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EXISTENCE OF A MEAN-SQUARE STABILIZING SOLUTION TO A MODIFIED ALGEBRAIC RICCATI EQUATION*

JIANYING ZHENG[†] AND LI QIU[‡]

Abstract. In this paper, the existence of a mean-square stabilizing solution to a discrete-time modified algebraic Riccati equation (MARE), which arises in the study of some stochastic linear quadratic optimal control problems, is investigated. The theory of cone-invariant operators is employed as the mathematical tool to tackle this problem. We provide some criteria for the cone stability, cone observability, and cone detectability of a class of cone-invariant systems in terms of the associated distinguished eigenvalues. Then two scenarios concerning the problem of existence in terms of the positive definiteness of the input weighting matrix are considered. We first study the MARE under the assumption that the input weighting matrix in the cost function is positive definite. Then an explicit necessary and sufficient condition is obtained. Such a condition is derived for the very first time and it indicates that the common condition of observability or detectability of certain stochastic systems is unnecessary. Only the observability of the distinguished eigenvalue at 1 of an associated cone-invariant operator is required. Hence this necessary and sufficient condition is compatible with the one ensuring the existence of a stabilizing solution to the standard algebraic Riccati equation. However, when the input weighting matrix is only positive semidefinite, this condition does not hold. In this case, we get a sufficient condition and a necessary condition, respectively. These two conditions coincide when the input weighting matrix is indeed positive definite.

Key words. modified algebraic Riccati equation, cone-invariant operators, stochastic optimal control

AMS subject classifications. 47L07, 49N10, 93E20

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1. Introduction. As one of the most fundamental and important problems in control theory, the linear quadratic (LQ) optimal control problem has been extensively studied, both for deterministic and stochastic systems. See [5, 8, 9, 13, 22, 27] and the references therein. In the solutions to the deterministic LQ optimal control problem, as well as to the deterministic optimal linear filtering problem, the well-known standard algebraic Riccati equations (AREs) arise and play an essential role. The solutions, properties, and applications of AREs are studied in a large number of research works; see, e.g., [15, 16, 27] and the references therein.

Likewise, in our previous study of LQ optimal control of a discrete linear time-invariant (LTI) system with random input gains [25], a modified algebraic Riccati equation (MARE), which plays a similar role to AREs, appears. In general, there are many solutions to the MARE. Among them, the one called mean-square stabilizing solution attracts our attention. With the static state feedback controller associated

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with the mean-square stabilizing solution, this stochastic LQ optimal control problem is solvable, i.e., the cost function is minimized and the closed-loop system is mean-square stable. Thus we are interested in the existence of such a mean-square stabilizing solution to the MARE. Note that similar MAREs arise in the stochastic LQ optimal control problems with different settings. In most existing research [4, 9, 10, 11, 12, 24], only sufficient conditions are provided for the existence of the mean-square stabilizing solution. These sufficient conditions are often given in terms of stabilizability and observability (or detectability) of certain stochastic systems. Such conditions are quite strong. In the book [5], one numerical necessary and sufficient condition is given in terms of the feasibility of some linear matrix inequalities (LMIs). However, such a condition does not have explicit interpretations with respect to the dynamical properties of the underlying stochastic system.

In this paper, we employ the theory of cone-invariant operators to investigate the existence of a mean-square stabilizing solution to the MARE. Cone-invariant operators have many appealing properties, e.g., the fundamental Krein–Rutman theorem and the special distinguished eigenvalues, and have been extensively studied by mathematicians [1, 2, 19]. We first study some fundamental properties of cone-invariant systems, including cone stability, cone observability, and cone detectability, and obtain some criteria given in terms of the distinguished eigenvalues of the associated cone-invariant operators. Then two scenarios concerning the problem of existence are considered. The first one is the case in which the input weighting matrix in the cost function is positive definite, which is a common assumption in many papers. Then an explicit necessary and sufficient condition ensuring the existence of the mean-square stabilizing solution to the MARE is obtained. Such a condition is compatible with the one ensuring the stabilizing solution to the standard ARE and it indicates that the common condition of observability or detectability of certain stochastic systems is unnecessary. Only the observability of the distinguished eigenvalue at 1 of an associated cone-invariant operator is required. This result can be generalized to a more general class of MAREs. In the second scenario, the input weighting matrix in the cost function could be positive semidefinite. Then the previous necessary and sufficient condition does not hold any more. Fortunately, we can still take advantage of cone-invariant operators theory and some classical results for the standard ARE to investigate the existence conditions of the mean-square stabilizing solution to the MARE. Specifically, we get a sufficient condition and a necessary condition, respectively. These two conditions coincide when the input weighting matrix is indeed positive definite. When the random input gains become deterministic, the sufficient condition is also necessary, which is exactly the one for the standard ARE.

The remainder of this paper is organized as follows. The problem is stated in section 2. In section 3, we introduce some basic concepts of cone-invariant operators and study some important properties of cone-invariant systems. In section 4, we consider the existence of a mean-square stabilizing solution to the MARE in two scenarios and provide the corresponding existence conditions. Note that the results for the scenario when the input weighting matrix in the cost function is positive definite were presented in the conference paper [26] but without detailed proofs of lemmas. A conclusion follows in section 5. All the proofs are given in Appendix A.

The notation is rather standard in this paper. Denote by $\mathbb{R}^n$ the n -dimensional Euclidean space and $\mathbb{R}_+^n$ the subset of $\mathbb{R}^n$ with nonnegative entries. The symbol $\mathbf{1}_{n \times n}$ stands for an $n \times n$ matrix with all elements being 1. Denote the real vector space of $n \times n$ real symmetric matrices by $\mathcal{S}_n$ and the subset of positive semidefinite matrices of $\mathcal{S}_n$ by $\mathcal{P}_n$. By $\text{diag}\{\cdot\}$, we mean a diagonal matrix with diagonal entries given in

FIG. 1. LQ optimal control of LTI systems with random input gains.

the bracket in the same order. The symbol $\odot$ stands for the Hadamard product and $(\cdot)^+$ the pseudoinverse. The spectral radius and kernel of matrices or operators are denoted by $\rho(\cdot)$ and $\ker(\cdot)$, respectively. Let $\mathbf{E}(\cdot)$ be the expectation of a random variable, $\text{int } \mathcal{V}$ and $\langle \cdot, \cdot \rangle$ be the topological interior and inner product of a vector space $\mathcal{V}$, respectively.

2. Problem formulation. In this paper, we focus on studying a MARE which arises in the LQ optimal control of a discrete-time LTI system with random input gains, as shown in Figure 1. The plant is described by

$$(1) \quad \begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ z(k) &= Cx(k) + Du(k), \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. Different from the standard LTI systems, there are random gains $\kappa(k)$ in the input channels, i.e., $u(k) = \kappa(k)v(k)$, where $v(k)$ is generated by an LTI state feedback controller $\mathbf{K}$ in the form of

$$(2) \quad \begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K x(k), \\ v(k) &= C_K x_K(k) + D_K x(k). \end{aligned}$$

The parallel transmission strategy for the multi-input system is adopted here, i.e., each element of $v(k)$ is sent across an individual input channel. Then $\kappa(k)$ can be described by a diagonal random matrix $\text{diag}\{\kappa_1(k), \dots, \kappa_m(k)\}$, whose diagonal elements $\kappa_i(k)$, $i = 1, \dots, m$, are mutually uncorrelated independent and identically distributed random processes with nonzero mean $\mu_i = \mathbf{E}[\kappa_i(k)]$ and variance $\sigma_i^2 = \mathbf{E}[(\kappa_i(k) - \mu_i)^2]$, respectively. The signal-to-noise ratio of the i th input channel is defined as $\text{SNR}_i \triangleq \mu_i^2 / \sigma_i^2$. Denote

$$\begin{aligned} M &\triangleq \text{diag}\{\mu_1, \dots, \mu_m\}, \quad \Sigma^2 \triangleq \text{diag}\{\sigma_1^2, \dots, \sigma_m^2\}, \\ \text{SNR} &\triangleq \text{diag}\{\text{SNR}_1, \dots, \text{SNR}_m\}, \end{aligned}$$

and

$$W = \mathbf{1}_{m \times m} + \text{SNR}^{-1}.$$

Let $\hat{x}(k) = \begin{bmatrix} x(k) \\ x_K(k) \end{bmatrix}$ be the overall system state. Then the closed-loop system can be written as

$$(3) \quad \hat{x}(k+1) = \left\{ \begin{bmatrix} A & 0 \\ B_K & A_K \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \kappa(k) \begin{bmatrix} D_K & C_K \end{bmatrix} \right\} \hat{x}(k).$$

The state feedback controller $\mathbf{K}$ is said to be mean-square stabilizing if the corresponding closed-loop system (3) is stable in the mean-square sense, i.e., $\lim_{k \rightarrow \infty} \mathbf{E}[\hat{x}(k)\hat{x}'(k)] = 0$ for any initial state $\hat{x}(0)$. Then the objective of this stochastic LQ optimal con-

trol problem is to find a mean-square stabilizing controller $\mathbf{K}$ to minimize the cost function $J(x(0), v(\cdot))$, defined as

$$\begin{aligned} J(x(0), v(\cdot)) &= \mathbf{E} \left[\sum_{k=0}^{\infty} z'(k)z(k) \right] = \mathbf{E} \left\{ \sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ \kappa(k)v(k) \end{bmatrix}' \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa(k)v(k) \end{bmatrix} \right\} \\ &= \mathbf{E} \left\{ \sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ Mv(k) \end{bmatrix}' \begin{bmatrix} C'C & C'D \\ D'C & W \odot (D'D) \end{bmatrix} \begin{bmatrix} x(k) \\ Mv(k) \end{bmatrix} \right\}. \end{aligned}$$

It is well known that the solvability of a deterministic LQ optimal control problem depends on the existence of a stabilizing solution to the associated ARE [27]. Similarly, a MARE defined as follows comes up in this stochastic LQ optimal control problem:

$$(4) \quad A'XA - X + C'C - (A'XB + C'D) \times [W \odot (B'XB + D'D)]^{-1} (B'XA + D'C) = 0.$$

A positive semidefinite solution X to the MARE (4) is said to be mean-square stabilizing in the sense that the associated state feedback gain

$$(5) \quad F(X) \triangleq -M^{-1}[W \odot (B'XB + D'D)]^{-1}(B'XA + D'C)$$

is a mean-square stabilizing controller. It is shown that this stochastic LQ optimal control problem has a unique optimal mean-square stabilizing controller if and only if there is a mean-square stabilizing solution X to the MARE (4) [25]. Moreover, $F(X)$ is exactly the optimal mean-square stabilizing controller.

Therefore, we are interested in the conditions ensuring the existence of the mean-square stabilizing solution to the MARE (4). We employ the tool of cone-invariant operators to deal with this problem. Two scenarios are considered. One is under the assumption that the input weighting matrix $W \odot D'D$ is positive definite, which is a weaker assumption than that D has full column rank. The other is the case when $W \odot D'D$ is singular, which also occurs in applications. In this situation, we only assume that

$$\ker(B) \cap \ker(W \odot D'D) = 0$$

to ensure that $\{X \in \mathcal{S}_n | W \odot (B'XB + D'D) > 0\}$ is nonempty.

3. Preliminaries. Before stating the results, we first introduce some preliminary knowledge from cone-invariant operators theory.

3.1. Cone-invariant operators. Let $\mathcal{V}$ be a finite-dimensional real vector space. A nonempty subset $\mathcal{V}_+ \subset \mathcal{V}$ is said to be a convex cone if $\mathcal{V}_+ + \mathcal{V}_+ = \mathcal{V}_+$ and $a\mathcal{V}_+ \subset \mathcal{V}_+$ for all real nonnegative numbers a . A convex cone is pointed if $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{0\}$ and solid if $\text{int } \mathcal{V}_+$ is not empty. A convex cone is said to be proper if it is closed, pointed, and solid. A proper cone induces a partial order on the vector space as follows: for $X, Y \in \mathcal{V}$, we write $X \geq_{\mathcal{V}_+} Y$ if $X - Y \in \mathcal{V}_+$, and $X >_{\mathcal{V}_+} Y$ if $X - Y \in \text{int } \mathcal{V}_+$.

Example 3.1.

- The set of nonnegative vectors $\mathbb{R}_+^n$ is a proper cone and induces the entrywise partial order in $\mathbb{R}^n$.
- The set of nonnegative polynomials on $[0, 1]$

$$\{x \in \mathbb{R}^n : x_1 t^{n-1} + x_2 t^{n-2} + \cdots + x_n \geq 0 \text{ for } t \in [0, 1]\}$$

is also a proper cone in $\mathbb{R}^n$.

- The space of symmetric matrices $\mathcal{S}_n$ is a real Hilbert space endowed with the Frobenius inner product $\langle X, Y \rangle = \text{Tr}XY$ and the Frobenius norm

$$\|X\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_{ij}^2}.$$

Then $\mathcal{P}_n$ is a proper cone in $\mathcal{S}_n$ and induces the partial order as follows: given $X, Y \in \mathcal{S}_n$, $X \geq_{\mathcal{P}_n} Y$ if and only if $v'(X - Y)v \geq 0 \forall v \in \mathbb{R}^n$.

DEFINITION 3.2. Let $\mathcal{U}, \mathcal{V}$ be finite-dimensional real vector spaces together with the proper cones $\mathcal{U}_+$ and $\mathcal{V}_+$, respectively. A linear operator $\mathbf{T} : \mathcal{U} \rightarrow \mathcal{V}$ is said to be $(\mathcal{U}_+, \mathcal{V}_+)$ -invariant if $\mathbf{T}\mathcal{U}_+ \subset \mathcal{V}_+$. When $\mathcal{U}_+ = \mathcal{V}_+$, we simply write $(\mathcal{U}_+, \mathcal{V}_+)$ -invariant as $\mathcal{U}_+$ -invariant.

A typical example of cone-invariant operators is nonnegative matrices. To be specific, an $m \times n$ nonnegative matrix $\mathbb{R}_+^{m \times n}$ is an $(\mathbb{R}_+^n, \mathbb{R}_+^m)$ -invariant operator. For the nonnegative square matrices, we have the famous Perron–Frobenius theorem to characterize their spectral properties. It was first extended to a compact cone-invariant operator in a Banach space by Krein and Rutman [14]. A finite-dimensional Krein–Rutman theorem is stated below.

THEOREM 3.3. Let $\mathcal{V}_+$ be a proper cone in the finite-dimensional real vector space $\mathcal{V}$ and $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{V}$ be a linear $\mathcal{V}_+$ -invariant operator. Then $\rho(\mathbf{T})$ is an eigenvalue of $\mathbf{T}$ with a corresponding eigenvector X in $\mathcal{V}_+ \setminus \{0\}$, i.e., $\mathbf{T}X = \rho(\mathbf{T})X$.

Besides the spectral radius, a cone-invariant operator may have other eigenvalues with some corresponding eigenvectors in the cone. We call these eigenvalues distinguished eigenvalues and the corresponding eigenvectors in the cone distinguished eigenvectors. It is easy to see that distinguished eigenvalues are always real and non-negative, and the spectral radius is the largest one. To tell whether an eigenvalue is distinguished or not, it is equivalent to solving a conic feasibility problem. Particularly, when the underlying cone is $\mathbb{R}_+^n$, the conic feasibility problem becomes a linear programming feasibility problem and when the underlying cone is $\mathcal{P}_n$, it corresponds to a semidefinite programming feasibility problem.

Analogously to the definition of generalized eigenvalues of a matrix pair, we can define generalized distinguished eigenvalues of a cone-invariant operator pair. Given two $\mathcal{V}_+$ -invariant operators $\mathbf{T}_1$ and $\mathbf{T}_2$, λ is a generalized distinguished eigenvalue of the operator pair $(\mathbf{T}_1, \mathbf{T}_2)$ if there exists $X \in \mathcal{V}_+$ such that $\mathbf{T}_1(X) = \lambda \mathbf{T}_2(X)$, and X is called a generalized distinguished eigenvector corresponding to λ .

3.2. Cone-invariant systems. Let $\mathcal{X}, \mathcal{U}$, and $\mathcal{Y}$ be finite-dimensional real vector spaces together with the proper cones $\mathcal{X}_+, \mathcal{U}_+$, and $\mathcal{Y}_+$, respectively. Then a discrete-time LTI system with the following form

$$(6) \quad \begin{aligned} X(k+1) &= \mathbf{A}X(k) + \mathbf{B}U(k), \\ Y(k) &= \mathbf{C}X(k) + \mathbf{D}U(k), \end{aligned}$$

where $X(k) \in \mathcal{X}$, $U(k) \in \mathcal{U}$, and $Y(k) \in \mathcal{Y}$, denoted by $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right]$ for brevity, is said to be a cone-invariant system if the partitioned linear operator

$$\left[\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{array} \right] : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X} \times \mathcal{Y}$$

is $(\mathcal{X}_+ \times \mathcal{U}_+, \mathcal{X}_+ \times \mathcal{Y}_+)$ -invariant. When $\mathcal{X} = \mathbb{R}^n$ with $\mathcal{X}_+ = \mathbb{R}_+^n$, $\mathcal{U} = \mathbb{R}^m$ with $\mathcal{U}_+ = \mathbb{R}_+^m$, and $\mathcal{Y} = \mathbb{R}^p$ with $\mathcal{Y}_+ = \mathbb{R}_+^p$, this cone-invariant system reduces to a positive system [7], and the partitioned linear operator is exactly a nonnegative matrix with dimension $(n+p) \times (n+m)$.

For a cone-invariant system, instead of the standard stability, observability, and detectability when we consider it as a linear system, one can define weaker notions of cone stability, cone observability, and cone detectability by restricting the state, input, and output to the respective cones.

DEFINITION 3.4. *The autonomous cone-invariant system $X(k+1) = \mathbf{A}X(k)$ is said to be $\mathcal{X}_+$ -stable if for any initial state $X(0) \in \mathcal{X}_+$, it holds that $\lim_{k \rightarrow \infty} X(k) = 0$.*

THEOREM 3.5. *The following statements are equivalent.*

- (i) *The cone-invariant system $X(k+1) = \mathbf{A}X(k)$ is $\mathcal{X}_+$ -stable.*
- (ii) $\rho(\mathbf{A}) < 1$.
- (iii) *There exists $X \in \text{int } \mathcal{X}_+$ such that $\mathbf{A}X <_{\mathcal{X}_+} X$.*

Proof. See section A.1 for the proof. □

Implied by the solidness of $\mathcal{X}_+$, which is equivalent to $\mathcal{X}_+ - \mathcal{X}_+ = \mathcal{X}$, we readily see that the autonomous cone-invariant system is stable in the whole vector space if and only if it is $\mathcal{X}_+$ -stable. Therefore the stability property is cone independent. However, such a conclusion does not hold for cone observability and cone detectability, which are defined as follows.

DEFINITION 3.6.

- (i) $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ *is said to be $\mathcal{X}_+$ -observable if for any $X(0) \in \mathcal{X}_+$ such that $Y(k) \equiv 0$ for all $k \geq 0$, it holds that $X(0) = 0$.*
- (ii) $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ *is said to be $\mathcal{X}_+$ -detectable if for any $X(0) \in \mathcal{X}_+$ such that $Y(k) \equiv 0$ for all $k \geq 0$, it holds that $\lim_{k \rightarrow \infty} X(k) = 0$.*

The distinguished eigenvalues of the $\mathcal{X}_+$ -invariant operator $\mathbf{A}$ play an important role in characterizing the system's $\mathcal{X}_+$ -observability and $\mathcal{X}_+$ -detectability.

DEFINITION 3.7. *A distinguished eigenvalue λ of $\mathbf{A}$ is said to be observable with respect to $\mathbf{C}$ if*

$$\mathbf{C}X \neq 0$$

for any distinguished eigenvector X of $\mathbf{A}$ corresponding to λ . We simply say that λ is an observable distinguished eigenvalue of $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$. Otherwise, it is said to be an unobservable distinguished eigenvalue.

THEOREM 3.8.

- (i) $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ *is $\mathcal{X}_+$ -observable if and only if all the distinguished eigenvalues of $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ are observable.*
- (ii) $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ *is $\mathcal{X}_+$ -detectable if and only if all the unstable distinguished eigenvalues of $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ are observable.*

Proof. See section A.2 for the proof. □

Remark 3.9. Similarly, given two $\mathcal{X}_+$ -invariant operators $\mathbf{A}_1$ and $\mathbf{A}_2$, a generalized distinguished eigenvalue λ of $(\mathbf{A}_1, \mathbf{A}_2)$ is said to be an observable generalized distinguished eigenvalue of $\begin{bmatrix} (\mathbf{A}_1, \mathbf{A}_2) \\ \mathbf{C} \end{bmatrix}$ if for any generalized distinguished eigenvector X corresponding to λ , it holds that $\mathbf{C}X \neq 0$.

4. Main results. In this section, we study the existence of a mean-square stabilizing solution to the MARE (4) and present the main results.

The LTI system with random input gains, denoted by $\{[A|B], \kappa(k)\}$, is said to be mean-square stabilizable if there exists an LTI state feedback controller $\mathbf{K}$ such that the closed-loop system (3) is mean-square stable. Let $v(k)$ be generated by a static state feedback gain F . Then the closed-loop system (3) is given by

$$(7) \quad \begin{aligned} x(k+1) &= (A + B\kappa(k)F)x(k), \\ z(k) &= (C + D\kappa(k)F)x(k). \end{aligned}$$

Let $X(k) \triangleq \mathbf{E}[x(k)x'(k)]$ and $Z(k) \triangleq \mathbf{E}[z(k)z'(k)]$. Then we obtain the following cone-invariant system:

$$\begin{aligned} X(k+1) &= (A + BMF)X(k)(A + BMF)' + B[\Sigma^2 \odot (FX(k)F')]B', \\ Z(k) &= (C + DMF)X(k)(C + DMF)' + D[\Sigma^2 \odot (FX(k)F')]D'. \end{aligned}$$

Denote the associated $\mathcal{P}_n$ -invariant and $(\mathcal{P}_n, \mathcal{P}_p)$ -invariant operators as follows:

$$\begin{aligned} \mathbf{A}_F : X \in \mathcal{S}_n &\mapsto (A + BMF)X(A + BMF)' + B[\Sigma^2 \odot (FXF')]B' \in \mathcal{S}_n, \\ \mathbf{C}_F : X \in \mathcal{S}_n &\mapsto (C + DMF)X(C + DMF)' + D[\Sigma^2 \odot (FXF')]D' \in \mathcal{S}_p. \end{aligned}$$

Then the above cone-invariant system can be denoted by $\left[\frac{\mathbf{A}_F}{\mathbf{C}_F}\right]$ for simplicity. The adjoint operator $\mathbf{A}_F^*$ satisfying $\langle \mathbf{A}_F X, Y \rangle = \langle X, \mathbf{A}_F^* Y \rangle$ for all $X, Y \in \mathcal{S}_n$ is given by

$$\mathbf{A}_F^* : X \in \mathcal{S}_n \mapsto (A + BMF)'X(A + BMF) + F'[\Sigma^2 \odot (B'XB)]F \in \mathcal{S}_n.$$

By Theorem 3.5 or Lemma 1 in [23], we have that $\{[A|B], \kappa(k)\}$ is mean-square stabilizable if and only if for any $Y > 0$, there exist F and $X > 0$ such that

$$(8) \quad X = \mathbf{A}_F^* X + Y.$$

Define the following set

$$\Gamma = \left\{ X \in \mathcal{S}_n \left| \begin{bmatrix} A'XA + Q - X & A'XB + S \\ B'XA + S' & W \odot (R + B'XB) \end{bmatrix} \geq 0, W \odot (R + B'XB) > 0 \right. \right\}.$$

It is easy to see that a solution to the MARE (4) is a member of Γ . Some other important solutions to the MARE (4) are defined as follows and their relationships with the mean-square stabilizing solution are characterized sequentially.

DEFINITION 4.1. A solution $X \in \mathcal{S}_n$ to the MARE (4) is said to be a maximal solution if $X \geq \tilde{X}$ for all $\tilde{X} \in \Gamma$. It is said to be a strong solution if $\rho(\mathbf{A}_{F(X)}) \leq 1$.

The maximal solution can be numerically computed by solving the following convex optimization problem

$$(9) \quad \begin{aligned} \max \quad & \text{tr}(X) \\ \text{subject to} \quad & X \in \Gamma. \end{aligned}$$

LEMMA 4.2 (see [5]).

- (i) When $\Gamma \neq \emptyset$ and $\{[A|B], \kappa(k)\}$ is mean-square stabilizable, there exists the unique maximal solution $X_+ \in \mathcal{P}_n$ to the MARE (4) and the minimal cost is given by $x'(0)X_+x(0)$. Moreover, X_+ is a strong solution.
- (ii) The mean-square stabilizing solution, if it exists, coincides with the maximal solution.

Now with all the preparation, we are ready to state the main results.

4.1. Scenario I: $W \odot D'D > 0$. The positive definiteness of the input weighting matrix in the cost function is often assumed by researchers. Hence, we first study the case when $W \odot D'D > 0$. Consider a specific state feedback gain

$$F = -M^{-1}[W \odot (D'D)]^{-1}D'C.$$

Then the corresponding cone-invariant system is given by

$$\begin{bmatrix} A_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \\ C_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \end{bmatrix}.$$

The following result, which itself is an interesting result but also plays an essential role in deriving the condition ensuring the existence of a mean-square stabilizing solution, is obtained.

LEMMA 4.3. *Given a solution $X \in \mathcal{P}_n$ to the MARE (4), any unstable distinguished eigenvalue of $A_{F(X)}$ is an unobservable distinguished eigenvalue of this particular cone-invariant system $\begin{bmatrix} A_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \\ C_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \end{bmatrix}$.*

Proof. See section A.3 for the proof. $\square$

By virtue of Lemma 4.3, we can obtain an explicit necessary and sufficient condition ensuring the existence of a mean-square stabilizing solution to the MARE (4).

THEOREM 4.4. *When $W \odot D'D > 0$, a mean-square stabilizing solution to the MARE (4) exists if and only if*

- (i) $\{[A|B], \kappa(k)\}$ is mean-square stabilizable;
- (ii) $\begin{bmatrix} A_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \\ C_{-M^{-1}[W \odot (D'D)]^{-1}D'C} \end{bmatrix}$ has no unobservable distinguished eigenvalue at point 1.

Moreover, if the mean-square stabilizing solution exists, it is unique.

Proof. See section A.4 for the proof. $\square$

Remark 4.5. One of the most important results for the following standard ARE,

$$(10) \quad A'XA - X + C'C - (A'XB + C'D)(B'XB + D'D)^{-1}(B'XA + D'C) = 0,$$

is that a stabilizing solution to ARE (10) exists if and only if

- (i) $[A|B]$ is stabilizable;
- (ii) $\begin{bmatrix} A - e^{j\omega}I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \in [0, 2\pi)$.

When D has full column rank, condition (ii) is equivalent to that

- (iii) the LTI system $\begin{bmatrix} A - B(D'D)^{-1}D'C \\ C - D(D'D)^{-1}D'C \end{bmatrix}$ has no unobservable eigenvalues on the unit circle.

Therefore, the necessary and sufficient condition in Theorem 4.4 for the MARE (4) is an extension of the one for the standard ARE. It indicates that the common assumption or condition of the observability or detectability of certain stochastic or cone-invariant system is unnecessary. Moreover, it shows that the distinguished eigenvalue at 1 of a cone-invariant system plays a similar role to the eigenvalues of an LTI system on the unit circle.

Remark 4.6. The mean-square stabilization involved in our problem can be analytically solvable under a framework called channel/controller codesign. Define the capacity of the i th input channel with random gain as [6]

$$\mathfrak{C}_i \triangleq \frac{1}{2} \log(1 + \text{SNR}_i).$$

The overall mean-square capacity is given by $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$. Clearly, the channel's capacity can be used to measure the information constraint. The channel/controller codesign framework was proposed in [17] to study the stabilization of multi-input networked control systems with the signal-to-error ratio channel model. It is assumed that the overall channel capacity is given and that the controller designer has the freedom to participate in the channel design by allocating the overall channel capacity, as desired, to the individual input channels. It is shown in [23] that under the parallel transmission strategy and the channel/controller codesign framework, $\{[A|B], \kappa(k)\}$ is mean-square stabilizable with capacity $\mathfrak{C}$ if and only if $\mathfrak{C} > h(A)$, where $h(A) = \log \prod_{i=1}^n \max\{1, |\lambda_i(A)|\}$ is the topological entropy [3] of A .

A numerical example is provided below to illustrate Theorem 4.4 and also Remarks 4.5 and 4.6.

Example 4.7. Consider a MARE (4) with

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let the mean and variance of the random input gains $\kappa(k)$ be $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$, respectively. Then $\text{SNR} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $W = \begin{bmatrix} 1.25 & 1 \\ 1 & 2 \end{bmatrix}$. By Remark 4.6 (or referring to [23] for more details), $\{[A|B], \kappa(k)\}$ is mean-square stabilizable since

$$\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 = \frac{1}{2} \log(1+4) + \frac{1}{2} \log(1+1) = \frac{1}{2} \log 10 > \log 2 = h(A),$$

and $[A|B]$ is in the Wonham decomposition form with $\mathfrak{C}_1 > \log 2$ and $\mathfrak{C}_2 > \log 1$.

On the other hand, we get

$$F_0 \triangleq -M^{-1}[W \odot (D'D)]^{-1}D'C = \begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix}$$

and the cone-invariant system $\begin{bmatrix} \mathbf{A}_{F_0} \\ \mathbf{C}_{F_0} \end{bmatrix}$ with

$$\begin{aligned} \mathbf{A}_{F_0} : X \in \mathcal{S}_3 &\mapsto \begin{bmatrix} 2 & -0.8 & 0 \\ 0 & 0.2 & -0.25 \\ 0 & 0 & 0.75 \end{bmatrix} X \begin{bmatrix} 2 & -0.8 & 0 \\ 0 & 0.2 & -0.25 \\ 0 & 0 & 0.75 \end{bmatrix}' + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \odot \left(\begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix} X \begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix}' \right) \right\} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}' \in \mathcal{S}_3, \\ \mathbf{C}_{F_0} : X \in \mathcal{S}_3 &\mapsto \begin{bmatrix} 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} X \begin{bmatrix} 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}' + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \odot \left(\begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix} X \begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix}' \right) \right\} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}' \in \mathcal{S}_2. \end{aligned}$$

It is easy to compute the spectrum of $\mathbf{A}_{F_0}$, i.e., $\{4, 0.4, 1.5, 0.15, 0.2, 0.625\}$, by applying the vectorization operator to $\mathbf{A}_{F_0}$ and making use of the Kronecker product. Since 1 is not an eigenvalue of $\mathbf{A}_{F_0}$, it follows from Theorem 4.4 that a mean-square stabilizing solution to the MARE (4) exists, which coincides with its maximal solution.

By solving the convex optimization problem (9), we get the maximal solution

$$X = \begin{bmatrix} 75.9593 & -13.0253 & 7.6301 \\ -13.0253 & 5.2438 & -2.7106 \\ 7.6301 & -2.7106 & 3.6209 \end{bmatrix}$$

with the associated state feedback gain

$$F(X) = \begin{bmatrix} -1.7776 & 0.0904 & -0.0787 \\ 0.3831 & -0.1528 & -0.2106 \end{bmatrix}.$$

Since $\rho(\mathbf{A}_{F(X)}) < 1$, X is indeed mean-square stabilizing.

It is worth noting that 4 is an unobservable distinguished eigenvalue of $\begin{bmatrix} \mathbf{A}_{F_0} \\ \mathbf{C}_{F_0} \end{bmatrix}$ since the associated eigenvector

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{P}_3 \quad \text{satisfies } \mathbf{C}_{F_0} \begin{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{pmatrix} = 0.$$

By Theorem 3.8, $\begin{bmatrix} \mathbf{A}_{F_0} \\ \mathbf{C}_{F_0} \end{bmatrix}$ is not $\mathcal{P}_3$ -detectable, even though a mean-square stabilizing solution to the MARE (4) exists. This demonstrates that only the distinguished eigenvalue at 1 matters concerning the existence of a mean-square stabilizing solution.

The result for the MARE (4) can be easily generalized to a more general class of MAREs. Consider a MARE of the following form:

$$(11) \quad \begin{aligned} & A'XA - X + Q + \sum_{i=1}^N A'_i X A_i - (A'XB + S + \sum_{i=1}^N A'_i X B_i) \\ & \times (B'XB + R + \sum_{i=1}^N B'_i X B_i)^{-1} (A'XB + S + \sum_{i=1}^N A'_i X B_i)' = 0, \end{aligned}$$

where $A, A_i \in \mathbb{R}^{n \times n}$, $B, B_i \in \mathbb{R}^{n \times m}$ for $i = 1, \dots, N$, and $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \in \mathcal{P}_{n+m}$ with $R > 0$. Such a MARE often arises in the stochastic LQ optimal control problem with multiple multiplicative white noises on both the system state and control input.

The associated cone-invariant operators are given by

$$\begin{aligned} \hat{\mathbf{A}}_F : X \in \mathcal{S}_n &\mapsto (A + BF)X(A + BF)' + \sum_{i=1}^N (A_i + B_i F)X(A_i + B_i F)' \in \mathcal{S}_n, \\ \hat{\mathbf{C}} : X \in \mathcal{S}_n &\mapsto (Q - SR^{-1}S')^{1/2}X(Q - SR^{-1}S')^{1/2} \in \mathcal{S}_n. \end{aligned}$$

Then we have the following result.

COROLLARY 4.8. *There exists a mean-square stabilizing solution to MARE (11) if and only if*

(i) *there exists F such that $\hat{\mathbf{A}}_F$ is mean-square stable;*

(ii) *$\begin{bmatrix} \hat{\mathbf{A}}_{-R^{-1}S'} \\ \hat{\mathbf{C}} \end{bmatrix}$ has no unobservable distinguished eigenvalue at 1.*

Corollary 4.8 is also stated in [21], which extends the result in [26] to a class of MAREs in infinite dimensions.

4.2. Scenario II: $W \odot D'D \geq 0$. In this subsection, we remove the assumption of the positive definiteness of the input weighting matrix. Evidently, Theorem 4.4 does not hold any more. Fortunately, we can still employ the tool of cone-invariant operators to investigate the conditions ensuring the existence of a mean-square stabilizing solution. Some classical results from the theory of standard AREs are also helpful. We recall them first.

Consider the standard ARE (10) when D loses the full column rank. Then the ARE (10) may have no solutions even though the corresponding LQ optimal control problem is solvable. Furthermore, a stabilizing solution to the ARE (10) exists if and only if the LQ optimal control has a unique optimal stabilizing controller. Note that any solution to the ARE (10) is also a solution to the following generalized algebraic Riccati equation (GARE)

$$(12) \quad \begin{aligned} & A'XA - X + C'C - (A'XB + C'D)(B'XB + D'D)^+(B'XA + D'C) = 0, \\ & \ker(B'XB + D'D) \subset \ker(A'XB + C'D). \end{aligned}$$

Denote $T(z) = C(zI - A)^{-1}B + D$ and $T^\sim(z) \triangleq T'(z^{-1})$. Then the so-called Popov function is given by

$$\Phi(z) \triangleq T^\sim(z)T(z) = \begin{bmatrix} (z^{-1}I_n - A)^{-1}B \\ I_n \end{bmatrix}' \begin{bmatrix} C'C & C'D \\ D'C & D'D \end{bmatrix} \begin{bmatrix} (zI_n - A)^{-1}B \\ I_n \end{bmatrix}.$$

An important observation is stated as follows.

LEMMA 4.9 (see [18]). *Let r be the normal rank of $\Phi(z)$.*

- (i) *If $X \in \mathcal{S}_n$ is a solution to the GARE (12), then the rank of $B'XB + D'D$ is equal to r .*
- (ii) *If a solution to the ARE (10) exists, then $\Phi(z)$ must be full normal rank.*

Evidently, statement (ii) is a special case of statement (i). This observation tells a remarkable fact, i.e., either all or none of the solutions to the GARE (12) will satisfy the invertibility of $B'XB + D'D$. Hence, the full normal rank of $\Phi(z)$ is necessary for the existence of a solution to the ARE (10). Moreover, implied by Lemma 2.12 in [18], it is easy to verify that condition (ii) given in Remark 4.5 guarantees the full normal rank of $\Phi(z)$. Thus the necessary and sufficient condition given in Remark 4.5 for the existence of a stabilizing solution to the ARE (10) still holds even though D does not have full column rank.

Note that the weaker assumption $\ker(B) \cap \ker(D) = \{0\}$ is also necessary for the existence of a solution to the ARE (10). Hence it is assumed to hold. Given $X_0 \in \mathcal{S}_n$ such that $B'X_0B + D'D > 0$, we have the following statements.

LEMMA 4.10. $\begin{bmatrix} A - e^{j\omega}I & B \\ C & D \end{bmatrix}$ has full column rank for all ω if and only if

$$\begin{bmatrix} A - B(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \\ C - D(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \end{bmatrix} - e^{j\omega} \begin{bmatrix} I - B(B'X_0B + D'D)^{-1}B'X_0 \\ -D(B'X_0B + D'D)^{-1}B'X_0 \end{bmatrix}$$

has full column rank for all ω .

Proof. See section A.5 for the proof. □

Two special cases are discussed. When D has full column rank, we simply let $X_0 = 0$. Then the latter condition in Lemma 4.10 is equivalent to condition (iii) in Remark 4.5. When $D = 0$, B has to be full column rank. Let $X_0 = I$. Then the

latter condition in Lemma 4.10 is equivalent to that the matrix pair

$$\left[\begin{array}{c} (A - B(B'B)^{-1}B'A, I - B(B'B)^{-1}B') \\ C \end{array} \right]$$

has no unobservable generalized eigenvalue on the unit circle.

Inspired by the above results for the ARE (10), we study the conditions ensuring the existence of the mean-square stabilizing solution to the MARE (4) without the positive definiteness of $W \odot D'D$. Denote

$$\tilde{D} = (W \odot D'D)^{1/2}.$$

We assume that $\ker(B) \cap \ker(\tilde{D}) = \{0\}$.

We start with an extreme case, i.e., $\tilde{D} = 0$ and B has full column rank. Then the MARE (4) has a simple form,

$$(13) \quad A'XA - X + C'C - A'XB[W \odot (B'XB)]^{-1}B'XA = 0.$$

A sufficient condition and a necessary condition are provided, respectively.

THEOREM 4.11. *The MARE (13) has a mean-square stabilizing solution if*

(i) $[A|B, \kappa(k)]$ *is mean-square stabilizable;*

(ii) $\begin{bmatrix} A - e^{j\omega}I & B \\ C & 0 \end{bmatrix}$ *has full column rank for all* ω ,

and only if

(iii) $[A|B, \kappa(k)]$ *is mean-square stabilizable;*

(iv) $\begin{bmatrix} A_{F_0} \\ C_{F_0} \end{bmatrix}$ *has no unobservable distinguished eigenvalue at point 1, where* $F_0 = -M^{-1}(B'B)^{-1}B'A$.

Proof. See section A.6 for the proof. $\square$

Remark 4.12. When the random input gains become deterministic, the sufficient condition is also necessary. When C has full column rank, conditions (ii) and (iv) in Theorem 4.11 are automatically satisfied. Then $[A|B, \kappa(k)]$ is mean-square stabilizable if and only if there exists $P > 0$ such that

$$A'PA - P - A'PB[W \odot (B'PB)]^{-1}B'PA < 0.$$

This mean-square stabilization criterion has been stated in [23].

Now consider a general $\tilde{D}$. Without loss of generality, the system matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be written as

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_1 & 0 \\ C_2 & 0 & 0 \end{array} \right],$$

where B_2 and D_1 have full column rank. A sufficient condition and a necessary condition are provided, respectively.

THEOREM 4.13. *The MARE (4) has a mean-square stabilizing solution if*

(i) $[A|B, \kappa(k)]$ *is mean-square stabilizable;*

(ii) $\Phi(z)$ *has full normal rank;*

(iii) the operator pair

$$\left[\begin{array}{c} (\bar{B}_2 \hat{A}_{F_1}(X) \bar{B}_2, \bar{B}_2 X \bar{B}_2) \\ \left[\begin{array}{c} \hat{C}_{F_1}(X) \\ C_2 X C_2' \end{array} \right] \end{array} \right],$$

where

$$\bar{B}_2 = I - B_2(B_2' B_2)^{-1} B_2', \quad F_1 = -M_1^{-1}(D_1' D_1)^{-1} D_1' C_1,$$

and

$$\begin{aligned} \hat{A}_F : X \in \mathcal{S}_n &\mapsto (A + B_1 M_1 F) X (A + B_1 M_1 F)' + B_1 [\Sigma_1^2 \odot (F X F')] B_1' \in \mathcal{S}_n, \\ \hat{C}_F : X \in \mathcal{S}_n &\mapsto (C_1 + D_1 M_1 F) X (C_1 + D_1 M_1 F)' \in \mathcal{S}_{p_1}, \end{aligned}$$

has no unobservable distinguished generalized eigenvalue at 1,
and only if

(iv) $[A|B, \kappa(k)]$ is mean-square stabilizable;

(v) $\left[\begin{array}{c} A_{F_2} \\ C_{F_2} \end{array} \right]$ has no unobservable distinguished eigenvalue at 1, where

$$F_2 = - \left[\begin{array}{c} M_1^{-1}(D_1' D_1)^{-1} D_1' C_1 \\ M_2^{-1}(B_2' B_2)^{-1} B_2' A \end{array} \right].$$

Proof. See section A.7 for the proof. $\square$

Remark 4.14. Condition (ii) in Theorem 4.13 ensures the existence of a symmetric solution to the MARE (4). When the random input gains become deterministic, the sufficient condition in Theorem 4.13 is also necessary. When D has full column rank, the sufficient condition and necessary condition coincide, which gives the result in Theorem 4.4.

Remark 4.15. We are still interested in finding a necessary and sufficient condition for the MARE in this scenario. It is worth noting that the full column rank of $\left[\begin{array}{c|c} A - e^{j\omega} I & B \\ \hline C & D \end{array} \right]$ is equivalent to that the system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ has no zeros at the points $e^{j\omega}$. Inspired by this, if we can properly define the zeros of a cone-invariant system, we speculate that a necessary and sufficient condition to the MARE (4), without the assumption that $W \odot (D' D) > 0$, exists if and only if $\{[A|B], \kappa(k)\}$ is mean-square stabilizable and an associated cone-invariant system has no zero at point 1. Hence, it is worth studying the zeros of cone-invariant systems in our future work.

5. Conclusion. In this paper, we study the existence of a mean-square stabilizing solution to a discrete-time MARE by employing the tool of cone-invariant operators. When the input weighting matrix of the cost function is positive definite, we can obtain an explicit necessary and sufficient condition, which is analogous to the one for the standard ARE. When the input weighting matrix is only positive semidefinite, we obtain a sufficient condition and a necessary condition, respectively. One natural but challenging question is whether we can also find an explicit necessary and sufficient condition in the latter case. As stated in Remark 4.15, the approach from the zeros of a cone-invariant system may be feasible. This will be the direction for future work.

Appendix A.

A.1. Proof of Theorem 3.5. The equivalence of (i) and (ii) can be easily verified. Here we show the equivalence of (ii) and (iii).

(ii) $\Rightarrow$ (iii): By Theorem 3.3, there exists $Y \in \mathcal{X}_+ \setminus \{0\}$ such that $\mathbf{A}Y = \rho(\mathbf{A})Y$. Then we have $\mathbf{A}Y < Y$ due to $\rho(\mathbf{A}) < 1$. This implies that

$$U = Y - \mathbf{A}Y \in \text{int } \mathcal{X}_+.$$

Hence, the linear iteration

$$X(k+1) = \mathbf{A}X(k) + U$$

with $X(0) = 0$, converges, and the solution $X = \lim_{k \rightarrow \infty} X(k)$ satisfies $X \in \text{int } \mathcal{X}_+$ and $\mathbf{A}X < X$.

(ii) $\Leftarrow$ (iii): Let $Y \in \mathcal{X}_+^* \setminus \{0\}$ be a distinguished eigenvector of the adjoint operator $\mathbf{A}^*$ corresponding to $\rho(\mathbf{A}^*)$ and $X \in \text{int } \mathcal{X}_+$ satisfy that $\mathbf{A}X < X$. Then by the definition of adjoint operator, we have that

$$\begin{aligned} \mathbf{A}X < X &\Rightarrow \langle \mathbf{A}X, Y \rangle < \langle X, Y \rangle \\ &\Rightarrow \langle X, \mathbf{A}^*Y \rangle < \langle X, Y \rangle \\ &\Rightarrow \langle X, \rho(\mathbf{A})Y \rangle < \langle X, Y \rangle \\ &\Rightarrow \rho(\mathbf{A})\langle X, Y \rangle < \langle X, Y \rangle \Rightarrow \rho(\mathbf{A}) < 1, \end{aligned}$$

which completes the proof.

A.2. Proof of Theorem 3.8. We first prove the criterion for cone observability. To show the necessity, suppose that there exists a distinguished eigenvalue $\lambda \geq 0$ with $X \in \mathcal{X}_+ \setminus \{0\}$ such that $\mathbf{A}X = \lambda X$, $\mathbf{C}X = 0$. Let $X(0) = X \neq 0$. Then

$$Y(k) = \mathbf{C}\mathbf{A}^k X = \lambda^k \mathbf{C}X = 0$$

for any $k \geq 0$. Hence $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ is not $\mathcal{X}_+$ -observable, which causes conflict. This shows the necessity.

Now assume that there exists $X(0) = X_0 \in \mathcal{X}_+$ such that $Y(k) = \mathbf{C}\mathbf{A}^k X_0 \equiv 0$ for any $k \geq 0$. It suffices to show that $X_0 = 0$. Denote by S_{X_0} the linear subspace $\text{span}\{X_0, \mathbf{A}X_0, \mathbf{A}^2X_0, \dots\}$. Then $\mathbf{C}X = 0$ for any $X \in S_{X_0}$. By Theorem 2.2. in [20], we have that $\rho(\mathbf{A}|_{S_{X_0}})$, where $\mathbf{A}|_{S_{X_0}}$ denotes the restriction of $\mathbf{A}$ to the invariant subspace S_{X_0} , is a distinguished eigenvalue of $\mathbf{A}$ with a distinguished eigenvector X in $S_{X_0} \cap \mathcal{X}_+$, i.e., $\mathbf{A}X = \rho(\mathbf{A}|_{S_{X_0}})X$. At the same time, $\mathbf{C}X = 0$. In order not to conflict with the cone-observability criterion, S_{X_0} has to be $\{0\}$, which implies $X_0 = 0$. Hence $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ is $\mathcal{X}_+$ -observable.

The proof of the cone-detectability criterion is similar. We only take a look at the sufficiency part. We need to show that $\lim_{k \rightarrow \infty} X(k) = 0$ for $X(0) = X_0 \in \mathcal{X}_+$ satisfying $Y(k) \equiv 0$. In this case, we also have that $\mathbf{A}X = \rho(\mathbf{A}|_{S_{X_0}})X$, $\mathbf{C}X = 0$ for some $X \in S_{X_0} \cap \mathcal{X}_+$. By the cone-detectability criterion, $\rho(\mathbf{A}|_{S_{X_0}})$ should be less than 1. Hence $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ is stable with X_0 . This implies the cone detectability of $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$.

A.3. Proof of Lemma 4.3. Denote

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C'C & C'D \\ D'C & D'D \end{bmatrix}, \tilde{R} \triangleq W \odot R.$$

The MARE (4) can be rewritten as

$$X = \mathbf{A}_{F(X)}^* X + H,$$

where

$$\begin{aligned} H &\triangleq [C + DMF(X)]' [C + DMF(X)] + F'(X)[\Sigma^2 \odot (D'D)]F(X) \\ &= Q - S\tilde{R}^{-1}S' + \left[\tilde{R}^{-1}S' + MF(X) \right]' \tilde{R} \left[\tilde{R}^{-1}S' + MF(X) \right]. \end{aligned}$$

Suppose that $\lambda \geq 1$ is a distinguished eigenvalue of $\mathbf{A}_{F(X)}$, and $Y \geq 0$ is an associated distinguished eigenvector, i.e., $\mathbf{A}_{F(X)}Y = \lambda Y$. Then

$$0 \leq \text{Tr}HY = \text{Tr}(X - \mathbf{A}_{F(X)}^* X)Y = \text{Tr}X(Y - \mathbf{A}_{F(X)}Y) = (1 - \lambda)\text{Tr}XY \leq 0,$$

i.e., $\text{Tr}HY = 0$. Because $\begin{bmatrix} Q & S \\ S' & \tilde{R} \end{bmatrix} \geq 0$ and $\tilde{R} > 0$, it follows that the Schur complement $Q - S\tilde{R}^{-1}S' \geq 0$. Therefore, we have

$$\begin{aligned} \text{Tr}(Q - S\tilde{R}^{-1}S')Y &= 0, \\ \text{Tr} \left[\tilde{R}^{-1}S' + MF(X) \right]' \tilde{R} \left[\tilde{R}^{-1}S' + MF(X) \right] Y &= 0. \end{aligned}$$

Moreover, with the assumption $\tilde{R} > 0$, it holds that

$$\left[\tilde{R}^{-1}S' + MF(X) \right] Y = 0,$$

i.e.,

$$MF(X)Y = -\tilde{R}^{-1}S'Y.$$

It follows that $\lambda Y = \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}(Y)$, i.e., λ is an unstable distinguished eigenvalue of $\mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}$. At the same time,

$$\begin{aligned} \text{Tr}\mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'} &= \text{Tr}\mathbf{E} \left[(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')' \right] \\ &= \text{Tr}\mathbf{E} \left[(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')'(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y \right] \\ &= \text{Tr}(Q - S\tilde{R}^{-1}S')Y = 0, \end{aligned}$$

which implies $\mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'} = 0$. Therefore, λ is an unobservable distinguished eigenvalue of $\begin{bmatrix} \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'} \\ \mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'} \end{bmatrix}$.

A.4. Proof of Theorem 4.4. For simplicity, denote

$$\begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \triangleq \begin{bmatrix} A'XA + C'C - X & A'XB + C'D \\ B'XA + D'C & B'XB + D'D \end{bmatrix}, \quad \tilde{\mathcal{N}}(X) \triangleq W \odot \mathcal{N}(X).$$

We start with the “if” part. By Lemma 4.2, the maximal solution X_+ exists and $\rho(\mathbf{A}_{F(X_+)}) \leq 1$. Because $\lambda = 1$ is not an unobservable distinguished eigenvalue of $\begin{bmatrix} \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'} \\ \mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'} \end{bmatrix}$, by Lemma 4.3, $\rho(\mathbf{A}_{F(X_+)}) < 1$. Therefore, the maximal solution is exactly the mean-square stabilizing solution.

Now take a look at the “only if” part. Evidently, the mean-square stabilizability of $\{[A|B], \kappa(k)\}$ is necessary. Suppose that $\begin{bmatrix} \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'} \\ \mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'} \end{bmatrix}$ has an unobservable distinguished eigenvalue at 1, i.e., there exists $Y \geq 0$ such that

$$\mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}Y = Y, \mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'}Y = 0.$$

We show that in this case, for any $X \in \mathcal{P}_n$ satisfying the MARE (4), it is not a mean-square stabilizing solution. Denote

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T'_{12} & T_{22} \end{bmatrix} \triangleq \begin{bmatrix} I & -S\tilde{R}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \tilde{\mathcal{N}}(X) \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{R}^{-1}S' & I \end{bmatrix}.$$

Hence,

$$\begin{aligned} T_{11} &= \mathcal{M}(X) - S\tilde{R}^{-1}\mathcal{L}'(X) - \mathcal{L}(X)\tilde{R}^{-1}S' + S\tilde{R}^{-1}\tilde{\mathcal{N}}(X)\tilde{R}^{-1}S' \\ &= \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}^*X - X + Q - S\tilde{R}^{-1}S', \\ T_{12} &= \mathcal{L}(X) - S\tilde{R}^{-1}\tilde{\mathcal{N}}(X), \\ T_{22} &= \tilde{\mathcal{N}}(X). \end{aligned}$$

It follows that

$$\begin{aligned} \text{Tr}(Q - S\tilde{R}^{-1}S')Y &= \text{Tr} \mathbf{E} \left[(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')'(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y \right] \\ &= \text{Tr} \mathbf{E} \left[(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')' \right] \\ &= \text{Tr} \mathbf{C}_{-M^{-1}\tilde{R}^{-1}S'}Y = 0, \\ \text{Tr}T_{11}Y &= \text{Tr}[\mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}^*X - X]Y + \text{Tr}(Q - S\tilde{R}^{-1}S')Y \\ &= \text{Tr}X[\mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}Y - Y] \\ &= 0, \end{aligned}$$

and

$$T_{22}^{-1}T'_{12} = -MF(X) - \tilde{R}^{-1}S'.$$

Since it holds that

$$\begin{aligned} &\mathcal{M}(X) - \mathcal{L}(X)\tilde{\mathcal{N}}(X)^{-1}\mathcal{L}'(X) \\ &= \begin{bmatrix} I \\ -\tilde{\mathcal{N}}(X)^{-1}\mathcal{L}'(X) \end{bmatrix}' \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \tilde{\mathcal{N}}(X) \end{bmatrix} \begin{bmatrix} I \\ -\tilde{\mathcal{N}}(X)^{-1}\mathcal{L}'(X) \end{bmatrix} \\ &= \begin{bmatrix} I \\ -T_{22}^{-1}T'_{12} \end{bmatrix}' T \begin{bmatrix} I \\ -T_{22}^{-1}T'_{12} \end{bmatrix} \\ &= T_{11} - T_{12}T_{22}^{-1}T'_{12} \end{aligned}$$

for any solution $X \in \mathcal{P}_n$ to the MARE (4), we have that

$$\begin{aligned} 0 &= \text{Tr}[\mathcal{M}(X) - \mathcal{L}(X)\tilde{\mathcal{N}}(X)^{-1}\mathcal{L}'(X)]Y \\ &= \text{Tr}(T_{11} - T_{12}T_{22}^{-1}T'_{12})Y \\ &= -\text{Tr}T_{12}T_{22}^{-1}T'_{12}Y, \end{aligned}$$

which implies $T_{12}T_{22}^{-1}T'_{12}Y = 0$. It follows that $T'_{12}Y = 0$ as $T_{22} > 0$ and $Y \geq 0$. Therefore, $T_{22}^{-1}T'_{12}Y = 0$, which directly implies that $MF(X)Y = -\tilde{R}^{-1}S'Y$. Hence,

$$\mathbf{A}_{F(X)}(Y) = \mathbf{A}_{-M^{-1}\tilde{R}^{-1}S'}Y = Y,$$

i.e., 1 is a distinguished eigenvalue of $\mathbf{A}_{F(X)}$, and thus $\rho(\mathbf{A}_{F(X)}) \geq 1$. Therefore the closed-loop system (7) is not mean-square stabilizing with the controller $F(X)$. This shows the necessity.

A.5. Proof of Lemma 4.10. We start with the necessity part. Suppose that there exist ω and $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$\begin{bmatrix} A - B(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \\ C - D(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \end{bmatrix} x = e^{j\omega} \begin{bmatrix} I - B(B'X_0B + D'D)^{-1}B'X_0 \\ -D(B'X_0B + D'D)^{-1}B'X_0 \end{bmatrix} x.$$

Then we have

$$\begin{aligned} (A - e^{j\omega})x &= B[(B'X_0B + D'D)^{-1}(B'X_0A + D'C) - e^{j\omega}(B'X_0B + D'D)^{-1}B'X_0]x, \\ Cx - D[(B'X_0B + D'D)^{-1}(B'X_0A + D'C) - e^{j\omega}(B'X_0B + D'D)^{-1}B'X_0]x &= 0. \end{aligned}$$

Let

$$y = -[(B'X_0B + D'D)^{-1}(B'X_0A + D'C) - e^{j\omega}(B'X_0B + D'D)^{-1}B'X_0]x.$$

Then we have

$$\begin{aligned} (A - e^{j\omega})x + By &= 0, \\ Cx + Dy &= 0, \end{aligned}$$

which is in contradiction with the full column rank of $\begin{bmatrix} A - e^{j\omega}I & B \\ C & D \end{bmatrix}$. This shows the necessity.

Next take a look at the sufficiency part. Suppose there exist ω and a nonzero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $\begin{bmatrix} A - e^{j\omega}I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$, i.e., $\begin{bmatrix} A - e^{j\omega}I \\ C \end{bmatrix} x = -\begin{bmatrix} B \\ D \end{bmatrix} y$. Here $x \neq 0$, otherwise we have $y = 0$.

Premultiplying the above equation by $\begin{bmatrix} B'X_0 & D' \end{bmatrix}$ gives

$$(B'X_0A - e^{j\omega}BX_0 + D'C)x = -(B'X_0B + D'D)y.$$

Therefore

$$y = -(B'X_0B + D'D)^{-1}(B'X_0A + D'C)x + e^{j\omega}(B'X_0B + D'D)^{-1}BX_0x.$$

This gives

$$\begin{bmatrix} A - B(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \\ C - D(B'X_0B + D'D)^{-1}(B'X_0A + D'C) \end{bmatrix} x = e^{j\omega} \begin{bmatrix} I - B(B'X_0B + D'D)^{-1}B'X_0 \\ -D(B'X_0B + D'D)^{-1}B'X_0 \end{bmatrix} x,$$

which causes a contradiction. This shows the sufficiency.

A.6. Proof of Theorem 4.11. We first prove the “if” part. Implied by the mean-square stabilizability of $[A|B, \kappa(k)]$, we know that $[A|B]$ is stabilizable. Combining with the full column rank of $\begin{bmatrix} A - e^{j\omega} I & B \\ C & 0 \end{bmatrix}$ for all ω , a stabilizing solution to

$$A'XA - X + C'C - A'XB(B'XB)^{-1}B'XA = 0$$

exists. Therefore the symmetric solution set to

$$\begin{bmatrix} A'XA - X + C'C & A'XB \\ B'XA & B'XB \end{bmatrix} \geq 0, \quad B'XB > 0,$$

is nonempty. Hence the symmetric solution set to

$$(14) \quad \begin{bmatrix} A'XA - X + C'C & A'XB \\ B'XA & W \odot (B'XB) \end{bmatrix} \geq 0, \quad W \odot (B'XB) > 0,$$

is also nonempty. By Lemma 4.2, the mean-square stabilizability of $[A|B, \kappa(k)]$ ensures the existence of a maximal solution X_+ to the LMI (14), which is a strong solution to the MARE (13).

Let $F = F(X_+)$. Then $\rho(\mathbf{A}_F) \leq 1$. Now suppose that $\rho(\mathbf{A}_F) = 1$, i.e., there exists $Y \in \mathcal{P}_n$ such that $\mathbf{A}_F(Y) = Y$. It follows that

$$\begin{aligned} 0 &\leq \text{Tr}CYC' = \text{Tr}C'CY \\ &= \text{Tr}[X_+ - \mathbf{A}_F^*(X_+)]Y = \text{Tr}X_+[Y - \mathbf{A}_F(Y)] = 0, \end{aligned}$$

i.e., $CYC' = 0$. Let $\bar{B} = I - B(B'B)^{-1}B'$. Then

$$\bar{B}B = [I - B(B'B)^{-1}B']B = 0.$$

This implies that

$$\begin{aligned} \bar{B}AY A' \bar{B} &= \bar{B}[(A + BMF)Y(A + BMF)' + B(\Sigma^2 \odot FYF')B'] \bar{B} \\ &= \bar{B}\mathbf{A}_F(Y)\bar{B} = \bar{B}Y\bar{B}. \end{aligned}$$

Then by Lemma 4.10, we have that $\begin{bmatrix} A - e^{j\omega} I & B \\ C & 0 \end{bmatrix}$ loses full column rank at point 1, which causes a contradiction with condition (ii). Hence $\rho(\mathbf{A}_F) < 1$, i.e., X_+ is mean-square stabilizing.

The proof of the “only if” part basically follows the same idea as that of Theorem 4.4 excepting that the MARE (13) can be rewritten as

$$X = \mathbf{A}_{F(X)}^* X + C'C,$$

and the matrix T is defined as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}' & T_{22} \end{bmatrix} \triangleq \begin{bmatrix} I & F_0' M \\ 0 & I \end{bmatrix} \begin{bmatrix} A'XA - X + C'C & A'XB \\ B'XA & W \odot (B'XB) \end{bmatrix} \begin{bmatrix} I & 0 \\ MF_0 & I \end{bmatrix}.$$

A.7. Proof of Theorem 4.13. We first prove the “if” part. The mean-square stabilizability of $[A|B, \kappa(k)]$ and full normal rank of $\Phi(z)$ ensures that the symmetric solution set to

$$\begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \geq 0, \mathcal{N}(X) > 0,$$

is nonempty. Hence the symmetric solution set to

$$\begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \tilde{\mathcal{N}}(X) \end{bmatrix} \geq 0, \tilde{\mathcal{N}}(X) > 0,$$

is also nonempty.

Furthermore, the mean-square stabilizability of $[A|B, \kappa(k)]$ ensures the existence of a strong solution X_+ to the MARE (4). Let $F = F(X_+)$. Suppose $\rho(\mathbf{A}_F) = 1$, i.e., there exists $Y \in \mathcal{P}_n$ such that $\mathbf{A}_F(Y) = Y$. Let $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$. By some computations, we have

$$\text{Tr}(C + DMF)'(C + DMF)Y = 0.$$

This gives

$$\begin{aligned} 0 &= \text{Tr} \begin{bmatrix} (C_1 + D_1 M_1 F_1)' & C_2' \\ C_2 & \end{bmatrix} \begin{bmatrix} C_1 + D_1 M_1 F_1 \\ C_2 \end{bmatrix} Y \\ &= \text{Tr}(C_1 + D_1 M_1 F_1)'(C_1 + D_1 M_1 F_1)Y + C_2' C_2 Y \\ &= \text{Tr} C_2 Y C_2' + \text{Tr}(Q_1 - S_1 R_1^{-1} S_1')Y + \text{Tr}[(R_1^{-1} S_1' + M_1 F_1)' R_1 (R_1^{-1} S_1' + M_1 F_1)]Y, \end{aligned}$$

where $\begin{bmatrix} Q_1 & S_1 \\ S_1' & R_1 \end{bmatrix} = \begin{bmatrix} C_1' \\ D_1' \end{bmatrix} \begin{bmatrix} C_1 & D_1 \end{bmatrix}$. It follows that

$$\begin{aligned} C_2 Y C_2 &= 0, \\ (Q_1 - S_1 R_1^{-1} S_1')Y &= 0, \\ (R_1^{-1} S_1' + M_1 F_1)Y &= 0. \end{aligned}$$

Since $M_1 F_1 Y = -R_1^{-1} S_1' Y$ and $\bar{B}_2 B_2 = 0$, we have that

$$\begin{aligned} \bar{B}_2 Y \bar{B}_2 &= \bar{B}_2 \mathbf{A}_F(Y) \bar{B}_2 \\ &= \bar{B}_2 \left\{ (A + B_1 M_1 F_1 + B_2 M_2 F_2)Y (A + B_1 M_1 F_1 + B_2 M_2 F_2)' \right. \\ &\quad \left. + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \Sigma_1^2 \odot (F_1 Y F_1') & 0 \\ 0 & \Sigma_2^2 \odot (F_2 Y F_2') \end{bmatrix} \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \right\} \bar{B}_2 \\ &= \bar{B}_2 \{ (A + B_1 M_1 F_1)Y (A + B_1 M_1 F_1)' + B_1 [\Sigma_1^2 \odot (F_1 Y F_1')] B_1' \} \bar{B}_2 \\ &= \bar{B}_2 \hat{\mathbf{A}}_{F_1}(Y) \bar{B}_2. \end{aligned}$$

At the same time,

$$\begin{aligned} \text{Tr} \hat{\mathcal{C}}_{F_1}(Y) &= \text{Tr}(C_1 - D_1 R_1^{-1} S_1')Y (C_1 + D_1 R_1^{-1} S_1')' \\ &= \text{Tr}(C_1 + D_1 R_1^{-1} S_1')'(C_1 - D_1 R_1^{-1} S_1')Y \\ &= \text{Tr}(Q_1 - S_1 R_1^{-1} S_1')Y = 0, \end{aligned}$$

which implies

$$\hat{\mathcal{C}}_{F_1}(Y) = 0.$$

Therefore, 1 is an unobservable generalized distinguished eigenvalue of

$$\begin{bmatrix} (\bar{B}_2 \hat{A}_{F_1}(X) \bar{B}_2, \bar{B}_2 X \bar{B}_2) \\ \begin{bmatrix} \hat{\mathcal{C}}_{F_1}(X) \\ C_2 X C_2' \end{bmatrix} \end{bmatrix}$$

with a generalized distinguished eigenvector Y . This causes a contradiction. Hence $\rho(\mathbf{A}_F) < 1$, i.e., X_+ is mean-square stabilizing.

The proof of the “only if” part follows the same idea as that of Theorem 4.4. One only needs to pay attention to that in this case, the matrix T is defined as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}' & T_{22} \end{bmatrix} \triangleq \begin{bmatrix} I & F_2' M \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \tilde{\mathcal{N}}(X) \end{bmatrix} \begin{bmatrix} I & 0 \\ M F_2 & I \end{bmatrix}.$$

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