \gamma^{o,d}(\bar{t}) \quad (27)$$

$$\Rightarrow \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{Q} = \sum_{\forall p} \sum_{\forall \bar{t}} \frac{\bar{c}_*^{o,d}(p, \bar{t})}{Q} - \sum_{\forall \bar{t}} \gamma^{o,d}(\bar{t}) \quad (28)$$

$$\Rightarrow \sum_{\forall \bar{t}} \gamma^{o,d}(\bar{t}) = 0 \quad (29)$$

Since $\gamma^{o,d}(\bar{t}) \geq 0$, we get $\gamma^{o,d}(\bar{t}) = 0$. Thus:

$$\sum_{\forall p \in \mathbf{P}^{o,d}} \bar{c}_*^{o,d}(p, \bar{t}) = \frac{Q}{D^{o,d}} \sum_{\forall p} \sum_{\forall \bar{t}} \bar{c}_*^{o,d}(p, \bar{t}) \quad (30)$$

$\square$

Remark 6. The proof of Lemma 2 is triggered by a simple example as below. Suppose there are three positive variables a, b , and c , with the constraint: $0 < a, b, c \leq d$, where $d = \frac{a+b+c}{3}$. The interest here is to prove that $a = b = c = d$. Firstly, let $a = d - \gamma_1$, $b = d - \gamma_2$, and $c = d - \gamma_3$, where $\gamma_{1,2,3} \geq 0$. Then $a + b + c = 3d - \sum_{i=1,2,3} \gamma_i$, which can be simplified to: $\sum_{i=1,2,3} \gamma_i = 0$. Thus $\gamma_{1,2,3} = 0$, and we have $a = b = c = d$. The proof of Lemma 2 is simply a generalized version of the proof of this example.

3.2. An algorithm to obtain an efficient and fair system state

Firstly, we present an important extra constraint as below:

$$c^{o,d}(p, \bar{t}) \geq r^{o,d}(p, \bar{t}) LB^{o,d}(k) \quad (31)$$

where $LB^{o,d}(k)$ is an exogenous constant, denoting the lower bound of the travel time at equilibrium for OD (o,d) at the k iteration.

Then we add this constraint to the LP relaxation. For the sake of brevity, we denote the domain of the LP relaxation plus constraint (31) as: $\mathbf{AY} \leq \mathbf{b}, \mathbf{Y} \geq \mathbf{0}$, where $\mathbf{Y}$ represents the set of variables of the LP relaxation, and $\mathbf{AY} \leq \mathbf{b}, \mathbf{Y} \geq \mathbf{0}$ represents all the linear constraints of the LP relaxation plus constraint (31). Thus, the updated LP formulation (denoted as LP2) can be written as:

$$\begin{aligned} \min z_2 &= \sum_{\forall(o,d)} \sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t}) & (32) \\ \text{S.t. :} & \quad \mathbf{AY} \leq \mathbf{b}, \mathbf{Y} \geq \mathbf{0} & (33) \end{aligned}$$

Next, it is vital to obtain an estimate of $LB^{o,d}(k)$. At the initial stage, let $k = 0$, $LB^{o,d}(k) = 0$. Then the LP2 formulation reduces to the LP relaxation. After solving the LP relaxation, we compute the average travel time of OD pair (o,d) as $l_{avg}^{o,d}(k) = \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}}$. Then update $LB^{o,d}(k+1) = \lceil l_{avg}^{o,d}(k) \rceil$, where $\lceil l_{avg}^{o,d}(k) \rceil$ denotes the nearest integer greater than or equal to $l_{avg}^{o,d}(k)$. Summary of the algorithm is presented in Algorithm 1.

Algorithm 1 Towards an efficient and fair system state

```

1: Initialize:  $k = 0, LB^{o,d}(k) = 0, check = 1$ 
2: while  $check > 0$  do
3:    $k = k + 1$ ;
4:   Solve LP2 to obtain  $l_{avg}^{o,d}(k) = \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}}$ ;
5:   if  $LB^{o,d}(k) < \lceil l_{avg}^{o,d}(k) \rceil$  then
6:     Update:  $LB^{o,d}(k+1) = \lceil l_{avg}^{o,d}(k) \rceil$ ;
7:   else
8:      $check = 0$ ;
9:   end if
10: end while

```

Proposition 1. At each iteration k , $LB^{o,d}(k)$ is a lower bound of the travel time of an efficient and fair system state for OD (o,d).

Proof. We prove this proposition by the principle of mathematical induction.

In the initial stage, $k = 1$, and the LP2 formulation reduces to the LP relaxation. It is obvious that the LP relaxation is a lower bound of the efficient and fair system state.

Assuming that $LB^{o,d}(k), k \geq 1$ is an lower bound of an efficient and fair system state. In the $(k+1)$ iteration, the LP relaxation is solved with the lower bound of $LB^{o,d}(k)$ and outputs a new lower bound $l_{avg}^{o,d}(k+1)$. $LB^{o,d}(k+1)$ takes the value of the nearest integer great than or equals to $l_{avg}^{o,d}(k+1)$, because the time space is discretized to time steps and the travel time is an integer multiple of the time steps. Thus, $LB^{o,d}(k+1)$ is also a lower bound. $\square$

Proposition 2. The number of iterations of Algorithm 1 is upper bounded by $\max\{l_*^{o,d}(p, \bar{t}) : \forall (o, d) \in \mathbf{U}, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]\}$, where $l_*^{o,d}(p, \bar{t})$ denotes the solution of an efficient and fair system state.

Proof. Firstly, it is easy to see that $\exists (o, d) \in \mathbf{U}$, such that $LB^{o,d}(k+1) \geq LB^{o,d}(k) + 1$. Otherwise, $LB^{o,d}(k) = \lceil l_{avg}^{o,d}(k) \rceil$, the iteration stops.

Thus, in the worst scenario, when $LB^{o,d}(k)$ reaches $\max\{l_*^{o,d}(p, \bar{t}) : \forall (o, d) \in \mathbf{U}, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]\}$, the algorithm ends. $\square$

Remark 7. Note that after the first iteration, $LB^{o,d}(1) = \lceil l_{avg}^{o,d}(1) \rceil$. Thus the remaining number of iterations equals to $\max\{l_*^{o,d}(p, \bar{t}), \forall (o, d) \in \mathbf{U}, p \in \mathbf{P}^{o,d}, \bar{t} \in [0, T_d]\} - \lceil l_{avg}^{o,d}(1) \rceil$. It is worthwhile to note that $\lceil l_{avg}^{o,d}(1) \rceil$ provides a tight lower bound on the solution, because of which this algorithm converges very fast in practice. Of all the tests in the numerical case studies of Section 4, the number of iterations is less than or equal to two.

Lemma 3. Algorithm 1 stops at $l_{avg}^{o,d}(k) = \lceil l_{avg}^{o,d}(k) \rceil$.

Proof. As shown in line 5, Algorithm 1 ends when $LB^{o,d}(k) = \lceil l_{avg}^{o,d}(k) \rceil$. Firstly, from (31), we have:

$$c^{o,d}(p, \bar{t}) \geq r^{o,d}(p, \bar{t}) \lceil \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}} \rceil \quad (34)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) \geq \sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t}) \lceil \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}} \rceil \quad (35)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) \geq Q \lceil \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}} \rceil \quad (\text{From Lemma 1}) \quad (36)$$

From (30) of Lemma 2, we have:

$$\frac{Q}{D^{o,d}} \sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t}) \geq Q \lceil \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}} \rceil \quad (37)$$

$$\Rightarrow \frac{Q}{D^{o,d}} \sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t}) = Q \lceil \frac{\sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t})}{D^{o,d}} \rceil \quad (38)$$

$$\Rightarrow l_{avg}^{o,d}(k) = \lceil l_{avg}^{o,d}(k) \rceil \quad (39)$$

$\square$

Theorem 1. Any sequence of the departure rate pattern generated by Algorithm 1 converges to the solution of an efficient and fair system state.

Proof.

(1) *Fair.*

From Lemma 3, when Algorithm 1 ends, $LB^{o,d}(k) = l_{avg}^{o,d}(k)$. From (31), we have:

$$l^{o,d}(p, \bar{t}) = \frac{c^{o,d}(p, \bar{t})}{r^{o,d}(p, \bar{t})} \geq l_{avg}^{o,d}(k) \quad (40)$$

Note that fairness is corresponding to user equilibrium (UE). To show that traffic flows are at UE, basically we need to prove that the travel cost of used routes are the same, and are equal to or less than the travel cost of unused routes. Firstly, consider the case of used routes, i.e., $r^{o,d}(p, \bar{t}) > \zeta$. Introduce a new set of variable, $\mu^{o,d}(p, \bar{t}) \geq 0$, such that:

$$\frac{c^{o,d}(p, \bar{t})}{r^{o,d}(p, \bar{t})} = l_{avg}^{o,d}(k) + \mu^{o,d}(p, \bar{t}) \quad (41)$$

$$\Rightarrow c^{o,d}(p, \bar{t}) = r^{o,d}(p, \bar{t})l_{avg}^{o,d}(k) + r^{o,d}(p, \bar{t})\mu^{o,d}(p, \bar{t}) \quad (42)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) = \sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t})l_{avg}^{o,d}(k) + \sum_{\forall p \in \mathbf{P}^{o,d}} r^{o,d}(p, \bar{t})\mu^{o,d}(p, \bar{t}) \quad (43)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) = Ql_{avg}^{o,d}(k) + Q\mu^{o,d}(p, \bar{t}) \quad (\text{Lemma 1}) \quad (44)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) = Q \frac{1}{D^{o,d}} \sum_{\forall p} \sum_{\forall \bar{t}} c^{o,d}(p, \bar{t}) + Q\mu^{o,d}(p, \bar{t}) \quad (\text{Lemma 2}) \quad (45)$$

$$\Rightarrow \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) = \sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) + Q\mu^{o,d}(p, \bar{t}) \quad (46)$$

$$\Rightarrow \mu^{o,d}(p, \bar{t}) = 0 \quad (47)$$

Thus, we have: $l^{o,d}(p, \bar{t}) = l_{avg}^{o,d}(k)$, where $r^{o,d}(p, \bar{t}) > \zeta$.

For unused routes, from (40), we have $l^{o,d}(p, \bar{t}) \geq l_{avg}^{o,d}(k)$, where $r^{o,d}(p, \bar{t}) = \zeta$.

(2) *Efficient.*

From Proposition 1, at every iteration, Algorithm 1 produces the lower bound of an efficient and fair system state. Thus the output of Algorithm 1 is the most efficient system state under the condition that the solution is fair. $\square$

4. Numerical studies

In this section, we construct two types of test networks: a single OD network (Ziliaskopoulos, 2000) and a multi-OD network (Nguyen and Dupuis, 1984) to demonstrate the performance of the proposed framework in modeling an efficient and fair transportation system for a general network. For the purpose of comparison, we compare the departure rate results of an efficient and fair system state with the results of the system optimal (SO) state. To obtain the solution of the SO state, we directly solve the LP relaxation formulation without the user equilibrium constraints (22) - (24).

4.1. Test network 1: (Ziliaskopoulos, 2000)'s network

The network representation of (Ziliaskopoulos, 2000)'s network is shown in Figure 2. The settings of this test case are as follows. The size of time interval is 2 s, and the saturation rate is 1800 veh/h. Thus the outflow capacity is 1 vehicle per time step. The jam density for link 1 and 10 is of a large value 100, while for other links, the holding capacity is 10 units. The outflow capacity of link 4 and 5 is time-variant to capture the

effect of signal (or accident): 0 for time steps (5-10), 1 for all other time steps. The ratio of free flow speed and shock wave speed is fixed at 1. The initial state for the network is empty. The total demand is 25 for a departure time period of 30 time steps. The presumed traffic (i.e., ζ) for unused routes is set at 0.0001. Details of the parameter setting are provided in Table 1.

Figure 2: Test network 1 (Ziliaskopoulos, 2000)

Table 2 presents the value of variables $r^{o,d}(p, \bar{t})$ (i.e., departure rate of path p time $\bar{t}$) and $l^{o,d}(p, \bar{t})$ (i.e., the average travel time cost corresponding to departure rate $r^{o,d}(p, \bar{t})$) of an efficient and fair system state. The departure rate and travel time of used routes are highlighted in dark. From Table 2, the values of $r^{o,d}(p, \bar{t})$ are all equal to either 0 or 1, indicating that the departure rate is either 0 or 1 (note that in the parameter setting, $\zeta = 0.0001$, i.e., the departure rate of unused routes is set at 0.0001. We denote it as 0 for the sake of presentation). This observation confirms Lemma 1 which states that the departure rate of all paths is either zero or the maximum outflow (one in this test case). Further, the average travel times corresponding to positive departure rates are all equal to 11, indicating that the travel time cost is the same for different positive departure rates. On the other hand, the average travel times corresponding to non-positive departure rates are all equal to or greater than 11. This observation confirms that the solution is at equilibrium (i.e., the complementarity constraint (18) is satisfied).

Similarly, Table 3 presents the value of variables $r^{o,d}(p, \bar{t})$ and $l^{o,d}(p, \bar{t})$ in the system optimal (SO) state. It is obvious that the values of $l^{o,d}(p, \bar{t})$ of different departure time are not the same, indicating that the traffic flows of different departure rates are not at equilibrium (i.e., the complementarity constraint (18) is not satisfied). Further, the total system travel time (TSTT) is 248, less than that (275) of an efficient and fair system state.

It is worthwhile to point out that Algorithm 1 converges fast. It takes only one iteration for all the tests in this test network. Even in test network 2 of the next section, it takes no more than two iterations for all the tests. The fast convergence of Algorithm 1 mainly thanks to the tight lower bound from the LP relaxation formulation.

Table 1: Parameter settings of test network 1

Links	1	2	3	4	5	6	7	8	9	10
L_i	2	2	2	4	2	2	2	2	2	2
N_i	100	10	10	10	10	10	10	10	10	100
Q_i	1	1	1	-	-	1	1	1	1	1

Table 2: Departure rate (i.e., $r^{o,d}(p, \bar{t})$) and average travel time (i.e., $l^{o,d}(p, \bar{t})$) of an efficient and fair system state (TSTT: 275) in test network 1

Departure time	Departure rate			Average travel time		
	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
1	0	1	0	12	11	16
2	1	0	0	11	12	15
3	1	0	0	11	13	15
4	1	0	0	11	12	15
5	1	0	0	11	11	13
6	1	0	0	11	11	13
7	0	0	1	11	12	11
8	0	0	1	11	12	11
9	0	0	1	14	11	11
10	1	0	0	11	13	11
11	0	1	0	14	11	14
12	1	0	0	11	14	14
13	1	0	0	11	13	11
14	1	0	0	11	12	11
15	0	0	1	11	12	11
16	0	1	0	11	11	11
17	0	0	1	16	12	11
18	1	0	0	11	15	13
19	0	1	0	11	11	11
20	0	0	1	11	12	11
21	0	0	1	12	12	11
22	1	0	0	11	11	11
23	1	0	0	11	12	13
24	0	0	1	12	16	11
25	1	0	0	11	17	11

Table 3: Departure rate (i.e., $r^{o,d}(p, \bar{t})$) and average travel time (i.e., $l^{o,d}(p, \bar{t})$) of the SO state (TSTT: 248) in test network 1

Departure time	Departure rate			Average travel time		
	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
1	1	0	0	7	11	20
2	1	0	0	7	30	34
3	1	0	0	7	14	18
4	1	0	0	7	19	31
5	0	0	1	12	18	10
6	0	0	1	10	30	10
7	0	0	1	7	16	10
8	0	0	1	7	15	10
9	0	0	1	12	26	10
10	0	0	1	8	12	10
11	0	0	1	8	12	10
12	0	0	1	7	12	10
13	0	0	1	7	19	10
14	0	0	1	9	22	10
15	0	0	1	7	18	10
16	0	1	0	8	11	15
17	1	0	0	13	15	18
18	1	0	0	15	17	17
19	1	0	0	10	16	14
20	1	0	0	14	14	15
21	1	0	0	7	16	15
22	1	0	0	10	13	14
23	1	0	0	12	13	13
24	1	0	0	7	12	11
25	1	0	0	11	12	11

4.2. Test network 2: (Nguyen and Dupuis, 1984)'s network

The second test network is from Nguyen and Dupuis (1984) with 13 nodes, 19 links and 4 OD pairs, as shown in Figure 3. The settings for the network are as following. The size of the time step is taken at 6 s. The saturation flow is still 1800 vph. The maximum accumulative departure rate Q is set at 3 vehicles/time step. Other parameters are the same as defined in Test network 1 unless specified otherwise. The route choices for the network are listed in Table 4.

Figure 3: Test network 2 (Nguyen and Dupuis, 1984)

Table 4: Route choices of different ODs of test network 2

OD	Route NO.	Sequence of links	OD	Route NO.	Sequence of links
(1,2)	1	1-4-13	(4,2)	1	5-6-7-8-13
	2	1-3-7-8-13		2	5-6-7-12-16
	3	1-3-11-15-16		3	5-6-11-15-16
	4	2-10-14-15-16		4	9-14-15-16
(1,3)	1	1-3-11-15-18	(4,3)	1	5-6-11-15-18
	2	2-6-11-15-18		2	5-10-14-15-18
	3	2-10-14-15-18		3	9-17-19

The purpose of constructing test network 2 is to demonstrate the proposed framework's capability of handling a generalized multi-OD network. In the experiment design, we consider two demand scenarios as presented in Table 5. Moreover, we consider that the outflow capacity of link 1, 7, and 10 drops to zero (due to traffic signal or accident) during time step (1~5), and the outflow capacity of link 4, 6, and 14 drops to zero during time step (10~15).

Table 6 and Table 7 present the value of departure rate $r^{o,d}(p, \bar{t})$ and average travel time $l^{o,d}(p, \bar{t})$ of path p and departure time $\bar{t}$ for demand scenario 2 (results of demand

scenario 1 are omitted for the sake of space) under the efficient and fair system state. The travel time of used routes (departure rates are positive) is highlighted in dark. Note that in this test network, $Q = 3$. From Table 6, the summation of values of $r^{o,d}(p, \bar{t})$ over all paths are equal to either 0 or 3, indicating that the accumulative departure rate is either zero or 3 over all paths (as expected from Lemma 1). Further, from Table 7, the travel time of used routes is different under different ODs but is the same under the same OD, and no greater than that of unused routes. This observation confirms that the traffic flows under the same OD are at equilibrium (the complementarity constraint of DUE is satisfied). The total system travel time is 2250 time steps, slightly less than that of SO state (which is 2130 as presented in Table 9). We expect that the system outcome of efficient and fair system state is no greater than that of SO state, as extra constraints on fairness is added to the efficient and fair state. Later in Table 10, the fairness constraints are gradually relaxed, and we see that the system outcome of efficient and ε -tolerant system state evolves to that of SO state (please refer to Appendix for the notion of ε -tolerant system state).

Table 5: Demand scenarios of test network 2

OD	Demand 1	Demand 2
(1,2)	30	45
(1,3)	9	15
(4,2)	30	45
(4,3)	9	15

Table 6: Departure rate (i.e., $r^{o,d}(p, \bar{t})$) for demand scenario 2 of an efficient and fair system state in test network 2

Departure time	OD(1,2)				OD(1,3)			OD(4,2)				OD(4,3)		
	R1	R2	R3	R4	R1	R2	R3	R1	R2	R3	R4	R1	R2	R3
1	2.17	0.83	0	0	1.12	1.88	0	3	0	0	0	0.78	0	2.22
2	3	0	0	0	3	0	0	3	0	0	0	0	0	3
3	3	0	0	0	3	0	0	3	0	0	0	0	0	3
4	0	3	0	0	3	0	0	1.69	1.31	0	0	0.51	0	2.49
5	1.17	1.83	0	0	3	0	0	3	0	0	0	0	0	3
6	3	0	0	0	0	0	0	3	0	0	0	0	0	0
7	3	0	0	0	0	0	0	3	0	0	0	0	0	0
8	3	0	0	0	0	0	0	3	0	0	0	0	0	0
9	3	0	0	0	0	0	0	3	0	0	0	0	0	0
10	3	0	0	0	0	0	0	3	0	0	0	0	0	0
11	3	0	0	0	0	0	0	3	0	0	0	0	0	0
12	3	0	0	0	0	0	0	3	0	0	0	0	0	0
13	3	0	0	0	0	0	0	3	0	0	0	0	0	0
14	3	0	0	0	0	0	0	3	0	0	0	0	0	0
15	3	0	0	0	0	0	0	3	0	0	0	0	0	0

Table 8 and Table 9 present the value of departure rate $r^{o,d}(p, \bar{t})$ and average travel time $l^{o,d}(p, \bar{t})$ for demand scenario 2 under the system optimal condition. From Table 8, the departure rates (i.e., $r^{o,d}(p, \bar{t})$) are equal to either 0 or 3, indicating that the departure rate is either 0 or 3. Table 9 shows that the average travel time (i.e., $l^{o,d}(p, \bar{t})$) are different for different departure rates (i.e., the solution is not at equilibrium). Further, the total system travel time (TSTT) is 2130, less than that (2250) of an efficient and fair system state.

Table 7: Average travel time (i.e., $l^{o,d}(p, \bar{t})$) for demand scenario 2 of an efficient and fair system state (TSTT: 2250) in test network 2

Departure time	OD(1,2)				OD(1,3)			OD(4,2)				OD(4,3)		
	R1	R2	R3	R4	R1	R2	R3	R1	R2	R3	R4	R1	R2	R3
1	18	18	20	18	17	17	17	22	22	22	22	13	14	13
2	18	18	18	20	17	17	20	22	22	22	22	13	20	13
3	18	18	18	20	17	17	19	22	22	22	22	13	19	13
4	18	18	18	20	17	17	18	22	22	22	22	13	18	13
5	18	18	18	20	17	19	17	22	22	22	22	19	17	13
6	18	20	18	18	17	18	17	22	22	22	22	18	16	13
7	18	20	18	18	17	17	17	22	22	22	22	17	15	13
8	18	19	18	19	17	17	17	22	22	22	22	16	14	13
9	18	20	18	18	17	17	17	22	22	22	22	15	14	13
10	18	18	18	18	17	17	17	22	22	22	22	14	14	13
11	18	18	18	20	17	17	17	22	22	22	22	13	14	13
12	18	19	19	18	17	17	17	22	22	22	22	13	14	13
13	18	18	18	18	17	17	17	22	22	22	22	13	14	13
14	18	18	18	18	17	17	17	22	22	22	22	13	14	13
15	18	18	18	18	17	17	17	22	22	22	22	13	14	13

Table 8: Departure rate (i.e., $r^{o,d}(p, \bar{t})$) for demand scenario 2 of the SO state in test network 2

Departure time	OD(1,2)				OD(1,3)			OD(4,2)				OD(4,3)		
	R1	R2	R3	R4	R1	R2	R3	R1	R2	R3	R4	R1	R2	R3
1	3	0	0	0	0	3	0	0	0	3	0	0	3	0
2	3	0	0	0	3	0	0	3	0	0	0	0	0	3
3	3	0	0	0	3	0	0	0	0	3	0	0	0	3
4	0	3	0	0	3	0	0	0	3	0	0	0	0	3
5	0	3	0	0	3	0	0	3	0	0	0	0	0	3
6	3	0	0	0	0	0	0	3	0	0	0	0	0	0
7	3	0	0	0	0	0	0	3	0	0	0	0	0	0
8	3	0	0	0	0	0	0	3	0	0	0	0	0	0
9	3	0	0	0	0	0	0	3	0	0	0	0	0	0
10	3	0	0	0	0	0	0	3	0	0	0	0	0	0
11	3	0	0	0	0	0	0	3	0	0	0	0	0	0
12	3	0	0	0	0	0	0	3	0	0	0	0	0	0
13	3	0	0	0	0	0	0	3	0	0	0	0	0	0
14	3	0	0	0	0	0	0	3	0	0	0	0	0	0
15	3	0	0	0	0	0	0	3	0	0	0	0	0	0

Table 9: Average travel time (i.e., $l^{o,d}(p, \bar{t})$) for demand scenario 2 of the SO state (TSTT: 2130) in test network 2

Departure time	OD(1,2)				OD(1,3)			OD(4,2)				OD(4,3)		
	R1	R2	R3	R4	R1	R2	R3	R1	R2	R3	R4	R1	R2	R3
1	22	18	16	14	25	13	22	15	15	12	40	23	22	15
2	10	16	15	33	18	16	21	14	14	13	31	22	21	14
3	11	15	15	32	19	15	20	32	13	12	35	21	19	14
4	23	14	13	31	17	14	19	15	13	25	30	20	18	14
5	19	14	14	30	14	20	18	32	29	25	35	19	18	10
6	21	29	13	29	13	19	17	31	34	18	27	18	17	12
7	24	26	27	28	14	18	16	26	26	32	29	17	16	10
8	12	25	25	27	13	18	15	31	26	25	27	16	15	10
9	21	26	25	26	13	16	14	27	26	14	26	15	14	10
10	15	25	24	25	16	16	14	24	28	13	30	14	14	10
11	10	24	23	24	15	15	15	21	30	28	27	13	14	10
12	10	21	22	23	14	13	14	29	24	27	19	13	14	10
13	15	20	21	22	13	13	14	27	25	20	23	13	14	10
14	12	19	20	21	13	13	14	24	27	21	24	13	14	10
15	14	20	19	20	13	13	14	20	19	19	19	13	14	10

Table 10 presents the total system travel time (TSTT) for different values of ε -tolerance. When $\varepsilon = 0$, there is no tolerance and the solution is exactly the same as the efficient and fair system state. When $0 < \varepsilon < 2$, the TSTT gradually improves with the increase of tolerance ε . When ε is increased to 2 or greater than 2, the TSTT is the same as the system optimal (SO) solution. The results are in accord with our expectation that the value of ε -tolerance governs the relaxation of the equilibrium constraint. The equilibrium constraint is relaxed more with the increase of the ε -tolerance. Eventually, when the ε -tolerance is increased to a certain threshold, the equilibrium constraint places no effect on the formulation due to the relaxation, and the efficient and fair system state formulation is reduced to the SO formulation.

Table 10: Efficient and ε -tolerant system states in test network 2

ε -tolerance (time steps)	TSTT (time steps)	
	Demand 1	Demand 2
0	1302	2250
0.5	1267	2175
1	1246	2155
1.5	1236	2140
≥ 2	1227 (SO)	2130 (SO)

5. Conclusions

With the advance in communication and computing technologies, personalized travel guidance to drivers is rapidly becoming a reality. To fully utilize the benefit of guidance devices and encourage users to comply with the guidance, there is a pressing need to provide efficient and fair guidance to attain the system state that is closest to system optimal and individual users do not worse off. Addressing the need, this paper models a transportation system that is both efficient (system optimal) and fair (user equilibrium) within the dynamic traffic assignment environment. The formulation is capable of accounting for dynamic departure time and route choice of a generalized multi-OD network. The *key contribution* of this paper is twofold: (1) we propose a linear programming formulation that produces a tight lower bound of an efficient and fair system state, (2) we develop an efficient iterative algorithm to obtain the exact solution of an efficient and system optimal state. Firstly, a bi-level optimization formulation based on a link-based traffic flow model is introduced. At the upper level, the objective is to optimize the total system travel time of the traffic flow at the network level (efficient). At the lower level, the objective is to equilibrium the travel cost of individual users (fair). Then the bi-level formulation is relaxed to a linear programming formulation which produces a lower bound of the efficient and fair system state. Based on the lower bound, we further develop an efficient algorithm to obtain the solution of an efficient and fair system state. It is shown that the number of iterations of the algorithm is bounded and the traffic flow output of the algorithm is at equilibrium. In the numerical case studies, we construct two test networks including a single OD network and a multi-OD network. For all tests, the algorithm converges within two iterations, indicating that the algorithm converges fast and the LP relaxation formulation produces a tight lower bound on the solution of an efficient and fair system state. Moreover, the results of various tests confirm that the average travel cost of every positive departure rate is the same and minimum under the same OD. For test work 2, we also test the efficient and ε -tolerant system state, it shows that the total system travel time decreases with the increase of the tolerance (less than two time steps). When the tolerance reaches two time steps or more, the total system travel time is the same as that of the system optimal state.

There are multiple future research directions of this study:

1. This study is motivated by the need of providing an efficient and fair personalized travel guidance to travelers. However, users' decision-making process is a complex process. Users may not necessarily comply with guidance even if the guidance is both efficient and fair. How to capture user's realistic decision-making behavior in the formulation will be an interesting topic.
2. We have not incorporated the impact of information guidance into the formulation. Providing more information does not necessarily lead to improved system outcome due to the self-interest driven reactions of individual agents. There is a need to understand at a fundamental level how information can be collected, synthesized and disseminated in ways that are easily consumable by a large number of self-interested agents to yield globally fair and efficient outcomes.
3. This study considers users as a homogeneous class. In the real world, users can be classified into multiple classes depending on the transportation mode (e.g., bus, truck, passenger car), various value of time, various tolerance levels of early or late

arrival, and so on. Hence it will be more realistic to incorporate the heterogeneity of users into the formulation in the future study.

Appendix: ε -tolerant fairness

The solution of Algorithm 1 requires that all the departure rates of the same OD experience the equal and minimum travel time. Here we propose the notion of tolerance-based fairness adapted from the behavioral notion of bounded-rationality (Simon, 1991). The tolerance-based fairness only requires the travel time of all departure rates of the same OD to be within an acceptable tolerance level (i.e., ε) from the minimum travel time (Szeto et al., 2015). The tolerance ε is an exogenous parameter capturing the network user's behavior of tolerating unfairness. To model the ε -tolerant fairness, we firstly obtain the solution of an efficient and fair system state, then solve the LP2 formulation with the modified constraints (23) and (31) as below:

$$\sum_{\forall p \in \mathbf{P}^{o,d}} c^{o,d}(p, \bar{t}) \leq \frac{Q}{D^{o,d}} \sum_{\forall p \in \mathbf{P}^{o,d}} \sum_{t=1}^{T_d} c^{o,d}(p, \bar{t}) + \varepsilon Q \quad (48)$$

$$c^{o,d}(p, \bar{t}) \leq r^{o,d}(p, \bar{t}) (LB_*^{o,d} + \varepsilon) \quad (49)$$

where $LB_*^{o,d}$ denotes the solution of Algorithm 1.

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