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Wong, Patricia J. Y.

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Discrete Quintic Spline for Boundary Value Problem in Plate Deflation Theory

Patricia J. Y. Wong

School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore. E-mail: ejywong@ntu.edu.sg

Abstract. We propose a numerical scheme for a fourth-order boundary value problem arising from plate deflation theory. The scheme involves a discrete quintic spline, and it is of order 4 if a parameter takes a specific value, else it is of order 2. We also present a well known numerical example to illustrate the efficiency of our method as well as to compare with other numerical methods proposed in the literature.

INTRODUCTION

We consider the fourth-order boundary value problem (BVP)

$$\begin{aligned} y^{(4)}(x) &= f(x)y(x) + g(x), \quad a \leq x \leq b \\ y(a) &= \alpha_1, \quad y(b) = \beta_1, \quad y''(a) = \alpha_2, \quad y''(b) = \beta_2 \end{aligned} \quad (1)$$

where f and g are continuous on $[a, b]$. BVP (1) arises from the physical problem of bending a rectangular simply supported beam resting on an elastic foundation [1, 2], here y is the vertical deflection of the plate.

Polynomial splines have been used in the numerical treatment of BVP (1), for example Usmani and Warsi [3] have used quintic and sextic splines respectively to develop second- and fourth-order convergent methods for (1). Thereafter, quartic splines are employed by Usmani [4] to formulate second-order convergent method. Also, Al-Said et al. have used cubic and quartic splines to obtain second-order convergent methods for (1) in [5, 6]. Further, non-polynomial splines have been proposed by Ramadan et al. [7] to obtain second- and fourth-order convergent methods for (1), these methods reduced to those of [3, 4, 5] when certain parameters take certain values.

On the other hand, discrete splines were first introduced by Mangasarian and Schumaker [8] in 1971 as solutions to constrained minimization problems in real Euclidean space, which are discrete analogs of minimization problems in Banach space whose solutions are generalized splines. Subsequent investigations on discrete splines can be found in [9, 10, 11, 12, 13, 14, 15].

Motivated by all the above research, in this paper we shall employ a *discrete quintic spline* to get a numerical solution of (1). Following [11, 12], the discrete spline we use will involve central differences. Our proposed method is fourth-order convergent when a parameter takes certain value, else it is second-order convergent. Through a well known numerical example, we illustrate that our method outperforms other spline methods for solving (1) in the literature [3, 4, 5, 6, 7].

DERIVATION OF NUMERICAL SCHEME

Let $P : a = x_0 < x_1 < \dots < x_n = b$ be a uniform mesh of $[a, b]$ with step size $p = \frac{b-a}{n}$. Let $h \in (0, p]$ be a given constant. Recall the *central difference operator* $D_h^{[k]}$ applying to a function $F(x)$ gives

$$\begin{aligned} D_h^{[0]} F(x) &= F(x); & D_h^{[1]} F(x) &= [F(x+h) - F(x-h)]/(2h); & D_h^{[2]} F(x) &= [F(x+h) - 2F(x) + F(x-h)]/h^2; \\ D_h^{[3]} F(x) &= [F(x+2h) - 2F(x+h) + 2F(x-h) - F(x-2h)]/(2h^3); \end{aligned}$$

$$D_h^{(4)}F(x) = [F(x+2h) - 4F(x+h) + 6F(x) - 4F(x-h) + F(x-2h)]/h^4.$$

Definition 1. Let $S(x; h)$ be a piecewise continuous function defined over $[a, b]$ (with mesh P) and $S_i(x)$ be its restriction in $[x_{i-1}, x_i]$, $1 \leq i \leq n$ passing through the points (x_{i-1}, s_{i-1}) and (x_i, s_i) . We say $S(x; h)$ is a *discrete quintic spline* if $S_i(x)$, $1 \leq i \leq n$ is a polynomial of degree 5 or less and

$$D_h^{[j]}S_i(x_i) = D_h^{[j]}S_{i+1}(x_i), \quad j = 0, 1, 2, 3, 4, \quad 1 \leq i \leq n-1.$$

Throughout, we shall use the notations $y_i^{(k)} = y^{(k)}(x_i)$, $f_i = f(x_i)$, $g_i = g(x_i)$, $s_i = S_i(x_i)$, $M_i = D_h^{[2]}S_i(x_i)$ and $F_i = D_h^{(4)}S_i(x_i)$, $0 \leq i \leq n$. We propose s_i 's to be the numerical solution of (1) at the mesh points, i.e.,

$$y_i \cong s_i, \quad 0 \leq i \leq n.$$

Discretizing (1) and noting the boundary conditions, we set

$$s_0 = y_0 = \alpha_1, \quad s_n = y_n = \beta_1, \quad M_0 = y_0'' = \alpha_2, \quad M_n = y_n'' = \beta_2, \quad F_i = f_i s_i + g_i, \quad 0 \leq i \leq n. \quad (2)$$

Next, we shall obtain an explicit expression of $S_i(x)$ in terms of its central differences. To begin, let the basis functions $q_i(x)$, $\bar{q}_i(x)$ and $\bar{\bar{q}}_i(x)$ satisfy the following for $0 \leq i, j \leq n$:

$$q_i(x_j) = D_h^{[2]} \bar{q}_i(x_j) = D_h^{(4)} \bar{\bar{q}}_i(x_j) = \delta_{ij}; \quad D_h^{[2]} q_i(x_j) = D_h^{(4)} q_i(x_j) = 0; \quad \bar{q}_i(x_j) = D_h^{(4)} \bar{q}_i(x_j) = 0; \quad \bar{\bar{q}}_i(x_j) = D_h^{[2]} \bar{\bar{q}}_i(x_j) = 0.$$

The explicit expressions of q_i , $\bar{q}_i$, $\bar{\bar{q}}_i$ can be obtained by direct computation. Clearly, $S_i(x)$, the restriction of $S(x; h)$ in $[x_{i-1}, x_i]$, can be expressed as

$$S_i(x) = s_{i-1}q_{i-1}(x) + s_i q_i(x) + M_{i-1} \bar{q}_{i-1}(x) + M_i \bar{q}_i(x) + F_{i-1} \bar{\bar{q}}_{i-1}(x) + F_i \bar{\bar{q}}_i(x), \quad x \in [x_{i-1}, x_i], \quad 1 \leq i \leq n. \quad (3)$$

Using (3), the 'continuity' requirements $D_h^{[1]}S_i(x_i) = D_h^{[1]}S_{i+1}(x_i)$ and $D_h^{[3]}S_i(x_i) = D_h^{[3]}S_{i+1}(x_i)$, $1 \leq i \leq n-1$ lead to the equations

$$(p^2 - h^2)M_{i-1} + 2(h^2 + 2p^2)M_i + (p^2 - h^2)M_{i+1} \\ = 6(s_{i-1} - 2s_i + s_{i+1}) + \frac{(p^2 - h^2)}{60} [(2h^2 + 7p^2)F_{i-1} + 4(4p^2 - h^2)F_i + (2h^2 + 7p^2)F_{i+1}], \quad (4)$$

$$M_{i-1} - 2M_i + M_{i+1} = \frac{1}{6} [(p^2 - h^2)F_{i-1} + 2(h^2 + 2p^2)F_i + (p^2 - h^2)F_{i+1}]. \quad (5)$$

From (4) and (5), we are able to eliminate M 's and get the F -equation as

$$a_1 F_{i-2} + a_2 F_{i-1} + a_3 F_i + a_2 F_{i+1} + a_1 F_{i+2} = s_{i-2} - 4s_{i-1} + 6s_i - 4s_{i+1} + s_{i+2}, \quad 2 \leq i \leq n-2 \quad (6)$$

where

$$a_1 = \frac{(p^2 - h^2)(p^2 - 4h^2)}{120}, \quad a_2 = \frac{2(p^2 - h^2)(8h^2 + 13p^2)}{120}, \quad a_3 = \frac{6(4h^4 + 5h^2 p^2 + 11p^4)}{120}. \quad (7)$$

Upon substituting $F_j = f_j s_j + g_j$ into (6), we see that (6) gives $(n-3)$ equations with $(n-1)$ unknowns s_i , $1 \leq i \leq n-1$. We need two more equations which we write as

$$b_1 F_0 + b_2 F_1 + b_3 F_2 + b_4 F_3 = p^2 M_0 + b_5 s_0 + b_6 s_1 + b_7 s_2 + b_8 s_3, \quad (8)$$

$$c_1 F_{n-3} + c_2 F_{n-2} + c_3 F_{n-1} + c_4 F_n = p^2 M_n + c_5 s_{n-3} + c_6 s_{n-2} + c_7 s_{n-1} + c_8 s_n \quad (9)$$

where b_i 's and c_i 's are numbers to be determined. Let the local truncation errors in both (8) and (9) be $O(p^8)$ (the reason will be clear when we present the convergence analysis). To fulfill this, we carry out Taylor series expansion in (8) about x_0 and set the coefficients of $s_0^{(k)}$, $0 \leq k \leq 7$ to zeros. This yields 8 equations which we can solve to get b_i , $1 \leq i \leq 8$. Similarly, in (9) we expand about x_n and set the coefficients of $s_n^{(k)}$, $0 \leq k \leq 7$ to zeros to get c_i , $1 \leq i \leq 8$. The resulting (8) and (9) are given as follows

$$\frac{p^4}{360} (28F_0 + 245F_1 + 56F_2 + F_3) - p^2 M_0 = -2s_0 + 5s_1 - 4s_2 + s_3, \quad (10)$$

$$\frac{p^4}{360} (F_{n-3} + 56F_{n-2} + 245F_{n-1} + 28F_n) - p^2 M_n = s_{n-3} - 4s_{n-2} + 5s_{n-1} - 2s_n. \quad (11)$$

Once again, we substitute $F_j = f_j s_j + g_j$ into (10) and (11) to give two equations in s_i , $i = 1, 2, 3, n-3, n-2, n-1$.

Noting (2), we can now solve (6), (10), (11) to obtain the values of s_i , $1 \leq i \leq n-1$. The solvability of the linear system will be given in the next section.

SOLVABILITY AND CONVERGENCE

Let $e_i = y_i - s_i$, $1 \leq i \leq n-1$ be the errors. Let $Y = [y_i]$, $S = [s_i]$, $W = [w_i]$, $T = [t_i]$ and $E = [e_i]$ be $(n-1)$ -dimensional column vectors. The system (6), (10), (11) can be written as

$$AS = W \quad (12)$$

where $A = A_0 + BF$, $F = \text{diag}(f_i)$, $i = 1, 2, \dots, n-1$, A_0 and B are $(n-1) \times (n-1)$ five-band symmetric matrices given by

$$A_0 = \begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & & & \ddots & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -4 \\ & & & & & & 1 & -4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{245p^4}{360} & -\frac{56p^4}{360} & -\frac{p^4}{360} & & & \\ -a_2 & -a_3 & -a_2 & -a_1 & & \\ -a_1 & -a_2 & -a_3 & -a_2 & -a_1 & \\ & & & \ddots & & \\ & & & & -a_1 & -a_2 & -a_3 & -a_2 & -a_1 \\ & & & & & -a_1 & -a_2 & -a_3 & -a_2 \\ & & & & & & -\frac{p^4}{360} & -\frac{56p^4}{360} & -\frac{245p^4}{360} \end{pmatrix}$$

and for the vector $W = [w_i]$, we have

$$w_i = \begin{cases} 2s_0 - p^2 M_0 + \frac{p^4}{360}(28f_0 s_0 + 28g_0 + 245g_1 + 56g_2 + g_3), & i = 1 \\ -s_0 + a_1 f_0 s_0 + a_1 g_0 + a_2 g_1 + a_3 g_2 + a_2 g_3 + a_1 g_4, & i = 2 \\ a_1 g_{i-2} + a_2 g_{i-1} + a_3 g_i + a_2 g_{i+1} + a_1 g_{i+2}, & 3 \leq i \leq n-3 \\ -s_n + a_1 g_{n-4} + a_2 g_{n-3} + a_3 g_{n-2} + a_2 g_{n-1} + a_1 g_n + a_1 f_n s_n, & i = n-2 \\ 2s_n - p^2 M_n + \frac{p^4}{360}(g_{n-3} + 56g_{n-2} + 245g_{n-1} + 28g_n + 28f_n s_n), & i = n-1. \end{cases}$$

The local truncation error in the i -th equation of the linear system (12) is computed to be

$$t_i = \frac{p^4(p^2 - 3h^2)}{12} y_i^{(6)} + \frac{p^4(4p^4 - 15p^2 h^2 + 8h^4)}{240} y_i^{(8)} + O(p^9), \quad 2 \leq i \leq n-2 \quad (13)$$

$$t_1 = t_{n-1} = -\frac{241}{60480} p^8 y_i^{(8)} + O(p^9). \quad (14)$$

Remark 1. For the special case $h = \frac{p}{\sqrt{3}}$, it is clear from (13) that $t_i = -\frac{p^8}{2160} y_i^{(8)} + O(p^9)$, $2 \leq i \leq n-2$. Thus, taking (14) into consideration, we have

$$\|T\| = \max_{1 \leq i \leq n-1} |t_i| = \frac{241}{60480} p^8 L \quad (15)$$

where $L = \max_x |y^{(8)}(x)|$.

Theorem 1. (Solvability) The system (12) has a unique solution if

$$\frac{489}{480} K \hat{f} < 1 \quad (16)$$

where $K = \frac{5(b-a)^4 + 4(b-a)^2 p^2}{384}$ and $\hat{f} = \max_{1 \leq i \leq n-1} |f_i|$.

Theorem 2. (Convergence) Suppose $\frac{489}{480} K \hat{f} < 1$. Then,

$$\|E\| = \max_{1 \leq i \leq n-1} |e_i| \cong O(p^4) \quad \text{if } h = \frac{p}{\sqrt{3}}$$

and $\|E\| \cong O(p^2)$ for other values of $h \in (0, p]$, i.e., the method (12) is fourth-order convergent if $h = \frac{p}{\sqrt{3}}$ and is second-order convergent otherwise.

NUMERICAL EXAMPLE

Consider the fourth-order BVP [3, 4, 5, 6, 7]

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \quad 0 \leq x \leq 1; \quad y(0) = y(1) = 0, \quad y''(0) = 0, \quad y''(1) = -4e. \quad (17)$$

The analytical solution of (17) is $y(x) = x(1-x)e^x$.

In this example, we have $a = 0$, $b = 1$, $f(x) = -x$ and $g(x) = -(8 + 7x + x^3)e^x$. So $K = \frac{5+4p^2}{384}$ and $\hat{f} < 1$. For any $p \in (0, 1)$, we have $\frac{489}{480}K\hat{f} < 1$ and hence it follows from Theorem 3.1 that our method gives a unique numerical solution for (17). First, we fix the mesh P (and hence the step size p) and choose $h = \frac{p}{\sqrt{3}}$. Then, we solve the system (6), (10), (11) to get s_i , $1 \leq i \leq n-1$, which approximates y_i .

The maximum absolute errors ($\|E\| = \max_i |y_i - s_i|$) obtained by our method as well as by other methods in the literature are presented in Table 1. From the table we can see that our method is fourth-order convergent when $h = \frac{p}{\sqrt{3}}$. Moreover, a clear comparison shows that our method *outperforms* continuous polynomial spline (cubic, quartic, quintic, sextic) and non-polynomial spline (quintic) methods.

TABLE 1. Maximum absolute errors $\|E\| = \max_i |y_i - s_i|$

Methods	$p = 1/8$	$p = 1/16$	$p = 1/32$
Our method	$7.48E-08$	$5.30E-09$	$4.91E-10$
Quintic nonpolynomial spline (4th order) [7]	$2.09E-07$	$7.92E-09$	$1.27E-09$
Sextic spline [3]	$1.26E-06$	$7.87E-08$	$4.91E-09$
Quintic nonpolynomial spline (2nd order) [7]	$9.42E-05$	$6.17E-06$	$3.95E-07$
Quartic spline [4]	$4.24E-04$	$1.08E-04$	$2.70E-05$
Cubic spline [6]	$5.69E-04$	$1.47E-04$	$3.71E-05$
Quintic spline [3]	$8.67E-04$	$2.16E-04$	$5.40E-05$
Quartic spline [5]	$1.62E-03$	$6.39E-04$	$5.88E-05$

CONCLUSION

We have developed a numerical scheme for a fourth-order BVP using discrete quintic spline. The convergence order is $O(p^4)$ when the parameter $h = \frac{p}{\sqrt{3}}$; and $O(p^2)$ when $h \in (0, p]$. Numerical simulation is presented to demonstrate the outperformance of the scheme over continuous spline methods in the literature.

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